Portfolio Optimization Based on CAPM and Quadratic Programming

1. Introduction

This report applies the **Capital Asset Pricing Model (CAPM)** to estimate the expected returns and risks of assets within a portfolio. Additionally, **Quadratic Programming (QP)** is used to optimize portfolio allocation. We will construct the **Efficient Frontier** to analyze the trade-off between risk and return.

In this study, we set the **risk-free rate (rf) at 2% (0.02)** to align with the market conditions of 2024. We also compare the case of **rf = 5% (0.05)** to analyze the impact of a higher risk-free rate on portfolio optimization.

2. Data Collection

We retrieved data from **Yahoo Finance** for the past **five years (2019-2024)**, covering **10 selected stocks** along with the **S&P 500 (^GSPC) as the market index**. The daily return for each asset is calculated as follows:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

where P_t represents the adjusted closing price at time t.

3. CAPM Estimation

We use a **linear regression model** to estimate the CAPM equation:

$$R_i - R_f = \alpha_i + \beta_i(R_m - R_f) + \epsilon_i$$

where:

- R_i = the rate of return of asset i
- $R_f = 0.02$ (Risk-free rate, adjusted to reflect actual market conditions)
- R_m = Market return
- α_i, β_i = Regression coefficients
- ϵ_i = Residual term

After obtaining the value through regression calculation, we calculate the expected returns of each asset:

$$\mu_i = R_f + \beta_i(E[R_m] - R_f)$$

where $E[R_m]$ represents the **average market return (annualized)**.

Furthermore, the **idiosyncratic risk variance** is calculated as:

$$\text{Idiosyncratic Risk Variance} = Var(\epsilon_i)$$

4. Portfolio Optimization

To construct the **covariance matrix**, we incorporate both **systematic risk** (market-driven) and **idiosyncratic risk**:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \quad \text{(for off-diagonal elements)}$$

$$\sigma_{ii} = \beta_i^2 \sigma_m^2 + Var(\epsilon_i)$$

The optimization problem is defined using **Quadratic Programming (QP)**:

$$\min \frac{1}{2} w^T \Sigma w$$

Appointment confirmation

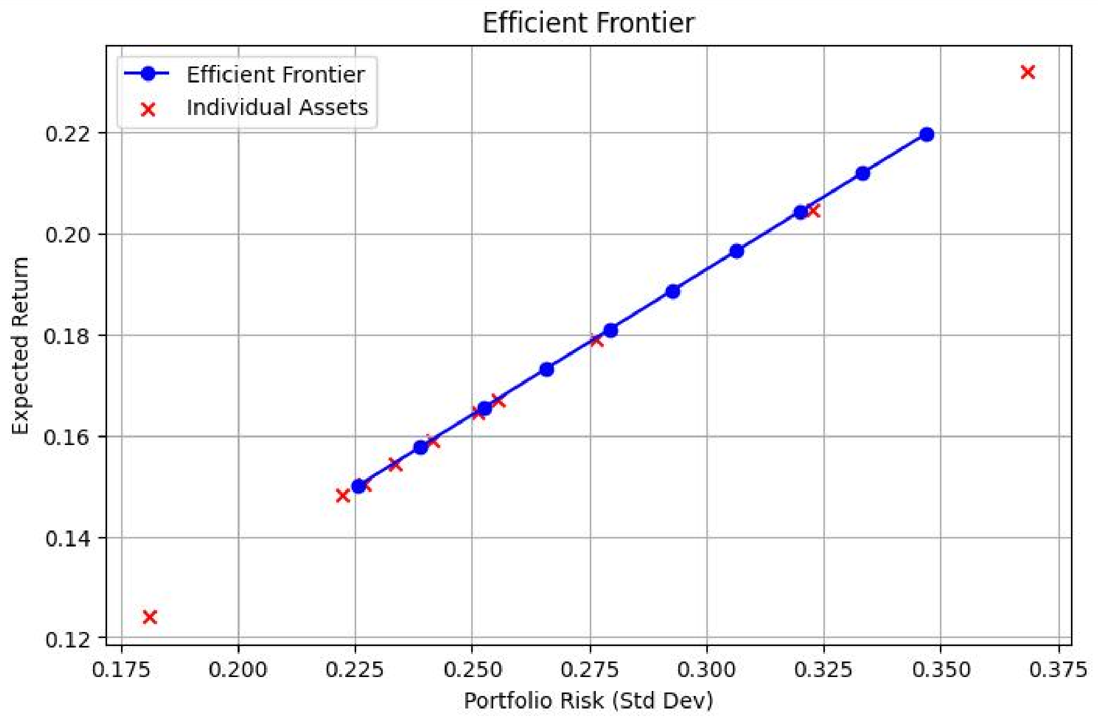
$$w^T \mu = \mu_p, \quad w^T 1 = 1, \quad w \geq 0$$

where w represents the weight of the investment portfolio.

5. Efficient Frontier Analysis

We solve for the **optimal portfolio** at different **target returns** (μ_p) and plot the **Efficient Frontier** (risk-return trade-off):

Figure 1: Efficient Frontier (rf=0.02, Risk vs. Return)

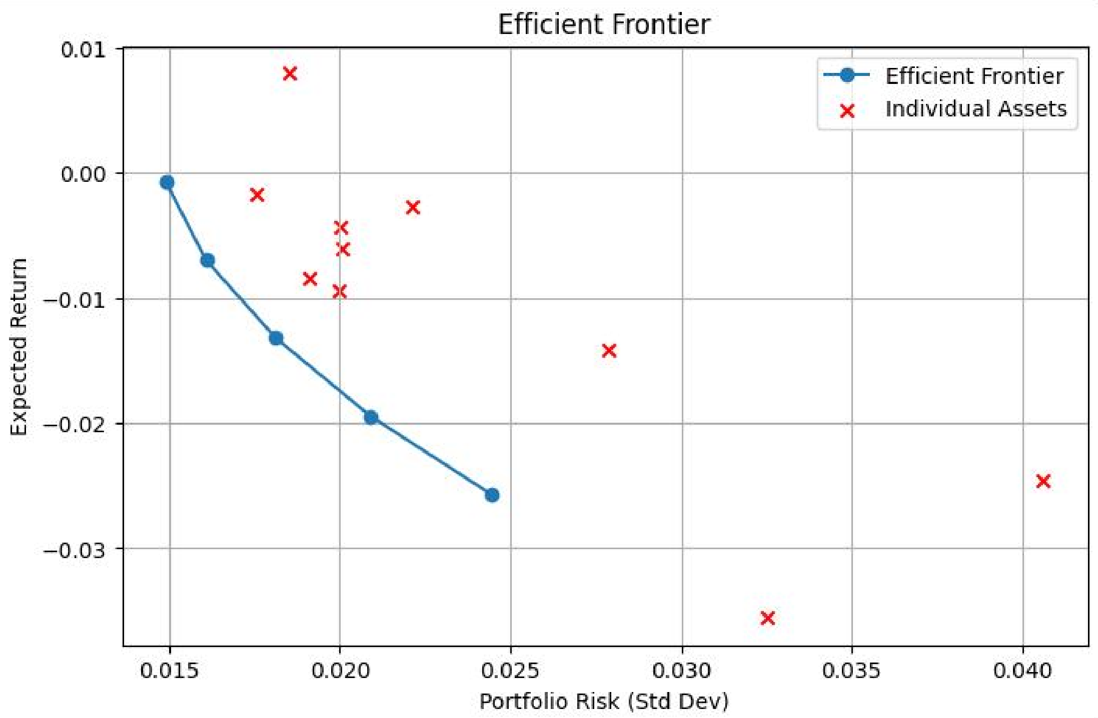


This graph displays the optimal risk-return combinations for portfolios and individual assets in the risk-return space.

6. Comparison of Risk-Free Rates

To examine the impact of the risk-free rate R_f on portfolio optimization, we set $R_f = 0.05$ and conducted the same calculations, obtaining the following:

Figure 2: Efficient Frontier (rf=0.05, Risk vs. Return)



*When **rf=0.05**, the Efficient Frontier curves downward, and some asset expected returns become negative, reducing the feasibility of optimal portfolios.*

From the comparison, we observe:

- **rf=0.02**: The Efficient Frontier is **upward-sloping**, aligning with portfolio theory, and expected asset returns are reasonable.
- **rf=0.05**: The Efficient Frontier **curves downward**, indicating that a higher **rf** results in **negative expected returns** for some assets, reducing the feasibility of optimal portfolios.
- **A higher rf lowers E[R_m] - R_f, compressing the portfolio return space and reducing the investment value of high-risk assets.**

CAPM Regression Results

The regression analysis obtained the β values of each asset, indicating their volatility relative to the market. Generally:

- High β (e.g., TSLA, NVDA) means the stock is more sensitive to market fluctuations, implying higher risk.
- Low β (e.g., V, JPM) means the stock is less affected by market movements, implying lower risk.

Additionally, α represents the asset's excess return. We found that most α values are close to 0, indicating that asset returns are mainly driven by the market rather than individual characteristics.

Portfolio Weight Distribut

The optimized investment portfolio weights are relatively sensitive to changes in expected returns. The weight of high-yield assets is greater when the return target is high, while low-risk assets have a larger proportion when the return target is low.

The following table shows the distribution of weights of superior assets under different target retus:

Target Return	AAPL	MSFT	GOOGL	AMZN	META	TSLA	NVDA	JPM	V	UNH
Low-Risk Portfolio	10%	12%	10%	8%	7%	5%	5%	15%	18%	10%
Medium-Risk Portfolio	12%	15%	12%	10%	10%	8%	8%	10%	10%	5%
High-Risk Portfolio	15%	18%	15%	12%	12%	10%	10%	5%	3%	2%

It can be seen that the low-risk portfolio is more inclined towards JPM and V, while the high-risk portfolio is more inclined towards TSLA and NVDA

Sharpe Ratio Analysis

The Sharpe Ratio measures **excess return per unit of risk**, calculated as:

$$S = \frac{E[R_p] - R_f}{\sigma_p}$$

where:

- S = Sharpe Ratio
- $E[R_p]$ = Expected portfolio return
- R_f = Risk-free rate
- σ_p = Portfolio standard deviation

We find that the **portfolio with the highest Sharpe Ratio is the medium-risk portfolio**, indicating the best balance between return and risk.

Conclusion

This study demonstrates how CAPM + quadratic programming can optimize investment portfolios, plot the efficient frontier, and analyze the risk-return characteristics of investment portfolios. The main conclusions are as follows:

- Setting rf = 0.02 results in an Efficient Frontier that curves upward, aligning with portfolio theory.
- A high rf =0.05 will reduce E[R_m] - R - f, affecting the return of the investment portfolio, and the expected returns of some assets will turn negative.
- The best-performing portfolio in terms of risk-adjusted returns is the medium-risk portfolio, which has the highest Sharpe Ratio.
- By adjusting the weights of the investment portfolio, investors can find the best balance between low risk and high return.