Analysis Intro hw2

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(1) (i) For any $x, y \in X$,

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq \frac{0}{1+0} = 0$$

with the equality only holding when $x = y$.

(ii) For any $x, y \in X$,

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$$

(iii) For any $x, y, z \in X$, notice that

$$\begin{aligned}
&(1 - \frac{1}{1+d(x,y)} - \frac{1}{1-d(y,z)}) + \frac{1}{1+d(x,y)+d(y,z)} \\
&\geq (1 - \frac{1}{1+d(x,y)} - \frac{1}{1-d(y,z)}) + \frac{1}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)} \\
&= \frac{d(x,y)d(y,z)}{(1+d(x,y)(1+d(y,z)))} \geq 0
\end{aligned}$$

Hence $(1 - \frac{1}{1+d(x,y)} - \frac{1}{1-d(y,z)}) \geq -\frac{1}{1+d(x,y)+d(y,z)}$

$$\begin{aligned}
d'(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\
&= 1 - \frac{1}{1+d(x,y)} \\
&\leq 1 - \frac{1}{1+d(x,y)+d(y,z)} \\
&\leq 1 + 1 - \frac{1}{1+d(x,y)} - \frac{1}{1-d(y,z)} \\
&= (1 + \frac{1}{1+d(x,y)} + (1 + \frac{1}{1+d(y,z)})) = d'(x,y) + d'(y,z)
\end{aligned}$$

Thus d' is a metric on X.

(2) (a) Since B is clearly a subset of C and $\overline{B} = B \cup \partial B$, it suffice to prove that $\partial B \subseteq C$. Assume the opposite, that is, suppose $\exists x \in \partial B$ such that $x \notin C$. Since $x \notin C$, we have $d(x, x_0) > r$. Let $r_0 := d(x, x_0) - r > 0$. Since $x \in \partial B$, $B(x, r_0) \cap B \neq \varnothing$. Let $y \in B(x, r_0) \cap B$. Then $d(y, x_0) < d(x, x_0) - r$ and $d(x_0, y) < r$. Adding the two equations together gives us $d(x_0, y) + d(y, x) < d(x, x_0)$, which contradicts the triangle inequality. Thus $\partial B \subseteq C$ and $\overline{B} \subseteq C$.

(b) Consider the metric space $(\mathbb{R}, d)$ where d is the discrete metric, and let $x_0 = 0$, $r = 1$. Since $d(0, x) = 1$ for any $x \neq 0$, we have $B = \{0\}$ and $C = \mathbb{R}$. Now see that for any $0 < r < 1$ and $x \neq 0$, we have $d(x, 0) = 1 \not< r$, thus $B(x, r) \cap B = \varnothing$ and $B(0, r) \cap \mathbb{R} \backslash B = \varnothing$. Hence $\partial B = \varnothing$ and $\overline{B} = 0 \neq \mathbb{R} = C$.

(3) (a) **Sufficiency:**
Suppose E is open in (X, d_2). Then $\text{int}(E) = E$ in (X, d_2). Thus for any $x \in E$, $\exists r > 0$ such that $B_{(X,d_2)}(x, C_2 r) \subseteq E$. Now see that since $C_2 d_2(x, y) \geq d_1(x, y)$, we have $B_{(X,d_1)}(x, r) \subseteq B_{(X,d_2)}(x, C_2 r) \subseteq E$ for any $x \in E$. Hence $\text{int}(E) = E \; in (X, d_1)$, and so E is open is (X, d_1).
Necessity:
Suppose E is open in (X, d_1). Note that $\frac{1}{C_1} d_1(x, y) \geq d_2(x, y)$. The rest of the proof is analogous to that of the sufficiency section.

(b) **Sufficiency:**
Assume E is closed in (X, d_2). Then $X \backslash E$ is open. By (a) we know that $X \backslash E$ is also open in (X, d_1), therefore E is closed in (X, d_1).
Necessity: The proof is analogous to that of the sufficiency section.

(c) Consider $(\mathbb{R}, d(x, y) := |x - y|)$ and $(\mathbb{R}, d'(x, y) := \min(1, |x - y|))$. Note that d is unbounded while d' is bounded, so they clearly are not Lipschitz equivalent. Now we prove that they are topologically equivalent.
For any $U \subset \mathbb{R}$ that is open in $(\mathbb{R}, d)$:
$\forall x \in U$, $\exists r > 0$ such that $B_d(x, r) \subseteq U$. Note we may choose $r < 1$. Then $B'_d(x, r) = B_d(x, r) \subset U$.
Following similar arguments we see that any subset open in $(\mathbb{R}, d')$ is also open in $(\mathbb{R}, d)$. Hence d and d' are topologically equivalent metrics.

(4) Let $L_C(x) := Cx$ for $x \in \mathbb{R}^n$, where $C \in \mathcal{M}_n$. Then L_C is linear and has the same rank as C.

Claim 1. $N(A - B) = \mathbb{R}^n \iff A = B$

Proof. Assume $N(A - B) = \mathbb{R}^n$. Then $(A - B)x = 0 \forall x \in \mathbb{R}^N$. Thus $Ax = Bx$ for any x and hence $A = B$. Now assume $A = B$. Then $A - B = 0$ and so $(A - B)x = 0 \forall x \in \mathbb{R}^n$. Hence $N(A - B) = \mathbb{R}^n$.

(i) By Claim 1, $\rho(A, B) = rank(A - B) = n - nullity(A - B) \geq 0$ where the equality only holds when $A = B$.

(ii) $\rho(A, B) = rank(A - B) = rank(B - A) = \rho(B, A)$.

(iii) Note that $rank(A + B) \leq rank(A) + rank(B)$. Hence $\rho(A, C) = rank(A - B + B - C) \leq rank(A - B) + rank(B - C) = \rho(A, B) + \rho(B, C)$.

Hence ρ is a metric on $\mathcal{M}_n$.

(5) (a) $\partial E = X \backslash (int(E) \cup ext(E))$, thus $X \backslash \partial E = int(E) \cup ext(E)$. Suppose that $\exists x \in \partial(\partial E)$ such that $x \notin \partial E$. Since $x \notin \partial E$, $x \in int(E)$ or $x \in ext(E)$. If $x \in int(E)$, then $\exists r > 0$ such that $B(x, r) \subseteq E$. But $x \in \partial(\partial E)$ so $\varnothing \neq B(x, r) \cap ext(E) \subseteq B(x, r) \cap X \backslash E$, arriving at a contradiction. If $x \in ext(E)$, we arrive at a contradiction in a similar manner. Thus $\partial(\partial E) \subseteq \partial E$ and ∂E is closed.

(b) Note that $\forall x \in \partial E$, $r > 0$, we have $(B(x,r) \cap E \neq \varnothing$ and $B(x,r) \cap X\backslash E \neq \varnothing) \iff (B(x,r) \cap X\backslash(X\backslash E) \neq \varnothing$ and $B(x,r) \cap X\backslash E \neq \varnothing$. Hence $\partial E = \partial(X\backslash E)$. Now see that $\overline{E} = int(E) \cup \partial E$ and $\overline{X\backslash E} = ext(E) \cup \partial(X\backslash E) = ext(E) \cup \partial E$. From this we obtain $\overline{E} \cap \overline{X\backslash E} = \partial E$.

(c) E is closed, so $\overline{E} = E$. E is open, so $X\backslash E$ is closed and $\overline{E\backslash X} = E\backslash X$. By (b), we have $\partial E = \overline{E} \cap \overline{X\backslash E} = E \cap (X\backslash E) = \varnothing$.

(d) We work with the metric $d = |x - y|$. Consider $S = \{1\}$. Then $\partial S = \{1\} \cap ([1, \infty] \cup [-\infty, 1]) = 1$. $\partial(\partial S) = \partial(\{1\}) = \{1\} \neq \varnothing$.

(6) (a) Clearly, $A \subseteq S \subseteq T$. It suffice to prove that $\overline{S} \subseteq \overline{A}$. Suppose $\exists x \in \overline{S}$ such that $x \notin \overline{A}$. Since $S \subseteq \overline{A}$, we must have $x \in \partial S$. By definition, for any $r > 0$, $B(x,r) \cap S \neq \varnothing \Rightarrow B(x,r) \cap A \neq \varnothing$. Hence $x \in \partial A \subseteq \overline{A}$, contradicting our assumption. Thus $\overline{S} \subseteq \overline{A}$ and $A \subseteq S \subseteq T \subseteq \overline{S} \subseteq \overline{A}$ and A is dense in T.

(b) Suppose that $B \not\subseteq \overline{A \cap B}$. Then $\exists x_0 \in B$, $r_0 > 0$ such that $B(x_0, r_0) \cap A \cap B = \varnothing$. Since $B \subseteq S \subseteq \overline{A}$, for any $x \in B$, either $x \in int(A)$ or $x \in \partial A$. Note $x_0 \in B(x_0, r_0) \cap B$. By our assumption, we must have $x_0 \notin A$ and so $x_0 \in \partial A$. Since B is open in S, $\exists r_1 > 0$ such that $B(x_0, r_1) \cap S \subseteq B$. Recall that $x_0 \in \partial A$, so $\exists x_1 \in B(x_0, r_1) \cap A$. Then $x_1 \in A \subseteq S \Rightarrow x_1 \in S\backslash B$. But this means that $B(x_0, r_1) \cap S\backslash B \neq \varnothing$, contradicting the statement that $B(x_0, r_1) \cap S \subseteq B$. Hence $B \subseteq \overline{A \cap B}$.

(c) $A \cap B \subseteq S$ is trivial, so it suffice to show that $S \subseteq \overline{A \cap B}$. Note that by (b) we have $B \subseteq \overline{A \cap B}$, so it remains to show $S\backslash B \subseteq \overline{A \cap B}$. Since $S \subseteq \overline{A}$, for any $x_0 \in S\backslash B$, either $x \in int(A)$ or $x \in \partial A$. Also, $S\backslash B \subseteq \overline{B}$, thus $S\backslash B \subseteq \partial B$.
If $x_0 \in int(A)$, then combined with $x_0 \in \partial B$ we have for any $r > 0$

$$B(x_0, r) \cap A \cap B \neq \varnothing$$

and so $x_0 \in \overline{A \cap B}$.
If $x_0 \in \partial A$, then for any $r_0 > 0 \, \exists x_1 \in B(x_0, r_0) \cap A$.
If $x_1 \in B$ then $x_0 \in \overline{A \cap B}$.
If $x_1 \in \partial B$ then $\forall r_1 > 0$ we have $B(x_1, r_1) \cap B \neq \varnothing$ and so

$$B(x_1, r_1) \cap A \cap B \neq \varnothing$$

Let $r_1 < r_0 - d(x_0, x_1)$. Then $B(x_1, r_1) \subseteq B(x_0, r_0)$. Hence $B(x_0, r_0) \cap A \cap B \neq \varnothing$ for any $r_0 > 0 \Rightarrow x_0 \in \overline{A \cap B}$.
We conclude that $A \cap B \subseteq S \subseteq \overline{A \cap B}$, and so $A \cap B$ is dense in S.