CS101 HW3

1) Prove $3^{2^n} = o(2^{3^n})$

$$\Rightarrow \lim_{n\to\infty} \frac{3^{2^n}}{2^{3^n}} \Rightarrow \lim_{n\to\infty} \frac{2^n \ln 3}{3^n \ln 2} \Rightarrow \lim_{n\to\infty} \frac{c_1 2^n}{c_2 3^n} = 0$$

$$\therefore 3^{2^n} = o(2^{3^n})$$

2) prove $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$ using stirling's formula

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \Theta\left(\frac{4^n}{\sqrt{n}}\right) \quad ; n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{a_n}$$

assume positive constants $c_1, c_2,$ and n_0 such that

$$c_1\left(\frac{4^n}{\sqrt{n}}\right) \le \frac{(2n)!}{n!(2n-n)!} \le c_2\left(\frac{4^n}{\sqrt{n}}\right) \quad \text{for all } n \ge n_0$$

then divide by $\left(\frac{4^n}{\sqrt{n}}\right)$ $\qquad \sqrt{2\pi}\,(n)\,($

$$c_1 \le \frac{\sqrt{n}\,(2n)!}{4^n n!(2n-n)!} \le c_2 \Rightarrow c_1 \le \frac{\sqrt{n}(2)}{4^n (2n-n)!} \le c_2$$

~~this holds true for n>0~~

we can hold this true by making $n=1$ & $c_2 = \frac{1}{2}$ & $c_1 = \frac{1}{4}$

$$\therefore \binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$$

3 Prove for all $n \geq 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

11a) I) Base Step prove $p(n_0)$ true

II) prove $\forall n \geq n_0 : P(n) \rightarrow P(n+1)$

<u>Base case</u> $P(1) \Rightarrow 1 = \left(\frac{1(1+1)}{2}\right)^2 = 1$

i.e. $1=1$ which true

$\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$ be chosen arbitrarily.

Assume $P(n)$ is true $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$(n^2+3n+2)(n^2+3n+2)$

$\cancel{n^4} + \cancel{3n^3} + 2n^2 + \cancel{3n^3} + 9n^2 + 6n$

$+ 2n^2 + 6n + 4$

$n^4 + 6n^2 + 13n^2 + 12n + 4$

prove $P(n+1) \Rightarrow \sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$

$$= \left(\frac{n^2+3n+2}{2}\right)^2 = \frac{n^4 + 6n^2 + 13n^2 + 12n + 4}{4}$$

So

$$\sum_{i=1}^{n+1} i^3 = \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \leftarrow \text{by induction hyp.}$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \left(\frac{n^2+n}{2}\right)^2 + n^3 + 3n^2 + 3n + 1$$

$$= \frac{n^4 + 2n^3 + n^2}{4} + \frac{4n^3 + 12n^2 + 12n + 4}{4}$$

$$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

By the P.M.I. $\forall n \geq 1$ $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$P(n) = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IIb) prove $\forall\, n>1 \quad P(n-1) \to P(n)$

Let $n>1$ Assume

$$P(n-1) \Rightarrow \sum_{i=1}^{n-1} i^3 = \left(\frac{n-1\,(n)}{2}\right)^2 \quad \text{is true}$$

$$\Downarrow$$

$$\left(\frac{n^2-n}{2}\right)^2 = \frac{n^4-2n^3+n^2}{4}$$

proving $P(n-1) \to P(n)$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n-1} i^3\right) + n^3$$

$$= \left(\frac{n^4-2n^3+n^2}{4}\right) + \frac{4n^3}{4}$$

$$\downarrow$$

math was done for ile

$$\left(\frac{n(n+1)^2}{2}\right) = \frac{n^4+2n^3+n^2}{4} \quad \checkmark$$

4 Define $S(n)$ for $n \in \mathbb{Z}^+$ by $S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil \frac{n}{2} \rceil) + 1 & \text{if } n \geq 2 \end{cases}$

Prove that $\underbrace{\boxed{S(n) \geq lg(n)}}_{= P(n)}$ for all $n \geq 1$ and hence $S(n) = \Omega(lg\, n)$

I $P(1)$ says $T(1) = \leq lg(1)$
i.e. $0 \leq 0$ ✓

II d $\forall n > 1$: $P(1) \wedge \ldots \wedge P(n-1) \to P(n)$

Let $n > 1$ assume for all K
in the range $1 \leq K < n$ that $S(K) \geq lg(K)$ is true

$$\begin{aligned} S(n) &= S(\lceil \tfrac{n}{2} \rceil) + 1 \quad \text{def of } S(n) \\ &\geq lg(\lceil \tfrac{n}{2} \rceil) + 1 \\ &\geq lg(\tfrac{n}{2}) + 1 \\ &= lg(n) - lg(2) + 1 \\ &= lg(n) \end{aligned}$$

$\therefore$ $S(n) \geq lg(n)$ for all $n \geq 1$
& hence $S(n) = \Omega(lg\, n)$

5. $T(n) = \begin{cases} 1 & n=1 \\ T(\lfloor \frac{n}{2} \rfloor) + n^2 & n \geq 2 \end{cases}$

Prove that for all $n \geq 1$ $T(n) \leq \frac{4}{3}n^2$

Base step: $T(1) \leq \frac{4}{3}(1)^2$

$1 \leq \frac{4}{3}$

Induction Step: Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $T(k) \leq \frac{4}{3}k^2$ is true

$$
\begin{aligned}
T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) + n^2 && \text{def of } T(n) \\
&\leq \tfrac{4}{3}(\lfloor \tfrac{n}{2} \rfloor)^2 + n^2 && \text{induction hyp.} \\
&\leq \tfrac{4}{3}(\tfrac{n}{2})^2 + n^2 \\
&= \tfrac{4}{3}(\tfrac{n^2}{4}) + n^2 \\
&= \tfrac{n^2}{3} + n^2 = \tfrac{4}{3}n^2
\end{aligned}
$$

$\therefore T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$

6. $T(n) = \begin{cases} 2 & n=1,2 \\ 9T(\lfloor \frac{n}{3} \rfloor) + 1 & n \geq 3 \end{cases}$

- show $\forall n \geq 1$: $T(n) \leq 3n^2 - 1$

Base Step: $T(1) \leq 3(1)^2 - 1$ $\quad$ $T(2) \leq 3(2)^2 - 1$

$2 \leq 3 - 1$ $\quad$ $2 \leq 12 - 1$

$2 \leq 2$ $\quad$ $2 \leq 11$

induction: Let $n > 2$ and assume for all k in the range $1 \leq k < n$ that $T(k) \leq 3k^2 - 1$

$$
\begin{aligned}
T(n) &= 9T(\lfloor \tfrac{n}{3} \rfloor) + 1 && \text{def. of } T(n) \\
T(n) &\leq 9(3(\lfloor \tfrac{n}{3} \rfloor)^2 - 1) + 1 && \text{induction hyp.} \\
&\leq 9(3(\tfrac{n}{3})^2 - 1) + 1 \\
&= 9(\tfrac{3n^2}{9} - 1) + 1 \\
&= 3n^2 - 8 \\
&\leq 3n^2 - 1
\end{aligned}
$$

$\therefore T(n) \leq 3n^2 - 1$ for all $n \geq 1$

7. $T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 1 \end{cases}$

Show $\forall n \ge 1 : T(n) \le 6n$

Base step $T(1) \le 6(1)$ $\quad T(2) \le 6(2$

$6 \le 6 \quad 6 \le 12$

induction step

Let $n > 2$ and assume for all k in the range $1 \le k < n$ that $T(k) \le 6k$

$$T(n) = 2T(\lfloor n/3 \rfloor) + n \quad \text{def } T(n)$$
$$\le 2(6(\lfloor n/3 \rfloor) + n \quad \text{by induction hyp}$$
$$\le 2(6(n/3)) + n$$
$$\le 2(2n) + n$$
$$= 4n + n$$
$$= 5n$$
$$\le 6n$$

$\therefore T(n) \le 6n$ for all $n \ge 1$