ODE HW3

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I. Give the solutions, where possible in terms of the Bessel functions, of the differential equations

(a) $x\frac{d^2y}{dx^2} + (x+1)^2 y = 0$,

(b) $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

Grading: [1 %]: State your method and explain why it works. [2 %]: Show your calculations.
Sketch: For (a), use the method of Frobenius, we would find that the roots of the indicial equation differ by 1. For (b), the equation is clearly a legendre equation. Use the formula for the legendre functions, then use the reduction of order to obtain the other solution.
Calculation:

(a) Start by assuming $y = \sum_{r=0}^{\infty} a_r x^{r+c}$. Plug it into the equation and obtain

$$\sum_{r=0}^{\infty} a_r(r+c)(r+c-1)x^{r+c-1} + (x^2+2x+1)\sum_{r=0}^{\infty} a_r x^{r+c} = 0$$

For easier notation let $a_{-1} = a_{-2} = 0$.

$$a_0(c)(c-1)x^{c-1} + \sum_{r=0}^{\infty} a_{r+1}(r+c+1)(r+c)x^{r+c} + \sum_{r=0}^{\infty} a_r x^{r+c} + \sum_{r=0}^{\infty} 2a_{r-1}x^{r+c} + \sum_{r=0}^{\infty} a_{r-2}x^{r+c} = 0$$

$a_0 c(c-1) = 0$

Note that if $c = 0$, this makes $(c)(c+1)a_1 + a_0 = 0$ for any a_1. This gives $a_0 = 0$ which we do not want. Thus we let $c = 1$.

(b) $P_n(x) = \sum_{r=0}^{m}(-1)^r \frac{(2n-2r)!x^{n-2r}}{2^n r!(n-r)!(n-2r)!}$

II. Determine the coefficients of the Fourier-Bessel series for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1, \\ -1 & \text{for } 1 \le x \le 2, \end{cases}$$

in terms of the Bessel function $J_0(x)$.
Grading: [1 %]: State your method and explain why it works. [2 %]: Show your calculations.
Sketch:
By [1], the coefficient C_j corresponding to $J_0(\lambda_j x)$ is $\frac{2}{a^2[J_0'(\lambda_j a)]^2}\int_0^a xJ_0(\lambda_j x)f(x)dx$.

Calculations:

$$
\begin{aligned}
C_j &= \frac{2}{2^2[J_0'(\lambda_j a)]^2}\int_0^a xJ_0(\lambda_j x)f(x)dx \\
&= \frac{1}{2[J_0'(\lambda_j a)]^2}(\int_0^1 xJ_0(\lambda_j x)dx - \int_1^2 xJ_0(\lambda_j x)dx) \\
&\text{Let } y = \lambda_j x. \\
C_j &= \frac{1}{2\lambda_j^2[J_0'(\lambda_j a)]^2}(\int_0^1 xJ_0(\lambda_j x)dx - \int_1^2 xJ_0(\lambda_j x)dx) \\
&= \frac{1}{2\lambda_j^2[J_0'(\lambda_j a)]^2}(\int_0^{\lambda_j} yJ_0(y)dy - \int_{\lambda_j}^{2\lambda_j} yJ_0(y)dy) \\
&\text{By [1], } \frac{x}{dx}(x^c J_c(x)) = x^c J_{c-1}(x) \\
C_j &= \frac{1}{2\lambda_j^2[J_0'(\lambda_j a)]^2}(\int_0^{\lambda_j} \frac{d}{dy}(yJ_1(y))dy - \int_{\lambda_j}^{2\lambda_j} \frac{d}{dy}(yJ_1(y))dy) \\
&= \frac{1}{2\lambda_j^2[J_0'(\lambda_j a)]^2}(\lambda_j J_1(\lambda_j) - (2\lambda_j J_1(2\lambda_j) - \lambda_j J_1(\lambda_j)) \\
&= \frac{1}{2\lambda_j^2[J_0'(\lambda_j a)]^2}(2\lambda_j J_1(\lambda_j) - 2\lambda_j J_1(2\lambda_j)) \\
&\text{By [1], } J_0'(x) = -J_1(x) \\
C_j &= \frac{1}{\lambda_j[J_1(2\lambda_j)]^2}(J_1(\lambda_j) - J_1(2\lambda_j))
\end{aligned}
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References

[1] A. C. King, J. Billingham, and S. R. Otto. *Differential Equations: Linear, Nonlinear, Ordinary, Partial.* Cambridge University Press, 2003.