PHY 509 Final Project:
Open Strings, Mesons, and Branes

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1 Introduction

In this paper we discuss the quantization and spectrum of the (bosonic) open string, i.e. a string which has endpoints. We assume knowledge of gauge fixing of the Polyakov action and closed string quantization, which are discussed in other presentations/papers for this class.

The motivation for open strings actually starts with QCD. In Section 2, we will find that, if we describe mesons as two quarks attached to the endpoints of a string, we can recover the famed “Regge trajectories”, i.e. the relationship between the spins and masses of mesonic states. This is our first hint that QCD is somehow related to string theory.

We then take a detour in Section 3 to discuss an important aspect of open string theory that gives rise to dynamical higher-dimensional objects known as D-branes. In particular, we will see that in the weak field limit, these branes give rise to particle theories.

The last section of this paper covers a generalization of QCD known as the ’t Hooft model and briefly explains how it can be interpreted as a theory of open strings. Although this particular case ends up not working perfectly, the same method applied to supersymmetric field theories leads to a formulation of the AdS/CFT correspondence.

2 Open string quantization

We follow [1] and [2] in this section.

The dynamics for open strings is all the same as for closed strings, so as usual we use the Polyakov action to describe it:

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial^\beta X^\nu \eta_{\mu\nu} \tag{2.1}$$

Let us choose conformal gauge right away, so the action becomes

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \; \partial_\alpha X^\mu \partial^\alpha X_\mu \tag{2.2}$$

Taking the variation, we get

$$\delta S = \frac{1}{2\pi\alpha'} \Big(-\int d^2\sigma \; \partial^2 X_\mu (\delta X^\mu) + \Big[\int d\sigma \; \partial_\tau X_\mu (\delta X^\mu) \Big]_{\tau=\tau_i}^{\tau=\tau_f} - \Big[\int d\tau \; \partial_\sigma X_\mu (\delta X^\mu) \Big]_{\sigma=0}^{\sigma=\pi} \Big) \tag{2.3}$$

The 2nd term vanishes, as usual, by $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$. However, the 3rd term is not one we had in the closed string case, because in that case it vanished due to the periodicity of σ, i.e. $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$. We no longer have this condition since the string is open, so we must impose some additional boundary conditions to make this term go away. There are two possibilities:

1. Neumann conditions, requiring $\partial_\sigma X^\mu = 0$ at $\sigma = 0$ and $\sigma = \pi$;
2. Dirichlet conditions, requiring $\delta X^\mu = 0$ at $\sigma = 0$ and $\sigma = \pi$. In other words, the endpoints are fixed to some location c^μ of spacetime.

Let us assume Neumann conditions for the first $p+1$ coordinates and Dirichlet conditions for the remaining $D-p-1$ coordinates: $\partial_\alpha X^i = 0$ for $i = 0, \ldots, p$ and $X^j = c^j$ for $j = p+1, \ldots, D-1$. How do these affect quantization?

Let us take light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ on the worldsheet, as well as on the spacetime, with $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$. We can also set

$$x^\pm(\tau) = \frac{1}{\pi}\int_0^\pi d\sigma\, X^\pm \tag{2.4}$$

i.e. the mean value of $X^\pm$, and

$$Y^-(\tau,\sigma) = X^-(\tau,\sigma) - x^-(\tau) \tag{2.5}$$

This last variable is not dynamical and can be eliminated by its equation of motion in the action. Gauge fixing further allows us to set

$$X^+ = x^+ + \alpha' p^+ \tau \tag{2.6}$$

In other words, we can get rid of X^+ by shifting τ.

As usual, we take the Fourier expansions of the X^i. For the components with Neumann conditions, we have

$$X^i(\tau,\sigma) = x^i(0) + \frac{p^i(0)}{p^+}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\tau}\cos(n\sigma), \quad i = 1, \ldots, p \tag{2.7}$$

$$X^j(\tau,\sigma) = \sqrt{2\alpha'}\alpha_0^j\,\sigma - \sqrt{2\alpha'}\sum_{n\neq 0} e^{-in\tau}\alpha_n^j \sin(n\sigma), \quad j = p+1, \ldots, D-2 \tag{2.8}$$

where

$$x^i(\tau) = \frac{1}{\pi}\int_0^\pi d\sigma\, X^i, \quad p^i(\tau) = \frac{p^+}{\pi}\int_0^\pi d\sigma\, \partial_\tau X^i \tag{2.9}$$

and we have $\alpha^i_{-n} = (\alpha^i_n)^\dagger$ to keep X^i real. The Hamiltonian becomes

$$H = \frac{p^i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\Big(\sum_{n>0} \alpha_{-n} \cdot \alpha_n - A\Big) \tag{2.10}$$

where A is some normal-ordering constant. Similarly, the mass-squared operator is

$$M^2 = 2p^+ H - p^i p^i = \frac{1}{\alpha'}\Big(\sum_{n>0} \alpha_{-n} \cdot \alpha_n - A\Big) \tag{2.11}$$

In order to retain Lorentz invariance, we set $A = 1$, same as in the closed string case.

To quantize, we promote the Poisson brackets to commutators:

$$[x^i, p^j] = i\,\delta^{ij}, \;\; [\alpha^i_m, \alpha^j_n] = m\,\delta^{ij}\delta_{m+n,0} \tag{2.12}$$

Thus we have a harmonic oscillator in each (m, i) mode.

Given a center-of-mass momentum $k = (k^+, k^i)$, let us postulate a vacuum state $|0; k\rangle$ defined by

$$p^+|0; k\rangle = k^+|0; k\rangle, \;\; p^i|0; k\rangle = k^i|0; k\rangle, \;\; \alpha^i_{n>0}|0; k\rangle = 0 \tag{2.13}$$

The rest of the Hilbert space is then built by acting with the $\alpha_{n<0}$ on this state. Hence we get particles in our theory. Indeed, we can define the number operator

$$N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n \tag{2.14}$$

whose eigenvalue is the sum of the occupation numbers of all the modes. Namely, we have the commutation relations

$$[N, \alpha^i_{-n}] = n\alpha^i_n \tag{2.15}$$

so α^i_{-n} raises N by n and α^i_n lowers it by n.

The mass-squared operator is simply

$$M^2 = \frac{N - A}{\alpha'} \tag{2.16}$$

Note that the ground state has mass $M^2 = -1$, so the mass is imaginary. A state with imaginary mass is called a tachyon.

We are also interested in the spins of the particles. Recall that, due to our choice of light-cone gauge, we now only have the SO$(D-2)$ symmetry in the transverse directions. We can define the spin operators

$$S^{ij} = -i \sum_{n>0} \frac{1}{n} \left(\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n \right) \tag{2.17}$$

which is antisymmetric in the i, j indices (hence there can be no added constant). These are the generators of the SO$(D-2)$ rotations. We have the commutation relations

$$[S^{ij}, \alpha^i_{-n}] = i\alpha^j_{-n}, \;\; [S^{ij}, \alpha^j_{-n}] = -i\alpha^i_{-n} \tag{2.18}$$

We can take linear combinations of the α^i_{-n}'s to get operators that raise or lower one of the spin generators by an integer.

Consider a state with occupation number $N = n$. From (2.16), recall that the mass-squared of such a state is

$$M^2 = \frac{n-1}{\alpha'} \tag{2.19}$$

On the other hand, S^{ij} can take on a range of values, but the maximum possible occurs if we choose all the spins to be in the same direction. For example, the operator $B^{ij} = \alpha^i_{-1} + i\alpha^j_{-1}$ has $[N, B^{ij}] = [S^{ij}, B^{ij}] = 1$, so a state $(B^{ij})^n |0; k\rangle$ has $N = S^{ij} = n$. In general, any state with $N = n$ has $S^{ij} \leq n$. Thus for any state we find the inequality

$$S^{ij} \leq \alpha' M^2 + 1 \tag{2.20}$$

This relationship between the spin and the mass of a string state is called a Regge trajectory, and the parameter α' is known as the Regge slope.

As an interesting sidenote, Regge trajectories appear in the spectra of meson masses. String theory actually originated as a model of mesons – two quarks connected by an open string. We will return to this idea in the last section.

3 D-branes

At the beginning of the previous section, we very briefly discussed two types of boundary conditions for the endpoints of open strings. Let us now study the effect of these more closely. We follow [1] in this section.

Neumann boundary conditions require $\partial_\sigma X^i = 0$ at $\sigma = 0$ and $\sigma = \pi$. From the equation of motion of the metric in (2.2), we must have $(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0$. Now choosing $X^0 = t = R\tau$ and labelling the spatial components $x^i = X^i$ where $i = 1, 2, 3$, the constraint becomes $(\partial_\tau x)^2 + (\partial_\sigma x)^2 = R^2$. Thus if $\partial_\sigma x^i = 0$, we must have $(dx/dt)^2 = 1$ at the endpoints.

The situation with Dirichlet boundary conditions seems innocuous in comparison, but in fact it is a rich source of new physics. Dirichlet conditions require $\delta X^i = 0$ at the endpoints, so in other words the endpoints must remain at a fixed location in space. If we assume Dirichlet conditions for $i = p + 1, \ldots, D - 1$, then we constrain the string endpoints to move on a $(p+1)$-dimensional surface in the D-dimensional spacetime. This surface is called a Dp-brane. Its group of isometries is SO$(1, p)$.

Let us briefly go back to the string spectrum. Recall that states with $N = 1$ are massless from (2.16). There are two such types of states which are qualitatively different. The first is of the form

$$|\psi\rangle = \sum_{i=0}^{p} \alpha^i_{-1}|0; k\rangle \tag{3.1}$$

These excitations are longitudinal with respect to the D-brane, and they transform in the fundamental representation of SO$(1, p)$. In contrast, we also have transverse excitations

$$|\chi\rangle = \sum_{j=p+1}^{D-1} \alpha^j_{-1}|0; k\rangle \tag{3.2}$$

which transform in the fundamental representation of SO$(D-1-p)$. We can think of the $|\psi\rangle$ states as the Hilbert space of a U(1) gauge field A_i and the $|\chi\rangle$ states as the Hilbert space of $D - p - 1$ scalar fields ϕ^j, all living on the brane.

We can make this precise by, as usual, postulating an action for the D-brane and identifying the above fields as the dynamical components of the brane. Let us first deal with the transverse components via the action

$$S_{\rm Dir} = -T_p \int d^{p+1}\xi \sqrt{-\det\gamma} \tag{3.3}$$

where ξ^i are the brane coordinates with $i = 0, \ldots, p$ and γ_{ij} is the pullback metric given by

$$\gamma_{ij} = \frac{\partial X^\mu}{\partial \xi^i}\frac{\partial X^\nu}{\partial \xi^j} g_{\mu\nu} \tag{3.4}$$

The dynamical fields are $X^\mu(\xi)$ with $\mu = 1, \ldots, D-1$. Why so many degrees of freedom? Actually, this is deceptive: the action has reparametrization invariance. We can use this to reparametrize away the longitudinal components by choosing the gauge $X^a = \xi^a$ for $a = 0, \ldots, p$. Now we can set the transverse fields to be $\phi^i = X^i(\xi)/(2\pi\alpha')$, $i = p+1, \ldots, D-1$.

What about the fluctuations corresponding to a gauge field on the brane? It turns out that the correct action for these components is

$$S_{\rm Neu} = -T_p \int d^{p+1}\xi \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})} \tag{3.5}$$

where ξ^i are again the brane coordinates and now we have the field strength F_{ab} arising from a gauge field $A_a(\xi)$ on the brane. If we expand in F_{ab} we find

$$S \approx -T_p \int d^{p+1}\xi \, \frac{(2\pi\alpha')^2}{4} F_{ab}F^{ab} \tag{3.6}$$

where we have gotten rid of the overall constant. This is the usual Maxwell action for a U(1) gauge field. Note that the linear term vanishes because $\mathrm{Tr}(F_{ab}) = 0$.

Now, putting these actions together, we arrive at the Dirac-Born-Infeld (DBI) action

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \tag{3.7}$$

Again we can use the reparametrization invariance to choose $X^a = \xi^a$ for $a = 0, \ldots, p$ and set $\phi^i = X^i/(2\pi\alpha')$. In this gauge the pullback metric is

$$\gamma_{ab} = \eta_{ab} + \frac{\partial X^i}{\partial \xi^a}\frac{\partial X^j}{\partial \xi^b}\delta_{ij} = \eta_{ab} + (2\pi\alpha')^2 \partial_a\phi^i\partial_b\phi^j\delta_{ij} \tag{3.8}$$

Again expanding the action to first order in F_{ij} and X^i, we find

$$S \approx -T_p \int d^{p+1}\xi \big(\frac{(2\pi\alpha')^2}{2} \partial_a \phi^i \partial^a \phi_i + \frac{(2\pi\alpha')^2}{4} F_{ab} F^{ab} \big) \tag{3.9}$$

Thus we see that when the fields are small we obtain $D-1-p$ non-interacting massless scalar fields in addition to the U(1) gauge field.

The logical next step is to consider multiple branes so that strings can begin or end on different branes. This corresponds to a non-abelian generalization of the DBI action. Instead of finding such a generalization of the full action, let us consider only the weak field limit. Suppose there are N branes lying at the same location in spacetime. The correct generalization of (3.9) in this case is

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \mathrm{Tr}\big(\frac{1}{2} D_a \phi^i D^a \phi_i + \frac{1}{4} F_{ab} F^{ab} - \frac{1}{4} \sum_{i \neq j} [\phi^i, \phi^j]^2 \big) \tag{3.10}$$

where the gauge group is U(N) and the covariant derivative acts as $D_a \phi^i = \partial_a \phi^i + i[A_a, \phi^i]$. Thus we see that we have the usual Yang-Mills action coupled to massless scalars in the adjoint representation interacting through a quartic term.

4 Large N QCD

This section draws from [3], [4], and [5].

Recall that, in Section 2, we alluded to an interpretation of mesons as quarks connected by an open string. In fact, one can draw a correspondence between interactions in QCD and string theory. To illustrate this, it is easier to work in the reverse direction, starting with a generalization of QCD known as the 't Hooft model.

Real-world QCD has gauge group SU(3) and 3 flavors. Let us instead consider gauge group U(N) and N_f flavors with masses m_a. (Sometimes we omit the flavor index because it just amounts to an extra factor of N_f in the end.) The action is

$$S = -\int d^4x \left(\mathrm{Tr}\big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \big) - \overline{\psi}_a (i\not{D} - m) \psi_a \right) \tag{4.1}$$

where the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and the covariant derivative acts as $D_\mu\psi = \partial_\mu\psi + igA_\mu\psi$. One must first carry out the Faddeev-Popov gauge fixing method that is standard for non-abelian gauge theories. Then, we can play the old game of writing down Feynman diagrams for interactions.

Now comes 't Hooft's great insight: in the limit where $N \to \infty$ and $g \to 0$ but $\lambda = g^2 N$ remains fixed, the only Feynman diagrams that contribute are the planar diagrams. Namely, we can turn a diagram into a 2-surface by thickening the lines into ribbons, and it turns out that the $1/N$-dependence of a diagram increases with its genus. Thus, only planar diagrams contribute to interaction amplitudes in the 't Hooft limit.

It is actually simple to derive this property. Let us first replace $g \to g/\sqrt{N}$ in the action and then rescale

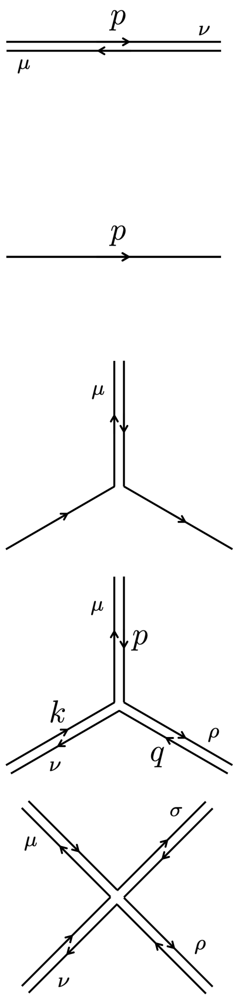
$$A_\mu \to \frac{\sqrt{N}}{g} A_\mu, \quad \psi \to \sqrt{N}\psi \tag{4.2}$$

Now the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and the covariant derivative is $D_\mu\psi = \partial_\mu\psi + iA_\mu\psi$. Importantly, we have gotten rid of the factors of g there. The action becomes

$$S = -N \int d^4x \left(\mathrm{Tr}\big(\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}\big) - \overline{\psi}_a (i\not{D} - m_a)\psi_a \big) \right) \tag{4.3}$$

The advantage is that the N-dependence is easy to extract. Feynman rules with this new convention are shown in Figure 1 (diagrams involving ghosts are not shown). Note the representation of gluons as a quark and an antiquark, each with its own $\mathrm{U}(N)$ index. One can think of this as splitting the adjoint representation of $\mathrm{U}(N)$ into the tensor product of the fundamental representation and its conjugate. Writing the Feynman rules in this way is called double line notation.

Suppose a diagram has P propagators, V vertices, and L loops. The cubic and quartic quark vertices both have a factor of N. Propagators have a factor of $1/N$. Loops contribute a factor of N, since tracing over the quark's $\mathrm{U}(N)$ indices gives $\sum_i \delta_{ii} = N$. Thus the overall N-dependence of the diagram is N^{V-P+L}. This is a topological invariant, the Euler characteristic! And in fact if we thicken the diagram into an orientable surface in the way mentioned previously, then we have $V - P + L = 2 - 2H - B$, with H the number of handles and B the number of boundaries of the surface. Riemann surfaces are classified by these characteristics. Therefore, the perturbative expansion of interactions in this theory corresponds to a sum over topologies.



$$= \frac{1}{N}\frac{i\eta_{\mu\nu}}{p^2 + i\epsilon}$$

$$= \frac{1}{N}\frac{i}{\not{p} - m + i\epsilon}$$

$$= -iN\gamma_\mu$$

$$= -iN\big(\eta_{\mu\nu}(q-p)_\rho + \eta_{\nu\rho}(k-q)_\mu + \eta_{\rho\mu}(p-k)_\nu\big)$$

$$= -2iN\big(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\big)$$

Figure 1: Feynman rules for gluon-gluon and gluon-quark interactions in the ’t Hooft model. The conventions for the coupling and fields follow (4.3).

This is precisely the same expansion that occurs in the calculation of string amplitudes! We will not go into the details of string scattering here, but the point is that the perturbative expansion for open strings in powers of the string coupling is a sum over topologies of the worldsheet, which is a smooth 2d surface embedded in the spacetime. Unfortunately, there is a caveat: the surfaces arising from QCD diagrams are not smooth, as they are triangulated by the edges and vertices of Feynman diagrams. As the number of edges increases, the ribbon diagram becomes a better approximation of a smooth surface. This hints to us that, in the $\lambda \gg 1$ limit, the 't Hooft model may be viewed as a string theory.

For various reasons, it turns out the string theory side of this picture is difficult to make sense of. However, our analysis of QCD in the $N \to \infty$ limit extends to a whole class of large N gauge theories. Among these is the maximally supersymmetric Yang-Mills theory, whose interpretation as the boundary CFT dual to a string theory on $\mathrm{AdS}_5 \times S^5$ is now well understood. This is one instance of the famed AdS/CFT correspondence.

5 Conclusion

In this paper we demonstrated a few interesting connections between field theories and open strings. These topics are foundational to several fruitful areas of ongoing research: for example, generalizing large N QCD to theories with different types of matter and other gauge groups like $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$ provides a wealth of toy models that can be used to test non-perturbative methods.

Acknowledgement of original work

This paper represents my work in accordance with University regulations.

/s/ Loki Lin

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