Generalizations on the Colonel Blotto Game

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Abstract

In this paper we characterize the equilibrium in a relaxed version of the Colonel Blotto game in which battlefield valuations maybe heterogeneous across battlefields and asymmetric across players and the, possibly asymmetric, budget constraints hold only on average, and then show how this characterization partially extends to the full Colonel Blotto game, in which the budget constraints must be satisfied with probability one. For the non-constant-sum generalization of the Colonel Blotto game examined, we find that there exist non-pathological parameter configurations with multiple, payoff inequivalent, equilibria.

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1 Introduction

The Colonel Blotto game is a two-player resource allocation game in which each player is endowed with a level of resources to allocate across a set of battlefields, within each battlefield the player that allocates the higher level of resources wins the battlefield, and each player's payoff is the sum of the valuations of the battlefields won. This simple game, that originates with Borel (1921), illustrates the fundamental strategic considerations that arise in multi-dimensional resource allocation competition such as: political campaign resource allocation, research and development competition where innovation involves obtaining a collection of interrelated patents, attack and defense of a collection of targets, etc. In this paper we examine a general formulation of the Colonel Blotto game in which battlefield valuations maybe heterogeneous across battlefields and asymmetric across players, and the players may face asymmetric resource constraints. We characterize the equilibrium in a relaxed version of the Colonel Blotto game in which budgets must only be satisfied on average and then, show how this characterization extends, over a subset of the parameter space, to the full Colonel Blotto game with budget constraints that hold with probability one. Unlike existing constant-sum formulations of the Colonel Blotto game, we find that uniqueness of the equilibrium sets of univariate marginal distributions does not extend to the generalized (non-constant-sum) version of the game examined here, where we show that there exist non-pathological parameter configurations with multiple, payoff inequivalent, equilibria.

There are a number of notable formulations of Blotto-type games including: Friedman (1954) which introduces a version of the game with the lottery contest success function,[1] Myerson (1993) which examines a political economy application of the game with a continuum of battlefields (and a relaxed budget constraint),[2] and Hart (2008) which introduces a version of the game in which resource allocations are restricted to integers,[3] but our focus here is on the branch of the Colonel Blotto literature that assumes an auction contest success function, a finite number of battlefields, and resource endowments that are continuously divisible. Within this branch, we further limit our scope to the case that resources are use-it-or-lose-it in the sense that unused resources have no value (i.e. the per unit cost of allocating resources is 0 up to the budget constraint)[4] and that each player's payoff is the sum

[1]For a more recent treatment of this game, see Robson (2005).

[2]Related political economy applications include Lizzeri and Persico (2001, 2005), Sahuguet and Persico (2006), Roberson (2008), and Crutzen and Sahuguet (2009). See also Bell and Cover (1980) and Washburn (2013) for a relaxed budget constraint version of the game.

[3]See also Hortala-Vallve and Llorente-Saguer (2012) and Dziubiński (2013).

[4]For alternative cost functions see Kvasov (2007) and Roberson and Kvasov (2012).

of the battlefield valuations in the battlefields won.[5] This particular slice of the literature is summarized in the following table, where the vertical axis specifies the type of objective and the horizontal axis denotes the cost structure. With regards to objective type, linear pure count refers to games in which each players' payoff is the sum of the battlefield valuations in the battlefields won, where battlefield valuations are homogeneous across battlefields and symmetric across players, so that each player's payoff is linear in the number, or pure count, of battlefields won. Linear heterogeneous symmetric (asymmetric) is similar, except that battlefield valuations are now heterogeneous across battlefields but symmetric (asymmetric) across players, and each player's payoff is equal to the sum of the battlefield valuations in the battlefields won.

Costs → Objective ↓	Symmetric Budget, Use-it-or-lose-it Resources	Asymmetric Budget, Use-it-or-lose-it Resources
Linear Pure Count	<u>Continuous</u> -Borel and Ville (1938) (n = 3) -Gross and Wagner (1950) (n ≥ 2) -Weinstein (2012) (n ≥ 3) <u>Discrete</u> -Hart (2008)	<u>Continuous</u> -Gross and Wagner (1950) (n=2) -Macdonell and Mastronardi (2013) (n = 2) -Roberson (2006) (n ≥ 3) <u>Discrete</u> -Hart (2008) (partial result)
Linear Heterogeneous Symmetric	<u>Continuous</u> -Gross (1950) -Laslier (2002) -Thomas (2012) <u>Discrete</u> -Hortala-Vallve and Llorente-Saguer (2012) (partial result)	<u>Continuous</u> ✔ (Partial Result)
Linear Heterogeneous Asymmetric	<u>Continuous</u> ✔ (Partial Result) <u>Discrete</u> -Hortala-Vallve and Llorente-Saguer (2012) (partial result)	<u>Continuous</u> ✔ (Partial Result)

Table 1: Blotto Game Variations with Linear Pure Count and Use-it-or lose it Resources

[5]For alternative definitions of success see Szentes and Rosenthal (2003a, 2003b), Golman and Page (2009), Kovenock and Roberson (2010), Tang, Shoham, and Lin (2010), and Rinott, Scarsini, and Yu (2012).

As shown in Table 1, this paper provides a partial result for each of the three checked cells: linear heterogeneous symmetric objective with asymmetric budget constraints, and linear heterogeneous asymmetric objective with both symmetric and asymmetric budget constraints. We also provide a complete characterization of the relaxed versions of these games, where the budget constraint must only hold on average. In discussing the literature in the three remaining cells of Table 1, we begin with a brief review of equilibrium in the linear pure-count game with symmetric budgets and $n = 3$ and the extension of this approach to $n > 3$ (top row of the symmetric budget column of Table 1), then look at how this approach can be modified for the linear heterogeneous symmetric objective game with symmetric budgets (second row of the symmetric budget column of Table 1), and, finally, examine how the equilibria in the asymmetric budget version of the linear pure-count game (top row of the asymmetric budget column of Table 1) differ and how this approach can be modified for the generalized Colonel Blotto games in this paper.

First, note that in the Colonel Blotto game a mixed strategy is an n-variate joint distribution function (with one dimension for the resource allocation to each of the n battlefields). Except in special cases,[6] equilibrium requires that each n-tuple in the support of a mixed strategy is budget balancing. Equilibrium also requires that each player has no incentive to deviate to an n-tuple outside the support of his joint distribution. If we consider the case of a single symmetric all-pay auction in which each player has a valuation of $v = (2/n)$ for the object, then it is well known[7] that the unique equilibrium involves each player randomizing uniformly over the interval $[0, 2/n]$. In the results section we explore the relationship between the all-pay auction and the Colonel Blotto game in more detail, but for now note that for the linear pure-count objective with n battlefields and symmetric budgets normalized to one unit of (use-it-or-lose-it) resources, a player facing an opponent using a joint distribution function in which each univariate marginal distribution is uniform on $[0, 2/n]$ has no payoff increasing deviations from the set of budget-balancing n-tuples in the n-box $[0, 2/n]^n$. That is, for a player facing such an opponent the expected payoff from any budget-balancing n-tuple in the n-box $[0, 2/n]^n$ is $\sum_{i=1}^{n} nx_i/2 = n/2$.

As all of the constructions in the first two rows of the symmetric budget column of Table 1 — with the exception of Weinstein (2012), which we will return to momentarily — involve randomizing on the surface of an n-gon, the two following properties of regular n-gons are worth noting: (1) the sum of the perpendiculars from any point in a regular n-gon to the sides

[6] See Roberson (2006) for more details.

[7] For the complete characterization of equilibrium in the all-pay auction see Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996).

of the regular n-gon is equal to n times the inradius, i.e. the radius of the incircle (the largest circle that can be inscribed in the n-gon) and (2) if each side of the regular n-gon has length $(2/n)\tan(\pi/n)$, then the inradius is equal to $(1/n)$. Normalizing the symmetric budget to one unit of (use-it-or-lose-it) resources, these two properties of regular n-gons imply that any arbitrary point in a regular n-gon with side length of $(2/n)\tan(\pi/n)$ is budget balancing in that the perpendiculars sum to one, and, for the case of $n = 3$, this is illustrated in panel A. of Figure 1 below where $x_1 + x_2 + x_3 = 1$. For $n = 3$ any distribution on the surface of a regular 3-gon with side lengths $(2/3)\tan(\pi/3)$ that generates uniform marginal distributions on $[0, 2/3]$ for each of the three battlefields is an equilibrium joint distribution, and Borel and Ville (1938) provide two such equilibrium joint distributions.[8] Gross and Wagner (1950), making use of the two properties of regular n-gons listed above, show that both types of equilibria in Borel and Ville (1938) for the linear pure-count objective game with symmetric budgets and $n = 3$ can be directly extended to $n > 3$ and they, also, provide a new fractal equilibrium.

For the case of symmetric budgets, the regular n-gon approach can be modified to allow for battlefield valuations to be symmetric across players but heterogeneous across battlefields, i.e. the linear heterogeneous symmetric objective game with $n \geq 3$ as in the second row of the first column of Table 1. This is exactly what is done in Gross (1950)/Laslier (2002)[9] where the modification involves partitioning the n battlefields into three sets, denoted $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and then randomizing on the surface of the irregular triangle with the three side lengths equal to the total valuations of each of the three sets of battlefields, henceforth denoted $V_{\mathcal{A}}$, $V_{\mathcal{B}}$, and $V_{\mathcal{C}}$, respectively.[10] Then, as illustrated in panel B. of Figure 1, note that for each point on the surface of this irregular triangle the sum across the three sides of the product of each perpendicular and the length of its corresponding side is equal to a constant. That is, $h_{\mathcal{A}}V_{\mathcal{A}} + h_{\mathcal{B}}V_{\mathcal{B}} + h_{\mathcal{C}}V_{\mathcal{C}}$ is equal to twice the surface area of the triangle, which with $V_{\mathcal{A}} + V_{\mathcal{B}} + V_{\mathcal{C}} = 1$ is equal to the inradius. Furthermore, note that $h_i \leq 2r$ for all i, where r denotes the inradius. Thus, for any tri-variate distribution on the incircle the

[8] Borel (1921), a paper on mixed strategies in zero-sum games, introduces the Colonel Blotto game as an example, but does not provide a solution.

[9] See also Thomas (2012) who provide a new construction method for the linear heterogeneous symmetric objective game, with symmetric budgets and $n \geq 3$, that also involves irregular n-gons, but that method differs in that it does not involve merging the battlefields into three groups, but rather utilizes an irregular n-gon in which the number of sides equals the number battlefields.

[10] This constriction, and the following discussion, is for the case that no battlefield has a value that is a majority of the total battlefield values and that it is not the case that the battlefields can be combined into four groups with equal sums of valuations. For more details on the remaining two special cases see Laslier (2002).

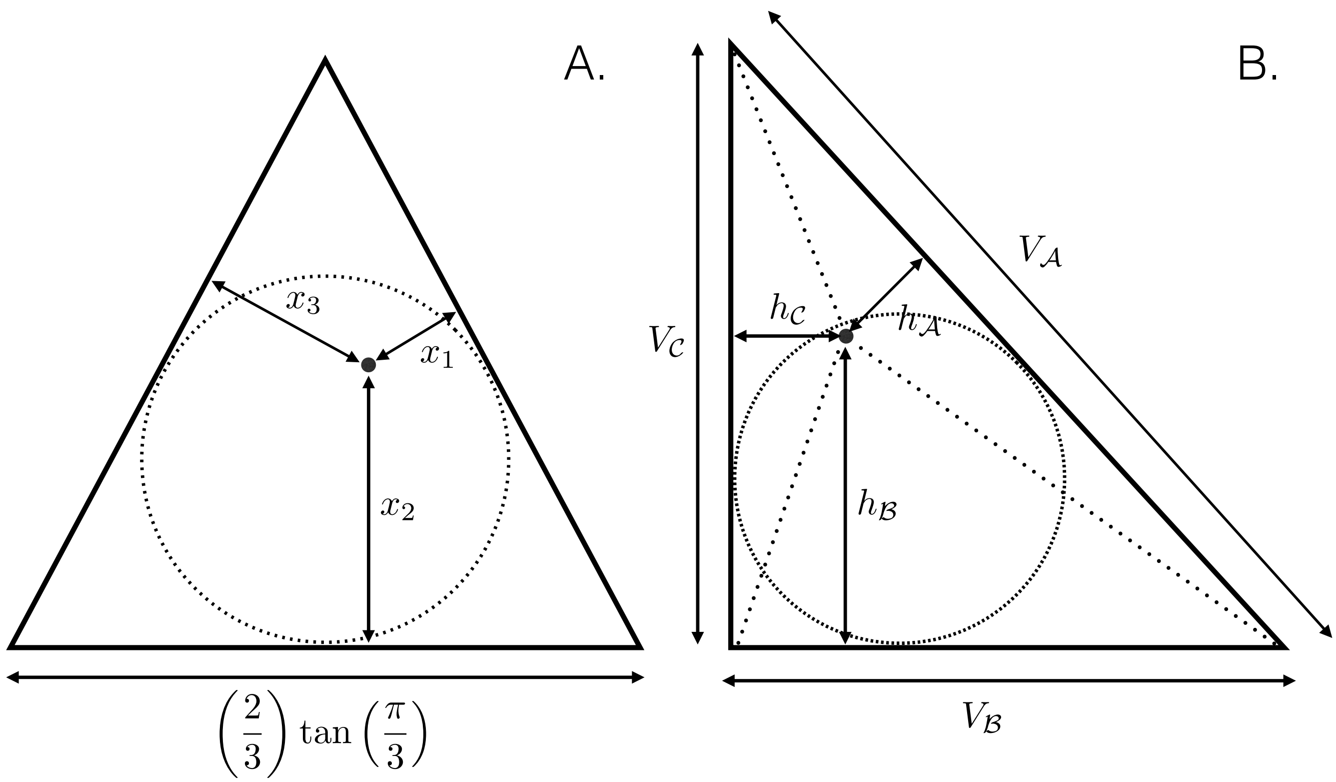

Figure 1: Arbitrary points in a regular and an irregular 3-gon

random variable $\tilde{h}_i$ is contained in the interval $[0, 2r]$ for each $i = \mathcal{A}, \mathcal{B}, \mathcal{C}$, and so we can construct an n-variate distribution function where the random variable $\tilde{h}_{\mathcal{A}}$ is transformed into $\tilde{x}_j \equiv \frac{\tilde{h}_{\mathcal{A}} v_j}{r}$ for each $j \in \mathcal{A}$, performing a corresponding transformation for each $j \in \mathcal{B}$ and $j \in \mathcal{C}$ the resulting n-variate distribution function is budget-balancing with probability one ($\sum_{j=1}^{n} x_j = 1$) and for each $j = 1, \ldots, n$, the random variable $\tilde{x}_j$ is contained in $[0, 2v_j]$. Lastly, as shown in Gross (1950)/Laslier (2002), one of the Borel and Ville (1938) solutions can be used for the tri-variate distribution of the $\tilde{h}_i$ variables; in which case, each $\tilde{h}_i$ is uniformly distributed on the interval $[0, 2r]$ — thus each $\tilde{x}_j$ is uniformly distributed on the interval $[0, 2v_j]$ on each battlefield j where v_j is the relative value of battlefield j — and with symmetric budgets, equilibrium in the linear heterogeneous symmetric objective game requires, following along similar lines as the linear-pure count game, that the univariate marginal distribution functions are uniform on $[0, 2v_j]$ for each battlefield j.

A drawback of using n-gons to construct budget-balancing joint distribution functions is that this reduces the dimensionality of the set of points which can be used to form the support of the joint distribution function. With symmetric budgets and symmetric battlefield valuations, this reduction is a non-binding constraint for the construction of equilibrium joint distribution functions, but with asymmetric budgets and/or asymmetric battlefield valuations it is easier, if not necessary, to work directly with the budget hyperplane in $\mathbb{R}^n$, as in Roberson (2006) and Weinstein (2012). Under this approach, the relative variables are n-tuples in $\mathbb{R}^n$, rather than the perpendiculars of the n-gon approach, and budget balancing requires that n-tuples lie on the budget hyperplane. In this paper, we utilize this full dimensionality approach to examine a subset of possible parameter configurations for each of the three checked cells in Table 1. The heterogeneity in the battlefield valuations creates some technical issues with regards to the existence of sufficient joint distributions. We completely characterize the equilibrium sets of univariate marginal distributions for the case that the budget constraint is relaxed so that it must only hold on average, as in Bell and Cover (1980), Myerson (1993), Hart's (2008) General Lotto game, and Dziubiński (2013). For a portion of the parameter space, we show that there exist budget-balancing joint distributions with the same set of univariate marginals as in the corresponding relaxed version of the game, and we provide a necessary condition for the existence of such a joint distribution. We find that uniqueness of the equilibrium sets of univariate marginal distributions does not extend to the linear heterogeneous asymmetric objective the game (with relaxed or binding budget), and we demonstrate that there exist non-pathological parameter configurations with multiple, payoff inequivalent, equilibria.

Lastly, the case of $n = 2$ places a severe restriction on the set of available joint distributions, which leads to a distinct set of strategic considerations. For both symmetric and asymmetric budgets and both homogeneous and heterogeneous battlefield valuations, the case of $n = 2$ was first examined in Gross and Wagner (1950). Macdonell and Masronardi (2014) complete the characterization for the case of $n = 2$ and provide a characterization of equilibrium in a generalized version of the heterogeneous battlefield valuation game with symmetric or asymmetric budgets. As a point of reference, Table 1 also includes the corresponding results for the discrete version of the Colonel Blotto game.

The rest of the paper is organized as follows. In section 2 we provide a description of the model. Section 3 provides the results and an example, and Section 4 concludes.

2 The Model

Two players, A and B, simultaneously allocate resources across a finite number, $n \geq 3$, of independent battlefields. Battlefield j has a (normalized) value of $v_{i,j}$, where $\sum_{j=1}^{n} v_{i,j} = 1$, for player $i = A, B$. Each player has a fixed level of available resources (or budget), X_i for $i = A, B$. Let $X_A \leq X_B$, and let $\mathbf{x}_i$ denote player i's allocation of resources $(x_{i,1}, \ldots, x_{i,j}, \ldots, x_{i,n})$ across the n-battlefields. If both players allocate X_A resources to a battlefield, then it is assumed that player B wins the battlefield. Otherwise, in the case of a tie, each player wins the battlefield with equal probability. As long as the asymmetry in the players' budgets is below a threshold $[X_B \leq (n-1)X_A]$, any tie-breaking rule which avoids the need to have the stronger player B provide a bid arbitrarily close to, but above, player A's maximal bid yields similar results.

In each battlefield j the payoff to player i for a resource expenditure of $x_{i,j}$ is given by

$$\pi_{i,j}\left(x_{i,j}, x_{-i,j}\right) = \begin{cases} v_{i,j} & \text{if } \; x_{i,j} > x_{-i,j} \\ 0 & \text{if } \; x_{i,j} < x_{-i,j} \end{cases}$$

where ties are handled as described above. Each player's payoff across all n battlefields is the sum of the payoffs across the individual battlefields.

The resources allocated to each battlefield must be nonnegative. For player i, the set of feasible resource allocations across the n battlefields is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}_+^n \middle| \sum_{j=1}^{n} x_{i,j} \leq X_i \right\}.$$

Strategies

A mixed strategy, which we term a *distribution of resources*, for player i is an n-variate distribution function $P_i : \mathbb{R}_+^n \to [0, 1]$ with support (denoted $\mathrm{Supp}(P_i)$) contained in the set of player i's set of feasible bids $\mathfrak{B}_i$ and with one-dimensional marginal distribution functions $\{F_{i,j}\}_{j=1}^n$, one univariate marginal distribution function for each battlefield j. To avoid confusion with the support of the joint distribution, when referring to the support of a given univariate marginal distribution — the smallest closed univariate interval whose complement has probability zero — we will make a slight abuse of terminology and use the term *domain* to denote the support of the given univariate marginal distribution function. Player i's allocation of resources across the n battlefields is a random n-tuple drawn from the n-variate distribution function P_i.

The Generalized Colonel Blotto game

The Generalized Colonel Blotto game, which we label

$$GB\big\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\big\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of resources subject to their budget constraints, each battlefield is won by the player that allocates the higher level of resources to that battlefield (where in the case of a tie the tie-breaking rule described above applies), and players' receive the sum of their payoffs across the individual battlefields.

It will also be convenient to define the relaxed Colonel Blotto game in which the resource constraints must only hold on average and so a strategy is a set of n-univariate marginal distributions $\{F_{i,j}\}_{j=1}^n$. This is a generalization of the relaxed versions of the Blotto game examined in Bell and Cover (1980), Myerson (1993), Hart's (2008) General Lotto game, and Dziubiński (2013). As those papers involve linear pure-count objectives, battlefield valuations are homogeneous across battlefields and symmetric across players, a strategy consists of a single univariate marginal distribution. In contrast, as we allow for more general battlefield value configurations, our relaxed version of the game a strategy involves a univariate marginal distribution for each battlefield.[11]

[11]Note that in the case that a player does not have a distinct battlefield valuation for each battlefield, a strategy involves a univariate marginal distribution for each distinct battlefield valuation.

The Relaxed Colonel Blotto game

The Relaxed Colonel Blotto game, which we label

$$RB\big\{X_A, X_B, n, \{v_{A,j}, v_{B,j}\}_{j=1}^n\big\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of resources subject to their budget constraints holding on average, each battlefield is won by the player that allocates the higher level of resources to that battlefield (where in the case of a tie the tie-breaking rule described above applies), and players' receive the sum of their payoffs across the individual battlefields.

3 Results

In order to provide intuition for our main results, we begin this section with a few informal insights regarding the necessary conditions for equilibrium in the generalized and relaxed Colonel Blotto games. First, note that any joint distribution may be broken into a set of univariate marginal distribution functions and an n-copula, the function that maps the univariate marginal distribution functions into a joint distribution function.[12] Then, given that player $-i$'s strategy is given by the n-variate distribution function P_{-i} with the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^n$, note that player i's expected payoff[13] for any feasible resource allocation $\mathbf{x}_i \in \mathbb{R}_+^n$,

$$\pi_i\left(\mathbf{x}_i, \{F_{-i,j}\}_{j=1}^n\right) = \sum_{j=1}^n \left[v_{i,j} F_{-i,j}\left(x_{i,j}\right)\right], \tag{1}$$

depends on P_{-i} through, only, the set of univariate marginal distribution functions $\{F_{-i,j}\}_{j=1}^n$. It follows directly that the set of equilibrium univariate marginal distributions in the generalized Colonel Blotto game will coincide with those in the relaxed Colonel Blotto game as long as there exists a mapping from the set of equilibrium univariate marginal distributions in the relaxed Colonel Blotto game into a joint distribution (an n-copula), C, in which the support of the resulting n-variate distribution function $C(F_{i,1}(x_1), \ldots, F_{i,n}(x_n))$ is contained in $\mathfrak{B}_i$. Additionally, noting that if the budget constraint holds with probability one then it

[12]See Nelsen (1999) or Schweizer and Sklar (1983) for an introduction to copulas.

[13]This expression is for the case that none of player $-i$'s univariate marginal distributions contains a mass point.

also holds in expectation, for both the generalized and relaxed Colonel Blotto games, player i's constrained optimization problem may be written as,

$$\max_{\left\{\{F_{i,j}\}_{j=1}^{n}\in\ \mathcal{C}_i\right\}} \sum_{j=1}^{n}\left[\int_0^{\infty}\left[v_{i,j}F_{-i,j}\left(x_{i,j}\right)-\lambda_i x_{i,j}\right]dF_{i,j}\right]+\lambda_i X_i. \tag{2}$$

where λ_i is the multiplier on player i's expected resource expenditure constraint and the generalized Colonel Blotto game has the side constraint that there exists a sufficient n-copula.[14] For each $j=1,\ldots,n$ the corresponding first variation provides a necessary condition for equilibrium and is given by

$$\frac{d}{dx}\left[v_{i,j}F_{-i,j}\left(x_{i,j}\right)-\lambda_i x_{i,j}\right]=0. \tag{3}$$

Dividing both sides of (3) by $\lambda_i>0$, we see that (3) is equivalent to the necessary condition for a single all-pay auction, without a budget constraint, and in which player i's value for the prize is $v_{i,j}/\lambda_i$, which we henceforth refer to as the *implicit value of the prize.* In such an all-pay auction, the unique equilibrium[15] is described as follows. If $v_{i,j}/\lambda_i \geq v_{-i,j}/\lambda_{-i}$, then

$$\begin{aligned} F_{-i,j}\left(x\right) &= \left(\frac{\frac{v_{i,j}}{\lambda_i}-\frac{v_{-i,j}}{\lambda_{-i}}}{\frac{v_{i,j}}{\lambda_i}}\right)+\frac{x}{\frac{v_{i,j}}{\lambda_i}} \quad && x\in\left[0,\frac{v_{-i,j}}{\lambda_{-i}}\right] \\ F_{i,j}\left(x\right) &= \frac{x}{\frac{v_{-i,j}}{\lambda_{-i}}} && x\in\left[0,\frac{v_{-i,j}}{\lambda_{-i}}\right] \end{aligned} \tag{4}$$

Next, to solve for the multipliers (λ_A,λ_B), let Ω_A denote the set of battlefields in which $v_{A,j}/\lambda_A > v_{B,j}/\lambda_B$. The combination of equation (4) and budget-balancing implies the following:

$$\sum_{j\in\Omega_A}\frac{v_{B,j}}{2\lambda_B}+\sum_{j\notin\Omega_A}\frac{\left(\frac{v_{A,j}}{\lambda_A}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B}\right)}=X_A \tag{5}$$

$$\sum_{j\in\Omega_A}\frac{\left(\frac{v_{B,j}}{\lambda_B}\right)^2}{2\left(\frac{v_{A,j}}{\lambda_A}\right)}+\sum_{j\notin\Omega_A}\frac{v_{A,j}}{2\lambda_A}=X_B \tag{6}$$

λ_A^* and λ_B^* are implicitly defined by equations (5) and (6), henceforth referred to as a solution

[14] Roughly speaking, this side constraint is non-binding if for each player the intersection of the hyperplane formed by the n-tuples which exhaust his respective budget and the n-box formed by the domains of each of the univariate marginal distributions for the corresponding battlefields is "well behaved."

[15] For more details see Baye, Kovenock, and de Vries (1996).

to system ($\star$).

Proposition 1. *For any feasible configuration of battlefield values* $\{v_{A,j}, v_{B,j}\}_{j=1}^{n}$ *and resource endowments* $\{X_A, X_B\}$ *there exists a solution to system ($\star$). For the case of the linear heterogeneous symmetric objective the game is constant-sum (i.e.* $v_{A,j} = v_{B,j}$ *for all* j*) and there exists a unique solution to system ($\star$).*

Proof. We begin with the proof that there exists a solution to system ($\star$). Let $\mu = \frac{\lambda_A}{\lambda_B}$. Multiplying both sides of (6) by λ_A and both sides of (5) by λ_B, and then dividing the former by the latter, we have:

$$\frac{X_B \mu}{X_A} = \frac{\mu^2 \sum_{j\in\Omega_A} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j\notin\Omega_A} v_{A,j}}{\sum_{j\in\Omega_A} v_{B,j} + \frac{1}{\mu^2}\sum_{j\notin\Omega_A} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}} \tag{7}$$

First, note that as μ varies the set Ω_A changes, but the right-hand side of (7) is continuous at each point where Ω_A changes. In particular, the changes in Ω_A occur at $\mu = \frac{v_{A,k}}{v_{B,k}}$ for each $k \in \{1, \ldots, n\}$, and for all $k \in \{1, \ldots, n\}$

$$\lim_{\mu\to\left(\frac{v_{A,k}}{v_{B,k}}\right)^+} \frac{\mu^2 \sum_{j\in\Omega_A} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j\notin\Omega_A} v_{A,j}}{\sum_{j\in\Omega_A} v_{B,j} + \frac{1}{\mu^2}\sum_{j\notin\Omega_A} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}} = \lim_{\mu\to\left(\frac{v_{A,k}}{v_{B,k}}\right)^-} \frac{\mu^2 \sum_{j\in\Omega'_A} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j\notin\Omega'_A} v_{A,j}}{\sum_{j\in\Omega'_A} v_{B,j} + \frac{1}{\mu^2}\sum_{j\notin\Omega'_A} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}}$$

where $\Omega'_A = \Omega_A - \{k\}$.

Next, note that if $\frac{X_B}{X_A}\sum_{j=1}^{n} \frac{(v_{A,j})^2}{v_{B,j}} > \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$ then $\mu = \frac{X_B}{X_A}\sum_{j=1}^{n} \frac{(v_{A,j})^2}{v_{B,j}}$ is a solution to (7), in this case $\Omega_A = \emptyset$ and the result follows directly. Similarly, if $\frac{X_B}{X_A}\left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1} < \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$ then $\mu = \frac{X_B}{X_A}\left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1}$ is solution to (7), where in this case $\Omega_A = \{1, \ldots, n\}$ and the result follows directly.

We now examine the case that $\frac{X_B}{X_A}\sum_{j=1}^{n} \frac{(v_{A,j})^2}{v_{B,j}} \leq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$ and $\frac{X_B}{X_A}\left(\sum_{j=1}^{n} \frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1} \geq \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$. Multiplying both sides of (7) by (X_A/X_B), we can construct the following continuous and increasing function:

$$\mu(\mu) = \left(\frac{X_A}{X_B}\right)\left(\frac{\mu^2 \sum_{j\in\Omega_A} \frac{\left(v_{B,j}\right)^2}{v_{A,j}} + \sum_{j\notin\Omega_A} v_{A,j}}{\sum_{j\in\Omega_A} v_{B,j} + \frac{1}{\mu^2}\sum_{j\notin\Omega_A} \frac{\left(v_{A,j}\right)^2}{v_{B,j}}}\right)$$

Then, as $\frac{X_B}{X_A}\sum_{j=1}^{n}\frac{(v_{A,j})^2}{v_{B,j}} \leq \max_j\{\frac{v_{A,j}}{v_{B,j}}\}$, it follows that

$$\mu\left(\max_j\{\frac{v_{A,j}}{v_{B,j}}\}\right) \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\} \tag{8}$$

and, as $\frac{X_B}{X_A}\left(\sum_{j=1}^{n}\frac{(v_{B,j})^2}{v_{A,j}}\right)^{-1} \geq \min_j\{\frac{v_{A,j}}{v_{B,j}}\}$, it follows that

$$\mu\left(\min_j\{\frac{v_{A,j}}{v_{B,j}}\}\right) \leq \min_j\{\frac{v_{A,j}}{v_{B,j}}\} \tag{9}$$

Combining (8), (9), with the continuity of μ, it follows that there exists at least one point $\mu^* \in [\min_j\{\frac{v_{A,j}}{v_{B,j}}\}, \max_j\{\frac{v_{A,j}}{v_{B,j}}\}]$ such that $\mu(\mu^*) = \mu^*$. This completes the proof of the existence of a μ that solves (7), and then given a solution μ we can use (5) and (6) to solve for λ_B and λ_A (a solution to system ($\star$)), respectively.

For uniqueness in the constant-sum game, note that when $v_{A,j} = v_{B,j}$ for all j then $\max_j\{\frac{v_{A,j}}{v_{B,j}}\} = \min_j\{\frac{v_{A,j}}{v_{B,j}}\} = 1$ for all j and so either $\Omega_A = \emptyset$ or $\Omega_A = \{1,\ldots,n\}$, and in either case (7) becomes $\mu = \frac{X_B}{X_A}$. But as $\frac{X_A}{X_B} \geq 1$, it follows that $\mu = \frac{X_B}{X_A} \geq \max_j\{\frac{v_{A,j}}{v_{B,j}}\} = 1$, and so $\Omega_A = \emptyset$. $\square$

Although there exists a unique solution to system ($\star$) when the game is constant-sum, there may be multiple solutions to system ($\star$) in non-constant-sum versions of the game, and these multiple solutions give rise to multiple equilibria. Following the statement and proof of Theorem 1, we provide an example in which there are multiple payoff inequivalent equilibria.

We now examine equilibrium in the general case of the linear heterogeneous asymmetric objective and then move on to the case of the linear heterogeneous symmetric objective.

Theorem 1. *For each solution to system ($\star$) there exists a unique Nash equilibrium of the relaxed Blotto game and the equilibrium univariate marginals are as follows: If $v_{i,j}/\lambda_i^* \geq v_{-i,j}/\lambda_{-i}^*$, then*

$$\begin{aligned} F_{-i,j}(x) &= \left(\frac{\frac{v_{i,j}}{\lambda_i^*}-\frac{v_{-i,j}}{\lambda_{-i}^*}}{\frac{v_{i,j}}{\lambda_i^*}}\right) + \frac{x}{\frac{v_{i,j}}{\lambda_i^*}} \quad && x \in \left[0, \frac{v_{-i,j}}{\lambda_{-i}^*}\right] \\ F_{i,j}(x) &= \frac{x}{\frac{v_{-i,j}}{\lambda_{-i}^*}} && x \in \left[0, \frac{v_{-i,j}}{\lambda_{-i}^*}\right] \end{aligned}$$

The expected payoff for player A is $\sum_{j\in\Omega_A}\left(v_{A,j} - \frac{\lambda_A^* v_{B,j}}{\lambda_B^*}\right) + \lambda_A^* X_A$ and the expected payoff for player B is $\sum_{j\notin\Omega_A}\left(v_{B,j} - \frac{\lambda_B^* v_{A,j}}{\lambda_A^*}\right) + \lambda_B^* X_B$.

Proof. In this proof we show that this is in fact an equilibrium. For the proof of uniqueness, it follows directly from Kovenock and Roberson's (2008) extension of Sahuguet and Persico (2006) that, for each solution to system ($\star$), there exists a strategic equivalence between the relaxed Blotto game and a unique set of appropriately chosen independent simultaneous two-bidder all-pay auctions. Uniqueness then follows from the characterization of the all-pay auction given by Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996).

We focus on player A, and note that the argument for player B is symmetric. First, observe that as (λ_A, λ_B) is a solution to ($\star$), this is a feasible strategy for player A:

$$\sum_{j=1}^{n} \int_0^{\infty} x dF_{A,j} = \sum_{j \in \Omega_A} \frac{v_{B,j}}{2\lambda_B^*} + \sum_{j \notin \Omega_A} \frac{\left(\frac{v_{A,j}}{\lambda_A^*}\right)^2}{2\left(\frac{v_{B,j}}{\lambda_B^*}\right)} = X_A.$$

Then given that player B is following the equilibrium strategy, player A's payoff from an arbitrary strategy $\left\{\bar{F}_A^j\right\}_{j=1}^n$ is:

$$\pi_A\left(\left\{\bar{F}_A^j, F_B^j\right\}_{j=1}^n\right) = \sum_{j=1}^{n} \int_0^{\infty} v_{A,j} F_B^j(x)\, d\bar{F}_A^j(x)$$

Because it is never a best response for player A to place strictly positive mass on zero in any battlefield $j \in \Omega_A$ nor to provide offers outside the domain of one of player B's univariate marginal distributions, we have:

$$\pi_A\left(\left\{\bar{F}_A^j, F_B^j\right\}_{j=1}^n\right) = \sum_{j \in \Omega_A} \left[\left(v_{A,j} - \frac{v_{B,j}\lambda_A^*}{\lambda_B^*}\right) + \int_0^{\frac{v_{B,j}}{\lambda_B^*}} x\lambda_A^* d\bar{F}_A^j(x)\right] + \sum_{j \notin \Omega_A} \int_0^{\frac{v_{A,j}}{\lambda_A^*}} x\lambda_A^* d\bar{F}_A^k(x).$$

But from the budget constraint, it follows that

$$\pi_A\left(\left\{\bar{F}_A^j, F_B^j\right\}_{j=1}^n\right) \le \sum_{j \in \Omega_A} \left(v_{A,j} - \frac{\lambda_A^* v_{B,j}}{\lambda_B^*}\right) + \lambda_A^* X_A$$

which holds with equality if $\{\bar{F}_A^j\}_{j=1}^n$ is the equilibrium strategy. This completes the proof that there are no payoff increasing deviations for player A. A symmetric argument applies to player B, and thus $\left(\left\{F_A^j, F_B^j\right\}_{j=1}^n\right)$ is an equilibrium. □

The following example provides a simple example in which multiple equilibria arise. It is clear from Proposition 1 that if the game is constant-sum, i.e. the players' battlefield valuations are symmetric for all battlefields, then there exists a unique set of equilibrium marginal distribution functions. Thus, for multiple payoff inequivalent equilibria to arise it must be the case that there exists a nonempty set of battlefields, termed the disagreement set, in which $v_{A,j} \neq v_{B,j}$. Example 1 is a special case of the linear heterogeneous asymmetric objective game in which the players' valuations in the disagreement set take a specific parametric form. Interestingly, even though the players' valuations are either in agreement or when they disagree take only two different values we find that there are five payoff inequivalent equilibria. The parametric form that is used for the battlefield valuations in the disagreement set is useful in that it simplifies the calculation of the set Ω_A, thereby simplifying the problem of solving system ($\star$). In moving from this example to an arbitrary configuration of battlefield valuations the calculation of the set Ω_A becomes more involved and the issue of multiple equilibria is potentially greater.

Example 1. *Consider a relaxed Colonel Blotto game in which $X_A = 1$, $X_B = 1$, and the battlefields may be partitioned into an agreement set, denoted $\mathcal{A}$, in which $v_{A,j} = v_{B,j}$ for each $j \in \mathcal{A}$ and $\sum_{j\in\mathcal{A}} v_{A,j} = (n_{\mathcal{A}}/n)$, where $n_{\mathcal{A}}$ is the number of battlefields in the agreement set, and a disagreement set, denoted $\mathcal{D}$, with an even number $n_{\mathcal{D}}$ of battlefields, where for the first $(n_{\mathcal{D}}/2)$ battlefields $v_{A,j} = \frac{2(1-\epsilon)}{n}$ and $v_{B,j} = \frac{2\epsilon}{n}$ and for the last $(n_{\mathcal{D}}/2)$ battlefields $v_{A,j} = \frac{2\epsilon}{n}$ and $v_{B,j} = \frac{2(1-\epsilon)}{n}$, with $\epsilon \in (0, .5)$. This configuration of battlefield values is illustrated in Figure 2 below.*

For all $\epsilon \in (0, .5)$, $n_D \geq 0$, and $n_A \geq 0$ equation (7) has a solution at $\mu = 1$, but depending on the values of ϵ, n_D, and n_A there may exist multiple solutions, and thus multiple equilibria. To solve for these equilibria, first note that with symmetric resource constraints it must be the case that either $\frac{\epsilon}{1-\epsilon} < \mu \leq 1$ or $1 < \mu \leq \frac{1-\epsilon}{\epsilon}$.[16] If $\frac{\epsilon}{1-\epsilon} < \mu \leq 1$, then Ω_A includes $\mathcal{A}$ and the portion of $\mathcal{D}$ in which $v_{A,j} = \frac{2(1-\epsilon)}{n}$ and $v_{B,j} = \frac{2\epsilon}{n}$ and (7) may be written as

$$\mu^3 \sum_{j\in\Omega_A} \frac{(v_{B,j})^2}{v_{A,j}} - \frac{X_B \mu^2}{X_A} \sum_{j\in\Omega_A} v_{B,j} + \mu \sum_{j\notin\Omega_A} v_{A,j} - \frac{X_B}{X_A} \sum_{j\notin\Omega_A} \frac{(v_{A,j})^2}{v_{B,j}} =$$
$$\mu^3 \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_{\mathcal{D}}}{n} + \frac{n_{\mathcal{A}}}{n}\right) - \mu^2 \left(\epsilon \cdot \frac{n_{\mathcal{D}}}{n} + \frac{n_{\mathcal{A}}}{n}\right) + \mu \left(\epsilon \cdot \frac{n_{\mathcal{D}}}{n}\right) - \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_{\mathcal{D}}}{n}\right) = 0 \quad (10)$$

[16]With symmetric budget constraints, it is clear that there exist no equilibria in which one player has a strictly higher effective valuation for all battlefields.

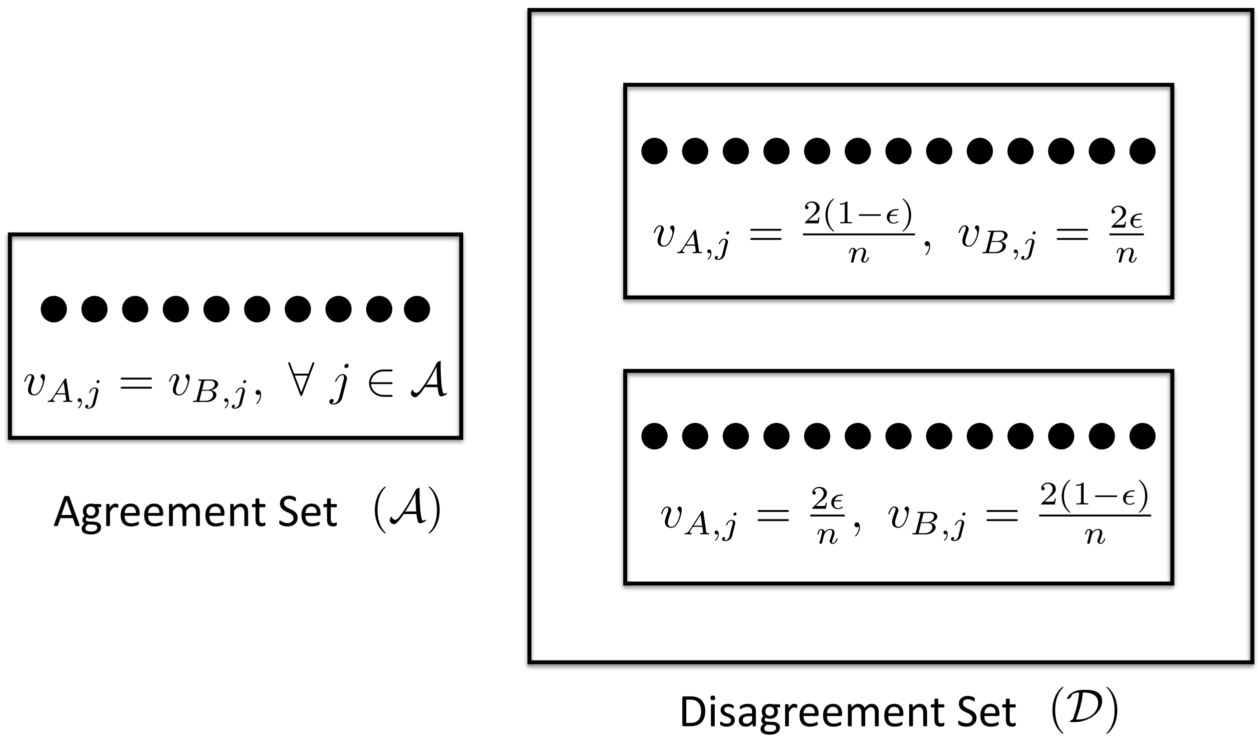

Figure 2: Example 1 Battlefield Configuration $[\epsilon \in (0, .5)]$

Similarly, if $1 < \mu \leq \frac{1-\epsilon}{\epsilon}$, then Ω_A includes only the portion of $\mathcal{D}$ in which $v_{A,j} = \frac{2(1-\epsilon)}{n}$ and $v_{B,j} = \frac{2\epsilon}{n}$ and (7) may be written as

$$\mu^3 \sum_{j\in\Omega_A} \frac{(v_{B,j})^2}{v_{A,j}} - \frac{X_B \mu^2}{X_A} \sum_{j\in\Omega_A} v_{B,j} + \mu \sum_{j\notin\Omega_A} v_{A,j} - \frac{X_B}{X_A} \sum_{j\notin\Omega_A} \frac{(v_{A,j})^2}{v_{B,j}} =$$

$$\mu^3 \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_{\mathcal{D}}}{n} \right) - \mu^2 \left(\epsilon \cdot \frac{n_{\mathcal{D}}}{n} \right) + \mu \left(\epsilon \cdot \frac{n_{\mathcal{D}}}{n} + \frac{n_{\mathcal{A}}}{n} \right) - \left(\frac{\epsilon^2}{1-\epsilon} \cdot \frac{n_{\mathcal{D}}}{n} + \frac{n_{\mathcal{A}}}{n} \right) = 0 \quad (11)$$

Letting $\epsilon = 0.10$, $(n_{\mathcal{A}}/n) = 0.1$, and $(n_{\mathcal{D}}/n) = 0.9$, equation (10) has three real roots for $\frac{\epsilon}{1-\epsilon} = \frac{1}{9} < \mu \leq 1$ and equation (11) has two real roots for $1 < \mu \leq 9 = \frac{1-\epsilon}{\epsilon}$: $\mu \approx 0.1604$, $\mu \approx 0.5668$, $\mu = 1$, and $\mu \approx 1.764$, $\mu \approx 6.234$ respectively. When $\mu = 1$, it follows from (5) and (6) that $\lambda_A = \lambda_B = 0.1$. Similarly when $\mu = 0.1604$, $\lambda_A \approx 0.05382$ and $\lambda_B \approx 0.2893$, when $\mu = 0.5668$, $\lambda_A \approx 0.06267$ and $\lambda_B \approx 0.1106$, when $\mu = 1.764$, $\lambda_A \approx 0.1106$ and $\lambda_B \approx 0.06267$, and when $\mu = 6.234$, $\lambda_A \approx 0.2893$ and $\lambda_B \approx 0.05382$.

For each of the five solutions to system ($\star$), Theorem 1 provides the unique set of equilibrium marginal distributions. For the three solutions with $\frac{1}{9} < \mu = \frac{\lambda_A}{\lambda_B} \leq 1$ equilibrium is described as follows: for all battlefields $j \in \mathcal{A}$ let $v_j \equiv v_{A,j} = v_{B,j}$

$$\begin{aligned} F_{B,j}(x) &= \left(1 - \tfrac{\lambda_A}{\lambda_B}\right) + \tfrac{x}{\frac{v_j}{\lambda_A}} \quad & x \in \left[0, \tfrac{v_j}{\lambda_B}\right] \\ F_{A,j}(x) &= \tfrac{x}{\frac{v_j}{\lambda_B}} \quad & x \in \left[0, \tfrac{v_j}{\lambda_B}\right] \end{aligned}$$

for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{9}{5n}$ and $v_{B,j} = \frac{1}{5n}$

$$\begin{aligned} F_{B,j}(x) &= \left(1 - \tfrac{\lambda_A}{9\lambda_B}\right) + \tfrac{x}{\frac{9}{5n\lambda_A}} \quad & x \in \left[0, \tfrac{1}{5n\lambda_B}\right] \\ F_{A,j}(x) &= \tfrac{1}{5n\lambda_B} \quad & x \in \left[0, \tfrac{1}{5n\lambda_B}\right] \end{aligned}$$

and for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{1}{5n}$ and $v_{B,j} = \frac{9}{5n}$

$$\begin{aligned} F_{A,j}(x) &= \left(1 - \tfrac{\lambda_B}{9\lambda_A}\right) + \tfrac{x}{\frac{9}{5n\lambda_A}} \quad & x \in \left[0, \tfrac{1}{5n\lambda_B}\right] \\ F_{B,j}(x) &= \tfrac{1}{5n\lambda_B} \quad & x \in \left[0, \tfrac{1}{5n\lambda_B}\right] \end{aligned}$$

The expected payoff for player A is $\frac{n_{\mathcal{A}}}{n}\left(1 - \frac{\lambda_A}{\lambda_B}\right) + \frac{n_{\mathcal{D}}}{n}\left(\frac{9}{10} - \frac{\lambda_A}{10\lambda_B}\right) + \lambda_A$ and the expected payoff for player B is $\frac{n_{\mathcal{D}}}{n}\left(\frac{9}{10} - \frac{\lambda_B}{10\lambda_A}\right) + \lambda_B$. Similarly, for the two solutions with $1 < \mu =$

$\frac{\lambda_A}{\lambda_B} \leq 9$ equilibrium is described as follows: for all battlefields $j \in \mathcal{A}$

$$\begin{aligned} F_{A,j}(x) &= \left(1 - \frac{\lambda_B}{\lambda_A}\right) + \frac{x}{\frac{v_j}{\lambda_B}} \quad & x \in \left[0, \frac{v_j}{\lambda_A}\right] \\ F_{B,j}(x) &= \frac{x}{\frac{v_j}{\lambda_A}} \quad & x \in \left[0, \frac{v_j}{\lambda_A}\right] \end{aligned}$$

for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{9}{5n}$ and $v_{B,j} = \frac{1}{5n}$

$$\begin{aligned} F_{B,j}(x) &= \left(1 - \frac{\lambda_A}{9\lambda_B}\right) + \frac{x}{\frac{9}{5n\lambda_A}} \quad & x \in \left[0, \frac{1}{5n\lambda_B}\right] \\ F_{A,j}(x) &= \frac{1}{5n\lambda_B} \quad & x \in \left[0, \frac{1}{5n\lambda_B}\right] \end{aligned}$$

and for $j \in \mathcal{D}$ such that $v_{A,j} = \frac{1}{5n}$ and $v_{B,j} = \frac{9}{5n}$

$$\begin{aligned} F_{A,j}(x) &= \left(1 - \frac{\lambda_B}{9\lambda_A}\right) + \frac{x}{\frac{9}{5n\lambda_A}} \quad & x \in \left[0, \frac{1}{5n\lambda_B}\right] \\ F_{B,j}(x) &= \frac{1}{5n\lambda_B} \quad & x \in \left[0, \frac{1}{5n\lambda_B}\right] \end{aligned}$$

The expected payoff for player A is $\frac{n_{\mathcal{D}}}{n}\left(\frac{9}{10} - \frac{\lambda_A}{10\lambda_B}\right) + \lambda_A$ and the expected payoff for player B is $\frac{n_{\mathcal{A}}}{n}\left(1 - \frac{\lambda_B}{\lambda_A}\right) + \frac{n_{\mathcal{D}}}{n}\left(\frac{9}{10} - \frac{\lambda_B}{10\lambda_A}\right) + \lambda_B$.

In moving from the relaxed Blotto game to the generalized Colonel Blotto game, we face the issue of the existence of joint distributions that spend the players budgets with probability one and that provide the necessary univariate marginal distributions. As the Theorem 1 sets of univariate marginals form an equilibrium in the relaxed version of the game without the constraint on the support of the joint distribution, it is clear that if in the full version of the game the constraint on the support of the joint distribution is non-binding then the results in Theorem 1 extend directly to the full version of the game. The following corollary provides a sufficient condition for the constraint on the support of the joint distribution to be non-binding, i.e the existence of budget-balancing joint distributions that provide the equilibrium sets of univariate marginal distributions given in Theorem 1. In the statement of the corollary and in the discussion that follows, it will be useful to partition the battlefields into subsets based on distinct pairs of valuations $v_{A,j}$ and $v_{B,j}$, let $n_j \geq 1$ denote the number of battlefields with this distinct pair of valuations, and let $j \in \{1, \ldots, k\}$ index the $k \leq n$ distinct pairs of battlefield valuations,

Corollary 1. *Given a solution to system ($\star$), if for each distinct pair of battlefield valuations $v_{A,j}$ and $v_{B,j}$ with $\frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*} \leq 1$, for some $i \in \{A, B\}$, it is the case that $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$, then* *there exists a Nash equilibrium of the generalized Colonel Blotto game with the same set of univariate marginals and equilibrium expected payoffs as in Theorem 1.*

Given the $k \leq n$ distinct pairs of battlefield valuations, we can form independent multivariate marginal distributions on these subsets where the budget constraint on each subset is equal to the expected expenditure from the corresponding set of univariate marginals. For example, if $v_{i,j}/\lambda_i \geq v_{-i,j}/\lambda_{-i}$, then from Theorem 1 it follows that player $-i$'s expected expenditure on the jth set of battlefields is $n_j\,(v_{-i,j}\lambda_i/v_{i,j}\lambda_{-i})\,(v_{-i,j}/2\lambda_{-i})$ and i's expected expenditure on the jth set of battlefields is $n_j\,(v_{-i,j}/2\lambda_{-i})$. Then, the problem of constructing an equilibrium joint distribution P_i, for each player i, that is budget balancing with probability one and that provides the univariate marginals given in Theorem 1 is broken up into the problem of constructing the n_j-variate marginal distributions, denoted $P_{i,j}$, for each of the k sets of battlefields with distinct valuations as described as follows, and the joint distribution $P_i(x) = \prod_{j=1}^{k} P_{i,j}(x_j)$ where x_j is the restriction of x to the battlefields in set j. For each of the n_j-variate marginal distributions, $P_{i,j}$, if $v_{i,j}/\lambda_i \geq v_{-i,j}/\lambda_{-i}$ and it is the case that $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i}{v_{i,j}\lambda_{-i}}$, then player i's n_j-variate marginal distribution $P_{i,j}$ may be formed by allocating $n_j\,(v_{-i,j}/2\lambda_{-i})$ resources to subset j of battlefields and then constructing $P_{i,j}$ using any of the existing construction methods from Gross and Wagner (1950), Roberson (2006), or Weinstein (2012), and player $-i$'s n_j-variate marginal distribution $P_{-i,j}$ may be formed by allocating $n_j\,(v_{-i,j}\lambda_i/v_{i,j}\lambda_{-i})\,(v_{-i,j}/2\lambda_{-i})$ resources to subset j of battlefields and then constructing $P_{-i,j}$ using Roberson (2006). Any such construction provides the necessary univariate marginals characterized in Theorem 1 and the resulting joint distributions P_i and P_{-i} are budget-balancing with probability one.

In the case of player $-i$'s n_j-variate marginal distribution $P_{-i,j}$, the construction in Roberson (2006) is required because each of the univariate marginals of $P_{-i,j}$ has a mass point of size $1 - (v_{-i,j}\lambda_i/v_{i,j}\lambda_{-i})$ at 0, which is not possible with the n-gon constructions described above. That is, if the support of the joint distribution is contained in the incircle of the corresponding n-gon, then placing strictly positive mass at 0 on any one battlefield implies directly that there exist corresponding mass points (at strictly positive values) in the univariate marginals of the remaining battlefields. This issue does not arise in the Roberson (2006) construction because that method focuses on the budget hyperplane rather than on the strict subset of budget-balancing points that are contained on the surface of the corresponding n-gon. For the case of $n_j = 3$ and $(v_{-i,j}\lambda_i/v_{i,j}\lambda_{-i}) > (2/n_j)$, the support of $P_{-i,j}$ is given in Figure 3 below. Note that the support of $P_{-i,j}$ is lies on the budget hyperplane $\sum_{i=1}^{3} x_i = 3\left(\frac{v_{-i,j}\lambda_i}{v_{i,j}\lambda_{-i}}\right)\left(\frac{v_{-i,j}}{2\lambda_{-i}}\right)$ and provides the set of univariate marginals specified by

Theorem 1.

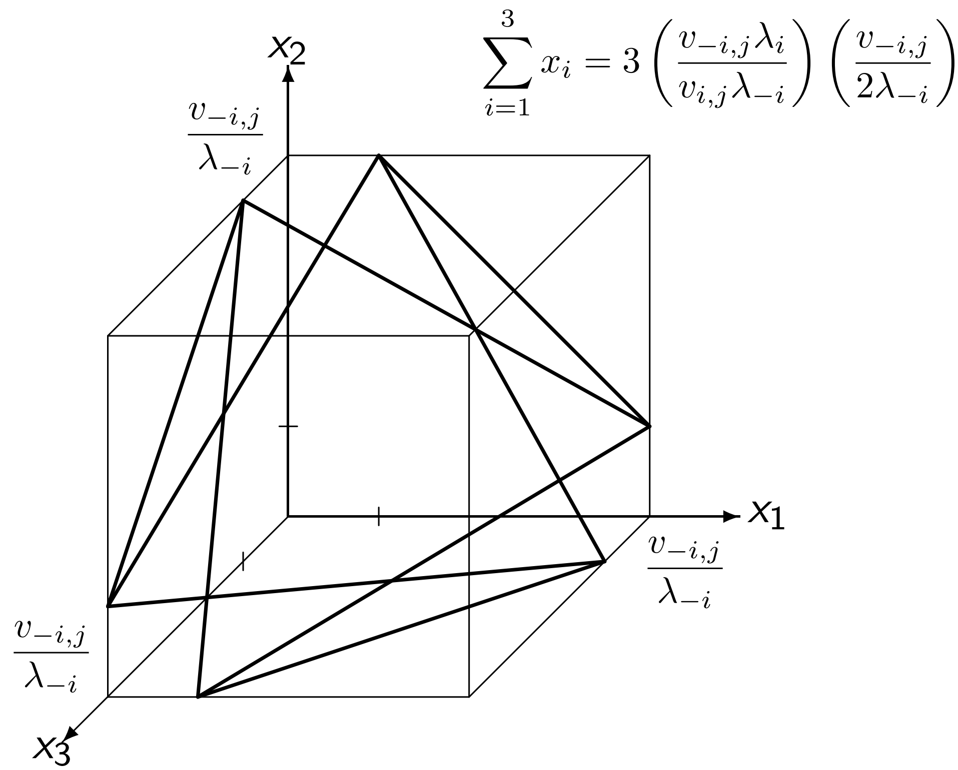

Figure 3: Support of $P_{-i,j} \quad (v_{i,j}/\lambda_i \geq v_{-i,j}/\lambda_{-i})$

The role of the condition that $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$ for each $j = 1, \ldots, k$ can be seen in Figure 3, where the condition implies that that budget hyperplane cuts through the n_j-box, or hypercube, above its intercepts and thus the upper bound of the support of each of the univariate marginals is feasible. Furthermore, Roberson (2006) shows that as long as the hyperplane cuts through the hypercube above its intercepts the constraint on the support of the joint distribution is non-binding and the results in Theorem 1 extend directly to the full version of the game. Thus, $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$ for each $j = 1, \ldots, k$ is a sufficient condition for the existence of a budget-balancing joint distribution that provides the set of equilibrium univariate marginal distributions in Theorem 1. It follows directly that a necessary condition for the existence of such a joint distribution is $X_i \geq \max_j \left\{\min\left\{\frac{v_{-i,j}}{\lambda_{-i}^*}, \frac{v_{i,j}}{\lambda_i^*}\right\}\right\}$ for each player i. This necessary condition requires only that the intersection of each player's total budget constraint cuts through the n-box, formed by the supports of each of the n univariate marginals specified by Theorem 1, above its intercepts.

In the case of the linear heterogeneous symmetric objective $v_{A,j} = v_{B,j} \equiv v_j$ for all j, and as $\frac{X_A}{X_B} \leq 1$ it must be the case that $\lambda_B \leq \lambda_A$. Thus, the unique solution to system $(\star)$ is $\lambda_A^* = \frac{1}{2X_B}$ and $\lambda_B^* = \frac{X_A}{2X_B^2}$ and we have the following corollary, which appears in a closely related form in Bell and Cover (1980), Sahuguet and Persico (2006), and Washburn (2013).

Corollary 2. *If $v_{A,j} = v_{B,j} \equiv v_j$ for all j, then the unique Nash equilibrium univariate marginals of the relaxed Blotto game are, for all $j \in \{1, \ldots, n\}$:*

$$\begin{array}{ll} F_{A,j}(x) = \left(1 - \frac{X_A}{X_B}\right) + \frac{x}{2v_j X_B}\left(\frac{X_A}{X_B}\right) & x \in [0, 2v_j X_B] \\ F_{B,j}(x) = \frac{x}{2v_j X_B} & x \in [0, 2v_j X_B] \end{array}$$

The expected payoff for player A is $\frac{X_A}{2X_B}$ and the expected payoff for player B is $1 - \frac{X_A}{2X_B}$.

Similar to the case of the linear heterogeneous asymmetric objective, the existing construction methods can be directly adapted to this generalized Colonel Blotto game with the linear heterogeneous symmetric objective.

Corollary 3. *If $v_{A,j} = v_{B,j} \equiv v_j$ for all j and it is the case that $\frac{2}{n_j} \leq \frac{X_A}{X_B} \leq 1$, then there exists a Nash equilibrium of the generalized Colonel Blotto game with the same set of univariate marginals and equilibrium expected payoffs as in Corollary 2.*

Given that, in the case of the linear heterogeneous symmetric objective, $v_{A,j} = v_{B,j} \equiv v_j$ for all j and $\lambda_A^* = \frac{1}{2X_B}$ and $\lambda_B^* = \frac{X_A}{2X_B^2}$, the Corollary 3 condition that $\frac{2}{n_j} \leq \frac{X_A}{X_B} \leq 1$ corresponds directly to the condition in Corollary 1, $\frac{2}{n_j} \leq \frac{v_{-i,j}\lambda_i^*}{v_{i,j}\lambda_{-i}^*}$. Note that, as before, $\frac{2}{n_j} \leq \frac{X_A}{X_B} \leq 1$ is a sufficient condition that allows us to use the construction method outlined above where we break the problem of constructing each P_i into the problem of construction each of the n_j-variate marginal distributions $P_{i,j}(x_j)$ and then setting $P_i(x) = \prod_{j=1}^{k} P_{i,j}(x_j)$ for each player i.

4 Conclusion

In this paper we have shown how existing solutions to the Colonel Blotto game can be extended to a subset of the possible parameter configurations in which battlefield valuations maybe heterogeneous across battlefields and asymmetric across players. Relaxing the constraint that the joint distribution must satisfy the budget constraint with probability one, we provide a complete characterization of equilibrium. When the budget constraint is required to hold with probability one, we find that existing methods of constructing the equilibrium joint distributions can be modified to cover a portion of the parameter space. The remaining subset of parameter valuations remains an open question.

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