TTIC 31230, Fundamentals of Deep Learning

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Some Information Theory

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Why Information Theory?

The fundamental equation involves cross-entropy.

Cross-entropy is an information-theoretic concept.

Information theory arises in many places and many forms in deep learning.

Entropy of a Distribution

The entropy of a distribution P is defined by

$$H(P) = E_{y \sim \text{Pop}} \; -\ln P(y) \quad \text{in units of "nats"}$$

$$H_2(P) = E_{y \sim \text{Pop}} \; -\log_2 P(y) \quad \text{in units of bits}$$

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Why Bits?

Why is $-\log_2 P(y)$ a number of bits?

Example: Let P be a uniform distribution on 256 values.

$$E_{y \sim P} -\log_2 P(y) = -\log_2 \frac{1}{256} = \log_2 256 = 8 \text{ bits} = 1 \text{ byte}$$

$$\textcolor{red}{1 \text{ nat} = \frac{1}{\ln 2} \text{ bits} \approx 1.44 \text{ bits}}$$

Shannon's Source Coding Theorem

Why is $-\log_2 P(y)$ a number of bits?

A prefix-free code for $\mathcal{Y}$ assigns a bit string $c(y)$ to each $y \in \mathcal{Y}$ such that no code string is prefix of any other code string.

For a probability distribution P on $\mathcal{Y}$ we consider the average code length $E_{y \sim P} |c(y)|$.

Theorem: For any c we have $E_{y \sim P} |c(y)| \geq H_2(P)$.

Theorem: There exists c with $E_{y \sim P} |c(y)| \leq H_2(P) + 1$.

Cross Entropy

Let P and Q be two distribution on the same set.

$$H(P, Q) = E_{y \sim P} \ -\ln\ Q(y)$$

$$\Phi^* = \underset{\Phi}{\mathrm{argmin}}\ H(\mathrm{Pop}, P_\Phi)$$

H(P,Q) also has a data compression interpretation.

$H(P, Q)$ can be interpreted as 1.44 times the number of bits used to code draws from P when using the imperfect code defined by Q.

Entropy, Cross Entropy and KL Divergence

Let P and Q be two distribution on the same set.

Entropy :
$$H(P) = E_{y \sim P} \; - \ln \; P(y)$$

CrossEntropy :
$$H(P,Q) = E_{y \sim P} \; - \ln \; Q(y)$$

KL Divergence :
$$KL(P,Q) = H(P,Q) - H(P)$$

$$= E_{y \sim P} \; \ln \; \frac{P(y)}{Q(y)}$$

We have $H(P,Q) \geq H(P)$ or equivalently $KL(P,Q) \geq 0$.

The Universality Assumption

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; H(\mathrm{Pop}, P_\Phi) = \operatorname*{argmin}_{\Phi} \; H(\mathrm{Pop}) + KL(\mathrm{Pop}, P_\Phi)$$

Universality assumption: P_Φ can represent any distribution and Φ can be fully optimized.

This is clearly false for deep networks. But it gives important insights like:

$$P_{\Phi^*} = \mathrm{Pop}$$

This is the motivatation for the fundamental equation.

Asymmetry of Cross Entropy

Consider

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; H(P, Q_\Phi) \qquad (1)$$

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; H(Q_\Phi, P) \qquad (2)$$

For (1) Q_Φ must cover all of the support of P.

For (2) Q_Φ concentrates all mass on the point maximizing P.
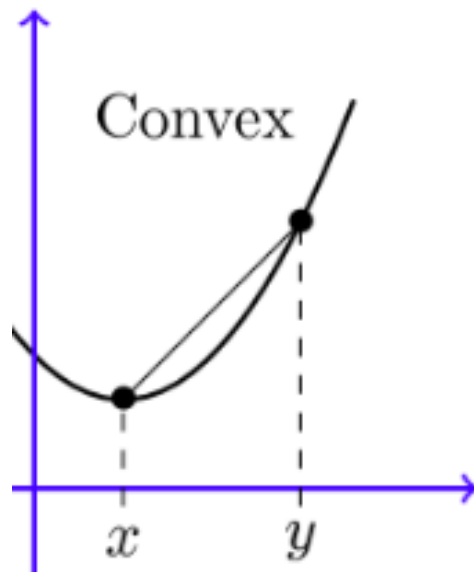
Asymmetry of KL Divergence

Consider

$$\Phi^* = \operatorname*{argmin}_{\Phi} \ KL(P, Q_\Phi)$$

$$= \operatorname*{argmin}_{\Phi} \ H(P, Q_\Phi) \qquad\qquad (1)$$

$$\Phi^* = \operatorname*{argmin}_{\Phi} \ KL(Q_\Phi, P)$$

$$= \operatorname*{argmin}_{\Phi} H(Q_\Phi, P) - H(Q_\Phi) \quad (2)$$

If Q_Φ is not universally expressive we have that (1) still forces Q_Φ to cover all of P (or else the KL divergence is infinite) while (2) allows Q_Φ to be restricted to a single mode of P (a common outcome).

Proving $KL(P, Q) \geq 0$: Jensen's Inequality

For f convex (upward curving) we have

$$E[f(x)] \geq f(E[x])$$

Proving $KL(P, Q) \geq 0$

$$KL(P, Q) = E_{y \sim P} \ -\ln \frac{Q(y)}{P(y)}$$

$$\geq -\ln E_{y \sim P} \frac{Q(y)}{P(y)}$$

$$= -\ln \sum_y P(y) \frac{Q(y)}{P(y)}$$

$$= -\ln \sum_y Q(y)$$

$$= 0$$

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Appendix: The Rearrangement Trick

$$KL(P, Q) = E_{x \sim P} \ln \frac{P(x)}{Q(x)}$$

$$= E_{x \sim P} \left[(-\ln Q(x)) - (-\ln P(x)) \right]$$

$$= (E_{x \sim P} - \ln Q(x)) - (E_{x \sim P} - \ln P(x))$$

$$= H(P, Q) - H(P)$$

In general $E_{x \sim P} \ln \left(\prod_i A_i \right) = E_{x \sim P} \sum_i \ln A_i$

Summary

$$\Phi^* = \text{argmin}_\Phi \; H(\text{Pop}, P_\Phi) \text{ unconditional}$$

$$\Phi^* = \text{argmin}_\Phi \; E_{x \sim \text{Pop}} \; H(\text{Pop}(y|x), P_\Phi(y|x)) \text{ conditional}$$

$$\text{Entropy}: \qquad H(P) = E_{y \sim P} \; -\ln \; P(y)$$

$$\text{CrossEntropy}: \quad H(P, Q) = E_{y \sim P} \; -\ln \; Q(y)$$

$$\text{KL Divergence}: KL(P, Q) = H(P, Q) - H(P)$$

$$= E_{y \sim P} \quad \ln \; \frac{P(y)}{Q(y)}$$

$$H(P, Q) \geq H(P), \quad KL(P, Q) \geq 0, \quad \text{argmin}_Q \; H(P, Q) = P$$

END