Chapter 1

Definition of the Markov Chain

In this chapter, we begin by defining the state space and subsequently provide a characterization of this space as the union of edge-disjoint cycles, leveraging Euler's theorem for multigraphs. Following this, we introduce two distinct types of simple square transformations and demonstrate their capability to systematically construct any state, commencing from the empty state. Together with their inverses these transformations are then employed to define an irreducible and aperiodic Markov Chain, converging to our chosen invariant distribution.

1.1 The space of link configurations and pairings

Definition 1.1.1. Let $G = (V, E)$ be a simple graph. A **link configuration** is an edge function $m : E \to \mathbb{N}$ such that $\forall x \in V$ we have $\sum_{y \sim x} m_{x,y} \in 2\mathbb{N}$.

Given a graph G, we denote the set of all its link configurations as Σ_G. Our focus will be specifically on instances where G is a box subset of $\mathbb{Z}^2$ with periodic boundary conditions.

Definition 1.1.2. A **multicolor link configuration** of N colors is a tuple $(m^1, \ldots, m^N)$ of link configurations.

Note that the parity constraints are independent for each color. We will now simply refer to them as link configurations and call single-color configuration as monochromatic.

Given a link configuration m, we will define a binary relation on each vertex which connects links of the same color.

Definition 1.1.3. A **pairing** for a link configuration m is a collection of symmetric binary realtions at each vertex on its incident links, $\pi = (\pi_x)_{x \in V}$ such that for all l_i, l_j, l_k incident links of vertex x:

1. $\neg(l_i \pi_x l_i)$,
2. if $l_i \pi_x l_j$ then they are the same color,
3. for any l_i there exists an l_j such that $l_i \pi_x l_j$,
4. if $l_i \pi_x l_j$ then for any $l_k \neq l_j$ $\neg(l_i \pi_x l_k)$.

In words, there are no self-pairings; only pairings between links of the same color are permitted. Every link is paired (this is possible since we always have an even number of links of the same color incident at each vertex), and each link can be paired only with one other link.

We denote the set of all possible pairing configurations of a link configuration m as $\mathcal{P}(m)$.
Our state space will be the set $\Omega := \{(m, \pi) : m \in \Sigma, \pi \in \mathcal{P}(m)\}$.

1.2 Edge-disjoint cycle representation

Consider monochromatic states. We can view them as a multigraph with even degree on all vertices. By the Euler's theorem adapted for multigraphs, we can represent G as a union of edged disjoint cycles. Thus it will suffice to show that we can construct any cycle.

Lemma 1.2.1. Let $G(V, E)$ be a multigraph with non-empty edge set, such that $\forall x \in V$ $d(x) \in 2\mathbb{N}$. Then G contains a cycle.

Proof. Since $|E| > 0$ there exsits a vertex $v \in V$ such that $d(v) \geq 2$. Starting from this vertex choose any edge, call it v, x_1, since $d(x_1)$ is even, there exists at least another edge unvisited edge. Keep going, since V is a finite set after enough steps we encounter an already visited vertex: we have found a cycle. $\square$

Proposition 1.2.1. If $G(V, E)$ is a multigraph in which $\forall v \in V$ $\deg(v) \in 2\mathbb{N}$ then the edge set E can be partitioned into (edge disjoint) cycles.

Proof. We prove it by strong induction on the number of cycles. The base case is a multigraph with $|E| = 0$. Such a graph consists of one or more isolated vertices and the (empty) edge set can clearly be partitioned into a union of zero cycles.

Now suppose the result is true for every multigraph $G(V, E)$ with $|E| \leq m$ edges whose vertices all have even degree. Consider a multigraph with $|E| = m + 1$: . From Lemma 1.2.1 we know that there is at least one cycle $C(\mathcal{V}, \mathcal{E})$ contained in G. Then we can form a new graph $G'(V, E')$ by removing the edges that appear in the cycle $E' := E \setminus \mathcal{E}$. Every vertex in the cycle has its degree reduced by two, vertices that didn't appear in the cycle maintain the same degree: parity is preserved. Since we removed at least two edges (the shortest cycle possible in a multigraph) $|E'| \leq m$ so that by the indcytion hypotesys we can partition the edge set of G' as disjoint cycles $E' = \bigcup_{i=1}^{N} C_i$. Then adding C to the partition of E' we obtain a partition for the original edge set E. $\square$

1.3 Square transformations

We define the following two types of square transformations, which are function that map a state m into a new state m' by modifing only link in a given square of the grid. The first are called **uniform** transformations, the second are called **single link**.

$$U_s^{\pm 1}(m)_e := \begin{cases} m_e \pm 1 & \text{if } e \text{ is an edge of square } s \\ m_e & \text{otherwise} \end{cases}$$

$$R_s^{\pm 2}(m)_e := \begin{cases} m_e \pm 2 & \text{if } e \text{ is the right edge of square } s \\ m_e & \text{otherwise} \end{cases}$$

$L_s^{\pm 2}, T_s^{\pm 2}$ and $B_s^{\pm 2}$ which are called single link left, top and bottom are defined in the obious way. We will use the notation X_s for any square transformation that acts on square s.

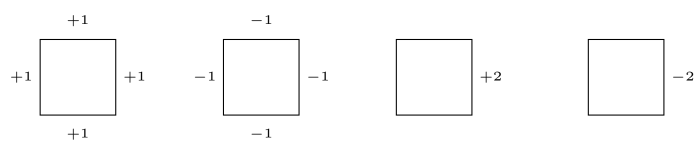

Figure 1.1: The square transformations U^{+1}, U^{-1}, R^{+2} and R^{-2}, from left to right.

We will prove that it is possible to construct any state from the empty state (zero links on all edges) using only the uniform +1 and the four single-link −2 transformations. To ensure reversibility, we will also need their inverses [1]. It is important to observe that each transformation, by construction, maintains the parity of the degree of each vertex, although the uniform +1 transformation disrupts the parity of each edge.

Since our state is in general multicolor, we have a set of square transformations for each of the possible N colors.

Proposition 1.3.1. Any state $m \in \Sigma$ can be constructed from the empty state $0 \in \Sigma$ just by using the trasformations uniform +1 and the four single link −2 defined above.

Proof. Since we can work with any color independently, we consider a monochromatic state without loss of generality. Since our state has even degree on all vertices of the lattice, we can apply proposition 1.2.1 on the multigraph obtained by keeping only non-isolated vertices of the lattice. Call the partition of the edges $\{C_i\}_{i \in [N]}$, where N is the number of cycles. Since the cycles are edge disjoint, we can construct them separately, thus we only need to prove that we can build a single cycle[2].

[1] We can apply masks that remove links only if there are enough links.
[2] This argument can be made precise through induction on N.

Constructing a cycle starting from the empty state is straightforward: if the cycle has more than 2 edges (the cycle is a simple graph) apply the uniform $+1$ mask to every square in the interior of the cycle (which is at least one square). Subsequently, eliminate the undesired double edges with the single link -2 transformations.

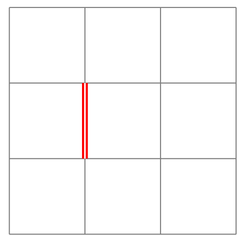

Figure 1.2: A two edge cycle with empty interior.

If the cycle has 2 edges (in this case the interior is empty), we can construct it by applying the uniform $+1$ transformation twice and then applying single link -2 to eliminate the three undesired edges. □

Remark 1.3.1. Clearly, with the inverse transformations uniform -1 and single link $+2$, we can reach the empty state starting from any state.

Example 1.3.1. Suppose we want to build the red cycle in the left picture of figure 1.3.

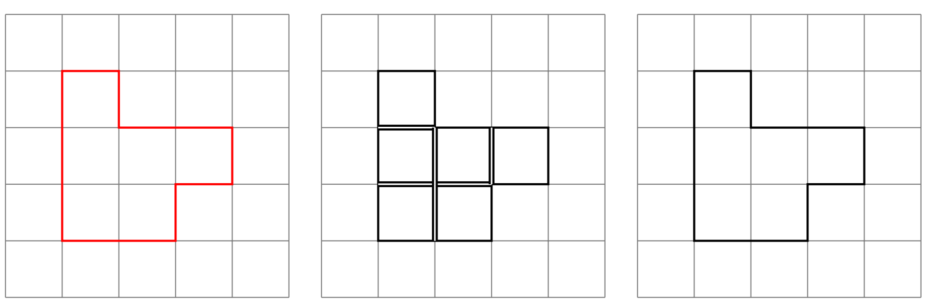

Figure 1.3: How to build any cycle.

1. Apply the uniform $+1$ mask on all squares on the interior of the cycle.

2. Use the -2 single link to remove the double edges inside.

1.4 Transition probabilities

We define a probability measure (1.1) on the set Ω, where each element is a link configuration with a compatible pairing, $\omega = (m, \pi)$.

$$\mathbb{P}(\omega) = \frac{1}{Z} \prod_{e \in E} \prod_{i=1}^{N} \frac{\beta^{m_e^i}}{m_e^i!} \prod_{x \in V} \frac{\Gamma(\frac{N}{2})}{2^{n_x(m)} \Gamma(\frac{N}{2} + n_x(m))} = \frac{1}{Z} \mu(m) \tag{1.1}$$

Where Z is a normalization constant, $\beta \in \mathbb{R}_{\geq 0}$, N is the number of colors and $n_x(m) := \frac{1}{2} \sum_{i=1}^{N} \sum_{y \sim x} m_{x,y}^i$ which we refer to as the *local time*.

Since it is independent of the pairing configuration, the marginal probability of the link configuration is simply

$$\rho(m) = \sum_{\pi \in \mathcal{P}(m)} \mathbb{P}(m, \pi) = \frac{1}{Z} \mu(m) |\mathcal{P}(m)| \tag{1.2}$$

where the number of possible pairings configuration is

$$|\mathcal{P}(m)| = \prod_{x \in V} \prod_{i=1}^{N} (2n_x(m) - 1)!!$$

To sample from the probability measure (1.1) we first sample from the marginal measure (1.2) and then uniformly sample a pairing configuration.

Definition 1.4.1. We say that states $m, m' \in \Sigma$ are neighbours and write $m \sim m'$ if we can find a square transformation X_s such that $X_s(m) = m'$.

From a starting state $m \in \Sigma$, we attempt to transition to a neighboring state m' by applying one of the possible square transformations for a randomly chosen color, which acts on a square s, both chosen uniformly at random.

The jump probability from state m to state m' is defined as

$$q(m' \mid m) = \begin{cases} \frac{1}{N} \frac{1}{|V|} \frac{1}{M_s(m)} & \text{if } m' \sim m \\ 0 & \text{otherwise} \end{cases}$$

where $M_s(m)$ is the number of possibile transformation of the chosen color that can be applied to square s with link configuration m, and $|V|$ is the total number of squares in the grid (which is equal to the number of vertices).

We define the transition probability

$$P(m \to m') = q(m' \mid m) A(m', m)$$

where $A(m', m)$ is the acceptance probability from state m to state m'. Imposing the detailed balance equations, we find the condition

$$\frac{A(m', m)}{A(m, m')} = \frac{q(m \mid m')}{q(m' \mid m)} \frac{\rho(m')}{\rho(m)} = \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)} \tag{1.3}$$

A common choice which satisfies condition (1.3) is the Metropolis-Hastings acceptance probability:

$$A(m', m) = \min\left(1, \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)}\right) \tag{1.4}$$

Since m and m' only differ by a transformation in one square s, we can compute efficiently the ratios $\rho(m')/\rho(m)$.
Putting it all together, our transition probabilities are:

$$P(m \to m') = \begin{cases} \frac{1}{N} \frac{1}{|V|} \frac{1}{M_s(m)} \min\left(1, \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)}\right) & \text{if } m' \sim m \\ \sum_{m' \sim m} \frac{1}{N} \frac{1}{|V|} \frac{1}{M_s(m)} \left[1 - \min\left(1, \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)}\right)\right] & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

The probability $P(m \to m)$ is defined to ensure the transition matrix is stochastic.

1.5 Irreducibiliy

The key ingredient is Proposition 1.3.1, which asserts that any link configuration can be constructed starting from the empty configuration by applying only the uniform $+1$ and single link -2 transformations. If we also include their inverses, uniform -1 and single link $+2$, we can reach the empty configuration starting from any state.

Remark 1.5.1. If m and m' are neighbours, then the transition probabilities $P(m \to m')$ and $P(m' \to m)$ are strictly positive. This follows from the definition of the acceptance probability and the fact that the probability measure (1.2) is striclty positive on all states.

Proposition 1.5.1. The Markov Chain with state space Σ and transition probabilities (1.5) is irreducibile.

Proof. Let $m, m' \in \Sigma$ be any states. According to Proposition 1.3.1, there exists a sequence of transformations allowing us to construct state m' from the empty state 0. This sequence has positive probability since it is the product of strictly positive terms, as stated in Remark 1.5.1. Consequently, we have proven that $P(0 \to m') > 0$.
Moreover, by Remark 1.3.1 and the same reasoning, we also have $P(m \to 0) > 0$, which implies that $P(m \to m') > 0$.
By Remark 1.3.1 and the same reasoning, we also have $P(m \to 0) > 0$, implying that $P(m \to m') > 0$.

□

1.6 Aperiodicity

Since the chain is irreducible, we only need to prove that a state has period one. By our choice, if we can always find a state m and one of its neighbors m' such that the acceptance probability $A(m, m')$ is strictly less than one, then one can deduce from (1.5) that $P(m \to m) > 0$, ensuring that the state m has period one.

Proposition 1.6.1. For any fixed $\beta > 0$ there exists a state $m \in \Sigma$ such that $A(m', m) < 1$ where m' is obtained by applying a single link +2 square trasnformation.

Proof. By the definition of the acceptance probability (1.4), we need to check

$$\frac{\rho(m')}{\rho(m)} \frac{M_s(m)}{M_s(m')} < 1$$

Since every transformation that we can apply at square s to state m can also be applied on state m', we have $\frac{M_s(m)}{M_s(m')} \leq 1$.

We can compute the probability ratio:

$$\frac{\rho(m')}{\rho(m)} = \frac{\beta^2}{(m_e + 2)(m_e + 1)} \frac{1}{4} \xrightarrow{m_e \to \infty} 0$$

so we can make the acceptance probability arbitrary small if we choose a state with enough links on a single edge.

□

Optional These masks should improve mixing times, since they make multiple steps in one.