Assignment - 2

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(1) It is given that $f(x)$ is the consumption rate at a given time x. The function $f: [-12, 12] \to [0, \infty)$ is modelled as a polynomial as:

$$\boxed{f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0} \quad (1)$$

(a) Firstly, it is mentioned that $f(12) = f(-12)$ as both corresspending to the consumption rate at 3 am
Using this fact and equation (1), we get

$$f(12) = a_4 (12)^4 + a_3 (12)^3 + a_2 (12)^2 + a_1 (12) + a_0$$
$$= a_4(-12)^4 + a_3(-12)^3 + a_2(-12)^2 + a_1(-12) + a_0$$
$$= f(-12)$$

Further simplifying, we get

$$\Rightarrow a_4(12)^4 - a_4(12)^4 + a_3(12)^3 + a_3(12)^3 + a_2(12)^2 - a_2(12)^2 + 12a_1 + 12a_1 + a_0 - a_0 = 0$$

$$\Rightarrow 2(12)^3 a_3 + 2(12) a_1 = 0$$

$$\Rightarrow \boxed{144 a_3 + a_1 = 0} \quad (2)$$

Secondly, it is mentioned that 9 am (ie $x = -6$) and 9 pm ($x = 6$) are local maxima of consumption rate
Since, at local maxima, $f'(x) = 0$, (Using definition of stationary points)
we can write that $f'(6) = 0$
and $f'(-6) = 0$

The above statement holds true only if $f'(x)$ exist and $f(x)$ is modelled as a polynomial, the derivative of $f(x)$ also exist in its domain $[-12, 12]$

Therefore, Using basic derivative rules and scalar and sum rule of derivatives, we get

$$f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$$

[If $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$
If $f(x) = c\,g(x)$, $f'(x) = c\,g'(x)$]

Therefore,
$f'(6) = 0$
implies $4(6)^3a_4 + 3(6)^2a_3 + 2(6)a_2 + a_1 = 0$

$$\Rightarrow \boxed{864a_4 + 108a_3 + 12a_2 + a_1 = 0} \quad \text{— (3)}$$

and $f'(-6) = 0$
implies $4(-6)^3a_4 + 3(-6)^2a_3 + 2(-6)a_2 + a_1 = 0$

$$\Rightarrow \boxed{-864a_4 + 108a_3 - 12a_2 + a_1 = 0} \quad \text{— (4)}$$

Thirdly, it is given that total energy consumption during the day equals 24 and it can be written as a definite integral of consumption rates from $x = -12$ to $x = 12$.

Therefore
$$\int_{-12}^{12} f(x)\,dx = 24$$

For finding the above integral, lets find the antiderivative of $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

So, antiderivative $F(x) = \frac{a_4x^5}{5} + \frac{a_3x^4}{4} + \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x$

[Using the sum and scalar rule of integration as well as basic integration of x^n] → If $f(x) = g(x) + h(x)$ then, $F(x) = G(x) + H(x) + C$
If $f(x) = c\,g(x)$, then $\boxed{F(x) = c\,G(x) + C_1}$

2

Using the Fundamental Theorem of Calculus

$$\int_{-12}^{12} f(x)\,dx = F(12) - F(-12) = 24$$

which implies,

$$\Rightarrow \frac{a_4(12)^5}{5} + \frac{a_3(12)^4}{4} + \frac{a_2(12)^3}{3} + \frac{a_1(12)^2}{2} + a_0(12)$$
$$- \frac{a_4(-12)^5}{5} - \frac{a_3(-12)^4}{4} - \frac{a_2(-12)^3}{3} - \frac{a_1(-12)^2}{2} - a_0(-12)$$
$$= 24$$

further simplifying, we get,

$$\Rightarrow \frac{2(12)^5 a_4}{5} + \frac{2(12)^3 a_2}{3} + 24a_0 = 24$$

$$\Rightarrow \frac{20736a_4}{5} + \frac{144}{3}a_2 + a_0 = 1$$

$$\Rightarrow \boxed{\frac{20736}{5}a_4 + 48a_2 + a_0 = 1} \quad (5)$$

$\therefore$ Linear system of equations will be:

$$\begin{cases} 144a_3 + a_1 = 0, \\ 864a_4 + 108a_3 + 12a_2 + a_1 = 0, \\ -864a_4 + 108a_3 - 12a_2 + a_1 = 0. \\ \frac{20736}{5}a_4 + 48a_2 + a_0 = 1 \end{cases}$$

(1)(b) Writing the above system of equations found in part (a) in matrix form, we get

$$\begin{bmatrix} 0 & 144 & 0 & 1 & 0 \\ 864 & 108 & 12 & 1 & 0 \\ -864 & 108 & -12 & 1 & 0 \\ 4147.2 & 0 & 48 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Using Gaussian elimination methods,
Applying row transformations

$$R_2 + R_3 \rightsquigarrow R_2$$

$$\begin{bmatrix} 0 & 144 & 0 & 1 & 0 \\ 0 & 216 & 0 & 2 & 0 \\ -864 & 108 & -12 & 1 & 0 \\ 4147.2 & 0 & 48 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Performing $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} -864 & 108 & -12 & 1 & 0 \\ 0 & 216 & 0 & 2 & 0 \\ 0 & 144 & 0 & 1 & 0 \\ 4147.2 & 0 & 48 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Performing $\frac{-1R_1}{864} \to R_1$

$$\begin{bmatrix} 1 & -1/8 & 1/72 & -1/864 & 0 \\ 0 & 216 & 0 & 2 & 0 \\ 0 & 144 & 0 & 1 & 0 \\ 4147.2 & 0 & 48 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Performing $R_4 - \frac{(12)^5}{5} R_1 \to R_4$

$$\begin{bmatrix} 1 & -1/8 & 1/72 & -1/864 & 0 \\ 0 & 216 & 0 & 2 & 0 \\ 0 & 144 & 0 & 1 & 0 \\ 0 & 518.4 & -9.6 & 4.8 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(5)

Performing $\frac{1}{216}R_2 \rightarrow R_2$

$$\begin{bmatrix} 1 & -1/8 & 1/72 & -1/864 & 0 \\ 0 & 1 & 0 & 1/108 & 0 \\ 0 & 144 & 0 & 1 & 0 \\ 0 & 518.4 & -9.6 & 4.8 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Performing $R_3 - 144R_2 \rightsquigarrow R_3$

$$\begin{bmatrix} 1 & -1/8 & 1/72 & -1/864 & 0 \\ 0 & 1 & 0 & 1/108 & 0 \\ 0 & 0 & 0 & -1/3 & 0 \\ 0 & 518.4 & -9.6 & 4.8 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Performing $R_4 - 518.4R_2 \rightsquigarrow R_4$

$$\begin{bmatrix} 1 & -1/8 & 1/72 & -1/864 & 0 \\ 0 & 1 & 0 & 1/108 & 0 \\ 0 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & -9.6 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore from the above matrix, we get that a_4 be a free variable and there are infinitely many solutions

Also, $a_4 - \frac{1}{8}a_3 + \frac{1}{72}a_2 - \frac{1}{864}a_1 = 0$

$$a_3 + \frac{1}{108}a_1 = 0$$

$$-\frac{1}{3}a_1 = 0 \implies \text{implies } \boxed{a_1 = 0}$$

$$-9.6a_2 + a_0 = 1$$

Putting $a_1 = 0$ in ~~first~~ second equation, we get

$\cancel{a_4 - \frac{1}{8}a_3 + \frac{1}{72}a_2 = 0}$ ~~and~~ $\boxed{a_3 = 0}$

Now, putting $a_1 = a_3 = 0$ in first equation, we get,

$$a_4 + \frac{1}{72} a_2 = 0 \Rightarrow \boxed{a_2 = -72 a_4}$$

Putting $a_2 = -72 a_4$ in last equation, we get

$$-9.6 \times -72 a_4 + a_0 = 1$$

$$\boxed{a_0 = 1 - 691.2 a_4}$$

The solution set for the given system of linear equations is

$$\left\{ \begin{pmatrix} K \\ 0 \\ -72K \\ 0 \\ 1 - 691.2K \end{pmatrix} : K \in \mathbb{R} \right\} \quad \left[\begin{matrix} \text{Taking} \\ a_4 = K \end{matrix}\right]$$

(1)(c) Given function is

$$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 .$$

Substituting the values of a_4, a_3, a_2, a_1 and a_0 in terms of K in the above equation, we get a function g which is a function of x and K.

$$g(x, K) = \cancel{Kx^4} \; Kx^4 + 0x^3 - 72Kx^2 + 0x + 1 - 691.2K$$

$$g(x, K) = Kx^4 - 72Kx^2 - 691.2K + 1$$

Now, the first data given is $f(12) = f(-12)$ and codomain of $f(x)$ is $[0, \infty)$ with domain $[-12, 12]$

which implies that

$$g(12, K) = g(-12, K) \quad \text{and} \quad \begin{matrix} g(12, K) \geq 0 \\ g(-12, K) \geq 0 \end{matrix}$$

From the equality $g(12,k) = g(-12,k)$,
we have

$$12^4 k - 12^2 \times 72k - 691.2k + 1$$
$$= (-12)^4 k - (-12)^2 \times 72k - 691.2k + 1$$

which is satisfied for all $k \in \mathbb{R}$.

It is also given that $x = 6$ and $x = -6$ are the local maxima for the function $f(x)$.

which gives that

$f'(x)$ at $x = 6$ and $x = -6$ should be zero.

Here, $f'(x) = 4kx^3 - 144kx = 0$ which has $x = 6$ and -6 as its solutions

So, $f'(6) = 0$ and $f'(-6) = 0$ satisfies for all $k \in \mathbb{R}$.

Now, since we have the fact that $x = 6$ & -6 are local maxima for $f(x)$, by second derivative test, we know that $f''(x) < 0$.

Here $f''(x) = 12kx^2 - 144k$

So, $f''(6) = 12k(6)^2 - 144k < 0$

which implies, $432k - 144k < 0$

$\Rightarrow 288k < 0$

$\Rightarrow \boxed{k < 0}$

and from $f''(-6) = 12k(-6)^2 - 144k < 0$

$\Rightarrow 12k(6)^2 - 144k < 0$

which gives the same condition that $\boxed{k < 0}$.

Also, we are given that the ~~range~~ codomain of function $f(x)$ is $[0, \infty)$ which implies that

$f(x) \geq 0$ for the domain of x as $[-12, 12]$

$(0 \leq x^2 \leq 144)$

So, $(kx^4 - 72kx^2 - 691.2k + 1) \geq 0$

$K(x^4 - 72x^2 - 691.2) + 1 \geq 0$

$$K(x^2 - 80572)(x^2 + 8.578) + 1 \geq 0 \quad \text{——} (*')$$

(The roots here are approximate)

For $0 \leq x^2 \leq 80.572$,

The value for $(x^2 - 80.572)(x^2 + 8.578) \cancel{\geq} \leq 0$

and we already know that $k < 0$,

So, the above condition in $(*')$ holds true for all $K < 0$.

and For $80.572 < x^2 \leq 144$,

The value for $(x^2 - 80.572)(x^2 + 8.578) > 0$

Thus, the above condition in $(*')$ can be written as

$$K \geq \frac{-1}{(x^2 - 80.572)(x^2 + 8.578)}$$

or $$K \geq \frac{-1}{x^4 - 72x^2 - 691.2}$$

For ~~the~~ the domain of x as $[-12, 12]$,

the maximum value for the right hand side of this inequality would be at $x = 12$ and $x = -12$

So, $K \geq \dfrac{-1}{(12)^4 - 72(12)^2 - 691.2}$

$$\boxed{K \geq \frac{-1}{9676.8}}$$

Also, the function (9) already satisfies the integral condition-

$$\int_{-12}^{12} f(x)\,dx = 24.$$

Thus, the function required is

$$f(x,K) = Kx^4 - 72Kx^2 - 691.2K + 1$$

where $x \in [-12, 12]$,

$$\frac{-1}{9676.8} \leq K < 0$$

and ~~range~~ codomain of the function is $[0, \infty)$.

(1)(d) We are given to check whether there is a possibility for total energy consumed from 1 am ($x = 10$) to 5 am ($x = -10$) or broken as 1 am to 3 am ($x = 10$ to 12) and 3 am to 5 am ($x = -12$ to -10).

∴ Total energy consumed for the given time will be

$$T = \int_{10}^{12} f(x,k)\,dx + \int_{-12}^{-10} f(x,k)\,dx$$

Here, the antiderivative of $f(x,k)$ is

$$F(x,k) = \frac{Kx^5}{5} - \frac{72Kx^3}{3} - 691.2Kx + x$$

So,

$$T = \frac{K(12)^5}{5} - \frac{72K(12)^3}{3} - 691.2K \times 12 + 12 - \frac{K(10)^5}{5} + \frac{72K(10)^3}{3}$$

$$+ 691.2k \times 10 - 10 + \frac{k(-10)^5}{5} - \frac{72k(-10)^3}{3}$$

$$- 691.2k \times (-10) + (-10) - \frac{k(-12)^5}{5} + \frac{72k(-12)^3}{3}$$

$$+ 691.2k(-12) - (-12)$$

Simplying, we get,

$$T = \frac{2k(12)^5}{5} - \frac{2 \times 72k(12)^3}{3} - 2 \times 691.2k \times 12 + 24$$

$$- \frac{2k(10)^5}{5} + \frac{2 \times 72k(10)^3}{3} + 2 \times 691.2k \times 10 - 20$$

$$\Rightarrow T = \frac{297664k}{5} - \frac{1456k \times 72}{3} - 2764.8k + 4$$

So, For the domain of K found in (1)(c) that

$$\frac{-1}{9676.8} \leq k < 0,$$

We deduce that the total energy consumed from 1 am to 5 am can be smaller than 2 for $f(x,k) = kx^4 - 72kx^2 - 691.2k + 1$ such that

$$\boxed{\frac{-1}{9676.8} \leq k < \frac{-1}{10912}}$$

We can take any example say $\boxed{k = -1/9676.8}$.