# TTIC 31230, Fundamentals of Deep Learning

David McAllester, Winter 2020

## Einstein Notation

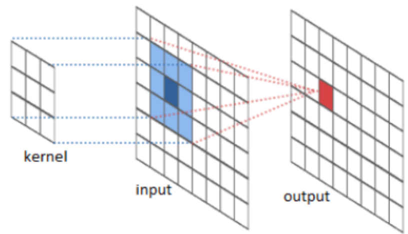
# Tensors

An array with more than two indeces is called a tensor.

In a convolutional neural network (CNN) for vision a "layer" in the model is a three index (third order) tensor $L[b, x, y, i]$.

$L[b, x, y, i]$ is the value of feature $i$ for batch element $b$ at image position $\langle x, y \rangle$.

# A Convolution Layer




$$W[\Delta x, \Delta y, i, j] \qquad L_\ell[b, x, y, i] \qquad L_{\ell+1}[b, x, y, j]$$

River Trail Documentation

$$L_{\ell+1}[b, x, y, j]$$

$$= \sigma\left(\left(\sum_{\Delta x, \Delta y, i} W[\Delta x, \Delta y, i, j]\; L_\ell[b, x + \Delta x, y + \Delta y, i]\right) - B[j]\right)$$

## Einstein Notation

The convolution equation can be written more compactly in Einstein notation.

Capital letter indices will used to indicate subtensors (slices) so that, for example, $M[I,J]$ denotes a matrix while $M[i,j]$ denotes one element of the matrix, $M[i,J]$ denotes the $i$th row, and $M[I,j]$ denotes the $j$th collumn.

By abuse of notation, capital letters will also be used to indicate the range of the indices. A vector $x[I]$ has $I$ dimensions with values $x[0]$, ..., $x[I-1]$. A Matrix $M[I,J]$ has $I \times J$ entries.

## Einstein Notation

$y = Wx$ abbreviates $y[i] = \sum_j W[i,j]x[j]$.

$y = x^\top W$ abbreviates $y[j] = \sum_i W[i,j]x[i]$.

$i$ is a "row index" and $j$ is a "column index" of $W$.

Linear algebra suppresses indeces.

Einstein notation uses explicit indeces.

# Einstein Notation

Einstein notation is the convention that repeated capital indeces in a product of tensors are implicitly summed.

$y = Wx$ abbreviates $y[i] = \sum_j W[i,j]x[j] = W[i,J]x[J]$.

$y = x^\top W$ abbreviates $y[j] = \sum_i W[i,j]x[i] = W[I,j]x[I]$.

# An MLP in Einstein Notation

$$h[j] = \sigma\left(W^0[j, I]\; x[I] - b^0[j]\right)$$

$$s[\hat{y}] = \sigma\left(W^1[\hat{y}, J]\; h[J] - b^1[\hat{y}]\right)$$

$$P_\Phi[\hat{y}] = \operatorname*{softmax}_{\hat{y}}\; s[\hat{y}]$$

# Convolution in Einstein Notation

$$L_{\ell+1}[b, x, y, j]$$

$$= \sigma\left(\left(\sum_{\Delta x, \Delta y, i} W[\Delta x, \Delta y, i, j]\; L_\ell[b, x + \Delta x, y + \Delta y, i]\right) - B[j]\right)$$

$$= \sigma\left(W[\Delta X, \Delta Y, I, j]\; L_\ell[b, x + \Delta X, y + \Delta Y, I] - B[j]\right)$$

## Why Einstein Notation?

We will need to work with tensors — arrays with more than two indeces.

For 2D CNNS the weight tensor and the data tensor each have four indeces (including the batch index of the data tensor).

For higher order tensors suppressing indeces becomes confusing.

Einstein went back to explicit index notation (Einstein notation) when working with the higher order tensors in his theory of gravitation.

## Why Einstein Notation?

Also, the indeces of tensors generally have types such as a "time index", "x coordinate", "y coordinate", "batch index", or "feature index".

Writing a matrix as $W[T, I]$ where $T$ is a time index and $I$ is a feature index makes the type of the matrix $W$ clear and clarifies the order of the indeces (disambiguates $W$ from $W^\top$).

END