On the robustness of the relation $\beta_{tr} = 2\beta$

Definitions:

1. $$C_i(t) = \mathrm{Tr}\left\{\rho_\infty(\hat{n}_i(t) - \frac{1}{2})(\hat{n}_{i_0}(t=0) - \frac{1}{2})\right\} \qquad R^2(L,t) = \sum_{i\in\Lambda} |i - i_0|^2 C_i(t)$$

   $$\hat{N}_{\mathcal{D}} = \sum_{i\in\mathcal{D}} \hat{n}_i \qquad w^2(L,t) = \left\langle \hat{N}_{\mathcal{D}}^2 \right\rangle - \left\langle \hat{N}_{\mathcal{D}} \right\rangle^2$$

2. Defining exponents $\{\alpha_{tr}, \beta_{tr}, z_{tr} = \alpha_{tr}/\beta_{tr}\}$ for the FV scaling of MSD $R^2(L,t)$ and exponents $\{\alpha, \beta, z = \alpha/\beta\}$ for the FV scaling of subsystem number variance $w^2(L,t)$, i.e.,:

   $$R^2(L,t) = \begin{cases} t^{\beta_{tr}} \text{ for } t << L^{z_{tr}} \\ L^{\alpha_{tr}} \text{ for } t >> L^{z_{tr}} \end{cases}$$

   $$w^2(L,t) = \begin{cases} t^{\beta} \text{ for } t << L^{z} \\ L^{\alpha} \text{ for } t >> L^{z} \end{cases}$$

Questions:

1. Do MSD $R^2(L,t)$ and number variance $w^2(L,t)$ saturate (to a system size dependent value) on the same time scale, i.e., is $z = z_{tr}$?
   We know that for a single particle subspace:

   $$4C_i(t) = |\psi(i,t)|^2 = \langle \hat{n}_i \rangle_t$$

   $R^2(L,t)$ saturating in the time scale $L^{z_{tr}}$, implies that 2-point correlators $\langle \hat{n}_i \rangle_t$ saturate in the same time scale.
   Whilst $w^2(L,t)$ is made up of summation of 4-point and 2-point correlators- $\langle \hat{n}_i \hat{n}_j \rangle_t$ and $\langle \hat{n}_i \rangle_t$. **Hence, $z = z_{tr}$ implies that the 4-point correlators saturate in a time scale which is at the most equal to $L^{z_{tr}}$.**

2. $\alpha_{tr} = 2$?
   Assuming that at long times (and for systems with finite size) $\langle \hat{n}(t) \rangle$ saturates to a constant value (say, $= n$)(independent of i and t), we get:

   $$\lim_{t\to\infty} R^2(L,t) = n \sum_{i\in\Lambda} |i - i_0|^2$$
   $$\sum_{i\in\Lambda} |i - i_0|^2 \propto L^3 + L^2 + L$$

   where L is the size of the 1-D system. This gives $\alpha_{tr} = 3$