Taylor's series for function of two variables.

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$f(x+h, y+k) = f(x, y+k) + h \frac{\partial f(x,y+k)}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 f(x,y+k)}{\partial x^2} + \frac{h^3}{3!} \frac{\partial^3 f(x,y+k)}{\partial x^3} + \cdots \quad (1)$$

$$f(x, y+k) = f(x,y) + k \frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^2 f(x,y)}{\partial y^3} + \cdots \quad (2)$$

Substitute (2) in (1).

$$f(x+h, y+k) = f(x,y) + k \frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f(x,y)}{\partial y^3} + \cdots$$

$$+ h \frac{\partial}{\partial x} \left\{ f(x,y) + k \frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f(x,y)}{\partial y^3} + \cdots \right\}$$

$$+ \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} \left\{ f(x,y) + k \frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f(x,y)}{\partial y^3} + \cdots \right\}$$

$$+ \frac{h^3}{3!} \frac{\partial^3}{\partial x^3} \left\{ f(x,y) + k \frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{k^3}{3!} \frac{\partial^3 f(x,y)}{\partial y^3} + \cdots \right\}$$

$$= f(x,y) + h \frac{\partial f(x,y)}{\partial x} + k \frac{\partial f(x,y)}{\partial y} + \frac{1}{2!} \left(h^2 \frac{\partial^2 f(x,y)}{\partial x^2} + 2hk \frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2 \frac{\partial^2 f(x,y)}{\partial y^2} \right)$$

$$+ \frac{1}{3!} \left(h^3 \frac{\partial^3 f(x,y)}{\partial x^3} + 3h^2 k \frac{\partial^3 f(x,y)}{\partial x \partial x \partial y} + 3hk^2 \frac{\partial^3 f(x,y)}{\partial x \partial y \partial y} + k^3 \frac{\partial^3 f(x,y)}{\partial y^3} \right)$$

$$= f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk + k^2 \frac{\partial^2}{\partial y^2}\right) f(x,y)$$

$$+ \frac{1}{3!}\left(\cancel{\frac{\partial^3}{\partial}}\, h^3\frac{\partial^3}{\partial x^3} + 3h^2k\frac{\partial^3}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3}{\partial x \partial y^2} + k^3\frac{\partial^3}{\partial y^3}\right) f(x,y)$$

$$+ \cdots$$

$$\boxed{= f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \cdots}$$

This is the Taylor's theorem for function of two variables.

Corollary-1. Putting $x=a$ and $y=b$, we get

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] +$$
$$\frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] +$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

Corollary-2. Putting $a+h = x$ and $b+k = y$.
i.e. $h = x-a$, $k = y-b$.

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] +$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] +$$

This is Taylor's expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$

Corollary-3:

Putting $a=0,\ b=0$ in corollary 2, we get.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

This is Maclaurin's ~~expansi~~ expansion of $f(x,y)$.

Ex:1 Expand $e^x \log(1+y)$ in powers of x and y upto third degree. ~~at (0,0)~~.

In powers of x and y implies $\begin{cases} a=0 \\ b=0. \end{cases}$

$f(x,y) = e^x \log(1+y)$ $\quad \therefore f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$ $\quad f_x(0,0) = 0$

$f_y(x,y) = e^x (1+y)^{-1}$ $\quad f_y(0,0) = 1.$

$f_{xx}(x,y) = e^x \log(1+y)$ $\quad f_{xx}(0,0) = 0$

$f_{xy}(x,y) = e^x (1+y)^{-1}$ $\quad f_{xy}(0,0) = 1.$

$f_{yy}(x,y) = -e^x (1+y)^{-2}.$ $\quad f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x \log(1+y)$ $\quad f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \dfrac{e^x}{1+y}.$ $\quad f_{xxy}(0,0) = 1.$

$f_{xyy} = -e^x (1+y)^{-2}$ $\quad f_{xyy}(0,0) = -1.$

$f_{yyy}(x,y) = +2e^x (1+y)^{-3}$ $\quad f_{yyy}(0,0) = 2.$

$$\therefore f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$$

$\therefore e^{x}\log(1+y) = 0 + [x \times 0 + y \times 1] + \frac{1}{2!}[x^2(0) + 2xy \times 1 + y^2 \times (-1)]$
$+ \frac{1}{3!}[x^3 \times (0) + 3x^2 y \times (1) + 3xy^2 \times (-1) + y^3 \times (2)] + \cdots$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2 y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \cdots$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2 y - xy^2) + \frac{1}{3}y^3 + \cdots$$

Ex-2.

Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem.

$f(x,y) = x^2y + 3y - 2$

In powers of $(x-1)$ and $(y+2)$ implies $\begin{cases} a = +1 \\ b = -2 \end{cases}$

Taylor's expansion of $f(x,y)$ in power of $(x-a)$ and $(y-b)$ is given by —

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y}\right] +$$
$$\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] +$$
$$\frac{1}{3!}\left[(x-a)^3\frac{\partial^3 f(a,b)}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3 f(a,b)}{\partial x^2 \partial y} + 3(x-a)(y-b)^2\frac{\partial^3 f(a,b)}{\partial x \partial y^2}\right.$$
$$\left. + (y-b)^3\frac{\partial^3 f(a,b)}{\partial y^3}\right] + \cdots$$

Hence $a = 1$, $b = -2$ and $f(x,y) = x^2y + 3y - 2$.

(2) $f(x,y) = x^2y+3y-2$ $\quad f(1,-2) = -2+3(-2)-2 = -10$

$f_x(x,y) = 2xy$ $\quad f_x(1,-2) = -4$

$f_y(x,y) = x^2+3$ $\quad f_y(1,-2) = 4$ $\quad f_{xx}(1,-2) = -4$.

$f_{xx}(x,y) = 2y = -4$ And all other higher order partial

$f_{xy}(x,y) = 2x = 2$ derivatives are zero.

~~$f_{yx}(x,y) = 2x$.~~

$f_{yy}(x,y) = 0$

~~$\therefore f(x,y)\ x^2+3y-2 = -7+(x-1)\times 2+(y+2)\times 3+$~~

$f_{xxx} = 0$. ~~$\frac{1}{2}(x-1)^2\times 2$~~

$f_{xxy} = 2$.

$f_{xyy} = 0$ ~~$x^2+3y-2 =$~~ ~~$f(x,y) =$~~

$f_{yyy} = 0$.

$$\therefore\ x^2y+3y-2 = -10+[(x-1)(-4)+(y+2)(4)]+$$

$$\frac{1}{2!}[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)]+$$

$$\frac{1}{6}[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$

$$= -10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2) + (x-1)^2(y+2)$$

H.W. Expand $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.