Problems for GANs.

Problem 1. Conditional GANs In a conditional GAN we model a conditional distribution $\text{Pop}(y|x)$ defined by a population distribution on pairs $\langle x, y \rangle$. For conditional GANs we consider the probability distribution over triples $\langle x, y, i \rangle$ defined by

$$
\begin{aligned}
\tilde{P}_\Phi(i = 1) &= 1/2 \\
\tilde{P}_\Phi(y|x, i = 1) &= \text{pop}(y|x) \\
\tilde{P}_\Phi(y|x, i = -1) &= p_\Phi(y|x)
\end{aligned}
$$

(a) Write the conditional GAN adversarial objective function for this problem in terms of $\tilde{P}(x, y, i)$, $P_\Phi(y|x)$ and $P_\Psi(i|y, x)$.

Solution:

$$
\Phi^* = \underset{\Phi}{\text{argmax}} \ \underset{\Psi}{\min} \ E_{x,y,i \sim \tilde{P}(x,y,i)} \quad -\ln P_\Psi(i|x, y)
$$

Problem 2. GAN instability

Consider the following adversarial objective where x and y are scalars (real numbers).

$$
\max_x \min_y \ xy
$$

(a) Write the differential equation for gradient flow of this adversarial objective.

Solution:

$$
\begin{aligned}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x
\end{aligned}
$$

(b) Give a general solution to your differential equation. (Hint: It goes in a circle). You solution should have parameters allowing for any given initial value of x and y.

$$x = r_0 \sin(t + \Theta_0)$$

$$y = r_0 \cos(t + \Theta_0)$$

Problem 3. Contrastive GANs.

A GAN can be built with a "contrastive" discriminator. Rather than estimate the probability that y is from the population, the discriminator must select which of $y_1, \ldots, y_N$ is from the population.

More formally, for $N \geq 2$ let $\tilde{P}_\Phi^{(N)}$ be the distribution on tuples $\langle i, y_1, \ldots, y_N \rangle$ defined by drawing one "positive" from Pop and $N - 1$ IID negatives from P_Φ; then inserting the positive at a random position among the negatives; and returning $(i, y_1, \ldots, y_N)$ where i is the index of the positive.

$$\Phi^* = \operatorname*{argmax}_\Phi \min_\Psi \ E_{(i, y_1, \ldots, y_{N+1}) \sim \tilde{P}_\Phi^{(N)}} \ -\ln p_\Psi(i | y_1, \ldots, y_{N+1}) \quad (1)$$

Restate the above definition of $\tilde{P}_\Phi^{(N)}$ and the GAN adversarial objective for the case of conditional constrastive GANs.