1. Satisfiability Modulo Theories

Prof. Roberto Amadini

Department of Computer Science and Engineering, University of Bologna, Italy

Combinatorial Decision Making and Optimization

2nd cycle degree programme in Artificial Intelligence

University of Bologna, Academic Year 2023/24

From SAT to SMT

- Boolean satisfiability problem (SAT) famous and well-studied
  - *NP-complete*
- Several applications in AI and other fields
  - *The Silent (R)evolution of SAT*
    `https://dl.acm.org/doi/10.1145/3560469`
- Based on propositional logic
  - Atomic proposition or atoms (facts) can be either true or false
  - Atoms combined via connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
  - Decidable, "efficient", but limited expressiveness
- Real-world applications often require a more expressive logic, e.g., First-Order Logic (FOL)

From SAT to SMT

- FOL formulas are *more general*, e.g., $(\forall x)(x < u \wedge (\nexists y)\ f(x, y) = u)$

- Many applications do not require "general-purpose" FOL satisfiability: only satisfiability w.r.t. a *fixed* background theory

$0 < z - x < 1$

$\begin{cases} y = x+1 \\ x < z < y \end{cases} \quad x < z < x+1 \quad x - x < z - x < x + 1 - x$

- E.g., given formula $y = x + 1 \wedge x < z \wedge z < y$ one typically checks satisfiability w.r.t. an arithmetical theory, unless otherwise specified...
  - But one might interpret 1 as '1', $+$ as concatenation, $<$ as lex...
  - ...What arithmetic theory? Integers or reals?

- Focusing on a particular theory enables a more efficient solving via specialized decision procedures
  - Especially for quantifier-free formulas

From SAT to SMT

- A formula ϕ can be undecidable!
  - We cannot prove neither ϕ nor $\neg\phi$ (e.g., *Peano arithmetic*)
- But we can restrict to decidable fragments of an undecidable theory
  - "subsets of the theory" (e.g., *Presburger arithmetic*)
- **Satisfiability Modulo Theory** (SMT) concerns the study of the satisfiability of formulas w.r.t. some backgrounds theories
  - theory of integer arithmetic, real numbers, arrays, strings, (multi-)sets, trees, lists, bit-vectors, . . .
- SMT solvers are used to determine the satisfiability of formulas through different procedures over different background theories
  - E.g., Z3 or CVC5 (formerly CVC4, CVC3, . . .)

SMT history

- The roots of SMT date back to late 70s - early 80s
  - Nelson and Oppen, Shostak, Boyer and Moore, ...
- Modern SMT research started in the 90s, following the developments in SAT solving
- From the 2000s up to now big development in SMT's foundational and practical aspects
  - SMT approaches integrated in various tools for theorem proving, program analysis and testing
  - Following big development in SAT solvers

SMT vs SAT/CP

- SMT extends SAT, and tackles combinatorial problems from an orthogonal perspective w.r.t. CP
- Like SAT solving, SMT historically specialised in theorem proving, software verification, model checking, automated test generation
  - CP more oriented to scheduling, resource allocation, optimization
- Like CP and unlike SAT, SMT employs domain-specific reasoning but, unlike CP, it doesn't generally require finite domains
- SMT solving uses SAT abstractions instead of propagators
  - It natively handles nogoods, which is not always true for CP solvers (but modern CP solvers do it $\rightarrow$ LCG solvers)

SMT Preliminaries

Formal preliminaries

- Let's recall/introduce the relevant FOL concepts and notation
  - We shall always assume FOL with equality
- The signature of a FOL is the set $\Sigma = \Sigma^F \cup \Sigma^P$ of its non-logical symbols, i.e., functions in Σ^F and predicates in Σ^P
  - We denote Σ^F_k (resp. Σ^P_k) the functions (predicates) with arity $k \geq 0$
  - So, $\Sigma^P = \bigcup_k \Sigma^P_k$ and $\Sigma^F = \bigcup_k \Sigma^F_k$
- The 0-arity functions of Σ^F_0 are constant symbols
- The 0-arity predicates of Σ^P_0 are propositional symbols
  - E.g., propositional logic has $\Sigma^F = \emptyset$ and $\Sigma^P = \Sigma^P_0 \supseteq \{\bot, \top\}$
- We will consider only quantifier-free fragments (no $\exists, \forall$)
  - All variables are free variables

Terms and Formulas

- The set $\mathbb{T}^\Sigma$ of terms Σ is defined as:
  - $c \in \Sigma_0^F \implies c \in \mathbb{T}^\Sigma$
  - $f \in \Sigma_k^F$ and $t_1, \ldots, t_k \in \mathbb{T}^\Sigma \implies f(t_1, \ldots, t_k) \in \mathbb{T}^\Sigma$
  - $\varphi \in \mathbb{F}^\Sigma$ and $t_1, t_2 \in \mathbb{T}^\Sigma \implies ite(\varphi, t_1, t_2) \in \mathbb{T}^\Sigma \longrightarrow$ if φ then: t_1 else: t_2

    iF, then, else
- The set $\mathbb{F}^\Sigma$ of formulas of Σ is defined as:
  - $\bot, \top \in \mathbb{F}^\Sigma$
  - $t_1, t_2 \in \mathbb{T}^\Sigma \implies t_1 = t_2 \in \mathbb{F}^\Sigma$
  - $A \in \Sigma_0^P \implies A \in \mathbb{F}^\Sigma$
  - $p \in \Sigma_k^P$ and $t_1, \ldots, t_k \in \mathbb{T}^\Sigma \implies p(t_1, \ldots, t_k) \in \mathbb{F}^\Sigma$

    Atomic formulas
  - $\varphi \in \mathbb{F}^\Sigma \implies \neg\varphi \in \mathbb{F}^\Sigma$
  - $\varphi_1, \varphi_2 \in \mathbb{F}^\Sigma \implies \varphi_1 \rightarrow \varphi_2,\ \varphi_1 \leftrightarrow \varphi_2,\ \varphi_1 \wedge \varphi_2,\ \varphi_1 \vee \varphi_2 \in \mathbb{F}^\Sigma$

Formal preliminaries

- An atomic formula is also called an atom
- A literal is either:
  - an atomic formula (positive literal), or
  - the negation of one (negative literal)
- A clause is a disjunction $\ell_1 \vee \cdots \vee \ell_k$ of literals
  - A unit clause is a clause consisting of a single literal $\neq \bot, \top$
- A formula is in Conjunctive Normal Form (CNF) if it is the conjunction $c_1 \wedge \cdots \wedge c_k$ of $k \geq 0$ clauses
  - Also denoted $\{c_1, \ldots, c_k\}$ or simply $c_1, \ldots, c_k$

Semantics

- The semantics of a formula denotes its *"meaning"*, i.e., a truth value in $\{true, false\}$, by means of a certain interpretation
- A model for Σ is a pair $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ where set M is the universe of $\mathcal{M}$ and a mapping $(\cdot)^{\mathcal{M}}$ such that:
  - $f^{\mathcal{M}} \in \{\varphi \mid \varphi : M^k \to M\}$ for each function $f \in \Sigma^F_k$
    - In particular $c^{\mathcal{M}} \in M$ for each constant $c \in \Sigma^F_0$
  - $p^{\mathcal{M}} \in \{\varphi \mid \varphi : M^k \to \{true, false\}\}$ for each predicate $f \in \Sigma^P_k$
    - In particular $B^{\mathcal{M}} \in \{true, false\}$ for each proposition $B \in \Sigma^P_0$
- The $(\cdot)^{\mathcal{M}}$ extension to terms and formulas is called interpretation:
  - $\bot^{\mathcal{M}} = false$, $\top^{\mathcal{M}} = true$, $(t_1 = t_2)^{\mathcal{M}} = true \iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$
  - $f(t_1, \dots, t_k)^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{M}, \dots, t_k^{\mathcal{M}})$
  - $p(t_1, \dots, t_k)^{\mathcal{M}} = p^{M}(t_1^{M}, \dots, t_k^{\mathcal{M}})$
  - $ite(\varphi, t_1, t_2)^{\mathcal{M}} = \begin{cases} t_1^{\mathcal{M}} & \text{if } \varphi^{\mathcal{M}} = true \\ t_2^{\mathcal{M}} & \text{if } \varphi^{\mathcal{M}} = false \end{cases}$

Satisfiability

- $\mathcal{M}$ satisfies (resp. falsifies) $\varphi \in \mathbb{F}^{\Sigma}$ if $\varphi^{\mathcal{M}} = \mathit{true}$ (resp. $\varphi^{\mathcal{M}} = \mathit{false}$)
- A Σ-theory is a (*possibly infinite*) set $\mathcal{T}$ of Σ-models
- $\varphi \in \mathbb{F}^{\Sigma}$ is $\mathcal{T}$-satisfiable if there exists a model $\mathcal{M} \in \mathcal{T}$ satisfying φ
  - $\{\varphi_1, \ldots, \varphi_k\} \subseteq \mathbb{F}^{\Sigma}$ is $\mathcal{T}$-consistent iff $\varphi_1 \wedge \cdots \wedge \varphi_k$ is $\mathcal{T}$-satisfiable
- $\Gamma \subseteq \mathbb{F}^{\Sigma}$ $\mathcal{T}$-entails φ iff every $\mathcal{M} \in \mathcal{T}$ that satisfies Γ also satisfies φ
  - If Γ $\mathcal{T}$-entails φ, we write $\Gamma \models_{\mathcal{T}} \varphi$
  - Γ is $\mathcal{T}$-consistent iff $\Gamma \not\models_{\mathcal{T}} \perp$
- $\varphi \in \mathbb{F}^{\Sigma}$ is $\mathcal{T}$-valid iff $\emptyset \models_{\mathcal{T}} \varphi$, i.e., every $\mathcal{M} \in \mathcal{T}$ satisfies φ
  - A $\mathcal{T}$-valid clause $c = \ell_1 \vee \cdots \vee \ell_k$ is called theory lemma
  - φ is $\mathcal{T}$-consistent $\iff$ $\neg\varphi$ is not $\mathcal{T}$-valid

Example

- Suppose Σ defined by $\Sigma_0^F = \{a, b, c, d\}, \Sigma_2^F = \{f, g\}, \Sigma_1^P = \{p\}$
- Let $\mathcal{M}_1, \mathcal{M}_2$ be 2 models having universe $\mathcal{P}(\mathbb{Z})$ and such that:
  - $a^{\mathcal{M}_1} = \emptyset, b^{\mathcal{M}_1} = \{2x \mid x \in \mathbb{Z}\}, c^{\mathcal{M}_1} = \{2x+1 \mid x \in \mathbb{Z}\}, d^{\mathcal{M}_1} = \mathbb{Z}$
  - $a^{\mathcal{M}_2} = \{0\}, b^{\mathcal{M}_2} = \{x \in \mathbb{Z} \mid x > 0\}, c^{\mathcal{M}_2} = \{x \in \mathbb{Z} \mid x < 0\}, d^{\mathcal{M}_2} = \mathbb{Z}$
  - $f^{\mathcal{M}_1} = f^{\mathcal{M}_2} = \cup, \quad g^{\mathcal{M}_1} = g^{\mathcal{M}_2} = \cap, \quad p^{\mathcal{M}_1}(X) = p^{\mathcal{M}_2}(X) \Leftrightarrow X = \emptyset$
- Consider theory $\mathcal{T} = \{\mathcal{M}_1, \mathcal{M}_2\}$ and provide example(s) of:
  - A formula $\mathcal{T}$-satisfiable and not atomic
  - A set of ≥ 2 formulas not $\mathcal{T}$-consistent
  - A set of ≥ 2 formulas that $\mathcal{T}$-entails a not $\mathcal{T}$-valid formula
  - A $\mathcal{T}$-lemma of ≥ 3 clauses

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- Consider theory $\mathcal{T} = \{\mathcal{M}_1, \mathcal{M}_2\}$ and provide example(s) of:
  - A formula $\mathcal{T}$-satisfiable and not atomic: $p(a) \vee p(g(b, c))$
  - A set of ≥ 2 formulas not $\mathcal{T}$-consistent: $\{p(a), p(d)\}$
  - A set of ≥ 2 formulas that $\mathcal{T}$-entails a not $\mathcal{T}$-valid formula $\{p(a), \neg p(c)\} \models_{\mathcal{T}} (a = g(b, c))$
  - A $\mathcal{T}$-lemma of ≥ 3 clauses: $p(b) \vee p(c) \vee d = f(f(a, b), c)$

Expansion

- We check the $\mathcal{T}$-satisfiability of formulas with quantifier-free variables
  - In other terms, we find a consistent assignment values $\Rightarrow$ variables
- Because no quantification is involved, variables can be seen as "additional constants" not in Σ_0^F
- More generally, given signature Σ we can consider formulas with uninterpreted symbols, i.e., symbols not in Σ
  - Variable $\equiv$ uninterpreted constant $\equiv$ uninterpreted function
- Given Σ-model $\mathcal{M} = \langle M, (\cdot)^{\mathcal{M}} \rangle$ and $\Sigma' \supseteq \Sigma$, an expansion $\mathcal{M}'$ to Σ' of that model is any Σ'-model $\mathcal{M}' = \langle M', (\cdot)^{\mathcal{M}'} \rangle$ such that:
  - $M' = M$
  - $s^{\mathcal{M}'} = s^{\mathcal{M}}$ for each $s \in \Sigma$

Expansion

- Instead of a Σ-theory $\mathcal{T}$, we (sometimes implicitly) consider the theory $\mathcal{T}' = \{\mathcal{M}' \mid \mathcal{M}' \text{ is an expansion of a } \Sigma\text{-model } \mathcal{M}\ \}$
- The ground $\mathcal{T}$-satisfiability problem is determining, given Σ-theory $\mathcal{T}$, the $\mathcal{T}$-satisfiability of ground formulas over a Σ-expansion $\mathcal{T}'$
  - ground formula $\equiv$ formula with no variables: because uninterpreted constants play the role of variables, our formulas are always ground
- Because φ is $\mathcal{T}$-satisfiable $\iff$ $\neg\varphi$ is not $\mathcal{T}$-valid, the ground $\mathcal{T}$-satisfiability problem has a dual validity problem
  - E.g., $x > 5$ satisfiable iff $x \leq 5$ not valid

Example

- Let's take Σ as $\Sigma_0^F = \{a, b, c, d\}, \Sigma_1^F = \{f, g\}, \Sigma_2^P = \{p\}$ and a Σ-model $\mathcal{M} = \langle [0, 2\pi), (\cdot)^{\mathcal{M}} \rangle$ s.t.
  - $a^{\mathcal{M}} = 0, b^{\mathcal{M}} = \frac{\pi}{2}, c^{\mathcal{M}} = \pi, d^{\mathcal{M}} = \frac{3}{2}\pi$,
  - $f^{\mathcal{M}} = \sin$, $g^{\mathcal{M}} = \cos$, $p^{\mathcal{M}}(x, y) \Leftrightarrow x > y$
- By expanding Σ with uninterpreted constants $x, y, z, \ldots$ we can check the satisfiability of arbitrarily complex ground formulas
  - $p(f(y), g(g(d))) \vee p(a, f(b))$, $g(x) \iff g(c) \wedge f(g(z))$, $\ldots$
- E.g., is $p(g(x), f(d))$ is $\mathcal{M}$-satisfiable?
  - Let $\Sigma' = \Sigma \cup \{x\}$, and expansion $\mathcal{M}'$ of $\mathcal{M}$ s.t. $x^{\mathcal{M}'} = \frac{1}{2}\pi$
  - $p^{\mathcal{M}'}(g(x), f(d)) \equiv g^{\mathcal{M}}(x^{\mathcal{M}'}) > f^{\mathcal{M}}(d^{\mathcal{M}}) \equiv \cos(\frac{1}{2}\pi) > \sin(\frac{3}{2}\pi) \equiv 0 > -1 \equiv \mathit{true}$

Axiomatic definition

- A theory can be defined axiomatically
- A (minimal) set of formulas $\Lambda \subseteq \mathbb{F}^{\Sigma}$ called axioms is given, and the corresponding theory is the set of all models of Λ
- E.g., Peano axioms. Given Σ with constant 0 e unary function S:
  - $(\forall x)\ \neg(S(x) = 0)$
  - $(\forall x)(\forall y)\ S(x) = S(y) \rightarrow x = y$
  - $(\varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow (\forall x)\ \varphi(x)$ for any $\varphi \in \mathbb{F}^{\Sigma}$
- A theory satisfying Peano axioms is called arithmetic theory
- By adding $+$ and $*$, we can prove many arithmetic theorems
  - But not all of them! See Gödel incompleteness theorems

Many-sorted logic

- Most of SMT applications involve different data types or sorts
- It may be convenient to formalize SMT problems with a many-sorted FOL having:
  - a set of sort symbols $\mathcal{S}$ (i.e., a set of types)
  - a set of sorted variables uniquely associated with a sort $\sigma \in \mathcal{S}$
  - a sorted signature Σ including a set $\Sigma^{\mathcal{S}} \subseteq \mathcal{S}$ of sort symbols
  - corresponding semantics for variables and sorted signatures...
- ...Let's just remember that sort $\equiv$ type without adding new formalism

Some theories of interest

Theories of interest

Let's see an overview of some SMT theories of interest:

- Uninterpreted functions
- Arithmetic
- Arrays
- Bit-vectors
- Strings

EUF Theory

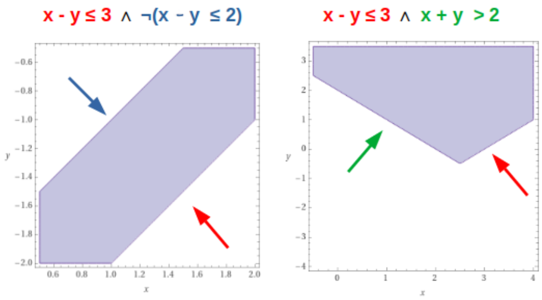
- EUF = Equality with Uninterpreted Functions theory ($\mathcal{T}_{EUF}$)
- No restrictions on how Σ-symbols should be interpreted: $\mathcal{T}_{EUF}$ includes all the possible Σ-models
  - Sometimes called empty theory, because its set of axioms is $\emptyset$
- Why $\mathcal{T}_{EUF}$? To abstract complex or "black-box" functions
- E.g., consider $a * (f(b) + f(c)) = d \land b * (f(a) + f(c)) \neq d \land a = b$
  We don't need any arithmetic theory to prove it unsatisfiable!
- Let's abstract $+$ and $*$ with fresh uninterpreted functions g and h:
  $h(a, g(f(b), f(c))) = d \land h(b, g(f(a), f(c))) \neq d \land a = b$
  - Congruence closure procedure easily detects unsatisfiability

Arithmetic Theories

- Theory over numbers are clearly very used and useful
- Let $\Sigma \equiv (0, 1, +, -, \leq)$ and $\mathcal{T}_{\mathcal{Z}}$ interpreting Σ symbols in the usual way. $\mathcal{T}_{\mathcal{Z}}$ is a.k.a. Presburger arithmetic
    - We can define $\mathcal{T}_{\mathcal{R}}$ interpreting Σ symbols over reals
- The satisfiability of ground formulas for $\mathcal{T}_{\mathcal{Z}}$ and $\mathcal{T}_{\mathcal{R}}$ is decidable
    - There exist procedures to decide if a formula is true/false
- Ground satisfiability in $\mathcal{T}_{\mathcal{R}}$ is decidable in polynomial time
    - Simplex method is exponential but works fine in practice
- $\mathcal{T}_{\mathcal{Z}}$-satisfiability is harder: NP-complete in general

Arithmetic Theories

- $\mathcal{T}_{\mathbb{Z}}$ fragments have more efficient decision procedures, e.g.:
- Difference logic: every atom must be $x - y \bowtie k$ with $\bowtie \in \{=, \leq\}$, x, y variables and k integer
- UTPVI ("unit two variable per inequality") every atom $x \pm y \bowtie k$



Arithmetic Theories

- Things are much harder with multiplication:
  - The integer case becomes undecidable
  - The real case becomes doubly-exponential
- Another non-trivial case is the floating-point arithmetic, e.g.
  - For $\mathcal{T}_{\mathcal{Z}}$ and $\mathcal{T}_{\mathcal{R}}$, $(x + y) + z = x + (y + z)$ is valid (*associativity*)
  - For IEEE754 floating points, this is no longer true! E.g. for a sound floating-point model $\mathcal{M}$ s.t. $x^{\mathcal{M}} = 1, y^{\mathcal{M}} = 10^{100}, z^{\mathcal{M}} = -10^{100}$ we have $((x + y) + z)^{\mathcal{M}'} = \cdots = 0 \neq 1 = \cdots = x + (y + z)^{\mathcal{M}'}$
    - *"Catastrophic cancellation"*
  - *Why?* FLOATING POINT PROBLEM $10^{100} + 1 = 10^{100}$

WE HAVE NO PROBLEM HERE, SINCE GIVEN A NUMBER IT'S ALWAYS REPRESENTIBLE THE OPPOS.

- For floating points, $+$ and $\cdot$ are still commutative but not necessarily associative nor distributive

Arithmetic Theories

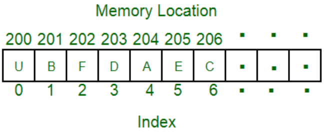
```python
x = 1
y = 1e100 #10^100
z = -1e100 #-10^100
print("x =", x, "y =", y, "z =", z)
print("(x+y) + z =", (x+y) + z)
print("x + (y+z) =", x + (y+z))
```

Shell

```
x = 1 y = 1e+100 z = -1e+100
(x+y) + z = 0.0
x + (y+z) = 1.0
```

Theory of Arrays

- Arrays are homogeneous and indexed collections of elements



- Let $\Sigma_{\mathcal{A}}$ be a signature with 2 interpreted functions read and write:
  - $read(a, i)$ returns the value of $a[i]$
  - $write(a, i, v)$ returns the array obtained by replacing $a[i]$ with v

- The theory of array $\mathcal{T}_{\mathcal{A}}$ is the set of all models of these axioms:
  (i) $(\forall a)(\forall i)(\forall v)\ read(write(a, i, v), i) = v$
  (ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \rightarrow read(write(a, i, v), j) = read(a, j)$
  (iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a, i) = read(a', i)) \rightarrow a = a'$ $\quad$ $(extensionality)$

Theory of Arrays

- Exercise: is the following formula $\mathcal{T}_{\mathcal{A}}$-satisfiable?

NOT SATISFIABLE

$$\underline{write(a,i,x) \neq b \ \wedge \ a = b} \ \wedge \ i = j \ \wedge$$
$$read(b,i) = y \ \wedge \ read(write(b,i,x),j) = y$$

- Hint: remember the axioms:
  - (i) $(\forall a)(\forall i)(\forall v)\ read(write(a,i,v),i) = v$
  - (ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \rightarrow read(write(a,i,v),j) = read(a,j)$
  - (iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a,i) = read(a',i)) \rightarrow a = a'$

Theory of Arrays

- Exercise: is the following formula $\mathcal{T}_{\mathcal{A}}$-satisfiable?

$$write(a, i, x) \neq b \ \wedge \ a = b \ \wedge \ i = j \ \wedge$$
$$read(b, i) = y \ \wedge \ read(write(b, i, x), j) = y$$

- Hint: remember the axioms:
  (i) $(\forall a)(\forall i)(\forall v)\ read(write(a, i, v), i) = v$
  (ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \rightarrow read(write(a, i, v), j) = read(a, j)$
  (iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a, i) = read(a', i)) \rightarrow a = a'$

Theory of Arrays

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- Hint: remember the axioms:

(i) $(\forall a)(\forall i)(\forall v)\ read(write(a, i, v), i) = v$
(ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \to read(write(a, i, v), j) = read(a, j)$
(iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a, i) = read(a', i)) \to a = a'$

Theory of Arrays

- Exercise: is the following formula $\mathcal{T}_{\mathcal{A}}$-satisfiable?

$$write(a, i, x) \neq a \ \wedge \ read(a, i) = y \ \wedge \ x = y$$

- Hint: remember the axioms:

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Theory of Arrays

- Exercise: is the following formula $\mathcal{T}_{\mathcal{A}}$-satisfiable?

$$write(a, i, x) \neq a \;\wedge\; read(a, i) = x$$

- Hint: remember the axioms:
  - (i) $(\forall a)(\forall i)(\forall v)\ read(write(a, i, v), i) = v$
  - (ii) $(\forall a)(\forall i)(\forall j)(\forall v)\ i \neq j \rightarrow read(write(a, i, v), j) = read(a, j)$
  - (iii) $(\forall a)(\forall a')\ ((\forall i)\ read(a, i) = read(a', i)) \rightarrow a = a'$

- We found a contradiction: let $a' = write(a, i, x)$, then we have $a' \neq a \xrightarrow{(iii)} (\exists j)\ read(a', j) \neq read(a, j) \xrightarrow{(ii)} i = j$. So it must be $read(a', i) \neq read(a, i) \equiv read(write(a, i, x), i) \neq read(a, i) \xrightarrow{(ii)} i \neq i$

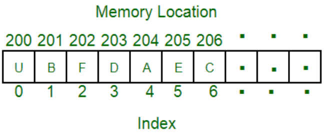
SMT-LIB Encoding

```
; Signature expansion.
(declare-fun a () (Array Int Real))
(declare-fun b () (Array Int Real))
(declare-fun i () Int)
(declare-fun j () Int)
(declare-fun x () Real)
(declare-fun y () Real)
; Formulas. select = read, store = write
(assert (not (= (store a i x) b)))
(assert (= (select b i) y))
(assert (= (select (store b i x) j) y))
(assert (= a b))
(assert (= i j))
; Checking satisfiability.
(check-sat)
```

arrays.smt2

Theory of Arrays

- The full $\mathcal{T}_\mathcal{A}$ theory is undecidable, but there are decidable fragments
- Useful theory for SW/HW verification
- In particular, arrays often used to abstract memory locations
  - Main advantage: the abstraction depends on the number of accesses to the memory rather than its size



Theory of bit-vectors

MODELS MEMORY → NOT FLOATING POINT PROBLEM

- The theory of bit-vectors $\mathcal{T}_{\mathcal{BV}}$ naturally handles verification of programs and circuits

- Constants of $\mathcal{T}_{\mathcal{BV}}$ are typically vectors of bits with fixed bit-width. Typical bit-vector operations:
  - string-like operations (selection, slicing, concatenation, ...),
  - logical operations (*bit-wise* NOT, OR, AND...)
  - arithmetic operations $(+, -, \cdot, \ldots, \ldots)$

- Straightforward reduction to SAT (bit-blasting)

- Exercise: if a, b, c are bit-vectors is $a[0:1] \neq b[0:1] \wedge (a \mid b) = c \wedge c[0] = 0 \wedge \underline{a[1] + b[1] = 0}$ a $\mathcal{T}_{\mathcal{BV}}$-satisfiable formula?

NOT SATISF

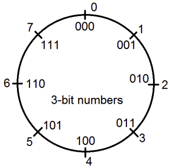
$\rightarrow 0a_2 \neq b_1 b_2$, $b_1 = 1$, $a_2 \neq b_2 \rightarrow a_2 + b_2 = 1$

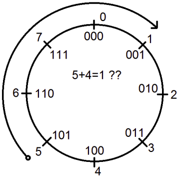
Theory of bit-vectors

- Bit-vectors better than integers to model machine operations

- E.g., $x = 200 \wedge y = x + 100 \wedge y > x$ with x, y unsigned 8-bit integers
  - The formula is valid if we consider *"classical"* arithmetic theories
  - But machines operate differently! 8-bit unsigned integers are enclosed in $[0, 2^8 - 1] = [0, 255]$ so $y = x + 100 = 300$ is out of range

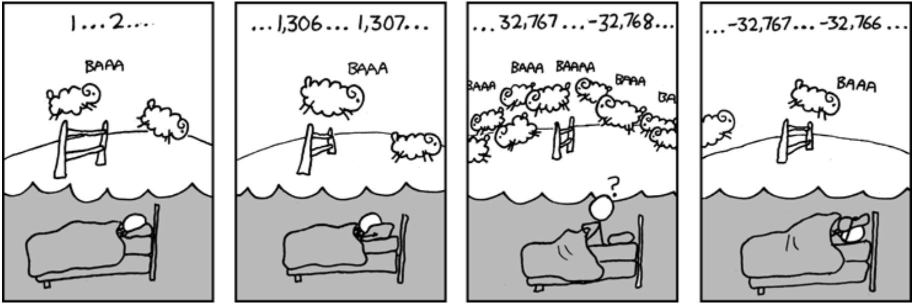
- In these cases typically $y = (x + 100) \mod 2^8 = 44$ so $y < x$
  - "wraparound"





Wraparound

- This may also happen for signed ints, e.g. a 16-bit signed int can only be in $[-2^{15}, 2^{15}-1] = [-32768, 32767]$ so e.g. $32767 + 3 = -32766$



https://imgs.xkcd.com/comics/cant_sleep.png

Theory of strings

- Over the last years the need for theory of strings emerged
  - Strings can enable vulnerabilities, especially in web programs
  - `https://mosca2023.github.io`
- Before, string solving typically handled with automata or bit-vectors
  - Automata limit the expressiveness and may be inefficient
  - Bit-vectors impose a limit on string length
- Theory of strings can handle complex operations on unbounded-length strings natively, often in conjunction with other theories
  - E.g. arithmetic theory for string length, or regular expressions
- Some CP proposals too (over bounded-length strings)

Theory of strings

- The theory of word equations is fundamental for string solving
- Fixed an alphabet $\mathcal{S}$, a word equation has form $L = R$ with L, R are concatenations of (uninterpreted) string constants
  - In other terms, L, R concatenate string variables and strings of $\mathcal{S}^*$
  - E.g. $X \cdot world \cdot Z = hello \cdot Y$
- The general theory of word equations is undecidable
  - Equivalent to arithmetic theory
- The quantifier-free theory of word equations is decidable
- Exercise: are the following formulas satisfiable?
  - $XY = YX \ \wedge \ X \neq Y$
  - $aX = Xb \ \wedge \ a \neq b$

Theory of strings

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- The general theory of word equations is undecidable
  - Equivalent to arithmetic theory
- The quantifier-free theory of word equations is decidable

- Exercise: are the following formulas satisfiable?
  - $XY = YX \ \wedge \ X \neq Y$ $\quad X = \epsilon, Y = a$
  - $aX = Xb \ \wedge \ a \neq b$ $\quad X$ starts with a and ends with b, so $X = aX_1b$, thus $aaX_1b = aX_1bb$, $aaaX_2bb = aaX_2bbb, \dots, a^{k+1}X_kb^k = a^kX_kb^{k+1}$ with $|X_k| < |a| + |b|$: unsat because both a (as prefix) and b (suffix) should fit into X_k

SMT in practice

- In practice, theories not isolated: real-world applications often need a combination of arithmetic, strings, arrays, ... e.g.
  - $a = b + 2 \wedge A = write(B, a, 4) \wedge (read(A, b + 3) = 2 \vee f(a - 1) \neq f(b))$
  - $aX = Yb \ \wedge \ X \in \mathcal{L}(d \mid c^* ab) \ \wedge \ |Y| = 2 \cdot |X|$
- The goal is to efficiently combine decision procedures for each theory
  - Efficient procedures already exist for many theories of interest
- Decidability issues
  - ...But we can always restrict to fragments

Take-home messages

- SMT extends SAT to solve formulas in (quantifier-free) FOL
  - functions, (constants), predicates with arity > 1
- Similar/orthogonal to CP, it tackles combinatorial problems from a "more logical" perspective (formulas)
  - More oriented to problems derived from software analysis
- Eventually, SMT solving eagerly or lazily relies on SAT solving
  - SAT $\sim$ machine language, SMT $\sim$ higher-level language
- Several theories of interest developed and studied over last decades
  - EUF, arithmetic, arrays, bit-vectors, strings, ...

Resources

- Handbook of Satisfiability – Chapter 12 "*Satisfiability Modulo Theories*" by C. Barrett, R. Sebastiani, S.A. Seshia, C. Tinelli
    - Search "Satisfiability Modulo Theories - EECS at UC Berkeley"
- Barrett, Clark, and Cesare Tinelli. "Satisfiability modulo theories." Handbook of model checking. Springer, Cham, 2018. 305-343.
- SAT/SMT schools
    - `https://sat-smt.in/`
- ...