TTIC 31230, Fundamentals of Deep Learning

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Gaussian Noisy Channel RDAs

Gaussian Noisy-Channel RDA

$$\Phi^* = \operatorname*{argmin}_{\Phi} E_{y,\epsilon}\ \ln \frac{p_\Phi(\tilde{z}|y)}{q_\Phi(\tilde{z})} + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}))$$

$$\tilde{z}[i] \;=\; z_\Phi(y)[i] + \sigma_\Phi(y)\epsilon[i] \quad \epsilon[i] \sim \mathcal{N}(0,1)$$

$$p_\Phi(\tilde{z}[i]|y) \;=\; \mathcal{N}(z_\Phi(y)[i], \sigma_\Phi(y)[i])$$

$$q_\Phi(\tilde{z}[i]) \;=\; \mathcal{N}(\mu_q[i], \sigma_q[i])$$

$$\mathrm{Dist}(y, \hat{y}) \;=\; ||y - \hat{y}||^2$$

Gaussian Noisy-Channel RDA

$$\Phi^* = \underset{\Phi}{\mathrm{argmin}}\, E_{y,\epsilon}\;\ln \frac{p_\Phi(\tilde{z}|y)}{q_\Phi(\tilde{z})} + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}))$$

We will show that in the Gaussian case can fix q_Φ

$$\begin{aligned} p_\Phi(\tilde{z}[i]|y) &= \mathcal{N}(z_\Phi(y)[i], \sigma_\Phi(y)[i]) \\ q_\Phi(\tilde{z}[i]) &= \mathcal{N}(0,1) \\ \mathrm{Dist}(y,\hat{y}) &= ||y-\hat{y}||^2 \end{aligned}$$

Gaussian Noisy-Channel RDA

$$\Phi^* = \underset{\Phi}{\mathrm{argmin}}\, E_{y,\epsilon}\ \ln \frac{p_\Phi(\tilde{z}|y)}{q_\Phi(\tilde{z})} + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}))$$

$$= \underset{\Phi}{\mathrm{argmin}}\ E_{y\sim\mathrm{Pop}} \left(\begin{array}{r} KL(p_\Phi(\tilde{z}|y), q_\Phi(\tilde{z})) \\ \\ +\lambda\ E_\epsilon\ \mathrm{Dist}(y,\ y_\Phi(\tilde{z})) \end{array}\right)$$

Closed Form KL-Divergence

$$KL(p_\Phi(\tilde{z}|y), q_\Phi(\tilde{z}))$$

$$= \sum_i \frac{\sigma_\Phi(y)[i]^2 + (z_\Phi(y)[i] - \mu_q[i])^2}{2\sigma_q[i]^2} + \ln \frac{\sigma_q[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

Standardizing $p_\Phi(z)$

$$KL(p_\Phi(\tilde{z}|y), p_\Phi(\tilde{z}))$$

$$= \sum_i \frac{\sigma_\Phi(y)[i]^2 + (z_\Phi(y)[i] - \mu_q[i])^2}{2\sigma_q[i]^2} + \ln \frac{\sigma_q[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

$$KL(p_{\Phi'}(\tilde{z}|y), \mathcal{N}(0, I))$$

$$= \sum_i \frac{\sigma^\epsilon_{\Phi'}(y)[i]^2 + z_{\Phi'}(y)[i]^2}{2} + \ln \frac{1}{\sigma^\epsilon_{\Phi'}(y)[i]} - \frac{1}{2}$$

Standardizing $p_\Phi(z)$

$$KL_\Phi = \sum_i \frac{\sigma_\Phi(y)[i]^2 + (z_\Phi(y)[i] - \mu_q[i])^2}{2\sigma_q[i]^2} + \ln \frac{\sigma_q[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

$$KL_{\Phi'} = \sum_i \frac{\sigma^\epsilon_{\Phi'}(y)[i]^2 + z_{\Phi'}(y)[i]^2}{2} + \ln \frac{1}{\sigma^\epsilon_{\Phi'}(y)[i]} - \frac{1}{2}$$

Setting Φ' so that

$$\begin{aligned} z_{\Phi'}(y)[i] &= (z_\Phi(y)[i] - \mu_q[i])/\sigma_q[i] \\ \sigma^\epsilon_{\Phi'}(y)[i] &= \sigma_\Phi(y)[i]/\sigma_q[i] \end{aligned}$$

gives $KL(p_\Phi(\tilde{z}|y), p_\Phi(\tilde{z})) = KL(p_{\Phi'}(\tilde{z}|y), \mathcal{N}(0, I))$.

Sampling

Sample $\tilde{z} \sim \mathcal{N}(0, I)$ and compute $y_\Phi(\tilde{z})$

[Alec Radford]

Summary: Rate-Distortion

RDA: y continuous, $\tilde{z}$ a bit string,

$$\Phi^* = \underset{\Phi}{\mathrm{argmin}}\, E_{y \sim \mathrm{Pop}}\, |\tilde{z}_\Phi(y)| + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y)))$$

Gaussian RDA: $\tilde{z} = z_\Phi(y) + \sigma_\Phi(y) \odot \epsilon, \qquad \epsilon \sim \mathcal{N}(0, I)$

$$\Phi^* = \underset{\Phi}{\mathrm{argmin}}\, E_{y \sim \mathrm{Pop}} \left(\begin{array}{rl} & KL(p_\Phi(\tilde{z}|y), \mathcal{N}(0, I)) \\ \\ + & \lambda E_\epsilon\, \mathrm{Dist}(y, y_\Phi(\tilde{z})) \end{array} \right)$$

END