# Application of Markov-switching Multifractal Model

Sisheng Chen sc3739
Jiaqi Chen jc4260
Simiao Gao sg3200
Yiyang Han yh2696
Chen Qian cq2171

## Introduction

In the project, we evaluate the performance of the ability of Markov-switching multifractal (MSM) predict realized volatility for S&P 100 index by comparing with implied, GARCH and historical volatilities. In the literature, implied, GARCH and historical are three widely used predictors for future volatility even though no consensus is reached regarding which one has the best performance. In 1997, Mandelbrot, Fishier and Calvet develop the Multifractal Model of Asset Returns (MMAR), which are the forerunning studies in multifractal measures. Binomial measure, the simplest multifractal measures, was used frequently to simulate MSM volatilities. MMAR model have some substantial effect, it captures Noah effect and Joseph effect, where Noah effect denotes the long-tails of the return distribution and Joseph effect indicates the long dependence in volatility, which is the property of a generalized Brownian motion. But MMAR model lacks the applicable statistical measures and Calvet and Fisher propose the Markov-switching multiftactal model in 2004. MSM is a model of asset returns that incorporates stochastic volatility components of heterogeneous durations. In financial industry, MSM is used to forecast volatility, compute value-at-risk, and price derivatives since it can properly capture the outliers, moment scaling and long memory exhibited in the financial volatility time series with a small number of parameters but an arbitrarily large number of frequencies.

## MSM Model in Discrete Time

First, the returns $r_t$ are modeled as:

$$r_t = \sigma(M_t)\varepsilon_t$$

with $\varepsilon_t$ drawn form a standard Normal distribution N(0,1).

Then stochastic volatility is determined by the $M_{k,t}$ and a constant scale factor $\bar{\sigma}$ :

$$\sigma(M_t) = \bar{\sigma}(\prod_{k=1}^{\bar{k}} M_{k,t}))^{1/2}$$

with $\bar{\sigma}$ a positive constant, the unconditional stantdard deviation of the return $r_t$.

The independent switching events are across k and t. A requirement for the distribution M is that it has a positive support, M≥0 and unit mean E[M]=1. In our case, we assume M to be a binomial distribution that can take the values $m_0$ and 2-$m_0$ with equal probabilities.

The dynamics of $M_{k,t}$ can be summarized as:

--$M_{k,t}$ drawn form distribution M with prob $\Upsilon_k$

--$M_{k,t}$ = $M_{k,t-1}$ with prob 1-$\Upsilon_k$

Each volatility component is renewed at time t with probability $\gamma_k$ depending on its rank within the hierarchy of multipliers and remains unchanged with probability 1- $\gamma_k$

The transition probability $\gamma_k$ are specified as:

$$\gamma_k = 1 - (1 - \gamma_1)^{(b^{k-1})}$$

$where\ \gamma_1 \in (0,1)\ and\ b \in (1, \infty)$

## Data description

Our sample is the S&P 100 index. The reason of using this index is that the S&P 100 index option market is the most active index option market in the world(see also Harvey & Whaley, 1993; Christensen & Prabhala, 1998). The data of the S&P index are obtained from Bloomberg, which provides the daily index, implied volatility, and historical volatility. The sample period covers from Jan 3rd, 2005 to December 31st, 2014.

MSM and GARCH volatilities are estimated using the daily returns of the index over the last one year. Consequently, our studying period is from Jan 3rd, 2006 through December 31st, 2014, which covers 2516 trading days.

## MSM Empirical Results

We start out with estimating the model on the 2516 observations of the S&P 100 returns we have. Table 1 shows the result of the parameter estimates for the volatility process.

Table 1

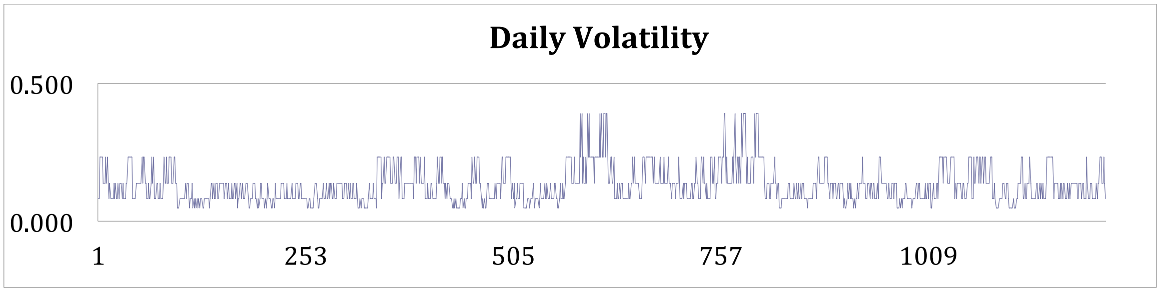
| MSM($\bar{k} = 5$) | $m_0$ | $\gamma_k$ | b | $\bar{\sigma}$ |
|---|---|---|---|---|
| | 1.4811 | 0.8683 | 7.7273 | 0.2474 |

With these parameters, we can simulate the MSM process using Excel spreadsheet.

The sequences of the 5 frequency components $M_{k,t}$, k=1, 2, 3, 4, 5, are:

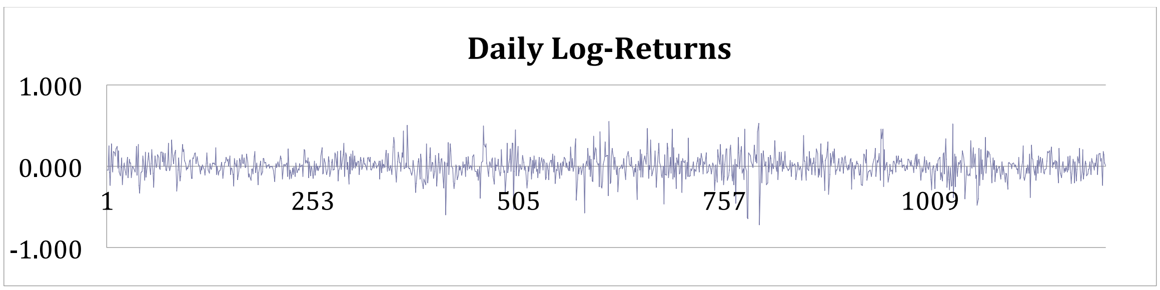

Each of the components can take on two values. In the figure we can see how the simulated components change over time. We see that the first component has a small transition probability while the fifth component has a large one. The product of all volatility components at time t times the unconditional volatility $\bar{\sigma}$ is the simulated volatility at time t.

Therefore, the daily volatility process is:




Noted that, this MSM volatility process is a product of a random simulation. It is unrelated to the real data and thus don't have to exhibit the same pattern of peaks. As expected, the MSM process shows sudden jumps from state to state.

Using this daily volatility, we can also simulate the daily log-return:




## GARCH (1,1) Model

Engle (1982) presented that the autoregressive conditional heteroskedasticity (ARCH) provides a good characterization for many financial time series. Afterward, Bollerslev (1986) extends Engle's (1982) ARCH model and proposes the generalized ARCH (GARCH) model. The GARCH model is specified as follows:

$$r_t = \mu + \varepsilon_t, \varepsilon_t \sim N(0, h_t)$$

$$h_t = \omega + \alpha(L)\varepsilon_{t-1}^2 + \beta(L)h_{t-1},$$

where $r_t$ is the index or equity return, $\mu$ is the mean of returns, $h_t$ is the conditional variance, L is the lag operator, and $\varepsilon_t$ is the random error.

Bollerslev (1987) proposes that the GARCH (1, 1) specification is an adequate representation for most economic and financial time series. Therefore, we also use GARCH (1,1) to measure and predict volatility. Hence, we rewrite the formula as follows:

$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + h_{t-1}$$

We obtain a daily s-step-ahead forecast of volatility as follows:

$$\hat{h}_{t+s} = \hat{\omega}\sum_{j=1}^{s-1}(\hat{\alpha} + \hat{\beta})^j + (\hat{\alpha} + \hat{\beta})^{s-1}\hat{h}_{t+1}\,, s = 1,2, \dots , T_M$$

where $\hat{h}_{t+s}$ denotes the s-day ahead volatility, $\hat{\omega}, \hat{\alpha},$ and $\hat{\beta}$ are estimated using daily underlying stock returns over the previous year. Finally, we generate the annualized GARCH volatility forecast (GV) as follows:

$$GV = (\frac{252}{T_M}\sum_{s=1}^{T_M}\hat{h}_{t+s})^{1/2}$$

where $T_M$ is the number of trading days until the expiration of the option contract.

## Regression results

## Rv～garch

Call:
lm(formula = rv[1:2510] ~ aa[30, 1:2510])

Residuals:

| Min | 1Q | Median | 3Q | Max |
|---|---|---|---|---|
| -0.2266 | -0.0469 | -0.0300 | 0.0232 | 0.2726 |

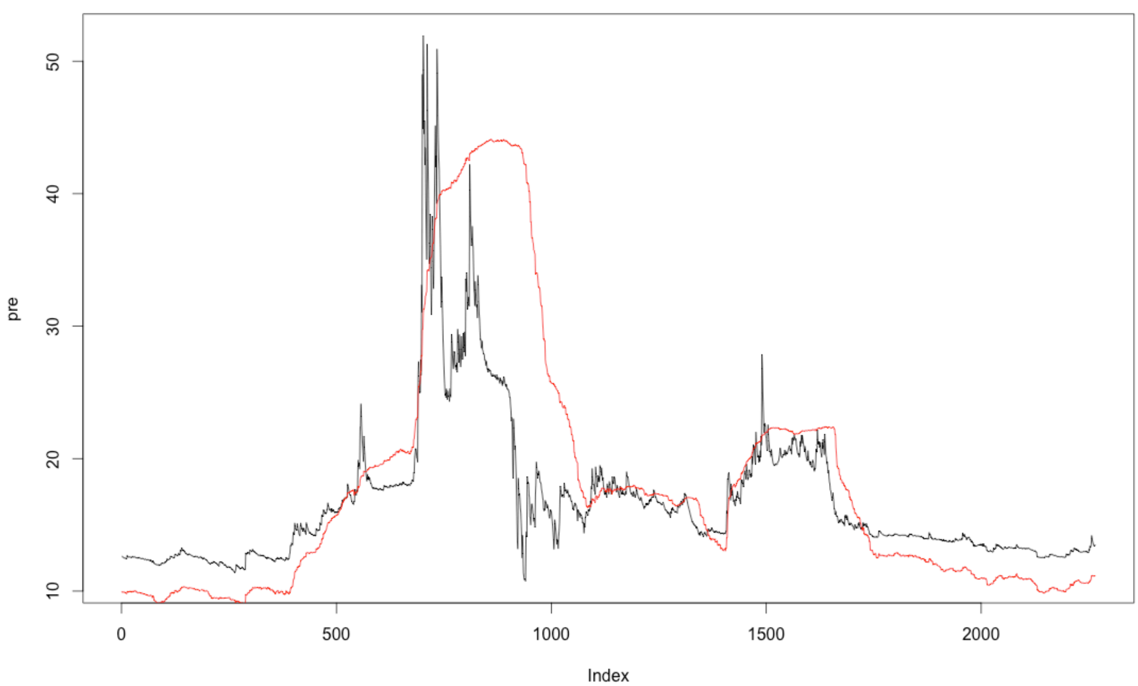
Coefficients:

| | Estimate | Std. Error | t value | Pr(>\|t\|) | |
|---|---|---|---|---|---|
| (Intercept) | 0.0593 | 0.00298 | 36.7 | <2e-16 | *** |
| aa[30, 1:2510] | 0.8329 | 0.02006 | 29.5 | <2e-16 | *** |

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.0821 on 2256 degrees of freedom
 (252 observations deleted due to missingness)

Multiple R-squared:  0.6244 Adjusted R-squared:  0.6235
F-statistic:  871 on 1 and 2256 DF p-value: <2e-16



Rv～msm

Residuals:

Call:

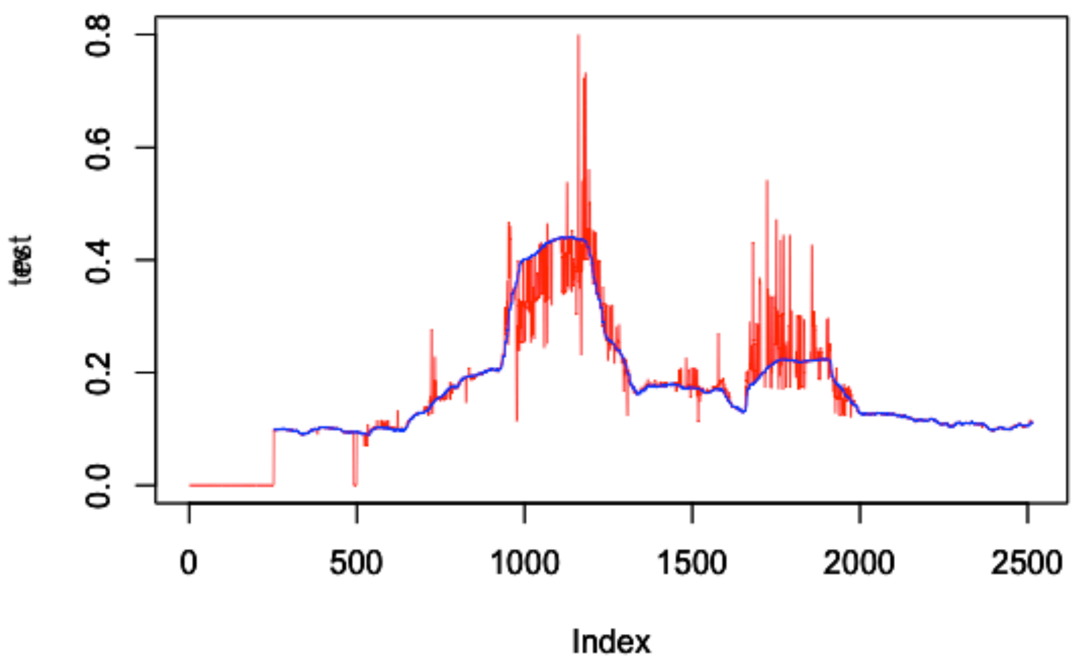
| Min | 1Q | Median | 3Q | Max |
|---|---|---|---|---|
| -0.3290 | -0.0051 | -0.0035 | 0.0012 | 0.2429 |

Coefficients:
Estimate Std. Error t value Pr(>|t|)

| | | | | |
|---|---|---|---|---|
| (Intercept) | 0.01070 | 0.00153 | 7.01 | 3.1e-12 *** |
| test[256:2516] | 0.94299 | 0.00746 | 126.34 | <2e-16 *** |

---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.034 on 2258 degrees of freedom
(1 observation deleted due to missingness)
Multiple R-squared: 0.876 Adjusted R-squared: 0.876
F-statistic: 1.6e+04 on 1 and 2258 DF p-value: <2e-16



Rv~iv

Residuals:
Call:

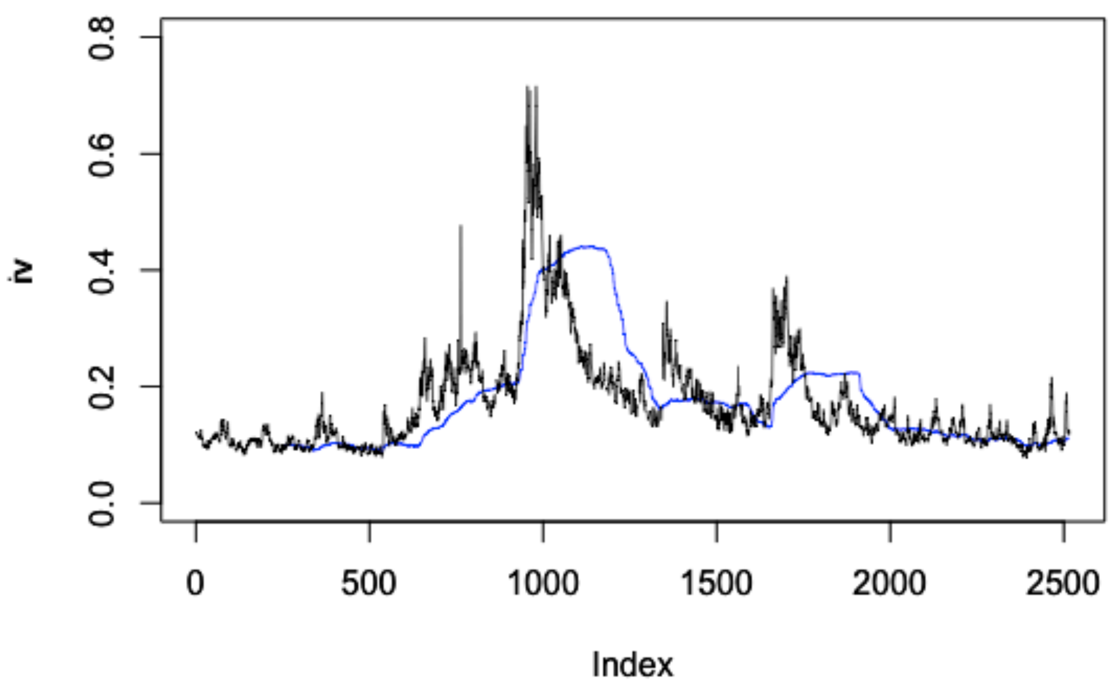
| Min | 1Q | Median | 3Q | Max |
|---|---|---|---|---|
| -0.2529 | -0.0353 | -0.0197 | 0.0194 | 0.2466 |

Coefficients:
Estimate Std. Error t value Pr(>|t|)

| | | | | |
|---|---|---|---|---|
| (Intercept) | 0.05388 | 0.00325 | 16.6 | <2e-16 *** |
| iv[256:2516] | 0.71274 | 0.01623 | 43.9 | <2e-16 *** |

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.071 on 2223 degrees of freedom
(36 observations deleted due to missingness)
Multiple R-squared: 0.465 Adjusted R-squared: 0.464
F-statistic: 1.93e+03 on 1 and 2223 DF p-value: <2e-16



Rv~hv

Residuals:

| Min | 1Q | Median | 3Q | Max |
|---|---|---|---|---|
| -0.1444 | -0.0434 | -0.0294 | 0.0209 | 0.2693 |

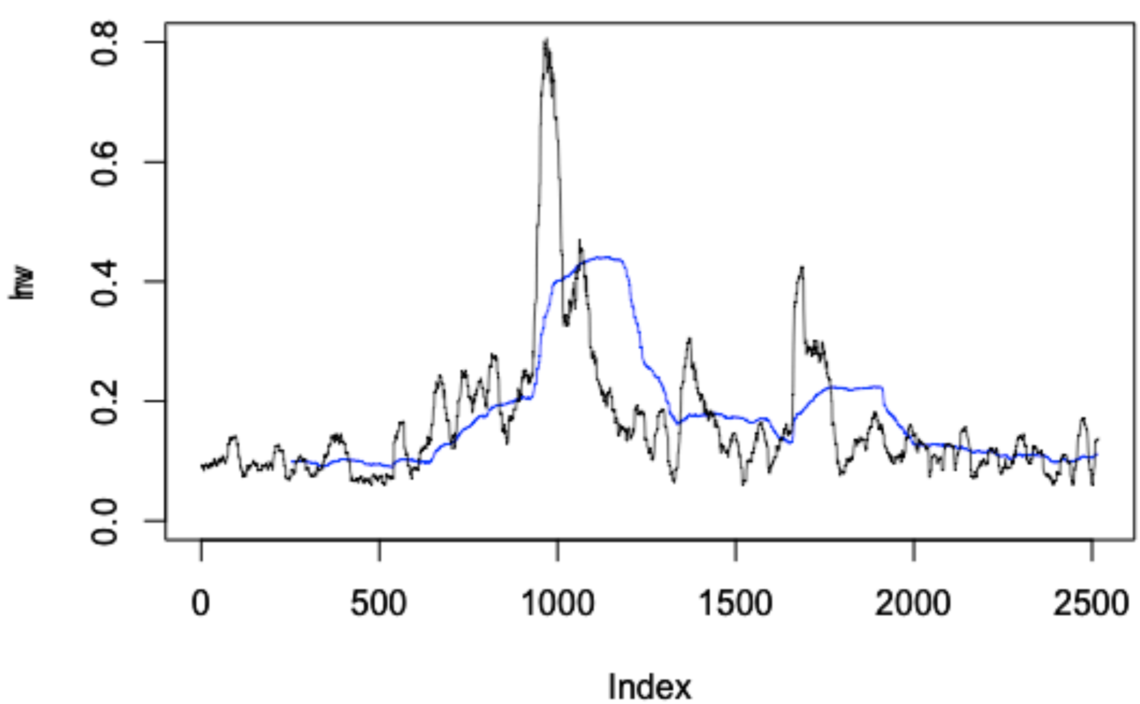
Coefficients:

| | Estimate | Std. Error | t value | Pr(>\|t\|) | |
|---|---|---|---|---|---|
| (Intercept) | 0.09794 | 0.00284 | 34.5 | <2e-16 | *** |
| hv[256:2516] | 0.48233 | 0.01352 | 35.7 | <2e-16 | *** |

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.0773 on 2259 degrees of freedom

Multiple R-squared: 0.36 Adjusted R-squared:  0.36
F-statistic: 1.27e+03 on 1 and 2259 DF p-value: <2e-16



## Regression analysis

We examine the predictive ability of the volatility measures for realized volatility by the following regression model for each volatility measure:

$$RV_{i,t} = \alpha + \beta VYM_{ij,t} + \varepsilon_t$$

where RV and VYM are the realized volatility and volatility measure, respectively, the subscript $i$ of RV denotes the S&P 100 index or equity options, and the subscript $j$ of VYM denotes four volatility measures: implied volatility (IV), MSM volatility (MV), GARCH volatility (GV), and historical volatility (HV). As in typical tests of market efficiency, if the volatility measure is an unbiased estimator of realized volatility, we would expect the intercept to be zero and the slope coefficient to be one. In addition, it is expected to see that more information contained in the volatility measure will result in a higher adjusted-$R^2$ in Eq.

Table 2:

| Volatility measure | α | β | Adjusted-$R^2$ |
|---|---|---|---|
| IV | 0.0539***<br>(16.6) | 0.7127***<br>(43.9) | 0.464 |
| HV | 0.0979***<br>(34.5) | 0.4823***<br>(35.7) | 0.360 |
| GV | 0.0593***<br>(36.7) | 0.8329***<br>(29.5) | 0.6235 |
| MV | 0.0107 ***<br>(7.01) | 0.9430***<br>(126.34) | 0.876 |

As for forecasting performance, it can be easily seen that the MSM model outperform than other models, with the highest Adjusted-$R^2$, which is 0.876, and with the highest β value, which is 0.943 close to 1. It can also be seen that the GARCH model also plays a good role in predicting the realized volatility, with β equal to 0.8329 and Adjusted-$R^2$ equal to 0.6235. The implied volatility and historical volatility measure is not as good as the GARCH model and MSM model, their adjusted-$R^2$ are all below 0.5.

Comparing the degree of predictability of the index, it shows that the ability of MSM volatilities to predict the realized volatilities of the index is better than that of implied, historical and GARCH volatilities.

## Conclusion

Predicting volatility is an important task for option pricing and financial asset and risk management. Implied, GARCH, and historical volatilities are three widely used predictors for future volatility in the literature. Using the Markov-switching Multifractal (MSM) model to predict volatility has received far less attention. The MSM model has the advantage over implied, GARCH, and historical volatilities in predicting future volatility.

In this project, firstly we study the theory of MSM model. Secondly, we use MSM model to simulate the sequence of volatility. Thirdly, we evaluate the predictable ability of MSM volatility compared with implied, GARCH, and historical volatilities to predict realized volatility for S&P 100 index. The result shows that MSM volatility performs better than the other volatility measures in predicting future volatility.

To conclude, MSM is a very useful and accurate model for volatility measure. It would be an interesting issue to see whether MSM volatility helps develop better hedge and risk management strategies. We leave these research topics for future studies.

## References

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31, 307–327.

Chuang, Wen., Huang, T.,Lin, B. (2013) Predicting volatility using the Markov-switching multifractal model: Evidence from S&P 100 index and equity options. The North American Journal of Economics and Finance 25, 168-187.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica, 50, 987–1007.