Portfolio Optimization
Financial Mathematics

→ Given n risky securities with μ as their mean vector and C where C_{ij} represents $\mathrm{Cov}(K_i, K_j)$.
→ Suppose each security assosciates with a weight W_i. Given that $W_i \geq 0$ since short-selling is not allowed.

$V = (W_1, W_2, \dots, W_n)$

Return on Portfolio $K_V = W^T K$ where $K = (K_1 \dots K_n)$

Optimization Problem: Maximize the return on Portfolio such that there is an upper bound on loss (d).

→ $\underset{W}{\text{Maximize}}$ $W^T K$ subject to

(loss) $-W^T K \leq d$

$W^T[1] = 1$ or $\Sigma W_i = 1$

$W_i \geq 0 \ \forall \ i \in \{1, \dots, n\}$

Our objective function here is a random variable, and it doesn't make sense to maximize such.
So, we use chance constraints here to help construct a constraint which holds the essence of our original constraint alongside making it mathematically convenient to solve. We maximize the expected value of oure objective function here.

Obj fnc: $E[W^T K] \Rightarrow W^T \mu$

Constraint 1: $\Sigma W_i = 1 \rightarrow W^T[1] = 1$

Constraint 2: $W_i \geq 0 \ \forall \ i \in \{1, 2, \dots n\}$

Constraint 3 → Chance Constraint $P(\Omega \mid -W^T K(\Omega) \leq d) \geq \alpha$

Note this α usually is in range $[0.9, 1]$

To simplify Constraint 3, we need to know distribution of K. Here, we work with Multivariate Normal Distribution

$K \sim N(\mu, C)$ [$C \rightarrow$ positive semi-definite Matrix] Also assume $\det C \neq 0$.

$-W^T K \sim N(-W^T \mu, W^T C W)$

$P(-W^T K \leq d) \geq \alpha$

We use CLT to simply this expression

$$P\left(\frac{-w^TK - \text{Mean}}{S.D} \le \frac{d - \text{Mean}}{SD}\right) \geq \alpha$$

$$\downarrow$$

$$\underbrace{\frac{-w^TK + w^T\mu}{\sqrt{w^TCw}}} \le \frac{d + w^T\mu}{\sqrt{w^TCw}}$$

A R.V. Z which follows Standard Normal Distribution

$$P\left(Z \le \frac{d + w^T\mu}{\sqrt{w^TCw}}\right) \geq \alpha$$

This is CDF and we know these values.

$$\rightarrow \quad \Phi\left(\frac{d + w^T\mu}{\sqrt{w^TCw}}\right) \geq \alpha$$

$$\frac{d + w^T\mu}{\sqrt{w^TCw}} \geq \Phi^{-1}(\alpha)$$

$$d \geq -w^T\mu + (w^TCw)\,\Phi^{-1}(\alpha)$$

We know $\sqrt{w^TCw} = \|C^{1/2}w\|$ where $C^{1/2} = X$ given, $X \cdot X = C$

Constraint 3 $\rightarrow$ $\|C^{1/2}w\|\,\Phi^{-1}(\alpha) - w^T\mu \le d$

Constraints form a convex set, so we have a convex optimization Problem.

Final Problem:

$$\text{Max} \quad w^T\mu$$

(i) $-w^T\mu + \|C^{1/2}w\|\,\Phi^{-1}(\alpha) \le d$

(ii) $\sum w_i = 1$

(iii) $w_i \geq 0 \quad \forall\, i \in \{1, 2 \ldots n\}$

Problem Formulation