9.9 Fourier Series

As we have seen, power series representations of functions make it possible to approximate those functions as closely as we want in intervals near a particular point of interest by using partial sums of the series, that is, polynomials. However, in many important applications of mathematics, the functions involved are required to be periodic. For example, much of electrical engineering is concerned with the analysis and manipulation of *waveforms*, which are periodic functions of time. Polynomials are not periodic functions, and for this reason power series are not well suited to representing such functions.

Much more appropriate for the representations of periodic functions over extended intervals are certain infinite series of periodic functions called Fourier series.

Periodic Functions

Recall that a function f defined on the real line is **periodic** with period T if

$$f(t+T)=f(t) \quad \text{for all real } t. \tag{$*$}$$

This implies that $f(t+mT)=f(t)$ for any integer m, so that if T is a period of f, then so is any multiple mT of T. The smallest positive number T for which $(*)$ holds is called the **fundamental period**, or simply **the period** of f.

The entire graph of a function with period T can be obtained by shifting the part of the graph in any half-open interval of length T (e.g., the interval $[0, T)$) to the left or right by integer multiples of the period T. Figure 9.6 shows the graph of a function of period 2.

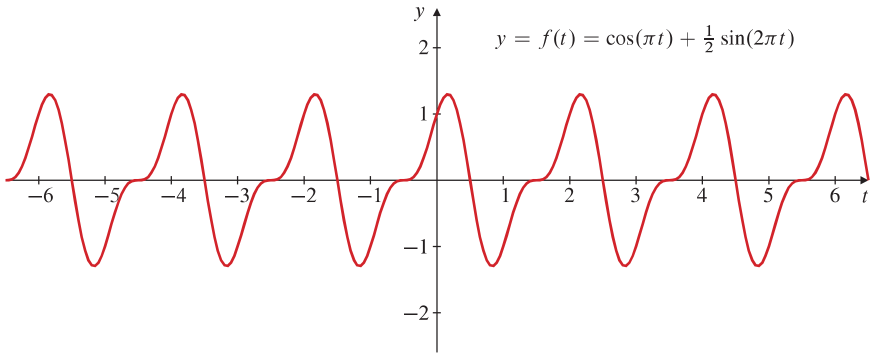

Figure 9.6 This function has period 2. Observe how the graph repeats the part in the interval $[0, 2)$ over and over to the left and right

EXAMPLE 1 The functions $g(t)=\cos(\pi t)$ and $h(t)=\sin(\pi t)$ are both periodic with period 2:

$$g(t+2)=\cos(\pi t+2\pi)=\cos(\pi t)=g(t).$$

The function $k(t)=\sin(2\pi t)$ also has period 2, but this is not its fundamental period. The fundamental period is 1:

$$k(t+1)=\sin(2\pi t+2\pi)=\sin(2\pi t)=k(t).$$

The sum $f(t)=g(t)+\frac{1}{2}k(t)=\cos(\pi t)+\frac{1}{2}\sin(2\pi t)$, graphed in Figure 9.6, has period 2, the least common multiple of the periods of its two terms.

EXAMPLE 2 For any positive integer n, the functions

$$f_n(t)=\cos(n\omega t) \qquad \text{and} \qquad g_n(t)=\sin(n\omega t)$$

both have fundamental period $T = 2\pi/(n\omega)$. The collection of all such functions corresponding to all positive integers n have common period $T = 2\pi/\omega$, the fundamental period of f_1 and g_1. T is an integer multiple of the fundamental periods of all the functions f_n and g_n. The subject of Fourier series is concerned with expressing general functions with period T as series whose terms are real multiples of these functions.

Fourier Series

It can be shown (but we won't do it here) that if $f(t)$ is periodic with fundamental period T, is continuous, and has a piecewise continuous derivative on the real line, then $f(t)$ is everywhere the sum of a series of the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(n\omega t) + b_n\sin(n\omega t)\big), \qquad (**)$$

called the **Fourier series** of f, where $\omega = 2\pi/T$ and the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are the **Fourier coefficients** of f. Determining the values of these coefficients for a given such function f is made possible by the following identities, valid for integers m and n, which are easily proved by using the addition formulas for sine and cosine. (See Exercises 49–51 in Section 5.6.)

$$\int_0^T \cos(n\omega t)\,dt = \begin{cases} 0 & \text{if } n \neq 0 \\ T & \text{if } n = 0 \end{cases}$$

$$\int_0^T \sin(n\omega t)\,dt = 0$$

$$\int_0^T \cos(m\omega t)\cos(n\omega t)\,dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_0^T \cos(m\omega t)\sin(n\omega t)\,dt = 0.$$

If we multiply equation $(**)$ by $\cos(m\omega t)$ (or by $\sin(m\omega t)$) and integrate the resulting equation over $[0, T]$ term by term, all the terms on the right except the one involving a_m (or b_m) will be 0. (The term-by-term integration requires justification, but we won't try to do that here either.) The integration results in

$$\int_0^T f(t)\cos(m\omega t)\,dt = \frac{1}{2}Ta_m$$

$$\int_0^T f(t)\sin(m\omega t)\,dt = \frac{1}{2}Tb_m.$$

(Note that the first of these formulas is even valid for $m = 0$ because we chose to call the constant term in the Fourier series $a_0/2$ instead of a_0.) Since the integrands are all periodic with period T, the integrals can be taken over any interval of length T; it is often convenient to use $[-T/2, T/2]$ instead of $[0, T]$. The Fourier coefficients of f are therefore given by

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt \quad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt \quad (n = 1, 2, 3, \ldots),$$

where $\omega = 2\pi/T$.

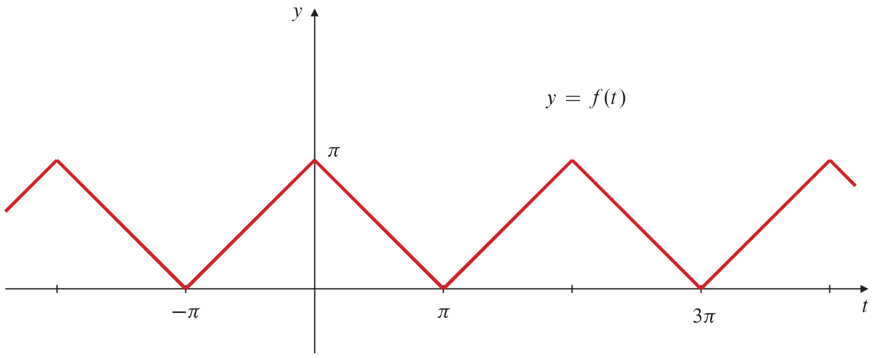

Figure 9.7 A sawtooth function of period 2π

EXAMPLE 3 Find the Fourier series of the sawtooth function $f(t)$ of period 2π whose values in the interval $[-\pi, \pi]$ are given by $f(t) = \pi - |t|$. (See Figure 9.7.)

Solution Here $T = 2\pi$ and $\omega = 2\pi/(2\pi) = 1$. Since $f(t)$ is an even function, $f(t)\sin(nt)$ is odd, so all the Fourier sine coefficients b_n are zero:

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = 0.$$

Also, $f(t)\cos(nt)$ is an even function, so

$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{4}{2\pi}\int_{0}^{\pi} f(t)\cos(nt)\,dt \\ &= \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos(nt)\,dt \\ &= \begin{cases} \pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \text{ and } n \text{ is even} \\ 4/(\pi n^2) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Since odd positive integers n are of the form $n = 2k - 1$, where k is a positive integer, the Fourier series of f is given by

$$f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cos\big((2k-1)t\big).$$

Convergence of Fourier Series

The partial sums of a Fourier series are called Fourier polynomials because they can be expressed as polynomials in $\sin(\omega t)$ and $\cos(\omega t)$, although we will not actually try to write them that way. The Fourier polynomial of order m of the periodic function f having period T is

$$f_m(t) = \frac{a_0}{2} + \sum_{n=1}^{m}\big(a_n\cos(n\omega t) + b_n\sin(n\omega t)\big),$$

where $\omega = 2\pi/T$ and the coefficients a_n $(0 \le n \le m)$ and b_n $(1 \le n \le m)$ are given by the integral formulas developed earlier.

EXAMPLE 4 The Fourier polynomial of order 3 of the sawtooth function of Example 3 is

$$f_3(t) = \frac{\pi}{2} + \frac{4}{\pi}\cos t + \frac{4}{9\pi}\cos(3t).$$

The graph of this function is shown in Figure 9.8. Observe that it appears to be a reasonable approximation to the graph of f in Figure 9.7, but, being a finite sum of differentiable functions, $f_3(t)$ is itself differentiable everywhere, even at the integer multiples of π where f is not differentiable.

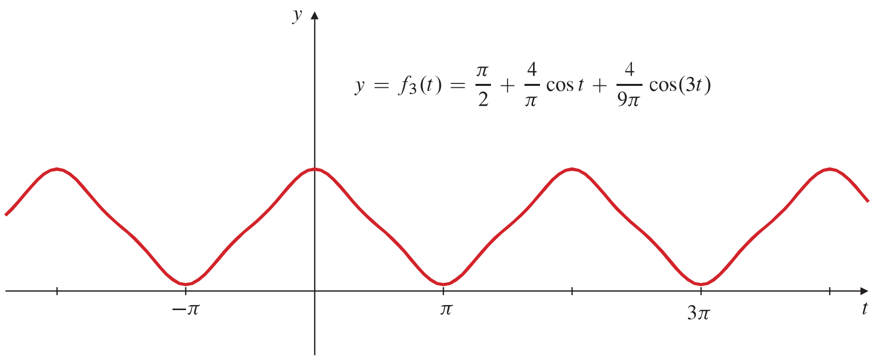

Figure 9.8 The Fourier polynomial approximation $f_3(t)$ to the sawtooth function of Example 3

As noted earlier, the Fourier series of a function $f(t)$ that is periodic, continuous, and has a piecewise continuous derivative on the real line converges to $f(t)$ at each real number t. However, the Fourier coefficients (and hence the Fourier series) can be calculated (by the formulas given above) for periodic functions with piecewise continuous derivative even if the functions are not themselves continuous, but only piecewise continuous.

Recall that $f(t)$ is piecewise continuous on the interval $[a, b]$ if there exists a partition $\{a = x_0 < x_1 < x_2 < \cdots < x_k = b\}$ of $[a, b]$ and functions $F_1, F_2, \ldots, F_k$, such that

(i) F_i is continuous on $[x_{i-1}, x_i]$, and

(ii) $f(t) = F_i(t)$ on (x_{i-1}, x_i).

The integral of such a function f is the sum of integrals of the functions F_i:

$$\int_a^b f(t)\,dt = \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} F_i(t)\,dt.$$

Since $f(t)\cos(n\omega t)$ and $f(t)\sin(n\omega t)$ are piecewise continuous if f is, the Fourier coefficients of a piecewise continuous, periodic function can be calculated by the same formulas given for a continuous periodic function. The question of where and to what the Fourier series converges in this case is answered by the following theorem, proved in textbooks on Fourier analysis.

THEOREM 25

The Fourier series of a piecewise continuous, periodic function f with piecewise continuous derivative converges to that function at every point t where f is continuous. Moreover, if f is discontinuous at $t = c$, then f has different, but finite, left and right limits at c:

$$\lim_{t\to c-} f(t) = f(c-), \qquad \text{and} \qquad \lim_{t\to c+} f(t) = f(c+).$$

The Fourier series of f converges at $t = c$ to the average of these left and right limits:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(n\omega c) + b_n\sin(n\omega c)\big) = \frac{f(c-) + f(c+)}{2},$$

where $\omega = 2\pi/T$.

EXAMPLE 5 Calculate the Fourier series for the periodic function f with period 2 satisfying

$$f(t) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Where does f fail to be continuous? To what does the Fourier series of f converge at these points?

Solution Here $T = 2$ and $\omega = 2\pi/2 = \pi$. Since f is an odd function, its cosine coefficients are all zero:

$$a_n = \int_{-1}^{1} f(t)\cos(n\pi t)\,dt = 0. \qquad \text{(The integrand is odd.)}$$

The same symmetry implies that

$$\begin{aligned} b_n &= \int_{-1}^{1} f(t)\sin(n\pi t)\,dt \\ &= 2\int_0^1 \sin(n\pi t)\,dt = -\frac{2\cos(n\pi t)}{n\pi}\bigg|_0^1 \\ &= -\frac{2}{n\pi}\big((-1)^n - 1\big) = \begin{cases} 4/(n\pi) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Odd integers n are of the form $n = 2k - 1$ for $k = 1, 2, 3, \ldots$. Therefore, the Fourier series of f is

$$\begin{aligned} &\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\big((2k-1)\pi t\big) \\ &= \frac{4}{\pi}\left(\sin(\pi t) + \frac{1}{3}\sin(3\pi t) + \frac{1}{5}\sin(5\pi t) + \cdots\right). \end{aligned}$$

Note that f is continuous except at the points where t is an integer. At each of these points f jumps from -1 to 1 or from 1 to -1, so the average of the left and right limits of f at these points is 0. Observe that the sum of the Fourier series is 0 at integer values of t, in accordance with Theorem 25. See Figure 9.9.

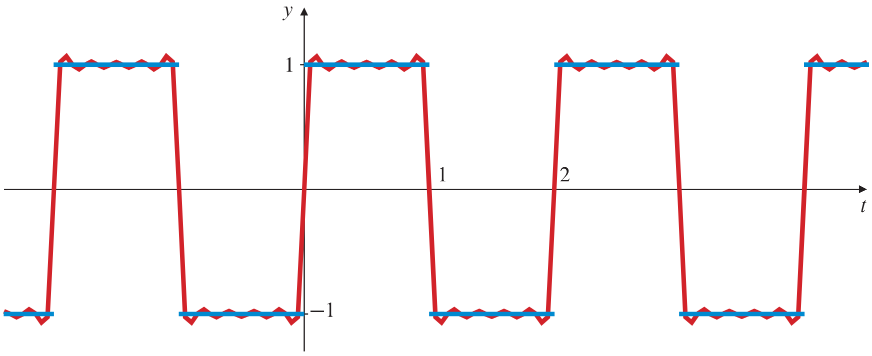

Figure 9.9 The piecewise continuous function f (blue) of Example 5 and its Fourier polynomial f_{15} (red)

$$f_{15}(t) = \sum_{k=1}^{8}\frac{4\sin\big((2k-1)\pi t\big)}{(2k-1)\pi}$$

Fourier Cosine and Sine Series

As observed in Example 3 and Example 5, even functions have no sine terms in their Fourier series, and odd functions have no cosine terms (including the constant term $a_0/2$). It is often necessary in applications to find a Fourier series representation of a given function defined on a finite interval $[0, a]$ having either no sine terms (a **Fourier cosine series**) or no cosine terms (a **Fourier sine series**). This is accomplished by extending the domain of f to $[-a, 0)$ so as to make f either even or odd on $[-a, a]$,

$$\begin{aligned} f(-t) &= f(t) \text{ if } -a \le t < 0 \text{ for the even extension} \\ f(-t) &= -f(t) \text{ if } -a \le t < 0 \text{ for the odd extension,} \end{aligned}$$

and then calculating its Fourier series considering the extended f to have period $2a$. (If we want the odd extension, we may have to redefine $f(0)$ to be 0.)

EXAMPLE 6 Find the Fourier cosine series of $g(t) = \pi - t$ defined on $[0, \pi]$.

Solution The even extension of $g(t)$ to $[-\pi, \pi]$ is the function f of Example 3. Thus, the Fourier cosine series of g is

$$\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2} \cos\big((2k-1)t\big).$$

EXAMPLE 7 Find the Fourier sine series of $h(t) = 1$ defined on $[0, 1]$.

Solution If we redefine $h(0) = 0$, then the odd extension of h to $[-1, 1]$ coincides with the function $f(t)$ of Example 5 except that the latter function is undefined at $t = 0$. The Fourier sine series of h is the series obtained in Example 5, namely,

$$\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\big((2k-1)\pi t\big).$$

Remark Fourier cosine and sine series are treated from a different perspective in Section 13.5.

EXERCISES 9.9

In Exercises 1–4, what is the fundamental period of the given function?

1. $f(t) = \sin(3t)$

2. $g(t) = \cos(3 + \pi t)$

3. $h(t) = \cos^2 t$

4. $k(t) = \sin(2t) + \cos(3t)$

In Exercises 5–8, find the Fourier series of the given function.

5. $f(t) = t$, $-\pi < t \le \pi$, f has period 2π.

6. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2, \end{cases}$ f has period 2.

7. $f(t) = \begin{cases} 0 & \text{if } -1 \le t < 0 \\ t & \text{if } 0 \le t < 1, \end{cases}$ f has period 2.

8. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2 \\ 3-t & \text{if } 2 \le t < 3, \end{cases}$ f has period 3.

9. What is the Fourier cosine series of the function $h(t)$ of Example 7?

10. Calculate the Fourier sine series of the function $g(t)$ of Example 6.

11. Find the Fourier sine series of $f(t) = t$ on $[0, 1]$.

12. Find the Fourier cosine series of $f(t) = t$ on $[0, 1]$.

13. Use the result of Example 3 to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

14. Verify that if f is an even function of period T, then the Fourier sine coefficients b_n of f are all zero and the Fourier cosine coefficients a_n of f are given by

$$a_n = \frac{4}{T} \int_0^{T/2} f(t)\cos(n\omega t)\,dt, \qquad n = 0, 1, 2, \ldots,$$

where $\omega = 2\pi/T$. State and verify the corresponding result for odd functions f.

CHAPTER REVIEW

Key Ideas

- **What does it mean to say that the sequence $\{a_n\}$**
  - ◇ is bounded above?
  - ◇ is ultimately positive?
  - ◇ is alternating?
  - ◇ is increasing?
  - ◇ converges?
  - ◇ diverges to infinity?
- **What does it mean to say that the series $\sum_{n=1}^{\infty} a_n$**
  - ◇ converges?
  - ◇ diverges?
  - ◇ is geometric?
  - ◇ is telescoping?
  - ◇ is a p-series?
  - ◇ is positive?
  - ◇ converges absolutely?
  - ◇ converges conditionally?