

CMPS 101

Homework Assignment 2

Solutions

1. p.52: 3.1-1

Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ -notation, prove that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$.

Proof:

Since $f(n)$ and $g(n)$ are asymptotically non-negative, there exists a positive constant n_0 such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$. For such n we have

$$\begin{aligned} 0 &\leq \max(f(n), g(n)) \\ &\leq \min(f(n), g(n)) + \max(f(n), g(n)) \\ &\leq 2 \cdot \max(f(n), g(n)). \end{aligned}$$

But $f(n) + g(n) = \min(f(n), g(n)) + \max(f(n), g(n))$, so for all $n \geq n_0$ we have

$$0 \leq 1 \cdot \max(f(n), g(n)) \leq f(n) + g(n) \leq 2 \cdot \max(f(n), g(n)).$$

Thus $f(n) + g(n) = \Theta(\max(f(n), g(n)))$, as required. ■

2. p.53: 3.1-3

Explain why the statement “The running time of algorithm A is at least $O(n^2)$ ” is meaningless.

Solution:

This statement is true under all circumstances, hence it conveys no useful information, and is therefore meaningless. To illustrate, let $T(n)$ be the running time of algorithm A. To say that $T(n)$ is “at least $O(n^2)$ ” is to say that $T(n)$ is bounded below by a function that is bounded above (asymptotically) by n^2 . If $T(n)$ is in the class $O(n^2)$, then $T(n)$ is bounded below by itself, which is bounded above asymptotically by n^2 , and hence the statement is true. If, on the other hand, $T(n)$ is in the class $\Omega(n^2)$, then $T(n)$ is bounded below by cn^2 (for sufficiently large n), which is bounded above asymptotically by n^2 , and again the statement is true. Even if $T(n)$ is not comparable to n^2 , $T(n)$ is bounded below by some positive constant, which is bounded above by n^2 . (This is true since even if algorithm A performs no operations, it takes some time to start and stop, so that $T(n)$ is greater than some, possibly small, positive number.) In all cases the statement is true, and therefore devoid of meaning. ■

3. p. 53: 3.1-4

Determine whether the following statements are true or false.

a. $2^{n+1} = O(2^n)$

Solution: True since $2^{n+1} = 2 \cdot 2^n = \text{const} \cdot 2^n = O(2^n)$. ■

b. $2^{2n} = O(2^n)$

Solution: False since $2^{2n} = (2^2)^n = 4^n = \omega(2^n)$ and since $\omega(2^n) \cap O(2^n) = \emptyset$. ■

4. p.61: 3-2abcdef

Indicate, for each pair of expressions (A, B) in the table below, whether A is O , o , Ω , ω , or Θ of B . Assume that $k \geq 1$, $\varepsilon > 0$, and $c > 1$ are constants. Place 'yes' or 'no' in each of the empty cells below, and justify your answers.

| | A | B | O | o | Ω | ω | Θ |
|----|-------------|-----------------|-----|-----|----------|----------|----------|
| a. | $\lg^k n$ | n^ε | yes | yes | no | no | no |
| b. | n^k | c^n | yes | yes | no | no | no |
| c. | \sqrt{n} | $n^{\sin n}$ | no | no | no | no | no |
| d. | 2^n | $2^{n/2}$ | no | no | yes | yes | no |
| e. | $n^{\lg c}$ | $c^{\lg n}$ | yes | no | yes | no | yes |
| f. | $\lg(n!)$ | $\lg(n^n)$ | yes | no | yes | no | yes |

Justifications:

- Applying l'Hopital's rule $\lceil k \rceil$ times, we get $\lim_{n \rightarrow \infty} \frac{(\lg n)^k}{n^\varepsilon} = 0$, whence $\lg^k n = o(n^\varepsilon)$.
- Again, by $\lceil k \rceil$ applications of l'Hopital's rule, $\lim_{n \rightarrow \infty} \frac{n^k}{c^n} = 0$, whence $n^k = o(c^n)$.
- Let $c > 0$, and observe that $n^{\sin n} > c\sqrt{n}$ for $n = 2\pi k + \frac{\pi}{2}$, where k is a sufficiently large positive integer. Therefore the inequality $n^{\sin n} \leq c\sqrt{n}$ is false for arbitrarily large n , and hence $n^{\sin(n)} \neq O(\sqrt{n})$. Also note that $n^{\sin n} < c\sqrt{n}$ for $n = 2\pi k + \frac{3\pi}{2}$, where k is sufficiently large, and therefore the inequality $n^{\sin n} \geq c\sqrt{n}$ is false for arbitrarily large n , whence $n^{\sin(n)} \neq \Omega(\sqrt{n})$.
- Observe $\lim_{n \rightarrow \infty} (2^n / 2^{n/2}) = \lim_{n \rightarrow \infty} (\sqrt{2})^n = \infty$, and therefore $2^n = \omega(2^{n/2})$.
- The identity $x^{\log_b y} = y^{\log_b x}$ implies that the two functions are equal: $n^{\lg c} = c^{\lg n}$.
- Note $\lg(n^n) = n \lg(n)$, and Stirling's formula implies $\lg(n!) = \Theta(n \lg(n))$.

5. p.62: 3-4deh

Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for sufficiently large n .) Prove or disprove the following statements.

- d. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$. **False**

Counter-Example:

Let $f(n) = 2n$ and $g(n) = n$. Then $2^{g(n)} = 2^n$ and $2^{f(n)} = 2^{2n} = 4^n = \omega(2^n)$, so $2^{f(n)} = \omega(2^{g(n)})$, and therefore $2^{f(n)} \neq O(2^{g(n)})$. ■

- e. $f(n) = O((f(n))^2)$. **False**

Counter-Example:

Let $f(n) = 1/n$. Then $f(n) = \omega((f(n))^2)$ since $\lim_{n \rightarrow \infty} \frac{f(n)}{f(n)^2} = \lim_{n \rightarrow \infty} \frac{1}{f(n)} = \lim_{n \rightarrow \infty} n = \infty$, and therefore $f(n) \neq O((f(n))^2)$. ■

h. $f(n) + o(f(n)) = \Theta(f(n))$. **True**

Proof:

In the above formula, $o(f(n))$ stands for some anonymous function $h(n)$ in the class $o(f(n))$,

whence $\lim_{n \rightarrow \infty} \frac{h(n)}{f(n)} = 0$. Thus $\lim_{n \rightarrow \infty} \frac{f(n) + h(n)}{f(n)} = \lim_{n \rightarrow \infty} \left(1 + \frac{h(n)}{f(n)} \right) = 1$, and $f(n) + h(n) = \Theta(f(n))$, as claimed. ■

6. Let $f(n) = \Theta(n)$. Prove that $\sum_{i=1}^n f(i) = \Theta(n^2)$. (See the hint at bottom of p.4 of the handout on asymptotic growth rates.)

Proof:

Since $f(n) = \Theta(n)$ there exist positive constants c_1 , c_2 , and n_0 such that for all $n \geq n_0$ the inequality $0 \leq c_1 n \leq f(n) \leq c_2 n$ holds. In particular, given $n_0 \leq i \leq n$ we have $c_1 i \leq f(i) \leq c_2 i$. Upon summing these inequalities from $i = n_0$ to $i = n$ we obtain

$$\sum_{i=n_0}^n c_1 i \leq \sum_{i=n_0}^n f(i) \leq \sum_{i=n_0}^n c_2 i$$

for any $n \geq n_0$. Define $A = \sum_{i=1}^{n_0-1} f(i)$. Then the above inequality can be rewritten as

$$\sum_{i=1}^n c_1 i - \sum_{i=1}^{n_0-1} c_1 i \leq \sum_{i=1}^n f(i) - A \leq \sum_{i=1}^n c_2 i - \sum_{i=1}^{n_0-1} c_2 i$$

Adding the constant A to all terms, and using the well known formula $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, we obtain

$$c_1 \left(\frac{n(n+1)}{2} \right) - c_1 \left(\frac{n_0(n_0-1)}{2} \right) + A \leq \sum_{i=1}^n f(i) \leq c_2 \left(\frac{n(n+1)}{2} \right) - c_2 \left(\frac{n_0(n_0-1)}{2} \right) + A$$

Now observe that since c_1 , c_2 , n_0 , and A are constants, the left hand side is in the class $\Omega(n^2)$ and the right hand side is in $O(n^2)$, and this inequality holds for all $n \geq n_0$. In other words, the inequality

$h_1(n) \leq \sum_{i=1}^n f(i) \leq h_2(n)$ holds for all sufficiently large n , where $h_1(n) = \Omega(n^2)$ and $h_2(n) = O(n^2)$. By

an exercise in the handout on asymptotic growth of functions, we conclude that $\sum_{i=1}^n f(i) = \Theta(n^2)$, as required. ■

7. Let $g(n)$ be an asymptotically non-negative function. Prove that $o(g(n)) \cap \Omega(g(n)) = \emptyset$.

Proof:

Assume to get a contradiction that $f(n) \in o(g(n)) \cap \Omega(g(n))$. Then since $f(n) = \Omega(g(n))$ we have

$$(1) \quad \exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1: \quad 0 \leq c_1 g(n) \leq f(n)$$

Also, since $f(n) = o(g(n))$ we have

$$(2) \quad \forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2: \quad 0 \leq f(n) < c_2 g(n)$$

Let $c_2 = c_1$. Then $c_2 > 0$, and by (2) there exists $n_2 > 0$ such that $0 \leq f(n) < c_1 g(n)$ for all $n \geq n_2$. Pick any $m \geq \max(n_1, n_2)$. Then by (1) and (2) we have $0 \leq c_1 g(m) \leq f(m) < c_1 g(m)$, and hence $c_1 g(m) < c_1 g(m)$, a contradiction. Our assumption was therefore false, and no such function $f(n)$ can exist. We conclude that $o(g(n)) \cap \Omega(g(n)) = \emptyset$. ■