

Chapter 1

Definition of the Markov Chain

In this chapter, we begin by defining the state space and subsequently provide a characterization of this space as the union of edge-disjoint cycles, making use of Euler's theorem for multigraphs. Following this, we introduce two distinct types of simple square transformations and demonstrate their capability to systematically construct any state, starting from the empty state. Together with their inverses these transformations are then employed to define an irreducible and aperiodic Markov Chain, converging to our chosen invariant distribution.

1.1 The space of link configurations and pairings

Definition 1.1.1. Let $G = (V, E)$ be a simple graph. A **link configuration** is an edge function $m : E \rightarrow \mathbb{N}$ such that $\forall x \in V$ we have $\sum_{y \sim x} m_{x,y} \in 2\mathbb{N}$.

Given a graph G , we denote the set of all its link configurations as Σ_G . Our focus will be specifically on instances where G is a grid subset of \mathbb{Z}^2 with periodic or free boundary conditions. In both cases, the vertex set is defined as

$$V_n := \{(i, j) \mid i, j \in \{0, \dots, n-1\}\}$$

while the edge sets for the free and the periodic boundary condition are, respectively:

$$E_n^f := \left\{ \{x, y\} \mid x, y \in V_n, \|x - y\| = 1 \right\}$$
$$E_n^p := E_n^f \cup \left\{ \{(i, 0), (i, n-1)\}, \{(0, j), (n-1, j)\} \mid i, j \in \{0, \dots, n-1\} \right\}$$

It's natural to define a *square* s to be an ordered tuple $s = (e_1, e_2, e_3, e_4)$ where the e_i s are edges in a cycle of length four. We will use the convention that e_1 is the top edge, e_2 the right, e_3 the bottom and e_4 the left. We will define s_v to be the set of its vertices.

We can identify any square s with one of its vertices, for example the top-right one, this implies that the number of squares is $|V_n| = n^2$.

We will call $B \subset V_n$ the set of vertices of degree less than 4 in (V_n, E_n^f) .

Definition 1.1.2. A **multicolor link configuration** of N colors is a tuple (m^1, \dots, m^N) of link configurations.

Note that the parity constraints are independent for each color. We will now simply refer to them as link configurations and call single-color configuration as monochromatic.

Given a link configuration m , we will define a binary relation on each vertex which connects links of the same color. TODO rewrite, less notation, simpler, we don't have links but only natural numbers on each edge!

Definition 1.1.3. A **pairing** for a link configuration m is a collection of symmetric binary relations at each vertex on its incident links, $\pi = (\pi_x)_{x \in V}$ such that for all l_i, l_j, l_k incident links of vertex x :

1. $\neg(l_i \pi_x l_i)$,
2. if $l_i \pi_x l_j$ then they are the same color,
3. for any l_i there exists an l_j such that $l_i \pi_x l_j$,
4. if $l_i \pi_x l_j$ then for any $l_k \neq l_j$ $\neg(l_i \pi_x l_k)$.

In words, there are no self-pairings; only pairings between links of the same color are permitted. Every link is paired (this is possible since we always have an even number of links of the same color incident at each vertex), and each link can be paired only with one other link.

We denote the set of all possible pairing configurations of a link configuration m as $\mathcal{P}(m)$. Our state space will be the set $\Omega := \{(m, \pi) : m \in \Sigma, \pi \in \mathcal{P}(m)\}$.

1.2 Edge-disjoint cycle representation

Consider monochromatic states. We can view them as a multigraph with even degree on all vertices. By the Euler's theorem adapted for multigraphs, we can represent G as a union of edged disjoint cycles.

Lemma 1.2.1. Let $G = (V, E)$ be a multigraph with non-empty edge set, such that $\forall x \in V$ $d(x) \in 2\mathbb{N}$. Then G contains a cycle.

Proof. Since $|E| > 0$ there exists a vertex $v \in V$ such that $d(v) \geq 2$. Starting from this vertex choose any edge, call it v, x_1 , since $d(x_1)$ is even, there exists at least another edge unvisited edge. Keep going, since V is a finite set after enough steps we encounter an already visited vertex: we have found a cycle. \square

Proposition 1.2.1. If $G = (V, E)$ is a multigraph in which $\forall v \in V$ $\deg(v) \in 2\mathbb{N}$ then the edge set E can be partitioned into edge-disjoint cycles.

Proof. We prove it by strong induction on the number of cycles. The base case is a multigraph with $|E| = 0$. Such a graph consists of one or more isolated vertices and the (empty) edge set can clearly be partitioned into a union of zero cycles.

Now suppose the result is true for every multigraph $G = (V, E)$ with $|E| \leq m$ edges whose vertices all have even degree. Consider a multigraph with $|E| = m + 1$. From Lemma 1.2.1 we know that there is at least one cycle $C = (\mathcal{V}, \mathcal{E})$ contained in G . Then we can form a new graph $G' = (V, E')$ by removing the edges that appear in the cycle $E' := E \setminus \mathcal{E}$. Every vertex in the cycle has its degree reduced by two, vertices that didn't appear in the cycle maintain the same degree: parity is preserved. Since we removed at least

two edges (the shortest cycle possible in a multigraph) $|E'| \leq m$ so that by the induction hypothesis we can partition the edge set of G' as disjoint cycles $E' = \bigcup_{i=1}^N C_i$. Then adding C to the partition of E' we obtain a partition for the original edge set E . \square

1.3 Square transformations

We define a class of transformations that map a state m to a new state m' by modifying only the links within a designated square on the grid. We limit ourselves to these transformations due to their local nature. Since our goal is to simulate a Markov chain on Σ , the locality of square transformations is crucial to make the exploration of the state space computationally tractable.

Before doing this, we define the subset S of state the domain $S \subseteq \Sigma$ of a square transformation X_s with associated integers $(\alpha, \beta, \gamma, \delta)$ is thus:

$$S := \{m \in \Sigma \mid m_{e_1} \geq \alpha, m_{e_2} \geq \beta, m_{e_3} \geq \gamma, m_{e_4} \geq \delta, (e_1, e_2, e_3, e_4) = s\}$$

Definition 1.3.1. Given a square $s = (e_1, e_2, e_3, e_4)$ and four integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with the same parity, we define the *square transformation*

$$X_{s,(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} : S \subseteq \Sigma \rightarrow \Sigma$$

that sends a state m to a new state m' defined by

$$m'_e = \begin{cases} m_{e_i} + \alpha_i & \text{if } e = e_i \\ m_e & \text{otherwise.} \end{cases}$$

$$\begin{cases} \alpha + \beta \equiv 0 \pmod{2}, \\ \beta + \gamma \equiv 0 \pmod{2}, \\ \gamma + \delta \equiv 0 \pmod{2}, \\ \delta + \alpha \equiv 0 \pmod{2}. \end{cases}$$

Each square transformation is characterized by the square it acts on and the four integers associated with it. The parity constraints, which can be more easily stated as α, β, γ and δ have the same parity, ensure that the parity number of links incident at each vertex is preserved, in accordance with definition 1.1.1.

Note also that we can apply a square transformation which removes links only when there are a sufficient number of links on the edge:

Given that our state is generally a multicolor link configuration, we require a distinct set of square transformations for each of the possible N colors, each acting only on a single color.

We define two special types of square transformations, the first are called *uniform*, the second are called *single*:

$$U_s^{\pm 1}(m)_e := \begin{cases} m_e \pm 1 & \text{if } e \text{ is an edge of square } s \\ m_e & \text{otherwise} \end{cases}$$

$$R_s^{\pm 2}(m)_e := \begin{cases} m_e \pm 2 & \text{if } e \text{ is the right edge of square } s \\ m_e & \text{otherwise} \end{cases}$$

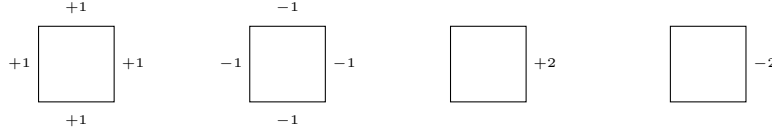


Figure 1.1: The square transformations U^{+1} , U^{-1} , R^{+2} and R^{-2} , from left to right.

$L_s^{\pm 2}$, $T_s^{\pm 2}$ and $B_s^{\pm 2}$ which are called *single left*, *top* and *bottom* are defined in the obvious way.

Proposition 1.3.1. All square transformations can be obtained by composing single and uniform transformations.

Proof. Let X_s denote any square transformation, and let $\alpha, \beta, \gamma, \delta$ be its associated integers. Choose the biggest integer, say α . Begin with the uniform transformation U_s^α , obtained by applying the uniform transformation with the sign of α $|\alpha|$ times. Since from the parity constraints also follows that $\alpha - \beta \equiv 0 \pmod{2}$, it makes sense to apply the right single -2 transformation a number $|\alpha - \beta|$ of times, we call this $R_s^{(\alpha - \beta)}$. Similarly, apply the left and the bottom single transformations $L_s^{(\delta - \alpha)}$ and $B_s^{(\gamma - \alpha)}$ respectively.

We found the (non unique) decomposition:

$$X_s = R_s^{(\beta - \alpha)} \circ B_s^{(\gamma - \alpha)} \circ L_s^{(\delta - \alpha)} \circ U_s^\alpha.$$

□

We will prove that it is possible to construct any link configuration starting from the empty state (zero links on all edges) using only the uniform $+1$ and the four single-link -2 transformations in the case of free boundary conditions.

We will also prove that this task is impossible if we consider periodic boundary conditions: this is obvious when our graph is $B(n)$ with n odd, since we can find states with an odd number of links, but square transformation preserves the parity of the total number of links in a state. The general case will follow from the topological properties of the torus.

Proposition 1.3.2. Any state $m \in \Sigma_G$, where $G = (V_n, E_n^f)$ is a box subset of \mathbb{Z}^2 with free boundary conditions, can be constructed from the empty state $0 \in \Sigma_G$ just by using the uniform $+1$ and the four single -2 transformations.

Proof. Since we can work with any color independently, we consider a monochromatic state without loss of generality. Since our state has even degree on all vertices of the lattice, we can apply proposition 1.2.1 on the multigraph obtained by keeping only non-isolated vertices of the lattice. Call the edge-disjoint cycle partition of the edges $\{C_i\}_{i \in [N]}$, where N is the number of cycles. Since the cycles are edge disjoint, we can construct them separately, thus we only need to prove that we can build a single cycle¹.

Constructing a cycle starting from the empty state is straightforward: the *interior*² $\text{Int}(C)$ of a cycle C is defined as a subset of squares such that:

$$\text{Int}(C) := \{s \mid \text{any path from } s_v \text{ to } B \text{ contains a vertex of } C\}$$

¹This argument can be made precise through induction on N .

²Here the assumption of free boundary conditions is necessary.

if the cycle has more than 2 edges (the cycle is a simple graph) apply the uniform $+1$ mask to every square in $\text{Int}(C)$ (which is at least one square). Subsequently, eliminate the undesired double edges with the single link -2 transformations.

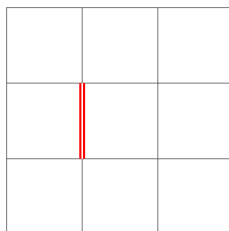


Figure 1.2: A two edge cycle with empty interior.

If the cycle has 2 edges (in this case the interior is empty), we can construct it by applying the uniform $+1$ transformation twice and then applying single link -2 to eliminate the three undesired edges. \square

Remark 1.3.1. Clearly, with the inverse transformations uniform -1 and single link $+2$, we can reach the empty state starting from any state.

Example 1.3.1. Suppose we want to build the red cycle in the left picture of figure 1.3.

1. Apply the uniform $+1$ mask on all squares on the interior of the cycle.
2. Use the -2 single link to remove the double edges inside.

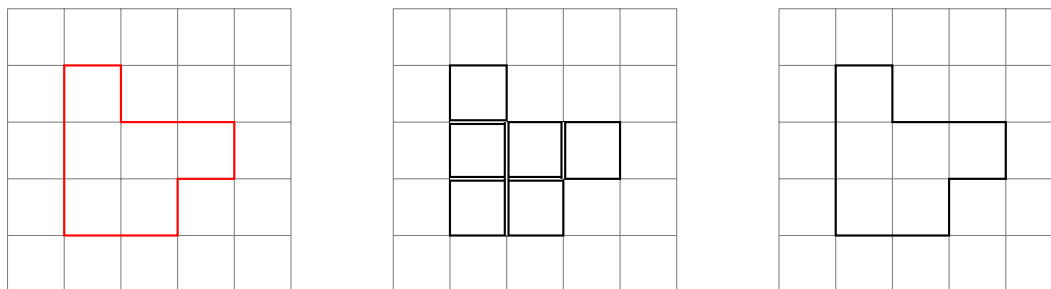


Figure 1.3: How to build any cycle with a well defined interior.

This procedure works on the plane, but not on the torus! In this case the boundary B is empty, and we can't define the interior/exterior of a closed curve as in the proof of proposition 1.3.2. Still there are some simple closed curves that separate the torus into two regions both of which have the curve as boundary; clearly we can build these curves using the same procedure applied to any of these two regions. However this does not work in general, in fact there exists closed curves which don't separate the torus into two regions, hence we can't apply the same procedure. We call these curve *non-separating*, see figure 1.4.

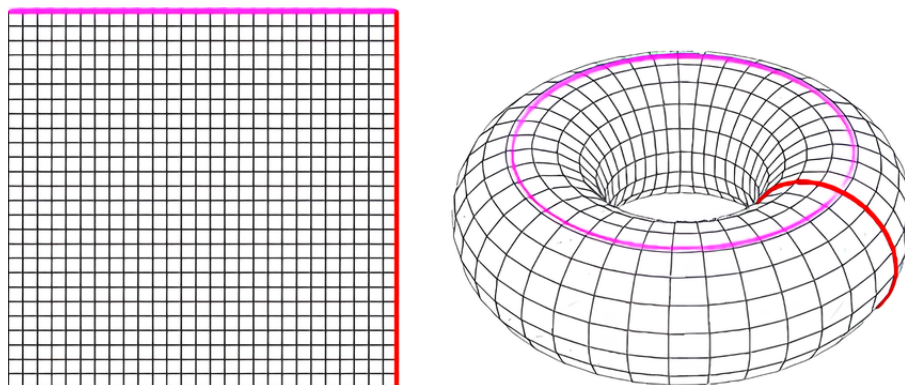


Figure 1.4: In pink an *horizontal non-separating* closed curve on the torus, in red a *vertical* one.

Being non-separating is a topological property, we will work with a characterization more suited for our setting.

We call an *horizontal loop* a simple closed curve on the torus that is parallel to horizontal links and does not intersect any vertex, like the curve in orange in figure 1.5. Their vertical counterpart is called *vertical loop*.

The characterization will exploit the fact that if a closed curve is separating, then the number of intersections with any horizontal and vertical loops l is even, call this number $\text{Cross}_l(\gamma)$.

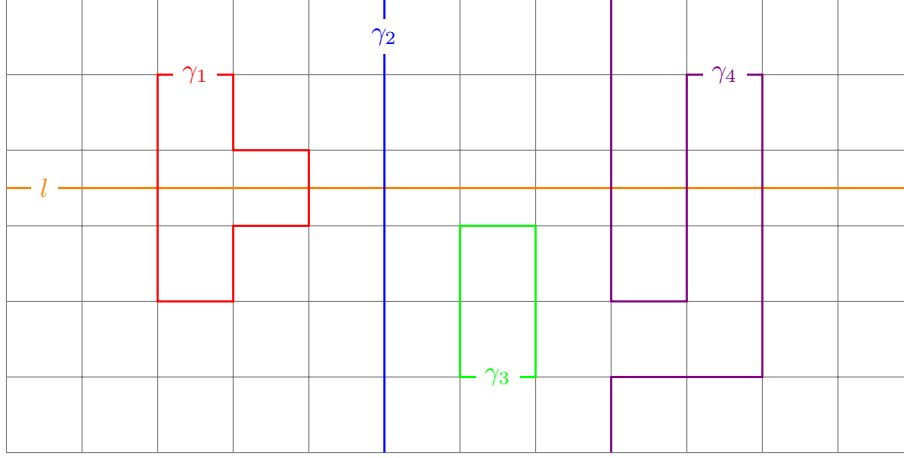


Figure 1.5: Four closed simple curves on the torus and an horizontal loop l in orange. Here $\text{Cross}_l(\gamma_1) = 2$, $\text{Cross}_l(\gamma_2) = 1$, $\text{Cross}_l(\gamma_3) = 0$ and $\text{Cross}_l(\gamma_4) = 3$.

Clearly if γ_1 and γ_2 are two edge-disjoint curves, then $\text{Cross}_l(\gamma \cup \eta) = \text{Cross}_l(\gamma) + \text{Cross}_l(\eta)$, so that any separating circuit, which can be written as the union of edge-disjoint cycles (which are closed simple curves), has even cross number.

Proposition 1.3.3. A closed curve γ on the torus is non-separating if and only if we can find an horizontal/vertical loop l such that $\text{Cross}_l(\gamma)$ is odd.

Proof. We will prove that γ is separating if and only if $\text{Cross}_l(\gamma)$ is even for every horizontal/vertical loop l .

This can be proven using the same argument used in the proof of the Jordan's curve theorem for polygons.

Assume γ is separating, call A and B the two disjoint regions of the torus such that $\partial A = \partial B = \gamma$. Choose a point x on any given any vertical/horizontal loop l which does not lie on γ . Then either $x \in A$ or $x \in B$. Suppose wlog that $x \in A$, if $\text{Cross}_l(\gamma)$ were odd, then also $x \in B$, but $A \cap B = \emptyset$.

Suppose now that for any vertical/horizontal loop l $\text{Cross}_l(\gamma)$ is even. Then we can define two disjoint subsets in the following way:

1. Pick point x in the center of an unassigned square s and an horizontal loop $x \in l$. Assign all the points of the square s in the set A .
2. Let s' be any other square which intersect with l , call x' the point at its center. Then we can split l into two curves l_1 from x to x' and l_2 from x' to x . Since $\text{Cross}_l(\gamma)$ is even, the intersection of the two curves l_1 and l_2 with γ has the same parity. If it's even, assign all points in the square s' to the set A , otherwise to B .
3. Repeat this until all squares have been assigned using a vertical loops.

It's easy to see that the sets $A \setminus \gamma$ and $B \setminus \gamma$ have the desired properties, hence γ is separating. \square

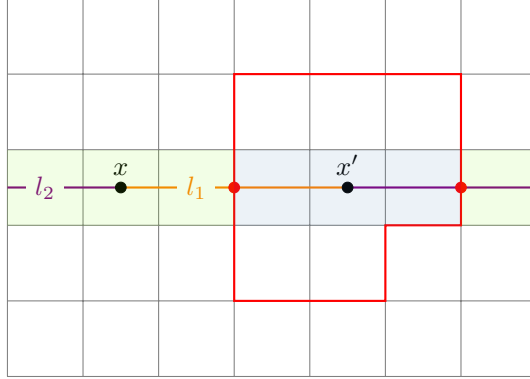


Figure 1.6: The procedure to define the subsets A and B discussed in the proof of proposition 1.3.3. The number of intersection of both l_1 and l_2 is odd.

We call a non-separating closed curve γ *vertical non-separating* if there exists an horizontal loop l such that $\text{Cross}_l(\gamma)$ is odd; we call it *horizontal non-separating* if the same is true for a vertical loop l . The curves γ_2 and γ_4 in figure 1.5 are both vertical non-separating.

Proposition 1.3.4. Let $m \in \Sigma_G$ with $G = (V_n, E_n^p)$ be a state consisting of only separating curves. If given a square transformation X_s the new state $m' = X_s(m)$ contains one non-separating curve γ , then it contains two.

Proof. We need to consider only transformation that add more links, namely uniform +1 and single +2. If the new state $m' = X_s(m)$ contains a non-separating curve γ , then one or two of its edges was introduced by the transformation X_s . This means that in the original state m , all the vertices in γ were already in the same connected component (a circuit is at least 2-edge-connected), then by Euler's theorem for multigraphs we can find in m a circuit C which visits every vertex in γ . This implies that if we remove any path P joining any two vertices in the cycle, then $C \setminus P$ contains a path visiting all edges (an eulerian path) joining the same vertices. We will employ this fact to build another non-separating curve. We need to check two cases:

- X_s is a *single* +2. Call e_1 and e_2 the two new added links with endpoints x and y . Call γ the non-separating curve, which we can write as the union of a path P_{xy} from x to y contained in the circuit C and one of the new edges e_1 :

$$\gamma = P_{xy} \cup \{e_1\}$$

Call \tilde{P}_{xy} the eulerian path joining x to y contained in $C \setminus P_{xy}$, then the closed curve $\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_2\}$ is also non-separating, since given an horizontal/vertical loop l :

$$\text{Cross}_l(\tilde{\gamma}) = \text{Cross}_l(C) - \text{Cross}_l(\gamma)$$

where $\text{Cross}_l(C)$ is even, since it's a circuit contained in the original state m , and $\text{Cross}_l(\gamma)$ is odd since γ is non-separating, this implies that $\text{Cross}_l(\tilde{\gamma})$ is also odd.

- X_s is a *uniform* +1. Call e_1, e_2, e_3 and e_4 the new added edges, x and y the two vertices on γ that also are affected by the transformation. Again we can write:

$$\gamma = P_{xy} \cup \{e_1, e_2\}$$

$$\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_3, e_4\}$$

the same parity of crossings argument applies here for $\tilde{\gamma}$.

□

A straightforward consequence of this proposition is that we cannot reach a state with an odd number of non-separating curves starting from the empty state, as they can only be produced in pairs. More generally:

Corollary 1.3.1. Each parity number of non-separating curves (vertical and horizontal) is invariant with respect to square transformations.

To overcome this problem we are forced to introduce new non square transformations that don't preserve the parity of non-separating curves, such as:

$$V_k^{+1}(m)_e := \begin{cases} m_e + 1 & \text{if } e \text{ connects two vertices on the vertical line } x = k \\ m_e & \text{otherwise} \end{cases}$$

$$H_k^{+1}(m)_e := \begin{cases} m_e + 1 & \text{if } e \text{ connects two vertices on the horizontal line } y = k \\ m_e & \text{otherwise} \end{cases}$$

Notice how we don't need their inverses, since we can apply them twice and then use the single -2 to remove all links. The downside is that they affect all the edges on a given vertical/horizontal line: they are not local, in the sense that the number of affected edges depends linearly on the size of the grid.

1.4 Transition probabilities

We define a probability measure (1.1) on the set Ω , where each element is a link configuration with a compatible pairing, $\omega = (m, \pi)$.

$$\mathbb{P}(\omega) = \frac{1}{Z} \prod_{e \in E} \prod_{i=1}^N \frac{\beta^{m_e^i}}{m_e^i!} \prod_{x \in V} \frac{\Gamma(\frac{N}{2})}{2^{n_x(m)} \Gamma(\frac{N}{2} + n_x(m))} = \frac{1}{Z} \mu(m) \quad (1.1)$$

Where Z is a normalization constant, $\beta \in \mathbb{R}_{\geq 0}$, N is the number of colors and $n_x(m) := \frac{1}{2} \sum_{i=1}^N \sum_{y \sim x} m_{x,y}^i$ which we refer to as the *local time*.

Since it is independent of the pairing configuration, the marginal probability of the link configuration is simply

$$\rho(m) = \sum_{\pi \in \mathcal{P}(m)} \mathbb{P}(m, \pi) = \frac{1}{Z} \mu(m) |\mathcal{P}(m)| \quad (1.2)$$

where the number of possible pairings configuration is

$$|\mathcal{P}(m)| = \prod_{x \in V} \prod_{i=1}^N (2n_x(m) - 1)!!$$

To sample from the probability measure (1.1) we first sample from the marginal measure (1.2) and then uniformly sample a pairing configuration.

Definition 1.4.1. We say that states $m, m' \in \Sigma$ are neighbours and write $m \sim m'$ if we can find a square transformation X_s such that $X_s(m) = m'$.

From a starting state $m \in \Sigma$, we attempt to transition to a neighboring state m' by applying one of the possible square transformations for a randomly chosen color, which acts on a square s , both chosen uniformly at random.

The jump probability from state m to state m' is defined as

$$q(m' | m) = \begin{cases} \frac{1}{N} \frac{1}{|V|} \frac{1}{M_s(m)} & \text{if } m' = X_s(m) \\ 0 & \text{otherwise} \end{cases}$$

where $M_s(m)$ is the number of possible transformation of the chosen color that can be applied to square s with link configuration m , and $|V|$ is the total number of squares in the grid (which is equal to the number of vertices).

We define the transition probability

$$P(m \rightarrow m') = q(m' | m) A(m', m)$$

where $A(m', m)$ is called the *acceptance probability* from state m to state m' . Imposing the detailed balance equations, we find the condition

$$\frac{A(m', m)}{A(m, m')} = \frac{q(m | m')}{q(m' | m)} \frac{\rho(m')}{\rho(m)} = \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)} \quad (1.3)$$

A common choice which satisfies condition (1.3) is the Metropolis-Hastings acceptance probability:

$$A(m', m) = \min \left(1, \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)} \right) \quad (1.4)$$

Another possible choice is the Glauber acceptance probability:

$$A(m', m) = \left(1 + \frac{M_s(m')}{M_s(m)} \frac{\rho(m)}{\rho(m')} \right)^{-1}$$

Since m and m' only differ by a transformation in one square s , we can compute efficiently the ratios $\rho(m')/\rho(m)$.

Putting it all together, our transition probabilities are:

$$P(m \rightarrow m') = \begin{cases} \frac{1}{N} \frac{1}{|V|} \frac{1}{M_s(m)} \min \left(1, \frac{M_s(m)}{M_s(m')} \frac{\rho(m')}{\rho(m)} \right) & \text{if } m' \sim m \\ \sum_{m'' \sim m} \frac{1}{N} \frac{1}{|V|} \frac{1}{M_s(m)} \left[1 - \min \left(1, \frac{M_s(m)}{M_s(m'')} \frac{\rho(m'')}{\rho(m)} \right) \right] & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases} \quad (1.5)$$

The probability $P(m \rightarrow m)$ is defined to ensure the transition matrix is stochastic:

$$\sum_{m' \in \Sigma} P(m \rightarrow m') = \frac{1}{N} \frac{1}{|V|} \sum_{m' \sim m} \frac{1}{M_s(m)} \quad (1.6)$$

$$= \frac{1}{N} \frac{1}{|V|} \sum_s \sum_{m' = X_s(m)} \frac{1}{M_s(m)} \quad (1.7)$$

$$= \frac{1}{N} \frac{1}{|V|} \sum_s N = 1 \quad (1.8)$$

TODO: better notation! define the graph as box subset of Z^2 .

We can compute the explicit acceptance probability for each of our possible neighbours. We call the edges of square s as s_E and its four vertices s_V .

- $m' = U_s^{+1}(m)$

$$\frac{\rho(m')}{\rho(m)} = \frac{\beta^4}{16} \prod_{e \in s_E} (m_e^i + 1)^{-1} \prod_{x \in s_V} \left(\frac{N}{2} + n_x \right)^{-1}$$

- $m' = U_s^{-1}(m)$

$$\frac{\rho(m')}{\rho(m)} = \frac{16}{\beta^4} \prod_{e \in s_E} m_e^i \prod_{x \in s_V} \left(\frac{N}{2} + n_x - 1 \right)$$

- $m' = R_s^{+2}(m)$

$$\frac{\rho(m')}{\rho(m)} = \frac{1}{4} \frac{\beta^2}{(m_e^i + 2)(m_e^i + 1)} \left(\frac{N}{2} + n_x \right)^{-1} \left(\frac{N}{2} + n_y \right)^{-1}$$

- $m' = R_s^{-2}(m)$

$$\frac{\rho(m')}{\rho(m)} = 4 \frac{m_e^i (m_e^i - 1)}{\beta^2} \left(\frac{N}{2} + n_x - 1 \right) \left(\frac{N}{2} + n_y - 1 \right)$$

- $m' = H_y^{+1}(m)$ or $m' = V_x^{+1}(m)$

$$\frac{\rho(m')}{\rho(m)} = \left(\frac{\beta}{2} \right)^{\text{grid_size}} \prod_{e \text{ fixed } y} (m_e + 1)^{-1} \prod_{(\cdot, y) \in V} \left(\frac{N}{2} + n_x \right)^{-1}$$

the last two transformation are non-local, the time to compute these ratios scales with the grid size, and the probability is so low that they never occur unless β is very high.

To improve convergence to equilibrium, in particular for low β , we introduce a set of additional square transformation. Note that all of these can be obtained by using the previous ones, we do it only for convergence purposes.

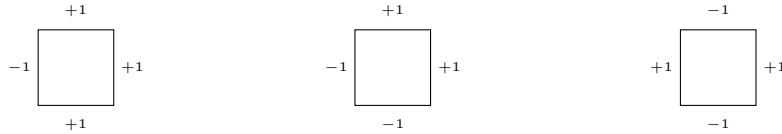


Figure 1.7: The square transformations *triple*, *swap-a* and *swap-b* from left to right.

1.5 Irreducibility

The key ingredient is Proposition 1.3.2, which asserts that with free boundary conditions any link configuration can be constructed starting from the empty configuration by applying only the uniform +1 and single link -2 transformations. If we also include their inverses, uniform -1 and single link +2, we can reach the empty configuration starting from any state.

Remark 1.5.1. If m and m' are neighbours, then the transition probabilities $P(m \rightarrow m')$ and $P(m' \rightarrow m)$ are strictly positive. This follows from the definition of the acceptance probability and the fact that the probability measure (1.2) is strictly positive on all states.

Proposition 1.5.1. The Markov Chain with state space Σ_G with $G = (V_n, E_n^f)$ and transition probabilities (1.5) is irreducible.

Proof. Let $m, m' \in \Sigma$ be any states. According to Proposition 1.3.2, there exists a sequence of transformations allowing us to construct state m' from the empty state 0. This sequence has positive probability since it is the product of strictly positive terms, as stated in Remark 1.5.1. Consequently, we have proven that $P(0 \rightarrow m') > 0$.

Moreover, by Remark 1.3.1 and the same reasoning, we also have $P(m \rightarrow 0) > 0$, which implies that $P(m \rightarrow m') > 0$. □

1.6 Aperiodicity

Since the chain is irreducible, we only need to prove that a state has period one. By our choice, if we can always find a state m and one of its neighbors m' such that the acceptance probability $A(m, m')$ is strictly less than one, then one can deduce from (1.5) that $P(m \rightarrow m) > 0$, ensuring that the state m has period one.

Proposition 1.6.1. For any fixed $\beta > 0$ there exists a state $m \in \Sigma$ such that $A(m', m) < 1$ where m' is obtained by applying a single link +2 square transformation.

Proof. By the definition of acceptance probability (1.4), we need to check

$$\frac{\rho(m')}{\rho(m)} \frac{M_s(m)}{M_s(m')} < 1$$

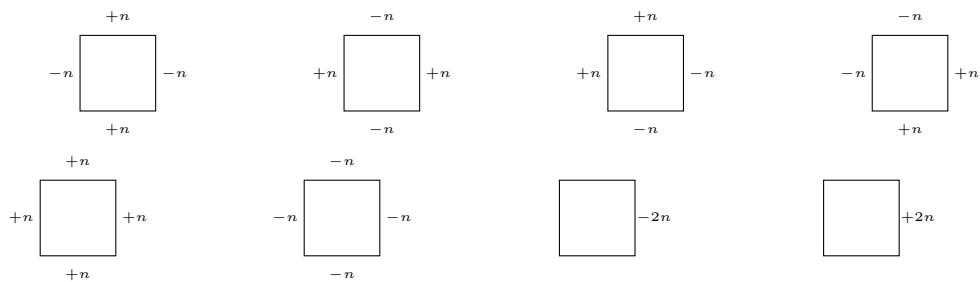
Since every transformation that we can apply at square s to state m can also be applied on state m' , we have $\frac{M_s(m)}{M_s(m')} \leq 1$.

We can compute the probability ratio:

$$\frac{\rho(m')}{\rho(m)} = \frac{1}{4} \frac{\beta^2}{(m_{\{x,y\}}^i + 2)(m_{\{x,y\}}^i + 1)} \left(\frac{N}{2} + n_x \right)^{-1} \left(\frac{N}{2} + n_y \right)^{-1} \xrightarrow{m_{\{x,y\}} \rightarrow \infty} 0$$

so we can make the acceptance probability arbitrary small if we choose a state with enough links on a single edge. □

Optional These masks should improve mixing times, since they make multiple steps in one.



1.7 Simulation

Heuristic for exploring the $\beta^{1/2}$ dependence of average number of link vs β .

Try to maximize the probability, assume uniform number of link x .

$$\approx \frac{\beta^x}{x!} \Gamma \approx \left(\frac{\beta e^3}{2x^2} \right)^x$$

\log is a concave function, thus is the same to maximize

$$x \log \beta e^3 - x \log 2x^2$$

which has stationary points

$$2x^2 = \beta e^3$$

$$x \sim \beta^{1/2}$$

1.8 useless stuff

Any cycle can be obtained as the boundary of the union of square boxes:

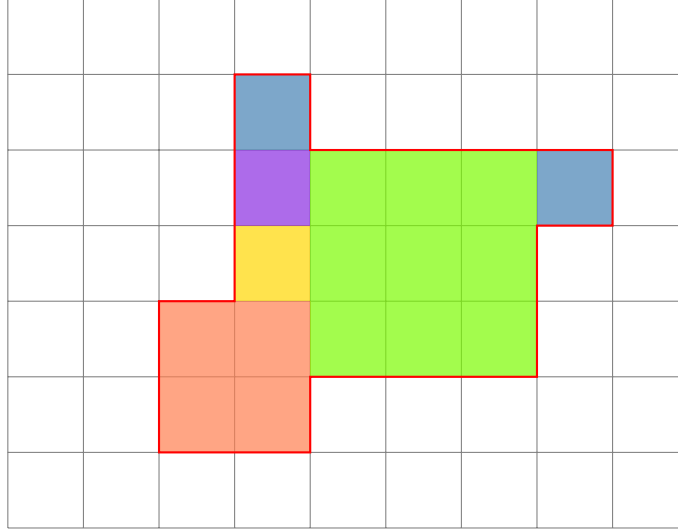


Figure 1.8: A cycle as the boundary of the union of squares.

Since any cycle Thus if we can build any $n \times n$ square we are done (there is the case with links on the same edge! easy just use mask +2, need to encompass this case better) Since every internal double edge can be removed using the single link -2 mask.

We prove it using induction: The base case with 1×1 squares are trivial, just use the uniform +1 mask. Suppose we have a $(n-1) \times (n-1)$ square, then we can add to a side all +1 uniform masks, then -2 to each internal boundary and obtain a $n \times n$ square. \square

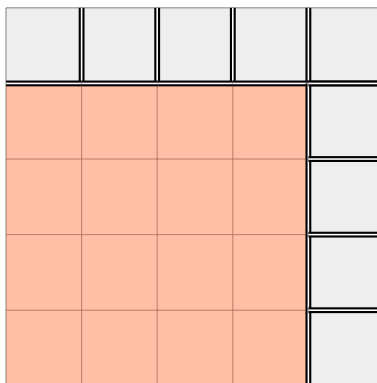


Figure 1.9: Add the missing part with the uniform $+1$ mask, then remove the double internal edges with the single link -2 .