ODE Homework 4

Chengyu Hsieh, B13201053

I.[2 %] Comment on the difficulties that you face when trying to construct the Green's function for the boundary value problem

$$y''(x) + y(x) = f(x) \quad \text{subject to} \quad y(a) = y'(b) = 0.$$

Grading: [1 %]: State your method and explain why it works. [1 %]: Show your results.
Method:
As shown in [1], we first find the solutions to the homogeneous equation, v_1, v_2, then construct the Green function by

$$\tilde{G}(x,s) = \begin{cases} v_1(s)v_2(x), & a \le s < x \\ v_1(x)v_2(s), & x < s \le b \end{cases}$$

and finally obtaining the Green function by $G(x,s) = \frac{\tilde{G}(x)}{Wp(x)} = \frac{\tilde{G}(x)}{W}$ to ensure that $C = 1$.
Calculation:
Notice that $v_1(x) = sin(x-a)$ and $v_2(x) = cos(x-b)$ are solutions to the homogeneous equation satisfying the boundary conditions.

$$\begin{aligned} W &= v_1(x)v_2'(x) - v_1'(x)v_2(x) \\ &= -cos(x-a)cos(x-b) - sin(x-a)sin(x-b) \\ &= -cos(b-a) \end{aligned}$$

Note that if $cos(b-a) = 0$ we cannot divide by the wronskian, hence this is the difficulty we face. Suppose that $cos(b-a) \neq 0$ we have

$$G(x,s) = \begin{cases} -\frac{v_1(s)v_2(x)}{cos(b-a)}, & a \le s < x \\ -\frac{v_1(x)v_2(s)}{cos(b-a)}, & x < s \le b \end{cases}$$

II.[2 %] Write the generalized Legendre equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} y = 0,$$

as a Sturm-Liouville equation.
Grading: [1 %]: State your method and explain why it works. [1 %]: Show your results.

Method: The Sturm-Liouville equation is of the form $(p(x)y')' + q(x)y = -\lambda r(x)y$. We use our

attention to notice the answer.
Calculation:
Notice that $(1-x^2)' = -2x$, hence our equation may be rearranged into

$$((1-x^2)y')' = -(n(n+1) - \frac{m^2}{1-\mu^2})y$$

III.[2 %] Show that

$$-(xy'(x))' = \lambda xy(x),$$

is self-adjoint on the interval $(0, 1)$, with $x = 0$ a singular endpoint and $x = 1$ a regular endpoint with the condition $y(1) = 0$.
Grading: [1 %]: State your method and explain why it works. [1 %]: State your proof.
Method:
For a linear operator L to be self adjoint we must have $\langle Lf, g\rangle = \langle f, Lg\rangle$ for any f, g. By [1], if L is of the form $\frac{d}{dx}(p(x)\frac{d}{dx}) + q(x)$, it is self adjoint iff $[p(y_1y_2' - y_1'y_2)]_a^b = 0$.
Calculations:
We let $L = -\frac{d}{dx}(x)$. Note that since for any $f, g \in C^2[0, 1]$ with $f(1) = g(1) = 0$, $[x(fg' - f'g)]_0^1 = 1(f(1)g'(1) - f'(1)g(1)) = 0$, we have $-L$ is self adjoint. Now for any $f, g \in C^2[0, 1]$, $\langle Lf, g\rangle = -\langle -Lf, g\rangle = -\langle f, -Lg\rangle = \langle f, Lg\rangle$, hence L is self adjoint.

References

[1] A. C. King, J. Billingham, and S. R. Otto. *Differential Equations: Linear, Nonlinear, Ordinary, Partial.* Cambridge University Press, 2003.