CMPS 101
Homework Assignment 3
Solutions

1. Prove that $3^{2^n} = o\left(2^{3^n}\right)$.

   Proof:

   Observe that $\ln\left(\frac{3^{2^n}}{2^{3^n}}\right) = 2^n(\ln 3) - 3^n(\ln 2) \to -\infty$, since $3^n = \omega(2^n)$. Therefore $\frac{3^{2^n}}{2^{3^n}} \to e^{-\infty} = 0$, whence $3^{2^n} = o\left(2^{3^n}\right)$. ■

2. The last exercise in the handout entitled *Some Common Functions*.

   Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

   Proof: By Stirling's formula

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{2\pi \cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta(1/2n)\right)}{\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^{n}\cdot\left(1+\Theta(1/n)\right)\right)^2}$$

$$= \frac{2^{2n}}{\sqrt{\pi n}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2} = \frac{1}{\sqrt{\pi}}\cdot\frac{4^n}{\sqrt{n}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2}$$

   so that

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}}\cdot\frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2} \to \frac{1}{\sqrt{\pi}} \quad \text{as} \quad n\to\infty$$

   The result now follows since $0 < \frac{1}{\sqrt{\pi}} < \infty$. ■

3. Exercise 1 from the induction handout.

   Prove that for all $n \ge 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

   a. Using form IIa of the induction step.
   b. Using form IIb of the induction step.

   Proof: Let $P(n)$ be the equation $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

   I. Observe that $\sum_{i=1}^{1} i^3 = 1^3 = 1^2 = \left(\frac{1\cdot(1+1)}{1}\right)^2$, whence $P(1)$ is true.

   IIa. Let $n \ge 1$ and assume $P(n)$ is true, i.e. for this *n*, we assume that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $P(n+1)$ holds: $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$. Thus

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{(by the induction hypothesis)} \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4n+4\right]}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

showing that $P(n+1)$ is true. ■

IIb. Let $n > 1$ and assume $P(n-1)$ is true, i.e. for this n, we assume that $\sum_{i=1}^{n-1} i^3 = \left(\frac{(n-1)n}{2}\right)^2$. We must show that $P(n)$ holds: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Thus

$$\begin{aligned}
\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n-1} i^3 + n^3 \\
&= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \qquad \text{(by the induction hypothesis)} \\
&= \frac{(n-1)^2 n^2 + 4n^3}{4} = \frac{n^2\left[(n^2-2n+1)+4n\right]}{4} \\
&= \frac{n^2\left[n^2+2n+1\right]}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

showing that $P(n)$ is true. ■

4. Exercise 2 from the induction handout.
Define $S(n)$ for $n \in Z^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

Proof: Let $P(n)$ be the inequality $S(n) \geq \lg(n)$.

I. The inequality $S(1) \geq \lg(1)$ reduces to $0 \geq 0$, which is obviously true, so $P(1)$ holds.

IId. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. Then

$$\begin{aligned}
S(n) &= S(\lceil n/2 \rceil) + 1 && \text{(by the definition of } S(n)\text{)} \\
&\geq \lg\lceil n/2 \rceil + 1 && \text{(by the induction hypothesis with } k = \lceil n/2 \rceil\text{)} \\
&\geq \lg(n/2) + 1 && \text{(since } \lceil x \rceil \geq x \text{ for any } x\text{)} \\
&= \lg(n) - \lg(2) + 1
\end{aligned}$$

$= \lg(n)$

showing that $P(n)$ holds. Therefore $S(n) \ge \lg(n)$ for all $n \ge 1$, as claimed. ■

5. Let $T(n)$ be defined by the recurrence formula:

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \ge 2 \end{cases}$$

Show that $\forall n \ge 1$: $T(n) \le \frac{4}{3} n^2$, and hence $T(n) = O(n^2)$. (Hint: follow Example 3 on page 3 of the induction handout.)

Proof:
Let $P(n)$ be the statement $T(n) \le (4/3)n^2$. Then $P(1)$ is true, since $T(1) = 1 \le 4/3 = (4/3) \cdot 1^2$, and the base case is satisfied.

Let $n > 1$ be chosen arbitrarily, and suppose for all k in the range $1 \le k < n$ that $T(k) \le (4/3)k^2$. We must show as a consequence that $T(n) \le (4/3)n^2$. Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\le (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\le (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. ■

6. Let $T(n)$ be defined by the recurrence formula:

$$T(n) = \begin{cases} 2 & n = 1, 2 \\ 9T(\lfloor n/3 \rfloor) + 1 & n \ge 3 \end{cases}$$

Show that $\forall n \ge 1$: $T(n) \le 3n^2 - 1$, and hence $T(n) = O(n^2)$. (Hint: emulate Example 4 on page 4 of the induction handout. I. Base: check the two cases $n = 1$, and $n = 2$. II. Induction step: show that for all $n \ge 3$, if for any k in the range $1 \le k < n$ we have $T(k) \le 3k^2 - 1$, then $T(n) \le 3n^2 - 1$.)

Proof:
Let $P(n)$ be the statement $T(n) \le 3n^2 - 1$. $P(1)$ is true since $T(1) = 2 = 3 \cdot 1^2 - 1$, and $P(2)$ is true because $T(2) = 2 \le 11 = 3 \cdot 2^2 - 1$.

Let $n > 2$ be arbitrary, and assume for all k in the range $1 \le k < n$ that $T(k) \le 3k^2 - 1$. Note that in particular $1 \le \lfloor n/3 \rfloor < n$ (since $n \ge 3 \Rightarrow n/3 \ge 1 \Rightarrow \lfloor n/3 \rfloor \ge 1$) and hence $T(\lfloor n/3 \rfloor) \le 3\lfloor n/3 \rfloor^2 - 1$. We must show as a consequence that $T(n) \le 3n^2 - 1$.

$$
\begin{aligned}
T(n) &= 9T(\lfloor n/3 \rfloor) + 1 && \text{by the recurrence formula for } T(n) \\
&\le 9\left(3\lfloor n/3 \rfloor^2 - 1\right) + 1 && \text{by the induction hypothesis} \\
&= 9 \cdot 3\lfloor n/3 \rfloor^2 - 9 + 1 && \\
&\le 9 \cdot 3(n/3)^2 - 9 + 1 && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\
&= 9 \cdot 3(n^2/3^2) - 9 + 1 && \\
&= 3n^2 - 8 && \\
&\le 3n^2 - 1 && \text{since } -8 \le -1
\end{aligned}
$$

and therefore $T(n) \le 3n^2 - 1$, as required. ■

7. Define $T(n)$ defined by the recurrence formula

$$
T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}
$$

Use induction to show that $\forall n \ge 1 \colon T(n) \le 6n$, and hence $T(n) = O(n)$. (Hint use strong induction with two base cases: $n = 1$ and $n = 2$.)

Proof:

I. $T(1) = 6 \le 6 \cdot 1$ and $T(2) = 6 \le 12 = 6 \cdot 2$, so both base cases are satisfied.

II. Let $n > 1$ and assume for all k in the range that $1 \le k < n$ that $T(k) \le 6k$. We must show that $T(n) \le 6n$. Observe

$$
\begin{aligned}
T(n) &= 2T(\lfloor n/3 \rfloor) + n && \\
&\le 2 \cdot 6\lfloor n/3 \rfloor + n && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\
&\le 12(n/3) + n && \text{since } \lfloor x \rfloor \le x \\
&= 4n + n && \\
&= 5n && \\
&\le 6n &&
\end{aligned}
$$

as required. ■