ESERCITAZIONE N.1: Integrali
Analisi 1, II modulo
Corso di laurea in Ingegneria Informatica
Universitá degli Studi di Roma "La Sapienza"
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1) Calcolare (integrando per parti) i seguenti integrali:

$$(i) \quad \int \frac{\log x}{(1+x)^2} dx$$

$$(ii) \quad \int e^x \sin(2x) dx$$

$$(iii) \quad \int x^3 e^{-x} dx$$

$$(iv) \quad \int x (\arctan x)^2 dx$$

2) Calcolare i seguenti integrali di funzioni razionali:

$$(i) \quad \int \frac{x^2+1}{x^2+x-2} dx$$

$$(ii) \quad \int \frac{x}{x^2+2x+2} dx$$

$$(iii) \quad \int \frac{12x}{(1+2x)^2} dx$$

$$(iv) \quad \int \frac{x+2}{x^2+4} \frac{dx}{x}$$

$$(v) \quad \int \frac{x}{(x^2+1)^2(x-1)^2} dx$$

$$(vi) \quad \int \frac{8x}{4x^2-8x+7} dx$$

3) Calcolare i seguenti integrali:

$$(i) \quad \int_1^2 \frac{1}{x^2} e^{-\frac{1}{x}} dx$$

$$(ii) \quad \int_{-1}^0 \frac{e^x+2}{e^{2x}+4} dx$$

$$(iii) \quad \int \sin(2x) \log(\sin x) dx$$

ricordando che $\sin(2x) = 2 \sin x \cos x$.

4) Calcolare i seguenti integrali:

$$(i) \quad \int \frac{x+1}{x(x-1)(x+2)} dx \qquad (ii) \quad \int \frac{dx}{x^2(x-1)} \qquad (iii) \quad \int \frac{dx}{x(x+1)}$$

$$(iv) \quad \int \frac{dx}{x^2(x+1)^2} \qquad (v) \quad \int \frac{dx}{(x+1)(x^2+1)} \qquad (vi) \quad \int \frac{dx}{x^2(x^2+x+1)}$$

$$(vii) \quad \int \frac{dx}{2x^2-1} \qquad (viii) \quad \int \frac{dx}{x(x+3)} \qquad (ix) \quad \int \frac{x^2+x}{x^2+x-2} dx$$

$$(x) \quad \int \frac{x^3}{x^2-3x+2} dx \qquad (xi) \quad \int \frac{dx}{(x^2+1)(x^2+2)}$$

5) Calcolare i seguenti integrali:

$$(i)\quad \int x^2 \ln x dx \qquad (ii)\quad \int x(\ln x)^2 dx$$

$$(iii)\quad \int \frac{(x+2)}{4}\text{sen}\frac{x}{2}dx \qquad (iv)\quad \int xe^x dx \qquad (v)\quad \int e^x\text{sen}(3x)dx$$

$$(vi)\quad \int x\arctan(2x)dx \qquad (vii)\int \frac{\ln x}{x^5}dx$$

6) Calcolare (con il metodo di sostituzione, in alcuni casi la sostituzione é esplicitamente suggerita, in altri no...) i seguenti integrali:

$$(i)\quad \int \frac{\text{sen}x\cos x}{1+\text{sen}x}dx \qquad [t=\text{sen}x]$$

$$(ii)\quad \int \frac{1+e^x}{e^{2x}+1}dx$$

$$(iii)\quad \int \frac{\ln x}{x(1+\ln x)}dx$$

$$(iv)\quad \int \frac{dx}{2\sqrt{x+1}+x+2} \qquad [t=\sqrt{x+1}]$$

$$(v)\quad \int \frac{x}{\sqrt{1+x^2}}dx \qquad [t=1+x^2]$$

$$(vi)\quad \int \frac{dx}{1+\text{sen}x} \qquad [t=\text{tg}\frac{x}{2}]$$

$$(vii)\quad \int \frac{dx}{1+\cos^2 x}$$

Soluzioni

1) (i) Si ha:

$$\int \frac{\log x}{(1+x)^2}dx = -\log x\frac{1}{x+1} + \int \frac{dx}{x(x+1)} = -\log x\frac{1}{x+1} + \int \frac{dx}{x} - \int \frac{dx}{x+1} =$$
$$= -\log x\frac{1}{x+1} + \log x - \log(x+1) + C\ .$$

(ii) Si ha:

$$\int e^x\sin(2x)dx = e^x\sin(2x) - 2\int e^x\cos(2x)dx = e^x\sin(2x) - 2e^x\cos(2x) - 4\int e^x\sin(2x)dx$$

quindi sommando l'ultimo termine del membro destro con il membro sinistro si ha:

$$5\int e^x\sin(2x) = e^x(\sin(2x) - 2\cos(2x))$$

cioe' anche

$$\int e^x\sin(2x) = \frac{1}{5}e^x(\sin(2x) - 2\cos(2x)) + C\ .$$

(iii) Si ha:

$$\int x^3e^{-x}dx = -e^{-x}x^3 + 3\int e^{-x}x^2dx = -e^{-x}x^3 - 3e^{-x}x^2 + 6\int e^{-x}xdx =$$
$$= -e^{-x}x^3 - 3e^{-x}x^2 - 6e^{-x}x + 6\int e^{-x}dx = -e^{-x}(x^3+3x^2+6x+6) + C\ .$$

(iv) Si ha:

$$\int x(\arctan x)^2dx = (\arctan x)^2\frac{x^2}{2} - \int \frac{x^2}{2}2\arctan x\frac{1}{x^2+1}dx =$$
$$= (\arctan x)^2\frac{x^2}{2} - \int\left(1-\frac{1}{x^2+1}\right)\arctan x dx =$$
$$= (\arctan x)^2\frac{x^2}{2} - \int \arctan x dx + \int\frac{1}{x^2+1}\arctan x dx =$$

$$= (\text{arctan}x)^2 \frac{x^2}{2} - x\text{arctan}x + \int \frac{x}{x^2+1} dx + \int \text{arctan}x d(\text{arctan}x) =$$
$$= (\text{arctan}x)^2 \frac{x^2}{2} - x\text{arctan}x + \frac{1}{2}\log(x^2+1) + \frac{(\text{arctan}x)^2}{2} + C \ .$$

2) (i) Si ha:

$$\int \frac{x^2+1}{x^2+x-2} dx = \int \frac{x^2+x-2-x+3}{x^2+x-2} dx = \int \left(1 - \frac{x-3}{x^2+x-2}\right) dx =$$
$$= x - \int \frac{x-3}{(x-1)(x+2)} dx = x + \frac{2}{3}\int \frac{dx}{x-1} - \frac{5}{3}\int \frac{dx}{x+2} =$$
$$= x + \frac{2}{3}\log|x-1| - \frac{5}{3}\log|x+2| + C \ .$$

(ii) Si ha:

$$\int \frac{x}{x^2+2x+2} dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+2} dx = \frac{1}{2}\log(x^2+2x+2) - \int \frac{1}{(x+1)^2+1} dx =$$
$$= \frac{1}{2}\log(x^2+2x+2) - \arctan(x+1) + C \ .$$

(iii) Si ha:

$$\int \frac{12x}{(1+2x)^2} dx = \int \frac{6}{(1+2x)} dx - \int \frac{6}{(1+2x)^2} dx = 3\int \frac{d(1+2x)}{(1+2x)} - 3\int \frac{d(1+2x)}{(1+2x)^2} =$$
$$= 3\log|1+2x| + \frac{3}{1+2x} + C \ .$$

(iv) Si ha:

$$\int \frac{x+2}{x^2+4}\frac{dx}{x} = \frac{1}{2}\int \frac{dx}{x} + \int \frac{-\frac{1}{2}x+1}{x^2+4} dx = \frac{1}{2}\log|x| - \frac{1}{4}\int \frac{2x}{x^2+4} dx + \frac{1}{4}\int \frac{1}{(\frac{x}{2})^2+1} dx =$$
$$= \frac{1}{2}\log|x| - \frac{1}{4}\log(x^2+4) + \frac{1}{2}\text{arctan}(\frac{x}{2}) + C \ .$$

(v) Si ha:

$$\int \frac{x}{(x^2+1)^2(x-1)^2} dx = \int \frac{-\frac{1}{4}}{x-1} dx + \int \frac{\frac{1}{4}(x-1)}{x^2+1} dx + \int \frac{d}{dx}\left(\frac{-\frac{1}{2}x^2+\frac{1}{4}x-\frac{1}{4}}{(x^2+1)(x-1)}\right) dx =$$
$$= -\frac{1}{4}\log|x-1| + \frac{1}{8}\log(x^2+1) - \frac{1}{4}\text{arctan}(x) + \frac{-\frac{1}{2}x^2+\frac{1}{4}x-\frac{1}{4}}{(x^2+1)(x-1)} + C \ .$$

(vi) Il denominatore di é un polinomio di secondo grado con radici complesse si ha

$$\int \frac{8x}{4x^2-8x+7} dx = \int \frac{8x-8+8}{4x^2-8x+7} dx = \log(4x^2-8x+7) + 8\int \frac{1}{4x^2-8x+7} dx =$$
$$= \log(4x^2-8x+7) + 8\int \frac{1}{(2x-2)^2+3} dx = \log(4x^2-8x+7) + \frac{8}{3}\int \frac{1}{\left(\frac{2x-2}{\sqrt{3}}\right)^2+1} dx =$$
$$= \log(4x^2-8x+7) + \frac{8\sqrt{3}}{3\times 2}\int \frac{1}{\left(\frac{2x-2}{\sqrt{3}}\right)^2+1} d\left(\frac{2x-2}{\sqrt{3}}\right) =$$
$$= \log(4x^2-8x+7) + \frac{4}{\sqrt{3}}\text{arctan}\left(\frac{2x-2}{\sqrt{3}}\right) + C$$

3) (i) Usando il metodo della sostituzione e ponendo $s := \frac{1}{x}$, si ha $ds = -\frac{1}{x^2}dx$. L'integrale indefinito associato diviene $\int -e^{-s}ds = e^{-s}$. Quindi si ha $(i) = e^{-\frac{1}{x}}\Big|_1^2 = e^{-\frac{1}{2}} - e^{-1}$.

(ii) Usando il metodo della sostituzione e ponendo $s := e^x$, si ha $ds = e^x dx$, cioé $dx = \frac{ds}{s}$. L'integrale indefinito associato diviene l'integrale di una funzione razionale $\int \frac{s+2}{s(s^2+4)}ds$, poiché $\frac{s+2}{s(s^2+4)} = \frac{\frac{1}{2}}{s} + \frac{\frac{-1}{2}s+1}{s^2+4}$ si ha:

$$\int \frac{s+2}{s(s^2+4)} ds = \int \frac{\frac{1}{2}}{s} + \frac{\frac{-1}{2}s+1}{s^2+4} ds = \frac{1}{2}\log s - \frac{1}{4}\int \frac{2s}{s^2+4} ds + \frac{1}{4}\int \frac{1}{\frac{s^2}{4}+1} ds =$$

$$= \frac{1}{2}\log s - \frac{1}{4}\log(s^2+4) + \frac{1}{4}\int \frac{1}{(\frac{s}{2})^2+1}ds = \frac{1}{2}\log s - \frac{1}{4}\log(s^2+4) + \frac{1}{2}\arctan\left(\frac{s}{2}\right) + C\ .$$

Quindi si ha $(iii) = \frac{1}{2}x - \frac{1}{4}\log(e^{2x}+4) + \frac{1}{2}\arctan\left(\frac{e^x}{2}\right)\Big|_{-1}^{0} = -\frac{1}{2} - \frac{1}{4}\log\left(\frac{5}{e^{-2}+4}\right) + \frac{1}{2}\arctan\left(\frac{1}{2}\right) - \frac{1}{2}\arctan\left(\frac{e^{-1}}{2}\right)$.

(iii) Usando il metodo della sostituzione e ponendo $s := \sin x$, si ha $ds = \cos x dx$. L'integrale diviene $\int 2s\log s ds$ che posso facilmente integrare per parti:

$$\int 2s\log s ds = s^2\log s - \int s^2\frac{1}{s}ds = s^2\log s - \int s ds = s^2(\log s - \frac{1}{2}) + C$$

quindi si ha $(iv) = (\sin x)^2(\log(\sin x) - \frac{1}{2}) + C$.

4) (i) $-\frac{1}{2}\ln|x| + \frac{2}{3}\ln|x-1| - \frac{1}{6}\ln|x+2| + C$

(ii) $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$

(iii) $\ln|x| - \ln|x+1| + C$

(iv) $-2\ln|x| - \frac{1}{x} + 2\ln|x+1| - \frac{1}{x+1} + C$

(v) $\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan x + C$

(vi) $-\ln|x| - \frac{1}{x} + \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$

(vii) $\frac{1}{2\sqrt{2}}\ln\left|\frac{x-\frac{1}{\sqrt{2}}}{x+\frac{1}{\sqrt{2}}}\right| + C$

(viii) $\frac{1}{3}\ln\left|\frac{x}{x+3}\right| + C$

(ix) $x + \frac{2}{3}\ln\left|\frac{x-1}{x+2}\right| + C$

(x) $\frac{x^2}{2} + 3x - \ln|x-1| + 8\ln|x-2| + C$

(xi) $\arctan x - \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$

5) (i) $\frac{x^3}{9}(3\ln x - 1) + C$

(ii) $\frac{x^2}{4}(1 + 2(\ln x)^2 - 2\ln x) + C$

(iii) $-\frac{(x+2)}{2}\cos\frac{x}{2} + \mathrm{sen}\frac{x}{2}$

(iv) $e^x(x-1) + C$

(v) $-\frac{e^x}{10}(3\cos(3x) - \mathrm{sen}(3x)) + C$

(vi) $\left(\frac{x^2}{2} + \frac{1}{8}\right)\arctan(2x) - \frac{1}{4}x + C$

(vii) $-\frac{1}{16x^4}(4\ln x + 1) + C$

6) (i) $\mathrm{sen}x - \ln|1 + \mathrm{sen}x| + C$

(ii) $x - \frac{1}{2}\ln(1+e^{2x}) + \arctan e^x + C$

(iii) $\ln\left|\frac{x}{1+\ln x}\right| + C$

(iv) $2\ln(1+\sqrt{x+1}) + \frac{2}{1+\sqrt{x+1}} + C$

(v) $\sqrt{1+x^2} + C$

(vi) $-\frac{2}{1+\mathrm{tg}\frac{x}{2}} + C$

(vii) $\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\mathrm{tg}x\right) + C$