CMPS 101
Homework Assignment 4
Solutions

1. Let T be a tree with n vertices and m edges. Prove that $m = n - 1$ by induction on m.

   Proof:
   This result was proved in the handout on Induction Proofs by induction on n. We prove it here by induction on m.

   I. If $m = 0$, then T can have only one vertex, since T is connected. Thus $n = 1$, establishing the base case.

   II. Let $m > 0$ and assume that any tree T' with fewer than m edges satisfies $|E(T')| = |V(T')| - 1$. We must show that if a tree T has m edges, then $|E(T)| = |V(T)| - 1$. Pick an edge $e \in E(T)$ and remove it. The resulting graph consists of two trees T_1 and T_2, each having fewer than m edges. Suppose T_i has m_i edges and n_i vertices (for $i = 1, 2$). Then the induction hypothesis implies $m_i = n_i - 1$ (for $i = 1, 2$). Also $n_1 + n_2 = n$, since no vertices were removed. Therefore

   $$m = m_1 + m_2 + 1 = (n_1 - 1) + (n_2 - 1) + 1 = (n_1 + n_2) - 1 = n - 1$$

   as required. ■

2. Let G be an acyclic graph with n vertices, m edges and k connected components. Use the result of the preceding problem to prove that $m = n - k$. (Hint: apply the preceding result to each of the k trees composing G.)

   Proof:
   Let the connected components of G (which are necessarily trees) be $T_1, T_2, \ldots, T_k$. Suppose T_i has m_i edges and n_i vertices (for $1 \le i \le k$). By the result of the preceding problem we have $m_i = n_i - 1$ $(1 \le i < n)$. Therefore

   $$m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

   as claimed. ■

3. Use the iteration method to find an exact solution to the recurrence:

   $$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + 5 & n \ge 3 \end{cases}$$

   Solution:
   Recurring down to the k^{th} level, we have

   $$T(n) = 5 + 2T(\lfloor n/3 \rfloor)$$

$$= 5 + 2\left(5 + 2T\left(\left\lfloor\frac{\lfloor n/3\rfloor}{3}\right\rfloor\right)\right) = 5 + 2\cdot 5 + 2^2T(\lfloor n/3^2\rfloor)$$
$$= 5 + 2\cdot 5 + 2^2\left(5 + 2T\left(\left\lfloor\frac{\lfloor n/3^2\rfloor}{3}\right\rfloor\right)\right) = 5 + 2\cdot 5 + 2^2\cdot 5 + 2^3T(\lfloor n/3^3\rfloor)$$
$$\vdots$$
$$= \sum_{i=0}^{k-1} 5\cdot 2^i + 2^kT(\lfloor n/3^k\rfloor)$$

The recursion terminates when the recursion depth k satisfies $1 \le \lfloor n/3^k\rfloor < 3$, which is equivalent to $k = \lfloor\log_3(n)\rfloor$. For this value of k we have $T(\lfloor n/3^k\rfloor) = 1$, and therefore

$$T(n) = \sum_{i=0}^{k-1} 5\cdot 2^i + 2^k = 5\left(\frac{2^k - 1}{2 - 1}\right) + 2^k = 6\cdot 2^k - 5$$

so the exact solution is $T(n) = 6\cdot 2^{\lfloor\log_3(n)\rfloor} - 5$ ■

4. Use the iteration method on the following recurrence

$$T(n) = \begin{cases} 3 & 1 \le n < 5 \\ 4T(\lfloor n/5\rfloor) + n & n \ge 5 \end{cases}$$

to show that

$$T(n) = \sum_{i=0}^{\lfloor\log_5(n)\rfloor - 1} 4^i\left\lfloor\frac{n}{5^i}\right\rfloor \quad + \quad 3\cdot 4^{\lfloor\log_5(n)\rfloor}$$

and hence $T(n) = \Theta(n)$.

Solution:
Recuring down to level k gives the expression:

$$T(n) = \sum_{i=0}^{k-1} 4^i\cdot\left\lfloor\frac{n}{5^i}\right\rfloor \;+\; 4^k\cdot T\left(\left\lfloor\frac{n}{5^k}\right\rfloor\right)$$

The recursion must terminate when $1 \le \lfloor n/5^k\rfloor < 5$, which is equivalent to $k = \lfloor\log_5 n\rfloor$. For this value of k we have $T(\lfloor n/5^k\rfloor) = 3$. This gives the expression

$$T(n) = \sum_{i=0}^{\lfloor\log_5(n)\rfloor - 1} 4^i\left\lfloor\frac{n}{5^i}\right\rfloor \quad + \quad 3\cdot 4^{\lfloor\log_5(n)\rfloor}$$

for $T(n)$. Estimating this summation upward we have

$$T(n) \le \sum_{i=0}^{\infty} 4^i\left(\frac{n}{5^i}\right) \;+\; 3\cdot 4^{\log_5 n}$$

$$
\begin{aligned}
&= n \cdot \sum_{i=0}^{\infty} \left(\frac{4}{5}\right)^{i} \; + \; 3 \cdot n^{\log_5 4} \\
&= n \cdot \left(\frac{1}{1 - \left(\frac{4}{5}\right)} \right) + 3n^{\log_5 4} \\
&= 5n + 3n^{\log_5 4} \\
&= O(n)
\end{aligned}
$$

The last step follows since $\log_5 4 < 1$, making the second term lower order. To find a lower bound, we can turn to the original recurrence: $T(n) = 4T(\lfloor n/5 \rfloor) + n \geq n = \Omega(n)$. It follows that $T(n) = \Theta(n)$.

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