Studies of covariance and nonlocality in Bohmian mechanics

Candidate: Leonardo Oleynik
Advisor: Prof. Dr. Renato Moreira Angelo

July of 2021

Group: Foundations of Quantum Mechanics

Abstract

According to the principle of general covariance, coordinates are mere event markers on which the ultimate physical laws should not rely. When we extend it to the quantum domain, we see that the resources encoding information about the state are varying quantities and, consequently, measured distinctly in each frame of reference. Of these features, nonlocality (NL) stands out since it is the genesis of seminal debates in Physics. We propose to quantify it through Bohmian mechanics, a theory in which the wave function represents a *physical field*. We distinguish between two types of NL — associated, respectively, with the non-inertiality of the frame (NL-x) and with the entanglement (NL-ψ). To identify the latter, we present a way to include the field directly into the force equations using conditional averages. We then verify that, for a system with two particles, NL-x is a necessary condition for the manifestation of NL-ψ. Moreover, we conclude that the principle of covariance employs a broader description of nonlocality within Bohmian mechanics.

Outline

Motivation

Motivation

- Principle of general covariance:
  - *The coordinates are mere event markers;*
- References:
  - In classical mechanics (MC);
  - In quantum mechanics (QM);
    1. *MQ can be consistently formulated from the point of view of a particle!*
    2. Thermodynamics, quantum gravity, entanglement detection, fundamental aspects of physics, among others.
- (Co)variance of (information) resources, e.g., entanglement, coherence and non-locality (NL);
- NL: foundational debates and the genesis of Bohmian mechanics (MB);
- *Would it be possible, through MB, to quantify aspects of NL?*

Nonlocality

Gravitational interaction in a system with 3 particles

- Three particles:
  - Environment: a, c;
  - Particle of interest: b.
- Laboratory R:

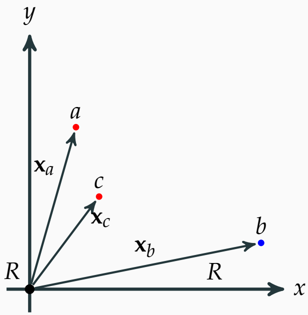
$$m_b \ddot{\mathbf{x}}_b = \mathbf{F}_{ab}(|\mathbf{x}_a - \mathbf{x}_b|) + \mathbf{F}_{cb}(|\mathbf{x}_c - \mathbf{x}_b|)$$

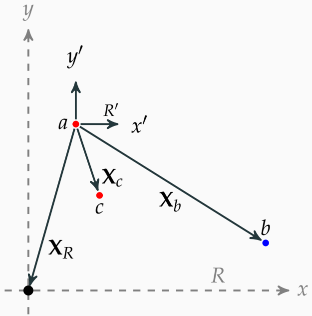
- Newtonian local causality (CLN): Let $\mathbf{F}_{ik} = \mathbf{F}(|\mathbf{x}_i - \mathbf{x}_k|)$, disturbances in $\mathbf{x}_i$ will only affect k in a time interval longer than $|\mathbf{x}_i - \mathbf{x}_k|/c$;
- No privileged frame of reference!
- R':

$$\mu \ddot{\mathbf{X}}_b = \mathbf{F}_{ab}(X_b) + \frac{m_a}{m_a + m_b}\mathbf{F}_{cb}(|\, Xc - \mathbf{X}_b|) + \frac{m_b}{m_a + m_b}\mathbf{F}_{ca}(X_c),$$

where $X_k = |\mathbf{X}_k|$ and $\mu = \frac{m_a m_b}{m_a + m_b}$;

- Fictitious force: *is not the result of the gravitational interaction between b and the environment.*





- Suppose: a =observer, b =distant star and c =Earth;
- R:

$$m_b\ddot{\mathbf{x}}_b = \mathbf{F}_{ab}(|\mathbf{x}_a - \mathbf{x}_b|) + \mathbf{F}_{cb}(|\mathbf{x}_c - \mathbf{x}_b|)$$
$$m_b\ddot{\mathbf{x}}_b \approx 0,$$

since $|\mathbf{x}_a - \mathbf{x}_b| \approx x_b \to \infty$ and $|\mathbf{x}_c - \mathbf{x}_b| \approx x_b\ to \infty;$

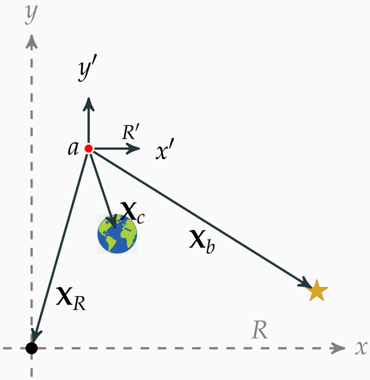
- R':

$$\mu\ddot{\mathbf{X}}_b = \mathbf{F}_{ab}(X_b) + \frac{m_a}{m_a+m_b}\mathbf{F}_{cb}(|\mathbf{X}_c - \mathbf{X}_b|) + \frac{m_b}{m_a+m_b}\mathbf{F}_{ca}(X_c)$$

$$m_a\ddot{\mathbf{X}}_b \approx \mathbf{F}_{ca}(X_c),$$

since $X_b \to \infty$ and $|\mathbf{X}_b - \mathbf{X}_c| \approx X_b \to \infty;$

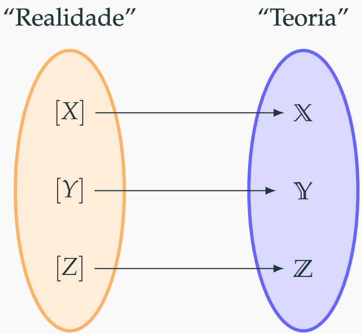
- *The responsible for this locale violation is the distant position!*



EPR Paradox: Conditions and Principles

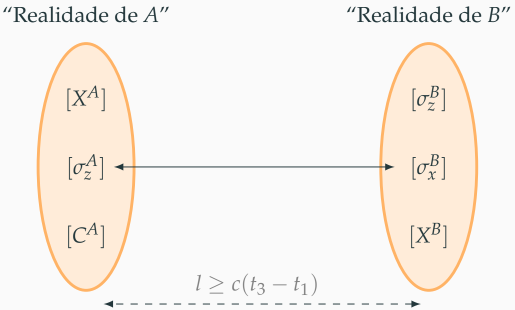
Required condition for completion

A theory is said to be complete if for every element of reality $[A]$ there is a counterpart in the theory (i.e., a physical quantity $\mathbb{A}$ that represents it);



Locality Principle

Elements of reality belonging to one system cannot be affected by measurements performed in another distant system.



EPR Paradox: Conditions and Principles (cont.)

Sufficient condition for Reality element

If it is possible to predict, with probability 1, the result of the measurement of a physical variable $\mathbb{A}$ at the instant t, we say that in t there is an element of reality $[A]$ corresponding to to the physical variable $\mathbb{A}$ with a value equal to that predicted for measurement;

$$\frac{|1\rangle + |0\rangle}{\sqrt{2}} \qquad |1\rangle \qquad |1\rangle$$

$$\nexists\, [\sigma_z] \qquad\qquad \exists\, [\sigma_z]$$

$$t_1 \qquad t_2 \qquad t_3 \qquad t$$

Previsão: ± 1 em $1/2$ vezes | Medição | Previsão: $+1$ sempre

EPR paradox: argument

- Consider a quantum system composed of two particles of spin 1/2 in the state:

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle);$$

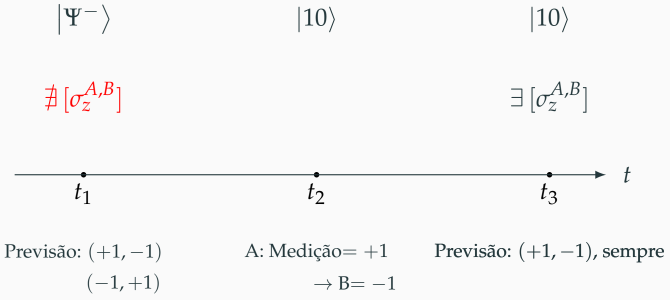
- $|0\rangle$ and $|1\rangle$ eigenvectors of σ_z: $+1$ and -1;
- Laboratories (A and B) separated by $l > c(t_3 - t_1)$;
- *Anti-correlated*: *given the spin of one of the particles, the other is inferred.*

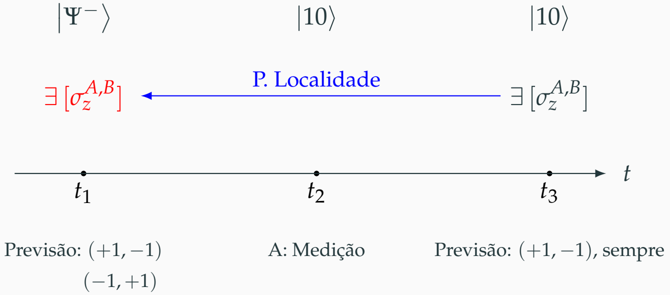
EPR paradox: argument (cont.)

Note that:

1. In t_1, we cannot predict the outcome with probability 1. Logo $\nexists[\sigma_z^{A,B}]$;
2. At t_2, Alice measures $+1$ and concludes from correlation that Bob's spin is -1. Due to the collapse, one can predict (with probability 1) the outcome of any future measurement of σ_z, so $\exists[\sigma_z^{A,B}]$;
3. Due to the distance l and P. locality, it follows that the elements already existed.

That is, either the first scenario is correct (MQ complete), or the second one (P. locality).





EPR paradox: conclusions

If the formalism is correct, then either MQ is incomplete, or the locality principle is false.

- EPR: ”MQ must be supplemented by hidden local variables”.
- Bell:
  - Local Causality Hypothesis (CL):

$$\wp(a, b \mid A, B) = \int_\lambda \wp(\lambda)\wp(a \mid A, \lambda)\wp(b \mid B, \lambda) d\lambda$$

    1. λ (factors in preparation) causally influences both outputs;
    2. Factorability;
  - Valid inequality $\Leftrightarrow$ $\wp$ satisfy CL;
  - MQ predictions are incompatible with CL;
- Bohm: ”MB completes MQ through non-local hidden variables!”

Bohmian mechanics MB

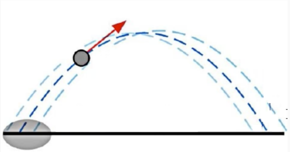
Hamilton-Jacobi formalism (HJ)

- Trajectory ensemble $\{\mathbf{x}_c(t), \varrho\}$;
- Velocity field $\mathbf{v}(\mathbf{x}, t) = \boldsymbol{\nabla} s/m$;
- Eq. HJ: $(\mathcal{V} = \mathcal{V}(\mathbf{x}, t))$

$$H(\mathbf{x}, \boldsymbol{\nabla} s, t) \equiv \partial_t s + \frac{(\boldsymbol{\nabla} s)^2}{2m} + \mathcal{V} = 0,$$

- Eq. of continuity:

$$\partial_t \varrho = -\boldsymbol{\nabla} \cdot (\varrho \mathbf{v}).$$



- Eq. from Schrödinger: $(\mathcal{V} = \mathcal{V}(\mathbf{x}, t))$

$$i\hbar \partial_t \psi(\mathbf{x}, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + \mathcal{V} \right) \psi(\mathbf{x}, t),$$

- Polar form:
  $\psi \equiv \psi(\mathbf{x}, t) = \sqrt{\rho}\, \mathrm{e}^{iS/\hbar}$,
  $\rho = |\psi(\mathbf{x}, t)|^2, \quad S = S(\mathbf{x}, t).$
- Eq. HJQ:

$$H(\mathbf{x}, \boldsymbol{\nabla} S, t) \neq \partial_t S + \frac{(\boldsymbol{\nabla} S)^2}{2m} + \mathcal{V} - frac\hbar^2 2m \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} = 0,$$

- Eq. of continuity:

$$\partial_t \rho = -\boldsymbol{\nabla} \cdot \left(\frac{\rho \boldsymbol{\nabla} S}{m} \right)$$

- Difference: *While s is the trajectory generator, S is only the phase.*

- Postulate 1: *The dynamics of a particle in a singular experiment is defined by the trajectory* $\mathbf{x}(t)$*, which moves continuously along the orientation of the wavefunction* ψ*, solution of the Schrödinger equation. The trajectory is obtained by integrating*

$$\dot{\mathbf{x}}(t) = \mathbf{v}^{[\psi]}(\mathbf{x}(t), t) \qquad e \qquad \mathbf{v}^{[\psi]}(\mathbf{x}, t) = \boldsymbol{\nabla} S/m$$

in time. The initial conditions $\mathbf{x}(t_0)$ *must be specified to fully determine the trajectory,*
or

$$\mathbf{F}^{[\psi]} \equiv \mathbf{F}^{[\psi]}(\mathbf{x}, t) = -\boldsymbol{\nabla}(Q^{[psi]} + \mathcal{V}), \qquad Q^{[\psi]} = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$$

- Postulate 2: *The starting position* $\mathbf{x}(t_0)$ *is inaccessible (cannot be known with arbitrary precision) and is distributed according to* $|\psi(\mathbf{x}, t_0)|^2$*.*

Example: free particle in one dimension

- Pure state prepared as $\psi(x,0) = \sqrt{G_{\delta_0}(x-\bar{x}_0)}e^{i\bar{p}_0/\hbar}$;
- For any t: $\psi(x,t) = \sqrt{G_\delta(x-\bar{x})}e^{iS(x,t)/\hbar}$, where $\bar{x} = \bar{x}_0 + \bar{p}t/m$ and $\delta = \delta_0\sqrt{1+(t/\tau_0)^2}$, with $\tau_0 = \frac{2m\delta_0^2}{\hbar}$
- Force field:

$$F^{[\psi]}(x,t) = -\partial_x Q^{[\psi]} = \frac{x-\bar{x}}{4m\delta^4}\hbar^2,$$

- By Newton's Second Law,

$$\ddot{x} = \frac{F^{[\psi]}(x(t),t)}{m} = \frac{x-\bar{x}}{\tau^2} \quad \text{com} \quad \tau := 2m\delta^2/\hbar = \tau_0\sqrt{1+(t/\tau_0)^2}$$

  - Particle is not "free";
  - On average, the classic result is recovered;
  - Acceleration does not encode the preparation or the ψ field. While $\lim_{(x-\bar{x})\to\infty}\ddot{x} = \infty$, $\lim_{(x-\bar{x})\to\infty}|\psi|^2 = 0$.

MB for two particles

- Polar form

$$\psi \equiv \psi(\mathbf{x}_a, \mathbf{x}_b, t) = \sqrt{\rho}\, \mathrm{e}^{iS/\hbar}, \quad \rho = |\psi(\mathbf{x}_a, \mathbf{x}_b, t)|^2 \quad S = S(\mathbf{x}_a, \mathbf{x}_b, t).$$

- *Eq. of continuity,*

$$\partial_t \rho = -\boldsymbol{\nabla} \cdot (\rho \mathbf{v}^{[\psi]}) = -\boldsymbol{\nabla}_a \cdot \left(\rho \mathbf{v}_a^{[\psi]}\right) - \boldsymbol{\nabla}_b \cdot \left(\rho \mathbf{v}_b^{[\psi]}\right)$$

  with velocity fields: $\mathbf{v}_k^{[\psi]} \equiv \boldsymbol{\nabla}_k S / m_k$ where $k = a, b$

- *Second Law for a*:

$$m_a \ddot{\mathbf{x}}_a = \mathbf{F}_a^{[\psi]}(\mathbf{x}_a, \mathbf{x}_b, t) = -\boldsymbol{\nabla}_a \left(\mathcal{V} + Q^{[\psi]}\right) \qquad \text{se} \qquad \psi \neq \psi_a(\mathbf{x}_a)\psi_b(\mathbf{x}_b)$$

Bohmian mechanics and reduced states

- Open systems: $\rho \in \mathcal{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}})$ with $\underbrace{\rho_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{A}}(\rho)}_{\textit{Interest}}$ and $\underbrace{\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho)}_{\textit{Environment}}$;
- Pure state $\rho = |\psi\rangle\langle\psi|$;
- Bohmian velocity field:

$$\mathbf{v}_b^{[\psi]} = \frac{\hbar}{m_b \rho_{\mathbf{x}_a,\mathbf{x}_b}} \mathrm{Im}\left(\boldsymbol{\nabla}_a\, \rho_{\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_a',\mathbf{x}_b}\right)_{\mathbf{x}_a'=\mathbf{x}_a},$$

with

$$\rho_{\mathbf{x}_a,\mathbf{x}_b} \equiv \langle \mathbf{x}_a, \mathbf{x}_b|\rho|\mathbf{x}_a, \mathbf{x}_b\rangle = |\psi(\mathbf{x}_a, \mathbf{x}_b)|^2, \quad \rho_{\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_a',\mathbf{x}_b'} \equiv \left\langle \mathbf{x}_a, \mathbf{x}_b \middle|\rho\middle| \mathbf{x}_a', \mathbf{x}_b'\right\rangle = \psi^*(\ xa', \mathbf{x}_b')\psi(\mathbf{x}_a, \mathbf{x}_b)$$

- Eq. Continuity:

$$\partial_t \rho_{\mathbf{x}_a,\mathbf{x}_b} = -\boldsymbol{\nabla}_a \cdot \left(\rho_{\mathbf{x}_a,\mathbf{x}_b} \mathbf{v}_a^{[\psi]}\right) - \mathit{grad}_b \cdot \left(\rho_{\mathbf{x}_a,\mathbf{x}_b} \mathbf{v}_b^{[\psi]}\right);$$

- Averaging over $\mathbf{x}_a$:

$$\partial_t \rho_{\mathbf{x}_b} = -\boldsymbol{\nabla}_b \cdot \left(\rho_{\mathbf{x}_b} \left\langle \mathbf{v}_b^{[\psi]} \right\rangle_{\mathbf{x}_b}\right) \qquad \text{com} \qquad \langle \mathbf{v}_b \rangle_{\mathbf{x}_b} := \int \rho_{\mathbf{x}_a|\mathbf{x}_b}\, \mathbf{v}_b^{[\psi]} \mathrm{d}\mathbf{x}_a.$$

- Equivalence:

$$\left\langle \mathbf{v}_b^{[\psi]} \right\rangle_{\mathbf{x}_b} = \frac{\hbar}{m_b \rho_{\mathbf{x}_b}} \mathrm{Im}(\boldsymbol{\nabla}_b \rho_{\ xb,\mathbf{x}_b'})_{\mathbf{x}_b'=\mathbf{x}_b} =: \mathbf{v}_b^{[\rho_{\mathcal{B}}]},$$

with

$$\rho_{\mathbf{x}_b} = \langle \mathbf{x}_b|\rho_{\mathcal{B}}|\mathbf{x}_b\rangle, \quad \rho_{\mathbf{x}_b,\mathbf{x}_b'} = \left\langle \mathbf{x}_b \middle|\rho_{\mathcal{B}}\middle| \mathbf{x}_b'\right\rangle.$$

Nonlocality in Bohmian dynamics

Nonlocality in Bohmian mechanics

- Review:
  - NL in MC: non-inertiality of the frame violates the hypothesis of CLN.
  - Bell's NL: state entanglement violates the CL hypothesis.
  - Bohmian Ontology: A priori existence of particles with well-defined positions x guided by the field ψ.
- New nomenclature: NL-x and NL-ψ;
- Objective: *quantify the NL-ψ through Bohmian dynamics;*
- Object in dynamics: local average of force.

Why is the NL-ψ quantified by the local average of the force?

- NL-ψ, entanglement and the physical preparation of the system;
- $\mathbf{F}^{[\psi]}$ does not encode preparation, typical force;
- $\left\langle \mathbf{F}^{[\psi]} \right\rangle$ doesn't work either: the dynamics are classic;
- Solution: conditioned averages $\left\langle \mathbf{F}^{[\psi]} \right\rangle_{x_k}$;
- Local average properties:
    1. Associates the subsystem with its respective field (shortened);
    2. Enhances the effects of NL-ψ;
    3. Displays terms associated with entanglement.

Considerations for a system with two particles a and b

- Consider: joint densities ρ_{x_a,x_b}, marginals $\rho_{x_{a(b)}}$ and conditionals $\rho_{x_a|x_b}$ and $\rho_{x_b|x_a}$.
- While $F_b^{[\psi]}(x_a, x_b)$,

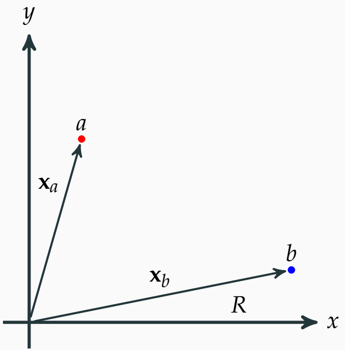
$$\left\langle F_b^{[\psi]} \right\rangle_{x_b} := \int F_b^{[\psi]} \rho_{x_a|x_b} \mathrm{d}x_a \equiv f(x_b),$$

- Rewriting it:

$$\left\langle F_b^{[\psi]} \right\rangle_{x_b} = \int F_b^{[\psi]}(\mathbf{x})\, \rho_{x_a} \mathrm{d}x_a + \; intF_b^{[\psi]}(\mathbf{x}) \left(\rho_{x_a|x_b} - \rho_{x_a}\right) \mathrm{d}x_a.$$

  This makes it explicit that the second term is *non-null* if:
  1. The variables are *dependent* on each other ($\rho_{x_a|x_b} \neq \rho_{x_a}$), i.e., if there is NL-$\psi$; and
  2. If $F_b^{[\psi]}$ depends on x_a, i.e. is NL-x.
- Proposal: *evaluate the proportionality of the second term with entanglement according to a classical and a quantum referential.*

Two free particles: analysis from a classical frame of reference R



Consider:

- State separable and prepared as $\psi(x_a, x_b, 0) = \psi_a(x_a, 0)\psi_b(x_b, 0)$, with $\psi_k(x_k, 0) = \sqrt{G_{\delta_{k,0}}(x_k - \bar{x}_{k,0})}\, e^{i\bar{p}_k/\hbar}$, where $\bar{p}_k$ is the average momentum of the kth particle;

Then:

- Each particle evolves independently: $\psi_k = \sqrt{G_{\delta_k}(x_k - \bar{x}_k)} e^{\frac{i}{\hbar}S_k}$; with densities: $\rho_{x_a} = \rho_{x_a|x_b}$ and $\rho_{x_b} = \rho_{x_b|x_a}$, i.e., without NL-$\psi$;
- Acceleration of the kth particle,

$$\ddot{x}_k = \frac{x_k - \bar{x}_k}{t_k^2}$$

with $t_k := t_{k,0}\sqrt{1 + (t/t_{k,0})^2}$ and $t_{k,0} = 2m_k\delta_{k,0}^2/\hbar$, ie, without NL-$x$

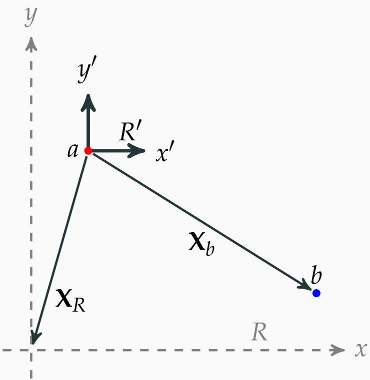
- The local average of the acceleration is innocuous:

$$\langle \ddot{x}_k \rangle_{x_k} = \int \frac{x_k - \bar{x}_k}{t_k^2} \rho_{x_j} dx_j + \int \frac{x_k - \bar{x}_k}{t_k^2} \left(\rho_{x_j|x_k} - \rho_{x_j} \right) dx_j$$
$$= \ddot{x}_k,$$

because:

1. The frame is inertial;
2. State is separable.

Two free particles: analysis from a quantum reference R'



Consider:

- Field observed by a, $\Psi(X_R, X_b) = \psi_a(-X_R)\psi_b(X_b - X_R)$,
- With densities $\varrho_{X_R,X_b} = |\Psi|^2$, $\varrho_{X_R|X_b} \neq \varrho_{X_R}$ and $\varrho_{X_b|X_R} \neq \varrho_{X_b}$.

So the

- Accelerations are:

$$\ddot{X}_R = \frac{X_R - \bar{X}_R}{t_a^2},$$

without NL-x and

$$\ddot{X}_b = \frac{X_b - \bar{X}_b}{t_b^2} - \frac{X_R - \bar{X}_R}{t_b^2} + \frac{X_R - \bar{X}_R}{t_a^2},$$

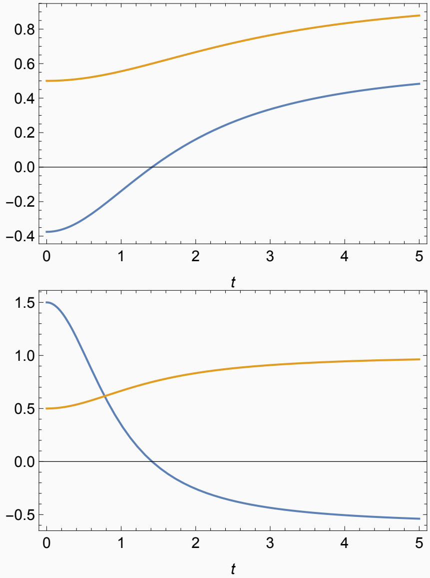
with NL-x.

- Local averages are: $\langle \ddot{X}_R \rangle_{X_R} = \ddot{X}_R$ and

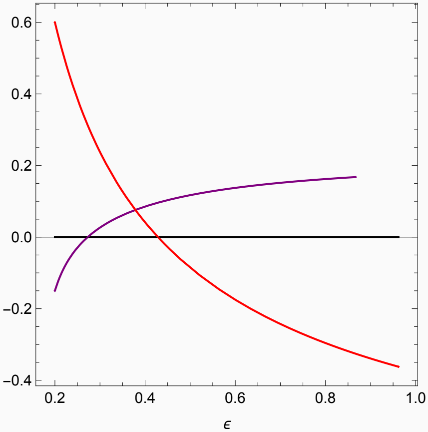
$$\langle \ddot{X}_b \rangle_{X_b} = \frac{X_b - \bar{X}_b}{t_b^2} \left\{ 1 + \underbrace{\left[\left(\frac{t_b}{t_a} \right)^2 - 1 \right] \Lambda^2}_{\eta} \right\},$$

where Λ is the statistical correlation between the variables.

Proportionality between entanglement ϵ and η



ϵ, η for $\delta_{a,0} = \delta_{b,0} = 1m$, $t_{a,0} = 2s$ and $t_{b,0} = 1s$.
Fig 1: $t_{a,0} > t_{b,0}$, Fig 2: $t_{a,0} < t_{b,0}$



- In black the accidental case: $t_{a,0} = t_{b,0}$;
- In red the case: $t_{a,0} < t_{b,0}$;
- In purple the case: $t_{a,0} > t_{b,0}$;

Conclusion

Conclusion

- Principle of general covariance
  - Quantum resources are variants by switching references;
  - NL.
- NL in MB: NL-x and NL-ψ;
  - Local averages encode ψ;
  - NL-$\psi \leftarrow$ NL-x;
  - η in $\langle \ddot{X}_b \rangle_{X_b}$ is a promising path.
- Future work
  - Enhance NL quantifier;
  - Apply it to more complex systems:
    1. Quantum frame of reference in superposition;
    2. Monogamy relationships.