ODE Homework 2 (3 %)

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Let P_n be the Legendre polynomial of degree n. Prove that $|P_n'(x)| < n^2$ and $|P_n''(x)| < n^4$ for $-1 < x < 1$.

Proof

For $n = 0$ or $n = 1$, one can check that the condition that the problem gave does not hold. We thus assume $n \geq 2$. It has been shown in [2] that $P_n(x) \leq 1$ in $[-1, 1]$. The Markov brothers' inequality[1] states that for a polynomial p with degree $\leq n$, we have

$$\max_{-1 \leq x \leq 1} |p^k(x)| \leq (\max_{-1 \leq x \leq 1} |T_n^k|)(\max_{-1 \leq x \leq 1} |p(x)|) = \frac{n^2(n^2 - 1^2)\dots(n^2 - (k-1)^2)}{1 \cdot 3 \dots (2k-1)}(\max_{-1 \leq x \leq 1} |p(x)|)$$

with the equality holding only if $p = \gamma T_n$ where $|\gamma| = 1$[1].

For the case of $|P_n''(x)|$, applying the Markov brothers' inequality on $|P_n''(x)|$ gives us

$$|P_n''(x)| \leq \frac{n^2(n^2 - 1^2)}{1 \cdot 3} \max_{-1 \leq x \leq 1} |P_n(x)| < n^2(n^2 - 1) < n^4$$

For the case of P_n', we again apply the Markov brothers' inequality to $|P_n'(x)|$. Note that $\nexists \gamma$ with $|\gamma| = 1$ such that $P_n'(x) = \gamma T_n$. Hence the equality does not hold. Thus

$$|P_n'(x)| < n^2 \max_{-1 \leq x \leq 1} |P_n(x)| = n^2$$

References

[1] A. Shadrin. Twelve proofs of the markov inequality. 2005.

[2] Gábor Szegő. *Orthogonal Polynomials*, volume 23 of *Colloquium Publications*. American Mathematical Society, 4th edition, 1939.