CMPS 101 HW5

1) $T(n) \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$

a) $T(n) = n + 2T(\lfloor \frac{n}{3} \rfloor)$

$= n + 2(\lfloor \frac{n}{3} \rfloor + 2T(\lfloor \frac{n}{3^2} \rfloor))$

$= \cancel{\frac{4n}{9} + 4(\frac{n}{9} + 2T(\lfloor \frac{n}{3^3} \rfloor))} \quad n + \frac{2}{3}n + 2^2 T(\lfloor \frac{n}{3^2} \rfloor)$

$= 2^3 T(\lfloor \frac{n}{3^3} \rfloor) + 2^2 \lfloor \frac{n}{3^2} \rfloor + 2 \lfloor \frac{n}{3} \rfloor + n$

$= 2^k T(\lfloor \frac{n}{3^k} \rfloor) + \sum_{i=0}^{k-1} 2^i \lfloor \frac{n}{3^i} \rfloor$

terminates $1 \le \lfloor \frac{n}{3^k} \rfloor < 3 \Rightarrow 3^k \le n < 3^{k+1}$

$k \le \log_3 n < k+1$

$k = \lfloor \log_3 n \rfloor$

b) ~~w/ $T(n)$ 6 $1 \le n < n_0$ $2T(\lfloor \frac{n}{3} \rfloor) + n$ $n \ge n_0$~~

using a), we get w/ $k = \lfloor \log_3 n \rfloor$

$T(n) = 2^{\lfloor \log_3 n \rfloor}(6) + \sum_{i=0}^{\lfloor \log_3 n - 1 \rfloor} 2^i \lfloor \frac{n}{3^i} \rfloor$

assuming summation is inf, assume $T(n) = O(n)$

$T(n) \le 2^{\log_3 n}(6) + \sum_{i=0}^{\infty} 2^i \left(\frac{n}{3^i}\right)$

$= n^{\log_3 2}(6) + n \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i$

$= 6n^{\log_3 2} + n\left(\frac{1}{1-\frac{2}{3}}\right) = 6n^{\log_3 2} + 3n = O(n)$

c) Master Theorem to prove $T(n) = \Theta(n)$

Letting $a=2, b=3$, $\log_b(a) = 0.63 < 1$

setting $\varepsilon = 1 - \log_3 2$ thus $\varepsilon > 0$

$\therefore f(n) = n = \Omega(n^{\log_3 2 + \varepsilon})$

$\boxed{\text{case } 3}$ $a f\left(\frac{n}{b}\right) \leq c f(n)$

$\Rightarrow \frac{2}{3} n \leq cn$ is true as long as c is chosen to satisfy $\frac{2}{3} < c < 1$

thus w/ case 3 of the Master Theorem $T(n) = \Theta(n)$

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a) $T(n) = 7T\left(\frac{n}{4}\right) + n$

w/ $a=7, b=4$, $\log_b(a) = 1.403 > 1$

then $\varepsilon = \log_b(a) - 1 \Rightarrow \varepsilon > 0$

$\therefore f(n) = n = O(n^{\log_4 7 - \varepsilon})$

with case (1) applies and $T(n) = \Theta(n^{\log_4 7})$

b) $T(n) = 9T(n/3) + n^2$

$a = 9, b = 3$ $\log_b(a) = 2$

$\therefore f(n) = n^2 = \Theta(n^{\log_b a})$

so by case (2)

$T(n) = \Theta(n^2 \log(n))$

c) $T(n) = 6T\left(\frac{n}{5}\right) + n^2$

$a=6, b=5 \qquad \log_b a = 1.113.. < 2$

let $\varepsilon = 2 - \log_5 6$

we get $f(n) = n^2 = \Omega(n^{\log_5 6 - \varepsilon})$

$6f\left(\frac{n}{5}\right) \le cf(n)$

$\frac{6}{25} n^2 \le cn^2 \qquad w/ \ \frac{6}{25} \le c < 1$

w/ case (3), $T(n) = \Theta(n^2)$

d) $T(n) = 6T\left(\frac{n}{5}\right) + n\log(n)$

$a=6, b=5 \qquad \log_b a = 1.113... > 1$

Let $\varepsilon = \log_5 6 - 1$

$f(n) = n\log(n) = O(n^{\log_5 a - \varepsilon})$

so case (1) applies

and $T(n) = \Theta(n^{\log_5 6})$

e) $T(n) = 7T\left(\frac{n}{2}\right) + n^2$

$a=7, b=2 \quad \log_b a = 0.301.. < 2$

set $\varepsilon = 2 - \log_2 7$; we have $\varepsilon > 0$ and
$f(n) = n^2 = \Omega(n^{\log_2 7 + \varepsilon})$

$7f\left(\frac{n}{2}\right) \le cf(n)$

$\frac{7}{4} n^2 \le cn^2 \qquad$ when $\frac{7}{4} \le c < 1$

by case (3) $T(n) = \Theta n^2$

f) $S(n) = aS\left(\frac{n}{4}\right) + n^2$

$a = a,\ b = 4,$ ~~$\log_b a$~~

now comparing n^2 to $n^{\log_4 a}$

case (1) ~~[illegible]~~ $a > 16$ must be

so $\log_4 a > 2$ and $T(n) = \Theta(n^{\log_4 a})$

case (2) $a = 16$ so $\log_4 16 = 2$

and

$T(n) = \Theta(n^2 \log(n))$

case (3) $a < 16$ then $\log_4 a < 2$

and $T(n) = \Theta(n^2)$

3 Alg. A $\rightarrow$ $T(n) = 7T\left(\frac{n}{2}\right) + n^2$

find a so $S(n) = o(T(n))$

Alg B $\rightarrow$ $S(n) = aS\left(\frac{n}{4}\right) + n^2$

by the master theorem, the recurrence for Alg. A's run time is $T(n) = \Theta(n^{\log_2 7})$

$\log_2 7 = \log_4 x \Rightarrow x = 49$

thus the largest integer value for 'a' such that B is a faster alg. than A (asymptotically speaking) is when

$a = 48$

4 Let $T(n) = aT(\frac{n}{b}) + f(n)$; $f(n)$ is a polynomial $\deg(f) > \log_b(a)$

~~case 3 for case 3 to apply $\log_b a$~~

Proof: Let $d = \deg(f)$ and replace $f(n)$ w/ n^d.
using Master Thrm. we compare n^d and $n^{\log_b(a)}$
Let $\varepsilon = d - \log_b(a)$ which is positive since $d > \log_b(a)$
$\therefore$ $d = \log_b(a) + \varepsilon$

thus $n^d = \Omega(n^d) = \Omega(n^{\log_b a + \varepsilon})$

verifying the first hypothesis of case ③. To prove the regularity cond.
~~see $d > \log_b(a)$~~ holds. see $d > \log_b(a) \Rightarrow b^d > a \Rightarrow a/b^d < 1$
Let c be in $a/b^d \leq c < 1$. Then for any $n \geq 1$ we have $a(n/b)^d = (a/b^d)n^d \leq cn^a$
thus holds the regularity condition

5) The Handshake Lemma says that any # of hands shaken at a party is twice the # of handshakes

~~Let there be $x, y \in V(G)$~~

Assume G is a graph containing an odd number of odd vertices.

knowing each edge has two ~~edges~~ vertices
by the lemma

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

The sum of all degrees of a graph is even. If we sum over only the vertices of odd degree we still get an even # since we would be subtracting out terms each of which are even