CS101 HW4

1) Let $T$ be a tree w/ $n$ vertices & $m$ edges.
prove $m=n-1$ ; induction on $m$

I Base Case: if $m=0$ then $T$ can only have 1 vertex since $T$ is connected. $\therefore n=1$, and $m=n-1$.

II

Let $m>0$ and assume any $T$, tree, w fewer than $m$ edges satisfies $E(T)=V(T)-1$.

Now remove any edge. Creating two new trees $T_1, T_2$ each w/ less than $m$ edges.

Suppose $T_i$ has $m_i$ edges & $n_i$ vertices $(i=1,2)$;
Induction hypothesis gives us $m_i=n_i-1$ $(i=1,2)$
Also note $n=n_1+n_2$ since no vertices were removed
$\therefore m=m_1+m_2+1=(n_1-1)+(n_2-1)+1 = n_1+n_2-1=n-1$
as required.

2 Let $G$ be an acyclic graph w/ $n$ vertices, $m$ edges, & $K$ connect components. Prove $m=n-k$

Let the connect components of $G$ be $T_1, T_2, \ldots, T_k$
suppose $T_i$ has $m_i$ edges and $n_i$ vertices $(1\le i\le k)$
using p1. we have $m_i=n_i-1$ $(1\le i\le k)$
$$\therefore m=\sum_{i=1}^{k} m_i=\sum_{i=1}^{k}(n_i-1)=\sum_{i=1}^{k} n_i-\sum_{i=1}^{k}1=n-k$$

3 find exact solution using iteration

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + 5 & n \ge 3 \end{cases}$$

$T(\because) = 2T(\lfloor \frac{\because}{3} \rfloor) + 5$

so

$T(n) = 5 + 2T(\lfloor \frac{n}{3} \rfloor)$

$= \cancel{5 + 5 \cdot 2T(\lfloor \frac{n/3}{3} \rfloor)}$

$= 5 + 2(5 + 2T\lfloor \frac{n/3}{3} \rfloor)$

$= \cancel{15 + 4T(\lfloor \frac{n}{3} \rfloor)}$

$= 5 + 2(5 + 2(5 + 2T\lfloor \frac{n}{3^3} \rfloor)$

$2^k(T(\lfloor \frac{n}{3^k} \rfloor) + 5) + 5$

recurrance chalks when $\lfloor \frac{n}{3^k} \rfloor < 3$

$1 \le \frac{n}{3^k} < 3$

$3^k \le n \le 3^{k+1}$

$k \le \log_3 n < k+1$

$\therefore T(n) = \cancel{\lfloor \log_3 n \rfloor + h}$

$= 2^{\log_3 n}(1 + 5) + 5$

$= 6 \cdot 2^{\lfloor \log_3 n \rfloor} + 5$

4

$$T(n) = \begin{cases} 3 & 1 \le n < 5 \\ 4T(\lfloor n/5 \rfloor) + n & n \ge 5 \end{cases}$$

to show that

$$T(n) = \sum_{i=0}^{\lfloor \log_5 n \rfloor - 1} 4^i \left\lfloor \frac{n}{5^i} \right\rfloor + 3 \cdot 4^{\lfloor \log_5 (n) \rfloor}$$

$$T(\ddot{u}) = \ddot{u} + 4T\left\lfloor \frac{\ddot{u}}{5} \right\rfloor$$

$$T(n) = n + 4T\left\lfloor \frac{n}{5} \right\rfloor$$

$$= \ddot{u} + 4\left(\ddot{u} + 4T\left\lfloor \frac{n/5}{5} \right\rfloor\right)$$

$$= n + 4\left(n + 4\left(n + 4T\left\lfloor \frac{n}{5^k} \right\rfloor\right)\right)$$

$$= 4^k\left(T\left\lfloor \frac{n}{5^k} \right\rfloor + n\right) + n$$

recursion halts when $\left\lfloor \frac{n}{5^k} \right\rfloor < 5$

$$1 \le \frac{n}{5^k} < 5$$

$$5^k \le n < 5^{k+1}$$

$$k \le \log_5 n < k+1$$

$$\therefore T(n) = 4^{\log_5 n}(3+n) + n$$

$$= n \cdot 4^{\lfloor \log_5 n \rfloor} + n + 3 \cdot 4^{\lfloor \log_5 n \rfloor}$$

$$= \sum_{i=0}^{\lfloor \log_5 n \rfloor - 1} 4^i \left\lfloor \frac{n}{5^i} \right\rfloor + 3 \cdot 4^{\lfloor \log_5 n \rfloor}$$

$$\lim_{n \to \infty} \frac{\lfloor \log_5 n \rfloor \, 4^{\lfloor \log_5 n \rfloor} \left\lfloor \frac{n}{\log_5 n} \right\rfloor + 3 \cdot 4^{\lfloor \log_5 n \rfloor}}{n}$$

$$=$$