ODE Homework 2 (3 %)

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Let P_n be the Legendre polynomial of degree n. Prove that $|P_n'(x)| < n^2$ and $|P_n''(x)| < n^4$ for $-1 < x < 1$.

Proof

It has been proven that $|P_n(x)| \leq 1$ for $x \in [-1, 1]$.
The Markov brothers' inequality states that, if P is a polynomial with degree $\leq n$, then for all nonnegative integers k we have

$$\max_{-1\leq x\leq 1} |P^{(k)}(x)| \leq \frac{n^2(n^2-(1)^2)(n^2-(2)^2)\ldots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\ldots(2k-1)} \max_{-1\leq x\leq 1} |P(x)|$$

Applying the markov brothers' inequality to $P_n'(x)$ and we obtain

$$|P_n'(x)| \leq \frac{n^2}{1} \max_{-1\leq x\leq 1} |P(x)| \leq n^2$$

for $x \in [-1, 1]$. We now prove that $|P_n'(x)$ the equality cannot be achieved in $(-1, 1)$

Applying the inequality to $|P''(x)|$ and we obtain

$$|P_n''(x)| \leq \frac{n^2(n^2(n^2-1^2))}{1\cdot 3} \max_{-1\leq x\leq 1} |P(x)| < \frac{n^2(n^2-1)}{1\cdot 1} < (n^2)(n^2) = n^4$$

for $x \in [-1, 1]$. Since $(-1, 1) \subset [-1, 1]$, the above inequality holds for $x \in (-1, 1)$ too.

Reference

G. Szegő, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, Vol. 23, 1939.