Abstract Algebra hw3

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1) (a) They are not isomorphic. In $(\mathbb{Z}/15\mathbb{Z})^\times$, we have $4^2 = 11^2 = 1$, but in $\mathbb{Z}/8\mathbb{Z}$, only $4+4=0$, hence they cannot be isomorphic.

   (b) They are isomorphic. Both are cyclic group of order 4, the generator in $\mathbb{Z}/4\mathbb{Z}$ is 1 and the generator in $\{z \in \mathbb{C}\backslash\{0\}\} : z^4 = 1$ is $e^{i(\frac{2\pi}{4})}$.

   (c) They are isomorphic. It is easy to verify that $\phi : n \mapsto 3n$ is a homomorphism. For any $3n \in 3\mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that $\phi(n) = 3n$, so ϕ is surjective. For $a, b \in \mathbb{Z}$, $a \neq b$, $\phi(a) = 3a \neq 3b = \phi(b)$, so ϕ is injective. Hence ϕ is an isomorphism.

   (d) They are not isomorphic. Note that any $x \in \mathbb{Z}$ has infinite order, while any $z \in \mathbb{C}_1 := \{z \in \mathbb{C}\backslash\{0\} : z^n = 1 \text{ for some } n \geq 1\}$ has finite order. We now verify $\mathbb{C}_1$ is a group. The identity, inverse and associativity are trivial. We prove that it is closed. Say $x, z \in \mathbb{C}_1$ with $x^a = 1$ and $z^b = 1$. then $(xz)^{ab} = (x^a)^b(z^b)^a = 1$. So it $\mathbb{C}_1$ is closed, and hence is a group.

   (e) They are isomorphic. Note that $(123)^3 = (1) = (12)^2$ and $(12)(123)(12) = (132) = (123)^{-1}$. Hence by mapping (123) to r and (12) to s we will obtain an isomorphism.

   (f) They are not isomorphic. $|S_4| = 4! = 24 \neq 8 = |D_4|$.

   (g) They are isomorphic. Note that $i^4 = 1$, $i^2 = -1 = j^2$, $ji = -ij = i^3 j$. Hence by mapping i to a, j to b we obtain an isomorphism.

   (h) They are isomorphic. Say a is the generator of G and a^m is the generator for H. Let $\phi : a \mapsto a^m$. This is clearly a homomorphism and is surjective. Now for $a^k \neq a^l$, we have $a^{mk} \neq a^{ml}$ so ϕ is also injective, hence it is an isomorphism.

2) It it obvious that φ is a bijection. We prove that it is a homomorphism. For $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $B = \begin{pmatrix} c & -d \\ c & d \end{pmatrix} \in G$, the sum of those two matrcies is $\begin{pmatrix} a+c & -(b+d) \\ a+c & b+d \end{pmatrix}$. Note $\varphi(A) + \varphi(B) = (a+c) + (b+d)i = \varphi(A+B)$, hence it is a homomorphism.

3) (a) Let $g \in G$. Let a be the generator for H. Let m_0 be the smallest positive integer such that $a^{m_0} \in N$. For any $a^n \in N$, let $n = bm_0 + r$, $0 \leq r < m_0$. Then $a^r = an - bm_0 \in N$. If $r \neq 0$ this would make r the smallest integer such that $a^r \in N$, contradicting our assumption. Hence $r = 0$ and a^{m_0} generates N. Since H is normal, let $g^{-1}ag = a^c$. Then $g^{-1}a^{m_0}g = a^{m_0 c} \in N$. It follows that any conjugation of $(a^{m_0})^b \in N$ with respect to g is in N. Hence N is normal.

   (b) Let $G = D_4 = \langle s^2 = r^4 = e, srs = r^{-1} \rangle$,
   $H = \{e, r^2, s, sr^2\} \cong K_4$, $N = \{e, s\}$. It is easy to verify $N \leq H \leq G$. Note that every element in H is its own inverse, hence by hw1 4) (c) we have H is abelian. By hw2 5) (c) we have N is normal in H. To prove that H is normal in G, it suffice to prove that

$rHr^{-1} = H = sHs^{-1}$. The latter is trivial since $s \in H$. See that $rr^2r^{-1} = r^2$, $rsr^{-1} = sr^2$, $rsr^2r^{-1} = sr$. Hence H is indeed normal in G. Now we prove that N is not normal in G. $rsr^{-1} = sr^2 \notin N$ so N is not normal in G.

4) Given any $n \in N$, $g \in G$, we have $f(g^{-1}ngn^{-1}) = e$. Hence $g^{-1}ngn^{-1} \in ker\, f \subseteq N$. Note that $g^{-1}ng = (g^{-1}ngn^{-1})n \in N$. We conclude that N is normal in G.