Dynamic Approximate Solutions of the HJ Inequality and of the HJB Equation for Input-Affine Nonlinear Systems

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***Abstract*—The solution of most nonlinear control problems hinges upon the solvability of partial differential equations or inequalities. In particular, disturbance attenuation and optimal control problems for nonlinear systems are generally solved exploiting the solution of the so-called Hamilton–Jacobi (HJ) inequality and the Hamilton–Jacobi–Bellman (HJB) equation, respectively. An explicit closed-form solution of this inequality, or equation, may however be hard or impossible to find in practical situations. Herein we introduce a methodology to circumvent this issue for input-affine nonlinear systems proposing a *dynamic*, i.e., time-varying, approximate solution of the HJ inequality and of the HJB equation the construction of which does not require solving any partial differential equation or inequality. This is achieved considering the *immersion* of the underlying nonlinear system into an *augmented* system defined on an *extended* state-space in which a (locally) positive definite *storage function*, or *value function*, can be explicitly constructed. The result is a methodology to design a *dynamic* controller to achieve $\mathcal{L}_2$-disturbance attenuation or approximate optimality, with asymptotic stability.**

***Index Terms*—$\mathcal{L}_2$-disturbance attenuation, Hamilton–Jacobi–Bellman partial differential equation, nonlinear systems, optimal Control.**

I. Introduction

In recent years, several nonlinear control problems have gained increasing interest. Often the methodologies proposed to solve these nonlinear control problems hinge upon the solution of (systems of) partial differential equations (pdes) or inequalities (pdis) the solution of which, when it exists, may however be hard or impossible to determine in specific examples. In addition the pdes arising in the above control problems are nonlinear. Therefore, the problem of finding approximate solutions to the partial differential equations or inequalities arising in nonlinear control problems has been addressed extensively. Note that partial solutions have already been proposed in specific contexts, see, e.g., [18] for the problems of observer design and adaptive control and [8], [25] for the stabilization problem of systems in strict-feedforward form.

Considering the class of input-affine smooth nonlinear systems, herein we focus on the disturbance attenuation and the optimal control problems, the classical solutions of which are given in terms of the solutions of the well-known Hamilton–Jacobi (HJ) partial differential inequality and the Hamilton–Jacobi–Bellman (HJB) partial differential equation, respectively. These are nonlinear first-order partial differential equations or inequalities. These problems are intrinsically related: the HJB pde can be obtained formally from the HJ partial differential inequality *relaxing* the disturbance attenuation constraint and replacing the inequality with the equality sign.

In most control applications (the output of) the dynamical system is affected by disturbances, which are in general unknown. An ideal achievement of the control law would be to render (the output of) the system insensitive to the disturbance. However, it has been shown that the conditions under which this goal can be achieved are seldom satisfied, see [16]. To avoid this issue a different approach has been recently pursued, namely the design of a control law that guarantees that the *effect* of the disturbance on (the output of) the system is kept under a desired level and that an equilibrium of the closed-loop system is (locally) asymptotically stable in the absence of disturbances. Obviously, the first issue to cope with to solve this problem is to define how the effect of an (unknown) disturbance on (the output of) a dynamical system can be measured. A natural choice is to consider the $\mathcal{L}_2$-induced norm of the nonlinear system, see for instance [10], [30], since for linear systems the $\mathcal{L}_2$-disturbance attenuation problem reduces to the time-domain formulation of the H_∞ control problem. The *infinite horizon* optimal control with stability deals with the problem of finding a control law such that the origin is an asymptotically stable equilibrium point of the closed-loop system and moreover a given criterion is minimized.

In recent years, several approaches to approximate, with a desired degree of accuracy, the solution of the Hamilton–Jacobi (Hamilton–Jacobi–Bellman, respectively) partial differential equation or inequality in a neighborhood possibly large of an equilibrium point have been proposed, see [3], [4], [9], [15], [19], [21], [23], [26], [32].

Most of these results rely either on the iterative computation of the coefficients of a local expansion of the solution, provided that all the functions of the nonlinear system are analytic (or at least smooth) or on the solution of the HJ inequality and of the HJB equation along the trajectories of the system. In particular,

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in [21], under the assumption of stabilizability of the zero equilibrium of the nonlinear system, which is supposed to be real analytic about the origin, an optimal stabilizing control law defined as the sum of a linear control law, that solves the linearized problem, and a convergent power series about the origin, starting with terms of order two, is proposed. Sufficient conditions that guarantee the convergence of the Galerkin approximation of the HJB equation over a compact set containing the origin are given in [4]. The local solution proposed in [32] hinges upon the technique of *apparent linearization* and the repeated computation of the steady-state solution of the Riccati equation. The basic idea of the state dependent Riccati equation (SDRE) is to solve the Riccati equation pointwise, using a state-dependent linear representation (SDLR), see [14]. The SDLR is not unique and for particular choices the SDRE not only provides a performance different from the optimal, but it may even fail to yield stability of the equilibrium point of the system. In [19], conditions for the existence of a local solution of the HJB equation, for a parameterized family of infinite horizon optimal control problems are given. In [23], it is shown that the solution of the HJB equation is the eigenfunction, relative to the zero eigenvalue, of the semigroup, which is max-plus linear, corresponding to the HJB equation, and a discrete-time approximation of the semigroup guarantees the convergence of the approximate solution to the actual one. Finally, a large effort has been devoted to avoid the hypothesis of differentiability of the storage function, interpreting the HJ inequality or the HJB equation in the viscosity sense, see [3], [9], and [26]. In particular the issue of existence, or uniqueness, of viscosity solutions has been extensively addressed and explored, see for instance [11]. A method to approximate viscosity solutions by means of a discretization in time and in the state variable has been proposed in [12] and using a domain decomposition without *overlapping* in [7].

The main contribution of this paper is a method to construct dynamically, i.e., by means of a dynamic extension, an exact solution of a (modified) HJ inequality or HJB equation, for input-affine nonlinear systems, without solving any partial differential equation. The conditions are given in terms of algebraic equations or inequalities, that can be shown to be solvable for specific classes of nonlinear systems including feedback linearizable, feedforward and fully actuated mechanical systems.

The rest of the paper is organized as follows. In Section II the description of the problem is given. Section III is dedicated to the definition of the notion of *algebraic $\bar{P}$ solution*. The $\mathcal{L}_2$-disturbance attenuation problem and the optimal control problem are discussed in Sections IV and V, respectively. In Section VI the particularization of these results to linear systems is proposed and the relation with the standard solution of the H_∞ disturbance attenuation and optimal control problems is investigated. A procedure to determine an *algebraic $\bar{P}$ solution* for classes of nonlinear systems is proposed in Section VII. The application of the method to the matched disturbance attenuation and the optimal control problems for fully actuated mechanical systems is presented in Section VIII. Finally, in the last two sections numerical examples are presented and conclusions are drawn, respectively.

Preliminary versions of this work have been published in [27], [28], and [29].

II. Definition of the Problem

A. $\mathcal{L}_2$-Disturbance Attenuation

Consider a nonlinear dynamical system affected by unknown disturbances described by equations of the form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u + p(x)d\\ z &= h(x) + l(x)u\end{aligned} \tag{1}$$

where the first equation describes a plant with state $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathbb{R}^m$, and exogenous input $d(t) \in \mathbb{R}^p$, while the second equation defines a penalty variable $z(t) \in \mathbb{R}^q$. In addition $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $p : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $l : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are smooth mappings.

Assumption 1: The mappings h and l in the system (1) are such that $h(x)^\top l(x) = 0$ and $l(x)^\top l(x) = I$, for all $x \in \mathbb{R}^n$.

The second condition of the Assumption 1 is needed to avoid a so-called *singular problem*, see for instance [2], [22].

Assumption 2: The mappings f and h are such that $f(0) = 0$ and $h(0) = 0$. Moreover, the nonlinear system (1), with $d(t) = 0$ for all $t \geq 0$ and output $y = h(x)$, is zero-state detectable, i.e., $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$ imply $\lim_{t\to\infty} x(t) = 0$.

As a consequence of Assumption 2 there exists some, possibly not unique, continuous function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$, for all x.

1) Problem 1: Consider the system (1) and let $\gamma > 0$. The *regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation* problem with stability consists in determining a nonnegative integer $\tilde{n}$, a dynamic control law of the form

$$\begin{aligned}\dot{\xi} &= \alpha(x, \xi)\\ u &= \beta(x, \xi)\end{aligned} \tag{2}$$

with $\xi(t) \in \mathbb{R}^{\tilde{n}}$, $\alpha : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{n}}$, $\beta : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^m$ smooth mappings and a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$, containing the origin[1] of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$, such that the closed-loop system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\beta(x, \xi) + p(x)d\\ \dot{\xi} &= \alpha(x, \xi)\\ z &= h(x) + l(x)\beta(x, \xi)\end{aligned} \tag{3}$$

has the following properties.

a) The zero equilibrium of the system (3) with $d(t) = 0$, for all $t \geq 0$, is asymptotically stable with region of attraction containing $\bar{\Omega}$.

b) For every $d \in \mathcal{L}_2(0, T)$ such that the trajectories of the system remain in $\bar{\Omega}$, the $\mathcal{L}_2$-gain of the system (3) from d to z is less than or equal to γ, i.e.,

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 dt \tag{4}$$

for all $T \geq 0$. $\diamond$

Note that if $\tilde{n} = 0$ then the problem is the standard static state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability.

It is well-known, see [30], that there exists a solution to the static state feedback $\mathcal{L}_2$-disturbance attenuation problem

[1]We say that the set Ω contains the origin to mean that it strictly contains the origin, i.e., $0 \in \Omega \setminus \partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω.

with stability, in some neighborhood of the origin, if there exists a smooth positive definite solution $V : \mathbb{R}^n \to \mathbb{R}_+$ of the Hamilton–Jacobi inequality[2]

$$V_x f(x) + \frac{1}{2} V_x \left[\frac{1}{\gamma^2} p(x) p(x)^\top - g(x) g(x)^\top \right] V_x^\top + \frac{1}{2} h(x)^\top h(x) \le 0 \quad (5)$$

for some $\gamma > 0$. Moreover the control law that solves the static state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability is given by $u = -g(x)^\top V_x^\top$.

In the linearized case the solution of the disturbance attenuation problem is given by a linear static state feedback of the form $u = -B_1^\top \bar{P} x$, where $\bar{P}$ is the symmetric positive definite solution of the algebraic Riccati equation

$$\bar{P} A + A^\top \bar{P} + \bar{P} \left[\frac{1}{\gamma^2} B_2 B_2^\top - B_1 B_1^\top \right] \bar{P} + H^\top H = 0 \quad (6)$$

where

$$\begin{aligned} A &\triangleq \frac{\partial f}{\partial x}\Big|_{x=0} = F(0) \\ H &\triangleq \frac{\partial h}{\partial x}\Big|_{x=0} \\ B_1 &\triangleq g(0) \\ B_2 &\triangleq p(0). \end{aligned} \quad (7)$$

Remark 1: As noted in [30] if the pair (H, A) is detectable, the solvability of the (6) guarantees the existence of a neighborhood $\mathcal{W}$ of the origin and of a smooth positive definite function V defined on $\mathcal{W}$ such that V is a solution of the Hamilton–Jacobi inequality (5). However, in most practical cases there is no *a priori* knowledge about the size of the neighborhood $\mathcal{W}$ in which the solution is defined. ▲

B. Optimal Control

Consider a nonlinear dynamical system described by an equation of the form

$$\dot{x} = f(x) + g(x) u \quad (8)$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ smooth mappings, where $x(t) \in \mathbb{R}^n$ denotes the state of the system and $u(t) \in \mathbb{R}^m$ the input. The *infinite horizon* optimal control problem with stability consists in finding u that minimizes the cost functional

$$J(x(0), u(t)) = \frac{1}{2} \int_0^\infty (q(x(t)) + u(t)^\top u(t)) dt \quad (9)$$

where $q : \mathbb{R}^n \to \mathbb{R}_+$ is positive semi-definite, subject to the dynamical constraint (8), the initial condition $x(0) = x_0$ and the requirement that the zero equilibrium of the closed-loop system be locally asymptotically stable.

Assumption 3: The vector field f is such that $f(0) = 0$, i.e., $x = 0$ is an equilibrium point for the system (8) when $u(t) = 0$ for all $t \ge 0$.

Assumption 4: The nonlinear system (8) with output $y = q(x)$ is zero-state detectable.

[2]In what follows the notation V_x denotes the row vector of the partial derivatives with respect to x of the scalar function $V : x \to \mathbb{R}$, $x \in \mathbb{R}^n$.

As already noted, by Assumption 3, $f(x) = F(x)x$, for some, possibly not unique, continuous function $F : \mathbb{R}^n \to \mathbb{R}^{n \times n}$. The classical optimal control design methodology relies on the solution of the HJB equation [5], [6], [31]

$$\min_u \left\{ V_x (f(x) + g(x) u) + \frac{1}{2} q(x) + \frac{1}{2} u^\top u \right\} = 0. \quad (10)$$

The solution of the HJB (10), if it exists, is the *value function* of the optimal control problem, i.e., it is a function which associates to every point in the state space, x_0, the optimal cost of the trajectory of system (8) with $x(0) = x_0$, i.e.,

$$V(x_0) = \min_u \frac{1}{2} \int_0^\infty (q(x(t)) + u(t)^\top u(t)) dt. \quad (11)$$

The knowledge of the value function on the entire state space allows to determine the minimizing input for all initial conditions. It is easy to check that the minimum of (10) with respect to u is attained for

$$u_o = -g(x)^\top V_x^\top. \quad (12)$$

Thus, if we are able to solve analytically the partial differential equation

$$V_x f(x) - \frac{1}{2} V_x g(x) g(x)^\top V_x^\top + \frac{1}{2} q(x) = 0 \quad (13)$$

we can design the optimal control law given by (12).

1) Problem 2: Consider system (8), with Assumptions 3 and 4, and the cost functional (9). The *regional dynamic optimal control* problem with stability consists in determining an integer $\tilde{n} \ge 0$, a dynamic control law described by (2) and a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x) \beta(x, \xi) \\ \dot{\xi} &= \alpha(x, \xi) \end{aligned} \quad (14)$$

has the following properties.

a) The zero equilibrium of the system (14) is asymptotically stable with region of attraction containing $\bar{\Omega}$.
b) For any $\bar{u}$ and any (x_0, ξ_0) such that the trajectory of the system (14) remain in $\bar{\Omega}$ $J((x_0, \xi_0), \beta) \le J((x_0, \xi_0), \bar{u})$, where β is defined in (2).

◇

2) Problem 3: Consider system (8), with Assumptions 3 and 4, and the cost functional (9). The *approximate regional dynamic optimal control* problem with stability consists in determining an integer $\tilde{n} \ge 0$, a dynamic control law described by (2), a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ and a function $c : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}_+$ such that the regional dynamic optimal control problem is solved with respect to the running cost $q(x) + c(x, \xi) + u^\top u$. ◇

Finally, recall that in the linearized case the solution of the optimal control problem is a linear static state feedback of the form $u = -B_1^\top \bar{P} x$, where $\bar{P}$ is the symmetric positive definite solution of the algebraic Riccati equation

$$\bar{P} A + A^\top \bar{P} - \bar{P} B_1 B_1^\top \bar{P} + Q = 0 \quad (15)$$

the matrices A and B_1 are defined in (7) and $Q \triangleq \frac{1}{2} \partial^2 q / \partial x^2 \big|_{x=0}$.

III. ALGEBRAIC $\bar{P}$ SOLUTION

Following [27] and [28], consider the HJ inequality (5) or the HJB (13) and suppose that they can be solved *algebraically*, as detailed in the following definition.

Definition 1: Consider system (1) (system (8), respectively), with Assumption 1 (Assumption 3, respectively), and let $\gamma > 0$. Let $\sigma(x) \triangleq x^\top \Sigma(x) x \geq 0$, with $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\Sigma(0) = 0$. A $\mathcal{C}^1$ mapping $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, zero at zero, is said to be a *$\mathcal{X}$-algebraic $\bar{P}$ solution* of inequality (5) [(13), respectively] if the following holds.

1) For all $x \in \mathcal{X} \subseteq \mathbb{R}^n$

$$P(x)f(x) + \frac{1}{2}P(x)\Pi(x)P(x)^\top + \frac{1}{2}\mathcal{H}(x) + \sigma(x) \leq 0 \quad (16)$$

where $\Pi(x) = (1/\gamma^2)p(x)p(x)^\top - g(x)g(x)^\top$, $\mathcal{H}(x) = h(x)^\top h(x)$ for the $\mathcal{L}_2$-disturbance attenuation problem and $\Pi(x) = -g(x)g(x)^\top$, $\mathcal{H}(x) = q(x)$ for the optimal control problem.

2) $P(x)$ is tangent at $x = 0$ to the symmetric positive definite solution of (6) ((15), respectively), i.e., $\partial P(x)^\top/\partial x\Big|_{x=0} = \bar{P}$.

If condition (i) holds for all $x \in \mathbb{R}^n$, i.e., $\mathcal{X} = \mathbb{R}^n$, then P is an *algebraic $\bar{P}$ solution*. $\diamond$

Remark 2: Let $P(x) = [P_1(x), \ldots, P_n(x)]$, with $P_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$, be an *algebraic $\bar{P}$ solution* of (16). Then there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that $V_x(x) = P(x)$ if and only if

$$\frac{\partial P_i}{\partial x_j}(x) = \frac{\partial P_j}{\partial x_i}(x) \quad (17)$$

for all x and $i, j = 1, 2, \ldots, n$. Obviously, since an arbitrary mapping that solves the inequality (16) is selected, the mapping P may not satisfy condition (17). ▲

Note that any $\mathcal{C}^2$ positive definite function σ can be written as $x^\top \Sigma(x) x$ for some continuous non-unique matrix valued function Σ. Moreover note that P is not assumed to be a gradient vector. A similar approach is proposed in [20] where the solutions of the Hamilton–Jacobi partial differential inequality are characterized in terms of nonlinear matrix inequalities (NLMI). Therein, however, it is assumed that the solution of the NLMI is a gradient vector.

Remark 3: For any given σ the condition (16) is an algebraic inequality in n unknowns, namely the components P_i $i = 1, \ldots, n$ of the mapping P. Moreover, any solution V of the HJ inequality or of the HJB inequality is such that V_x is a solution of the inequality (16). Finally, since $\Sigma(0) = 0$, the solvability of the algebraic Riccati (6) [(15), respectively] implies the existence of a *$\mathcal{X}$-algebraic $\bar{P}$ solution* for some non-empty set $\mathcal{X} \subseteq \mathbb{R}^n$. ▲

In what follows we assume the existence of an *algebraic $\bar{P}$ solution*, i.e., we assume $\mathcal{X} = \mathbb{R}^n$. Note that all the statements can be modified accordingly if $\mathcal{X} \subset \mathbb{R}^n$. Using the *algebraic $\bar{P}$ solution* P, define the function

$$V(x,\xi) = P(\xi)x + \frac{1}{2}\|x - \xi\|_R^2 \quad (18)$$

with $\xi(t) \in \mathbb{R}^n$ and $R = R^\top \in \mathbb{R}^{n\times n}$ positive definite, where $\|v\|_R^2$ denotes the squared Euclidean norm of the vector v weighted by the matrix R, i.e., $\|v\|_R^2 = v^\top R v$.

To provide concise statements of the main results, define the mapping

$$\Delta(x,\xi) = (R - \Phi(x,\xi))\Lambda(\xi)^\top \quad (19)$$

with $\Lambda(\xi) = \Psi(\xi)R^{-1}$, where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a continuous matrix valued function such that

$$P(x) - P(\xi) = (x - \xi)^\top \Phi(x,\xi)^\top \quad (20)$$

$\Psi(\xi) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of the mapping $P(\xi)$ and $A_{cl}(x) = F(x) + \Pi(x)N(x)$ with $N(x)$ such that $P(x) = x^\top N(x)^\top$. Note that the mappings Φ and N always exist since $P(x) - P(\xi) = 0$ for $x = \xi$ and $P(x) = 0$ for $x = 0$, respectively.

Remark 4: Consider V as in (18) and note that there exist a non-empty compact set $\Omega_1 \subseteq \mathbb{R}^{2n}$ containing the origin and a positive definite matrix $\bar{R}$ such that for all $R \geq \bar{R}$ the function V in (18) is positive definite for all $(x,\xi) \in \Omega_1 \subseteq \mathbb{R}^{2n}$. In fact, since P is tangent at $x = 0$ to the solution of the algebraic Riccati equation, the function $P(x)x : \mathbb{R}^n \to \mathbb{R}$ is, locally around the origin, quadratic and moreover has a local minimum for $x = 0$. Hence, the existence of $\bar{R}$ can be proved noting that the function $P(\xi)x$ is (locally) quadratic in (x,ξ) and, restricted to the manifold $\mathcal{M} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = x\}$, is positive definite. ▲

IV. $\mathcal{L}_2$-DISTURBANCE ATTENUATION PROBLEM

The following statement provides a solution to the regional dynamic state feedback $\mathcal{L}_2$—disturbance attenuation problem with stability, namely Problem 1.

Theorem 1: Consider system (1), with Assumptions 1 and 2, and let $\gamma > 0$. Let P be an *algebraic $\bar{P}$ solution* of (5). Let the matrix $R = R^\top > 0$ be such that the function V defined in (18) is positive definite in a set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and such that

$$\frac{1}{2}A_{cl}(x)^\top \Delta + \frac{1}{2}\Delta^\top A_{cl}(x) + \frac{1}{2}\Delta^\top \Pi(x)\Delta < \Sigma(x) \quad (21)$$

for all $(x,\xi) \in \Omega \setminus \{0\}$, with $\Pi(x) = 1/\gamma^2 p(x)p(x)^\top - g(x)g(x)^\top$. Then there exists $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the function V satisfies the Hamilton–Jacobi inequality

$$\begin{aligned}\mathcal{HJ}(x,\xi) &\triangleq V_x f(x) + V_\xi \dot{\xi} + \frac{1}{2}h(x)^\top h(x) \\ &+ \frac{1}{2}V_x\left[\frac{1}{\gamma^2}p(x)p(x)^\top - g(x)g(x)^\top\right]V_x^\top \leq 0 \quad (22)\end{aligned}$$

for all $(x,\xi) \in \Omega$, with $\dot{\xi} = -kV_\xi^\top = -k(\Psi(\xi)^\top x - R(x-\xi))$. Hence,

$$\begin{aligned}\dot{\xi} &= -k(\Psi(\xi)^\top x - R(x - \xi)) \\ u &= -g(x)^\top[P(x)^\top + (R - \Phi(x,\xi))(x - \xi)] \quad (23)\end{aligned}$$

solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability for all $(x,\xi) \in \bar{\Omega}$, where $\bar{\Omega}$ is the largest level set of V contained in Ω.

Remark 5: The control input u defined in (23) contains the *algebraic input*, i.e., the term $-g(x)^\top P(x)$, obtained from the solution of the (16) and a *dynamic compensation* term, i.e., $-g(x)^\top(R - \Phi(x,\xi))(x - \xi)$. ▲

Proof: To begin with note that the partial derivatives of the function V defined in (18) are

$$\begin{aligned} V_x &= P(x) + P(\xi) - P(x) + (x-\xi)^\top R \\ &= P(x) + (x-\xi)^\top (R - \Phi(x,\xi))^\top \\ V_\xi &= x^\top \Psi(\xi) - (x-\xi)^\top R. \end{aligned} \tag{24}$$

The Hamilton–Jacobi inequality (22), considering the partial derivatives of V as in (24), the dynamics of the controller as in (23) and recalling that the mapping P is an *algebraic* $\bar{P}$ *solution* of the inequality (16) reads

$$\begin{aligned} &- x^\top \Sigma(x) x + (x-\xi)^\top (R-\Phi)^\top F(x) x \\ &+ \frac{1}{2}(x-\xi)^\top (R-\Phi)^\top \Pi(x)(R-\Phi)(x-\xi) \\ &+ x^\top N(x)^\top \Pi(x)(R-\Phi)(x-\xi) \\ &- k(x^\top \Psi(\xi) - (x-\xi)^\top R)(\Psi(\xi)^\top x - R(x-\xi)) \le 0. \end{aligned} \tag{25}$$

Rewriting the inequality (25) as a *quadratic* form in x and $(x-\xi)$ yields

$$-[x^\top (x-\xi)^\top][M(x,\xi) + kC(\xi)^\top C(\xi)] \begin{bmatrix} x \\ (x-\xi) \end{bmatrix} \le 0 \tag{26}$$

where $C(\xi) = [\Psi(\xi)^\top \; -R]$ is a $n \times 2n$ matrix with constant rank n, since R is positive definite, and

$$M(x,\xi) = \begin{bmatrix} \Sigma(x) & \Gamma_1(x,\xi) \\ \Gamma_1(x,\xi)^\top & \Gamma_2(x,\xi) \end{bmatrix}$$

where $\Gamma_1(x,\xi) = -(1/2)A_{cl}(x)^\top (R - \Phi(x,\xi))$ and $\Gamma_2(x,\xi) = -(1/2)(R - \Phi(x,\xi))^\top \Pi(x)(R - \Phi(x,\xi))$. Note that the null space of $C(\xi)$ is spanned by the columns of the matrix

$$Z(\xi) = \begin{bmatrix} I \\ R^{-1}\Psi(\xi)^\top \end{bmatrix} = \begin{bmatrix} I \\ \Lambda(\xi)^\top \end{bmatrix}. \tag{27}$$

Exploiting the results in [1], consider the restriction of the matrix $M(x,\xi)$ to the subspace defined by the columns of the matrix $Z(\xi) \in \mathbb{R}^{2n\times n}$ and note that the inertia of the matrix $Z(\xi)^\top M(x,\xi) Z(\xi)$ does not depend on the choice of the basis for the null space of $C(\xi)$. Finally, condition (21) implies that the matrix $Z(\xi)^\top M(x,\xi) Z(\xi)$ is positive definite. This proves that there exists $\bar{k} \ge 0$ such that the Hamilton–Jacobi inequality (22) is satisfied for all $(x,\xi) \in \Omega \setminus \{0\}$ and for all $k > \bar{k}$. Furthermore, $\mathcal{HJ}(0,0) = 0$; hence, by continuity of the mappings in system (1), $\mathcal{HJ}(x,\xi)$ is continuous and smaller than or equal to zero for all $(x,\xi) \in \Omega$. Moreover, note that, by the condition (22), V is a non-strict Lyapunov function for the closed-loop system (3) with $d(t) = 0$ for all $t \ge 0$. In fact, $V(x,\xi) > 0$ for all $(x,\xi) \in \Omega \setminus \{0\}$ and $\dot{V} \le 0$. Moreover, the system (3), with $d(t) = 0$ for all t and $\alpha(x,\xi)$ and $\beta(x,\xi)$ defined in (23), is zero-state detectable with respect to the output $y = h(x)$. To prove this claim consider system (3) and note that $d(t) = 0$, $\beta(x(t),\xi(t)) = 0$, $h(x(t)) = 0$, for all $t \ge 0$, imply, by Assumption 2, that $x(t)$ asymptotically converges to zero while $\xi(t)$ belongs, by (22), to the compact set $\{(x,\xi) : V(x,\xi) \le V(x(0),\xi(0))\}$ for all $t \ge 0$. Therefore, standard arguments allow to conclude that also $\xi(t)$ tends to zero for t that goes to infinity.

Hence by LaSalle's invariance principle and zero-state detectability, the feedback (23) asymptotically stabilizes the zero equilibrium of the closed-loop system. Thus the dynamic control law $\dot{\xi} = -kV_\xi(x,\xi)^\top$, $u = -g(x)^\top V_x(x,\xi)^\top$ solves the regional dynamic $\mathcal{L}_2$-disturbance attenuation problem with stability for the system (1). Finally, since the condition (4) is satisfied for all the trajectories of system (1) that remain in the set Ω driven by the dynamic control law (23), then it is straightforward to conclude that $u, y \in \mathcal{L}_2(0,\infty)$ for all such disturbances $d \in \mathcal{L}_2(0,\infty)$. □

Remark 6: The vector field $A_{cl}(x)x = (F(x) + \Pi(x)N(x))x$ describes the closed-loop nonlinear system when the *algebraic* feedback control law $u = -g(x)^\top P(x)^\top$ and the *worst case disturbance* $d = (1/\gamma^2)p(x)^\top P(x)^\top$ are *implemented*. ▲

Remark 7: The conditions (16) and (21) imply

$$\mathcal{P}(x,\xi)^\top f(x) + \frac{1}{2}h(x)^\top h(x) + \frac{1}{2}\mathcal{P}(x,\xi)^\top \Pi \mathcal{P}(x,\xi) < 0 \tag{28}$$

for all $x \in \Omega \setminus \{0\}$, where $\mathcal{P}(x,\xi) = P(x) + x^\top \Delta(x,\xi)$, which highlights that inequality (16) has to be satisfied *robustly*. Note that (28) guarantees that the conditions (16) and (21) are independent of the choice of the matrices $N(x)$ and $F(x)$. ▲

The following statement provides an alternative solution to the regional $\mathcal{L}_2$—disturbance attenuation problem with stability.

Theorem 2: Consider system (1), with Assumptions 1 and 2, and let $\gamma > 0$. Let P be an *algebraic* $\bar{P}$ *solution* of (5) and suppose that $\Phi(x,\xi) = \Phi(x,\xi)^\top > \bar{R}$, for all $(x,\xi) \in \Omega$, where $\bar{R}$ is such that the function $\bar{V}(x,\xi) = P(\xi)x + (1/2)(x-\xi)^\top \bar{R}(x-\xi)$ is positive definite in Ω. Assume additionally that

$$\Lambda(\xi)\dot{\Phi}(x,\xi)\Lambda(\xi)^\top < \Sigma(x) \tag{29}$$

for all $(x,\xi) \in \Omega \setminus \{0\}$. Then $u = -g(x)^\top P(x)$ solves the regional dynamic $\mathcal{L}_2$-disturbance attenuation problem with stability.

Remark 8: The closed-loop system $\dot{x} = f(x) - g(x)g(x)^\top P(x)$ is independent of the dynamic extension ξ, i.e., Theorem 2 provides a *static* control law that solves the regional dynamic $\mathcal{L}_2$—disturbance attenuation problem with stability. ▲

Proof: To begin with, define

$$V(x,\xi,t) = P(\xi)x + \frac{1}{2}(x-\xi)^\top R(t)(x-\xi) \tag{30}$$

where $R(t) = \Phi(x(t),\xi(t))$, for all $t \ge 0$. Note that, by the assumptions on the mapping Φ, $R(t)$ is such that the function V in (30) is uniformly positive definite for all $(x,\xi,t) \in \Omega \times \mathbb{R}$. Consider now the Hamilton–Jacobi partial differential inequality

$$V_t + \mathcal{HJ}(x,\xi) \le 0. \tag{31}$$

Following the same steps of the proof of Theorem 1, it can be shown that the condition (29), similarly to the condition (21) of

Theorem 1, implies that the function V in (30) satisfies the HJ inequality (31) for all $(x,\xi,t)\in\Omega\times\mathbb{R}$.

Hence the control law

$$\begin{aligned}\dot{\xi} &= -k(\Psi(\xi)^\top x - R(t)(x-\xi))\\ u &= -g(x)^\top P(x)^\top\end{aligned}\tag{32}$$

solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability for all $(x,\xi)\in\Omega$. □

The conditions of Theorem 2 can be further relaxed under additional assumptions. To this end, consider the class of *algebraic $\bar{P}$ solutions* yielding a matrix $\Phi(x,\xi)$ such that[3]

$$(x-\xi)^\top(\Phi(x,\xi)-\bar{R})(x-\xi)\geq 0\tag{33}$$

for all $(x,\xi)\in\mathbb{R}^{2n}$. Condition (33) is sufficient to guarantee that the function V in (30) is uniformly positive definite in the set Ω and for all $t\geq 0$.

Lemma 1: Suppose that $\Phi(x,\xi)=\Phi(x,\xi)^\top$, then

$$\left(\frac{\partial\mathcal{Q}^\top}{\partial x}\right)+\left(\frac{\partial\mathcal{Q}^\top}{\partial x}\right)^\top\geq\bar{R}-\bar{P}\tag{34}$$

for all $x\in\mathbb{R}^n$, where $\mathcal{Q}(x)=P(x)-x^\top\bar{P}$, if and only if condition (33) holds for all $(x,\xi)\in\mathbb{R}^{2n}$.

Proof: Since $\bar{P}>\bar{R}=(1/2)\bar{P}$, the condition (34) is equivalent to requiring that the mapping $P(x)-x^\top\bar{R}$ is monotone, i.e.,

$$(P(x)-x^\top\bar{R}-P(\xi)+\xi^\top\bar{R})(x-\xi)\geq 0\tag{35}$$

for all $(x,\xi)\in\mathbb{R}^{2n}$. Recalling (20)

$$(P(x)-x^\top\bar{R}-P(\xi)+\xi^\top\bar{R})=(x-\xi)^\top(\Phi(x,\xi)-\bar{R}).\tag{36}$$

Necessity follows immediately right multiplying both sides of (36) by $(x-\xi)$. To show sufficiency, suppose, by contradiction, that there exists $(\bar{x},\bar{\xi})$ such that (33) does not hold. Then

$$\begin{aligned}0&>(\bar{x}-\bar{\xi})^\top(\Phi(\bar{x},\bar{\xi})-\bar{R})(\bar{x}-\bar{\xi})\\ &=(P(\bar{x})-\bar{x}^\top\bar{R}-P(\bar{\xi})+\bar{\xi}^\top\bar{R})(\bar{x}-\bar{\xi})\geq 0\end{aligned}$$

where the last inequality is obtained by monotonicity of $P(x)-x^\top\bar{R}$, hence the hypothesis of existence of $(\bar{x},\bar{\xi})$ is contradicted. □

We summarize the above discussion in the following result.

Corollary 1: Consider system (1), with Assumptions 1 and 2, and let $\gamma>0$. Let P be an *algebraic $\bar{P}$ solution* of (5) and suppose that condition (34) holds. Assume that the condition (29) is satisfied for all $(x,\xi)\in\mathbb{R}^{2n}\backslash\{0\}$. If $\bar{V}$ is positive definite and radially unbounded for all $(x,\xi)\in\mathbb{R}^{2n}$, then the control law $u=-g(x)^\top P(x)$ globally solves the dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability.

Remark 9: Selecting $R=\Phi(0,0)>0$ the matrix valued function Δ is such that $\Delta(0,0)=0$; hence, since it is continuous, it is sufficiently small in a neighborhood of the origin. Moreover, assume additionally that $\Sigma(0)=\bar{\Sigma}>0$ in the definition of *algebraic $\bar{P}$ solution*. Then, by continuity of the left-hand side of inequality (21), there exists a non-empty subset $\Omega_2\subset\mathbb{R}^{2n}$ containing the origin such that the condition (21) is satisfied for all $(x,\xi)\in\Omega_2$. Therefore, the *algebraic $\bar{P}$ solution* of (16), with $\Sigma(0)=\bar{\Sigma}$, solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability for all $(x,\xi)\in\Omega_1\cap\Omega_2$, with Ω_1 defined in Remark 4. In the optimal control problem, however, an additional cost is *paid* by the linearized solution $\bar{P}$, as discussed in the following. ▲

Corollary 2: Let P be an *algebraic $\bar{P}$ solution* of (16) with $\Sigma(0)>0$. Then, letting $R=\Phi(0,0)$, there exist a neighborhood of the origin $\Omega\subseteq\mathbb{R}^{2n}$ and $\bar{k}\geq 0$ such that for all $k>\bar{k}$ the function $V>0$ in (18) satisfies the partial differential inequality (22) for all $(x,\xi)\in\Omega$.

V. Optimal Control

In this section, the approximate regional dynamic nonlinear optimal control problem is solved.

Theorem 3: Consider system (8), with Assumptions 3 and 4 and the cost defined in (9). Let P be an *algebraic $\bar{P}$ solution* of (13). Let the matrix $R=R^\top>0$ be such that V defined in (18) is positive definite in a set $\Omega\subseteq\mathbb{R}^{2n}$ containing the origin and such that

$$\frac{1}{2}A_{cl}(x)^\top\Delta+\frac{1}{2}\Delta^\top A_{cl}(x)<\Sigma(x)+\frac{1}{2}\Delta^\top g(x)g(x)^\top\Delta\tag{37}$$

for all $(x,\xi)\in\Omega\backslash\{0\}$, with Δ defined in (19). Then there exists $\bar{k}\geq 0$ such that for all $k>\bar{k}$ the function V in (18) satisfies the Hamilton–Jacobi–Bellman inequality

$$\begin{aligned}\mathcal{HJB}(x,\xi)\triangleq{}&V_x f(x)+V_\xi\dot{\xi}+\frac{1}{2}q(x)\\ &-\frac{1}{2}V_x g(x)g(x)^\top V_x^\top\leq 0\end{aligned}\tag{38}$$

for all $(x,\xi)\in\Omega$, with $\dot{\xi}=-kV_\xi^\top=-k(\Psi(\xi)^\top x-R(x-\xi))$. Hence,

$$\begin{aligned}\dot{\xi}&=-k(\Psi(\xi)^\top x-R(x-\xi))\\ u&=-g(x)^\top[P(x)^\top+(R-\Phi(x,\xi))(x-\xi)]\end{aligned}\tag{39}$$

solves the approximate regional dynamic optimal control problem with $c(x,\xi)\geq 0$ such that $\mathcal{HJB}(x,\xi)+(1/2)c(x,\xi)=0$

Proof: The proof follows the same arguments as those of the proof of Theorem 1. The Hamilton–Jacobi–Bellman inequality for the *extended* system (14), i.e., the inequality (38), considering the partial derivatives of V as in (24), the controller as in (39) and recalling that the mapping P is an *algebraic $\bar{P}$ solution* of the (16), can be written as

$$\begin{aligned}&-x^\top\Sigma(x)x+(x-\xi)^\top(R-\Phi)^\top F(x)x\\ &-\frac{1}{2}(x-\xi)^\top(R-\Phi)^\top g(x)g(x)^\top(R-\Phi)(x-\xi)\\ &-x^\top N(x)^\top g(x)g(x)^\top(R-\Phi)(x-\xi)\\ &-k(\Psi(\xi)^\top x-R(x-\xi))^\top(\Psi(\xi)^\top x-R(x-\xi))\leq 0.\end{aligned}\tag{40}$$

Rewriting (40) as a *quadratic* form in x and $(x-\xi)$ yields (26) where $C(\xi)$ and $M(x,\xi)$ are defined in the proof of Theorem 1, with $\Pi(x)=-g(x)g(x)^\top$. Let the space spanned by the columns of the matrix $Z(\xi)$ be the null space of $C(\xi)$. Consider

[3]Condition (33) is milder than requiring $(\Phi(x,\xi)-\bar{R})\geq 0$, i.e., $v^\top(\Phi(x,\xi)-\bar{R})v\geq 0$ for all $v\in\mathbb{R}^n$.

now the restriction of the matrix $M(x,\xi)$ to the subspace defined by the columns of the matrix $Z(\xi) \in \mathbb{R}^{2n\times n}$. The condition (37) guarantees that the matrix $Z(\xi)^\top M(x,\xi)Z(\xi)$ is positive definite. This proves, by [1], that there exists $\bar{k} \geq 0$ such that the Hamilton–Jacobi–Bellman inequality (38) is satisfied for all $(x,\xi) \in \Omega \setminus \{0\}$ and for all $k > \bar{k}$. Furthermore, $\mathcal{HJB}(0,0) = 0$; hence, by continuity of the mappings in system (8) and in the control (39), the function $\mathcal{HJB}$ is continuous and smaller than or equal to zero for all $(x,\xi) \in \Omega$. Hence, the dynamic control law $\dot{\xi} = -kV_\xi(x,\xi)^\top$, $u = -g(x)^\top V_x(x,\xi)^\top$ is a dynamic optimal control for the system (8) and the running cost $q(x) + c(x,\xi) + u^\top u$. Finally, $V > 0$, by assumption, and $\dot{V} \leq 0$, by the condition (38), for all $(x,\xi) \in \Omega \setminus \{0\}$. Hence, by LaSalle's invariance principle and zero-state detectability, the feedback (39) asymptotically stabilizes the zero equilibrium of the closed-loop system. □

Remark 10: The problem solved herein is intrinsically different from the so-called *inverse optimal control problem* [13], where it is shown that optimality with respect to any meaningful cost functional guarantees several robustness properties of the closed-loop system. In fact, herein optimality of the control law is ensured with respect to a cost functional which upperbounds the original one, i.e., the approximate optimal control is determined with respect to the original cost and an *extra-cost*. ▲

In what follows two different approaches to reduce the *approximation error* of the solution of the approximate regional dynamic optimal control problem are proposed. The running cost imposed on the state of the extended system, namely $q(x) + c(x,\xi)$, can be *shaped* to approximate the original cost, on one hand, and the initial condition of the dynamic extension $\xi(0)$ can be selected to obtain the minimum value of the cost *paid* by the solution on the other hand, as detailed in the following results.

Theorem 4: Consider system (8), with Assumptions 3 and 4 and the cost defined in (9). Let P be an *algebraic* $\bar{P}$ *solution* of (13). Let $\mathcal{M}_C \triangleq \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \Psi(\xi)x - R(x-\xi) = 0\}$ and the matrix $R = R^\top > 0$ be such that V defined in (18) is positive definite in a set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and such that the condition

$$0 < \Sigma(x) + \frac{1}{2}\Delta^\top g(x)g(x)^\top \Delta - \frac{1}{2}\left(A_{cl}(x)^\top \Delta + \Delta^\top A_{cl}(x)\right) \leq \varepsilon I \quad (41)$$

is satisfied for all $(x,\xi) \in \Omega \setminus \{0\}$ and for some $\varepsilon \in \mathbb{R}_+$. Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be defined as $K(x,\xi) = (CC^\top)^{-1}\mathcal{G}(x,\xi)(CC^\top)^{-1}$, where $\mathcal{G} = CMZ\left(Z^\top MZ\right)^{-1} Z^\top MC^\top - CMC^\top$ and suppose that K is continuous at the origin. Then the control

$$\begin{aligned}\dot{\xi} &= -K(x,\xi)(\Psi(\xi)^\top x - R(x-\xi))\\ u &= -g(x)^\top [P(x)^\top + (R - \Phi(x,\xi))(x-\xi)]\end{aligned} \quad (42)$$

solves the approximate regional dynamic optimal control problem with $c(x,\xi) = 0$ for all $(x,\xi) \in \Omega \setminus \mathcal{M}_C$ and $0 \leq c(x,\xi) \leq \varepsilon y_1^\top y_1$, for all $(x,\xi) \in \mathcal{M}_C$, with

$$y_1 = \begin{bmatrix} I_n & (Z^\top MZ)^{-1}Z^\top MC^\top \end{bmatrix} \begin{bmatrix} Z^\top \\ C \end{bmatrix}^{-1} \begin{bmatrix} x \\ x-\xi \end{bmatrix}.$$

Proof: Let $T = [Z^\top C]^\top$ and note that the matrix T is full rank. Multiply, to the left and to the right, the matrix $M(x) + C(\xi)^\top K(x,\xi)C(\xi)$ by $T^\top$ and T, respectively, yielding

$$\begin{bmatrix} Z^\top MZ & Z^\top MC^\top \\ CMZ & CMC^\top + CC^\top KCC^\top \end{bmatrix} \quad (43)$$

where we have used the fact that $C(\xi)Z(\xi) = 0$ and that $C(\xi)C(\xi)^\top \in \mathbb{R}^{n\times n}$ is non-singular, for all $\xi \in \mathbb{R}^n$. Note that the quadratic forms associated to the matrices (43) and $M + C^\top KC$ are congruent. By Sylvester's law of inertia, the inertia of the matrix (43) is equal to the inertia of the matrix $M + C^\top KC$. The matrix function K is selected to maximize the number of zero eigenvalues of the *quadratic* form c. In particular the matrix (43) loses rank if and only if the Schur complement of the element $Z^\top MZ$, i.e., $CMC^\top + CC^\top KCC^\top - CMZ(Z^\top MZ)^{-1}Z^\top MC^\top$, is zero. Note that $Z^\top MZ$ is positive definite by condition (41), as shown in the proof of Theorem 3; hence the Schur complement is well-defined. Thus the matrix function K is selected to zero the Schur complement of the term $Z^\top MZ$ and it is such that the additional cost c is identically zero for all $(x,\xi) \in \Omega \setminus \mathcal{M}_C$ whereas the condition (41) implies that c is upper-bounded by $\varepsilon y_1^\top y_1$ for all $(x,\xi) \in \mathcal{M}_C$, proving the claim. □

Remark 11: If the closed-loop system (14), with α and β as in (42), is such that $(x(t),\xi(t)) \in \Omega \setminus \mathcal{M}_C$ for all $t \geq 0$, then the control law (42) is the optimal solution with respect to the original cost (9) since $c(x(t),\xi(t)) = 0$ for all $t \geq 0$. ▲

We conclude the section noting that, by definition of value function, $V(x(0),\xi(0))$ is the cost achieved by the dynamic control law defined in (42) for the system (8) initialized in $(x(0),\xi(0))$. Therefore, to minimize the cost, for a given initial condition $\bar{x}(0)$ of system (8), it is possible to select the initial condition of the dynamic extension $\xi(0)$ such that

$$\xi(0) = \arg\min_{\xi} V(\bar{x}(0),\xi). \quad (44)$$

VI. Linear Systems

In this section, we consider linear systems affected by an unknown disturbance described by equations of the form

$$\begin{aligned}\dot{x} &= Ax + B_1 u + B_2 d\\ z &= Hx + Lu\end{aligned} \quad (45)$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n\times p}$, $H \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$. The following statements represent a particularization of the results presented in Theorem 1 and Theorem 3 to linear systems. It is interesting to note that the methodology proposed therein yields the standard solutions to the $\mathcal{L}_2$-disturbance attenuation and infinite horizon quadratic optimal control problems, respectively, for linear time-invariant systems.

Assumption 5: The coefficient matrices of system (45) are such that $H^\top L = 0$ and $L^\top L = I$.

Proposition 1: Consider system (45), with Assumption 5, and let $\gamma > 0$. Suppose that there exists a positive definite matrix $\bar{P} = \bar{P}^\top \in \mathbb{R}^{n\times n}$ such that[4]

$$\bar{P}A + A^\top \bar{P} + \bar{P}\left[\frac{1}{\gamma^2}B_2B_2^\top - B_1B_1^\top\right]\bar{P} + H^\top H \leq 0. \quad (46)$$

Then there exists R such that the conditions in Theorem 1 hold for all $(x,\xi) \in \mathbb{R}^{2n}$. Moreover the control law (23) reduces to

$$\begin{aligned}\dot{\xi} &= -k\bar{P}\xi \\ u &= -B_1^\top \bar{P}x\end{aligned} \quad (47)$$

i.e., the standard solution of the $\mathcal{L}_2$-disturbance attenuation problem for linear systems is recovered.

Proof: To begin with select $R = \bar{P}$ and note that in the linear case the function V in (18) is quadratic and is given by

$$V(x,\xi) = \frac{1}{2}\left[x^\top \quad \xi^\top\right]\begin{pmatrix}\bar{P} & 0 \\ 0 & \bar{P}\end{pmatrix}\begin{bmatrix}x \\ \xi\end{bmatrix}.$$

To complete the proof it remains to show that the linear inequality equivalent to the condition (21) is satisfied. To this end, it is enough to note that, with the selection $R = \bar{P}$, $\Delta(x,\xi) = [I - \bar{P}R^{-1}]\bar{P} = 0$. In addition the left-hand side of the condition (21) is identically equal to zero; hence, the matrix M is only positive semi-definite in the null space of $C = [\bar{P} - R]$. However, the matrix $M = 0$ and this guarantees that the condition $Ker(Z^\top MZ) = Ker(MZ) = \mathbb{R}^n$ holds. Thus, by [1], there exists a positive constant $\bar{k} \geq 0$ such that the left-hand side of the inequality (26) is negative semi-definite and singular for all $k \geq \bar{k}$. Finally note that the dynamic control law (23) *reduces* to (47) yielding the standard static state feedback solution of the disturbance attenuation problem for linear systems. □

Proposition 2: Consider system (45), with Assumption 5, let $d(t) = 0$ for all $t \geq 0$ and $q(x) = x^\top Qx$ in (9). Suppose that there exists a positive definite matrix $\bar{P} = \bar{P}^\top \in \mathbb{R}^{n\times n}$ such that

$$\bar{P}A + A^\top \bar{P} - \bar{P}B_1B_1^\top \bar{P} + Q = 0. \quad (48)$$

Then there exists R such that the conditions in Theorem 3 hold for all $(x,\xi) \in \mathbb{R}^{2n}$. Moreover the control law (39) *reduces* to (47), hence the standard solution to the infinite horizon optimal control problem for linear systems with quadratic cost is recovered.

Proof: The proof of Proposition 2 follows the same steps of the proof of Proposition 1. □

Remark 12: The choice of the initial condition of the dynamic extension ξ is determined as the solution of the minimization problem (44). Note that in the linear case the minimizer can be determined explicitly as a function of the initial condition of the system, namely

$$\xi(0) = R^{-1}(R - \bar{P})\bar{x}(0) = 0 \quad (49)$$

which yields $\xi(t) = 0$ for all $t \geq 0$. ▲

[4]Since $\Sigma(0) = 0$, the equation is the linear version of condition (16) in the definition of the *algebraic* $\bar{P}$ *solution*.

VII. Algebraic $\bar{P}$ Solution for Classes of Nonlinear Systems

In this section, a systematic procedure to compute *algebraic* $\bar{P}$ *solutions* for classes of nonlinear systems with matched disturbance is presented.

A. Feedback Linearizable Systems With Matched Disturbance

Consider the class of nonlinear systems described by $\dot{x} = f(x) + g(x)(u+d)$, with

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + f_2(x_2) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}(x_2,\dots,x_{n-1}) \\ \dot{x}_n &= f_n(x_1,x_2,\dots,x_n) + b(x_1,x_2,\dots,x_n)(u+d)\end{aligned} \quad (50)$$

where $x(t) = (x_1(t),\dots,x_n(t)) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. The (50) describe a class of nonlinear systems in *feedback form* in which the functions $f_2,\dots,f_{n-1}$ do not depend on the variable x_1. Suppose that the system (50) has an equilibrium at $x = 0$; hence, $f_2(0) = f_3(0,0) = \cdots = f_n(0,\dots,0) = 0$, and moreover that $b(x) \neq 0$, for all $x \in \mathbb{R}^n$ and let $b(0) = b_1$. Hence, $b(x) = b_1 + b_2(x)$, for some continuous function b_2 with $b_2(0) = 0$. Since the origin of the state space is an equilibrium point there exist continuous functions $\phi_{i,j}$ and constants $a_{i,j}$, $i = 2,\dots,n$, $j = 2,\dots,i$ such that the functions $f_2,\dots,f_n$ of the system (50) can be expressed as

$$\begin{aligned}f_2(x) &= a_{2,2}x_2 + \phi_{2,2}(x)x_2 \\ f_3(x) &= a_{3,2}x_2 + a_{3,3}x_3 + \phi_{3,2}(x)x_2 + \phi_{3,3}(x)x_3 \\ &\vdots \\ f_n(x) &= a_{n,1}x_1 + \cdots + a_{n,n}x_n + \phi_{n,1}(x)x_1 + \phi_{n,2}(x)x_2 \\ &\quad + \cdots + \phi_{n,n}(x)x_n.\end{aligned}$$

In particular, note that $\partial f_2(x_2)/\partial x_2(0) = a_{2,2},\dots,\partial f_n(x)/\partial x(0) = [a_{n,2},\dots,a_{n,n}]$.

Consider the definition of *algebraic* $\bar{P}$ *solution* for the optimal control problem and the matched $\mathcal{L}_2$-disturbance attenuation problem. In order to deal simultaneously with both problems, define $\epsilon^2 = -(1/\gamma^2 - 1)$, where $\gamma \in (1,\infty]$ is the desired disturbance attenuation level. Note that $\epsilon \in (0,1]$, where the case $\epsilon = 1$, that is $\gamma = \infty$, represents the optimal control problem since no restriction on the attenuation level is imposed. Let

$$\bar{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{1,2} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & p_{n,n}\end{bmatrix}$$

be the symmetric solution of the algebraic Riccati equation for the linearized problem, i.e., $\bar{P}A + A^\top \bar{P} - \epsilon^2 \bar{P}BB^\top \bar{P} + H^\top H = 0$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & a_{2,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & a_{n-1,2} & a_{n-1,3} & \cdots & 1 \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}\end{bmatrix}, \quad B = \begin{bmatrix}0 \\ 0 \\ \vdots \\ 0 \\ b_1\end{bmatrix}$$

and H defines a penalty variable or a cost to minimize. To determine an *algebraic* $\bar{P}$ *solution* let $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, $\bar{f}(x) = f(x) - Ax$ and consider the *algebraic* inequality in the unknown $\mathcal{Q}(x) = [\mathcal{Q}_1(x), \ldots, \mathcal{Q}_n(x)]$

$$\begin{aligned} &2(x^\top \bar{P} + \mathcal{Q}(x))(Ax + \bar{f}(x)) + x^\top H^\top Hx + 2x^\top \Sigma(x)x \\ &\quad -\epsilon^2(x^\top \bar{P} + \mathcal{Q}(x))g(x)g(x)^\top(x^\top \bar{P} + \mathcal{Q}(x))^\top \leq 0 \end{aligned} \quad (51)$$

where $\Sigma(x) = diag\{\sigma_i(x)\}_{i=1,\ldots,n}$, with $\sigma_i(x) > 0$, for all $x \neq 0$.

Proposition 3: Consider a nonlinear system of the form (50) and the HJ inequality (51) and suppose that

$$a_{n,1} + \phi_{n,1}(x) - \epsilon^2 p_{1,n} b(x)^2 \neq 0 \quad (52)$$

for all $x \in \mathbb{R}^n$. Then the mapping $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, with

$$\begin{aligned} \mathcal{Q}_n(x) = &\frac{1}{a_{n,1} + \phi_{n,1}(x) - \epsilon^2 p_{1,n} b(x)^2}\Big[-x_1\sigma_1(x) \\ &- \phi_{n,1}(x_1 p_{1,n} + \cdots + x_n p_{n,n}) \\ &- \epsilon^2 p_{1,n} b_1(x_1 p_{1,n} + \cdots + x_n p_{n,n}) b_2(x)\Big] \quad (53) \\ \mathcal{Q}_i(x) = &\epsilon^2 p_{i+1,n} b_1(x_1 p_{1,n} + \cdots + x_n p_{n,n}) b_2(x) \\ &- x_{i+1}\sigma_{i+1} + \epsilon^2 p_{i+1,n} b(x)^2 \mathcal{Q}_n(x) \\ &- \sum_{j=i+1}^{n} \Big[(a_{j,i+1} + \phi_{j,i+1}(x))\mathcal{Q}_j(x) \\ &+ \phi_{j,i+1}(x)(x_1 p_{1,j} + \cdots + x_n p_{j,n})\Big] \quad (54) \end{aligned}$$

$i = 1, \ldots, n-1$, is an *algebraic* $\bar{P}$ *solution* of the (50).

Remark 13: Since the ith element of $\mathcal{Q}$ depends on the components $\mathcal{Q}_j$ with $j > i$, the solution can be determined iteratively from $\mathcal{Q}_{n-1}$ to $\mathcal{Q}_1$. ▲

Proof: The proof is by direct substitution. Considering the definition of the matrices A, Σ and H and of the vectors $\bar{f}$ and g, straightforward computations show that the mapping $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, where $\bar{P}$ is the solution of the algebraic Riccati equation and $\mathcal{Q}$ is defined in (53)–(54), is a solution of the algebraic inequality (51). □

B. Strict Feedforward Form

Consider a nonlinear system in *strict feedforward form* described by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_2, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_3, \ldots, x_n) \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_n) \\ \dot{x}_n &= u + d \end{aligned} \quad (55)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $d(t) \in \mathbb{R}$ and $f = [f_1, f_2, \ldots, f_{n-1}, 0]^\top$. Suppose that the origin is an equilibrium point for the system (55), i.e., $f_i(0) = 0$ $i = 1, \ldots, n-1$; hence, there exist continuous functions $\phi_{i,j}$ and constants $a_{i,j}$, $i = 2, \ldots, n$, $j = 2, \ldots, i$ such that

$$f(x) = \begin{pmatrix} a_{1,2}x_2 + \cdots + a_{1,n}x_n + \phi_{1,2}x_2 + \cdots + \phi_{1,n}x_n \\ a_{2,3}x_3 + \cdots + a_{2,n}x_n + \phi_{2,3}x_3 + \cdots + \phi_{2,n}x_n \\ \vdots \\ a_{n-1,n}x_n + \phi_{n-1,n}x_n \\ 0 \end{pmatrix}.$$

Proposition 4: Consider a nonlinear system of the form (55) and suppose that

$$a_{i,i+1} + \phi_{i,i+1}(x) \neq 0 \quad (56)$$

for all $x \in \mathbb{R}^n$, for $i = 1, \ldots, n-1$. Then the mapping $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, with $\mathcal{Q}_n(x) = x_1\sigma_1(x)/\epsilon^2 p_{1,n}$ and

$$\begin{aligned} \mathcal{Q}_i(x) = &-\frac{1}{a_{i,i+1} + \phi_{i,i+1}(x)}\Big[-\epsilon^2 p_{i+1,n}\mathcal{Q}_n(x) \\ &+ \sum_{j=1}^{i-1}\Big(\phi_{j,i}(x)(x_1 p_{1,j} + \cdots + x_n p_{j,n}) \\ &+ \mathcal{Q}_j(x)(a_{j,i+1} + \phi_{j,i+1}(x))\Big) \\ &- x_{i+1}\sigma_{i+1}(x)\Big] \end{aligned} \quad (57)$$

$i = 1, \ldots, n-1$, solves the inequality (51) and is an *algebraic* $\bar{P}$ *solution* for the nonlinear system (55).

Proof: As in the proof of Proposition 3 the claim is proved by direct substitution. □

Remark 14: The condition (56) is trivially satisfied if $f_i(x) = x_{i+1} + \bar{f}_i(x_{i+2}, \ldots, x_n)$, for $i = 1, \ldots, n-1$. ▲

VIII. Fully Actuated Mechanical Systems

Consider fully actuated mechanical systems described by the Euler–Lagrange equation [24], namely

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tilde{\tau} \quad (58)$$

where $G(q)$ contains the potential forces and the matrix $C(q, \dot{q})$, which is linear in the second argument, represents the Coriolis and centripetal forces and is computed using the Christoffel symbols of the first kind.

Defining the variables $x_1 = q$ and $x_2 = \dot{q}$, with $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^n$ yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= M(x_1)^{-1}[\tilde{\tau} - C(x_1, x_2)x_2 - G(x_1)]. \end{aligned} \quad (59)$$

Assume that the preliminary feedback $\tilde{\tau} = \hat{\tau} + G(0)$, $\hat{\tau} \in \mathbb{R}^n$, is applied to compensate for the effect of gravity at the origin. Under this assumption, the gravitational term becomes $\tilde{G}(x_1) = G(x_1) - G(0)$. Therefore there exists a continuous matrix valued function $\bar{G} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that $\tilde{G}(x_1) = \bar{G}(x_1)x_1$, for all $x_1 \in \mathbb{R}^n$.

Suppose that the variables to minimize in the optimal control problem or for which a desired attenuation level needs to be guaranteed in the disturbance attenuation problem are the positions x_1 and the velocities x_2 of the joints. Finally, let $\hat{\tau} = u + d$ in the disturbance attenuation problem, whereas let $\hat{\tau} = u$ in the optimal control problem.

The optimal control problem for mechanical systems has been addressed in [17]. In particular the same assumptions are considered herein and in [17], namely the equations are perfectly known and the positions and the velocities of the body are measurable. Therein, however, a preliminary feedforward term is designed to compensate for the effect of gravity, hence only the components of the control forces that affect the kinetic energy are considered in the optimization neglecting the gravitation-dependent torques.

The definition of an *algebraic $\bar{P}$ solution* for the system (59) requires the computation of the positive definite matrix $\bar{P}$. To this end consider the first-order approximation of the nonlinear system (59), namely

$$\dot{x} = \begin{bmatrix} 0 & I_n \\ -M(0)^{-1}D & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ M(0)^{-1} \end{bmatrix} u$$

where $D = D^\top \in \mathbb{R}^{n\times n}$ is a constant matrix defined as $D = \partial G/\partial x_1(0)$. Note that D is the Hessian matrix of the potential energy $\mathcal{U}(x_1)$ evaluated in $x_1 = 0$; therefore, if the potential energy has a strict local minimum at the origin then D is positive definite. The corresponding algebraic Riccati equation is given by

$$\bar{P}A + A^\top \bar{P} - \epsilon^2 \bar{P}BB^\top \bar{P} + H^\top H = 0 \tag{60}$$

where $H = diag\{\mu_1 I_n, \mu_2 I_n\}$. The positive constants μ_1 and μ_2 are weighting factors for the positions and the velocities of the joints. Partitioning the matrix $\bar{P}$ as

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^\top & \bar{P}_3 \end{bmatrix}$$

let the matrices $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3 \in \mathbb{R}^{n\times n}$ be defined as the solutions of the system of quadratic matrix equations

$$\begin{aligned} \mu_1^2 I_n &= \bar{P}_2 M(0)^{-1} D + DM(0)^{-1}\bar{P}_2^\top + \epsilon^2 \bar{P}_2 M(0)^{-2}\bar{P}_2^\top \\ \bar{P}_1 &= D^\top M(0)^{-1}\bar{P}_3 + \epsilon^2 \bar{P}_2 M(0)^{-2}\bar{P}_3 \\ \bar{P}_3 &= \frac{1}{\epsilon} M(0)\left[\bar{P}_2 + \bar{P}_2^\top + \mu_2^2 I_n\right]^{1/2}. \end{aligned} \tag{61}$$

Note that the positive definite solution $\bar{P}$ of the matrix (61) exists and is unique.

The following result provides the solutions to Problems 1 and 3 for fully actuated mechanical systems described by the (59).

Proposition 5: Consider the mechanical system (59). Suppose that $\Sigma_i(0) > 0$, $i = 1, 2$, let $\gamma \in (1, \infty]$. Let $x_i^\top \Upsilon_i^\top(x)\Upsilon_i(x)x_i = x_i^\top\left(\mu_i^2 I_n + \Sigma_i(x)\right)x_i > 0$, $i = 1, 2$, let $W_1(x_1, x_2)$ be such that

$$\begin{aligned} \Upsilon_1^\top(x)\Upsilon_1(x) = {} & W_1 M(x_1)^{-1}\bar{G}(x_1) \\ & + \bar{G}(x_1)^\top M(x_1)^{-1} W_1 + \epsilon^2 W_1 M(x_1)^{-2} W_1 \end{aligned}$$

and let

$$\begin{aligned} V_1(x) = {} & W_1 M(x_1)^{-1} C(x) + \bar{G}(x_1)^\top M(x_1)^{-1} W_2 \\ & + \epsilon^2 W_1 M(x_1)^{-2} W_2 \end{aligned}$$

$$\begin{aligned} V_2(x) = {} & W_2 M(x_1)^{-1} C(x) + \frac{\epsilon^2}{2} W_2 M(x_1)^{-2} W_2 \\ & - \frac{1}{2}\Upsilon_2(x)^\top \Upsilon_2(x) \end{aligned}$$

with $W_2(x_1, x_2)$ such that $W_2(0, 0) = \bar{P}_3$. Then there exist a matrix $R > 0$, a neighborhood of the origin $\Omega \subseteq \mathbb{R}^{4n}$ and $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the function $V > 0$ as in (18), with

$$P(x) = \left[x_1^\top V_1 + x_2^\top V_2, x_1^\top W_1 + x_2^\top W_2\right] \tag{62}$$

satisfies for all $(x, \xi) \in \Omega$ the Hamilton–Jacobi partial differential inequality (22), hence the dynamic control law defined in (23) solves Problem 1 if $\epsilon \in (0, 1)$ and Problem 3 if $\epsilon = 1$.

Proof: Let $\Sigma_i(0) > 0$, $i = 1, 2$, $r(x) = 0$:

$$f(x) \triangleq \begin{bmatrix} x_2 \\ -M(x_1)^{-1}(C(x_1, x_2)x_2 + \tilde{G}(x_1)) \end{bmatrix} \tag{63}$$

and

$$\Pi(x) \triangleq -\epsilon^2 \begin{bmatrix} 0 & 0 \\ 0 & M(x_1)^{-2} \end{bmatrix} \tag{64}$$

$\epsilon \in (0, 1)$, and note that P as in (62) is an *algebraic $\bar{P}$ solution* of the (16), i.e.,

$$\begin{aligned} & 2x_1^\top V_1 x_2 + 2x_2^\top V_2 x_2 - 2x_1^\top W_1 M^{-1} C x_2 \\ & - 2x_2^\top W_2 M^{-1} C x_2 + x_1^\top \Upsilon_1^\top \Upsilon_1 x_1 + x_2^\top \Upsilon_2^\top \Upsilon_2 x_2 \\ & - 2\left(x_1^\top W_1 + x_2^\top W_2\right) M^{-1}\tilde{G} \\ & - \epsilon^2\left(x_1^\top W_1 + x_2^\top W_2\right) M^{-2}(W_1 x_1 + W_2 x_2) = 0. \end{aligned} \tag{65}$$

Then, by Theorem 1 and Corollary 2 there exist k, R and a set $\Omega \subseteq \mathbb{R}^{4n}$ such that the dynamic control law (23) solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability. If $\epsilon = 1$ in (64) then the dynamic control law (23) solves the approximate regional dynamic optimal control problem. □

In the case of planar mechanical systems, i.e., systems with $G(x_1) = 0$ for all $x_1 \in \mathbb{R}^n$, the solution of the (65) can be given in closed-form. The solution of the algebraic Riccati (60) is

$$\begin{aligned} \bar{P}_1 &= \mu_1\left[\frac{2\mu_1}{\epsilon} M(0) + \mu_2^2 I_n\right]^{1/2} \\ \bar{P}_2 &= \frac{\mu_1}{\epsilon} M(0) \\ \bar{P}_3 &= \frac{1}{2}[S + S^\top] \end{aligned} \tag{66}$$

with $S = \epsilon^{-1} M(0)\left[2\mu_1 \epsilon^{-1} M(0) + \mu_2^2 I_n\right]^{1/2}$.

Corollary 3: Consider the mechanical system (59) with $G(x_1) = 0$ for all $x_1 \in \mathbb{R}^n$. Suppose that $\Sigma_i(0) > 0$, $i = 1, 2$, let $\gamma \in (1, \infty]$. Let $W_1(x_1, x_2) = (1/\epsilon)\Upsilon(x)^\top M(x_1)$ and

$$V_1(x_1, x_2) = \Upsilon(x)\left[\frac{1}{\epsilon} C(x_1, x_2) + \epsilon M(x_1)^{-1} W_2\right]$$

$$\begin{aligned} V_2(x_1, x_2) = {} & W_2 M(x_1)^{-1}\left[C(x_1, x_2) + \frac{\epsilon^2}{2} M(x_1)^{-1} W_2\right] \\ & - \frac{1}{2}\Upsilon_2(x)^\top \Upsilon_2(x) \end{aligned}$$

with $W_2(x_1, x_2)$ such that $W_2(0,0) = \bar{P}_3$. Then there exist a matrix $R > 0$, a neighborhood $\Omega \subset \mathbb{R}^{4n}$ containing the origin and $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the dynamic control law defined in (23), with P as in (62), solves Problem 1 if $\epsilon \in (0,1)$ and Problem 3 if $\epsilon = 1$.

IX. Examples

A. 2 Dof Planar Robot

Consider a planar fully actuated robot with two rotational joints and let $x_1 = [\chi_1, \chi_2] \in \mathbb{R}^2$ be the relative positions of the joints and $x_2 = [\chi_3, \chi_4] \in \mathbb{R}^2$ be the corresponding velocities. The dynamics of the mechanical system can be described by equations of the form (59) with

$$M = \begin{bmatrix} a_1 + 2a_2\cos(\chi_2) & a_2\cos(\chi_2) + a_3 \\ a_2\cos(\chi_2) + a_3 & a_3 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & -a_2\sin(\chi_2)(\chi_4 + 2\chi_3) \\ a_2\sin(\chi_2)\chi_3 & 0 \end{bmatrix}$$

where $a_1 = I_1 + m_1 d_1^2 + I_2 + m_2 d_2^2 + m_2 l_1^2$, $a_2 = m_2 l_1 d_2$ and $a_3 = I_2 + m_2 d_2^2$, with I_i, m_i, l_i and d_i the moment of inertia, the mass, the length and the distance between the center of mass and the tip of the ith joint, $i = 1, 2$, respectively. In the simulations we let $m_1 = m_2 = 0.5$ kg, $l_1 = l_2 = 0.3$ m, $d_1 = d_2 = 0.15$ m and $I_1 = I_2 = 0.0037\,\mathrm{kg \cdot m^2}$. To begin with, suppose that the action of the actuators is corrupted by white noise and consider a desired attenuation level on the position of the joints close to $\gamma = 1$, e.g., define $\epsilon = 0.1$. Let $\Sigma_1(\chi_1, \chi_2) = diag\{10^{-3}(1+\chi_1^2), 10^{-3}(1+\chi_2^2)\}$ and determine the *algebraic* $\bar{P}$ *solution* as described in Corollary 3. Since the Hamilton–Jacobi partial differential equation or inequality that yields the solution of the matched disturbance attenuation problem for the considered planar robot does not admit a closed-form solution, the performance of the dynamic control law defined in (23) is compared with the solution of the linearized problem, i.e., the static state feedback given by $u_o = -B_1^\top \bar{P} x$. In the dynamic control law, the matrix R is selected as $R = \alpha\Phi(0,0)$, with $\alpha \in \mathbb{R}_+$. Let the initial condition of the planar robot be $[\chi_1(0), \chi_2(0), \chi_3(0), \chi_4(0)] = [\pi/2, \pi/2, 0, 0]$. Fig. 1 displays the time histories of the angular positions of the joints for different values of the parameter α when the linearized control law u_o and the dynamic control law (23) are applied, dashed and solid lines, respectively. In all the plots the same disturbance affects the actuators. The behavior of the joints with $\alpha = 0.8$ is displayed in the top graph and it can be noted that the dynamic control law guarantees *worse* rejection of the matched disturbances than the control law u_o. Increasing the value of the parameter α improves the performance of the dynamic control law. In particular, the choice $R = \Phi(0,0)$ (middle graph) yields a solution almost identical to u_o whereas selecting $\alpha = 5$ the disturbance attenuation is significantly improved (bottom graph).

Consider the ideal case of absence of disturbances and let $\epsilon = 1$, i.e., $\gamma = \infty$. Fig. 2 displays, for different initial conditions, the ratio between the costs yielded by the dynamic control law—considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0,0]$—and by the optimal static state feedback for the linearized system, i.e., $\rho = V_d(\chi(0), \xi(0))/V_o(\chi(0))$, where V_d is the value function defined in (18), with P as in (62), and V_o is the quadratic value function for the linearized problem. Obviously, $\rho < 1$ implies that the cost paid by the dynamic control law is smaller than the cost of the optimal static state feedback for the linearized system. In Fig. 2, the ratio ρ is greater than 1 in a neighborhood of the origin since $\Sigma_1(0) > 0$, as explained in Remark 9.

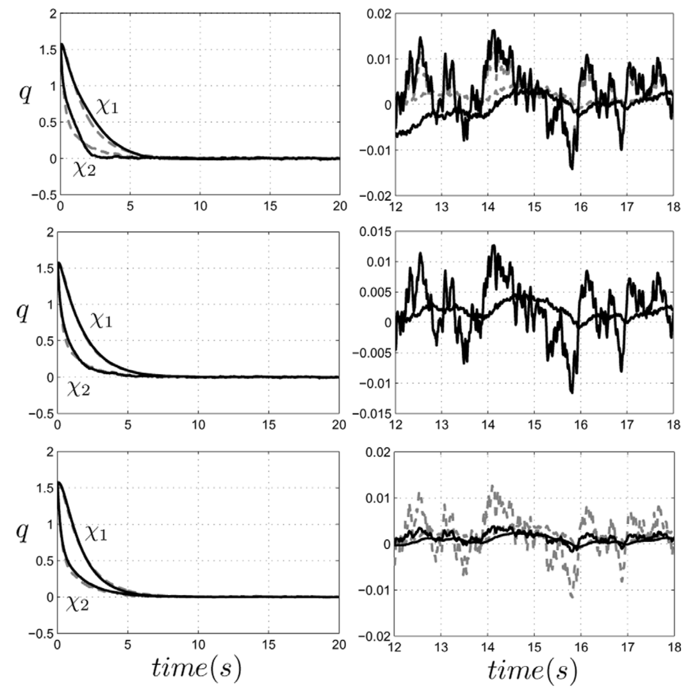

Fig. 1. Left graphs: time histories of the angular positions of the joints for different values of the parameter α (Top graph: $\alpha = 0.8$. Middle graph: $\alpha = 1$. Bottom graph: $\alpha = 5$) when the linearized control law u_o and the dynamic control law are applied, dashed and solid lines, respectively. Right graphs: restriction of the angular positions in the left graphs to the time interval $t \in [12, 18]$.

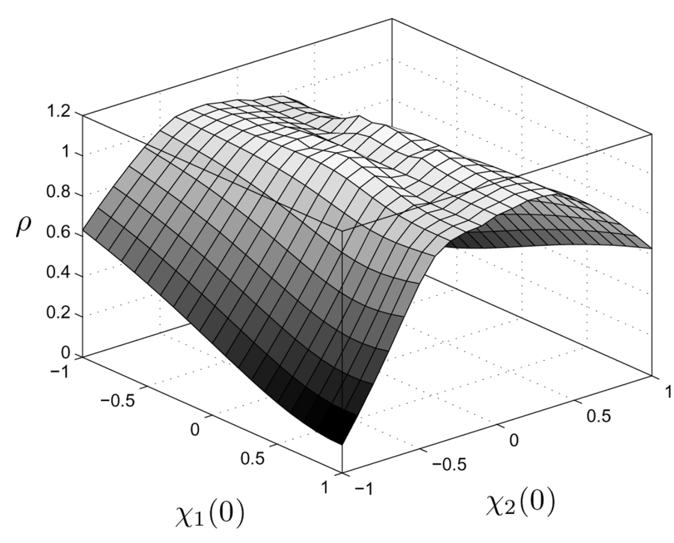

Fig. 2. Ratio ρ between the costs yielded by the dynamic control law—considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$ and letting $[\xi_3(0), \xi_4(0)] = [0,0]$—and by the optimal static state feedback for the linearized system.

Let $\alpha = 1.2$ and $\chi(0) = [\pi/4, \pi/4, 0, 0]$. As above, the dynamic control law is compared with the optimal solution of the linearized problem obtained with $\epsilon = 1$, the optimal cost

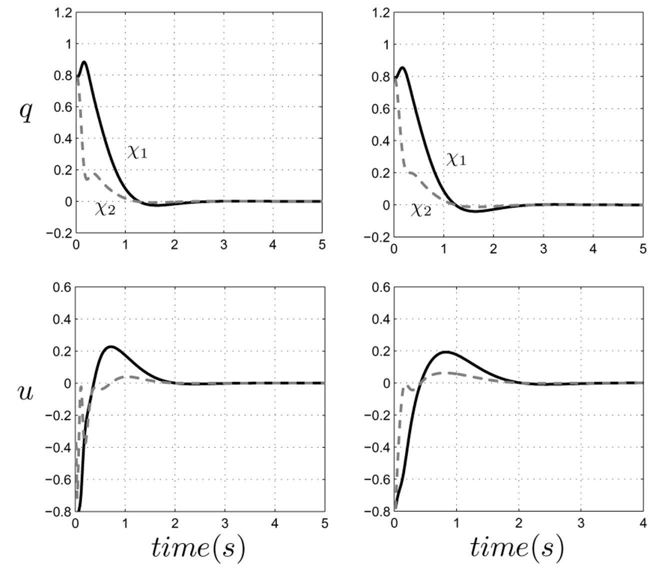

Fig. 3. Top graphs: time histories of the positions of the two joints when the dynamic control law and the optimal state feedback for the linearized problem are applied, left and right graph, respectively. Bottom graphs: time histories of the dynamic control law and of the optimal state feedback for the linearized problem, left and right graph, respectively.

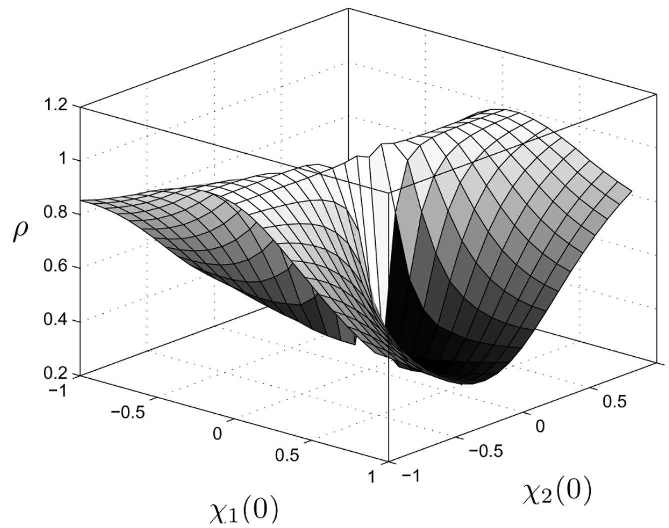

Fig. 4. Rratio ρ between the cost paid by the dynamic control law (39)—considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$ and letting $[\xi_3(0), \xi_4(0)] = [0, 0]$—and the control law proposed in [17].

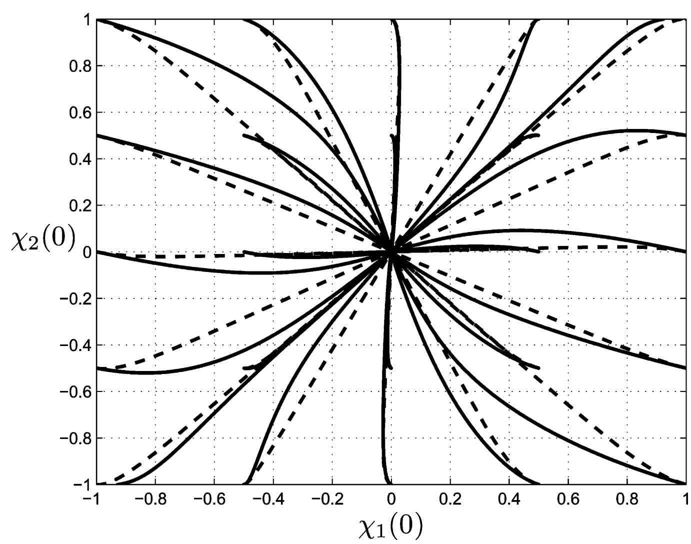

Fig. 5. Position of the joints for various initial condition when the dynamic control law and the linear control law proposed in [17] are applied, solid and dashed lines, respectively.

of which is $V(\chi(0)) = (1/2)\chi(0)^\top \bar{P}\chi(0) = 0.2606$. Set $[\xi_3(0), \xi_4(0)] = [0, 0]$. The optimization of the value function $V(\chi(0), \xi(0))$, which gives the optimal cost *paid* by the solution, with respect to $\xi_1(0)$ and $\xi_2(0)$, yields the values $\xi_1^o(0) = 0.3$, $\xi_2^o(0) = -0.9$, and the corresponding value attained by the function is $V(\chi(0), \xi_1^o(0), \xi_2^o(0), 0, 0) = 0.2081$. The top graph of Fig. 3 shows the time histories of the angular positions of the two joints when the dynamic control law, with $\alpha = 1.2$, and the optimal state feedback u_o are applied, left and right graph, respectively. The time histories of the dynamic control law and the optimal local state feedback are displayed in the bottom graph of Fig. 3, left and right graph, respectively.

In the last simulation, a comparison between the dynamic control law (39) and the linear control law proposed in [17] is performed. In the following the variables to minimize are the positions together with the velocities of the joints, since the conditions of existence for the control law in [17] cannot be satisfied considering only the minimization of the positions of the joints, as in the previous numerical example. Fig. 4 displays the ratio between the cost paid by the dynamic control law (39), considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0, 0]$, and the linear control law proposed in [17]. It can be noted that the cost paid by the dynamic solution is lower than the cost by the linear control law of [17]. Fig. 5 shows, for different initial conditions, the time histories of the positions of the joints when the dynamic control law (39) and the linear control law proposed in [17] are applied (solid and dashed lines, respectively).

B. Controlled Van Der Pol Oscillator

Consider the single-input, single-output, nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \mu\left(1 - x_1^2\right)x_2 + x_1 u \end{aligned} \qquad (67)$$

known as the controlled Van der Pol oscillator, with $x(t) = (x_1(t), x_2(t))^\top \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$. The parameter μ describes the strength of the damping effect and in the following is selected as $\mu = 0.5$, hence the oscillator has a stable but linearly uncontrollable equilibrium at the origin surrounded by an unstable limit cycle. Define the positive definite cost $q(x) = x_2^2$, i.e., the control action minimizes the *speed* of the oscillator together with the control effort.

Note that the solution of the HJB equation for system (67) with the given cost can be explicitly determined, namely $V_o(x_1, x_2) = (1/2)\left(x_1^2 + x_2^2\right)$, and the resulting optimal control is $u_o = -x_1 x_2$. The interesting aspect of the existence of the optimal solution is in allowing the explicit comparison between the optimal control feedback, the dynamic solution and the optimal solution of the linearized problem, as detailed in the following.

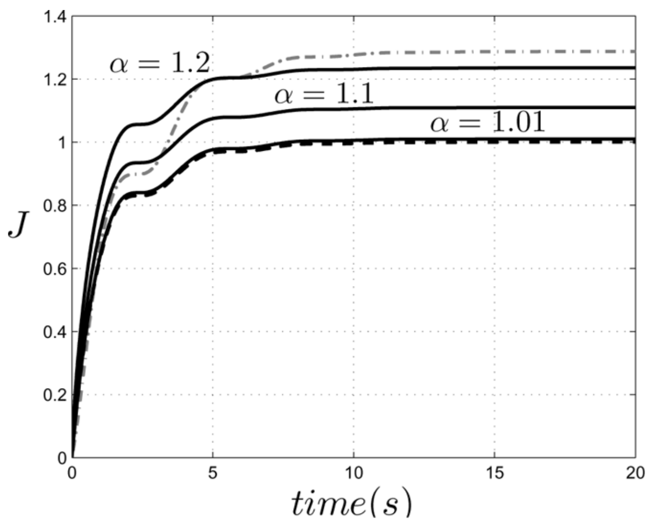

Fig. 6. Time histories of the integral of the running cost applying the control law u_o from the initial condition $x(0) = (1, -1)$ (dashed line) and the control law $u = 0$, i.e., the optimal solution of the linearized problem, (dash-dotted line) and the dynamic control from the initial condition $(x(0), \xi(0)) = (1, -1, 0, 0)$, for different values of α, (solid lines).

The linearization of the nonlinear system around the origin is given by $\dot{x} = Ax$, i.e., it is not affected by u, with A Hurwitz, and the solution of the algebraic Riccati (15) corresponding to the linearized problem is $\bar{P} = I$. The selection $\Sigma(x) = \lambda/2diag\left\{x_2^2, x_1^2\right\}$ with $\lambda > 0$, is such that $P(x) = [x_1(1 - \lambda x_1 x_2), x_2]$ is an *algebraic $\bar{P}$ solution* of the Hamilton–Jacobi–Bellman equation. Note that for this solution the condition (17) is not satisfied, hence P is not an exact differential. Let $R = diag(\alpha, \alpha)$ with $\alpha > 1$, then the function

$$V(x,\xi) = x_1\xi_1(1-\xi_1\xi_2) + x_2\xi_2 + \frac{\alpha}{2}(x_1-\xi_1)^2 + \frac{\alpha}{2}(x_2-\xi_2)^2 \tag{68}$$

is locally positive definite in the set $\Omega_1 = \left\{ (x,\xi) \in \mathbb{R}^4 : \xi_1\xi_2 \le 1, \xi_1\xi_2 > 1 - 2\alpha \right\}$ and moreover the condition (37) is strictly satisfied for all $(x,\xi) \in \Omega \setminus \{0\}$, where $\Omega = \Omega_1 \cap \Omega_2$, with $\Omega_2 \triangleq \left\{(x,\xi) \in \mathbb{R}^4 : \zeta_\alpha(x,\xi) > 0\right\}$, ζ_α being a known continuous function parameterized by α. Note that the set Ω_2 is non-empty for any $\alpha > 1$ and it is empty for $\alpha = 1$. Thus, it is possible to determine a value $\bar{k} \ge 0$ such that the *extended* Hamilton-Jacobi-Bellman inequality (38) holds. Since $R - \Phi(x,\xi)$ is not zero in $Im(g(x))$ then the dynamic control law $u(x,\xi) = -\alpha x_1 x_2 + (\alpha - 1)x_1\xi_2$ solves the dynamic optimal control problem with running cost $x_2^2 + u^2 + c(x,\xi)$, where $c(x,\xi) \ge 0$ is defined as in Theorem 3.

Interestingly, the control law $u(x,\xi)$ is such that for any $\epsilon > 0$ there exists $\alpha > 1$ such that the ratio $\rho = V(x(0),0)/V_o(x(0))$ is smaller than ϵ. Fig. 6 shows the time histories of the integral of the running cost applying the control law u_o from the initial condition $x(0) = (1,-1)$ (dashed line), the control law $u = 0$, i.e., the optimal solution of the linearized problem, (dash-dotted line) and the dynamic control from the initial condition $(x(0),\xi(0)) = (1,-1,0,0)$, for different values of α, (solid line). The top graph of Fig. 7 displays the trajectories of the system (67) driven by the control law $u(x,\xi)$ and by the optimal solution of the linearized problem, i.e., $u = 0$ (solid and dashed lines, respectively). The bottom graph shows the time histories of the control input $u(x(t),\xi(t))$, with $\alpha = 1.01$, (solid line) and the input $u = 0$ (dashed line). It is important to stress that the value function V_o is determined in principle as the solution of a partial differential equation while only algebraic conditions are involved in the computation of V in (68).

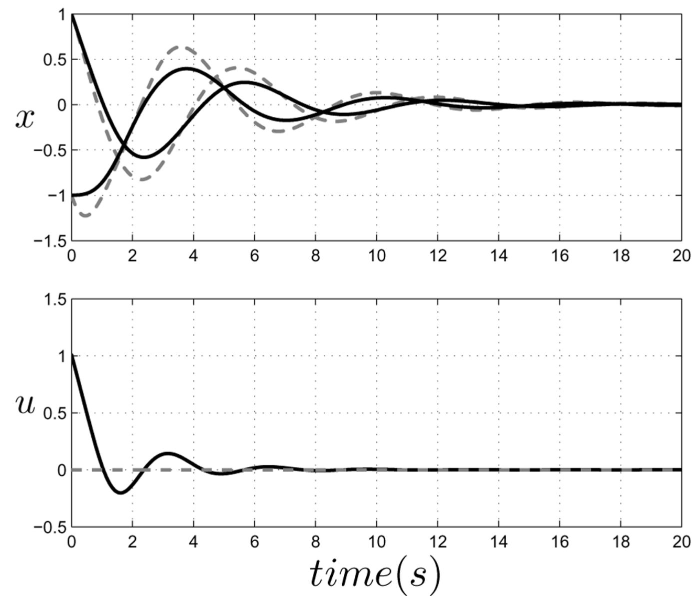

Fig. 7. Top graph: time histories of the state of system (67) driven by the control law $u(x,\xi)$, with $\alpha = 1.01$, and by the optimal solution of the linearized problem, i.e., $u = 0$, solid and dashed lines, respectively. Bottom Graph: time histories of the control input $u(x(t),\xi(t))$ (solid line) and the input $u = 0$ (dashed line).

X. Conclusion

The $\mathcal{L}_2$-disturbance attenuation and the optimal control problems are studied. It is shown that, using the notion of *algebraic $\bar{P}$ solution*, the issue of solving the Hamilton–Jacobi partial differential inequality, arising in the $\mathcal{L}_2$-disturbance attenuation problem, is resolved by means of a *dynamic extension*, that allows to construct (locally) an exact positive definite solution of an extended Hamilton–Jacobi pdi. Then, it is proved that the solution of the Hamilton–Jacobi–Bellman pde can be avoided in the optimal control of nonlinear systems, provided an additional cost is *paid*. Moreover, the additional cost can be reduced shaping the running cost, on one hand, and initializing the dynamic extension, on the other hand. The conditions imposed for the feasibility of the methodology are algebraic inequalities that must be satisfied (locally) in the extended state-space, and no partial derivatives are involved. Finally, a systematic procedure to determine an *algebraic $\bar{P}$ solution* for different classes of nonlinear systems has been proposed. To conclude the paper, the performances of the methodology have been tested with two numerical examples. Future work includes the solution of the optimal control and the $\mathcal{L}_2$-disturbance attenuation problems for non-affine systems and the computation of *algebraic $\bar{P}$ solutions* for wider classes of nonlinear systems.

References

[1] K. M. Anstreicher and M. H. Wright, "A note on the augmented Hessian when the reduced Hessian is semidefinite," *SIAM J. Optimiz.*, vol. 11, no. 1, pp. 243–253, 2000.

[2] A. Astolfi, "Singular H_∞ control for nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 7, pp. 727–740, 1997.
[3] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions to Hamilton-Jacobi-Bellman Equations*. Boston, MA: Birkhauser, 1997.
[4] R. W. Beard, G. N. Saridis, and J. T. Wen, "Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equation," *Automatica*, vol. 33, no. 12, pp. 2159–2177, 1997.
[5] D. P. Bertsekas, *Dynamic Programming and Optimal Control*. Nashua, NH: Athena Scientific, 1995.
[6] A. E. Bryson and Y. C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. New York: Taylor & Francis, 1975.
[7] F. Camilli, M. Falcone, P. Lanucara, and A. Seghini, "A domain decomposition method for Bellman equation," *Contemporary Math.*, vol. 180, pp. 477–483, 1994.
[8] D. Carnevale and A. Astolfi, "Integrator forwarding without PDEs," in *Proc. 48th IEEE Conf. Decision Control and 28th Chinese Control Conf.*, Shanghai, China, 2009, pp. 33–38.
[9] M. G. Crandall and P. L. Lions, "Viscosity solutions of Hamilton-Jacobi equations," *Trans. Amer. Math. Soc.*, vol. 227, pp. 1–42, 1983.
[10] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic, 1975.
[11] L. C. Evans, *Partial Differential Equations*. Providence, RI: American Mathematical Society, 1998.
[12] M. Falcone, "A numerical approach to the infinite horizon problem of deterministic control theory," *Appl. Math. Optimiz.*, vol. 15, pp. 1–13, 1987.
[13] R. A. Freeman and P. V. Kokotovic, "Inverse optimality in robust stabilization," *SIAM J. Control Optimiz.*, vol. 34, no. 4, pp. 1365–1391, 1996.
[14] Y. Huang and W. M. Lu, "Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation," in *Proc. 35th Conf. Decision Control*, Kobe, Japan, Dec. 1996, vol. 4, pp. 3942–3947.
[15] T. Hunt and A. J. Krener, "Improved patchy solution to the Hamilton-Jacobi-Bellman equations," in *Proc. 49th Conf. Decision Control*, Atlanta, GA, 2010.
[16] A. Isidori, *Nonlinear Control Systems*. New York: Springer-Verlag, 1995.
[17] R. Johansson, "Quadratic optimization of motion coordination and control," *IEEE Trans. Autom. Control*, vol. 35, no. 11, pp. 1197–1208, Nov. 1990.
[18] D. Karagiannis, M. Sassano, and A. Astolfi, "Dynamic scaling and observer design with application to adaptive control," *Automatica*, vol. 45, no. 12, pp. 2883–2889, 2009.
[19] A. J. Krener, "The existence of optimal regulators," in *Proc. 37th Conf. Decision Control*, Tampa, FL, 1998, vol. 3, pp. 3081–3086.
[20] W. M. Lu and J. C. Doyle, "H_∞ control of nonlinear systems: A convex characterization," *IEEE Trans. Autom. Control*, vol. 40, no. 9, pp. 1668–1675, Sep. 1995.
[21] D. L. Lukes, "Optimal regulation of nonlinear dynamical systems," *SIAM J. Control*, vol. 7, no. 1, pp. 75–100, 1969.
[22] W. C. A. Maas and A. J. van der Schaft, "Singular nonlinear H_∞ optimal control by state feedback," in *Proc. 33rd Conf. Decision Control*, Orlando, FL, Dec. 1994, vol. 2, pp. 1415–1420.
[23] W. M. McEneaney, "A curse-of-dimensionality-free numerical method for solution of certain HJB PDEs," *SIAM J. Control Optimiz.*, vol. 46, no. 4, pp. 1239–1276, 2007.
[24] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez, *Passivity-Based Control of Euler-Lagrange Systems*. London, U.K.: Springer-Verlag, 1998.
[25] L. Praly, D. Carnevale, and A. Astolfi, "Integrator forwarding via dynamic scaling," in *Proc. 49th IEEE Conf. Decision Control*, Atlanta, GA, 2010, pp. 5955–5960.
[26] L. Rosier and E. D. Sontag, "Remarks regarding the gap between continuous, Lipschitz and differentiable storage functions for dissipation inequalities appearing in H_∞ control," *Syst. Control Lett.*, vol. 41, pp. 237–249, 2000.
[27] M. Sassano and A. Astolfi, "Dynamic solution of the Hamilton-Jacobi inequality in the L_2-disturbance attenuation problem," in *Proc. 8th IFAC Symp. Nonlinear Control Syst.*, Bologna, Italy, 2010, pp. 3081–3086.
[28] M. Sassano and A. Astolfi, "Dynamic solution of the HJB equation and the optimal control of nonlinear systems," in *Proc. 49th Conf. Decision Control*, Atlanta, GA, 2010.
[29] M. Sassano and A. Astolfi, "Dynamic disturbance attenuation and approximate optimal control for fully actuated mechanical systems," in *Proc. Amer. Control Conf.*, San Francisco, CA, 2011.
[30] A. J. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state-feedback H_∞ control," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 770–784, Jun. 1992.
[31] R. B. Vinter, *Optimal Control*. Boston, MA: Birkhauser, 2000.
[32] A. Wernli and G. Cook, "Successive control for the nonlinear quadratic regulator problem," *Automatica*, vol. 11, pp. 75–84, 1975.

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