Group Coursework Submission Form (PA)
Specialist Masters Programme

Please list all names of group members: (Surname, first name) 1. Theo Cadier 2. Gael Chen 3. Lorenzo Rossi	4. Stephane Leboyer 5. **GROUP NUMBER:** 3
MSc in: Quantitative Finance	
Module Code: SMM272	
Module Title: Risk Analysis	
Lecturer: Gianluca Fusai	**Submission Date:20.03.2025**

Declaration:

By submitting this work, we declare that this work is entirely our own except those parts duly identified and referenced in my submission. It complies with any specified word limits and the requirements and regulations detailed in the coursework instructions and any other relevant programme and module documentation. In submitting this work we acknowledge that we have read and understood the regulations and code regarding academic misconduct, including that relating to plagiarism, as specified in the Programme Handbook. We also acknowledge that this work will be subject to a variety of checks for academic misconduct.

We acknowledge that work submitted late without a granted extension will be subject to penalties, as outlined in the Programme Handbook. Penalties will be applied for a maximum of five days lateness, after which a mark of zero will be awarded.

Marker's Comments (if not being marked on-line):

Deduction for Late Submission:

Final Mark:

For Students:

Once marked please refer to Moodle for your final coursework grade, including your Peer Assessment grade.

Risk Analysis

Gael Chen, Lorenzo Rossi, Theo Cadier, Stéphane Leboyer

March 20, 2025

Word count: 956

Contents

1 Question 1

1.1 Statistical Analysis of the portfolio returns

Statistic	Value
Mean	0.09198%
Standard Deviation	1.5085%
Kurtosis	5.77128
Skewness	-0.4400
Jarque-Bera Statistic	3930.8

Table 1: Summary Statistics

1.1.1 Mean and Standard Deviation

- The mean daily return of your portfolio is approximately 0.09198%, indicating a positive average daily return.
- The standard deviation, measuring the volatility of returns, is about 1.51%, suggesting moderate daily price movement.

1.1.2 Skewness and Kurtosis

- The skewness is -0.44, indicating that the return distribution is slightly skewed to the left, meaning it has a longer tail on the left side of the distribution.
- The kurtosis is approximately 5.77, which is significantly higher than 3 (the kurtosis of a normal distribution). This indicates a leptokurtic distribution, suggesting that the returns are prone to exhibiting extreme values more than a normal distribution would.

1.1.3 Jarque-Bera Test

- The Jarque-Bera statistic is 3930.82 with a p-value of 0.0, strongly rejecting the null hypothesis that the data is normally distributed. This result is significant as it confirms the non-normality of returns.

1.1.4 Visual Analysis (from the histograms)

- The first histogram of daily returns shows a bell-shaped curve but with visible deviations from symmetry, primarily reflected in the slight left skewness.
- The second histogram comparing the standardized returns to the normal distribution shows that while the distribution of returns follows the general shape of a normal distribution, there are discrepancies particularly in the tails, aligning with the results of kurtosis and skewness.

The Q-Q plot further shows that portfolio returns significantly deviate from normality, displaying pronounced fat tails; a characteristic of leptokurtosis. The extreme left tail deviations suggest that normal-based VaR models might underestimate the true risk of extreme negative returns in the portfolio.

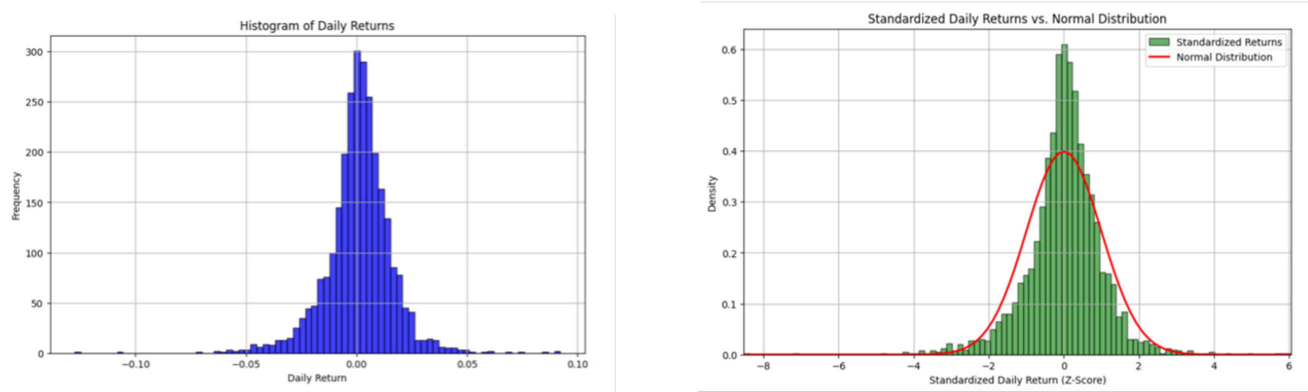


Figure 1: Histograms

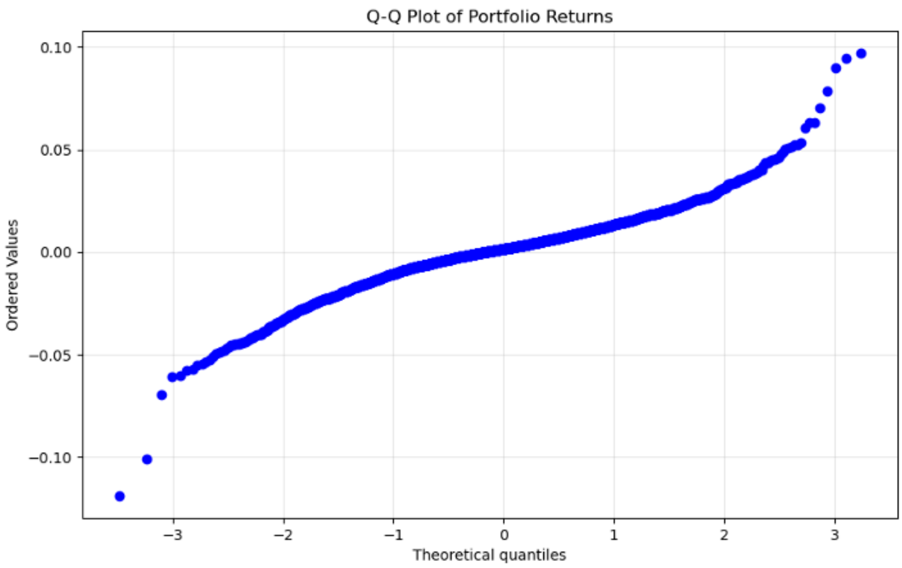


Figure 2: Q-Q plot

1.2 Different approaches

1.2.1 Top-down approach – normal (AP1_VaR)

In the parametric (variance-covariance) approach to Value at Risk (VaR) calculation, returns are assumed to follow a normal distribution, simplifying the estimation of VaR. This method calculates VaR by using the rolling mean and standard deviation of the returns over a specified window, which accounts for recent fluctuations in portfolio performance. VaR is then computed as the negative sum of the rolling mean and the product of the z-score and the rolling standard deviation, providing a measure of the maximum expected loss under normal market conditions on a given day.

$$r(t, t+\Delta) \sim \mathcal{N}(\mu_\Delta, \sigma_\Delta^2), \tag{1}$$

$$\widehat{VaR}_{1-\alpha}(t, t+\Delta) = -\left(\hat{\mu}_\Delta + z_{1-\alpha}\hat{\sigma}_\Delta\right). \tag{2}$$

1.2.2 Top-down approach t-distribution (AP1t_VaR)

As the statistical analysis indicates, there is a leptokurtosis of the distribution, but skewness is relatively subtle. Thus, the t-distribution is used for the alternative parametric approach as it could better cover fatter tails and higher peak than a normal distribution, also the kurtosis is a bit higher.

1.2.3 Delta-Gamma Approach (AP1B_VaR)

The Delta-Gamma approach to estimating VaR improves upon the previous parametric approach with the inclusion of higher moments: Skewness and kurtosis. Furthermore, this approach can more accurately estimate VaR when the distribution is non-gaussian by utilising the Cornish-Fisher-Expansion (Jaschke and Jiang, 2002). The expansion modifies the z-score to account for the higher moments; using these higher moments the approach can more accurately capture the non-normal return distribution.

$$z_{cf} = q_p + \frac{(q_p^2 - 1)S(X)}{6} + \frac{(q_p^3 - 3q_p)K(X)}{24} - \frac{(2q_p^3 - 5q_p)S^2(X)}{36} \tag{3}$$

1.2.4 Non-Parametric Bootstrap Approach for VaR Estimation (AP2_VaR)

The non-parametric bootstrap approach to estimating VaR involves creating numerous bootstrap samples from historical return data, which are then used to empirically estimate the distribution of returns. For each day in the dataset, after the initial window of 124 days, 1000 bootstrap samples are drawn with replacement from the windowed returns. The VaR is calculated by averaging the VaR estimates from these samples, determined by the percentile corresponding to the confidence level (90% and 99%). This technique, by relying on the actual data distribution rather than theoretical distribution assumptions, provides a more accurate and tailored risk assessment, capturing the tails of the distribution more effectively than standard parametric methods.

1.2.5 Bottom-up Monte Carlo Simulation Approach (AP3_VaR)

This bottom-up approach to VaR estimation utilizes a Monte Carlo simulation framework integrated with the RiskMetrics model for estimating the Exponentially Weighted Moving Average (EWMA) covariance matrix. By setting a decay factor (λ=0.94), the method emphasizes more recent observations, thereby adapting more quickly to changes in market volatility than standard covariance estimations. With the mean vector and covariance matrix derived, 2000 simulations of portfolio returns are generated for each day, using the multivariate normal distribution. This high number of simulations ensures that the distribution of simulated portfolio returns is well-defined and robust. For each simulation, the exact formula for portfolio logarithmic return is applied: $r_p(t) = \ln\left(\sum_{i=1}^{N} w_i e^{r_i(t)}\right)$ This formula considers the compounding effects of returns, which is crucial for accurately assessing performance over time. The final VaR is determined by calculating the required quantile (10th for 90% confidence and 1st for 99% confidence) from the distribution of the simulated portfolio returns.

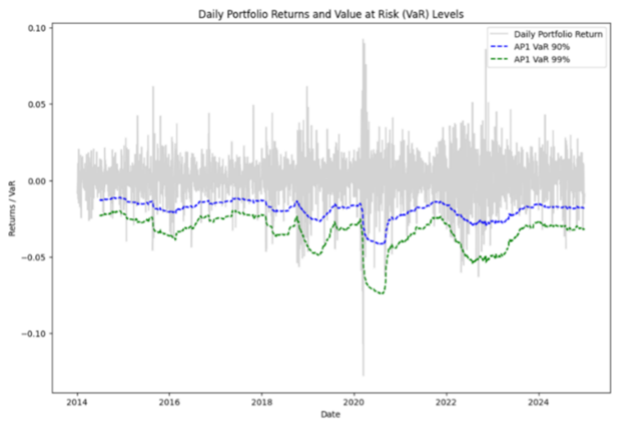
1.2.6 Analytical VaR Estimation Using Taylor Approximation and RiskMetrics (AP4_-VaR)

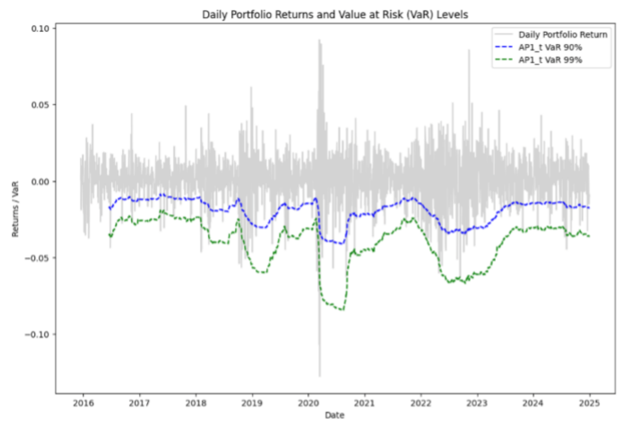
This approach employs a bottom-up parametric method using the Taylor approximation and RiskMetrics to estimate VaR. By assuming a normal distribution for portfolio returns, the method begins with the calculation of a rolling EWMA covariance matrix (λ=0.94). For each period beyond an initial 124-day window, it computes the portfolio's mean and standard deviation from the mean vector and covariance matrix of the log returns. VaR is then analytically derived by adjusting the portfolio mean by a z-score multiplier (corresponding to the desired confidence level) times the portfolio standard deviation.

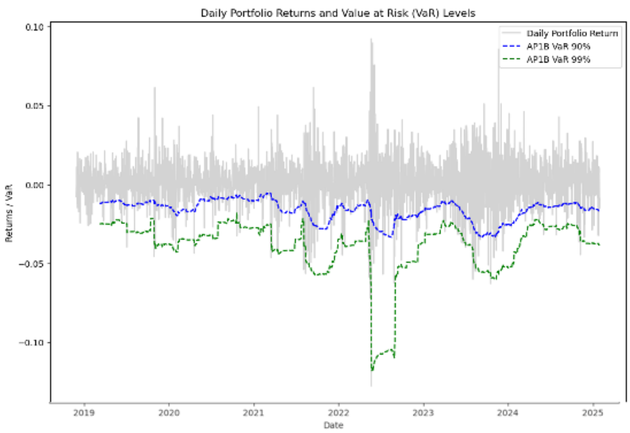
1.3 Violation of each forecast model:

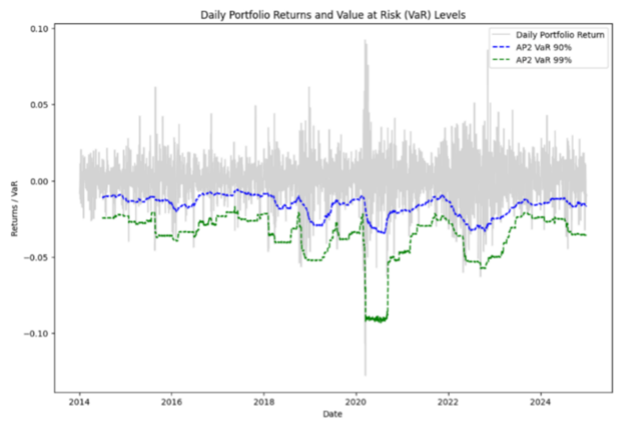
Model	Total Violations	Total Non-Violation	Violation Rate (%)
AP1_VaR_90 (90%)	257	2387	9.72
AP1_VaR_99 (99%)	75	2569	2.84
AP1t_VaR_90 (90%)	182	2094	7.996
AP1t_VaR_99 (99%)	39	2237	1.714
AP1B_VaR_90	270	2373	11.38
AP1B_VaR_99	35	2608	1.34
AP2_VaR_90 (90%)	280	2364	10.59
AP2_VaR_99 (99%)	55	2589	2.08
AP3_VaR_90 (90%)	265	2379	10.0
AP3_VaR_99 (99%)	73	2571	2.76
AP4_VaR_90 (90%)	260	2384	9.84
AP4_VaR_99 (99%)	73	2571	2.76

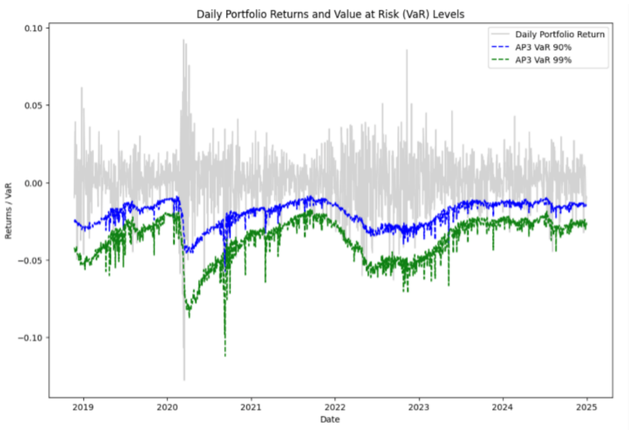
Table 2: VaR Violation Statistics

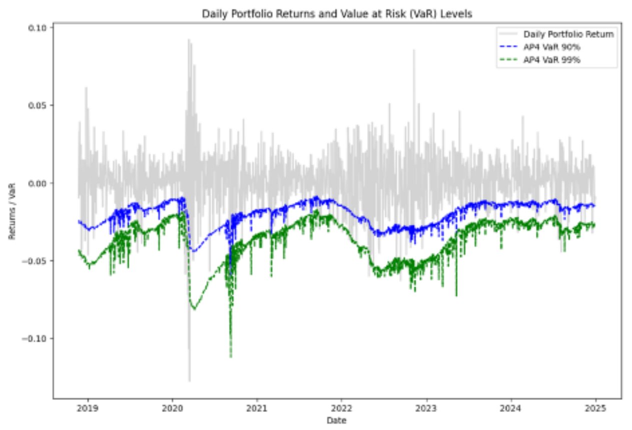












1.4 Statistical analysis

As the daily VaR forecast is achieved, the forecast is compared with a day ahead to see if the actual return is lower than the forecast. If so, the violation is counted. The unconditional test from Christoffersen (2003) is used. Unconditional tests test if the observed frequency of VaR breaches matches the expected frequency over a given period. However, as we have a long period of ten years and makes the underflow of the pi-hat value and made the likelihood ratio invalid to make the conclusion. Thus, the adjusted version of unconditional test is used for testing the case with 90% VaR instead of the test statistic stated on the book. From the table below, we could see that, all the 90% VaR has good unconditional coverage but none of 99% VaR has good conditional coverage. However, the binomial distribution assumes the independence between periods, and it might not be the case for the return, thus the conditional coverage test is needed for testing the case where there is clustering.

Model	LR_uc	P-value	Good unconditional coverage
AP1_VaR_90 (90%)	0.2321	0.6300	Yes
AP1_VaR_99 (99%)	60.1780	8.66e-15	No
AP1t_VaR_90 (90%)	10.8234	0.0010	No
AP1t_VaR_99 (99%)	9.6448	0.0019	No
AP1B_VaR_90 (90%)	0.1357	0.7126	Yes
AP1B_VaR_99 (99%)	2.5475	0.1105	Yes
AP2_VaR_90 (90%)	1.2769	0.2585	Yes
AP2_VaR_99 (99%)	20.8643	4.93e-06	No
AP3_VaR_90 (90%)	0.0015	0.9690	Yes
AP3_VaR_99 (99%)	60.1780	8.66e-15	No
AP4_VaR_90 (90%)	0.0818	0.7749	Yes
AP4_VaR_99 (99%)	55.9880	7.29e-14	No

Table 3: Likelihood Ratio Test for Unconditional Coverage

The conditional coverage tests additionally check for the independence of breach occurrences in the likelihood ratio calculation. The test is crucial as serial correlation in violations could indicate clustering of extreme losses. Compared to the unconditional coverage, the conditional one is more robust. From the result, we could see that only two models with bottom-up approaches at 90% have good conditional coverage at 95% significance level.

Model	LR	P-value	Good conditional coverage
AP1_VaR_90 (90%)	4.484	0.0342	No
AP1_VaR_99 (99%)	10.429	0.0012	No
AP1t_VaR_90 (90%)	5.054	0.0246	No
AP1t_VaR_99 (99%)	5.054	0.0246	No
AP1B_VaR_90 (90%)	4.466	0.0346	No
AP1B_VaR_99 (99%)	6.515	0.0107	No
AP2_VaR_90 (90%)	8.310	0.0039	No
AP2_VaR_99 (99%)	8.230	0.0041	No
AP3_VaR_90 (90%)	2.401	0.1213	Yes
AP3_VaR_99 (99%)	5.119	0.0237	No
AP4_VaR_90 (90%)	2.472	0.1159	Yes
AP4_VaR_99 (99%)	5.581	0.0182	No

Table 4: Likelihood Ratio Test for Conditional Coverage

The transformed probability is achieved by calculating the probability of getting the return less than or equal to the actual return using the risk model's forecasted distribution, Ft. If the risk model adopted the right distribution, the transformed probability should follow be uniformly distributed between 0 and 1 over time, a uniform distribution. From the histogram of transformed probabilities, it's evident that these probabilities are uniformly distributed. The high frequency at high and relatively high frequency close to 1 indicate that the model might not be able to capture the tail risks adequately and the distributional assumptions are mis-specified. The t-distribution with 10 degrees of freedom is also plotted and we could see that it follows a similar trait as the normal distribution, which is not close to an uniform distribution. It also means that p-distribution is not well-specified to predict the VaR in this case.

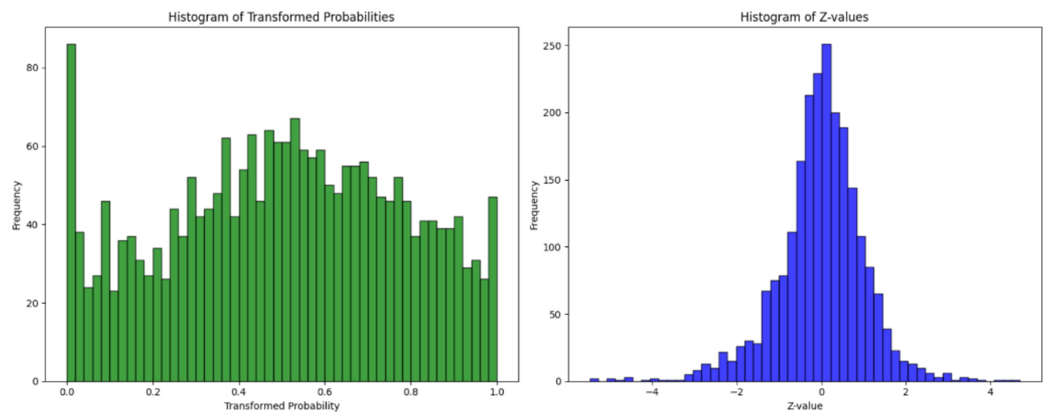


Figure 3: Transformed probability and frequency

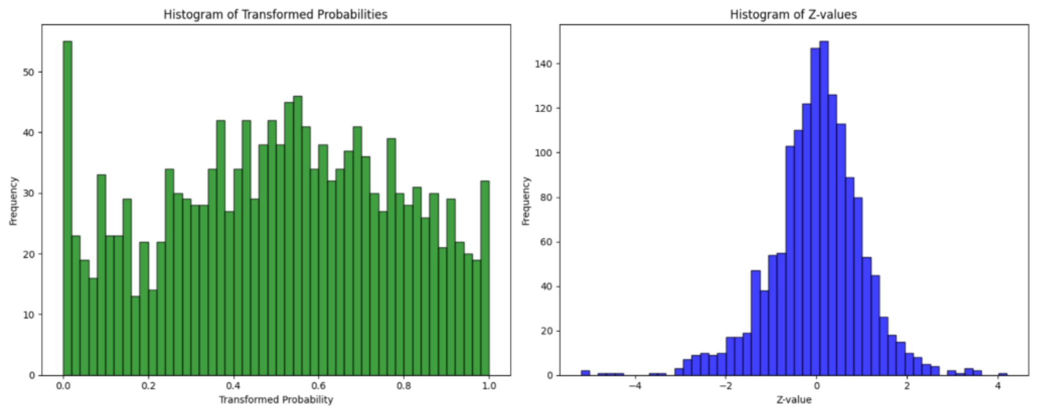


Figure 4: Transformed probability and frequency

1.5 Discussion

Approach 1 (top-down parametric approach) assumes a specific probability distribution for the asset return (normal in this case), and by imposing a strong distributional assumption, estimation error could be reduced since the entire data sample informs the parametric fit (Jorion, 2012). However, in the case of forecasting, the normal distribution assumptions are not met as the J&B normality rejected and there exists the leptokurtic distribution. As Ballotta and Fusai (2017) observed, a Gaussian VaR cannot capture heavy tails or asymmetry in returns, so it can underestimate the tail risk. Parametric approach is a "model dependent" approach and it only deliver reliable forecasts only if the return distribution is correctly specified. Hence, the top-down parametric approach is computationally convenient and statistically efficient, but it carries the risk of error from incorrect distribution assumptions.

Approach 1B can better capture the non-normal distribution by using higher moments. The Cornish-Fisher expansion accounts for skewness and excess kurtosis by adjusting the quantiles of the normal distribution. Since are return distribution is leptokurtic, the delta-gamma approach offers better accuracy for estimating the VaR (Aktaş and Sjöstrand, 2011). The Kupiec test shows that the model provides an accurate estimation for the VaR at both confidence levels implying that the model has good unconditional coverage however the model fails the conditional coverage test at both confidence levels implying that the violations are not independently distributed over time and therefore tend to cluster. Ultimately, the delta-gamma model with the Cornish-Fisher expansion captures the overall level of risk well but fails to adequately capture volatility persistence. As such the model would not be suitable during periods of market stress where violations are more likely to occur consecutively than in isolation. Therefore, the delta-gamma approach may perform better if combined with a GARCH model that is more suited for volatility clustering.

Approach 2 makes makes no assumption about the return distribution – instead, it uses the empirical distribution of past returns as the forecast for future risk. Bootstrapping techniques can enhance this by resampling the historical returns with replacement many times to build a more robust loss distribution or to compute confidence intervals for VaR estimates. The key advantage of historical/bootstrapped VaR is that it does not impose a parametric model for returns. This means it naturally captures whatever distributional features the data have, including fat tails, skewness, and volatility patterns, without having to explicitly model them. Ammann and Reich (2000) stated that historical simulation handles portfolio non-linearities naturally. However, parametric approach usually smooths them out (also could be observed from the VaR plot of Approach 1). The flip side is that historical/bootstrapped VaR is only as good as the data window. It assumes the past will represent the future. As we can see from the plot, especially the 99% VaR, the VaR adjusted pretty much but overestimated the risk for the following time because there was a past shock which might not happen again. Also, Jorion (2007) points out that an empirical quantile has much higher uncertainty – another sample of data could give a quite different VaR – whereas a parametric VaR (if valid) will be more stable. In summary, AP2 methods excel in capturing the true distribution (especially tail behavior) of past data , but they can lag current risk and suffer higher variability in VaR estimates when forecasting future risk

Approach 3 adopted the Bottom-up approach with normality assumption. It got relatively higher violation rate at 99% compared to the other approaches. The simulation is done based on the covariance matrix and mean calculated. There could be volatility clustering and presence of outliers which might make the covariance measure not accurate. The amount of observation (window of 124 days) for 6 assets might not be long enough to establish a reliable covariance. Furthermore, because thousands or millions of scenarios may be required for a stable VaR estimate (especially at very high confidence levels), Monte Carlo can be computationally intensive and slower than analytical or historical methods (Maxwell & Vuurel, 2014). Also, Monte Carlo with EWMA assumes a specific decay factor (like λ =0.94). If market volatility shifts more rapidly than the EWMA can adapt (λ=0.94 too high making it too slow), the VaR can be "too passive for recent market conditions" (Oanea & Anghelache, 2015). A fixed λ might not capture abrupt volatility changes, so there is a risk that the model underestimates risk in fast-changing markets.

The final approach is also light in computation and easy to apply. However, it also based on the Gaussian distribution assumption and VaR is calculated by approximating the portfolio's loss distribution analytically. Approximation error could be a big issue – it may underestimate extreme losses because it truncates the risk representation. Not reliable for portfolios with significant optionality during large market moves (Mina & Xiao, 2001). Also, as the normality assumption is chosen in the case, the model risk still exists.

Most of these approaches hinge on specific distributional assumptions, yet real-world returns often reflect a wide range of influences. Incorporating additional variables, especially leading indicators, to the forecast the price movement could substantially improve the model's predictive accuracy. Moreover, exploring alternative distributions may yield a more trustworthy representation of the actual return dynamics.

2 Question 2

2.1 Introduction

Portfolio optimization is a cornerstone of quantitative finance, aiming to balance risk and return through systematic allocation strategies. Among the various approaches, the **Risk Parity Portfolio** stands out as a method that ensures each asset contributes equally to the portfolio's total risk, measured through **Component Value at Risk (CVaR)**. By constructing weights that minimize the dispersion of individual **Conditional VaRs (CVaRs)**, this methodology offers an alternative to traditional allocation frameworks that may overly concentrate risk in a few high-volatility assets.

To gain a comprehensive understanding of portfolio risk allocation, we contrast the Risk Parity Portfolio with two alternative strategies:

- **The Maximum Diversification Portfolio**, which seeks to maximize the **diversification ratio**—a measure of risk dispersion—by optimizing asset weights relative to their volatilities and correlations.
- **The Equally-Weighted Portfolio**, a simple yet widely used benchmark where all assets receive equal weights, often leading to suboptimal risk-adjusted returns due to disproportionate exposure to volatile assets.

To empirically assess these strategies, we will first compute **Component VaR** using both **parametric (covariance-based) and non-parametric** methods. We will then evaluate the performance of each portfolio over an out-of-sample period using **key financial risk metrics**, including:

- **Sharpe Ratio** – measuring risk-adjusted returns under the assumption of a zero risk-free rate.
- **Maximum Drawdown** – assessing the worst peak-to-trough decline in portfolio value.
- **VaR Violations** – counting instances where losses exceed the estimated Value at Risk (VaR) at a 95% confidence level.
- **Skewness and Excess Kurtosis**, such as volatility, turnover, or tail risk metrics, to gain deeper insights into portfolio behavior under different market conditions.

Through this analysis, we aim to quantify the trade-offs inherent in each approach, evaluating their robustness in managing portfolio risk while maintaining return efficiency. Our findings will provide insight into the practical applications of **risk parity**, its advantages over traditional weighting schemes, and its effectiveness in real-world risk management.

2.2 Theoretical Framework

2.2.1 Risk Parity Portfolio

The **Risk Parity Portfolio** is constructed such that each asset contributes equally to the overall portfolio risk. A common measure of risk contribution is the **Component Value at Risk (CVaR)**, which represents the marginal risk contribution of each asset to the portfolio's total Value at Risk (VaR).

Given a portfolio with weights w, asset returns r, and covariance matrix Σ, the portfolio's Value at Risk (VaR) at confidence level α is given by:

$$\mathrm{VaR}_\alpha = \Phi^{-1}(\alpha) \cdot \sigma_p, \tag{4}$$

where $\Phi^{-1}(\alpha)$ is the quantile function of the standard normal distribution, and σ_p is the portfolio's standard deviation:

$$\sigma_p = \sqrt{w'\Sigma w}. \tag{5}$$

The **Component VaR** for asset i is defined as:

$$\text{CVaR}_i = w_i \frac{\partial \text{VaR}}{\partial w_i}. \tag{6}$$

In a Risk Parity Portfolio, the goal is to find weights w such that each asset contributes equally to the total risk:

$$\text{CVaR}_i = \text{CVaR}_j, \quad \forall i, j. \tag{7}$$

This condition is typically solved using numerical optimization techniques, minimizing the dispersion of individual asset risk contributions.

2.2.2 Maximum Diversification Portfolio

The **Maximum Diversification Portfolio (MDP)** seeks to maximize the diversification ratio, which is defined as:

$$D(w) = \frac{w'\sigma}{\sqrt{w'\Sigma w}}, \tag{8}$$

where σ is the vector of individual asset volatilities. The optimization problem is formulated as:

$$\max_w \frac{w'\sigma}{\sqrt{w'\Sigma w}} \tag{9}$$

subject to the constraints:

$$w'1 = 1, \quad w \geq 0. \tag{10}$$

This portfolio maximizes diversification by ensuring that risk is spread optimally across assets.

2.2.3 Equally-Weighted Portfolio

The **Equally-Weighted Portfolio (EWP)** is the simplest allocation strategy, assigning equal weights to all assets:

$$w_i = \frac{1}{N}, \quad \forall i = 1, \ldots, N. \tag{11}$$

While easy to implement, this approach does not account for differences in asset volatility or correlation, often leading to suboptimal risk-adjusted returns.

2.3 Portfolio Construction

In this section, we construct three portfolio strategies based on the dataset from Question 1:

- **Risk Parity Portfolio (RPP)** – weights are determined by ensuring equal risk contribution using Component Value at Risk (CVaR).
- **Maximum Diversification Portfolio (MDP)** – weights are optimized to maximize the diversification ratio.
- **Equally-Weighted Portfolio (EWP)** – each asset is assigned an equal weight.

Table 5: Portfolio Weights for Different Strategies

Asset	Risk Parity (RPP)	Max Diversification (MDP)	Equally-Weighted (EWP)
AAPL	0.17636735	0.19987679	0.16666667
MSFT	0.16894164	0.04035726	0.16666667
IBM	0.23478263	0.34855106	0.16666667
NVDA	0.1122478	0.15305913	0.16666667
GOOGL	0.16927668	0.09502316	0.16666667
AMZN	0.1383839	0.1631326	0.16666667

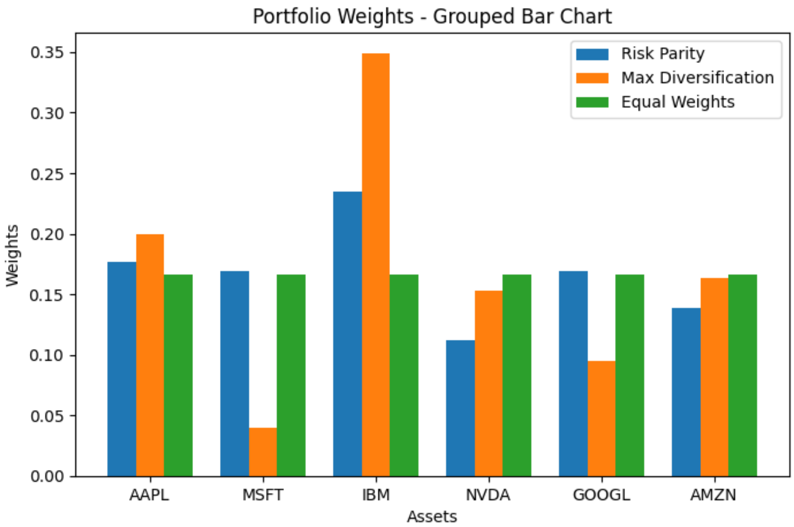


Figure 5: Portfolio Weights Comparison Across Strategies

The asset weights for each portfolio strategy are summarized in Table 5.

To provide a visual representation of the portfolio allocations, Figure 5 displays a bar plot of the asset weights for each portfolio.

The Risk Parity Portfolio weights are computed by solving an optimization problem to equalize the risk contribution of each asset. The Maximum Diversification Portfolio is obtained by maximizing the diversification ratio, while the Equally-Weighted Portfolio simply assigns equal weights to all assets.

2.4 Performance Evaluation

In this section, we evaluate the performance of the three portfolio strategies using key financial risk metrics: **Cumulative Portfolio Returns**, **Drawdowns**, and **Value-at-Risk (VaR) Violations**. These metrics help assess the trade-offs between risk and return and provide insights into the stability of each portfolio strategy.

2.4.1 Cumulative Portfolio Returns

Cumulative returns measure the **total growth of an initial investment** over time. This metric allows us to compare the **long-term performance** of different portfolios. Higher cumulative returns indicate superior performance, but they should be interpreted alongside risk measures to ensure they are achieved efficiently.

Figure 6 displays the cumulative returns for each portfolio strategy.

The plot reveals how each strategy performed over the evaluation period, highlighting differences in return stability and overall profitability.

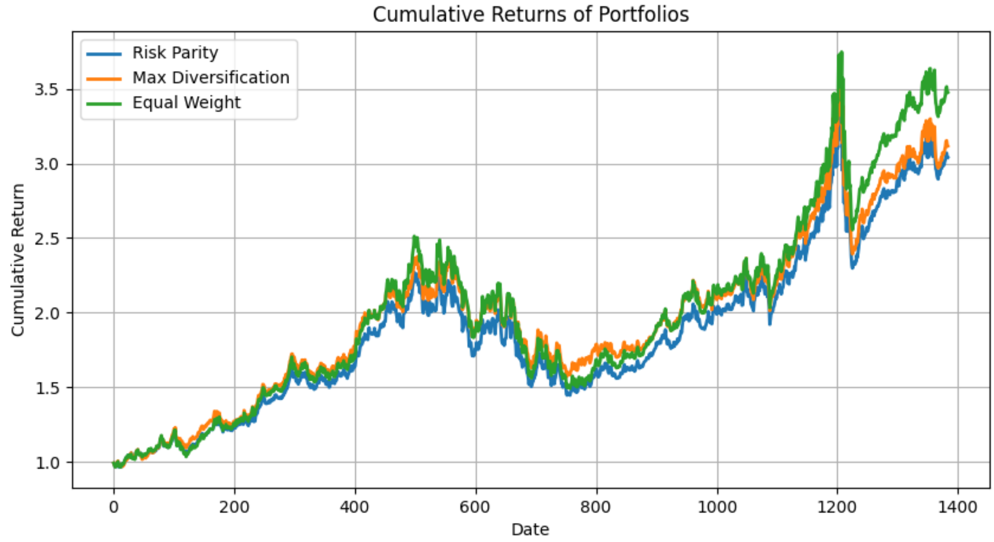


Figure 6: Cumulative Portfolio Returns Over Time

2.4.2 Drawdowns

Drawdowns quantify the **largest decline from a portfolio's peak value to its subsequent trough** before recovering. This metric is essential for understanding **risk exposure and capital preservation**, as larger drawdowns indicate higher downside risk.

Figure 7 illustrates the drawdowns experienced by each portfolio.

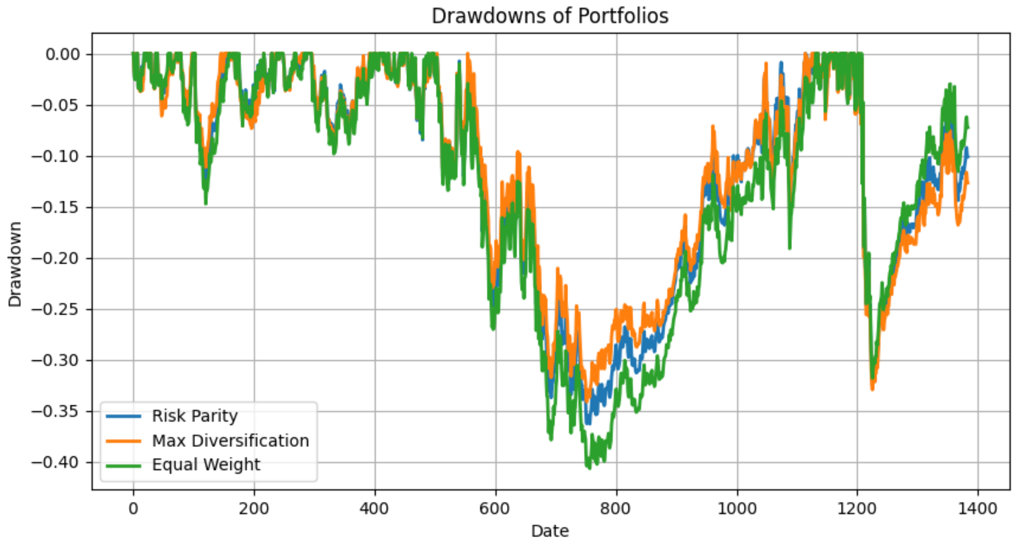


Figure 7: Drawdowns for Each Portfolio Strategy

A portfolio with smaller drawdowns suggests better **risk management and resilience during market downturns**.

2.4.3 Value-at-Risk (VaR) Violations

Value-at-Risk (VaR) is a statistical measure that estimates the **potential loss in a portfolio over a given time horizon at a specific confidence level**. VaR violations occur when **actual losses**

exceed the estimated VaR threshold, indicating potential underestimation of risk.

Figure 8 presents the number of VaR violations for each portfolio.

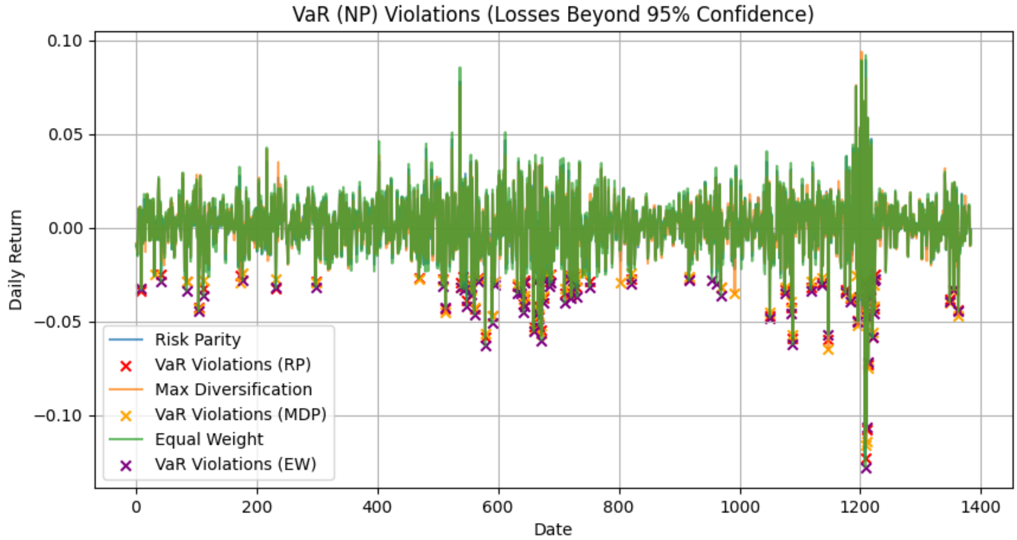


Figure 8: Number of VaR Violations Across Portfolios

Frequent VaR violations suggest that the risk model may not be capturing tail risk accurately, which could lead to unexpected losses.

2.5 Summary of Performance Metrics

To quantitatively compare the three portfolio strategies, we summarize key risk and return metrics in Table 6. These metrics provide insights into the trade-offs between risk-adjusted returns, downside risk, and distributional characteristics.

Table 6: Performance Metrics for Each Portfolio Strategy

Portfolio	Sharpe Ratio	Max Drawdown	VaR Violations		Skewness	Excess Kurtosis
			Non-Param	Param		
Risk Parity (RPP)	0.916951	-0.703953	106	121	-0.519803	6.588561
Max Diversification (MDP)	0.941901	-6.547856	96	106	-0.551991	6.693772
Equally-Weighted (EWP)	0.965797	-19.292575	121	126	-0.466187	5.528489

Sharpe Ratio: Measures risk-adjusted returns by evaluating return per unit of volatility. Higher values indicate better reward for risk taken.

Maximum Drawdown: Represents the largest observed peak-to-trough decline, assessing downside risk.

VaR Violations: Count of instances where actual losses exceed Value-at-Risk (VaR). We compute violations using both **non-parametric** (historical simulation) and **parametric** (based on normal distribution assumptions) approaches.

Skewness: Measures the asymmetry of return distributions. Negative values indicate a heavier left tail (higher downside risk), while positive values suggest a right-skewed distribution.

Excess Kurtosis: Measures the "tailedness" of the return distribution. Higher kurtosis indicates a greater probability of extreme returns (both gains and losses).

2.6 Normality Analysis

Assessing the normality of portfolio returns is crucial in risk management and financial modeling. Many traditional risk measures, such as Value-at-Risk (VaR) and the Sharpe Ratio, assume normally distributed returns. However, empirical evidence suggests that financial returns often exhibit skewness, excess kurtosis, and heavy tails.

To test the normality of returns for each portfolio, we employ the following statistical tests:

- **Shapiro-Wilk Test**: A widely used test that assesses whether a sample follows a normal distribution. A low p-value ($p < 0.05$) suggests non-normality.
- **Kolmogorov-Smirnov (KS) Test**: Compares the empirical distribution of returns to a theoretical normal distribution using the sample mean and standard deviation. A significant result indicates deviation from normality.
- **Jarque-Bera Test**: Checks for deviations from normality by examining skewness and kurtosis. A high test statistic and low p-value indicate non-normality.

To visually assess normality, we plot both the **histogram with kernel density estimate (KDE)** and the **Q-Q plot** for each portfolio.

2.6.1 Risk Parity Portfolio (RPP)

The test results for the Risk Parity Portfolio are as follows:

- **Shapiro-Wilk Test**: Statistic = 0.9548, p-value = 0.0000
- **Kolmogorov-Smirnov Test**: Statistic = 0.0801, p-value = 0.0000
- **Jarque-Bera Test**: Statistic = 545.5297, p-value = 0.0000

Since all p-values are below 0.05, we reject the null hypothesis of normality. Figure 9 presents the histogram and Q-Q plot for the Risk Parity Portfolio.

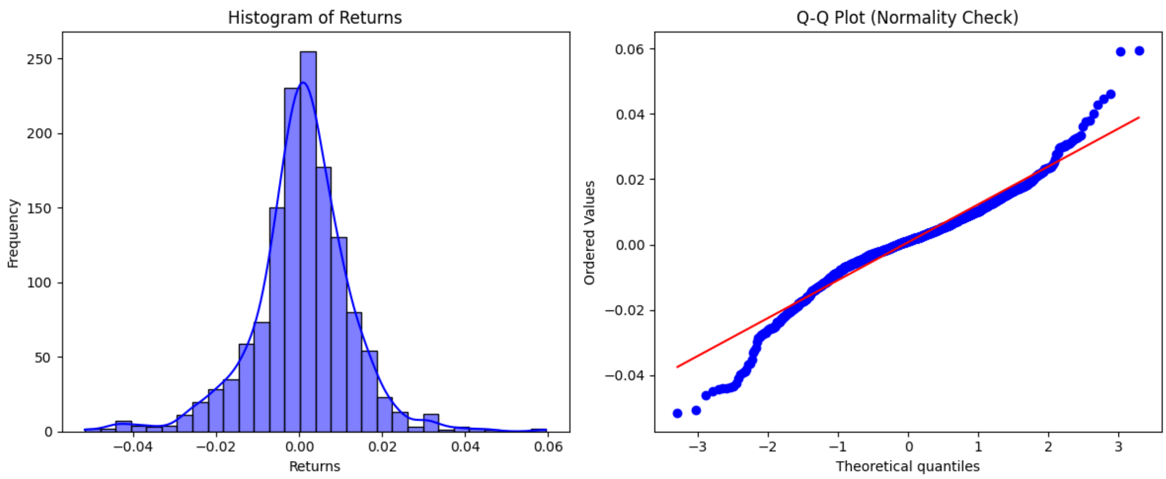


Figure 9: Normality Check for Risk Parity Portfolio (Histogram and Q-Q Plot)

2.6.2 Maximum Diversification Portfolio (MDP)

The test results for the Maximum Diversification Portfolio are as follows:

- **Shapiro-Wilk Test**: Statistic = 0.9613, p-value = 0.0000
- **Kolmogorov-Smirnov Test**: Statistic = 0.0777, p-value = 0.0000
- **Jarque-Bera Test**: Statistic = 425.4686, p-value = 0.0000

Again, all p-values indicate that the returns do not follow a normal distribution. Figure 10 illustrates the histogram and Q-Q plot for the Maximum Diversification Portfolio.

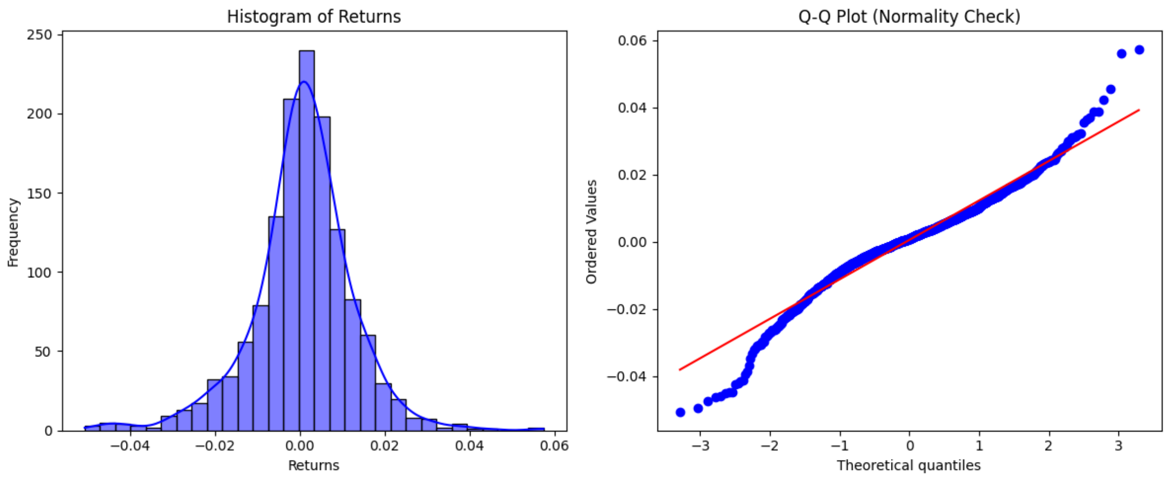


Figure 10: Normality Check for Maximum Diversification Portfolio (Histogram and Q-Q Plot)

2.6.3 Equally-Weighted Portfolio (EWP)

The test results for the Equally-Weighted Portfolio are as follows:

- **Shapiro-Wilk Test**: Statistic = 0.9530, p-value = 0.0000
- **Kolmogorov-Smirnov Test**: Statistic = 0.0813, p-value = 0.0000
- **Jarque-Bera Test**: Statistic = 567.5649, p-value = 0.0000

The results confirm that the Equally-Weighted Portfolio also does not exhibit normally distributed returns. Figure 11 presents its normality plots.

2.6.4 Conclusion

For all three portfolios, the **Shapiro-Wilk, Kolmogorov-Smirnov, and Jarque-Bera tests reject the null hypothesis of normality**, as all p-values are effectively zero.

Additionally, the histograms reveal **fat tails** in the return distributions, and the Q-Q plots show **significant deviations from the normal distribution line**, particularly in the extremes. These findings confirm that none of the portfolios exhibit normally distributed returns.

This has important implications for risk management, as traditional models assuming normality (such as parametric VaR) may underestimate tail risk. Future analysis may benefit from alternative risk models, such as those based on **extreme value theory (EVT)** or **non-Gaussian distributions**.

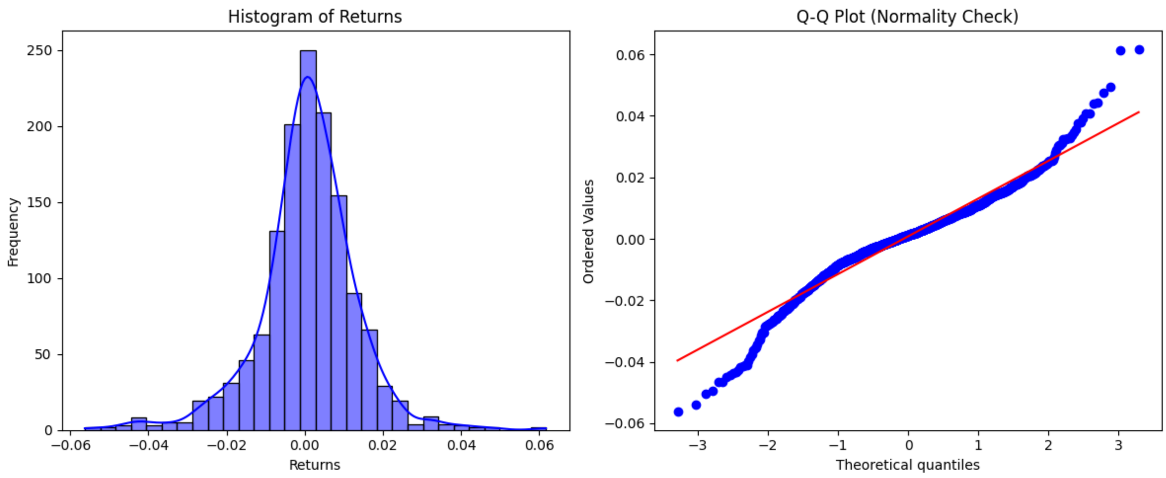


Figure 11: Normality Check for Equally-Weighted Portfolio (Histogram and Q-Q Plot)

2.7 Discussion and Interpretation

2.7.1 Portfolio Performance Comparison

The results indicate significant differences in performance across the three portfolio strategies.

- The **Sharpe Ratio** suggests that the Maximum Diversification Portfolio (MDP) had the best risk-adjusted returns, slightly outperforming the Risk Parity (RPP) and Equally-Weighted (EWP) portfolios.
- However, the **Maximum Drawdown** of MDP was substantially higher, indicating greater downside risk exposure. In contrast, the Risk Parity Portfolio exhibited better drawdown control.
- The **VaR Violations** highlight that the Equally-Weighted Portfolio experienced the highest number of breaches, suggesting it may not be well-optimized for risk management. The Risk Parity Portfolio had fewer violations, implying more effective risk allocation.

These findings suggest that while the Maximum Diversification Portfolio offers superior returns, it comes with higher drawdowns, making it less attractive for risk-averse investors. The Risk Parity Portfolio, by ensuring equal risk contribution, appears to provide a more balanced trade-off between risk and return.

2.7.2 Implications of Non-Normal Returns

The normality tests confirm that none of the portfolio returns follow a normal distribution. This has several implications:

- Traditional risk measures, such as parametric VaR and the Sharpe Ratio, may not fully capture the tail risks.
- The presence of **fat tails** and **negative skewness** suggests that extreme losses are more frequent than a normal distribution would predict.
- Given these findings, alternative risk models such as **Extreme Value Theory (EVT)**, **GARCH models**, or **semi-parametric VaR approaches** may provide more robust risk estimations.

2.7.3 Trade-Offs Between Portfolio Strategies

Each portfolio strategy presents unique strengths and weaknesses:

- **Risk Parity Portfolio (RPP)**: Provides a stable risk allocation and fewer VaR violations, but may sacrifice some return potential.
- **Maximum Diversification Portfolio (MDP)**: Achieves the highest Sharpe Ratio but exhibits higher drawdowns, making it less suitable for risk-averse investors.
- **Equally-Weighted Portfolio (EWP)**: Simple and intuitive, but fails to optimize risk allocation, leading to higher VaR violations.

2.7.4 Conclusion and Future Considerations

This study highlights the importance of selecting a portfolio strategy that aligns with an investor's risk-return profile. While the Maximum Diversification Portfolio maximized returns, its high drawdowns indicate potential stability issues. The Risk Parity approach, on the other hand, demonstrated a more balanced risk management framework.

Given the observed non-normality in returns, future research could explore the use of **non-Gaussian risk models** and **dynamic allocation strategies** to further enhance portfolio resilience. Additionally, stress testing under extreme market conditions could provide deeper insights into portfolio robustness.

3 Question 3

3.1 Introduction

This report presents the computation of Value at Risk (VaR) for a bond using various methodologies. The bond has a face value of 100, a 10-year maturity, and an annual coupon rate of 5%. The current price of the bond is 99, and daily fluctuations in yield to maturity (YTM) are assumed to follow an independent and identically distributed Gaussian process with a mean of 0 and standard deviation of 0.006.

3.2 Methodology

We compute VaR at a 99% confidence level across various time horizons (1, 10, 20, ..., 90 days) using the following methods:

1. Exact formula via full revaluation.
2. Exact formula via delta approximation.
3. Exact formula via delta-gamma approximation.
4. Monte Carlo simulation with delta approximation.
5. Monte Carlo simulation with delta-gamma approximation.
6. Monte Carlo simulation with full revaluation.

The computations for delta, gamma, and theta are performed using Monte Carlo simulations with 10,000 iterations. The VaR estimates are derived based on the worst-case YTM scenarios, revaluing the bond accordingly.

3.2.1 VaR Computation for the Monotonic Decreasing Case

If the bond price is a decreasing function of the YTM, we proceed as follows:

1. Compute the current bond price $C(P(t), t, T)$.
2. Determine the best YTM level at the assigned VaR horizon, assuming Gaussian returns:
$$YTM_{best}(t + \Delta n) = YTM(t)e^{\mu\Delta n + z_\alpha \sigma \sqrt{n}} \tag{12}$$
3. Revaluate the bond price at the best scenario:
$$C(YTM_{best}, t + \Delta n, T) \tag{13}$$
4. The bond VaR is obtained by:
$$VaR_\alpha(t, t + \Delta n) = |C(YTM_{best}, t + \Delta n, T) - C(P(t), t, T)| \tag{14}$$

3.2.2 VaR Computation using Delta-Gamma Approximation

Using Taylor's expansion, we approximate the bond price change as:

$$\Delta C \approx \Theta \Delta t + \Delta \frac{dP}{P} + \frac{1}{2}\Gamma \left(\frac{dP}{P}\right)^2 \tag{15}$$

where:

- $\Theta = \frac{\partial C}{\partial t}$ (Theta effect)
- $\Delta = \frac{\partial C}{\partial P} P$ (Delta effect)
- $\Gamma = \frac{\partial^2 C}{\partial P^2} P^2$ (Gamma effect)

3.3 Results

3.3.1 Yield to Maturity (YTM) Computation

The Yield to Maturity (YTM) is computed by solving the bond pricing equation:

$$P = \sum_{t=1}^{10} \frac{C}{(1+y)^t} + \frac{F}{(1+y)^{10}} \tag{16}$$

where P is the current bond price (99), C is the annual coupon payment (5), F is the face value (100), and y is the unknown YTM. Solving this numerically, we obtain a YTM of approximately **5.1303%**.

3.3.2 Probability of a 10% Price Decline within 30 Days

To estimate the probability that the bond price declines by 10% within 30 days, we model price changes using the first-order Taylor approximation:

$$\frac{\Delta P}{P} \approx -D^* \cdot \Delta y \tag{17}$$

where D^* is the modified duration, computed as:

$$D^* = \frac{D}{1 + YTM} \approx 7.7017 \tag{18}$$

Given this duration, the required increase in YTM to trigger a 10% price drop is:

$$\Delta y_{req} = \frac{0.10}{D^*} \approx 1.2984\% \tag{19}$$

Since Δy_{30} follows a normal distribution:

$$\Delta y_{30} \sim N(0, \sigma\sqrt{30}) \tag{20}$$

where $\sigma = 0.006$, we compute the probability of this event occurring as **34.6387%**.

3.3.3 VaR Computation

Table 7 summarizes the computed VaR values across different methods and time horizons.

Horizon (days)	Worst-case YTM (High)	Exact	Δ	$\Delta - \Gamma$	MC Δ	MC $\Delta - \Gamma$	MC Full
1	0.065331	10.00	10.56	9.84	10.70	9.97	9.96
10	0.095664	27.59	33.33	26.18	33.62	26.35	27.31
20	0.114039	36.08	47.10	32.80	47.20	33.00	35.72
30	0.128138	41.72	57.64	36.21	58.22	36.52	41.20
40	0.140025	45.95	66.52	37.95	67.05	38.02	45.29
50	0.150497	49.34	74.33	38.63	74.28	38.50	48.41
60	0.159964	52.16	81.38	38.55	81.86	38.62	51.67
70	0.168670	54.55	87.86	37.90	88.53	38.59	53.81
80	0.176774	56.63	93.89	36.80	95.35	38.55	55.77
90	0.184385	58.46	99.55	35.32	98.83	38.47	57.58

Table 7: Computed VaR values (99% confidence level) across different methodologies.

The plot is made based on the price to the yield change based on Monte Carlo Simulations. The blue line representing the full revaluation method is the benchmark as it recalculates the bond price fully for

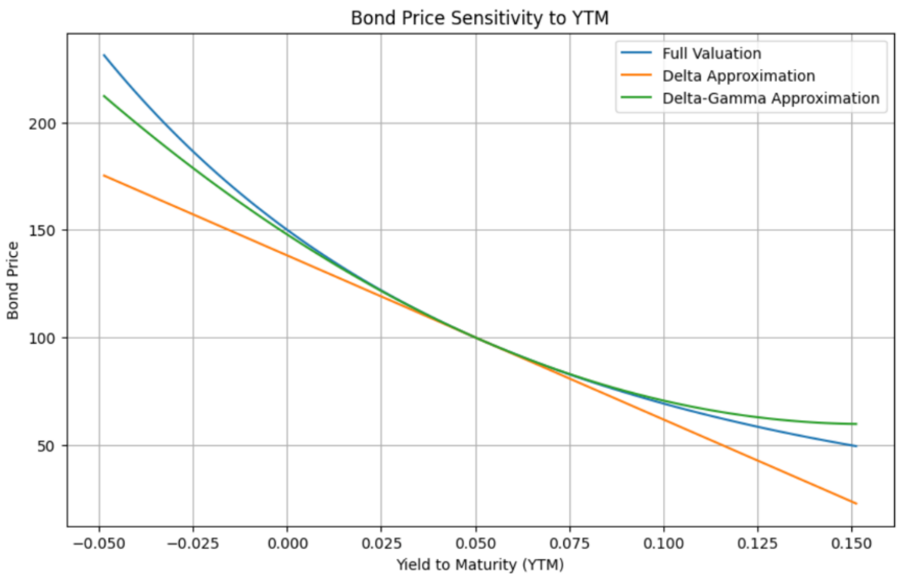


Figure 12

each change in YTM. This line is smoother and probably captures non-linear effects more accurately than the other methods. The orange line, representing the delta approximation, shows a linear relationship between bond price and YTM. As expected, this method does not adjust for the curvature of the bond price changes, only the parallel shift, and tends to overestimate the sensitivity (i.e. the slope) as YTM increases, leading to a steeper slope and higher price volatility. The green line, illustrating the delta-gamma approximation, lies between the full valuation and delta approximation lines. This method accounts for some curvature by including the second derivative (gamma), which makes it more accurate than the delta approximation, particularly for moderate changes in YTM. It includes the curvurture as part of the price calculation. It tracks closer to the full valuation method than the delta approximation, especially for smaller changes in YTM. However, it underestimates the price drop while there is huge increase in the yield.

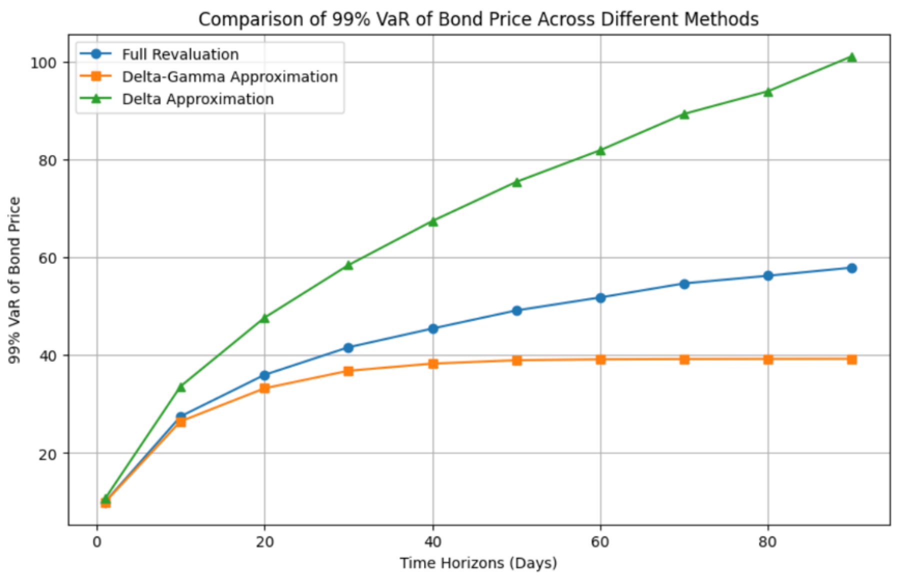


Figure 13

3.3.4 Expected Shortfall Computation

Table 8 summarizes the computed Expected Shortfall (ES) values for the Monte Carlo full revaluation.

Horizon (days)	MC Full
1	11.33
10	30.22
20	39.17
30	45.10
40	49.15
50	52.66
60	55.29
70	57.65
80	59.63
90	61.46

Table 8: Computed Expected Shortfall (99% confidence level) using the MC full revaluation.

For the expected shortfall, the full revaluation Monte Carlo simulation is adopted as based on the time cost of simulation, it is a good trade-off for higher accuracy with small amount of additional time needed. Also, with the exact formula of revaluation, it would be hard to calculate the expected shortfall as it is hard to integrate over a non-linear loss distribution. In our approach, it begins by simulating daily yield fluctuations, which are accumulated over the given time horizon to reflect potential changes in interest rates. These simulated yields are then used to fully revalue the bond using a full revaluation approach that accounts for the bond's coupon payments and time-to-maturity adjustments under a 360-day convention. The 99% VaR is determined as the difference between the initial bond price and the 1st percentile of the simulated bond prices, representing the worst expected loss at the confidence level. Expected Shortfall (ES) is then computed by averaging all losses beyond the VaR threshold, providing a more comprehensive measure of tail risk. This method captures non-linear price behavior due to bond convexity and allows for a detailed risk analysis across multiple time horizons, making it a robust approach for assessing downside risk in bond portfolios.

3.4 Discussion

Both the delta approximation and delta-gamma approximation got the delta and gamma value at the current time of the bond. They both assume the small yield movements. Britten-Jones & Schaefer (1999) emphasize that linear VaR is unlikely to be robust for options, also for bonds, and they advocate a quadratic approximation as a better solution. The delta-gamma approximation might better include the dynamic of the curvature change of the price curve however, on the other hand, researhers like Glasserman (2003) cautioned that even delta-gamma approximation has limits in accuracy. In this question, we simply assume the daily fluctuations of the yield as i.i.d. normal distribution but extreme market moves, and yield movement might not follow normal distribution. Though it's possible for yields to go negative, there are larger space moving more positive than negative. The delta-gamma method might not be accurate enough to replace simulation entirely in general. Delta-gamma approach helps for the moderate VaR levels, or for short period of time, however, it might underestimate the loss at more extreme cases (as shown by the orange curve in the VaR plot). Also, as shown in the VaR table, the VaR calculated through Delta-Gamma approximation diminishes when the time getting longer. It is due to the parabolic characteristic of the approximation. It also indicates that this approach might underestimate the loss at time with large yield change. But at short period of time, the Delta-Gamma approach gives very similar results to the exact formula approach and the Monte Carlo full revaluation approach.

4 References

- Aktaş, Ö. and Sjöstrand, M. (2011). Cornish-Fisher Expansion and Value-at-Risk Method in Application to Risk Management of Large Portfolios. *Available at:* https://www.academia.edu/40288783/Cornish_Fisher_Expansion_and_Value_at_Risk_method_in_application_to_risk_management_of_large_portfolios
- Jaschke, S.R. and Jiang, Y. (2002). Approximating Value at Risk in Conditional Gaussian Models. *Springer eBooks*, pp.3–33. https://doi.org/10.1007/978-3-662-05021-7_1
- Jorion, P. (2012). Risk Management for Alternative Investments. *Available at:* https://www.semanticscholar.org/paper/Risk-Management-for-Alternative-Investments-Jorion/befb74f14314e1
- Ballotta, L. and Fusai, G. (2017). A Gentle Introduction to Value at Risk. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.2942138
- Ammann, M. and Reich, C. (2001). Var for Nonlinear Financial Instruments — Linear Approximation or Full Monte Carlo? *Financial Markets and Portfolio Management*, 15(3), pp. 363–378. https://doi.org/10.1007/s11408-001-0306-9
- Maxwell, D. and van Vuuren, G. (2014). A Review and Update of Value at Risk. *Studies in Economics and Econometrics*, 38(3), pp. 79–102. https://doi.org/10.1080/10800379.2014.12097273
- Oanea, D.-C. and Anghelache, G. (2015). Value at Risk Prediction: The Failure of RiskMetrics in Preventing Financial Crisis. Evidence from Romanian Capital Market. *Procedia Economics and Finance*, 20, pp. 433–442. https://doi.org/10.1016/s2212-5671(15)00094-5
- Mina, J. and Xiao, J. (2007). Return to RiskMetrics: The Evolution of a Standard. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.1524810
- Britten-Jones, M. and Schaefer, S. M. (1999). Non-Linear Value-at-Risk. *European Finance Review*, 2, pp. 161–187.
- Glasserman, P. (2003). Monte Carlo Methods in Financial Engineering. *Springer-Verlag*, New York.