Lecture Notes in Statistical and Mathematical Methods for Artificial Intelligence 91255

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Contents

1 Numerical Computation and Finite Numbers

Without going into details, we define *numerical method* a mathematical tool which can be implemented on a computer and designed to solve numerical problems. The implementation of a numerical method is called a *numerical algorithm*, or simply *algorithm*.

When working with numerical methods and algorithms we go through approximations and errors which we can classify in the following way:

- *measure errors*, caused by the measuring instrument;
- *algorithmic errors*, caused by the propagation of rounding errors of each operation when the algorithm is executed on a computer;
- *truncation errors*, caused by the truncation of an infinite procedure to a finite one (e.g. the approximation of an infinite series with a finite sum);
- *inherent errors*, caused by the finite representation of the data of a problem.

How can we measure the whole error produced by the algorithm execution?

Definition 1.1. Let $\tilde{x}$ be an approximation of a datum x. We define the *absolute error* as

$$E_x = |x - \tilde{x}|$$

and the *relative error* as

$$R_x = \left| \frac{x - \tilde{x}}{x} \right|, \ x \neq 0$$

Definition 1.2. *Accuracy* is the number of correct significant digits in approximating some quantity. Accuracy means that the errors are small relative to a fixed tolerance. This quantity is not limited by the machine *precision*, which is the number of digits a number is expressed with.

Definition 1.3. The number $\tilde{x}$ is said to approximate x to d significant digits if d is the largest non-negative integer such that

$$\left| \frac{x - \tilde{x}}{x} \right| < \frac{10^{1-d}}{2}$$

Example 1.1. Let $x = 3.141592$, $\tilde{x} = 3.14$. Then we have

$$\left|\frac{x - \tilde{x}}{x}\right| = 0.000507 < \frac{10^{1-\mathbf{3}}}{2} = 0.5 \cdot 10^{-2}$$

Then we say that $\tilde{x}$ approximates x to $d = 3$ significant digits.

Suppose we want to compute the value of a function $f : \mathbb{R} \to \mathbb{R}$, where:

- x is the true input value
- $f(x)$ is the desired (true) output
- $\tilde{x}$ is the approximate input
- $\tilde{f}$ is the approximate computed function

Then we define the *total error* as

$$\tilde{f}(\tilde{x}) - f(x) = \tilde{f}(\tilde{x}) - f(\tilde{x}) + f(\tilde{x}) - f(x)$$

We call $\tilde{f}(\tilde{x}) - f(\tilde{x})$ the *algorithmic error* and $f(\tilde{x}) - f(x)$ the *inherent error*. It can be seen that the choice of algorithm has no effect on the propagated data error.

1.1 Machine representation of number

Let $\beta \in \mathbb{N}$ be fixed with $\beta \geq 2$, and $x \in \mathbb{R}$ with a finite number of digits $0 \leq x_k < \beta$ for $k = -m, \ldots, n$. The *positional representation* of x with respect to the base β is

$$x_\beta = \text{sign}(x)(x_n\beta^n + x_{n-1}\beta^{n-1} + \ldots + x_1\beta + x_0 + x_{-1}\beta^{-1} + \ldots + x_{-m}\beta^{-m}),\ x_n \neq 0$$

where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Another way to express real numbers is the *floating-point representation*, given by:

$$x = \text{sign}(x) \cdot (x_0.x_1 \ldots x_{t-1}) \cdot \beta^e = \text{sign}(x) \cdot m \cdot \beta^{e-t+1}$$

The integer number $m = x_0x_1 \ldots x_{t-1}$, $(0 \leq x_i \leq \beta - 1)$ is called *mantissa* and is such that $0 \leq m \leq \beta^t - 1$. The integer number e is called *exponent* and t is the number of digits x_i.

A system of floating-point numbers depends on the following parameters:

- β, base
- t, precision (number of digits)
- $[L, U]$, exponent range

It is denoted by

$$\mathcal{F}(\beta, t, L, U) = \{0\} \cup \left\{ x \in \mathbb{R} : x = \text{sign}(x)\beta^e \sum_{i=0}^{t-1} x_i \beta^{-i} \right\}$$

To guarantee the uniqueness of the representation and to store one bit less, it is assumed that the leading bit is different than 0 (i.e. $x_0 \neq 0$). This is called *normalized representation*. Therefore, the mantissa is in the range $[\beta^{t-1}, \beta^t - 1]$.

Example 1.2. The layout of a 32-bit floating point number

$$0\ 01111100\ 01000000000000000000000 = 0.15625$$

We have:

- sign $= 0$ (i.e. it's a positive number)
- $e = -127 + 124 = -3$
- $m = 10100000000000000000000$ (including the hidden bit)

We can convert from basis $\beta = 2$ to basis $\beta = 10$ in the following way:

$$(1.01)_2 \cdot 2^{-3} = (1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2}) \cdot 2^{-3} = 0.15625$$

The total number of normalized floating-point numbers in the set $\mathcal{F}(\beta, t, L, U$ is:

$$2(\beta - 1)\beta^{t-1}(U - L + 1) + 1$$

The smallest positive normalized number is

$$\text{UFL} = \beta^L$$

while the largest normalized number is

$$\text{OFL} = (\beta^t - 1)\beta^{U-t+1} = (1 - \beta^{-t})\beta^{U+1}$$

The approximation of a real number x to a floating-point number $fl(x)$ can be achieved by applying one of the following *rounding rules*:

- **round-by-chop**: truncate the base-β expansion of x after t digits (also called round toward zero)
- **round-to-nearest**: $fl(x)$ is set to the nearest floating-point number to x; when there is a tie, the floating-point number whose last digit is even is used

Definition 1.4. Accuracy of floating-point systems is characterized by the *roundoff unit* (or machine precision, or machine epsilon) ϵ_{mach}. When using round-by-chop we have $\epsilon_{mach} = \beta^{1-t}$, while when using round-to-nearest we have $\epsilon_{mach} = \frac{1}{2}\beta^{1-t}$.

Another definition of roundoff unit is the following:

Definition 1.5. The roundoff unit ϵ_{mach} is the smallest positive number satisfying:

$$fl(1 + \epsilon_{mach}) > 1$$

Proposition 1.1. The maximum relative error in representing a real number x in a floating-point system is given by

$$\left| \frac{fl(x) - x}{x} \right| \leq \epsilon_{mach}$$

Proof. Let's define $x \in \mathbb{R}$ as $\text{sign}(x)(d_0.d_1d_2 \ldots d_{t-1}d_td_{t+1} \ldots)\beta^e$ and $fl(x) = \text{sign}(x)(d_0.d_1d_2 \ldots \tilde{d}_{t-1})\beta^e$. We have:

$$|x - fl(x)| = |(d_{t-1} - \tilde{d}_{t-1}).d_t \ldots \cdot \beta^{e-t+1}| \leq \frac{1}{2}\beta^{e-t+1}$$

since $|d_{t-1} - \tilde{d}_{t-1}| \leq \frac{1}{2}$, and

$$|x| \geq \beta^e$$

Thus,

$$\left|\frac{x - fl(x)}{x}\right| \leq \frac{1}{2}\beta^{1-t}$$

□

2 Linear Algebra basics for AI

2.1 Vector spaces

Definition 2.1. A *vector space* over a field F ($F = \mathbb{R}$ or $F = \mathbb{C}$) is a set closed under finite *vector addition* and *scalar multiplication*. The elements of V are called *vectors* while the elements of F are called *scalars*. The two operations must satisfy the following properties:

1. Commutativity of addition

$$\forall \mathbf{v}, \mathbf{w} \in V,\ \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

2. Associativity of addition

$$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,\ \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

3. Existence of the identity element of addition, $\mathbf{0} \in V$, such that

$$\forall \mathbf{v} \in V,\ \mathbf{v} + \mathbf{0} = \mathbf{v}$$

4. Existence of the additive inverse

$$\forall \mathbf{v} \in V,\ \exists -\mathbf{v} \in V,\ \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

5. Existence of the identity element of scalar multiplication, $1 \in F$, such that

$$\forall \mathbf{v} \in V,\ 1\mathbf{v} = \mathbf{v}$$

6. Compatibility of scalar multiplication with field multiplication

$$\forall a, b \in F,\ \forall \mathbf{v} \in V,\ (ab)\mathbf{v} = a(b\mathbf{v})$$

7. Distributivity of scalar multiplication with respect to field addition

$$\forall a, b, \in F,\ \forall \mathbf{v} \in V,\ (a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

8. Distributivity of scalar multiplication with respect to vector addition

$$\forall a \in F,\ \forall \mathbf{v}, \mathbf{w} \in V,\ a(\mathbf{v}+\mathbf{w}) = a\mathbf{v} + a\mathbf{w}$$

Example 2.1. Some notable examples of vector spaces are:

- $\mathbb{R}^n$, the set of n-tuples of real numbers
- $\mathbb{C}^n$, the set of n-tuples of complex numbers
- $\mathbb{P}_n$, the set of polynomials having degree less or equal to n
- $C^n([a,b])$, the set of real (or complex)-valued functions continuous on $[a,b]$ up to their n-th derivative, $n \in [0,\infty)$

Definition 2.2. Given V vector space over the field F, the set W is a *subspace* of V if and only if:

- $W \subset V$
- W is a vector space over F

Example 2.2. Given $V = \mathbb{R}^3$ over $\mathbb{R}$, the set

$$W = \{(\alpha, 0, 0),\ \alpha \in \mathbb{R}\}$$

is a subspace of V because

- $\forall \mathbf{w}_1, \mathbf{w}_2 \in W,\ \mathbf{w}_1 + \mathbf{w}_2 \in W$
- $\forall \alpha \in \mathbb{R},\ \forall \mathbf{w} \in W,\ \alpha\mathbf{w} \in W$

Example 2.3. Given $V = \mathbb{R}^3$ over $\mathbb{R}$, the set

$$U = \{(1,0,0), (1,2,3)\}$$

is not a subset of V because it is not a vector space, since

$$(1,0,0) + (1,2,3) \notin U$$

2.1.1 Linear independence

Definition 2.3. Given a vector space V over F, the set W of all finite linear combinations of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\},\ \mathbf{v}_i \in V$, is called the *subspace spanned by* $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\},\ \mathbf{v}_i$ and it is written as

$$W = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} = \left\{ \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \;\middle|\; \mathbf{v}_i \in V,\ \alpha_i \in F \right\}$$

The system $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a *system of generators* for V.

Example 2.4. $\{(-1,0,0),(0,1,0),(0,0,2),(-1,0,4)\}$ is a system of generators for $\mathbb{R}^3$

Definition 2.4. Given a vector space V over F, a system of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\},\ \mathbf{v}_i \in V$ is called *linearly independent* if

$$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m = \mathbf{0} \implies \alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$$

with $\alpha_1, \ldots, \alpha_m \in F$. Otherwise, the system is called *linearly dependent*.

From a geometrical point of view, n vectors are linearly dependent if they lie on the same $(n-1)$-dimensional hyperplane.

Example 2.5. Given $V = \mathbb{R}^2,\ \mathbf{v}_1 = (1,2),\ \mathbf{v}_2 = (3,4)$

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \mathbf{0} \implies$$

$$\implies \alpha_1(1,2) + \alpha_2(3,4) = (0,0) \implies$$

$$\implies (\alpha_1 + 3\alpha_2, 2\alpha_1 + 4\alpha_2) = (0,0) \implies$$

$$\implies \begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ 2\alpha_1 + 4\alpha_2 = 0 \end{cases} \implies \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Definition 2.5. We call a *basis* for a vector space V any system of linearly independent generators of V.

Example 2.6. $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$.

Proposition 2.1. Let V be a vector space which admits a basis of n vectors. Then every system of linearly independent vectors has at most n elements and any other basis of V has exactly n elements. The number n is called *dimension* of V and is denoted by $\dim(V) = n$.

2.2 Matrices

Definition 2.6. Let m and n be two positive integers. We call *matrix* the rectangular array having m rows and n columns of elements in a field F. It is represented as

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If the $F = \mathbb{R}$ (respectively $F = \mathbb{C}$) we write $\mathrm{A} \in \mathbb{R}^{m \times n}$ (respectively $\mathrm{A} \in \mathbb{C}^{m \times n}$). If $m = n$ the matrix is called *square*. The set of entries A_{ij} with $i = j$ is called *main diagonal* of the matrix A.

We can write the above matrix as $A(m \times n)$ or $A = (a_{ij})$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Definition 2.7. Let $\mathrm{A} \in F^{m \times n}$. The maximum number of linearly independent columns (or rows) of A is called *rank*, denoted by rank(A). A is said to have *complete* or *full rank* if $\mathrm{rank(A)} = \min(m, n)$.

Definition 2.8. A *lower triangular matrix* is a matrix L where $l_{ij} = 0$ if $i < j$. An *upper triangular matrix* is a matrix U where $u_{ij} = 0$ if $i > j$.

$$\mathrm{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}, \ \mathrm{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

2.2.1 Operations with matrices

Let $\mathrm{A} \in (R)^{m \times p}$, $\mathrm{B} \in (R)^{m \times p}$, $\mathrm{C} \in (R)^{p \times n}$, $\lambda \in F$. We define the following operations:

– *matrix addition*

$$\mathrm{A} + \mathrm{B} = (a_{ij} + b_{ij}), \ \mathrm{A} + \mathrm{B} \in (R)^{m \times p}$$

The identity element of matrix addition is the *null matrix*, denoted by 0 and made up only by null entries;

– *matrix multiplication by a scalar*

$$\lambda \mathrm{A} = (\lambda a_{ij}),\ \lambda \mathrm{A} \in (R)^{m \times p}$$

– *matrix multiplication*

$$\mathrm{AC} = \left(\sum_{k=1}^{p} a_{ik} b_{kj} \right),\ \mathrm{A} \cdot \mathrm{C} \in (R)^{m \times n}$$

Notice how matrix multiplication is defined only when the number of columns of the first matrix is equal to the number of rows of the second matrix. This operation results in a matrix with size (m, n). Matrix product is associative and distributive with respect to matrix addition, but it is not in general commutative

$$\mathrm{A} \cdot \mathrm{C} \neq \mathrm{C} \cdot \mathrm{A}$$

We call *commutative* the square matrices for which $\mathrm{A} \cdot \mathrm{C} = \mathrm{CA}$ holds.

– *transposition*

$$\mathrm{A}^T = (a_{ji}),\ \mathrm{A}^T \in F^{p \times m}$$

Transposition enjoys the following properties:

$$\left(\mathrm{A}^{\mathrm{T}}\right)^T = A,\ (\mathrm{A} + \mathrm{B})^T = \mathrm{A}^T + \mathrm{B}^T,\ (\mathrm{AC})^T = \mathrm{C}^T \mathrm{A}^T,\ (\lambda \mathrm{A})^T = \lambda \mathrm{A}^T$$

Example 2.7. Let

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

Then we have

$$\mathrm{A} + \mathrm{B} = \begin{bmatrix} 1+7 & 2+8 & 3+9 \\ 4+1 & 5+2 & 6+3 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 5 & 7 & 9 \end{bmatrix}$$

$$\mathrm{B}^T = \begin{bmatrix} 7 & 1 \\ 8 & 2 \\ 9 & 3 \end{bmatrix}$$

$$\mathrm{AB}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 8 & 2 \\ 9 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 & 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \\ 4 \cdot 7 + 5 \cdot 8 + 6 \cdot 9 & 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 50 & 14 \\ 122 & 32 \end{bmatrix}$$

Definition 2.9. A *diagonal matrix* of order n is a square matrix of the type $A = (d_{ii}\delta_{ij})$, where δ_{ij} denotes the *Kronecker symbol* equal to 1 if $i = j$ and 0 otherwise. It can be written as $A = \text{diag}(d_{11}, d_{22}, \ldots, d_{nn})$ or $A = \text{diag}(\mathbf{d})$, where $\mathbf{d} = (d_{11}, d_{22}, \ldots, d_{nn})$.

Definition 2.10. We call *identity matrix of order n* the square matrix of size (n, n) given by $\mathrm{I}_n = \text{diag}(\mathbf{1})$. Usually, we write the identity matrix as I.

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The identity matrix is, by definition, the only matrix such that $\mathrm{AI}_n = \mathrm{I}_n\mathrm{A} = \mathrm{A}$, for all $\mathrm{A}(n \times n)$.

Definition 2.11. A matrix $\mathrm{A}(n \times n)$ is called *invertible* (or *nonsingular*) if there exists a matrix $\mathrm{B}(n \times n)$ such that $\mathrm{AB} = \mathrm{BA} = \mathrm{I}$. B is called *inverse matrix* of A and is denoted by A^{-1}. A non-invertible square matrix is called *singular*.

The inverse of an invertible matrix is also invertible, $\left(\mathrm{A}^{-1}\right)^{-1} = \mathrm{A}$, and if two square matrices A and B are both invertible, their product is also invertible, $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$. Moreover, if a square matrix A is invertible, then $\left(\mathrm{A}^{\mathrm{T}}\right)^{-1} = \left(\mathrm{A}^{-1}\right)^{T} = \mathrm{A}^{-T}$

Proposition 2.2. A square matrix is invertible iff its column vectors are linearly independent.

Example 2.8. Let

$$\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Let's check if its column vectors are linearly independent

$$\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ 3\alpha_1 + 4\alpha_2 = 0 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$$

Since the column vectors are independent we know that A is invertible. A method to compute the inverse will be given in the next section.

Definition 2.12. A square matrix A is called *symmetric* if $\mathrm{A} = \mathrm{A}^T$, *anti-symmetric* if $\mathrm{A} = -\mathrm{A}^T$, *orthogonal* if $\mathrm{A}^{-1} = \mathrm{A}^T$, i.e. $\mathrm{AA}^T = \mathrm{A}^T\mathrm{A} = \mathrm{I}$.

Example 2.9. The following matrix is symmetric

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

The following matrix is antisymmetric

$$\mathrm{A} = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

The following matrix is orthogonal

$$\mathrm{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2.2.2 Determinant of a matrix

Definition 2.13. Let $\mathrm{A} \in (C)^{n\times n}$. We call the *determinant* of A, denoted by $\det A$, the scalar defined through the following formula

$$\det(\mathrm{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ \sum_{j=1}^{n} (-1)^{i+j} \det(\mathrm{A}_{ij}) a_{ij} = \sum_{i=1}^{n} (-1)^{i+j} \det(\mathrm{A}_{ij}) a_{ij} & \text{if } n > 1 \end{cases}$$

with $i \in \{1, \ldots, n\}$ (or $j \in \{1, \ldots, n\}$) fixed, where A_{ij} is the submatrix of order $n-1$ obtained from A by eliminating row i and column j. This is known as *Laplace's rule*

If A is diagonal or triangular, then

$$\det(\mathrm{A}) = \prod_{i=1}^{n} a_{ii}$$

The determinant enjoys the following properties

$$\det(\mathrm{A}) = \det(\mathrm{A}^T),\ \det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B}),\ \det(\mathrm{A}^{-1}) = \frac{1}{\det(\mathrm{A})},$$

$$\det(\alpha \mathrm{A}) = \alpha^n \det(\mathrm{A}),\ \forall \alpha \in F$$

Proposition 2.3. Every orthogonal matrix A is invertible and

$$\det(\mathrm{A}) = \pm 1$$

In the following examples we illustrate methods to compute a matrix determinant that do not require the use of Laplace's rule.

Example 2.10. Let

$$\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The determinant of a 2×2 matrix can be computed as

$$\det(\mathrm{A}) = a_{11}a_{22} - a_{21}a_{12} = 1 \cdot 4 - 3 \cdot 2 = -2$$

Example 2.11. Let

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 3 & 0 & 4 \end{bmatrix}$$

The determinant of a 3×3 matrix can be computed using *Sarrus' rule*

$$\det(\mathrm{A}) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11} =$$

$$= 1 \cdot 4 \cdot 4 + 2 \cdot 5 \cdot 3 + 3 \cdot 0 \cdot 0 - 0 \cdot 4 \cdot 3 - 2 \cdot 3 \cdot 4 - 5 \cdot 0 \cdot 1 = 22$$

Example 2.12. Let

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & 3 & 0 & -2 \\ 0 & -2 & -2 & 3 \end{bmatrix}$$

Let's compute the determinant of A with Laplace's rule, fixing $i = 1$ ($|\cdot|$ denotes the determinant)

$$\det(\mathrm{A}) = 1 \cdot \begin{vmatrix} 4 & 1 & 0 \\ 3 & 0 & -2 \\ -2 & -2 & 3 \end{vmatrix} \cdot 1 + (-1) \cdot \begin{vmatrix} 3 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & 3 \end{vmatrix} \cdot 2 +$$

$$+1 \cdot \begin{vmatrix} 3 & 4 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{vmatrix} \cdot 0 + (-1) \cdot \begin{vmatrix} 3 & 4 & 1 \\ -1 & 3 & 0 \\ 0 & -2 & -2 \end{vmatrix} \cdot 0 =$$
$$= -21 + 18 + 0 + 0 = -3$$

Proposition 2.4. If A is invertible then

$$\mathrm{A}^{-1} = \frac{\mathrm{C}^T}{\det(\mathrm{A})}$$

where C is the *cofactor matrix* having elements $c_{ij} = (-1)^{i+j}\det(\mathrm{A}_{ij})$, called *cofactors*.

Proposition 2.5. For a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$ the following properties are equivalent:

- A is nonsingular
- $\det(\mathrm{A}) \neq 0$
- $\mathrm{rank}(\mathrm{A}) = n$
- A has linearly independent columns and rows

Example 2.13. Let

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 3 & 0 & 4 \end{bmatrix}$$

We already know (see Example 2.11) that $\det(\mathrm{A}) = 22$. Let's compute the cofactor matrix C

$$\det(\mathrm{A}_{11}) = \begin{vmatrix} 4 & 5 \\ 0 & 4 \end{vmatrix} = 16,\ \det(\mathrm{A}_{12}) = \begin{vmatrix} 3 & 5 \\ 3 & 4 \end{vmatrix} = -3,\ \det(\mathrm{A}_{13}) = \begin{vmatrix} 3 & 4 \\ 3 & 0 \end{vmatrix} = -12$$

$$\det(\mathrm{A}_{21}) = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8,\ \det(\mathrm{A}_{22}) = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4,\ \det(\mathrm{A}_{23}) = \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = -6$$

$$\det(\mathrm{A}_{31}) = \begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = 10,\ \det(\mathrm{A}_{32}) = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5,\ \det(\mathrm{A}_{33}) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

So we have

$$\mathrm{C} = \begin{bmatrix} 16 & 3 & -12 \\ -8 & 4 & 6 \\ 10 & -5 & -2 \end{bmatrix}$$

Thus,

$$\mathrm{A}^{-1} = \frac{\mathrm{C}^T}{\det(\mathrm{A})} = \begin{bmatrix} \frac{8}{11} & -\frac{4}{11} & \frac{5}{11} \\ \frac{3}{22} & \frac{2}{11} & -\frac{5}{22} \\ -\frac{6}{11} & \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

2.2.3 Eigenvalues and eigenvectors

Definition 2.14. Let $\mathrm{A} \in \mathbb{C}^{n\times n}$. The number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of A if

$$\exists \mathbf{x} \in \mathbb{C}^n,\ \mathbf{x} \neq \mathbf{0} \text{ such that } \mathrm{A}\mathbf{x} = \lambda \mathbf{x}$$

The vector $\mathbf{x}$ is called the *eigenvector* associated with the eigenvalue λ. The set of eigenvalues of A is called *spectrum* of A and is denoted by $\sigma(\mathrm{A})$. The eigenvalues of A are the solutions of the *characteristic equation*

$$p_{\mathrm{A}}(\lambda) = \det(\mathrm{A} - \lambda \mathrm{I}) = 0.$$

where $p_{\mathrm{A}}(\lambda)$ is called *characteristic polynomial*. For the fundamental theorem of algebra the characteristic polynomial, which has degree n, has exactly n solutions. Hence, a matrix with real or complex entries has n eigenvalues, counted with their multiplicity.

Definition 2.15. Let a square matrix A have an eigenvalue λ_i. The algebraic multiplicity of λ_i is the number of times the root appears in the characteristic polynomial.

Definition 2.16. The maximum eigenvalue in module of a matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$ is called the *spectral radius* of A and is denoted by

$$\rho(\mathrm{A}) = \max_{\lambda \in \sigma(\mathrm{A})} |\lambda|$$

The set of all the eigenvalues is called the spectrum of A.

Remark. The eigenvectors of a matrix A are not unique. If $\mathbf{x}$ is an eigenvector of A associated with an eigenvalue λ, then for any $c \in \mathbf{R}\ \{0\}$ $c\mathbf{x}$ is an eigenvector of A with the same eigenvalue.

$$\mathrm{A}(c\mathbf{x} = c\mathrm{A}\mathbf{x} = c\lambda\mathbf{x} = \lambda(c\mathbf{x})$$

Proposition 2.6. Let $\mathrm{A} \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, then:

$$\det(\mathrm{A}) = \prod_{i=1}^{n} \lambda_i$$

Proposition 2.7. A matrix is singular iff it has at least one null eigenvalue.

Proposition 2.8. Let $\mathrm{A} \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. If A is diagonal or triangular, then $\lambda_i = a_{ii}$, $i = 1, \ldots, n$.

Proposition 2.9. A symmetric positive (semi)definite matrix A has eigenvalues λ_i greater (equal) than zero.

Definition 2.17. Two matrices $\mathrm{A} \in \mathbb{C}^{n\times n}$ and $\mathrm{B} \in \mathbb{C}^{n\times n}$ are said *similar* if there exists a non singular matrix P so that $\mathrm{B} = \mathrm{PAP}^{-1}$.

Definition 2.18. (A) and (B) are similar iff they have the same eigenvalues.

Example 4.5 in MML.

2.3 Scalar Product and Norms in Vector Spaces

Definition 2.19. Let V be a vector space over the field F. We define the map $||\cdot|| : V \to F$ as a *norm* on V if the following properties are satisfied:

- $||\mathbf{v}|| \geq 0,\ \forall \mathbf{v} \in V$ and $||\mathbf{v}|| = 0 \iff \mathbf{v} = \mathbf{0}$
- $||\alpha\mathbf{v}|| = |\alpha|||\mathbf{v}||,\ \forall \alpha \in F,\ \forall \mathbf{v} \in V$
- $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||,\ \forall \mathbf{v}, \mathbf{w} \in V$

where $|\alpha|$ denotes the absolute value of α if $F = \mathbb{R}$, the module of α if $F = \mathbb{C}$. $(V, ||\cdot||)$ is called a *normed space*.

Example 2.14. Let $\mathbf{v} = (1, 2, 3)$. The Euclidean norm of $\mathbf{v}$ is

$$||\mathbf{v}||_2 = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{14}$$

Example 2.15. Let $\mathbf{v} = (1, 2, 3)$. The *one norm* of $\mathbf{v}$ is

$$||\mathbf{v}||_1 = \sum_{i=1}^{n} |v_i| = 6$$

Example 2.16. Let $\mathbf{v} \in \mathbb{R}^n$. The *p-norm* of $\mathbf{v}$ is

$$||\mathbf{v}||_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}, \ 1 \leq p < \infty$$

Notice how the p-norm with $p = 1$ is the one-norm and with $p = 2$ is the Euclidean norm.

Example 2.17. Let $\mathbf{v} = (-8, 2, 5)$. The *infinity norm* of $\mathbf{v}$ is

$$||\mathbf{v}||_\infty = \max_{i=1,\dots,n} |v_i| = 8$$

Definition 2.20. Two norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are said *equivalent* if there exists tow positive constants c_{pq} and C_{pq}:

$$c_{pq}\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq C_{pq}\|\mathbf{x}\|_q \forall \mathbf{x} \in V$$

Result. In a vector space all the p norms are equivalent.

2.3.1 Matrix norms

Definition 2.21. A *matrix norm* is a map $|| \cdot || : \mathbb{R}^{m \times n} \to \mathbb{R}$ satisfying the following properties:

- $||\mathrm{A}|| \geq 0, \ \forall \mathrm{A} \in \mathbb{R}^{m \times n}$ and $||\mathrm{A}|| = 0 \iff \mathrm{A} = 0$
- $||\alpha \mathrm{A}|| = |\alpha| ||\mathrm{A}||, \ \forall \alpha \in \mathbb{R}, \ \forall \mathrm{A} \in \mathbb{R}^{m \times n}$
- $||\mathrm{A} + \mathrm{B}|| \leq ||\mathrm{A}|| + ||\mathrm{B}||, \ \forall \mathrm{A}, \mathrm{B} \in \mathbb{R}^{m \times n}$

Definition 2.22. Let $\mathrm{A} \in \mathbb{R}^{n \times n}$. We say the norm $|| \cdot ||$ is compatible with a vector norm $|| \cdot ||$ if

$$||\mathrm{A}\mathbf{x}|| \leq ||\mathrm{A}|| ||\mathbf{x}||$$

Example 2.18. Let $\mathrm{A} \in \mathbb{R}^{m \times n}$. The *spectral norm* of A is

$$||\mathrm{A}||_2 = \sqrt{\rho(\mathrm{A}^T A)}$$

The spectral norm of the identity matrix is

$$||\mathrm{I}||_2 = 1$$

Example 2.19. Let $\mathrm{A} \in \mathbb{R}^{m \times n}$. The *p-norm* of A with $p = 1$ is

$$||\mathrm{A}||_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}|$$

The 1-norm of the identity matrix is

$$||\mathrm{I}||_1 = 1$$

Example 2.20. Let $\mathrm{A} \in \mathbb{R}^{m \times n}$. The *infinity norm* of A is

$$||\mathrm{A}||_\infty = \max_{i=1,\ldots,n} \sum_{j=1}^{n} |a_{ij}|$$

Notice how if A is symmetric $||\mathrm{A}||_1 = ||\mathrm{A}||_\infty$. The infinity norm of the identity matrix is

$$||\mathrm{I}||_\infty = 1$$

Example 2.21. Let $\mathrm{A} \in \mathbb{R}^{m \times n}$. The *Frobenius norm* of A is

$$||\mathrm{A}||_F = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

The Frobenius norm of the identity matrix is

$$||\mathrm{I}_n||_F = \sqrt{n}$$