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WIP Random loops model and correlations WIP

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Definition of the Markov Chain

In this chapter, we begin by defining the state space and subsequently provide a characterization of this space as the union of edge-disjoint cycles, making use of Euler's theorem for multigraphs. Following this, we introduce two distinct types of simple square transformations and demonstrate their capability to systematically construct any state, starting from the empty state. Together with their inverses these transformations are then employed to define an irreducible and aperiodic Markov Chain, converging to our chosen invariant distribution.

4.1 The space of link configurations and pairings

Definition 4.1.1. Let $G = (V, E)$ be a simple graph. A **link configuration** is an edge function $m : E \to \mathbb{N}$ such that $\forall x \in V$ we have $\sum_{y \sim x} m_{x,y} \in 2\mathbb{N}$.

Given a graph G, we denote the set of all its link configurations as Σ_G. Our focus will be specifically on instances where G is a grid subset of $\mathbb{Z}^2$ with periodic or free boundary conditions. In both cases, the vertex set is defined as

$$V_n := \{(i, j) \mid i, j \in \{0, \dots, n-1\}\}$$

while the edge sets for the free and the periodic boundary condition are, respectively:

$$E_n^f := \Big\{\{x, y\} \mid x, y \in V_n,\ \|x - y\| = 1\Big\}$$
$$E_n^p := E_n^f \cup \Big\{\{(i, 0), (i, n-1)\},\ \{(0, j), (n-1, j)\} \mid i, j \in \{0, \dots, n-1\}\Big\}$$

We define a *square* s to be an ordered tuple $s = (e_1, e_2, e_3, e_4)$ where the e_is are edges in a cycle of length four. We will use the convention that e_1 is the top edge, e_2 the right, e_3 the bottom and e_4 the left. We will define s_v to be the set of its vertices.

In the case of periodic boundary conditions we can identify any square s with one of its vertices, for example the top-right one, this implies that the number of squares is $|V_n| = n^2$. With free boundary conditions the number of squares is $(n-1)^2$.

In the case of free boundary conditions the graph has a boundary $B \subseteq V_n$, which is the set of vertices of degree less than 4 in (V_n, E_n^f).

Definition 4.1.2. A **multicolor link configuration** of N colors is a tuple $(m^1, \ldots, m^N)$ of link configurations.

Note that the parity constraints are independent for each color. We will now simply refer to them as link configurations and call single-color configuration as monochromatic.

Given a link configuration, there is a natural way of building a multigraph $G_m = (V, E_m)$: use the value m_e as the number of times the edge e appears in the edge multi-set E_m of the multigraph. To avoid confusion with the original graph G, we refer to edges in the multigraph as links. The advantage of working with a multigraph is that statements like "state m contains a cycle" make sense now. When it doesn't cause confusion we will refer to a link configuration and its associated multigraph as m.

Definition 4.1.3. A **pairing** for a link configuration m is a collection of partition of the set of links of the same color incident to a vertex, such that every set in the partition has exactly two elements.

In simpler terms, at each vertex we are pairing links of the same color. By doing so, we are in fact partitioning the whole link set in cycles, which we will refer to as loops.

We denote the set of all possible pairing configurations of a link configuration m as $\mathcal{P}(m)$.
Our state space will be the set $\Omega := \{(m, \pi) : m \in \Sigma, \pi \in \mathcal{P}(m)\}$.

4.2 Edge-disjoint cycle representation

Consider a monochromatic state m. As pointed out in the preceding section, we can view it as a multigraph. Then the parity constraint in the definition of a link configuration is equivalent to requiring that the multigraph has even degree on all vertices. By Euler's theorem for multigraphs, we can represent G_m as a union of edge-disjoint cycles.

Lemma 4.2.1. Let $G = (V, E)$ be a multigraph with non-empty edge set, such that $\forall x \in V \;\; d(x) \in 2\mathbb{N}$. Then G contains a cycle.

Proof. Since $|E| > 0$, there exists a vertex $v \in V$ such that $d(v) \geq 2$. Starting from this vertex, choose any edge, say $\{v, x_1\}$. Since $d(x_1)$ is even, there exists at least one more unvisited edge incident to x_1. We continue this process iteratively, exploring unvisited edges and vertices. Since V is a finite set, after a finite number of steps, we must encounter a vertex that has already been visited. Consequently, we have found a cycle. □

Proposition 4.2.1. If $G = (V, E)$ is a multigraph in which $\forall v \in V \;\; \deg(v) \in 2\mathbb{N}$ then the edge set E can be partitioned into edge-disjoint cycles.

Proof. We prove it by strong induction on the number of cycles. The base case is a multigraph with $|E| = 0$. Such a graph consists of one or more isolated vertices and the (empty) edge set can clearly be partitioned into a union of zero cycles.

Now suppose the result is true for every multigraph $G = (V, E)$ with $|E| \leq m$ edges whose vertices all have even degree. Consider a multigraph with $|E| = m + 1$: . From Lemma 4.2.1 we know that there is at least one cycle $C = (\mathcal{V}, \mathcal{E})$ contained in G. Then we can form a new graph $G' = (V, E')$ by removing the edges that appear in the cycle $E' := E \setminus \mathcal{E}$. Every vertex in the cycle has its degree reduced by two, vertices that didn't appear in the cycle maintain the same degree: parity is preserved. Since we removed at least two edges (the shortest cycle possible in a multigraph) $|E'| \leq m$ so that by the induction hypothesis we can partition the edge set of G' as disjoint cycles $E' = \bigcup_{i=1}^{N} C_i$. Then adding C to the partition of E' we obtain a partition for the original edge set E. □

4.3 Square transformations

We define a class of transformations that map a state m to a new state m' by modifying only the links within a designated square on the grid. We limit ourselves to these transformations due to their local nature. Since our goal is to simulate a Markov chain on Σ, the locality of square transformations is crucial to make the exploration of the state space computationally tractable.

Definition 4.3.1. Given a square $s = (e_1, e_2, e_3, e_4)$ and four integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with the same parity, we define the *square transformation*

$$X_{s,(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} : S \subseteq \Sigma \to \Sigma$$

that sends a state m to a new state m' defined by

$$m'_e = \begin{cases} m_{e_i} + \alpha_i & \text{if } e = e_i \\ m_e & \text{otherwise.} \end{cases}$$

The domain S of the square transformation is defined to ensure the absence of negative links after applying the transformation:

$$S := \{m \in \Sigma \mid m_{e_i} \geq \alpha_i, \text{ for } i = 1, 2, 3, 4\}$$

Each square transformation is characterized by the square it acts on and the four integers associated with it. We will ofter represent them using the square diagram in figure 4.1:

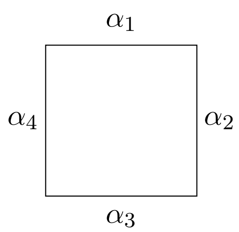

Figure 4.1. The diagram of the square transformation $X_{s,(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$.

The parity constraints on these integers ensure that the parity number of links incident at each vertex is preserved since in accordance with definition 4.1.1.

Given that our state is generally a multicolor link configuration, we require a distinct set of square transformations for each of the possible N colors, each acting only on a single color. When needed, we will explicitly indicate the color on which a transformation acts as X_s^c with $c = 1, \ldots, N$. A general square transformation will be denoted simply as X_s.

We define two special types of square transformations. The first type consists of a pair called *uniform* transformations $U_s^{\pm1}$. The second type, called *single* transformations, includes a pair for each side of the square: $R_s^{\pm2}$, $L_s^{\pm2}$, $T_s^{\pm2}$, and $B_s^{\pm2}$, corresponding to *right, left, top,* and *bottom*, respectively. The associated integers for these transformations are presented in Table 4.1. In figure 4.7, the diagrams of some uniform and single right transformations are drawn.

$U_s^{\pm1}$	$(\pm1, \pm1, \pm1, \pm1)$
$T_s^{\pm2}$	$(\pm2, 0, 0, 0)$
$R_s^{\pm2}$	$(0, \pm2, 0, 0)$
$B_s^{\pm2}$	$(0, 0, \pm2, 0)$
$L_s^{\pm2}$	$(0, 0, 0, \pm2)$

Table 4.1. Associated integers of uniform and single square transformations.

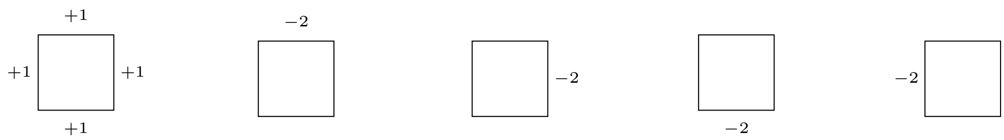

Figure 4.2. Diagrams of square transformations U^{+1}, T^{-2}, R^{-2}, B^{-2}, and L^{-2}, from left to right.

The importance of these simple transformations resides in the following proposition:

Proposition 4.3.1. All square transformations can be obtained by composing single -2 and uniform $+1$ transformations (the five transformation in figure 4.7).

Proof. Let X_s denote any square transformation, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be its associated integers. Since they have the same parity, we can always find non-negative integers $k, \beta_1, \beta_2, \beta_3, \beta_4$ such that:

$$\begin{cases} \alpha_1 = k - 2\beta_1 \\ \alpha_2 = k - 2\beta_2 \\ \alpha_3 = k - 2\beta_3 \\ \alpha_4 = k - 2\beta_4 \end{cases}$$

We can use these integers to get the following decomposition:

$$X_s = L_s^{\beta_4} \circ B_s^{\beta_3} \circ R_s^{\beta_2} \circ T_s^{\beta_1} \circ U_s^k$$

where the composition is defined as $R_s^n := R_s^{-2} \circ \cdots \circ R_s^{-2}$ n times.

□

4.4 State construction

We will prove that it is possible to construct any link configuration starting from the empty state (zero links on all edges) and the converse by applying only a sequence of uniform $+1$ and the four single -2 transformations (figure 4.7), in the case of free boundary conditions.

We will also prove that this task is impossible if we consider periodic boundary conditions: this is obvious when we our grid has odd length sides, since we can find states with an odd number of links, but square transformation preserves the parity of the total number of links in a state. The general case will follow from the topological properties of the torus.

Proposition 4.4.1. Any state $m \in \Sigma_G$, where $G = (V_n, E_n^f)$ is a box subset of $\mathbb{Z}^2$ with free boundary conditions, can be constructed from the empty state $0 \in \Sigma_G$ by using uniform $+1$ and the four single -2 transformations. Moreover, with the same set of transformations one can also reach the empty state starting from any state. This means there exists sequences of transformations depicted in figure 4.7 $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_l$ such that:

$$0 \xrightarrow{X_1} m_1 \xrightarrow{X_2} m_2 \ldots \xrightarrow{X_k} m \qquad\qquad m \xrightarrow{Y_1} n_1 \xrightarrow{Y_2} n_2 \ldots \xrightarrow{Y_l} 0$$

where $m_1, m_2, \ldots, m_{k-1} \in \Sigma$ and $n_1, n_2, \ldots, n_{l-1} \in \Sigma$ are intermediate states.

Proof. Since we can work with any color independently, we consider a monochromatic state without loss of generality.

Since our state has even degree on all vertices of the lattice, we can apply proposition 4.2.1 on the multigraph G_m. Call the edge-disjoint cycle partition of the edges $\{C_i\}_{i\in[N]}$, where N is the number of cycles. Since the cycles are edge disjoint, we can construct them separately, thus we only need to prove that we can build a single cycle[1].

Constructing a cycle starting from the empty state is straightforward: the *interior*[2] $\mathrm{Int}(C)$ of a cycle C is defined as a subset of squares such that:

$$\mathrm{Int}(C) := \{s \mid \text{ any path from } s_v \text{ to } B \text{ contains a vertex of } C\}$$

Recall that s_v is the set of vertices of square s and B is the set of boundary vertices of the graph.

If the cycle has more than 2 edges (the cycle is a simple graph) apply the uniform $+1$ to every square in $\mathrm{Int}(C)$ (which is at least one square). Subsequently, eliminate the undesired double edges with the single link -2 transformations.
If the cycle has 2 edges (in this case the interior is empty), we can construct it by applying the uniform $+1$ transformation twice and then applying single -2 to eliminate the three undesired edges. □

Example 4.4.1. Suppose we want to build the red cycle in the left picture of figure 4.3.

[1] This argument can be made precise through induction on N.
[2] Here the assumption of free boundary conditions is necessary.

1. Apply the uniform +1 mask on all squares on the interior of the cycle.
2. Use the −2 single link to remove the double edges inside.

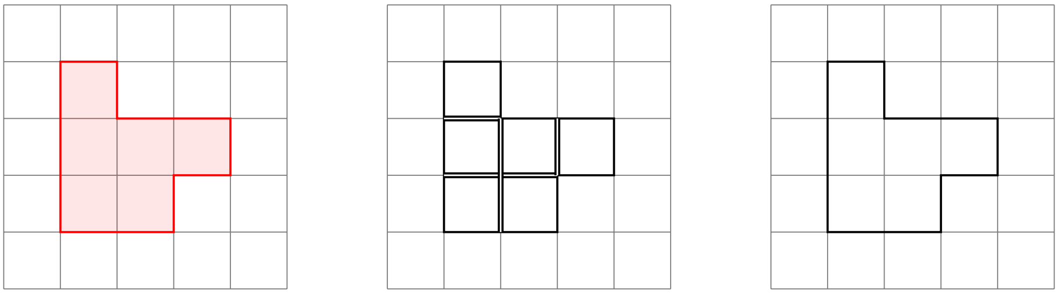

Figure 4.3. How to build/destroy any cycle with a well defined interior.

This procedure works on the plane, but not on the torus! In this case, the boundary B is empty, and we can't define the interior/exterior of a closed curve as in the proof of Proposition 4.4.1. Still, there are some simple closed curves that separate the torus into two regions, both of which have the curve as a boundary. Clearly, we can build these curves using the same procedure applied to any of these two regions.

However, this does not work in general. In fact, there exist closed curves that don't separate the torus into two regions, hence we can't apply the same procedure. We call these curves *non-separating*, see Figure 4.4.

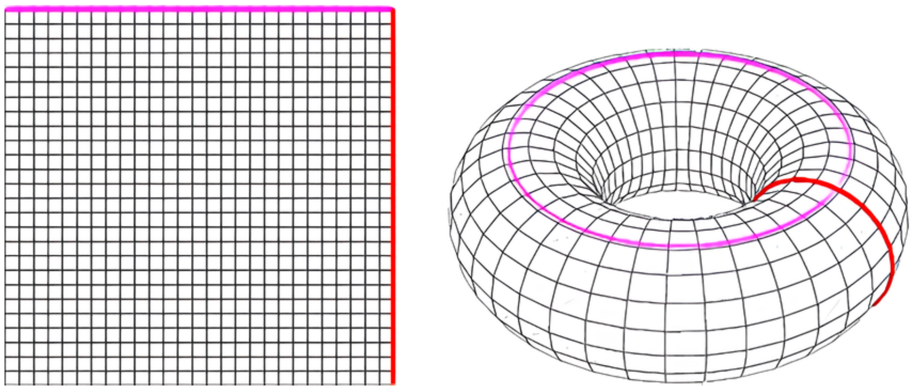

Figure 4.4. In pink an *horizontal non-separating* closed curve on the torus, in red a *vertical* one.

Being non-separating is a topological property, we will work with a characterization more suited for our setting.

We call an *horizontal loop* a simple closed curve on the torus that is parallel to horizontal links and does not intersect any vertex, like the curve in orange in figure 4.5. Their vertical counterpart is called *vertical loop*.

The characterization will exploit the fact that if a closed curve γ is separating, then the number of intersections with any horizontal and vertical loops l is even. We denote this number as $\mathrm{Cross}_l(\gamma)$.

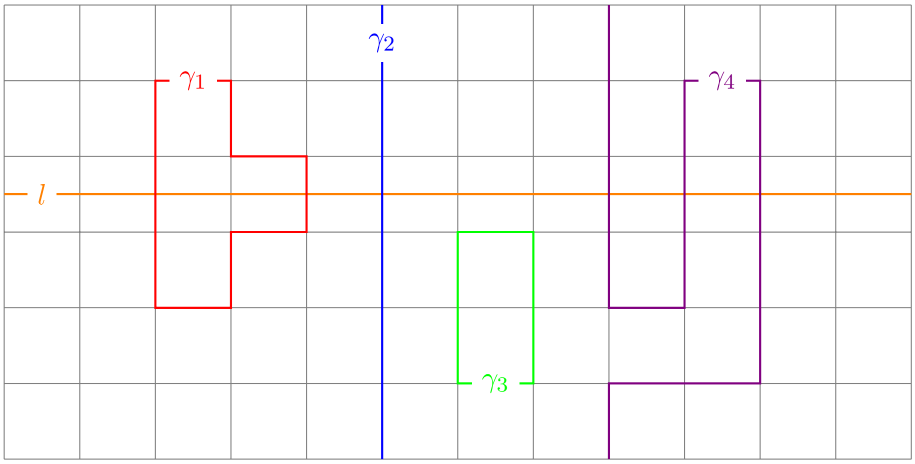

Figure 4.5. Four closed simple curves on the torus and an horizontal loop l in orange. Here $\text{Cross}_l(\gamma_1) = 2$, $\text{Cross}_l(\gamma_2) = 1$, $\text{Cross}_l(\gamma_3) = 0$ and $\text{Cross}_l(\gamma_4) = 3$.

Clearly if γ_1 and γ_2 are two edge-disjoint curves, then $\text{Cross}_l(\gamma \cup \eta) = \text{Cross}_l(\gamma) + \text{Cross}_l(\eta)$, so that any separating circuit, which can be written as the union of edge-disjoint cycles (which are closed simple curves), has even cross number.

We call a non-separating closed curve γ *vertical non-separating* if there exists an horizontal loop l such that $\text{Cross}_l(\gamma)$ is odd; we call it *horizontal non-separtating* if the same is true for a vertical loop l. The curves γ_2 and γ_4 in figure 4.5 are both vertical non-separating.

Proposition 4.4.2. A closed curve γ on the torus is non-separating if and only if we can find an horizontal/vertical loop l such that $\text{Cross}_l(\gamma)$ is odd.

Proof. We will prove that γ is separating if and only if $\text{Cross}_l(\gamma)$ is even for every horizontal/vertical loop l.

This can be proven using an argument similar to the one used in proving Jordan's curve theorem for polygons.

Assume γ is separating, call A and B the two disjoint regions of the torus such that $\partial A = \partial B = \gamma$. Choose a point x on any given any vertical/horizontal loop l which does not lie on γ. Then either $x \in A$ or $X \in B$. Suppose wlog that $x \in A$, if $\text{Cross}_l(\gamma)$ were odd, then also $x \in B$, but $A \cap B = \emptyset$.

Suppose now that for any vertical/horizontal loop l $\text{Cross}_l(\gamma)$ is even. We can then define two distinct subsets as follows:

1. Choose a point x at the center of a square s and an horizontal loop l that intersect x. Assign all points within square s to the set A.

2. Consider another square s' intersecting with l, and denote its center as x'. Divide loop l into two curves l_1 from x to x' and l_2 from x' to x. Since $\text{Cross}_l(\gamma)$ is even, the intersection of curves l_1 and l_2 with γ has the same parity. If it's even, assign all points in square s' to set A, otherwise to B.

3. Apply this procedure to the centers of the squares already assigned, using vertical loops to assign the remaining squares.

It's easy to see that the sets $A \setminus \gamma$ and $B \setminus \gamma$ have the desired properties, hence γ is separating. $\square$

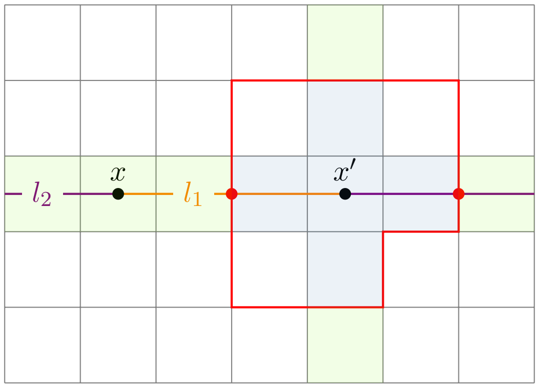

Figure 4.6. The procedure to define the subsets A and B discussed in the proof of proposition 4.4.2. The number of intersection of both l_1 and l_2 is odd.

Proposition 4.4.3. Let $m \in \Sigma_G$ with $G = (V_n, E_n^p)$ be a state consisting of only separating curves. If given a square transformation X_s the new state $m' = X_s(m)$ contains one non-separating curve γ, then it contains two.

Proof. We need to consider only transformation that add more links, and since by proposition 4.3.1 any square transformations can be written using uniform ± 1 and single -2, we are left with only uniform $+1$.

If the new state $m' = X_s(m)$ contains a non-separating curve γ, then one or two of its edges was introduced by the transformation X_s. This means that in the original state m, all the vertices in γ were already in the same connected component H (a circuit is at least 2-edge-connected), then by Euler's theorem for multigraphs we can find in H a circuit C which visits every vertex γ. If we remove the edges of any path P joining any two vertices in H, then $H \setminus P$ contains a path visiting all edges (an eulerian path) joining the same vertices. We will employ this fact to build another non-separating curve.

Call e_1, e_2, e_3 and e_4 the four new added links to the square s, x and y the vertices of this square that where also in circuit H.

Call γ the non-separating curve, which we can write as the union of a path P_{xy} from x to y contained in the circuit H and one or two of the new edges:

$$\gamma = P_{xy} \cup \{e_1\} \qquad \text{of} \qquad \gamma = P_{xy} \cup \{e_1, e_2\}$$

Call $\tilde{P}_{xy}$ the eulerian path joining x to y contained in $H \setminus P_{xy}$, then the closed curves

$$\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_2, e_3, e_4\} \qquad \tilde{\gamma} = \tilde{P}_{xy} \cup \{e_3, e_4\}$$

are also non-separating, since given an horizontal/vertical loop l:

$$\mathrm{Cross}_l(\tilde{\gamma}) = \mathrm{Cross}_l(C) - \mathrm{Cross}_l(\gamma)$$

where $\mathrm{Cross}_l(C)$ is even, since it's a circuit contained in the original state m, and $\mathrm{Cross}_l(\gamma)$ is odd since γ is non-separating, this implies that $Cross(\tilde{\gamma})$ is also odd. $\square$

A straightforward consequence of this proposition is that we cannot reach a state with an odd number of non-separating curves starting from the empty state, as they can only be produced in pairs. More generally:

Corollary 4.4.1. Each parity number of non-separating curves (vertical and horizontal) is invariant with respect to square transformations.

To overcame this problem we are forced to introduce new non square transformations that don't preserve the parity of non-separating curves, such as:

$$V_k^{+1}(m)_e := \begin{cases} m_e + 1 & \text{if } e \text{ connects two vertices on the vertical line } x = k \\ m_e & \text{otherwise} \end{cases}$$

$$H_k^{+1}(m)_e := \begin{cases} m_e + 1 & \text{if } e \text{ connects two vertices on the horizontal line } y = k \\ m_e & \text{otherwise} \end{cases}$$

Notice how we don't need their inverses, since we can apply them twice and then use the single -2 to remove all links. The downside is that they affect all the edges on a given vertical/horizontal line: they are not local, in the sense that the number of affected edges depends linearly on the size of the grid.

4.5 Transition probabilities

We define a probability measure (4.1) on the set Ω, where each element is a link configuration with a compatible pairing, $\omega = (m, \pi)$.

$$\mathbb{P}(\omega) = \frac{1}{Z} \prod_{e \in E} \prod_{i=1}^{N} \frac{\beta^{m_e^i}}{m_e^i!} \prod_{x \in V} \frac{\Gamma(\frac{N}{2})}{2^{n_x(m)} \Gamma(\frac{N}{2} + n_x(m))} = \frac{1}{Z} \mu(m) \tag{4.1}$$

Where Z is a normalization constant, $\beta \in \mathbb{R}_{\geq 0}$, N is the number of colors and $n_x(m) := \sum_{i=1}^{N} n_x^i(m) = \sum_{i=1}^{N} \frac{1}{2} \sum_{y \sim x} m_{x,y}^i$ which we refer to as the *local time*.

Since it is independent of the pairing configuration, the marginal probability of the link configuration is simply

$$\rho(m) = \sum_{\pi \in \mathcal{P}(m)} \mathbb{P}(m, \pi) = \frac{1}{Z} \mu(m) |\mathcal{P}(m)| \tag{4.2}$$

where the number of possible pairings configurations is

$$|\mathcal{P}(m)| = \prod_{x \in V} \prod_{i=1}^{N} (2n_x^i(m) - 1)!!$$

To sample from probability measure (4.1) we first sample from the marginal measure (4.2) and then uniformly sample a pairing configuration.

Definition 4.5.1. We say that states $m, m' \in \Sigma$ are neighbours and write $m \sim m'$ if we can find a square transformation X_s^c such that $X_s^c(m) = m'$.

From a starting state $m \in \Sigma$, we attempt to transition to a neighboring state m' by applying one of the possible square transformations for a randomly chosen color c, which acts on a square s, both chosen uniformly at random.

The jump probability from state m to state m' is defined as

$$q(m' \mid m) = \begin{cases} \frac{1}{N}\frac{1}{(n-1)^2}\frac{1}{\mathcal{M}_{s,c}(m)} & \text{if } m' = X_s^c(m) \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{M}_{s,c}(m)$ is the number of possibile transformation of color $c = 1, \ldots, N$ that can be applied to square s with link configuration m, and $(n-1)^2$ is the total number of squares in the grid.

We look for transition probability of the form:

$$P(m \to m') = q(m' \mid m)A(m', m)$$

where $A(m', m)$ is called the *acceptance probability* from state m to state m'. Imposing the detailed balance equations, we find the condition

$$\frac{A(m', m)}{A(m, m')} = \frac{q(m \mid m')}{q(m' \mid m)}\frac{\rho(m')}{\rho(m)} = \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')}\frac{\rho(m')}{\rho(m)} \tag{4.3}$$

A common choice which satisfies condition (4.3) is the Metropolis-Hastings acceptance probability:

$$A(m', m) = \min\left(1, \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')}\frac{\rho(m')}{\rho(m)}\right) \tag{4.4}$$

Another option is the Glauber acceptance probability:

$$A(m', m) = \left(1 + \frac{\mathcal{M}_{s,c}(m')}{\mathcal{M}_{s,c}(m)}\frac{\rho(m)}{\rho(m')}\right)^{-1}$$

Since m and m' only differ by a transformation in one square s, we can compute efficiently the ratios $\rho(m')/\rho(m)$.

Putting it all together, in the case of Metropolis acceptance probability the transition probabilities are:

$$P(m \to m') = \begin{cases} \frac{1}{N}\frac{1}{|V|}\frac{1}{\mathcal{M}_{s,c}(m)} \min\left(1, \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')}\frac{\rho(m')}{\rho(m)}\right) & \text{if } m' \sim m \\ \sum_{m'' \sim m} \frac{1}{N}\frac{1}{|V|}\frac{1}{\mathcal{M}_{s,c}(m)}\left[1 - \min\left(1, \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m'')}\frac{\rho(m'')}{\rho(m)}\right)\right] & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

The probability $P(m \to m)$ is defined to ensure the transition matrix is stochastic:

$$\sum_{m' \in \Sigma} P(m \to m') = \frac{1}{N}\frac{1}{(n-1)^2}\sum_{m' \sim m}\frac{1}{\mathcal{M}_{s,c}(m)} \tag{4.6}$$

$$= \frac{1}{N}\frac{1}{(n-1)^2}\sum_{s}\sum_{m' = X_s(m)}\frac{1}{\mathcal{M}_{s,c}(m)} \tag{4.7}$$

$$= \frac{1}{N}\frac{1}{(n-1)^2}\sum_{s} N = 1 \tag{4.8}$$

Transition	Probability Ratio $\rho(m')/\rho(m)$
$m' = U_s^{+1}(m)$	$\frac{\beta^4}{16}\prod_{e\in s}(m_e^i+1)^{-1}\prod_{x\in s_v}\left(\frac{N}{2}+n_x\right)^{-1}$
$m' = U_s^{-1}(m)$	$\frac{16}{\beta^4}\prod_{e\in s}m_e^i\prod_{x\in s_v}\left(\frac{N}{2}+n_x-1\right)$
$m' = R_s^{+2}(m)$	$\frac{\beta^2}{4(m^i_{\{x,y\}}+2)(m^i_{\{x,y\}}+1)}\left(\frac{N}{2}+n_x\right)^{-1}\left(\frac{N}{2}+n_y\right)^{-1}$
$m' = R_s^{-2}(m)$	$\frac{4m^i_{\{x,y\}}(m^i_{\{x,y\}}-1)}{\beta^2}\left(\frac{N}{2}+n_x-1\right)\left(\frac{N}{2}+n_y-1\right)$

Table 4.2. Probability ratios for various transitions.

We can compute the explicit the probability rations used in the acceptance probability for each of our possible neighbours.

To improve convergence to equilibrium, especially for low values of β, we introduce a supplementary set of square transformations. It is noteworthy that all of these transformations can be derived from the previous ones by proposition 4.3.1; we introduce them solely to speed up convergence based on numerical experimentation.

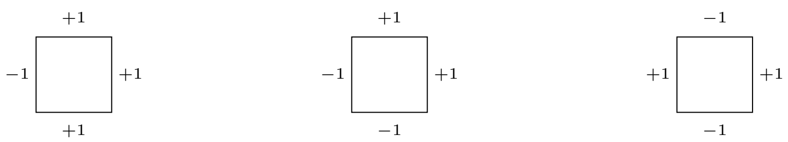

Figure 4.7. The square transformations *triple*, *swap-a* and *swap-o* from left to right.

4.6 Irreducibiliy

The key ingredient to show irreducibility of the Markov Chain with state space Σ and transition probabilities (4.5) is proposition 4.4.1, which asserts that in the case of free boundary conditions, any link configuration can be constructed starting from the empty state by applying the uniform $+1$ and single link -2 transformations and vice versa.

Remark 4.6.1. If m and m' are neighbours, then the transition probabilities $P(m \to m')$ and $P(m' \to m)$ are strictly positive. This follows from the definition of the acceptance probability and the fact that the probability measure (4.2) is strictly positive on all states.

Proposition 4.6.1. The Markov Chain with state space Σ_G with $G = (V_n, E_n^f)$ and transition probabilities (4.5) is irreducible.

Proof. Let $m, m' \in \Sigma$ be any states. By Proposition 4.4.1, there exists a sequence of transformations $X_1, X_2, \ldots, X_k$ such that $0 \xrightarrow{X_1} m_1 \xrightarrow{X_2} m_2 \ldots \xrightarrow{X_k} m'$, where $m_1, m_2, \ldots, m_{k-1} \in \Sigma$ are intermediate states. By Remark 4.6.1, each transformation X_i has a positive probability. Therefore, the probability of the sequence is also positive since it is the product of positive numbers, this implies that $P(0 \to m') > 0$.

Similarly, by Proposition 4.4.1, there exists a sequence of transformations $Y_1, Y_2, \ldots, Y_l$ such that $m \xrightarrow{Y_1} n_1 \xrightarrow{Y_2} n_2 \ldots \xrightarrow{Y_l} 0$, where $n_1, n_2, \ldots, n_{l-1} \in \Sigma$ are intermediate states. By Remark 4.6.1 and the same reasoning $P(m \to 0) > 0$, which toghter with $P(0 \to m') > 0$ implies that $P(m \to m') > 0$. □

4.7 Aperiodicity

Since the chain is irreducible, we only need to prove that a state has period one. By our choice of transition probabilities, if we can always find a state m and one of its neighbors m' such that the acceptance probability $A(m, m')$ is strictly less than one, then one can deduce from (4.5) that $P(m \to m) > 0$, ensuring that state m has period one.

Proposition 4.7.1. For any fixed $\beta > 0$ there exists a state $m \in \Sigma$ such that $A(m', m) < 1$ where $m' = U^{+1}(m)$.

Proof. By the definition of acceptance probability (4.4), we need to check

$$\frac{\rho(m')}{\rho(m)} \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')} < 1$$

Since every transformation that we can apply at square s to state m can also be applied on state m', we have $\frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')} \leq 1$.

We can compute explicitly the probability ratio:

$$\frac{\rho(m')}{\rho(m)} = \frac{\beta^4}{16} \prod_{e \in s} (m_e^i + 1)^{-1} \prod_{x \in s_v} \left(\frac{N}{2} + n_x \right)^{-1} \xrightarrow{m_e \to \infty} 0$$

so we can make the acceptance probability arbitrary small if we choose a state with enough links on the same edge.

□

Chapter 5

Numerical simulations

5.1 Mixing time

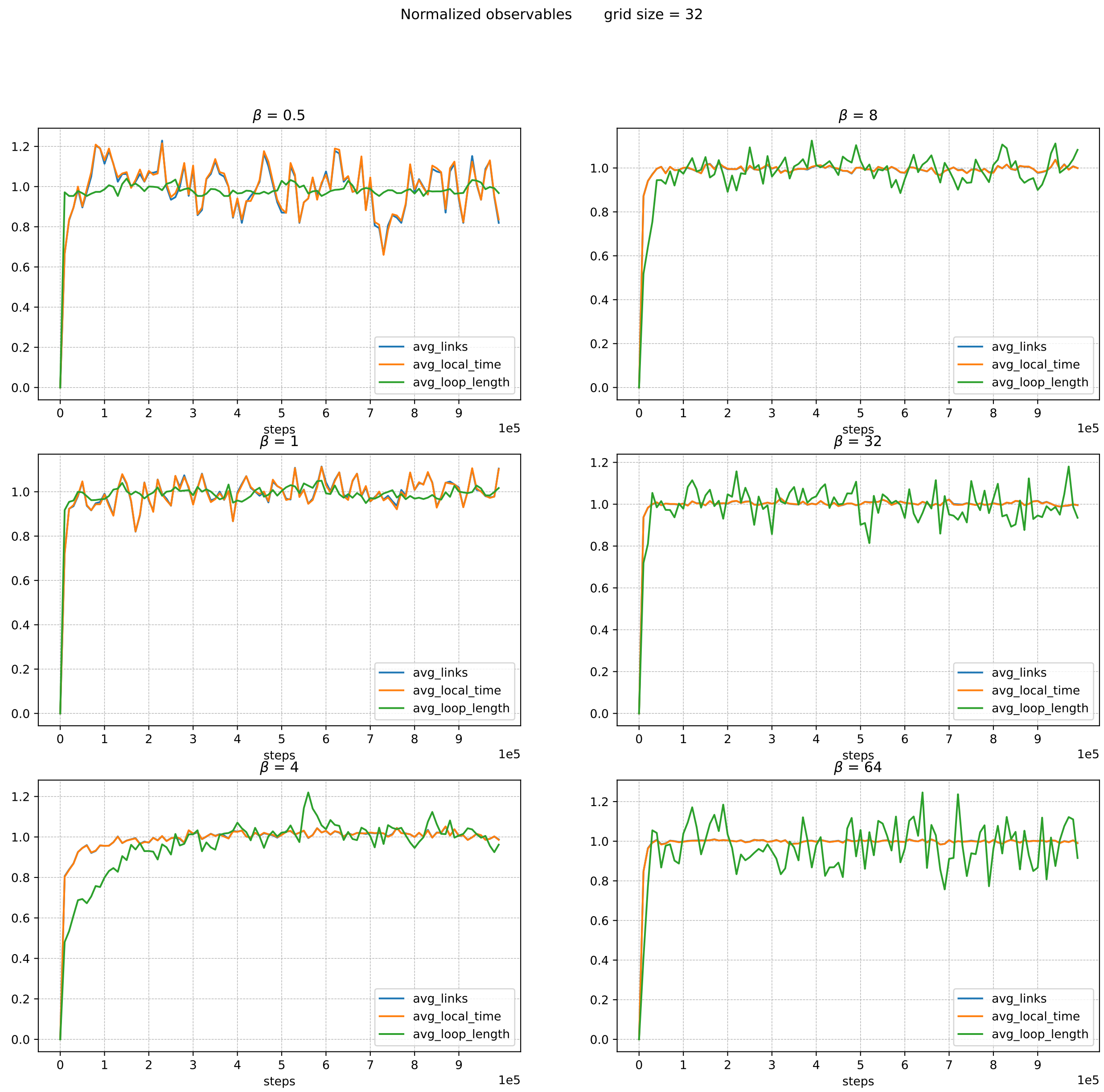

Figure 5.1. Some observables at variuous β, sampled every $10k$ steps, normalized by their average values of the last $100k$ steps.

5.2 Average number of links

Heuristic for explaining the $\beta^{1/2}$ dependence of the average number of link vs β. We use the approximation $m_e^i \simeq \frac{\langle m_e \rangle}{N}$, that is, links are distributed uniformly all over the grid, and that $m_e \geq 2$ so that we can apply any square transformation. If we are at equilibrium the expected number of links should stay constant, from this we obtain the equation:

$$2A(U^{+1}(m), m) = A(R^{-2}(m), m)$$

Using the acceptance probabilities, and the fact that in this approximation the local time is $n_x = 2\langle m_e \rangle$, we find

$$\beta^4 \langle m_e \rangle^{-8} \propto \beta^{-2} \langle m_e \rangle^4$$

rearranging we find

$$\langle m_e \rangle \propto \beta^{\frac{1}{2}}$$

5.3 useless stuff

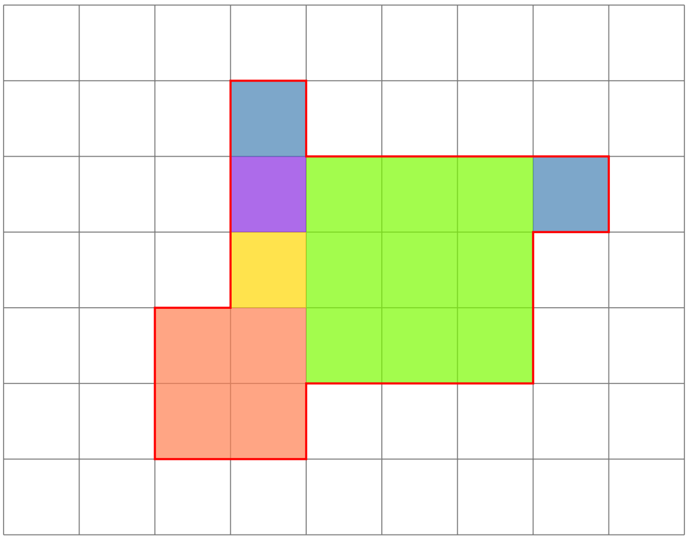

Figure 5.2. Caption.

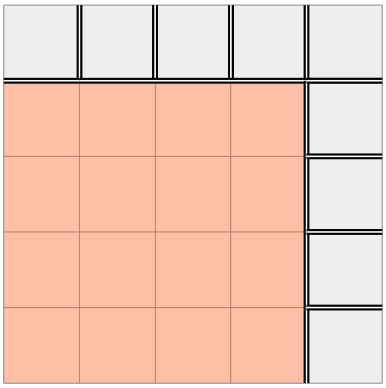

Figure 5.3. Caption.

Chapter 6

Conclusion