

Controllability_and_Observability

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0.1 Lateral Dynamics

State Matrix (\mathbf{A}_{lat})

$$\mathbf{A}_{lat} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4C_\alpha}{m\dot{x}} & \frac{4C_\alpha}{m} & -\frac{2C_\alpha(l_f-l_r)}{m\dot{x}} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2C_\alpha(l_f-l_r)}{I_z\dot{x}} & \frac{2C_\alpha(l_f-l_r)}{I_z} & -\frac{2C_\alpha(l_f^2+l_r^2)}{I_z\dot{x}} \end{bmatrix}$$

Input Matrix (\mathbf{B}_{lat})

$$\mathbf{B}_{lat} = \begin{bmatrix} 0 & 0 \\ \frac{2C_\alpha}{m} & 0 \\ 0 & 0 \\ \frac{2C_\alpha l_f}{I_z} & 0 \end{bmatrix}$$

Controllability matrix (\mathbf{P}_{lat})

$$\mathbf{P}_{lat} = [\mathbf{B}_{lat} \quad \mathbf{A}_{lat}\mathbf{B}_{lat} \quad \mathbf{A}_{lat}^2\mathbf{B}_{lat} \quad \mathbf{A}_{lat}^3\mathbf{B}_{lat}]$$

Observability matrix (\mathbf{Q}_{lat})

$$\mathbf{Q}_{lat} = \begin{bmatrix} \mathbf{C}_{lat} \\ \mathbf{C}_{lat}\mathbf{A}_{lat} \\ \mathbf{C}_{lat}\mathbf{A}_{lat}^2 \\ \mathbf{C}_{lat}\mathbf{A}_{lat}^3 \end{bmatrix}$$

Working

$$\mathbf{A}_{lat}\mathbf{B}_{lat} = \begin{bmatrix} \frac{2C_\alpha}{m} & 0 \\ -\frac{8C_\alpha^2}{m^2\dot{x}} - \frac{4C_\alpha^2 l_f(l_f-l_r)}{mI_z\dot{x}} & 0 \\ \frac{2C_\alpha l_f}{I_z} & 0 \\ -\frac{4C_\alpha^2(l_f-l_r)}{I_z m\dot{x}} - \frac{4C_\alpha^2 l_f(l_f^2+l_r^2)}{I_z^2\dot{x}} & 0 \end{bmatrix}$$

$$A_{lat}^2 B_{lat} = \begin{bmatrix} -\frac{8C_\alpha^2}{m^2 \dot{x}} - \frac{4C_\alpha^2 l_f (l_f - l_r)}{m I_z \dot{x}} & 0 \\ \frac{32C_\alpha^3}{m^3 \dot{x}^2} + \frac{32C_\alpha^3 l_f (l_f - l_r)}{m^2 I_z \dot{x}^2} + \frac{8C_\alpha^3 l_f^2 (l_f - l_r)^2}{m I_z^2 \dot{x}^2} + \frac{16C_\alpha^3 l_f (l_f^2 + l_r^2)(l_f - l_r)}{m I_z^2 \dot{x}^2} & 0 \\ \frac{4C_\alpha^2 l_f (l_f - l_r)}{I_z m \dot{x}} - \frac{4C_\alpha^2 l_f (l_f^2 + l_r^2)}{I_z^2 \dot{x}} & 0 \\ \frac{16C_\alpha^3 l_f (l_f - l_r)}{I_z^2 m \dot{x}^2} + \frac{16C_\alpha^3 l_f^2 (l_f^2 + l_r^2)}{I_z^3 \dot{x}^2} & 0 \end{bmatrix}$$

$$A_{lat}^3 B_{lat} = \begin{bmatrix} \frac{32C_\alpha^3}{m^3 \dot{x}^2} + \frac{32C_\alpha^3 l_f (l_f - l_r)}{m^2 I_z \dot{x}^2} + \frac{8C_\alpha^3 l_f^2 (l_f - l_r)^2}{m I_z^2 \dot{x}^2} + \frac{16C_\alpha^3 l_f (l_f^2 + l_r^2)(l_f - l_r)}{m I_z^2 \dot{x}^2} & 0 \\ -\frac{128C_\alpha^4}{m^4 \dot{x}^3} - \frac{128C_\alpha^4 l_f (l_f - l_r)}{m^3 I_z \dot{x}^3} - \frac{32C_\alpha^4 l_f^2 (l_f - l_r)^2}{m^2 I_z^2 \dot{x}^3} - \frac{64C_\alpha^4 l_f (l_f^2 + l_r^2)(l_f - l_r)}{m^2 I_z^2 \dot{x}^3} & 0 \\ \frac{16C_\alpha^3 l_f (l_f - l_r)}{I_z^2 m \dot{x}^2} + \frac{16C_\alpha^3 l_f^2 (l_f^2 + l_r^2)}{I_z^3 \dot{x}^2} & 0 \\ -\frac{64C_\alpha^4 l_f (l_f - l_r)}{I_z^3 m \dot{x}^3} - \frac{64C_\alpha^4 l_f^2 (l_f^2 + l_r^2)}{I_z^4 \dot{x}^3} & 0 \end{bmatrix}$$

$$C_{lat} A_{lat} = \begin{bmatrix} 0 & -\frac{4C_\alpha}{m \dot{x}} & \frac{4C_\alpha}{I_z} & -\frac{2C_\alpha (l_f - l_r)}{I_z \dot{x}} \\ 0 & -\frac{2C_\alpha (l_f - l_r)}{I_z \dot{x}} & \frac{2C_\alpha}{I_z} & -\frac{2C_\alpha (l_f^2 + l_r^2)}{I_z \dot{x}} \end{bmatrix}$$

$$C_{lat} A_{lat}^2 = \begin{bmatrix} 0 & \frac{16C_\alpha^2}{m^2 \dot{x}^2} + \frac{8C_\alpha^2 (l_f - l_r)}{m I_z \dot{x}^2} & -\frac{16C_\alpha^2}{m^2} - \frac{8C_\alpha^2 (l_f - l_r)}{m I_z} & \frac{8C_\alpha^2 (l_f^2 + l_r^2)}{m I_z^2 \dot{x}^2} + \frac{4C_\alpha^2 l_f (l_f - l_r)^2}{m I_z^2 \dot{x}^2} \\ 0 & \frac{8C_\alpha^2 (l_f - l_r)}{I_z m \dot{x}^2} + \frac{4C_\alpha^2 l_f (l_f^2 + l_r^2)}{I_z^2 \dot{x}^2} & -\frac{8C_\alpha^2 (l_f - l_r)}{I_z m} - \frac{4C_\alpha^2 l_f (l_f^2 + l_r^2)}{I_z^2} & \frac{4C_\alpha^2 (l_f^2 + l_r^2)(l_f - l_r)}{I_z^2 \dot{x}^2} + \frac{2C_\alpha^2 l_f (l_f - l_r)^3}{I_z^2 \dot{x}^2} \end{bmatrix}$$

$$C_{lat} A_{lat}^3 = \begin{bmatrix} 0 & -\frac{64C_\alpha^3}{m^3 \dot{x}^3} - \frac{32C_\alpha^3 (l_f - l_r)}{m^2 I_z \dot{x}^3} & \frac{64C_\alpha^3}{m^3 \dot{x}^2} + \frac{32C_\alpha^3 (l_f - l_r)}{m^2 I_z \dot{x}^2} & -\frac{32C_\alpha^3 (l_f^2 + l_r^2)}{m I_z^2 \dot{x}^3} - \frac{16C_\alpha^3 l_f (l_f - l_r)^2}{m I_z^2 \dot{x}^3} \\ 0 & -\frac{32C_\alpha^3 (l_f - l_r)}{I_z m^2 \dot{x}^3} - \frac{16C_\alpha^3 l_f (l_f^2 + l_r^2)}{I_z^2 \dot{x}^3} & \frac{32C_\alpha^3 (l_f - l_r)}{I_z m \dot{x}^2} + \frac{16C_\alpha^3 l_f (l_f^2 + l_r^2)}{I_z^2 \dot{x}^2} & -\frac{16C_\alpha^3 (l_f^2 + l_r^2)(l_f - l_r)}{I_z^2 \dot{x}^3} - \frac{8C_\alpha^3 l_f (l_f - l_r)^3}{I_z^2 \dot{x}^3} \end{bmatrix}$$

```
[2]: import sympy as sp
from sympy import Matrix, symbols, simplify, pprint, init_printing
sp.init_printing(use_latex=True)

# Symbols
C_alpha, m, I_z, l_f, l_r, x_dot = sp.symbols(r'C_{\alpha} m I_z l_f l_r \dot{x}', real=True)

# Assign numerical values (except x_dot)
param_values = {
    m: 1888.6,
    l_r: 1.39,
    l_f: 1.55,
    C_alpha: 20000,
    I_z: 25854
}

# State and input matrices
A_lat = sp.Matrix([
    [0, 1, 0, 0],
```

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    [0, -4*C_alpha/(m*x_dot), 4*C_alpha/m, -2*C_alpha*(l_f - l_r)/(m*x_dot)],
    [0, 0, 0, 1],
    [0, -2*C_alpha*(l_f - l_r)/(I_z*x_dot), 2*C_alpha*(l_f - l_r)/I_z,
    ↪ -2*C_alpha*(l_f**2 + l_r**2)/(I_z*x_dot)]
])

B_lat = sp.Matrix([
    [0, 0],
    [2*C_alpha/m, 0],
    [0, 0],
    [2*C_alpha*l_f/I_z, 0]
])

C_lat = sp.Matrix([
    [1, 0, 0, 0],
    [0, 0, 1, 0]
])

P_lat = B_lat.row_join(A_lat * B_lat).row_join(A_lat**2 * B_lat).
    ↪ row_join(A_lat**3 * B_lat)
Q_lat = C_lat.col_join(C_lat * A_lat).col_join(C_lat * A_lat**2).col_join(C_lat
    ↪ * A_lat**3)

# Substitute numerical values
A_lat_num = A_lat.subs(param_values)
B_lat_num = B_lat.subs(param_values)
C_lat_num = C_lat.subs(param_values)
P_lat_num = P_lat.subs(param_values)
Q_lat_num = Q_lat.subs(param_values)

# Simplify after substitution
A_lat_num = sp.simplify(A_lat_num)
B_lat_num = sp.simplify(B_lat_num)
P_lat_num = sp.simplify(P_lat_num)
Q_lat_num = sp.simplify(Q_lat_num)

# Display
print("A_lat (numeric except x_dot):")
display(A_lat_num)

print("B_lat (numeric):")
display(B_lat_num)

print("C_lat:")
display(C_lat_num)

print("P_lat (numeric except x_dot):")

```

```
display(P_lat_num)

print("Q_lat (numeric except x_dot):")
display(Q_lat_num)
```

A_lat (numeric except x_dot):

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{42.3594196759504}{\dot{x}} & 42.3594196759504 \\ 0 & 0 & 0 \\ 0 & -\frac{0.24754390036358}{\dot{x}} & 0.24754390036358 \end{bmatrix}$$

B_lat (numeric):

$$\begin{bmatrix} 0 & 0 \\ 21.1797098379752 & 0 \\ 0 & 0 \\ 2.39808153477218 & 0 \end{bmatrix}$$

C_lat:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

P_lat (numeric except x_dot):

$$\begin{bmatrix} 0 & 0 & 21.1797098379752 & 0 & -\frac{905.286725013534}{\dot{x}} & 0 & 101.581342148562 + \frac{38419.6858176627}{\dot{x}^2} \\ 21.1797098379752 & 0 & -\frac{905.286725013534}{\dot{x}} & 0 & 101.581342148562 + \frac{38419.6858176627}{\dot{x}^2} & 0 & -\frac{5208.2571924939}{\dot{x}} - \frac{1623}{\dot{x}^2} \\ 0 & 0 & 2.39808153477218 & 0 & -\frac{21.325099086717}{\dot{x}} & 0 & 0.593630456507386 + \frac{367.110157814573}{\dot{x}^2} \\ 2.39808153477218 & 0 & -\frac{21.325099086717}{\dot{x}} & 0 & 0.593630456507386 + \frac{367.110157814573}{\dot{x}^2} & 0 & -\frac{34.4057881556766}{\dot{x}} - \frac{119}{\dot{x}^2} \end{bmatrix}$$

Q_lat (numeric except x_dot):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{42.3594196759504}{\dot{x}} & 42.3594196759504 & -\frac{3.38875357407604}{\dot{x}} \\ 0 & -\frac{0.24754390036358}{\dot{x}} & 0.24754390036358 & -\frac{6.70627369072484}{\dot{x}} \\ 0 & \frac{1795.1593005604}{\dot{x}^2} & -\frac{1795.1593005604}{\dot{x}} & 42.3594196759504 + \frac{166.27154376084}{\dot{x}^2} \\ 0 & \frac{12.1459131100303}{\dot{x}^2} & -\frac{12.1459131100303}{\dot{x}} & 0.24754390036358 + \frac{45.812972092006}{\dot{x}^2} \end{bmatrix}$$

[3]: x_dot_values = [2, 5, 8]

```
for x_val in x_dot_values:
    # Substitute x_dot
    P_sub = P_lat_num.subs(x_dot, x_val)
    Q_sub = Q_lat_num.subs(x_dot, x_val)

    # Compute ranks
    rank_P = P_sub.rank()
```

```

rank_Q = Q_sub.rank()

print(f"x_dot = {x_val} m/s:")
print(f"  Rank of P_lat (controllability) = {rank_P}")
print(f"  Rank of Q_lat (observability) = {rank_Q}\n")

```

```

x_dot = 2 m/s:
  Rank of P_lat (controllability) = 4
  Rank of Q_lat (observability) = 4

```

```

x_dot = 5 m/s:
  Rank of P_lat (controllability) = 4
  Rank of Q_lat (observability) = 4

```

```

x_dot = 8 m/s:
  Rank of P_lat (controllability) = 4
  Rank of Q_lat (observability) = 4

```

The Lateral system is controllable and observable for the respective speeds

0.2 Longitudinal Dynamics

State Matrix (\mathbf{A}_{lon})

$$\mathbf{A}_{lon} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Input Matrix (\mathbf{B}_{lon})

$$\mathbf{B}_{lon} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$$

Controllability matrix (\mathbf{P}_{lon})

$$\mathbf{P}_{lon} = [\mathbf{B}_{lon} \quad \mathbf{A}_{lon}\mathbf{B}_{lon}]$$

Observability matrix (\mathbf{Q}_{lat})

$$\mathbf{Q}_{lon} = \begin{bmatrix} \mathbf{C}_{lon} \\ \mathbf{C}_{lon}\mathbf{A}_{lon} \end{bmatrix}$$

Working

$$\mathbf{A}_{lon}\mathbf{B}_{lon} = \begin{bmatrix} 0 & \frac{1}{m} \\ 0 & 0 \end{bmatrix}$$

$$A_{lon}^2 B_{lon} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_{lon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_{lon} A_{lon} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C_{lon} A_{lon}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

```
[4]: sp.init_printing(use_latex=True)

m = sp.symbols('m', real=True)

A_lon = sp.Matrix([[0, 1],
                   [0, 0]])

B_lon = sp.Matrix([[0, 0],
                   [0, 1/m]])

C_lon = sp.Matrix([[1, 0],
                   [0, 1]])

# Controllability matrix
P_lon = B_lon.row_join(A_lon*B_lon).subs(m, 1888.6)

# Observability matrix
Q_lon = C_lon.col_join(C_lon*A_lon).subs(m, 1888.6)

print("A_lon:")
display(A_lon)

print("B_lon:")
display(B_lon)

print("P_lon (controllability matrix):")
display(P_lon)

print("Q_lon (observability matrix):")
display(Q_lon)

print(f"P and Q don't depend on x_dot")
print(f"Rank of P_lon (controllability) = {P_lon.rank()}")
```

```
print(f" Rank of Q_lon (observability) = {Q_lon.rank()}\n")
```

A_lon:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

B_lon:

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$$

P_lon (controllability matrix):

$$\begin{bmatrix} 0 & 0 & 0 & 0.000529492745949381 \\ 0 & 0.000529492745949381 & 0 & 0 \end{bmatrix}$$

Q_lon (observability matrix):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

P and Q don't depend on x_dot

Rank of P_lon (controllability) = 2

Rank of Q_lon (observability) = 2

The Longitudinal system is controllable and observable for the respective speeds