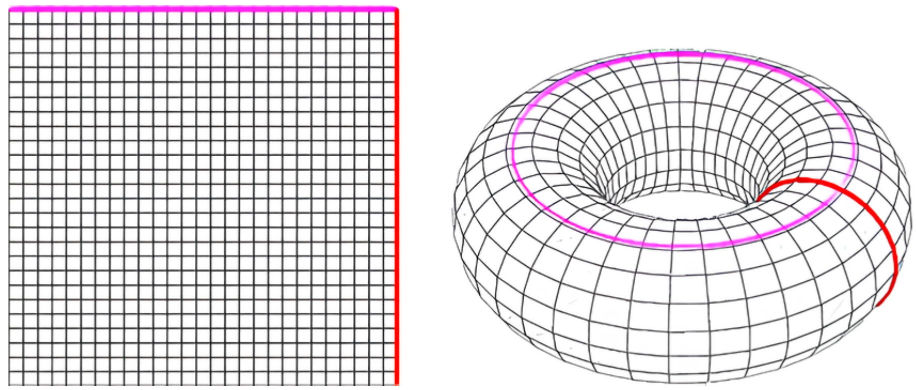

Figure 1.4: In pink an *horizontal non-separating* closed curve on the torus, in red a *vertical* one.

Beeing non-separating is a topological property, we will work with a characterization more suited for our setting.

We call an *horizontal loop* a simple closed curve on the torus that is composed only of horizontal links, like the curve in pink in figure 1.4. Their vertical counterpart is called *vertical loop*, see the curve in red in figure 1.4.

The characterization will exploit the fact that if a closed curve is separating, then the number of crossings with any horizontal and vertical loops l is even, call this number $Cross_l(\gamma)$.

Proposition 1.3.3. A closed curve γ on the torus is non-separating if and only if we can find an horizontal/vertical loop l such that $Cross_l(\gamma)$ is odd.

Proof. If γ is separating, meaning that is it's the boundary beetween two path disconnected regions of the torus, then $Cross_l(\gamma)$ is always even for any horizontal/vertical loop l. This can be proven using the same argument used in the proof of the Jordan's curve theorem for polygons. □

Proposition 1.3.4. Let $m \in \Sigma_G$ with $G = (V_n, E_n^p)$ be a state consisting of only separating curves. If given a square transformation X_s the new state $m' = X_s(m)$ contains one non-separating curve c, then it contains two.

Proof. We need to consider only transformation that add more links, namely uniform $+1$ and single $+2$. If the new state $m' = X_s(m)$ contains a non-separating curve γ, then one or two of its edges was introduced by the transformation X_s. This means that in the original state m, all the vertices in γ were already in the same connected component (a circuit is 2-connected), then by the Euler theorem for multigraphs we can find in m a circuit C which visits every vertex in γ. This implies that there exists exactly two edge-disjoint paths joining any pair of vertices. We will employ these paths to build two non-separating curves. We need to check two cases:

- *X_s is a single link $+2$.* Call e_1 and e_2 the two added links with endpoints x and y. Call γ the non-separating curve, which we can write as the union of a path P_{xy} from x to y contained in the circuit C and one of the new edges e_1:

$$\gamma = P_{xy} \cup \{e_1\}$$

Call $\tilde{P}_{xy}$ the other xy-path in C, then the closed curve $\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_2\}$ is also non-separating, since given an horizontal/vertical loop l:

$$Cross_l(\tilde{\gamma}) = Cross_l(C) - Cross_l(\gamma)$$

where $Cross_l(C)$ is even, since it's a closed curve contained in the original state m, and $Cross_l(\gamma)$ is odd since γ is non-separating, this implies that $Cross(\tilde{\gamma})$ is also odd.

- X_s *is a uniform* $+1$. Call e_1, e_2, e_3 and e_4 the new added edges, x and y the two vertices on γ that also are affected by the transformation. Again we can write:

$$\gamma = P_{xy} \cup \{e_1, e_2\}$$
$$\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_3, e_4\}$$

the same parity of crossings argument applies here for $\tilde{\gamma}$.

$\square$