HW2 CMPS 101

1. Prove $f(n)+g(n)=\Theta(\max(f(n),g(n)))$.

assuming ~~knowing~~ $f(n)$ and $g(n)$ are asymptotically non-negative functions.

assume $\max(f(n), g(n)) = \Theta(f(n)+g(n))$

↳ this is only true if and only if there exists $c_1, c_2, n_0 > 0$

so ⇒

$$0 \le c_1(f(n)+g(n)) \le \max(f(n), g(n)) \le c_2(f(n)+g(n)).$$

for all $n \ge n_0$

we also get $f(n)+g(n) \ge f(n) \ge 0$ & $f(n)+g(n) \ge g(n) \ge 0$

combining the two inequalities it becomes $f(n)+g(n) \ge \max(f(n), g(n))$

thus $\max(f(n),g(n)) \le c(f(n)+g(n))$ w/ $n \ge n_0$ & $c=1$

it is also true that $\max(f(n),g(n)) \ge f(n)$ $\max(f(n),g(n)) \ge g(n)$ $n \ge n_0$

$$2\max(f(n),g(n)) \ge (g(n)+f(n))$$

$$= \max(f(n),g(n)) \ge \tfrac{1}{2}(g(n)+f(n)) \quad \text{for all } n \ge n_0$$

$\therefore \max(f(n), g(n)) = \Omega(g(n)+f(n))$ w/ constant $c = 1/2$

2. Explain why "The running time of algorithm A is at least $O(n^2)$" is meaningless.

To say that statement is the running time, $T(n)$, could be ~~[illegible]~~ = or asymptotically > than $O(n^2)$

ex) ~~A plan~~ A rocketship goes at least 2 meters off the ground.

3 determine true or false.

$2^{n+1} = O(2^n)$

$\hookrightarrow 2^{n+1} = 2(2^n) \Rightarrow 2(2^n) \leq c \cdot 2^n$

$\Rightarrow$ True if $c \geq 2$

$2^{2n} = O(2^n)$

$\hookrightarrow$ is $2^{2n} \leq c \cdot 2^n$

$\Rightarrow$ false due to ratio

$\hookrightarrow \dfrac{2^{2n}}{c \cdot 2^n} = \dfrac{(2^n)^{\cancel{2}}}{c \cdot \cancel{2^n}} = \dfrac{2^n}{c}$

$c \cdot 2^n$ can never be an upper bound

4	A	B	O	o	Ω	ω	Θ
a	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b	n^k	c^n	yes	yes	no	no	no
c	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d	2^n	$2^{n/2}$	no	no	yes	yes	no
e	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

a) using L'Hospital's rule $\lim_{n \to \infty} \dfrac{(\lg n)^k}{n^\epsilon} = \lim_{n \to \infty} \dfrac{k(\lg n)^{k-1}}{\epsilon n^{\epsilon - 1}}$

$= \lim_{n \to \infty} \dfrac{k(k-1)(\lg n)^{k-2}}{\epsilon^2 n^{\epsilon-1}} \dfrac{1}{n}$

must do k times repeat here

$\hookrightarrow = \lim_{n \to \infty} \dfrac{k(k-1)(k-2)\ldots 1}{\epsilon^2 n^\epsilon} = 0$

b) L'Hospital $\Rightarrow \lim_{n \to \infty} \dfrac{n^k}{c^n} = 0$; applying the rule repeatedly to see $n^k = o(c^n)$

c) I can see B, $n^{\sin n}$, contains $\sin n$ which goes ~~back~~ between 1 & -1. Meaning ~~that~~ at it's maximum value $n^{\sin n} > c\sqrt{n}$ and so $n^{\sin n} \neq O(\sqrt{n})$. Also when at its minimum $n^{\sin n} < c\sqrt{n}$ thus $n^{\sin n} \neq \Omega(\sqrt{n})$.

d. $\lim_{n \to \infty} \frac{2^n}{2^{\frac{n}{2}}} = \infty$ meaning $2^n = \omega(2^{\frac{n}{2}})$

e $n^{\lg c} = c^{\lg n}$ equal

f. $\lg(n^n) = n(\lg n)$ & using stirling's formula $\lg(n!) = \Theta(n \lg(n))$

5. a. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$
false. if $f(n) = 2n$; $g(n) = n$ $\Rightarrow f(n) = O(g(n))$ but $2^{2n} = 4^n \neq O(2^n)$

e. $f(n) = O((f(n))^2)$

false to make this true, there must be positive constants c and n_0 such that for all $n \geq n_0$
$$0 \leq f(n) \leq c\,(f(n))^2$$

unless $f(n) = \frac{1}{n}$ then it doesn't hold. since the no positive constant c and n_0 such that all $n \geq n_0$
$$0 \leq \frac{1}{n} \leq c\frac{1}{n^2}.$$

h. $f(n) + o(f(n)) = \Theta(f(n))$
in definition $0 \leq o(f(n)) \leq f(n)$, so $f(n) \leq f(n) + o(f(n)) \leq 2f(n)$

$\therefore$ $f(n) + o(f(n)) = \Theta(f(n))$

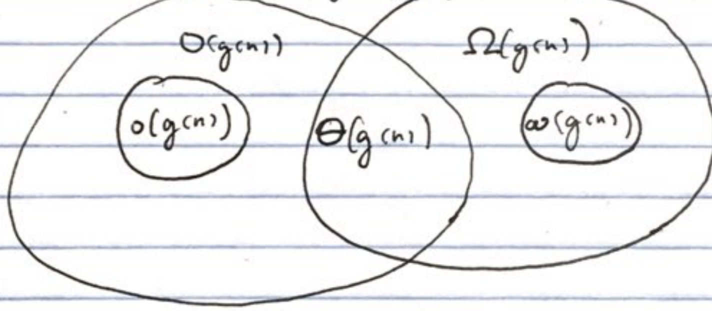
6. $f(n) = \Theta(n)$, Prove $\sum_{i=1}^{n} f(i) = \Theta(n^2)$

~~If~~ $f(n) = \Theta(n)$ iff $\lim_{n \to \infty} \frac{f(n)}{n} = c$

$f(n) = o(g(n))$ if and only if $\forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{n} < c$

7. Let $g(n)$ be a asymptotically non-negative function.

Prove $o(g(n)) \cap \Omega(g(n)) = \emptyset$



no element in $o(g(n))$ cannot be of $\Omega(g(n))$

Let $f(n) = o(g(n))$ then for any $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$. Meaning there are no positive constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$

$\therefore o(g(n)) \cap \Omega(g(n)) = \emptyset$