CMPS 101
Algorithms and Abstract Data Types
Fall 2017
Midterm Exam 1 Solutions

1. (20 Points) Determine whether the following statements are true or false. Prove or disprove each statement accordingly.

   a. (10 Points) If $h_1(n) = \Theta(f(n))$ and $h_2(n) = \Theta(f(n))$, then $h_1(n) + h_2(n) = \Theta(f(n))$. **True**

   Proof:
   By hypothesis there exist positive constants n_1, n_2, a_1, b_1, a_2 and b_2 such that

   $$\forall n \geq n_1: \ \ 0 \leq a_1 f(n) \leq h_1(n) \leq b_1 f(n)$$

   and

   $$\forall n \geq n_2: \ \ 0 \leq a_2 f(n) \leq h_2(n) \leq b_2 f(n).$$

   Let $c = a_1 + a_2, d = b_1 + b_2$ and $n_0 = \max(n_1, n_2)$. Then c, d and n_0 are positive constants, and for any $n \geq n_0$ both of the above inequalities are true. Adding the inequalities term by term gives

   $$0 \leq (a_1 + a_2) f(n) \leq h_1(n) + h_2(n) \leq (b_1 + b_2) f(n).$$

   Therefore

   $$\forall n \geq n_0: \ \ 0 \leq c f(n) \leq h_1(n) + h_2(n) \leq d f(n),$$

   and hence $h_1(n) + h_2(n) = \Theta(f(n))$ as required. ■

   b. (10 Points) $2^{\ln(n)} = o(n)$. **True.**

   Proof:
   By an identity proved in class $\left(a^{\log_b(x)} = x^{\log_b(a)}\right)$ we have $2^{\ln(n)} = n^{\ln(2)}$. Since $2 < e$, we also have $\ln(2) < 1$, and hence $n^{\ln(2)} = o(n^1)$. Therefore $2^{\ln(n)} = o(n)$ as claimed. ■

   Alternate Proof:

   $$\frac{2^{\ln(n)}}{n} = \frac{n^{\ln(2)}}{n} = n^{\ln(2)-1} \to 0 \ \text{ as } \ n \to \infty,$$

   again using $\ln(2) < 1$. Therefore $2^{\ln(n)} = o(n)$. ■

2. (20 Points) Use Stirling's formula to prove that $\log(n!) = \Theta(n\log(n))$.

Proof:
Taking log (any base) of both sides of Stirling's formula, and using the laws of logarithms, we get

$$\begin{aligned}
\log(n!) &= \log\left(\sqrt{2\pi n}\cdot(n/e)^n\cdot\big(1+\Theta(1/n)\big)\right)\\
&= \log\sqrt{2\pi n} + \log(n/e)^n + \log\big(1+\Theta(1/n)\big)\\
&= \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) + \log\big(1+\Theta(1/n)\big).
\end{aligned}$$

Therefore

$$\frac{\log(n!)}{n\log(n)} = \frac{\log(2\pi)}{2n\log(n)} + \frac{1}{2n} + 1 - \frac{\log(e)}{\log(n)} + \frac{\log\big(1+\Theta(1/n)\big)}{n\log(n)},$$

and hence

$$\lim_{n\to\infty}\left(\frac{\log(n!)}{n\log(n)}\right) = 1.$$

Thus $\log(n!) = \Theta(n\log(n))$, as claimed. ■

3. (20 Points) Consider the following algorithm that wastes time.

```
WasteTime(n)   (pre: n ≥ 1)
if n = 1
    waste 2 units of time
else
    WasteTime(⌈n/2⌉)
    WasteTime(⌊n/2⌋)
    waste 5 units of time
```

a. (10 Points) Write a recurrence relation for the number of units of time $T(n)$ wasted by this algorithm.

Solution:

$$T(n) = \begin{cases} 2 & n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5 & n \geq 2 \end{cases}$$

b. (10 Points) Show that $T(n) = 7n - 5$ is the solution to this recurrence. (Hint: you may use without proof the fact that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.)

Proof:
First observe that if $T(n) = 7n - 5$, then $T(1) = 7 - 5 = 2$. Second, if $n \geq 2$ we have

$$\begin{aligned} \text{RHS} &= T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 5 \\ &= (7\lceil n/2 \rceil - 5) + (7\lfloor n/2 \rfloor - 5) + 5 \\ &= 7(\lceil n/2 \rceil + \lfloor n/2 \rfloor) - 5 \\ &= 7n - 5 = T(n) = \text{LHS}, \end{aligned}$$

showing that $T(n) = 7n - 5$ solves the recurrence. ■

4. (20 Points) Prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$. (Hint: use weak induction.)

Proof:

Let $P(n)$ be the formula $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Base step

$P(1)$ says that $\sum_{i=1}^{1} i^3 = \left(\frac{1(1+1)}{2}\right)^2$, i.e. that $1^3 = 1^2$, i.e. $1 = 1$, which is true.

Induction Step (IIa)

Let $n \geq 1$ be chosen arbitrarily. Assume for this n that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$, i.e. $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$. Now observe that

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \qquad \text{(by the induction hypothesis)} \\
&= (n+1)^2\left(\frac{n^2}{2^2} + (n+1)\right) \\
&= (n+1)^2\left(\frac{n^2 + 4(n+1)}{4}\right) \\
&= (n+1)^2\left(\frac{n^2 + 4n + 4}{4}\right) \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}
$$

as required. It follows from the first principle of mathematical induction that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geq 1$. ■

5. (20 Points) Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

a. (4 Points) Determine the values $T(2), T(3), T(4)$ and $T(5)$.

Solution:

$T(2) = T(1) + 2^2 = 1 + 4 = 5$

$T(3) = T(1) + 3^3 = 1 + 9 = 10$

$T(4) = T(2) + 4^2 = 5 + 16 = 21$

$T(5) = T(2) + 5^2 = 5 + 25 = 30$

b. (16 Points) Prove that $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. (Hint: use strong induction.)

Proof:

Base Step

Observe that $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$, which establishes the base case.

Induction Step (IId)

Let $n > 1$ be chosen arbitrarily. Assume for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. We must show as a consequence that $T(n) \leq (4/3)n^2$. Observe

$$\begin{aligned} T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\ &\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\ &= n^2/3 + n^2 \\ &= (4/3)n^2, \end{aligned}$$

as required. It follows from the second principle of mathematical induction that $T(n) \leq \dfrac{4}{3}n^2$ for all $n \geq 1$. ■