Analysis HW3

(1) Let L be a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$. Consider $B(L,\epsilon)$ for any $\epsilon > 0$. Then $\exists n \geq m$ such that $d(x^{(n)}, L) < \epsilon \Rightarrow x^{(n)} \in B(L,\epsilon) \cap S$. So $B(L,\epsilon) \cap S \neq \varnothing$ and L is an adherent point of S.
Now consider the sequence $x_1 = 1, x_i = 0$ for $i \geq 2$. Then $\overline{S} = \{1, 0\}$ Set $L = 1$. For $\epsilon = \frac{1}{2} > 0$, $N = 2$, see that $\forall n \geq N$ we have

$$d(x^{(n)}, L) = d(0,1) = 1 \not< \frac{1}{2}$$

. Hence L is not a limit point of this sequence and the converse is false.

(2) (a) **Reflexive:**
For some Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, $\lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} 0 = 0$. Hence $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n$.
Symmetry:
For some Cauchy sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in X such that $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} y_n$, $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n)$. Hence $\mathrm{LIM}_{n\to\infty} y_n = \mathrm{LIM}_{n\to\infty} x_n$.
Transitive:
For some Cauchy sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$ in X such that $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} y_n$ and $\mathrm{LIM}_{n\to\infty} y_n = \mathrm{LIM}_{n\to\infty} z_n$, note that $0 \leq \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty}(d(x_n, y_n) + d(y_n, z_n)) = 0$ by the triangle inequality. Hence $\lim_{x\to\infty} d(x_n, z_n) = 0$ and $\mathrm{LIM}_{x\to\infty} x_n = \mathrm{LIM}_{n\to\infty} z_n$.
We conclude that the equality relation of the formal limit is an equivalence relation.

(b) We first verify that the limit exists. Given two Cauchy sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in X, by definition we can find $m > n$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$ and $d(y_n, y_m) < \frac{\epsilon}{2}$. Then, by the triangle inequality we obtain $d(x_m, y_m) - d(x_n, y_n) \leq d(x_n, x_m) + d(y_n, y_m) < \epsilon$. Thus $(d(x_n, y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ with the usual metric, which is complete, so $\lim_{n\to\infty} d(x_n, y_n)$ exists.
Next we show that this distance does not depend on the choice of representatives. Let $(x_n)_{n=1}^{\infty}$, $(x'_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in X such that $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x'_n$. Then $\lim_{n\to\infty} d(x'_n, y_n) \leq \lim_{n\to\infty}(d(x'_n, x) + d(x_n, y_n)) = \lim_{n\to\infty} d(x_n, y_n)$. But also, $\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty}(d(x_n, x'_n) + d(x'_n, y_n)) = \lim_{n\to\infty} d(x'_n, y_n)$. Hence we have $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n)$
$\Rightarrow d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x'_n, \mathrm{LIM}_{n\to\infty} y_n)$.
Now we show that $d_{\overline{X}}$ is a metric.
For any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_n) = \lim_{n\to\infty} d(x_n, x_n) = 0$.
For any Cauchy sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in X such that $\mathrm{LIM}_{n\to\infty}(x_n) \neq \mathrm{LIM}_{n\to\infty}(y_n)$, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) = \lim_{n\to\infty} d(x_n, y_n) > 0$.

For any Cauchy sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in X, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n) = \lim_{n\to\infty} d(x_n, y_n)$ $= \lim_{n\to\infty} d(y_n, x_n) = d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}y_n, \mathrm{LIM}_{n\to\infty}x_n)$

(c)

(d)

(e)

(f)

(3) (a) $\partial(A\cup B) \supseteq \partial A \cup \partial B$ is trivial. We prove that $\partial(A\cup B) \subseteq \partial A\cup\partial B$. Suppose $\exists x \in \partial(A\cup B)$ such that $x \notin \partial A \cup \partial B$. $\forall r > 0$, $B(x, r) \cap (A \cup B) \neq \varnothing$. Also, $\overline{A} \cap \overline{B} = \varnothing$, so for a given r, exactly one of $B(x, r) \cap A \neq \varnothing$, $B(x, r) \cap B \neq \varnothing$ must be true. Since $x \notin \partial A$, $\exists r_1 > 0$ such that $B(x, r_1) \cap A = \varnothing$. Then $B(x, r_1) \cap B \neq \varnothing$. However, $B(x, r_1) \subseteq B(x, r)$ for any $r \geq r_1$. Thus $B(x, r) \cap B \neq \varnothing$ and $B(x, r) \cap A = \varnothing$ for any $r \geq r_1$. Thus $\partial(A\cup B) \subseteq \partial A\cup\partial B$ and so $\partial(A\cup B) = \partial A\cup\partial B$. Now note that $x \notin \partial B$, so $\exists 0 < r_2 < r_1$ such that $B(x, r)\cap B = \varnothing$. But then $B(x, r_2)\cap A \neq \varnothing$ and $B(x, r_1)\cap A \supseteq B(x, r_2)\cap A \neq \varnothing$, arriving at a contradiction. Hence $\partial(A \cup B) \subseteq \partial A \cup \partial B$ and $\partial(A \cup B) = \partial A \cup \partial B$.

(b)

(4)

(5)

(6)