TTIC 31230 Fundamentals of Deep Learning
Quiz 2

Problem 1: Running Averages. Consider a sequence of vectors x_0, x_1, ... and two running averages y_t and z_t defined by as follows for $0 < \beta < 1$ and $\gamma > 0$.

$$\begin{aligned} y_0 &= 0 \\ y_{t+1} &= \beta y_t + (1-\beta)x_t \end{aligned}$$

$$\begin{aligned} z_0 &= 0 \\ z_{t+1} &= \beta z_t + \gamma x_t \end{aligned}$$

(a) Suppose that the values x_t are drawn IID from a distribution with mean vector $\overline{x} = E\ x_t$. Give values for

$$\overline{y} = \lim_{t\to\infty} E\ y_t$$

and

$$\overline{z} = \lim_{t\to\infty} E\ z_t$$

as functions of β, γ and $\overline{x}$
Hint: Solve for $E\ y_{t+1}$ as a function of $E\ y_t$ and assume that a limiting expectation exists.

Solution:

$$\begin{aligned} E\ y_{t+1} &= \beta\ E\ y_t + (1-\beta)\ E\ x_t \\ \overline{y} &= \beta\ \overline{y} + (1-\beta)\ \overline{x} \\ (1-\beta)\ \overline{y} &= (1-\beta)\ \overline{x} \\ \overline{y} &= \overline{x} \end{aligned}$$

$$\begin{aligned} E\ z_{t+1} &= \beta\ E\ z_t + \gamma\ E\ x_t \\ \overline{z} &= \beta\ \overline{z} + \gamma\ \overline{x} \\ (1-\beta)\ \overline{z} &= \gamma\ \overline{x} \\ \overline{z} &= \frac{\gamma}{1-\beta}\ \overline{x} \end{aligned}$$

(b) Express z_t as a function of y_t, β and γ.

Solution:

$$\begin{aligned} z_{t+1} &= \beta\, z_t + \gamma\, x_t \\ &= \sum_{t'=0}^{t} \gamma \beta^{t-t'} x_{t'} \\ &= \frac{\gamma}{1-\beta} \sum_{t'=0}^{t} (1-\beta)\beta^{t-t'} x_t \\ &= \frac{\gamma}{1-\beta}\, y_t \end{aligned}$$

Problem 2. Adaptive SGD. This problem considers the question of whether the convergence theorem hold for adaptive methods — in the limit as the learning rate goes to zero do adaptive methods converge to a local minimum of the loss.

Consider a generalization of RMSProp where we allow an arbitrary adaptation with with different learning rates for ifferent parameter values. More specifically consider the SGD update equation

$$(1) \quad \Phi_{t+1} = \Phi_t - \eta \left(A(\Phi_t, x_t, y_t) \odot \nabla_\Phi \mathcal{L}(\Phi_t, x_t, y_t) \right)$$

where $\langle x_t, y_t \rangle$ is the tth training pair, $A(\Phi_t, x_t, y_t)$ is an adaptation vector, and $\odot$ is the Haddamard product $(x \odot y)[i] = x[i]\; y[i]$.

Consider the special case given by

$$\begin{aligned} A(\Phi, x, y)[i] &= \frac{1}{\sqrt{s(\Phi, x, y)} + \epsilon} \\ s(\Phi, x, y) &= \frac{1}{d} ||\nabla_\Phi\; \mathcal{L}(\Phi, x, y)||^2 \end{aligned}$$

where d is the dimension of Φ.

(a) For the given interpretation of $A(\Phi, x, y)$, let Φ^* be a parameter setting that is a stationary point of the update equation (1) in the sense that expected update over a random draw from the population is zero. Write this stationary condition on Φ^* explicitly as an expectation equaling zero under the given interpretation of $A(\Phi, x, y)$.

Solution:

$$E_{\langle x, y \rangle \sim \mathrm{Pop}}\; \frac{1}{\sqrt{s(\Phi^*, x, y) + \epsilon}}\; \nabla_\Phi\; \mathcal{L}(\Phi, x, y) = 0$$

(b) Is Φ^* as defined in part (b) a stationary point of the original loss — a point where the expected gradient of $\mathcal{L}(\Phi^*, x, y)$ equal to zero?

Solution: No, the average a weighted sum is different from the average of an unweighted sum and hence the fact that the weighted average is zero does not imply that the average is zero.

(c) Do these observations have implications for the adaptive methods described in this class. Explain your answer.

Solution: Yes, the example considered here is just a special case of RMSProp or Adam which are in fact not guaranteed to converge to a stationary point (or local optimum) of the loss function.