5. Linear Programming

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Combinatorial Decision Making and Optimization

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Operations research

- Let's face (combinatorial) decision making and optimization from a different perspective
  - different paradigm, same goal: modeling and solving hard real-world optimization problems subject to different constraints
- Less "AI-oriented" and "logic-oriented", more "math-oriented"
- Less "constraints-centered", more "inequalities-centered"
- Based on relaxations and cutting-planes rather than propagation and search

Operations research

- Operation Research (OR) is a well-established field based on mathematical techniques for enhancing complex decision-making
- Originated in first half of 20^{th} century for military purposes, nowadays OR finds application in several fields, e.g.:
  - Finance
  - Manufacturing and Logistics
  - Simulations and stochastic models
  - Transportation
  - ...
- OR strongly influenced by linear programming techniques and its variants (ILP, MIP, NLP...)
  - As for CP, "programming" does not mean "coding" in this context...

Linear programming

- Linear programming (LP) is based on systems of linear (in-)equalities
- We typically resort to LP when we need an optimal allocation for a limited number of resources
- LP is among the most relevant scientific advances of last century: several applications in disparate fields — not only scientific fields
  - Agriculture, sports, marketing, environment etc.
  - Delta claimed 100.000.000$ saving per year using LP
  - H. Markowitz won Nobel prize for using LP to optimize portfolio profit
- Let's start with a toy example from MiniZinc tutorial
  - `https://www.minizinc.org/doc-2.5.5/en/modelling.html#an-arithmetic-optimisation-example`

Baking cakes

```
% We know how to make two sorts of cakes. A banana cake which takes 250g of
% self-raising flour, 2 mashed bananas, 75g sugar and 100g of butter, and a
% chocolate cake which takes 200g of self-raising flour, 75g of cocoa, 150g
% sugar and 150g of butter. We can sell a chocolate cake for $4.50 and a
% banana cake for $4.00. And we have 4kg self-raising flour, 6 bananas,
% 2kg of sugar, 500g of butter and 500g of cocoa. How many of each sort of
% cake should we bake for the fete to maximise the profit

var 0..100: b; % no. of banana cakes
var 0..100: c; % no. of chocolate cakes

% flour
constraint 250*b + 200*c <= 4000;
% bananas
constraint 2*b  <= 6;
% sugar
constraint 75*b + 150*c <= 2000;
% butter
constraint 100*b + 150*c <= 500;
% cocoa
constraint 75*c <= 500;

% maximize our profit
solve maximize 400*b + 450*c;

output ["no. of banana cakes = \(b)\n",
         "no. of chocolate cakes = \(c)\n"];
```

Geometric interpretation

- This is an example of linear problem: all the constraints are linear inequalities and we optimize a linear function

- Because only two variables are involved, we can geometrically represent the problem on the Cartesian plane
  - In 2 dim: equalities $\equiv$ straight-lines, inequalities $\equiv$ half-plane
  - In n dim: equalities $\equiv$ hyperplanes, inequalities $\equiv$ half-spaces

- Feasible solution $\equiv$ assignment satisfying all the constraints $\equiv$ points within the intersection of all half-spaces defined by inequalities
- Set of all solutions $\equiv$ set of all feasible points $\equiv$ feasible region
  - It is a convex polyhedron: it may be empty, bounded or unbounded

- GOAL: find a point within the feasible region where the objective function has maximal value

Geometric interpretation

Where is the feasible region? What point is optimal?

Drawn with `https://www.desmos.com/calculator`

Geometric interpretation

- By "*tuning the isolines*" of the objective function we find optimal solution $(3, 4/3)$, having optimal value $z = 400 \cdot 3 + 450 \cdot 4/3 = 1800\$$
- This is inconsistent with the model specification, where $b, c \in \mathbb{Z}$
- No worries, for now let's assume that we can sell slices of cake
  - If not, what would be the optimal solution?

Brewery problem

- Let's see another toy example: the brewery problem
  - From https://www.cs.princeton.edu/courses/archive/spr03/cs226/lectures/lp-4up.pdf
- A small brewery needs to produce ale and beer with limited resources:

Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Q.ty available	480	160	1190	

- How can they maximize profits?
  - Devote all resources to ale: 34 barrels of ale $\rightarrow$ 442$
  - Devote all resources to beer: 32 barrels of beer $\rightarrow$ 736$
  - 7.5 barrels of ale, 29.5 barrels of beer $\rightarrow$ 776$
  - 12 barrels of ale, 28 barrels of beer $\rightarrow$ 800$
  - ...

Brewery problem

- Let's see another simple example: the brewery problem
  - From https://www.cs.princeton.edu/courses/archive/spr03/cs226/lectures/lp-4up.pdf
- A small brewery needs to produce ale and beer with limited resources:

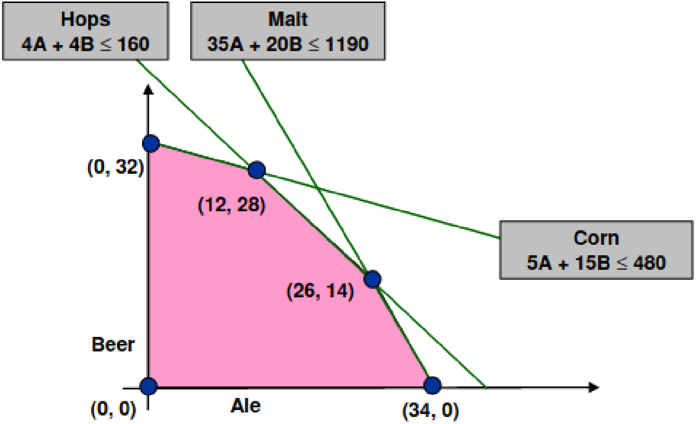
Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Q.ty available	480	160	1190	

- Let's formulate this as a LP problem:

$$\begin{array}{lrcrcl} \text{maximize} & 13A & + & 23B & & \\ \text{subject to} & 5A & + & 15B & \leq & 480 \\ & 4A & + & 4B & \leq & 160 \\ & 35A & + & 20B & \leq & 1190 \\ & A & , & B & \geq & 0 \end{array}$$

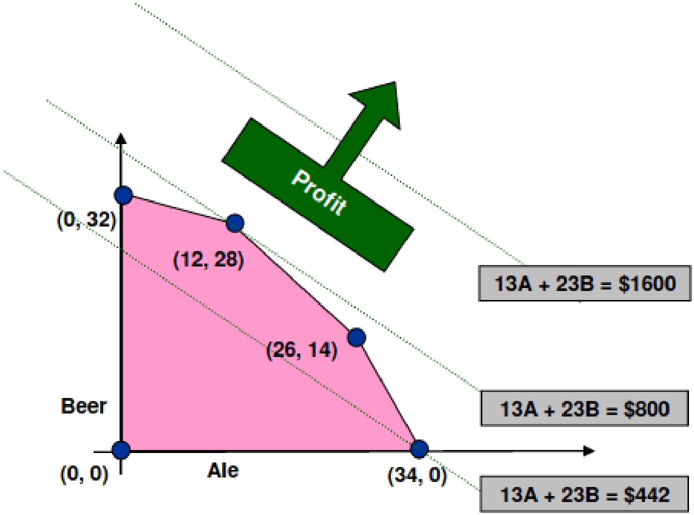
Geometry

The feasible region is:

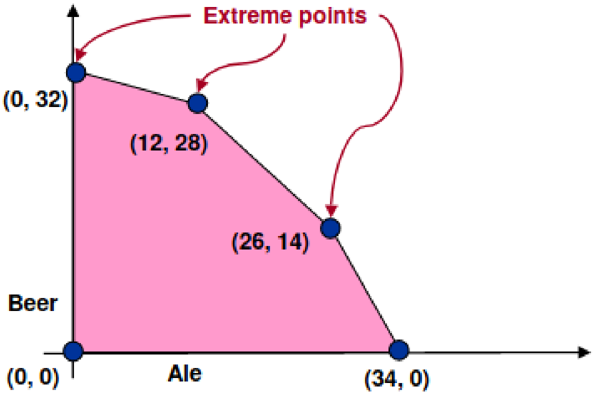


Geometry

We need to maximize $13A + 23B$:



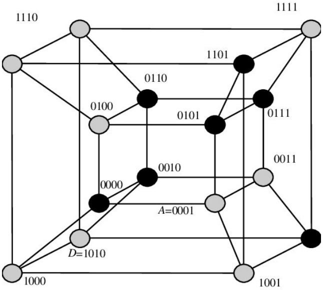
Observation: the optimal solution necessarily occurs at extreme point



Geometry

- Extreme point property: if an optimal solution for a LP problem exists, then there is one at the extreme point of its feasible region

- Good news: the number of extreme points is finite
- Bad news: the number of extreme points can be exponential
  - E.g., the *n*-dimensional hypercube has exactly 2^n vertices



Canonical form

- A LP problem in canonical form has the form:

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i & 1 \leq i \leq m \\ & x_j \geq 0 & 1 \leq j \leq n \end{array}$$

- m = no. of linear constraints, n = no. of non-negative variables
- $a_{i,j}, b_i, c_j \in \mathbb{R}$ are known parameters
- $\sum_{j=1}^{n} c_j x_j$ is the objective function to maximize
  - subject to m linear inequalities $\sum_{j=1}^{n} a_{i,j} x_j \leq b_i$

- Matrix form: $\max c \cdot x$ s.t. $Ax \leq b$ and $x \geq 0$
  - $c = \langle c_1, \ldots, c_n \rangle$, $x = \langle x_1, \ldots, x_n \rangle^t$, $b = \langle b_1, \ldots, b_m \rangle^t$,

    $$A = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots \\ a_{m,1} & \ldots & a_{m,n} \end{pmatrix}$$

Standard form

- A LP problem in standard form has the form:

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{i,j} x_j = b_i & 1 \le i \le m \\ & x_j \ge 0 & 1 \le j \le n \end{array}$$

- Matrix form: $\max c \cdot x$ s.t. $Ax = b$ and $x \geq 0$
- We can easily convert from canonical to equivalent standard form with m slack variables $y_1, \ldots, y_m \geq 0$
  - From n-dimensional to $(n+m)$-dimensional problem
- $\sum_{j=1}^{n} a_{i,j} x_j \leq b_i \implies \sum_{j=1}^{n} a_{i,j} x_j + y_i = b_i,\ y_i \geq 0$ for $i = 1, \ldots, m$
  - Objective function does not change

Example

- The above brewery problem is already in canonical form:

$$\begin{array}{lrcrcl} \max & 13A & + & 23B & & \\ \text{s.t.} & 5A & + & 15B & \leq & 480 \\ & 4A & + & 4B & \leq & 160 \\ & 35A & + & 20B & \leq & 1190 \\ & A & , & B & \geq & 0 \end{array}$$

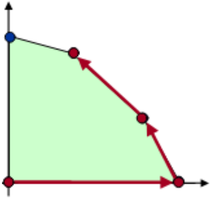
- The corresponding standard form is:

$$\begin{array}{lrcrcccccccl} \max & 13A & + & 23B & & & & & & & & \\ \text{s.t.} & 5A & + & 15B & + & S_C & & & & & = & 480 \\ & 4A & + & 4B & & & + & S_H & & & = & 160 \\ & 35A & + & 20B & & & & & + & S_M & = & 1190 \\ & A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0 \end{array}$$

- How to find an optimal solution with $n > 2$ variables?

Symplex algorithm

- **Symplex algorithm** developed by G. Dantzig in 1947
- General idea: start at some extreme point and iteratively move to a neighboring one that doesn't decrease the objective value
  - If no such extreme point exists, we found an optimal solution



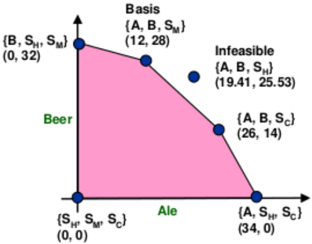
- It exploits linear algebra properties

Basis

- Given a LP problem P in standard form, a basis of P is a subset $\mathcal{B} = \{x_{i_1}, \dots, x_{i_m}\}$ of $m \leq n$ variables s.t. columns $A^{i_1}, \dots, A^{i_m}$ form a $m \times m$ invertible matrix $A_{\mathcal{B}}$
  - E.g., if $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ then $\mathcal{B} = \{x_1, x_2\}$ is not a basis because $A_{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ is not invertible, while $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are basis
- We can rewrite P by separating basic from non-basic variables: $\max(c_{\mathcal{B}} x_{\mathcal{B}} + c_{\mathcal{N}} x_{\mathcal{N}})$ s.t. $A_{\mathcal{B}} x_{\mathcal{B}} + A_{\mathcal{N}} x_{\mathcal{N}} = b$ and $x_{\mathcal{B}}, x_{\mathcal{N}} \geq 0$
  - $\mathcal{N} = \{x_1, \dots, x_n\} - \mathcal{B}$ are the non-basic variables of P
- By setting $x_{\mathcal{N}} = 0$, P becomes $\max(c_{\mathcal{B}} x_{\mathcal{B}})$ s.t. $A_{\mathcal{B}} x_{\mathcal{B}} = b$ hence $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b \in \mathbb{R}^m$ with objective value $c_{\mathcal{B}} A_{\mathcal{B}}^{-1} b$. This solution is called a basic solution for $\mathcal{B}$

Basis

- A basic solution for $\mathcal{B}$ is feasible iff $(\forall_{i=1}^{m})\ (x_{\mathcal{B}})_i \geq 0$
  - Basic feasible solution (BFS) $\equiv$ extreme point of feasible region
- A BFS for $\mathcal{B}$ is non-degenerate iff $(\forall_{i=1}^{m})\ (x_{\mathcal{B}})_i > 0$
  - If BFS non-degenerate, then it's represented by a unique basis
- Simplex method iteratively considers BFS $\widetilde{x}_1, \widetilde{x}_2, \ldots$ s.t. $c\widetilde{x}_k \geq c\widetilde{x}_{k-1}$



Brewery example

- The brewery example in standard form is:

$$\begin{array}{lrcrcccccccl}
\max & 13A & + & 23B & & & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_C & & & & & = & 480 \\
& 4A & + & 4B & & & + & S_H & & & = & 160 \\
& 35A & + & 20B & & & & & + & S_M & = & 1190 \\
& A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}$$

- Let's first pick an arbitrary feasible basis, e.g., $\mathcal{B} = \{S_C, S_H, S_M\}$
  - $A_{\mathcal{B}}$ is the 3×3 identity matrix, $\mathcal{N} = \{A, B\}$ so $A = B = 0$
- The BFS for $\mathcal{B}$ is $S_C = 480, S_H = 160, S_M = 1190$ with obj. value 0
  - Feasible, but we may improve it with a new basis $\mathcal{B}'$ adjacent to $\mathcal{B}$
  - $\mathcal{B}' = \mathcal{B} \cup \{x^{in}\} - \{x^{out}\}$ with $x^{in} \notin \mathcal{B}$ and $x^{out} \in \mathcal{B}$

Pivoting

- How to select entering variable x^{in} and leaving variable x^{out}?

- We can choose the x^{in} which *"increases more"* the objective value
  - x^{in} will increase from 0 ($x^{in} \in \mathcal{N}$) to a value ≥ 0 ($x^{in} \in \mathcal{B}'$)
  - In the brewery example, a unit increase in A increases the obj. value of 13; a unit increase in B increases the obj. value of 23: $x^{in} = B$

- Then, choose x^{out} by ensuring that $\mathcal{B}' = \mathcal{B} \cup \{x^{in}\} - \{x^{out}\}$ is a feasible basis
  - Minimum ratio rule: x^{out} is in the row i mimimizing ratio $\beta_i^{in}/\alpha_i^{in}$ where $\alpha_i^{in}, \beta_i^{in}$ are the x^{in} coefficient and known term in the i-th row
    - Otherwise, $\mathcal{B}'$ not feasible
  - In the brewery ex. $\min\{480/15, 160/4, 1190/20\} = \min\{32, 40, 59.5\} = = 32$ so $x^{out} = S_C$: the new basis will be $\mathcal{B}' = \{B, S_H, S_M\}$
  - Then, x^{in} is derived and its value replaced in all other equations

Pivoting

- Selecting x^{in}, x^{out} resp. means choosing a pivot column and a pivot row from the tableau representation of $\max(Z)$ s.t.:

$$
\begin{array}{rcrcrcrcrcl}
13A & + & 23B & & & & & - & Z & = & 0 \\
\hline
5A & + & 15B & + & S_C & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & = & 1190 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

- From 2nd row we get $B = \frac{480-5A-S_C}{15} = 32 - \frac{1}{3}A - \frac{1}{15}S_C$ and we substitute it in all other equations:

$$
\begin{array}{rcrcrcrcrcl}
\frac{16}{3}A & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & = & 550 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Pivoting

- Current basis $\{B, S_H, S_M\}$. What row/column should we select now?

$$
\begin{array}{rcrcrcrcrcl}
\frac{16}{3}A & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & = & 550 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Pivoting

- Current basis $\{B, S_H, S_M\}$. What row/column should we select now?

$$
\begin{array}{rcrcrcrcrcl}
\frac{16}{3}A & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & = & 550 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

- A contributes more than S_C in increasing Z: 1st column chosen ($x^{in} = A$)
  - The coefficients in the obj. function row are called reduced costs

Pivoting

- Current basis $\{B, S_H, S_M\}$. What row/column should we select now?

$$
\begin{array}{rcrcrcrcrcl}
\frac{16}{3}A & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & = & 550 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

- A contributes more than S_C in increasing Z: $x^{in} = A$
- The ratios for A are $\{32 \cdot 3, 32 \cdot \frac{3}{8}, 550 \cdot \frac{3}{85}\}$: 2^{nd} row chosen ($x^{out} = S_H$)

Pivoting

- Current basis $\{B, S_H, S_M\}$. What row/column should we select now?

$$
\begin{array}{ccccccccccc}
\frac{16}{3}A & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & = & 550 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

- A contributes more than S_C in increasing Z: $x^{in} = A$
- The ratios for A are $\{32 \cdot 3, 32 \cdot \frac{3}{8}, 550 \cdot \frac{3}{85}\}$: $x^{out} = S_H$
- New basis: $\{A, B, S_M\}$. We derive $A = \frac{3}{8} \cdot (32 + \frac{4}{15}S_C - S_H) = = 12 + \frac{1}{10}S_C - \frac{3}{8}S_H$ and substitute it in the other equations

Pivoting

- Current basis $\{A, B, S_M\}$. What row/column should we select now?

$$
\begin{array}{rrrrrrrrrrl}
 & - & S_C & - & 2S_H & & & - & Z & = & -800 \\
\hline
 & & B & + & \frac{1}{10}S_C & + & \frac{1}{8}S_H & & & = & 28 \\
A & & & - & \frac{1}{10}S_C & + & \frac{3}{8}S_H & & & = & 12 \\
 & & & - & \frac{25}{6}S_C & - & \frac{85}{8}S_H & + & S_M & = & 210 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Optimality

- Current basis $\{A, B, S_M\}$. What row/column should we select now?

$$
\begin{array}{rcrcrcrcrcl}
 & & & - & S_C & - & 2S_H & & & - & Z = -800 \\
\hline
 & & B & + & \frac{1}{10}S_C & + & \frac{1}{8}S_H & & & & = 28 \\
A & & & - & \frac{1}{10}S_C & + & \frac{3}{8}S_H & & & & = 12 \\
 & & & - & \frac{25}{6}S_C & - & \frac{85}{8}S_H & + & S_M & & = 210 \\
\hline
A & , & B & , & S_C & , & S_H & , & S_M & & \geq 0
\end{array}
$$

- All the reduced costs are ≤ 0: increasing the value of corresponding variables won't increase the obj. value
- We cannot improve the current feasible solution $\rightarrow$ we reached an optimal solution:
  - $S_C = S_H = 0$
  - $A = 12$, $B = 28$, $S_M = 210$
  - $-S_C - 2S_H - Z = -800 \implies Z = 800 - S_C - 2S = 800$

Optimality

- The simplex method performs an "optimality check": if all the reduced costs are ≤ 0, we reached an optimal solution
- This condition is sufficient: for any optimal solution there is at least a basis s.t. all the reduced costs are ≤ 0
- ...But it's not necessary: we may reach an optimal solution even if some reduced cost is >0
- E.g., $\max(x_1)$ s.t. $x_3 = 1 - x_2, x_4 = -x_1, x_i \geq 0$ with basis $\mathcal{B} = \{x_3, x_4\}$ corresponds to solution $x_3 = 1, x_1 = x_3 = x_4 = 0.$
- If we switch to $\mathcal{B} = \{x_3, x_1\}$ we get $\max(-x_4)$ s.t. $x_3 = 1 - x_2$, $x_1 = -x_4$: optimality condition is OK but last solution not improved

Optimal region

- The feasible region for a LP problem P in canonical form is a set $\mathcal{F}_P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ denoting a convex polyhedron
- The optimal region for a LP problem P in standard form is a set of solutions $\mathcal{O}_P = \{x^* \in \mathcal{F}_P \mid cx^* \geq cx, \forall x \in \mathcal{F}_P\}$
- Clearly $\mathcal{O}_P \subseteq \mathcal{F}_P$ and $\mathcal{F}_P = \emptyset \Rightarrow \mathcal{O}_P = \emptyset$
- If $\mathcal{O}_P$ is finite, then $|\mathcal{O}_P| = 1$ (hence $|\mathcal{O}_P| > 1 \Rightarrow \mathcal{O}_P$ infinite)
  - if $x_1, x_2 \in \mathcal{O}_P$ and $x_1 \neq x_2$ then all points in segment $\overline{x_1 x_2}$ are in $\mathcal{O}_P$ because $x_1, x_2 \in \mathcal{F}_P$ which is convex
- Is there any problem P such that $\mathcal{F}_P \neq \emptyset \ \wedge \ \mathcal{O}_P = \emptyset$?

Unboundedness

- Simply consider $\max(x)$ s.t. $x \geq 0$: we have $\mathcal{F}_P = [0, +\infty)$ but there is no $x^* \in \mathcal{R}$ s.t. $x^* \geq x$ forall $x \in \mathcal{F}_P$: $\mathcal{O}_P = \emptyset$
- In these cases P is said unbounded: no optimal solution exists
  - $\mathcal{F}_P$ is an unbounded polyhedron
- Simplex method also performs an "unboundedness check"
- With the tableau method seen above, we must ensure that no column j has reduced cost $\gamma_j > 0$ and coefficients $\alpha_{i,j} \leq 0$ for $i = 1, \ldots, m$
  - In the literature you can find different but equivalent formulations

Unboundedness

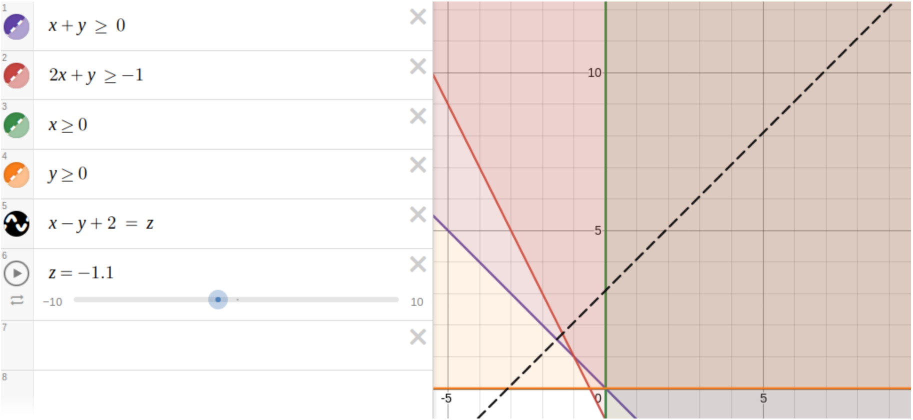
- For example, consider $\max(x - y + 2)$ s.t. $x + y \geq 0$, $2x + y \geq -1$, $x, y \geq 0$. The tableau of the problem can be written as:

$$
\begin{array}{rrrrrrrrrrrr}
 & x & - & y & & & & & - & Z & = & -2 \\
\hline
- & x & - & y & + & S_1 & & & & & = & 0 \\
- & 2x & - & y & & & + & S_2 & & & = & 1 \\
\hline
 & x & , & y & , & S_1 & , & S_2 & & & \geq & 0
\end{array}
$$

- With $\mathcal{B} = \{S_1, S_2\}$ we have a BFS with $x = y = 0$ and obj. value 2
  - Degenerate solution ($S_1 = 0$)

- Reduced cost of x is $1 > 0$ and its row coefficients are $-1, -2 \leq 0$: the problem is unbounded: x can be arbitrarily increased

Unboundedness



- E.g., from each feasible solution $(\tilde{x}, \tilde{y})$ we can always derive a better solution $(\tilde{x} + k, \tilde{y})$ for each $k > 0$

Simplex steps

Given LP P in standard form, we can (roughly) summarize the simplex method as:

0. Let $k \leftarrow 0$, let $\mathcal{B}_0$ a feasible base for P and go to 1.
1. If BFS of $\mathcal{B}_k$ is optimal then STOP, else go to 2.
2. If P unbounded then STOP, else go to 3.
3. Select an entering variable $x^{in} \notin \mathcal{B}_k$ and go to 4.
4. Select a leaving variable $x^{out} \in \mathcal{B}_k$ and go to 5.
5. Let $\mathcal{B}_{k+1} = \mathcal{B}_k \cup \{x^{in}\} - \{x^{out}\}$ and reformulate P accordingly. Let $k \leftarrow k + 1$ and go back to 1.

Baking cakes

- The former baking example in standard form is:

$$
\begin{array}{rrcrcrl}
\max & 400B & + & 450C & & & \\
\text{s.t.} & 250B & + & 200C & + & S_1 & = 4000 \\
 & 2B & & & + & S_2 & = 6 \\
 & 75B & + & 150C & + & S_3 & = 2000 \\
 & 100B & + & 150C & + & S_4 & = 500 \\
 & & & 75C & + & S_5 & = 500 \\
 & B & , & C & , & S_i \geq 0 & \quad i = 1, \ldots, 5
\end{array}
$$

- Exercise: Find the optimal solution through the simplex method.

Simplex properties

- If all the possible BFS are non-degenerate, the simplex method always terminates in a finite number of steps
  - The no. of vertices is finite and at each step we move from one vertex to another always strictly improving the obj. value
- Otherwise, possible stalling: we repeatedly change base without improving the obj. value (common in large scale applications)
  - Termination can be guaranteed by setting rules preventing possible (but unlikely) infinite loops $\mathcal{B}_k \rightarrow \mathcal{B}_{k+1} \rightarrow \cdots \rightarrow B_k$
- Worst-case time-complexity of simplex method is $O(2^n)$ but in practice is typically polynomial
- Alternative: interior point methods (polynomial) or approximation algorithms
  - Interior point better than simplex for problems with lots of vertices

Two-phase method

- So far we assumed that we can always find a feasible base
  - How to choose an initial feasible base $\mathcal{B}_0$?
  - What if the problem is unsatisfiable?

- The two-phase method finds (if any) an initial base for a standard problem P by first solving an "artificial" problem P' derived from P by adding fresh variables $s_1, \ldots, s_m$

- 2nd phase problem P : $\max(cx)$ s.t. $Ax = b, x \geq 0$
- 1st phase problem P' : $\max(-\sum_{i=1}^{m} s_i)$ s.t.
  - $\sum_{j=1}^{n} a_{i,j} x_j \; + s_i \; = b_i$ for $i \in \{k \in \{1, \ldots, m\} \mid b_k \geq 0\}$
  - $\sum_{j=1}^{n} a_{i,j} x_j \; - s_i \; = b_i$ for $i \in \{k \in \{1, \ldots, m\} \mid b_k < 0\}$
  - $s_i, x_j \geq 0$

Two-phase method

- 1^{st} phase problem P' : $\max(-\sum_{i=1}^{m} s_i)$ s.t.
  - $\sum_{j=1}^{n} a_{i,j}x_j \; + s_i \; = b_i$ for $i \in \{k \in \{1,\ldots,m\} \mid b_k \geq 0\}$
  - $\sum_{j=1}^{n} a_{i,j}x_j \; - s_i \; = b_i$ for $i \in \{k \in \{1,\ldots,m\} \mid b_k < 0\}$
  - $s_i, x_j \geq 0$

- E.g., if P is:

$$\begin{aligned} &\max(x_1 + 2x_2) \quad \text{s.t.} \\ &-x_1 - x_2 + x_3 = -1 \\ &x_1 + x_2 + x_4 = \; 2 \\ &x_1, \ldots, x_4 \geq 0 \end{aligned}$$

- Then P' is:

$$\begin{aligned} &\max(-s_1 - s_2) \text{ s.t.} \\ &-x_1 - x_2 + x_3 - s_1 = -1 \\ &x_1 + x_2 + x_4 + s_2 = \; 2 \\ &x_1, \ldots, x_4, s_1, s_2 \geq 0 \end{aligned}$$

Two-phase method

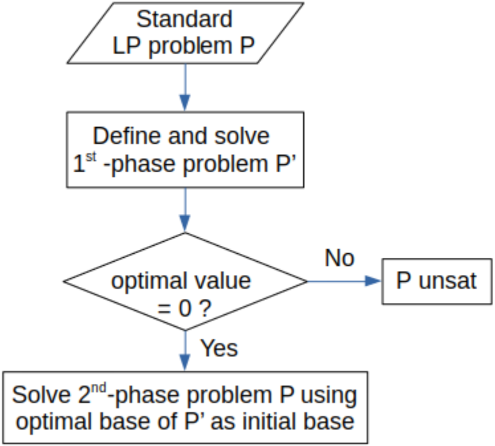
- Note that objective $-\sum_{i=1}^{m} s_i$ is always ≤ 0 and $\mathcal{B}' = \{s_1, \ldots, s_m\}$ is always a feasible basis corresponding to BFS $x_j = 0, s_i = |b_i|$
  - Hence, $\mathcal{F}_{P'} \neq \emptyset$ and $\mathcal{O}_{P'} \neq \emptyset$ (P' upper-bounded by 0)

- So we reformulate P' w.r.t. $\mathcal{B}'$. In the example above we get:

$$
\begin{aligned}
&\max(-3 - 2x_1 + 2x_2 - x_3 + x_4) \text{ s.t.} \\
&s_1 = 1 - x_1 - x_2 + x_3 \\
&s_2 = 2 - x_1 - x_2 - x_4 \\
&x_1, \ldots, x_4, s_1, s_2 \geq 0
\end{aligned}
$$

- Then we solve P' with the simplex method. A nice property is that $\mathcal{F}_P \neq \emptyset \iff \sum_{i=1}^{m} s_i = 0$:
  - If the optimal value of P' is < 0, then P is unsatisfiable,
  - Otherwise, from the basis corresponding to the optimal solution of P' we get an initial basis for P by removing s_i variables

Two-phase method



Duality

- Let's now tackle the brewery problem from a different angle

Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Q.ty available	480	160	1190	

- An entrepreneur wants to buy individual resources (corn, hops, malt) from brewer at minimum cost

- The brewer won't sell resources if $5C + 4M + 35H < 13$ (Ale profit) and $15C + 4M + 20H < 23$ (Beer profit)

- What would be the minimum unit cost for corn, hops, malt given the resource availability and the brewer's constraints?

Duality

- Let's now tackle the brewery problem from a different angle

Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Q.ty available	480	160	1190	

- The entrepreneur LP problem can be formulate as:

$$\begin{array}{rrcrcrcl} \text{minimize} & 480C & + & 160H & + & 1190M & & \\ \text{subject to} & 5C & + & 4H & + & 35M & \geq & 13 \\ & 15C & + & 4H & + & 20M & \geq & 23 \\ & C & , & H & , & M & \geq & 0 \end{array}$$

- Optimal solution: $C = 1, H = 2, M = 0$ with total cost 800\$
  - Exercise: transform in canonical and standard form

Duality

- The entrepreneur problem is the dual of the brewery problem
- Resource evaluation rather than allocation
- Two different perspectives, but same optimal value
  - Brewer knows that she can earn at most 800$
  - Entrepreneur knows that she has to spend at least 800$
- The duality concept is important (not only) in LP and can be extended to general (N)LP problems

Duality

- Let $P : \max(cx)$ s.t. $Ax = b, x \geq 0$ with $b \in \mathbb{R}^m, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$ the primal problem

- Its dual problem $\mathcal{D}(P) : \min(by)$ s.t. $A^t y \geq c$ with $y \in \mathbb{R}^m$ has:
  - a variable y_i for each constraint $\sum_{j=1}^{n} a_{i,j} x_j = b_i$ of P, $\quad i = 1, \ldots, m$
  - a constraint $\sum_{i=1}^{m} a_{j,i} y_i \leq c_j$ for each variable x_j of P, $\quad j = 1, \ldots, n$

- Exercise: find the dual of following primal problem:

$$
\begin{array}{rrcrcrcrcrl}
\max & x_1 & + & x_2 & & & & & & & \\
\text{s.t.} & 3x_1 & + & 2x_2 & + & x_3 & & & & & = 5 \\
 & 4x_1 & + & 5x_2 & & & + & x_4 & & & = 4 \\
 & & + & x_2 & & & & & + & x_5 & = 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq 0
\end{array}
$$

Duality

- Let $P : \max(cx)$ s.t. $Ax = b, x \geq 0$ with $b \in \mathbb{R}^m, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$ the primal problem

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- Exercise: find the dual of following primal problem:

$$
\begin{array}{rrrrrrrrrrrl}
\max & x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & 3x_1 & + & 2x_2 & + & x_3 & & & & & = 5 & (y_1) \\
 & 4x_1 & + & 5x_2 & & & + & x_4 & & & = 4 & (y_2) \\
 & & + & x_2 & & & & & + & x_5 & = 2 & (y_3) \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq 0 & \\
 & (con_1) & , & (con_2) & , & (con_3) & , & (con_4) & , & (con_5) & &
\end{array}
$$

Duality

- Let $P : \max(cx)$ s.t. $Ax = b, x \geq 0$ with $b \in \mathbb{R}^m, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$ the primal problem

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  - a variable y_i for each constraint $\sum_{j=1}^{n} a_{i,j} x_j = b_i$ of P, $\quad i = 1, \ldots, m$
  - a constraint $\sum_{i=1}^{m} a_{j,i} y_i \leq c_j$ for each variable x_j of P, $\quad j = 1, \ldots, n$

- Exercise: find the dual of following primal problem:

$$\begin{array}{rrcr}
\max \; x_1 + x_2 \text{ s.t.} & \Longrightarrow & & \min \; 5y_1 + 4y_2 + 2y_3 \text{ s.t.} \\
3x_1 + 2x_2 + x_3 \qquad = 5 & (y_1) & & 3y_1 + 4y_2 \qquad \geq 1 \\
4x_1 + 5x_2 + \quad x_4 \quad = 4 & (y_2) & & 2y_1 + 5y_2 + y_3 \geq 1 \\
x_2 + \qquad x_5 = 2 & (y_3) & & y_1 \geq 0 \\
x_1, x_2, x_3, x_4, x_5 \geq 0 & & & y_2 \geq 0 \\
 & & & y_3 \geq 0
\end{array}$$

Duality

- Let $P : \max(cx)$ s.t. $Ax = b, x \geq 0$ with $b \in \mathbb{R}^m, x \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}$ the primal problem

- Its dual problem $\mathcal{D}(P) : \min(by)$ s.t. $A^t y \geq c$ with $y \in \mathbb{R}^m$ has:
  - a variable y_i for each constraint $\sum_{j=1}^n a_{i,j}x_j = b_i$ of P, $\quad i = 1, \ldots, m$
  - a constraint $\sum_{i=1}^m a_{j,i}y_i \leq c_j$ for each variable x_j of P, $\quad j = 1, \ldots, n$

- What about the dual of the dual?

$$
\begin{array}{rcrc}
\max\ x_1 + x_2 \text{ s.t.} & \Longrightarrow & \min\ 5y_1 + 4y_2 + 2y_3 \text{ s.t.} & \Longrightarrow \\
3x_1 + 2x_2 + x_3 = 5 & (y_1) & 3y_1 + 4y_2 \geq 1 & (x_1) \\
4x_1 + 5x_2 + x_4 = 4 & (y_2) & 2y_1 + 5y_2 + y_3 \geq 1 & (x_2) \\
x_2 + x_5 = 2 & (y_3) & y_1 \geq 0 & (x_3) \\
x_1, x_2, x_3 \geq 0 & & y_2 \geq 0 & (x_4) \\
 & & y_3 \geq 0 & (x_5)
\end{array}
$$

Duality properties

- $\mathcal{D}(\mathcal{D}(P)) = P$: the dual of the dual is the primal
- Weak duality: the cost of any feasible solution of the primal is $\leq$ the cost of any solution of the dual: $(\forall x \in \mathcal{F}_P, \forall y \in \mathcal{F}_{\mathcal{D}(P)})\ cx \leq by$
  - by is an upper bound for obj. value of P
  - cx is a lower bound for obj. value of $\mathcal{D}(P)$
  - by "decreases" until a minimum value eventually reached $\searrow$
  - cx "increases" until a maximum value eventually reached $\nearrow$
  - If P unbounded, $\mathcal{D}(P)$ unfeasible: $(\mathcal{F}_P \neq \emptyset \wedge \mathcal{O}_P = \emptyset) \Rightarrow \mathcal{F}_{\mathcal{D}(P)} = \emptyset$
  - if $\mathcal{D}(P)$ unbounded, P unfeasible: $(\mathcal{F}_{\mathcal{D}(P)} \neq \emptyset \wedge \mathcal{O}_{\mathcal{D}(P)} = \emptyset) \Rightarrow \mathcal{F}_P = \emptyset$
- Strong duality: if primal and dual are feasible they have same optimal values: $\mathcal{F}_P, \mathcal{F}_{\mathcal{D}(P)} \neq \emptyset \Rightarrow (\forall x^* \in \mathcal{O}_P, \forall y^* \in \mathcal{O}_{\mathcal{D}(P)})\ cx^* = by^*$
  - Remember brewery example?

Possible cases

- Note that primal unbounded $\implies$ dual unfeasible but in general primal unfeasible $\not\Longrightarrow$ dual unbounded
  - E.g., $P : \max(2x_1 - x_2)$ s.t. $x_1 - x_2 \leq 1, -x_1 + x2 \leq -2, x_1, x_2 \geq 0$ is unfeasible, and so is $\mathcal{D}(P)$
  - Exercise: prove it!

- In summary (✓= possible, ✗= impossible):

	$\mathcal{D}(P)$ bounded	$\mathcal{D}(P)$ unbounded	$\mathcal{D}(P)$ unfeasible
P bounded	✓	✗	✗
P unbounded	✗	✗	✓
P unfeasible	✗	✓	✓

Dual simplex

- We can avoid to compute $\mathcal{D}(P)$ to apply the (primal) simplex on $\mathcal{D}(P)$ by running on P the dual simplex (*C.E. Lemke, 1954*)
  - C.E. Lemke, 1954
- Primal simplex: from feasible to optimal basis, preserving feasibility
- Dual simplex: from optimal basis (reduced costs ≤ 0, not necessarily feasible) to feasible basis, while preserving optimality
  - x^{out} = variable with minimum negative value
  - x^{in} = variable with maximum negative ratio
- Primal-dual: hybrid approach

Why duality?

- Theoretical purposes
  - E.g., finding a feasible solution is as hard as finding the optimal one: if we can find a solution for $\max(cx)$ s.t. $Ax = b, x \geq 0$ in $T(n, m)$ time, we can find one for $Ax = b, x \geq 0, A^t y \geq c, cx = by$ in $O(T(n, m))$
- Prove infeasibility of the primal problem (via dual simplex)
- Bounding the objective function (maybe with parallel solving)
  - Primal gives lower bound, dual gives upper bound
- Exploiting alternative/hybrid approaches
  - (primal-)dual simplex, (logic-based) Benders' decomposition
- Sensitivity analysis

Sensitivity analysis

- Sensitivity analysis refers to how the optimal solution of a problem is affected by changes in the input parameters
  - Post-optimality analysis
- If x^* is the optimal solution for standard LP problem P, will x^* be still feasible and/or optimal for a perturbed problem $\overline{P} \neq P$?
- $\overline{P}$: $\max(cx)$ s.t. $Ax = b, x \geq 0$ can be obtained from P by altering:
  - The known term: $b \rightsquigarrow \overline{b} = b + \Delta b$
  - The objective function coefficients: $c \rightsquigarrow \overline{c} = c + \Delta c$
  - The variables coefficients: $A \rightsquigarrow \overline{A} + \Delta A$

Sensitivity analysis

- Known term: Changing $b \rightsquigarrow \overline{b} = b + \Delta b$ can affect both the feasibility and optimality of current solution

Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Q.ty available	480	160	1190	

- E.g., for brewery problem if corn availability decreases from 480 to 479 the optimal solution $A = 12, B = 28$ is not feasible anymore and the objective value drops from 800 to 790 ($A = 13, B = 27$)
  - If hops decreases 1 unit, objective value drops to 787 ($A = 11, B = 28$)
  - If malt decreases up to 210 units, $A = 12, B = 28$ still optimal
  - In fact, $S_C = S_H = 0$ and $S_M = 210$ in the original optimal solution

Changing known term

- For brewery problem, increasing b doesn't affect feasibility, but can improve the obj. value
  - $\Delta b < 0 \rightarrow$ narrowing $\mathcal{F}_P$, $\Delta b > 0 \rightarrow$ extending $\mathcal{F}_P$
- In this case:
  - Increasing corn of ≥ 10 units improves the obj. value
  - Increasing hops of ≥ 4 units improves the obj. value
  - Increasing malt only never improves the obj. value
- Changing known term of $P \equiv$ changing objective function of $\mathcal{D}(P)$
- We can compute the impact of Δb on P without re-solving P: if $\mathcal{B}$ is an optimal basis for P, $\overline{x}_{\mathcal{B}} = A_{\mathcal{B}}^{-1}\overline{b} = A_{\mathcal{B}}^{-1}b + A_{\mathcal{B}}^{-1}\Delta b$
  - If feasible ($\overline{x}_{\mathcal{B}} \geq 0$), the objective value $c_{\mathcal{B}}A_{\mathcal{B}}^{-1}\overline{b}$ can change as well

Geometric interpretation

Baking example: what ingredients should we (*not*) buy first to increase profit? If we can't buy, which ones should we sell without altering profit?

Changing costs

- Cost coefficients: Changing $c \rightsquigarrow \overline{c} = c + \Delta c$ can't affect feasibility, but may involve:
  - Loss of optimality of current solution
  - Different objective value for current solution (still optimal)
- E.g., in the brewery example if the profit of beer is 39 instead of 23 the optimal solution will be still $A = 12, B = 28$
  - But total profit would be 1248 instead of 800!
  - The same applies if beer profit drops to 14 (total profit 548)
- If beer profit is outside $[14, 39]$ the solution changes as well
  - E.g., if beer profit $= 13$ then $A = 26, B = 14$ is more convenient
- Again, we don't need to re-solve P to assess the impact of Δc

Changing constraints

- The impact of changing a coefficient $A \rightsquigarrow \overline{A} = A + \Delta A$ depends on whether $\overline{a}_{i,j}$ refers to a variable x_j in the optimal basis:
- If not, $x_j = 0$ so the current solution is still feasible and its value won't change. But the obj. function can change: we may lose optimality
- If yes, we can't say much: we need to re-solve with $\overline{A}$
- E.g., if the ale production would now require 4 units of corn instead of 5, then $A = 12, B = 28$ still feasible but no more optimal
  - $A = 11, B = 29$ would be better in this case (total profit 810)

Take-home messages

- Linear programming (LP) is one of the main areas of Operations Research (OR) field
- LP is about solving problems with linear constraints/objective function in a canonical or standard form
- The feasible region for a LP is a convex polyhedron
  - it can be empty, bounded, unbounded
- The simplex algorithm is a well-known method to tackle LP problems
  - Worst-case exponential, typically polynomial
  - Worst-case polynomial algorithms exist for LP problems

Take-home messages

- Simplex works by moving from one extreme point of the feasible region to a "not-worse one" up to an optimal point
  - Interior point methods traversing feasible region may scale better
- Two-phase method to check feasibility and get initial basis
- We can solve the dual of a LP problem: different perspective, same optimal value
- Sensitivity analysis to assess the effect of perturbations of the original problem
  - Post-optimality analysis, often doable without re-solving the problem

Resources

- https://www.cs.princeton.edu/courses/archive/spr03/cs226/lectures/lp-4up.pdf
- CDMO course a.y. 2020/21
- ...