Chapter IV

Stochastic Ising Models

Stochastic Ising models can be thought of loosely as reversible spin systems with strictly positive rates. (For a more precise version of this statement, see Theorem 2.13.) The measures with respect to which they are reversible are the Gibbs states of classical statistical mechanics. Thus stochastic Ising models can be viewed as models for nonequilibrium statistical mechanics. These were among the first spin systems to be studied, because of their close connections with physics. While no prior knowledge of statistical mechanics will be assumed in this chapter, the reader may find it useful to refer occasionally to more comprehensive treatments of that subject. Recommended for this purpose are Ruelle (1969, 1978), Griffiths (1972), Preston (1974b), Sinai (1982), and Simon (1985).

This chapter is devoted to the study of stochastic Ising models, and their relation to the Gibbs states. The first section contains the definition and some of the elementary properties of Gibbs states. The concept of phase transition is introduced there, and a number of inequalities are proved which serve to clarify that concept. Stochastic Ising models are defined in Section 2. It is shown there that in the finite range case, they coincide with the spin systems with positive rates which are reversible with respect to some probability measure. Furthermore, the reversible measures are exactly the corresponding Gibbs states. Hence phase transition for the Gibbs states is seen to imply nonergodicity for the associated stochastic Ising models. The reverse implication is correct as well in certain cases. This is the first connection between these two important concepts of phase transition and ergodicity.

The third section is devoted to results which assert the presence or absence of phase transition in special cases. In particular, we will see that phase transition does not occur in one-dimensional models with finite range, but that it does occur in higher dimensions at sufficiently low temperatures. There will be an interplay in this section between stochastic Ising models and the corresponding Gibbs states. Results developed here or earlier about one of these will yield results about the other.

The main idea in Section 4 is to use the spectral theorem to study stochastic Ising models. This tool is available to us because reversibility corresponds exactly to the self-adjointness of the generator of the spin system in the L_2 space of the corresponding Gibbs state. The principal results are that convergence in $L_2(\nu)$ for an appropriate Gibbs state ν is equivalent to ν being an extremal Gibbs state, and that exponential convergence in $L_2(\nu)$ is equivalent to there being a gap in the spectrum of $-\Omega$ to the right of the origin.

Section 5 is devoted to the use of relative entropy techniques in the study of the invariant measures of stochastic Ising models. The main result asserts under appropriate conditions that in one and two dimensions, every invariant measure is a Gibbs state. In higher dimensions, every invariant measure which is translation invariant is a Gibbs state.

Throughout this chapter, we will adopt the setting of Chapter III. In particular, the flip rates of all spin systems will be assumed to be uniformly bounded and to satisfy (0.3) of that chapter.

Sections 1, 2, and 3 should be read in that order. Following that, Sections 4 and 5 can be read in either order. Sections 5 and 4 of Chapter II should be read before reading Sections 2 and 5 of this chapter respectively.

1. Gibbs States

Before giving the definition of Gibbs states, we need to introduce the concept of a potential. A potential describes the interactions among the particles located at the sites in S.

Definition 1.1. A *potential* is a collection $\{J_R\}$ of real numbers indexed by finite subsets R of S which satisfies

$$\sum_{R \ni x} |J_R| < \infty, \qquad x \in S. \tag{1.2}$$

For finite $R \subset S$ and $\eta \in X = \{0, 1\}^S$, let

$$\chi_R(\eta) = \prod_{x \in R} [2\eta(x) - 1]. \tag{1.3}$$

In order to lead up to the definition of the Gibbs states corresponding to a potential, consider first the case in which S is finite. The Gibbs state corresponding to the potential $\{J_R, R \subset S\}$ is then the unique probability measure ν on X which is given by

$$\nu\{\eta\} = K \exp\left\{\sum_R J_R \chi_R(\eta)\right\}, \tag{1.4}$$

where K is the normalizing constant

$$K = \left[\sum_\eta \exp\left\{\sum_R J_R \chi_R(\eta)\right\}\right]^{-1}.$$

In this case, of course, any probability measure on X which assigns strictly positive probabilities to all points is a Gibbs state for some potential. In order to see this, it suffices to express the logarithm of $\nu\{\eta\}$ as a linear combination of $\{\chi_R(\eta), R \subset S\}$. This can be done, since this set forms a basis for the $2^{|S|}$ dimensional vector space of all functions on X.

When S is countable, the definition of Gibbs states is considerably more subtle. After all, the summation in (1.4) is meaningless in general, and in fact we expect $\nu\{\eta\}$ to be zero in this case. There are two ways to overcome these difficulties. The first is to specify certain conditional probabilities of ν instead of the probabilities given in (1.4). The second is to define Gibbs states as in (1.4), but for configurations on finite subsets of S only, and then to pass to a limit. We will adopt the first of these approaches as the definition, and will then show that the second procedure yields the same class of Gibbs states.

Definition 1.5. A probability measure ν on X is said to be a *Gibbs state* relative to the potential $\{J_R\}$ provided that for all $x \in S$, a version of the conditional probability

$$\rho_x(\zeta) = \nu\{\eta: \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \text{ for all } u \neq x\} \tag{1.6}$$

is given by

$$\frac{1}{1+\exp\left[-2 \sum_{R \ni x} J_R \chi_R(\zeta)\right]}. \tag{1.7}$$

Note that the function in (1.7) is well defined and continuous on X for each x by (1.2). The class of all Gibbs states relative to a fixed potential $\{J_R\}$ will be denoted by $\mathscr{G}$.

Of course it is important to check that this definition coincides with (1.4) in case S is finite. This follows from the next result, which will be needed again later.

Proposition 1.8. *Suppose that S is finite and that ν is a probability measure on X. Then ν is given by* (1.4) *if and only if it is a Gibbs state relative to $\{J_R\}$ in the sense of Definition* 1.5.

Proof. Clearly $\nu\{\eta\} > 0$ for all $\eta \in X$ in either case. So, suppose (1.4) holds. Then

$$\begin{aligned}
\nu\{\eta: \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \text{ for all } u \neq x\} \\
&= \frac{\nu\{\zeta\}}{\nu\{\zeta\} + \nu\{\zeta_x\}} \\
&= \frac{\exp\left\{\sum_R J_R \chi_R(\zeta)\right\}}{\exp\left\{\sum_R J_R \chi_R(\zeta)\right\} + \exp\left\{\sum_R J_R \chi_R(\zeta_x)\right\}}
\end{aligned}$$

$$=\frac{1}{1+\exp\left\{\sum_R J_R[\chi_R(\zeta_x)-\chi_R(\zeta)]\right\}}$$

$$=\frac{1}{1+\exp\left\{-2\sum_{R\ni x} J_R\chi_R(\zeta)\right\}},$$

since

(1.9)
$$\chi_R(\eta_x)=\begin{cases}-\chi_R(\eta) & \text{if } x\in R,\\ \chi_R(\eta) & \text{if } x\notin R.\end{cases}$$

For the converse, suppose ν is Gibbs in the sense of Definition 1.5. By the comment following (1.4), ν satisfies (1.4) for some possibly different potential $\{\tilde{J}_R\}$. But then by the first part of this proof,

$$\sum_{R\ni x} J_R\chi_R(\eta)=\sum_{R\ni x}\tilde{J}_R\chi_R(\eta).$$

Since this is true for all x, it follows that $J_R=\tilde{J}_R$ for all $R\neq\varnothing$. But changing $J_\varnothing$ simply amounts to changing the normalizing constant K, so the proof is complete. □

Next we will see that any sufficiently nice measure is a Gibbs state for some potential.

Proposition 1.10. *Suppose that ν is a probability measure on X such that for all $x\in S$, there is a version of the conditional probabilities $\rho_x(\zeta)$ given in* (1.6) *which can be written in the form*

(1.11)
$$\rho_x(\zeta)=\frac{1}{1+\exp\left[-2\sum_R J_R^x\chi_R(\zeta)\right]}$$

for some family $\{J_R^x\}$ which satisfies

$$\sum_R |J_R^x|<\infty$$

for each $x\in S$. Then ν is a Gibbs state relative to some potential $\{J_R\}$.

Proof. Comparing (1.7) and (1.11), it is clear that we need only show that

(1.12)
$$J_R^x=0 \quad \text{if } x\notin R, \quad \text{and}$$

(1.13)
$$J_R^x=J_R^y \quad \text{if } x,y\in R,$$

since then we can define $\{J_R\}$ by

$$J_R = J_R^x \quad \text{for } x \in R.$$

By definition, $\rho_x(\zeta) + \rho_x(\zeta_x) = 1$ for all $x \in S$ and $\zeta \in X$. Therefore

$$\sum_R J_R^x \chi_R(\zeta) = -\sum_R J_R^x \chi_R(\zeta_x),$$

so that (1.12) follows from (1.9) and the fact that the functions χ_R are linearly independent. In order to prove (1.13), we will first show that the expression

$$\left[\frac{1}{\rho_x(\zeta)} - 1\right]\left[\frac{1}{\rho_y(\zeta_x)} - 1\right] \tag{1.14}$$

is symmetric in x and y. To do this, let T be a finite subset of S which contains neither x nor y. Note by (1.11), that ν assigns positive probability to every finite-dimensional set. Then

$$\begin{aligned}&\left[\frac{1}{\nu\{\eta\colon \eta(x) = \zeta(x) | \eta = \zeta \text{ on } T \cup \{y\}\}} - 1\right]\\ &\qquad\times\left[\frac{1}{\nu\{\eta\colon \eta(y) = \zeta(y) | \eta = \zeta_x \text{ on } T \cup \{x\}\}} - 1\right]\\ &\qquad = \frac{\nu\{\eta\colon \eta(x) \neq \zeta(x),\ \eta(y) \neq \zeta(y),\ \eta = \zeta \text{ on } T\}}{\nu\{\eta\colon \eta = \zeta \text{ on } T \cup \{x, y\}\}},\end{aligned}$$

which is symmetric in x and y, and converges to (1.14) as T increases to $S\backslash\{x, y\}$. Now, by (1.9), (1.11), (1.12), and the fact that (1.14) is symmetric in x and y, it follows that

$$\begin{aligned}\sum_{R \ni x} J_R^x \chi_R(\zeta) + \sum_{R \ni y} J_R^y \chi_R(\zeta_x) &= \sum_{R \ni x, y} (J_R^x - J_R^y)\chi_R(\zeta)\\ &\quad + \sum_{\substack{R \ni x\\ R \not\ni y}} J_R^x \chi_R(\zeta) + \sum_{\substack{R \ni y\\ R \not\ni x}} J_R^y \chi_R(\zeta)\end{aligned}$$

is symmetric in x and y. Since the sum of the second and third terms on the right is symmetric, it follows that the first must be also. But this implies that

$$\sum_{R \ni x, y} (J_R^x - J_R^y)\chi_R(\zeta) = 0,$$

which gives (1.13). □

Note that the hypothesis of Proposition 1.10 is satisfied whenever the conditional probabilities in (1.6) are strictly positive and depend on ζ through only finitely many coordinates. Thus in particular, the result applies to Markov random fields (see, for example, Spitzer (1971a, b)). The hypothesis of Proposition 1.10 can be thought of as saying that ν is nearly a Markov random field. Sullivan (1973) has shown that if $S = Z^d$ and $\nu \in \mathcal{S}$, then this hypothesis can be replaced by the assumption that the conditional probabilities $\rho_x(\zeta)$ are strictly positive and continuous in ζ.

The next step is to carry out the limiting procedure described just before Definition 1.5, and to show that it leads to the same concept of a Gibbs state. For finite $T \subset S$ and $\zeta \in \{0, 1\}^{S \setminus T}$, let $\nu_{T,\zeta}$ be the probability measure on $\{0, 1\}^T$ given by

$$\nu_{T,\zeta}\{\eta\} = K(T, \zeta) \exp\left[\sum_{R \cap T \neq \varnothing} J_R \chi_R(\eta^\zeta)\right], \tag{1.15}$$

where $K(T, \zeta)$ is again a normalizing constant and $\eta^\zeta \in X$ is defined by

$$\eta^\zeta(x) = \begin{cases} \eta(x) & \text{if } x \in T, \\ \zeta(x) & \text{if } x \notin T. \end{cases}$$

This expression is clearly analogous to (1.4). The summation converges by (1.2), since T is finite. Terms corresponding to R's which are disjoint from T are not included in the sums since their effect can be incorporated into the normalizing constant. The measure $\nu_{T,\zeta}$ can be regarded as a probability measure on X by setting

$$\nu_{T,\zeta}\{\eta \in X: \eta(x) = \zeta(x) \text{ for all } x \notin T\} = 1.$$

It is to be viewed as the Gibbs state with boundary condition ζ. Let $\mathcal{G}(T)$ be the closed convex hull of

$$\{\nu_{T,\zeta}: \zeta \in \{0, 1\}^{S \setminus T}\}.$$

Theorem 1.16. (a) $T_1 \subset T_2$ *implies* $\mathcal{G}(T_1) \supset \mathcal{G}(T_2)$.

(b) *If* $\nu \in \mathcal{G}$, *then for finite* $T \subset S$ *and* $\zeta \in \{0, 1\}^{S \setminus T}$,

$$\nu(\cdot \mid \eta(u) = \zeta(u) \text{ for all } u \notin T) = \nu_{T,\zeta}(\cdot).$$

(c)
$$\mathcal{G} = \bigcap_T \mathcal{G}(T).$$

(d) $\mathcal{G}$ *is nonempty, convex, and compact.*

Proof. To prove (a), suppose that $T_1 \subset T_2$, and take $\zeta \in \{0,1\}^{S\backslash T_2}$. For $\gamma \in \{0,1\}^{S\backslash T_1}$ such that $\gamma = \zeta$ on $S\backslash T_2$,

$$\nu_{T_2,\zeta}(\cdot \mid \eta = \gamma \text{ on } T_2\backslash T_1) = \nu_{T_1,\zeta}(\cdot) \tag{1.17}$$

as measures on $\{0,1\}^{T_1}$, as can easily be seen from (1.15). Therefore

$$\nu_{T_2,\zeta} = \sum_{\substack{\gamma:\gamma=\zeta \\ \text{on } S\backslash T_2}} \nu_{T_2,\zeta}(\eta\colon \eta = \gamma \text{ on } T_2\backslash T_1)\nu_{T_1,\gamma}, \tag{1.18}$$

which exhibits $\nu_{T_2,\zeta}$ as a convex combination of elements of $\mathcal{G}(T_1)$. Therefore $\mathcal{G}(T_2) \subset \mathcal{G}(T_1)$. For part (b), use (1.17) with $T_2 = T$ and $T_1 = \{x\}$ where $x \in T$ to write

$$\begin{aligned}
\nu_{T,\zeta}\{\eta\colon \eta(x) = \zeta(x) \mid \eta(u) = \zeta(u) \text{ for all } u \in T\backslash\{x\}\} \\
&= \nu_{\{x\},\zeta}\{\zeta(x)\} \\
&= \frac{\exp\left[\sum_{R\ni x} J_R\chi_R(\zeta)\right]}{\exp\left[\sum_{R\ni x} J_R\chi_R(\zeta)\right] + \exp\left[\sum_{R\ni x} J_R\chi_R(\zeta_x)\right]} \\
&= \frac{1}{1+\exp\left[-2\sum_{R\ni x} J_R\chi_R(\zeta)\right]}
\end{aligned} \tag{1.19}$$

for $\zeta \in X$ by (1.9). Comparing this with Definition 1.5, one sees that in order to prove part (b), it suffices to show that $\nu_{T,\zeta}$ is the only probability measure on $\{0,1\}^T$ whose one point conditional probabilities are given by the right side of (1.19). But this follows from Proposition 1.8. For part (c), note that $\mathcal{G} \subset \mathcal{G}(T)$ follows from (b), and $\bigcap_T \mathcal{G}(T) \subset \mathcal{G}$ follows from (1.19) and Definition 1.5. Part (d) is an immediate consequence of parts (a) and (c) and the compactness of X. □

The elements of $\mathcal{G}$ are interpreted as the possible phases of the physical system described by the potential $\{J_R\}$. By Theorem 1.16, there is always at least one such phase. One of the most important problems in classical statistical mechanics is to determine when there is more than one. We are therefore led to the following definition.

Definition 1.20. The potential $\{J_R\}$ is said to exhibit a *phase transition* if $\mathcal{G}$ contains more than one element.

In the physics literature, this concept is called a first-order phase transtion. Higher-order phase transitions refer to lack of smoothness of the Gibbs state as a function of the potential. These will not play a role in this book.

A number of partial answers to the question of when phase transition occurs will be given in Section 3. For now it suffices to say that in some sense, phase transition seldom occurs in one dimension, but frequently occurs in higher dimensions. The remainder of this section is devoted to the proofs of several inequalities and monotonicity statements which will help clarify our later discussion of phase transition. The first is a pair of correlation inequalities which we will use to show that if certain potentials exhibit phase transition, then so do many others. These inequalities have many other important applications. Some of them are described briefly in Section VIB of Griffiths (1972).

Theorem 1.21. *Suppose that S is finite, and that $\{J_R, R \subset S\}$ is a potential which is ferromagnetic in the sense that $J_R \geq 0$ for all R. Let ν be the corresponding Gibbs state which is given by* (1.4). *Then*

$$\int \chi_A \, d\nu \geq 0 \quad \text{for all } A \subset S, \quad \text{and} \tag{1.22}$$

$$\frac{\partial}{\partial J_B} \int \chi_A \, d\nu = \int \chi_A \chi_B \, d\nu - \int \chi_A \, d\nu \int \chi_B \, d\nu \geq 0 \tag{1.23}$$

for all $A, B \subset S$.

Proof. Since ν is the product measure with density $\frac{1}{2}$ when $J_R = 0$ for all R, (1.22) for general ferromagnetic potentials is an immediate consequence of (1.23). However, the proof of (1.23) uses (1.22), so we must prove (1.22) first. In order to do so, write

$$\begin{aligned}
\int \chi_A \, d\nu &= K \sum_\eta \chi_A(\eta) \exp\left\{\sum_R J_R \chi_R(\eta)\right\} \\
&= K \sum_\eta \chi_A(\eta) \sum_{n=0}^\infty \frac{1}{n!} \left\{\sum_R J_R \chi_R(\eta)\right\}^n \\
&= K \sum_{n=0}^\infty \frac{1}{n!} \sum_{R_1, \ldots, R_n} \left[\prod_{k=1}^n J_{R_k}\right] \sum_\eta \chi_A(\eta) \prod_{k=1}^n \chi_{R_k}(\eta).
\end{aligned}$$

To see that this is a sum of nonnegative terms, it suffices to note that

$$\chi_A(\eta) \prod_{k=1}^n \chi_{R_k}(\eta) = \chi_B(\eta)$$

where

$$B = \{x \in S \colon x \text{ is in an odd number of the sets } A, R_1, \ldots, R_n\}$$

and that for any $B \subset S$,

$$\sum_{\eta} \chi_B(\eta) = \begin{cases} 2^{|S|} & \text{if } B = \emptyset, \\ 0 & \text{if } B \neq \emptyset. \end{cases}$$

Turning to the proof of (1.23), use the explicit expression for K to write

$$\int \chi_A \, d\nu = \frac{\sum_{\eta} \chi_A(\eta) \exp\left\{\sum_R J_R \chi_R(\eta)\right\}}{\sum_{\eta} \exp\left\{\sum_R J_R \chi_R(\eta)\right\}}.$$

Therefore the equality in (1.23) is the result of a simple differentiation. To check the inequality, write

$$\begin{aligned} &\int \chi_A \chi_B \, d\nu - \int \chi_A \, d\nu \int \chi_B \, d\nu \\ &= K^2 \sum_{\eta,\zeta} [\chi_A(\eta)\chi_B(\eta) - \chi_A(\eta)\chi_B(\zeta)] \exp\left\{\sum_R J_R[\chi_R(\eta) + \chi_R(\zeta)]\right\}. \end{aligned} \tag{1.24}$$

Let C be the symmetric difference $A \Delta B$ and let $\gamma \in X$ be defined by

$$\gamma(x) = \begin{cases} 1 & \text{if } \eta(x) = \zeta(x), \\ 0 & \text{if } \eta(x) \neq \zeta(x). \end{cases}$$

Then

$$\chi_A(\eta)\chi_B(\eta) = \chi_C(\eta),$$

$$\chi_A(\eta)\chi_B(\zeta) = \chi_B(\gamma)\chi_C(\eta), \quad \text{and}$$

$$\chi_R(\eta) + \chi_R(\zeta) = \chi_R(\eta)[1 + \chi_R(\gamma)].$$

Making these substitutions in (1.24) yields

$$\begin{aligned} &\int \chi_A \chi_B \, d\nu - \int \chi_A \, d\nu \int \chi_B \, d\nu \\ &= K^2 \sum_{\eta,\gamma} \chi_C(\eta)[1 - \chi_B(\gamma)] \exp\left\{\sum_R J_R \chi_R(\eta)[1 + \chi_R(\gamma)]\right\}. \end{aligned} \tag{1.25}$$

For fixed γ, we can define a new potential by

$$J_R^\gamma = J_R[1 + \chi_R(\gamma)],$$

which is again ferromagnetic. By (1.22) applied to this potential, we see that

$$\sum_{\eta} \chi_C(\eta) \exp\left\{\sum_R J_R^\gamma \chi_R(\eta)\right\} \geq 0.$$

Thus (1.23) follows from (1.25) by summing first on η and then on γ. □

For the rest of this section, we will specialize to the important case in which $S = Z^d$ for some $d \geq 1$ and

$$(1.26) \qquad \begin{cases} J_{\{x\}} = \beta H & \text{for each } x \in S, \\ J_{\{x,y\}} = \beta J(y-x) & \text{for distinct } x, y \in S, \quad \text{and} \\ J_R = 0 & \text{for } |R| \geq 3, \end{cases}$$

where H is a real number, β is a nonnegative real number, and $J(x) \geq 0$ for all $x \neq 0$. The two parameters β and H represent the reciprocal of the temperature and the strength of the external magnetic field respectively. The famous Ising model, which gives this chapter its name, is the special case in which $J(x) = 1$ for the $2d$ neighbors of the origin, and $J(x) = 0$ otherwise. Under the above assumptions, Theorem 2.9 of Chapter II will provide additional monotonicity statements which are very useful. The theory in this case is quite analogous to the theory of attractive spin systems which was developed in Theorem 2.7 and Corollary 2.11 of Chapter III. The connection between them will be made more explicit in the next section. In particular, stochastic Ising models could be used in place of Theorem 2.9 of Chapter II in order to prove the next theorem and its corollaries.

Theorem 1.27. *Suppose that the potential is given as in* (1.26) *with* $\beta \geq 0$ *and* $J(x) \geq 0$. *Then* $\zeta_1 \leq \zeta_2$ *implies that* $\nu_{T,\zeta_1} \leq \nu_{T,\zeta_2}$ *for any finite* $T \subset S$, *where the last inequality is interpreted in the sense of Definition* 2.1 *of Chapter II.*

Proof. By Theorem 2.9 of Chapter II, it suffices to check that for $\eta_1, \eta_2 \in \{0, 1\}^T$,

$$\sum_{R \cap T \neq \varnothing} J_R[\chi_R(\eta_1^{\zeta_1} \wedge \eta_2^{\zeta_1}) + \chi_R(\eta_1^{\zeta_2} \vee \eta_2^{\zeta_2})] \geq \sum_{R \cap T \neq \varnothing} J_R[\chi_R(\eta_1^{\zeta_1}) + \chi_R(\eta_2^{\zeta_2})]$$

whenever $\zeta_1 \leq \zeta_2$. Using the special form for J_R which we have assumed, this can be rewritten as the statement that the expression

$$\begin{aligned} 2\beta H \sum_{x \in T} &[(\eta_1 \wedge \eta_2)(x) + (\eta_1 \vee \eta_2)(x) - \eta_1(x) - \eta_2(x)] \\ +2\beta \sum_{\substack{x,y \in T \\ x \neq y}} &J(y-x)[(\eta_1 \wedge \eta_2)(x)(\eta_1 \wedge \eta_2)(y) \\ &\qquad + (\eta_1 \vee \eta_2)(x)(\eta_1 \vee \eta_2)(y) \\ &\qquad - \eta_1(x)\eta_1(y) - \eta_2(x)\eta_2(y)] \end{aligned}$$

$$+4\beta \sum_{\substack{x\in T\\ y\notin T}} J(y-x)[(\eta_1\wedge\eta_2)(x)\zeta_1(y)+(\eta_1\vee\eta_2)(x)\zeta_2(y) \\ -\eta_1(x)\zeta_1(y)-\eta_2(x)\zeta_2(y)]$$

is nonnegative. The terms in the first sum are all zero, so the sign of H is irrelevant in verifying the nonnegativity of this expression. The term in brackets in the second sum is zero unless $\eta_1(x)=\eta_2(y)=0$ and $\eta_2(x)=\eta_1(y)=1$ or $\eta_1(x)=\eta_2(y)=1$ and $\eta_2(x)=\eta_1(y)=0$, in which case it is equal to 1. The term in brackets in the third sum is zero unless $\eta_1(x)=1$ and $\eta_2(x)=0$, in which case it is equal to $\zeta_2(y)-\zeta_1(y)$. So, since $\beta\geq 0$ and $J(y-x)\geq 0$, the required sums are nonnegative whenever $\zeta_1\leq\zeta_2$. □

For the next result, let $\underline{\nu}_T$ and $\bar{\nu}_T$ be defined by (1.15) with $\zeta\equiv 0$ and $\zeta\equiv 1$ respectively. By Theorem 1.27,

$$\underline{\nu}_T\leq \nu_{T,\zeta}\leq\bar{\nu}_T \tag{1.28}$$

for all $\zeta\in\{0,1\}^{S\setminus T}$.

Corollary 1.29. *Under the assumptions of Theorem 1.27, $T_1\subset T_2$ implies that*

$$\underline{\nu}_{T_1}\leq\underline{\nu}_{T_2}\quad\textit{and}\quad \bar{\nu}_{T_1}\geq\bar{\nu}_{T_2}.$$

Proof. The two statements are similar, so we will prove only the latter. By Theorem 1.27 $\nu_{T_1,\gamma}\leq\bar{\nu}_{T_1}$ for all γ. Therefore $\bar{\nu}_{T_2}\leq\bar{\nu}_{T_1}$ follows from (1.18). □

Corollary 1.30. *Under the assumptions of Theorem 1.27,*

(a) $\underline{\nu}=\lim_{T\uparrow S}\underline{\nu}_T\in\mathcal{S}\cap\mathcal{G}$ *exists,*
(b) $\bar{\nu}=\lim_{T\uparrow S}\bar{\nu}_T\in\mathcal{S}\cap\mathcal{G}$ *exists,*
(c) $\underline{\nu}\leq\nu\leq\bar{\nu}$ *for all* $\nu\in\mathcal{G}$,
(d) *phase transition occurs if and only if* $\underline{\nu}\neq\bar{\nu}$, *and*
(e) *phase transition occurs if and only if* $\underline{\nu}\{\eta\colon\eta(x)=1\}\neq\bar{\nu}\{\eta\colon\eta(x)=1\}$.

Proof. The fact that the limits exist in (a) and (b) and are translation invariant follows from Corollary 1.29. That these limits are in $\mathcal{G}$ follows from parts (a) and (c) of Theorem 1.16. To prove part (c), use (1.28) to show that

$$\underline{\nu}_T\leq\nu\leq\bar{\nu}_T$$

for all $\nu\in\mathcal{G}(T)$, and then pass to the limit as $T\uparrow S$ using part (c) of Theorem 1.16. Part (d) is an immediate consequence of part (c) and Definition 1.20. Part (e) follows from part (d) and Corollary 2.8 of Chapter II. □

Theorem 1.31. *In addition to the assumptions of Theorem 1.27, suppose that $H=0$. Then*

(a) $\underline{\nu}\{\eta\colon\eta(x)=1\}+\bar{\nu}\{\eta\colon\eta(x)=1\}=1$,

(b) $\bar{\nu}\{\eta: \eta(x)=1\}$ *is an increasing function of* β,
(c) *there is a critical value* $0 \le \beta_c \le \infty$ *such that there is no phase transition if* $\beta < \beta_c$ *and there is phase transition if* $\beta > \beta_c$, *and*
(d) β_c *is a decreasing function of the numbers* $J(x)$.

Proof. Since $H=0$, $\bar{\nu}_T$ is obtained from $\underline{\nu}_T$ by interchanging the roles of 0 and 1. Therefore

$$\underline{\nu}_T\{\eta: \eta(x)=1\}+\bar{\nu}_T\{\eta: \eta(x)=1\}=1.$$

Part (a) follows from this and Corollary 1.30. For part (b), note that if $\zeta \equiv 1$,

$$\chi_R(\eta^\zeta)=\chi_{R\cap T}(\eta),$$

so that (1.15) takes the form (1.4). Therefore we can apply (1.23) with $A=\{x\}$ to conclude that

$$\bar{\nu}_T\{\eta: \eta(x)=1\}-\bar{\nu}_T\{\eta: \eta(x)=0\}=2\bar{\nu}_T\{\eta: \eta(x)=1\}-1$$

is an increasing function of β for each $T \ni x$. Therefore part (b) follows from part (b) of Corollary 1.30. Part (c) is an immediate consequence of parts (a) and (b), together with the characterization of phase transition in part (e) of Corollary 1.30. Part (d) follows from (1.23) in the same way that part (b) did. □

It is of course an important problem to decide when $0<\beta_c<\infty$. It is relatively easy, as we will see in Section 3, to show under very weak assumptions that $\beta_c>0$. This says that phase transition does not occur at high temperatures (i.e., for small β). It is more difficult to show that phase transition does occur often at low temperatures, so that $\beta_c<\infty$. The importance of part (d) of Theorem 1.31 is that it implies that once we have shown $\beta_c<\infty$ for one choice of $\{J(x), x\in S\}$, we can conclude that $\beta_c<\infty$ for many other choices as well.

2. Reversibility of Stochastic Ising Models

We begin this section with the definition of a stochastic Ising model. While the definition may appear a bit strange at first glance, it will be motivated shortly.

Definition 2.1. Given a potential $\{J_R\}$, a spin system with strictly positive rates $c(x, \eta)$ is said to be a *stochastic Ising model* relative to that potential provided that for each $x \in S$, the function

$$c(x, \eta) \exp\left[\sum_{R \ni x} J_R \chi_R(\eta)\right] \tag{2.2}$$

does not depend on the coordinate $\eta(x)$.

The statement that (2.2) does not depend on $\eta(x)$ is often referred to as the condition of detailed balance. Special cases of stochastic Ising models have appeared in various guises earlier in this book. In Examples 3.1(a) and 4.3(a), of Chapter I, the potential is given by $J_R = \beta$ if R consists of a nearest-neighbor pair, and $J_R = 0$ otherwise. In Theorem 1.12 of Chapter III, the potential is identically zero. In Example 2.12 of Chapter III, $J_R = \frac{1}{4}\log(1-\delta)/\delta$ if R is a nearest-neighbor pair, and $J_R = 0$ otherwise.

In order to further motivate Definition 2.1, recall that in this chapter we are interested in finding spin systems which are reversible with respect to the Gibbs states associated with the potential $\{J_R\}$. Suppose that S is finite so that the Gibbs state ν is given by (1.4). By Proposition 5.3 of Chapter II, the spin system with rates $c(x, \eta)$ is reversible with respect to ν if and only if

$$c(x, \eta)\nu\{\eta\} = c(x, \eta_x)\nu\{\eta_x\} \tag{2.3}$$

for all $x \in S$ and $\eta \in X$. But that is exactly the statement that (2.2) is independent of $\eta(x)$.

There are of course infinitely many stochastic Ising models corresponding to a given potential. Two commonly used ones are

$$c(x, \eta) = \exp\left[-\sum_{R \ni x} J_R \chi_R(\eta)\right] \quad \text{and} \tag{2.4}$$

$$c(x, \eta) = \left\{1 + \exp\left[2 \sum_{R \ni x} J_R \chi_R(\eta)\right]\right\}^{-1}. \tag{2.5}$$

In the nearest-neighbor translation invariant case, these two versions appeared earlier in Example 4.3(a) of Chapter I. The second of these is particularly useful since then $c(x, \eta)$ is automatically bounded in x and η, and in fact satisfies

$$c(x, \eta) + c(x, \eta_x) = 1.$$

A sufficient condition for (2.4) to be uniformly bounded and for both versions to satisfy (0.3) of Chapter III is that

$$\sup_{x \in S} \sum_{R \ni x} |R|\,|J_R| < \infty, \tag{2.6}$$

where $|R|$ denotes the cardinality of R. The computation required to check this is elementary, and will in any case be carried out for version (2.5) in the proof of Theorem 3.1. While we will not assume (2.6) explicitly in this section, recall that we are assuming throughout this chapter that $c(x, \eta)$ is uniformly bounded and satisfies (0.3) of Chapter III.

There appear to be no physical grounds for isolating one particular version of the stochastic Ising model for study. Therefore it is important that the results obtained apply to as general a class of stochastic Ising models as possible. Sometimes, though, it will be convenient to assume that the processes we are working with have certain properties such as that of attractiveness. Note in this connection that if the potential is given by (1.26) with $\beta \geq 0$ and $J(x) \geq 0$, then both versions (2.4) and (2.5) of the stochastic Ising model are attractive.

The purpose of this section is to study the relationships among reversible spin systems with strictly positive rates, stochastic Ising models, and Gibbs states. Definition 5.1 and Propositions 5.2 and 5.3 of Chapter II should be recalled at this point.

Proposition 2.7. *Suppose that ν is a probability measure on X and that $c(x, \eta)$ are the rates for a spin system. Then ν is reversible for the spin system if and only if*

$$\int c(x, \eta)[f(\eta_x) - f(\eta)]\, d\nu = 0 \tag{2.8}$$

for all $x \in S$ and $f \in C(X)$. If the rates are strictly positive, then this is equivalent to the statement that ν has the following conditional probabilities:

$$\nu\{\eta\colon \eta(x) = \zeta(x) | \eta(u) = \zeta(u) \text{ for all } u \neq x\} = \frac{c(x, \zeta_x)}{c(x, \zeta) + c(x, \zeta_x)}. \tag{2.9}$$

Proof. If (2.8) holds for all $f \in C(X)$, then it can be applied to the function $f(\eta_x)g(\eta)$ for $f, g \in D(X)$ to obtain

$$\int c(x, \eta) f(\eta) g(\eta_x)\, d\nu = \int c(x, \eta) f(\eta_x) g(\eta)\, d\nu,$$

or equivalently

$$\int c(x, \eta) f(\eta)[g(\eta_x) - g(\eta)]\, d\nu = \int c(x, \eta) g(\eta)[f(\eta_x) - f(\eta)]\, d\nu.$$

Summing on x and using Proposition 5.3 of Chapter II leads to the conclusion that ν is reversible for the spin system. To prove the converse, assume that ν is reversible. For a finite subset T of S and an $x \in T$, let

$f(\eta)=\prod_{y\in T}\eta(y)$ and $g(\eta)=f(\eta_x)$. Then

$$g(\eta)\Omega f(\eta)=f(\eta_x)\sum_{y\in T}c(y,\eta)[f(\eta_y)-f(\eta)]=c(x,\eta)f(\eta_x),\quad \text{and}$$

$$f(\eta)\Omega g(\eta)=f(\eta)\sum_{y\in T}c(y,\eta)[g(\eta_y)-g(\eta)]=c(x,\eta)f(\eta),$$

so that (2.8) holds for that f by Proposition 5.3 of Chapter II. By linearity, it holds for all $f\in\mathscr{D}$ (the set of functions depending on finitely many coordinates). Since $\mathscr{D}$ is dense in $C(X)$, (2.8) holds for all $f\in C(X)$. Now assume that $c(x,\eta)>0$ for all $x\in S$ and $\eta\in X$. Fix an $x\in S$, and let $c_0(\eta)$ and $c_1(\eta)$ be the unique functions on X which do not depend on $\eta(x)$ such that

$$c(x,\eta)=\begin{cases}c_0(\eta) & \text{if } \eta(x)=0,\\ c_1(\eta) & \text{if } \eta(x)=1.\end{cases}$$

Then (2.9) can be rewritten as the statement that

$$\int \eta(x)f(\eta)\,d\nu=\int\frac{c_0(\eta)}{c_0(\eta)+c_1(\eta)}f(\eta)\,d\nu$$

for all functions $f\in C(X)$ which do not depend on $\eta(x)$. Since $c_0(\eta)+c_1(\eta)$ does not depend on $\eta(x)$ and is strictly positive, this is equivalent to the statement that

$$\int \eta(x)g(\eta)[c_0(\eta)+c_1(\eta)]\,d\nu=\int c_0(\eta)g(\eta)\,d\nu$$

for all $g\in C(X)$ which do not depend on $\eta(x)$. But this can be rewritten as

$$\int g(\eta)\{\eta(x)c_1(\eta)-[1-\eta(x)]c_0(\eta)\}\,d\nu=0,\quad\text{or}$$

$$\int c(x,\eta)g(\eta)[2\eta(x)-1]\,d\nu=0. \tag{2.10}$$

On the other hand, if $f\in C(X)$ is written as

$$f(\eta)=f_0(\eta)[1-\eta(x)]+f_1(\eta)\eta(x),$$

where $f_0(\eta)$ and $f_1(\eta)$ do not depend on $\eta(x)$, then

$$f(\eta_x)-f(\eta)=[f_0(\eta)-f_1(\eta)][2\eta(x)-1],$$

so that (2.8) can be rewritten as

$$\int c(x, \eta)[f_0(\eta)-f_1(\eta)][2\eta(x)-1]\,d\nu=0. \tag{2.11}$$

The proof of the proposition is completed by comparing (2.10) and (2.11). □

The first characterization of reversibility in Proposition 2.7 points out again how much stronger reversibility is than invariance. By Proposition 2.13 of Chapter I, μ is invariant if and only if

$$\sum_{x\in S}\int c(x, \eta)[f(\eta_x)-f(\eta)]\,d\nu=0 \tag{2.12}$$

for all $f\in D(X)$. Proposition 2.7 asserts that reversibility is equivalent to the vanishing of each term in (2.12), rather than the vanishing of the sum.

Theorem 2.13. *Suppose that $c(x, \eta)$ is strictly positive, and that for each x, $c(x, \eta)$ depends on only finitely many coordinates. If the spin system is reversible with respect to some probability measure ν, then it is a stochastic Ising model relative to some potential $\{J_R\}$.*

Proof. By Proposition 2.7, ν has conditional probabilities given by (2.9). By the finite dependence assumption on the rates and Proposition 1.10, ν is a Gibbs state relative to some potential $\{J_R\}$. Using (2.9) again and Definition 1.5, we see that

$$\frac{c(x, \zeta_x)}{c(x, \zeta)+c(x, \zeta_x)}=\frac{1}{1+\exp\left[-2\sum_{R\ni x} J_R\chi_R(\zeta)\right]},$$

or equivalently,

$$\frac{c(x, \zeta)}{c(x, \zeta_x)}=\exp\left[-2\sum_{R\ni x} J_R\chi_R(\zeta)\right].$$

Using (1.9), this can be rewritten as

$$c(x, \zeta)\exp\left[\sum_{R\ni x} J_R\chi_R(\zeta)\right]=c(x, \zeta_x)\exp\left[\sum_{R\ni x} J_R\chi_R(\zeta_x)\right],$$

which says that the spin system is a stochastic Ising model relative to the potential $\{J_R\}$. □

As a result of the previous theorem, we are justified in concentrating our attention on stochastic Ising models. Recall from Section 5 of Chapter II that $\mathscr{R}$ denotes the class of all reversible measures for a process.

Theorem 2.14. *Suppose that $c(x, \eta)$ are the rates for a stochastic Ising model relative to the potential $\{J_R\}$. Then $\mathscr{R} = \mathscr{G}$, where $\mathscr{G}$ denotes the set of all Gibbs states relative to the same potential.*

Proof. By Proposition 2.7 and Definition 1.5, it suffices to show that for a stochastic Ising model,

$$\frac{c(x, \zeta_x)}{c(x, \zeta)+c(x, \zeta_x)} = \frac{1}{1+\exp\left[-2 \sum\limits_{R \ni x} J_R \chi_R(\zeta)\right]}.$$

But this computation is the same (but in reverse order) as the one carried out in the proof of Theorem 2.13. □

We are now in a position to present two simple results which make the connection between the concepts of ergodicity of a stochastic Ising model as treated in Chapters I and III on the one hand, and phase transition as discussed in the first section of this chapter on the other.

Theorem 2.15. *Consider a stochastic Ising model relative to the potential $\{J_R\}$, and let $\mathscr{G}$ be the corresponding Gibbs states. Then $\mathscr{G} \subset \mathscr{I}$. In particular, if the stochastic Ising model is ergodic, then there is no phase transition for that potential.*

Proof. The containment is an immediate consequence of Proposition 5.2 of Chapter II, and Theorem 2.14. If the process is ergodic, then $\mathscr{I}$ is a singleton by Definition 1.9 of Chapter I. Therefore $\mathscr{G}$ is a singleton as well, so $\{J_R\}$ does not exhibit phase transition by Definition 1.20. □

Theorem 2.16. *Consider an attractive stochastic Ising model relative to the potential $\{J_R\}$, and let $\underline{\nu}$, $\bar{\nu}$ be defined as in Theorem 2.3 of Chapter III. Then $\underline{\nu}, \bar{\nu} \in \mathscr{G}$. In particular, the stochastic Ising model is ergodic if and only if there is no phase transition for that potential.*

Proof. Let S_n be finite sets which increase to S, and let $c_i^n(x, \eta)$ be the rates for the approximating spin systems which are defined just before the statement of Theorem 2.7 of Chapter III. By checking (2.3), or using Theorem 2.14 applied to these approximating spin systems, we see that $\nu_{S_n,\zeta}$ as defined in (1.15) is invariant for $c_0^n(x, \eta)$ if $\zeta \equiv 0$ and for $c_1^n(x, \eta)$ if $\zeta \equiv 1$. So, by the convergence theorem for irreducible finite-state Markov chains, $\underline{\nu}^n$ and $\bar{\nu}^n$ as defined in Theorem 2.7 of Chapter III are equal to $\nu_{S_n,\zeta}$ with

$\zeta \equiv 0$ and $\zeta \equiv 1$ respectively. Therefore $\underline{\nu}^n$, $\bar{\nu}^n \in \mathscr{G}(S_n)$, so that $\underline{\nu}$, $\bar{\nu} \in \mathscr{G}$ by Theorem 2.7 of Chapter III and Theorem 1.16 of this chapter. For the final statement, note that one direction follows from Theorem 2.15. The other follows from Corollary 2.4 of Chapter III, together with the fact that $\underline{\nu}$, $\bar{\nu} \in \mathscr{G}$. $\square$

Corollary 2.17. *Two attractive stochastic Ising models with respect to the same potential are either both ergodic or neither ergodic.*

For an interesting application of this corollary, recall Example 4.3(a) of Chapter I. There we took rates of the form (2.4) and (2.5) on Z^d with the potential given by $J_{\{x,y\}} = \beta \geq 0$ for nearest neighbors x and y, and $J_R = 0$ otherwise. Both versions are attractive, and Theorem 4.1 of that chapter gave ergodicity in one dimension for the second version for all β, but for the first version only for sufficiently small β. By Corollary 2.17, the first version is in fact ergodic for all β as well. Note however that this argument does not yield exponentially fast convergence to the invariant measure for the first version with large β. This is in fact the case, as was proved by Holley (1985). For more on this topic of exponentially fast convergence of stochastic Ising models, see Theorem 4.16 and Corollary 4.18.

Corollary 2.18. *Consider the stochastic Ising models with rates given by either* (2.4) *or* (2.5) *with the potential given by* (1.26) *with* $\beta \geq 0$, $H = 0$ *and* $J(x) \geq 0$ *for all* x. *Let* β_c *be as in Theorem* 1.31. *Then the process is ergodic if* $\beta < \beta_c$ *and not ergodic if* $\beta > \beta_c$.

3. Phase Transition

This section is devoted to a number of results which show when phase transition occurs, and when it does not. The most important ones will be proved, while some of the more refined ones will only be stated. The ones which are not proved will not be formally used in the sequel. Applications to the stochastic Ising model will be given.

The first theorem is both one of the most elementary and one of the most generally applicable of these results. It implies, for example, that there is no phase transition in the Ising model at high temperatures in any dimension. In particular, it follows that the β_c defined in Theorem 1.31 is strictly positive. The proof of this result is based on the connection between the Gibbs states and stochastic Ising models.

Theorem 3.1. *Let* $\{J_R\}$ *be a potential. Then there is no phase transition provided either of the following conditions is satisfied*:

(3.2) $$\sup_x \sum_{y \neq x} \sup_\eta |\rho_x(\eta) - \rho_x(\eta_y)| < 1,$$

where

$$\rho_x(\eta)=\left\{1+\exp\left[-2\sum_{R\ni x}J_R\chi_R(\eta)\right]\right\}^{-1}.$$

$$\text{(3.3)}\qquad \sup_x\sum_{R\ni x}[|R|-1]|J_R|<\frac{\log 2}{4}.$$

In particular, if the potential satisfies (2.6), *then there is no phase transition for the potential* $\{\beta J_R\}$ *for sufficiently small* β.

Proof. Consider the stochastic Ising model with rates given by (2.5). Then

$$c(x,\eta)=\rho_x(\eta_x),\quad\text{and}$$

$$c(x,\eta)+c(x,\eta_x)=1$$

for all $x\in S$ and $\eta\in X$. Therefore (3.2) is exactly condition (0.6) of Chapter III. By Theorem 4.1 of Chapter I, (3.2) then implies that this stochastic Ising model is ergodic, so that by Theorem 2.15, it implies that there is no phase transition for this potential. To complete the proof, it suffices to show that (3.3) implies (3.2). In order to do so, use (1.9) to write

$$\rho_x(\eta)-\rho_x(\eta_y)=\frac{\exp\left\{-2\sum_{R\ni x}J_R\chi_R(\eta)\right\}\left[\exp\left\{+4\sum_{R\ni x,y}J_R\chi_R(\eta)\right\}-1\right]}{\left[1+\exp\left\{-2\sum_{R\ni x}J_R\chi_R(\eta)\right\}\right]\left[1+\exp\left\{-2\sum_{R\ni x}J_R\chi_R(\eta_y)\right\}\right]}$$

for $x\neq y$, so that

$$\sup_\eta|\rho_x(\eta)-\rho_x(\eta_y)|\le\exp\left[4\sum_{R\ni x,y}|J_R|\right]-1$$

for $x\neq y$. Therefore

$$\begin{aligned}\sum_{y\neq x}\sup_\eta|\rho_x(\eta)-\rho_x(\eta_y)|&\le\sum_{y\neq x}\left\{\exp[4\sum_{R\ni x,y}|J_R|]-1\right\}\\&\le\exp\left\{4\sum_{y\neq x}\sum_{R\ni x,y}|J_R|\right\}-1\\&=\exp\left\{4\sum_{R\ni x}|J_R|[|R|-1]\right\}-1,\end{aligned}$$

so that (3.3) implies (3.2) as required. □

Other techniques give other similar sufficient conditions for the absence of phase transition. For example, Theorem 5.1 of Chapter III can be used in place of Theorem 4.1 of Chapter I in order to show that the condition

$$\sup_x \sum_{R \ni x} |J_R| < \frac{\pi}{4} \tag{3.4}$$

is sufficient to conclude that there is no phase transition. The computations needed to verify the assumption of Theorem 5.1 under that condition are given in the appendix of Holley and Stroock (1976b).

The technique illustrated in Theorem 3.1 of using the stochastic Ising model to deduce properties of the Gibbs states can be carried much further. For example, Theorem 4.20 of Chapter I can be used to show immediately that for finite range potentials, condition (3.2) implies that the unique Gibbs state has exponentially decaying correlations.

The next result and its corollaries will prepare us to prove that there is no phase transition in one dimension unless the potential has very long range. They will also be useful in Section 4.

Proposition 3.5. *Let $\{J_R\}$ be a potential, and $\mathcal{G}$ be the corresponding Gibbs states.*

(a) *Suppose $\mu_1, \mu_2 \in \mathcal{G}$ and μ_1 is absolutely continuous with respect to μ_2. Let $h = d\mu_1/d\mu_2$ be the Radon–Nikodym derivative. Then*

$$h(\eta_x) = h(\eta) \text{ a.e. } (\mu_2) \quad \text{for each } x \in S. \tag{3.6}$$

(b) *Suppose $\mu_2 \in \mathcal{G}$, $h \geq 0$, $\int h\, d\mu_2 = 1$, and h satisfies* (3.6). *Then $\mu_1 = h\mu_2$ is in $\mathcal{G}$.*

Proof. To prove part (a), it suffices to show that

$$\int f(\eta)h(\eta_x)\, d\mu_2 = \int f(\eta)h(\eta)\, d\mu_2$$

for all $x \in S$ and $f \in C(X)$. The right side is just $\int f(\eta)\, d\mu_1$. To compute the left side, use Definition 1.5 as follows:

$$\begin{aligned}\int f(\eta)h(\eta_x)\, d\mu_2 &= \int f(\eta_x)h(\eta)\frac{\rho_x(\eta_x)}{\rho_x(\eta)}\, d\mu_2\\ &= \int f(\eta_x)\frac{\rho_x(\eta_x)}{\rho_x(\eta)}\, d\mu_1\\ &= \int f(\eta)\, d\mu_1\end{aligned}$$

as required. The change of variables $\eta \to \eta_x$ is used in the first and last steps. The proof of part (b) is similar. □

Corollary 3.7. *Suppose that $\mu_1, \mu_2 \in \mathcal{G}$ implies that μ_1 is absolutely continuous with respect to μ_2. Then there is no phase transition.*

Proof. Suppose $\mu_1, \mu_2 \in \mathcal{G}$. We need to show that $\mu_1 = \mu_2$. By assumption, μ_1 and μ_2 are absolutely continuous with respect to each other. Therefore $h = d\mu_1/d\mu_2 > 0$ a.e. (μ_2). In order to show that $h = 1$, assume on the contrary that $0 < \mu_2(A) < 1$, where $A = \{\eta: h(\eta) > c\}$ for some $c > 0$. Let

$$g(\eta) = \begin{cases} \dfrac{1}{\mu_2(A)} & \text{if } \eta \in A, \\ 0 & \text{if } \eta \notin A. \end{cases}$$

By Proposition 3.5(a), h and hence g satisfies (3.6). Therefore $\mu_3 = g\mu_2$ is in $\mathcal{G}$ by Proposition 3.5(b). But μ_2 is not absolutely continuous with respect to μ_3, which contradicts the hypothesis of the corollary. □

Corollary 3.8. *A $\mu \in \mathcal{G}$ is extremal in $\mathcal{G}$ if and only if the only measurable functions on X which satisfy*

$$h(\eta_x) = h(\eta) \text{ a.e. } (\mu) \quad \text{for each } x \in S$$

are those which are constant a.e. (μ).

Proof. Suppose that $\mu = \alpha\mu_1 + \beta\mu_2$, where $\alpha + \beta = 1$, $0 < \alpha < 1$, and $\mu_1, \mu_2 \in \mathcal{G}$. Then μ_1 is absolutely continuous with respect to μ. So by part (a) of Proposition 3.5,

$$\frac{d\mu_1}{d\mu}(\eta_x) = \frac{d\mu_1}{d\mu}(\eta) \text{ a.e. } (\mu) \quad \text{for all } x \in S.$$

If $\mu_1 \neq \mu$, then of course $d\mu_1/d\mu$ is not constant. This gives one direction. For the converse, use part (b) of Proposition 3.5. □

Theorem 3.9. *Suppose that $S = Z^1$, $J_{R+x} = J_R$ for all $x \in S$ and all R, and*

$$\sum_{R \ni 0} \frac{(\text{diameter } R)|J_R|}{|R|} < \infty. \tag{3.10}$$

Then there is no phase transition for the potential $\{J_R\}$.

Proof. We will verify the hypothesis of Corollary 3.7. Let T be a finite connected subset of S and let $\zeta_1, \zeta_2 \in \{0, 1\}^{S \setminus T}$. Then

$$\frac{\exp\left[\sum_{R \cap T \neq \varnothing} J_R \chi_R(\eta^{\zeta_1})\right]}{\exp\left[\sum_{R \cap T \neq \varnothing} J_R \chi_R(\eta^{\zeta_2})\right]} = \exp\left[\sum_{\substack{R \cap T \neq \varnothing \\ R \cap T^c \neq \varnothing}} J_R[\chi_R(\eta^{\zeta_1}) - \chi_R(\eta^{\zeta_2})]\right]$$

$$\leq \exp\left[2 \sum_{\substack{R \cap T \neq \varnothing \\ R \cap T^c \neq \varnothing}} |J_R|\right]$$

$$\leq \exp\left[4 \sum_{\substack{R \cap (-\infty, 0] \neq \varnothing \\ R \cap (0, \infty) \neq \varnothing}} |J_R|\right] \leq c$$

where

$$c = \exp\left[4 \sum_{R \ni 0} \frac{(\text{diameter } R)|J_R|}{|R|}\right] < \infty.$$

The factor (diameter R) comes from the fact that for fixed R, there are that many translates of R which contain both positive and nonpositive points. The factor $|R|$ comes from the fact that for fixed R, there are that many translates of R which contain 0. By (1.15), it then follows that

$$c^{-2} \leq \frac{\nu_{T,\zeta_1}\{\eta\}}{\nu_{T,\zeta_2}\{\eta\}} \leq c^2$$

for all $\eta \in \{0, 1\}^T$. Therefore if $\mu_1, \mu_2 \in \mathscr{G}(T)$,

$$c^{-2} \leq \frac{\mu_1\{\zeta : \zeta = \eta \text{ on } T\}}{\mu_2\{\zeta : \zeta = \eta \text{ on } T\}} \leq c^2$$

for all $\eta \in \{0, 1\}^T$. Since this estimate is uniform in T, Theorem 1.16 can then be used to show that any two elements of $\mathscr{G}$ are absolutely continuous with respect to each other. In fact, c^{-2} and c^2 give bounds on the corresponding Radon–Nikodym derivatives. □

Of course, hypothesis (3.10) is satisfied for any finite range potential. It is of interest to see more precisely what happens when the potential has infinite range. Suppose the potential is given in the form (1.26) with $d = 1$, $H = 0$ and $J(n) \geq 0$. Then Theorem 3.9 shows that there is no phase transition when $\sum_{n=1}^{\infty} nJ(n) < \infty$. Rogers and Thompson (1981) have improved this result, showing that there is no phase transition in this case provided that

$$(3.11)\qquad \lim_{N\to\infty} \frac{1}{\sqrt{\log N}} \sum_{n=1}^{N} nJ(n) = 0.$$

On the other side of the picture, Fröhlich and Spencer (1982) showed that phase transition does occur for sufficiently large positive β if $J(n) = n^{-2}$. It then follows from Theorem 1.31 that the conclusion is valid whenever

$$(3.12)\qquad \inf_n n^2 J(n) > 0.$$

We can now apply Theorem 3.9 to the stochastic Ising models.

Theorem 3.13. *Every attractive translation invariant finite range stochastic Ising model in one dimension is ergodic.*

Proof. Since the process has finite range (recall Definition 4.17 of Chapter I), the corresponding potential satisfies the assumptions of Theorem 3.9. By that theorem, there is no phase transition in this case. Therefore the process is ergodic by Theorem 2.16. □

Theorem 3.13 is the main reason for believing that the positive rates conjecture is true. (See Problem 6 in Chapter III.) To make the connection clear, recall from Theorem 2.13 that in the finite range case, a stochastic Ising model is simply a spin system with positive rates which has a reversible invariant measure. While reversibility is an extremely useful tool, it is not at all clear why the loss of reversibility should make it possible for a spin system in one dimension to be nonergodic. There are those who doubt the positive rates conjecture. Reasons for this skepticism range from the fact that the proof of Theorem 3.14 of Chapter III depends so heavily on the nearest-neighbor assumption, to the fact that counterexamples to similar conjectures have been proposed in recent years. (See Gacs (1985) for an example of a discrete time system with a large but finite number of states per site which is said to be nonergodic. It appears to be very unlikely that his construction can be modified to yield a monotone example.) Of course the finite range assumption is needed in the positive rates conjecture, since without it a stochastic Ising model with a potential satisfying (3.12) would be a counterexample.

We turn now to the two-dimensional Ising model, in order to find more interesting examples of phase transition. This model is the one with potential given as in (1.26) with $J(x) = 1$ for all four neighbors of the origin, and $J(x) = 0$ otherwise. It has been shown by Ruelle (1972) and Lebowitz and Martin-Löf (1972) that phase transition for this model does not occur if $H \neq 0$. (See also Preston (1974c).) Therefore we will take $H = 0$. Note that the presence of phase transition is not very robust in this model, since the phase transition disappears when an arbitrarily small change is made in the potential. This implies an analogous statement for the stochastic Ising model. Arbitrarily small changes in the rates can change a nonergodic system into an ergodic one.

Theorem 3.14. *Suppose $S = Z^2$, $J_R = \beta$ if $R = \{x, y\}$ with $|y-x| = 1$, and $J_R = 0$ otherwise. For sufficiently large positive β, this potential exhibits phase transition.*

Proof. For $n \geq 1$, define ν_n as in (1.15) with $\zeta \equiv 1$ and $T = [-n, n]^2 \subset Z^2$. By Corollary 1.30(e) and Theorem 1.31(a), it suffices to prove that

$$\lim_{\beta \to \infty} \nu_n\{\eta: \eta(0) = 0\} = 0 \tag{3.15}$$

uniformly in n, since phase transition will occur for any β such that

$$\lim_{n \to \infty} \nu_n\{\eta: \eta(0) = 0\} < \tfrac{1}{2}.$$

To carry out the proof, it is important to visualize a configuration $\eta \in \{0, 1\}^T$ in a certain way. Write $+$ for 1 and $-$ for 0, and agree to draw vertical and horizontal lines of unit length between adjacent sites which have opposite signs. An illustration with a particular configuration in case $n = 3$ is given in Figure 1. The outer edge consists entirely of $+$ since the boundary condition is $\zeta \equiv 1$. Let $B(\eta)$ be the union of all these vertical and horizontal lines. Note that the configuration can be reconstructed from $B(\eta)$ since the outer edge is always $+$. Also, $B(\eta)$ is a disjoint union of contours, where a contour is a closed non-self-intersecting polygonal curve. The length $|\gamma|$ of a contour γ is the number of unit edges in γ. The sum of the lengths of all the contours which make up $B(\eta)$ will be denoted by $|B(\eta)|$. With this notation, we can proceed to prove (3.15). If $\eta(0) = 0$, then 0 is surrounded

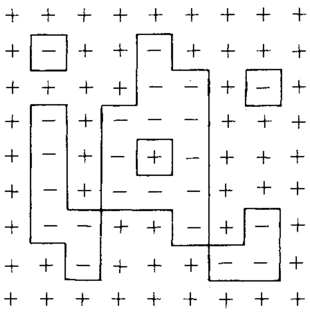

Figure 1

by at least one contour γ. Let Γ be the set of contours surrounding 0. Then

$$\nu_n\{\eta\colon \eta(0)=0\} \le \sum_{\gamma\in\Gamma} \nu_n\{\eta\colon \gamma\in B(\eta)\}, \tag{3.16}$$

so we need to estimate $\nu_n\{\eta\colon \gamma\in B(\eta)\}$ for fixed $\gamma\in\Gamma$. To do so, use (1.15) to write

$$\nu_n\{\eta\colon \gamma\in B(\eta)\} = \frac{\sum_{\eta:\gamma\in B(\eta)} \exp[-2\beta|B(\eta)|]}{\sum_{\eta} \exp[-2\beta|B(\eta)|]}. \tag{3.17}$$

If η is such that $\gamma\in B(\eta)$, define $\tilde\eta$ by

$$\tilde\eta(x) = \begin{cases} 1-\eta(x) & \text{if } \gamma \text{ surrounds } x, \\ \eta(x) & \text{otherwise.} \end{cases}$$

Then $B(\tilde\eta)$ is obtained from $B(\eta)$ by removing γ, so that $|B(\eta)| = |\gamma| + |B(\tilde\eta)|$. Therefore by (3.17),

$$\nu_n\{\eta\colon \gamma\in B(\eta)\} = \exp[-2\beta|\gamma|] \frac{\sum_{\eta:\gamma\in B(\eta)} \exp[-2\beta|B(\tilde\eta)|]}{\sum_{\eta} \exp[-2\beta|B(\eta)|]}.$$

Since the map $\eta\to\tilde\eta$ is one-to-one, each term in the numerator on the right side appears also in the denominator. Therefore we can conclude that

$$\nu_n\{\eta\colon \gamma\in B(\eta)\} \le \exp[-2\beta|\gamma|].$$

Using this in (3.16) gives

$$\nu_n\{\eta\colon \eta(0)=0\} \le \sum_{k=4}^{\infty} e^{-2\beta k} N(k,n), \tag{3.18}$$

where $N(k,n)$ is the number of contours $\gamma\in\Gamma$ of length k. But $N(k,n)\le k3^k$ for all n, since each contour $\gamma\in\Gamma$ of length k must cross the positive horizontal axis at at least one of k places, and such a contour can be continued at each point in at most three ways. Thus we obtain the estimate

$$\nu_n\{\eta\colon \eta(0)=0\} \le \sum_{k=4}^{\infty} k3^k e^{-2\beta k}, \tag{3.19}$$

from which (3.15) follows by the Dominated Convergence Theorem. □

We now know from Theorems 3.1 and 3.14 that for the Ising model considered in the latter theorem, $0<\beta_c<\infty$. While it will not be of great importance to us, it is interesting to note that the exact value of β_c is known in this case: $\beta_c=\frac{1}{2}\log(1+\sqrt{2})$. In fact, it is even known that

$$\bar{\nu}\{\eta\colon \eta(x)=1\}-\tfrac{1}{2}=[1-(\sinh 2\beta)^{-4}]^{1/8}$$

for $\beta\geq\beta_c$, so that this model does not exhibit phase transition at $\beta=\beta_c$. This formula for the "spontaneous magnetization" was obtained by Onsager in 1944. For more modern and mathematically precise versions of this result, see Abraham and Martin-Löf (1973) and Benettin, Gallavotti, and Jona-Lasinio (1973). The exact critical value for the Ising model in dimensions greater than 2 is not known. A great deal of additional information is available for the two-dimensional Ising model. For example, it was proved independently by Aizenman (1980) and Higuchi (1979) that all Gibbs states are convex combinations of $\underline{\nu}$ and $\bar{\nu}$ if $\beta\geq 0$. This statement is not correct in dimensions greater than two. (See, for example, Van Beijeren (1975).)

If β is sufficiently large and negative, the two-dimensional Ising model has Gibbs states which are not translation invariant. This is an easy consequence of Theorem 3.14, since the transformation $\eta\to\tilde{\eta}$ where

$$\tilde{\eta}(x)=\begin{cases}\eta(x) & \text{if } x \text{ is an even number of steps from the origin,}\\ 1-\eta(x) & \text{otherwise}\end{cases}$$

maps the Ising model with parameter β into the Ising model with parameter $-\beta$.

Finally, we summarize some of the main results of this section in the following way.

Corollary 3.20. *Suppose the potential satisfies the assumptions of Theorem* 1.31, *and let β_c be defined as in that result. Then*

(a) $\beta_c>0$,
(b) $\beta_c=\infty$ *if* $d=1$ *and* $\sum_{n=1}^{\infty} nJ(n)<\infty$,
(c) $\beta_c<\infty$ *if* $d\geq 2$ *and* $J(x)>0$ *for all* $2d$ *nearest neighbors of the origin.*

Proof. Parts (a) and (b) are consequences of Theorems 3.1 and 3.9 respectively. Part (c) follows from Theorem 3.14 and part (d) of Theorem 1.31. □

Corollary 3.21. *Under the assumptions of Corollary* 3.20, *consider an attractive stochastic Ising model with respect to the given potential. If* $\beta<\beta_c$, *the system is ergodic, while if* $\beta>\beta_c$, *it is not.*

Proof. This result is essentially a restatement of Corollary 2.18. □

4. L_2 Theory

Throughout most of this book, the semigroup associated with a spin system is regarded as operating on $C(X)$. A stochastic Ising model is reversible with respect to appropriate Gibbs states. This implies that its semigroup is self-adjoint on the L_2 spaces of those Gibbs states. Thus it is natural to regard the semigroup as acting on these spaces as well. This section is devoted to the exploitation of this point of view. An important result will be a general L_2 convergence theorem for stochastic Ising models.

As was pointed out in Section 2, the physics of a situation dictates the use of a particular potential. On the other hand, there appear to be no physical arguments which can be used to select one of the many stochastic Ising models corresponding to that potential. Thus it would be good to know that the behavior of a stochastic Ising model depends in some sense only on the potential, and not otherwise on the choice of rates. Theorems 2.14 and 2.16 and Corollary 2.17 are results of this type. Further results of this type will be obtained in the next section (see Theorems 5.12 and 5.14). Open Problems 1 and 3 raise similar issues. In this section, we will see that the L_2 theory of the stochastic Ising model is largely independent of the choice of rates (see Corollary 4.18, for example).

In order to describe the context in which we will operate, fix a potential $\{J_R\}$, a Gibbs state ν relative to it, and a stochastic Ising model η_t relative to it with rates $c(x, \eta)$ and semigroup $S(t)$ (see Definitions 1.1, 1.5, and 2.1 respectively). The norm in $L_2(\nu)$ will be denoted by $\|\cdot\|_\nu$. The theory of contraction semigroups and their generators was described in Section 2 of Chapter I for the Banach space $C(X)$. This theory, and in particular Theorem 2.9 in that chapter, applies equally well to the space $L_2(\nu)$. The only change needed is that (c) of Definition 2.1 in that chapter should be replaced by the statement that

$$\int f\Omega f\, d\nu \leq 0 \quad \text{for all } f \in \mathscr{D}(\Omega).$$

Proposition 4.1. *The semigroup $S(t)$ extends by continuity to a Markov semigroup on $L_2(\nu)$. Its generator Ω_ν is the closure of Ω in $L_2(\nu)$. The operator Ω_ν is self-adjoint on $L_2(\nu)$.*

Proof. $S(t)f(\eta) = E^\eta f(\eta_t)$ for $f \in C(X)$, so that

$$[S(t)f(\eta)]^2 \leq S(t)f^2(\eta)$$

by the Schwarz inequality. By Theorems 2.14 and 2.15, $\mathscr{G} = \mathscr{R} \subset \mathscr{I}$.

Therefore, since $\nu \in \mathscr{G}$,

$$\|S(t)f\|_\nu^2 = \int [S(t)f]^2 \, d\nu$$
$$\leq \int S(t)f^2 \, d\nu$$
$$= \int f^2 \, d\nu = \|f\|_\nu^2.$$

$C(X)$ is dense in $L_2(\nu)$, so $S(t)$ extends to a Markov semigroup on $L_2(\nu)$. Let Ω_ν be its generator. Clearly Ω_ν is an extension of Ω. Since $\mathscr{R}(I - \lambda\Omega) = C(X)$, which is dense in $L_2(\nu)$, Ω_ν is the closure of Ω. Therefore by Proposition 5.3 of Chapter II,

$$\int f\Omega_\nu g \, d\nu = \int g\Omega_\nu f \, d\nu \tag{4.2}$$

for all $f, g \in \mathscr{D}(\Omega_\nu)$. This says that Ω_ν is a symmetric operator on $L_2(\nu)$. Therefore the adjoint of Ω_ν is an extension of Ω_ν. This adjoint is also a Markov generator on $L_2(\nu)$, and hence must agree with Ω_ν. Therefore Ω_ν is self-adjoint. □

The main idea of this section is to use the spectral representation of $-\Omega_\nu$ to study the convergence of $S(t)f$ in $L_2(\nu)$. In doing so, it will be important to have a more explicit expression for the symmetric bilinear form which appears in (4.2).

Lemma 4.3. *If $f, g \in \mathscr{D}(\Omega_\nu)$, then*

$$-\int f\Omega_\nu g \, d\nu = \tfrac{1}{2}\sum_x \int c(x, \eta)[f(\eta_x) - f(\eta)][g(\eta_x) - g(\eta)] \, d\nu,$$

where the series converges absolutely.

Proof. Define bilinear forms

$$L(f, g) = -\int f\Omega_\nu g \, d\nu \quad \text{for } f, g \in \mathscr{D}(\Omega_\nu)$$

and

$$R(f, g) = \tfrac{1}{2}\sum_x \int c(x, \eta)[f(\eta_x) - f(\eta)][g(\eta_x) - g(\eta)] \, d\nu$$

for all f, g such that $R(f,f)<\infty$ and $R(g,g)<\infty$. Note that the series converges absolutely by the Schwarz inequality. For $f, g \in \mathscr{D}$,

$$\begin{aligned} L(f,g) &= -\int f\Omega_\nu g \, d\nu \\ &= -\sum_x \int c(x,\eta)f(\eta)[g(\eta_x)-g(\eta)]\, d\nu \\ &= \sum_x \int c(x,\eta)f(\eta_x)[g(\eta_x)-g(\eta)]\, d\nu, \end{aligned} \tag{4.4}$$

where of course there are only finitely many nonzero terms in each sum. The last equality follows from Proposition 2.7, as can be seen by applying (2.8) to the function

$$\eta \to [f(\eta_x)+f(\eta)]g(\eta).$$

Taking the average of the last two terms in (4.4), we see that

$$\begin{aligned} L(f,g) &= -\tfrac{1}{2}\sum_x \int c(x,\eta)f(\eta)[g(\eta_x)-g(\eta)]\, d\nu \\ &\quad +\tfrac{1}{2}\sum_x \int c(x,\eta)f(\eta_x)[g(\eta_x)-g(\eta)]\, d\nu \\ &= \tfrac{1}{2}\sum_x \int c(x,\eta)[f(\eta_x)-f(\eta)][g(\eta_x)-g(\eta)]\, d\nu \\ &= R(f,g), \end{aligned} \tag{4.5}$$

which proves the result for $f, g \in \mathscr{D}$. For $f \in \mathscr{D}(\Omega_\nu)$, take $f_n \in \mathscr{D}$ so that $f_n \to f$ and $\Omega_\nu f_n \to \Omega_\nu f$ in $L_2(\nu)$. Then

$$\lim_{n\to\infty} L(f_n, f_n) = L(f,f), \quad \text{and} \tag{4.6}$$

$$\liminf_{n\to\infty} R(f_n, f_n) \geq R(f,f),$$

the latter coming from Fatou's Lemma. Therefore

$$R(f,f) \leq L(f,f), \tag{4.7}$$

so in particular, $R(f,f)<\infty$ for $f \in \mathscr{D}(\Omega_\nu)$. On the other hand,

$$0 \leq L(f-f_n, f-f_n) \leq \|f-f_n\|_\nu \|\Omega_\nu f - \Omega_\nu f_n\|_\nu$$

which tends to zero as $n \to \infty$, so that by (4.7),

$$\lim_{n\to\infty} R(f-f_n, f-f_n) = 0.$$

Therefore

$$\lim_{n\to\infty} R(f_n, f_n) = R(f, f). \tag{4.8}$$

Putting (4.5), (4.6), and (4.8) together yields $R(f,f) = L(f,f)$ for all $f \in \mathscr{D}(\Omega_\nu)$. The desired result follows by polarization. $\square$

One consequence of Lemma 4.3 is that $-\Omega_\nu$ is not only self-adjoint, but also positive semidefinite. (This follows also from the fact that Ω_ν is a Markov generator on $L_2(\nu)$.) The spectral theorem then takes the following form:

$$-\Omega_\nu = \int_{[0,\infty)} \lambda \, dG(\lambda), \tag{4.9}$$

where $G(\lambda)$ is a resolution of the identity. Recall that this means that $\{G(\lambda), \lambda \geq 0\}$ is a family of projections on $L_2(\nu)$ which satisfy

$$G(\lambda_1)G(\lambda_2) = G(\lambda_1 \wedge \lambda_2),$$

$$\lim_{\lambda\uparrow\infty} G(\lambda)f = f,$$

and

$$\lim_{\gamma\downarrow\lambda} G(\gamma)f = G(\lambda)f$$

for all $f \in L_2(\nu)$. Treatments of the spectral theorem can be found in Chapter 13 of Rudin (1973) and Chapter XI of Yosida (1980). The semigroup $S(t)$ on $L_2(\nu)$ can then be written as

$$S(t) = \int_{[0,\infty)} e^{-\lambda t} \, dG(\lambda), \quad \text{or} \tag{4.10}$$

$$S(t) = G(0) + \int_{(0,\infty)} e^{-\lambda t} \, dG(\lambda) \tag{4.11}$$

where $G(0)$ is the projection on the eigenspace corresponding to the eigenvalue 0 for Ω_ν. Since $\Omega_\nu 1 = 0$, the 0 eigenvalue always has multiplicity at least one. In particular, we obtain the following convergence result:

$$\lim_{t\to\infty}\|S(t)f-G(0)f\|_\nu=0 \tag{4.12}$$

for all $f\in L_2(\nu)$. This is the general L_2 convergence theorem for stochastic Ising models.

As it stands, (4.12) depends on the particular choice of a stochastic Ising model for the given potential, since $G(0)$ depends on that choice. The rest of the results in this section show, however, that a number of statements can be made which are independent of that choice, subject perhaps to uniform positivity assumptions on the rates. The first deals with convergence, and the second with exponentially fast convergence. Recall that a criterion for a Gibbs state to be extremal was given in Corollary 3.8.

Theorem 4.13. *The following statements are equivalent.*

(a) $\lim_{t\to\infty} S(t)f=\int f\,d\nu$ *in* $L_2(\nu)$ *for all* $f\in L_2(\nu)$.
(b) 0 *is a simple eigenvalue for* Ω_ν.
(c) ν *is extremal in* $\mathcal{G}$.

Proof. By (4.12), (a) is equivalent to

$$G(0)f=\int f\,d\nu \quad \text{for all } f\in L_2(\nu), \tag{4.14}$$

which is clearly equivalent to (b). For the equivalence of (b) and (c), note that (b) is equivalent to the statement that

$$\Omega_\nu f=0 \quad \text{implies} \quad f=\text{constant a.e. } (\nu).$$

But by Lemma 4.3, $\Omega_\nu f=0$ is equivalent to

$$\int [f(\eta_x)-f(\eta)]^2\,d\nu=0 \quad \text{for all } x\in S. \tag{4.15}$$

Therefore (b) is equivalent to the statement that (4.15) implies $f=$ constant a.e. (ν). The equivalence to (c) then follows from Corollary 3.8. □

Theorem 4.16. *Suppose that the equivalent conditions of Theorem* 4.13 *are satisfied. Then the following statements are equivalent.*

(a) *There exists an* $\alpha>0$ *so that*

$$\left\|S(t)f-\int f\,d\nu\right\|_\nu\le e^{-\alpha t}\|f\|_\nu \quad \textit{for all } f\in L_2(\nu).$$

(b) *There exists an* $\alpha > 0$ *so that* $G(\alpha) = G(0)$.
(c) *There exists an* $\alpha > 0$ *so that*

$$\sum_x \int c(x, \eta)[f(\eta_x) - f(\eta)]^2 \, d\nu \geq \alpha \left\| f - \int f \, d\nu \right\|_\nu^2$$

for all $f \in L_2(\nu)$.

Proof. By (4.11) and (4.14),

$$S(t)f - \int f \, d\nu = \int_{(0,\infty)} e^{-\lambda t} \, dG(\lambda)f,$$

so that

$$\left\| S(t)f - \int f \, d\nu \right\|_\nu^2 = \int_{(0,\infty)} e^{-2\lambda t} \, d\|G(\lambda)f\|_\nu^2.$$

The equivalence of (a) and (b) is immediate from this. By Lemma 4.3, (c) is equivalent to the existence of an $\alpha > 0$ so that

$$(4.17) \qquad -\int f \Omega_\nu f \, d\nu \geq \alpha \left\| f - \int f \, d\nu \right\|_\nu^2$$

for all $f \in \mathscr{D}(\Omega_\nu)$. But by (4.9), (4.17) is the same as

$$\int_{(0,\infty)} \lambda \, d(G(\lambda)f, f)_\nu \geq \alpha \int_{(0,\infty)} d(G(\lambda)f, f)_\nu$$

where $(\ ,\ )_\nu$ denotes the $L_2(\nu)$ inner product. The equivalence of (b) and (c) is now clear. □

Corollary 4.18. *Suppose that* $S_1(t)$ *and* $S_2(t)$ *are the semigroups for two stochastic Ising models relative to a potential with rates* $c_1(x, \eta)$ *and* $c_2(x, \eta)$, *and that* ν *is one of the corresponding Gibbs states. Then*

$$\lim_{t\to\infty} S_1(t)f = \int f \, d\nu \quad \textit{for all } f \in L_2(\nu)$$

if and only if

$$\lim_{t\to\infty} S_2(t)f = \int f \, d\nu \quad \textit{for all } f \in L_2(\nu).$$

If this is the case and

$$0 < \inf_{x,\eta} \frac{c_1(x, \eta)}{c_2(x, \eta)} \leq \sup_{x,\eta} \frac{c_1(x, \eta)}{c_2(x, \eta)} < \infty, \tag{4.19}$$

then there is an $\alpha > 0$ *so that*

$$\left\| S_1(t)f - \int f\,d\nu \right\|_\nu \leq e^{-\alpha t}\|f\|_\nu \quad \textit{for all } f \in L_2(\nu)$$

if and only if there is an $\alpha > 0$ *so that*

$$\left\| S_2(t)f - \int f\,d\nu \right\|_\nu \leq e^{-\alpha t}\|f\|_\nu \quad \textit{for all } f \in L_2(\nu).$$

Proof. The first statement follows from Theorem 4.13 and the fact that statement (c) there depends only on ν and not on the process. The second statement follows from Theorem 4.16 in the same way, since under (4.19), statement (c) of that theorem depends only on ν and not on the process. □

Corollary 4.18 shows that to a large extent, the L_2 behavior of the stochastic Ising model is independent of the particular choice of rates. Therefore one way of showing L_2 exponential convergence for one model is to verify (0.6) of Chapter III for another model with the same potential, and then use Corollary 4.18. Even in one dimension, this works only in special cases. One such case is given in Example 4.3(a) of Chapter I. For a one-dimensional finite range stochastic Ising model, the statements in Theorem 4.13 all hold since $\mathscr{G}$ is a singleton by Theorem 3.5. Holley (1985) has shown that the statements in Theorem 4.16 hold as well in this case. In the same paper, he also showed that if the process is attractive, the $L_2(\nu)$ exponential convergence can be replaced by exponential convergence in the $C(X)$ norm.

Theorem 4.13 has other interesting consequences. For the following one, let $\mathscr{G}_e$ be the set of extreme points of $\mathscr{G}$.

Corollary 4.20. $\mathscr{G}_e \subset \mathscr{I}_e$.

Proof. Suppose $\nu \in \mathscr{G}_e$ and $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$, where $0 < \alpha < 1$ and $\nu_1, \nu_2 \in \mathscr{I}$. By Theorem 4.13,

$$\lim_{t\to\infty} S(t)f = \int f\,d\nu \quad \text{in } L_2(\nu) \quad \text{for all } f \in L_2(\nu).$$

Therefore

$$\lim_{t\to\infty} S(t)f = \int f\,d\nu \quad \text{in } L_2(\nu_1) \quad \text{for all } f \in L_2(\nu).$$

Since $\nu_1 \in \mathscr{I}$,

$$\int S(t)f\,d\nu_1 = \int f\,d\nu_1$$

for all $f \in C(X)$ and all $t \geq 0$. Therefore

$$\int f\,d\nu = \int f\,d\nu_1$$

for all $f \in C(X)$, and hence $\nu_1 = \nu$. $\square$

Corollary 4.21. *Suppose that $c(x, \eta_x) = c(x, \eta)$ for all $x \in S$ and $\eta \in X$, and that*

$$0 < \inf_{x,\eta} c(x, \eta). \tag{4.22}$$

Let ν be the product measure on X with $\nu\{\eta\colon \eta(x) = 1\} = \frac{1}{2}$ for all $x \in S$. Then there is an $\alpha > 0$ so that

$$\left\| S(t)f - \int f\,d\nu \right\|_\nu \leq e^{-\alpha t} \|f\|_\nu$$

for all $f \in L_2(\nu)$.

Proof. Since $c(x, \eta_x) = c(x, \eta)$, this spin system is a stochastic Ising model relative to the potential $J_R \equiv 0$. Of course, $\mathscr{G} = \{\nu\}$ in this case. By (4.22) and Theorem 4.16, it suffices to check that for some $\alpha > 0$,

$$\sum_x \int [f(\eta_x) - f(\eta)]^2\,d\nu \geq \alpha \left\| f - \int f\,d\nu \right\|_\nu^2 \tag{4.23}$$

for all $f \in L_2(\nu)$. The collection $\{\chi_R\}$, where R ranges over the finite subsets of S is an orthonormal basis in $L_2(\nu)$. Therefore, we can write

$$f = \sum_R c_R \chi_R,$$

where the series converges in $L_2(\nu)$. Now

$$f(\eta_x) - f(\eta) = -2 \sum_{R \ni x} c_R \chi_R,$$

so that

$$\int [f(\eta_x) - f(\eta)]^2\,d\nu = 4 \sum_{R \ni x} c_R^2.$$

Since

$$\left\| f - \int f\,d\nu \right\|_\nu^2 = \sum_{R \neq \varnothing} c_R^2,$$

(4.23) holds with $\alpha = 4$. □

5. Characterization of Invariant Measures

For a stochastic Ising model, every Gibbs state is invariant by Theorem 2.15. Since there are no known examples in which a stochastic Ising model has invariant measures which are not Gibbs, it is reasonable to conjecture that $\mathscr{I} = \mathscr{G}$, at least under some regularity conditions. At this point, we only know this to be the case if the process is attractive and $\mathscr{G}$ is a singleton (Theorem 2.16). In this section, we will use the relative entropy technique to obtain further results of this type.

The underlying idea is to define the relative entropies of the finite-dimensional distributions of a probability measure ν on X, and then to find a convenient expression for the time derivatives of these quantities corresponding to the distribution of the process at time t. If the measure is invariant, these time derivatives must be zero. This observation leads to a collection of identities which can be used to try to show that the measure is Gibbs.

Of course this technique could be used with functionals other than the relative entropy. The advantage of using the relative entropy is that it has certain monotonicity properties which are described in a Markov chain context in Section 4 of Chapter II. In the present context, this monotonicity is evidenced by the fact that many of the terms in the basic identities have a fixed sign. The relative entropy technique works extremely well in situations in which both the process and the measure are translation invariant. While it is less effective in the absence of translation invariance, it still yields important and useful conclusions which apparently cannot be obtained otherwise.

Throughout this section, $\{J_R\}$ will be a potential, and $c(x, \eta)$ will be the rates for a stochastic Ising model relative to that potential (see Definitions 1.1 and 2.1). In the first part of the section, T will be a fixed finite subset of S. For $\zeta \in \{0, 1\}^T$, define $f_\zeta \in \mathscr{D}$ by

$$f_\zeta(\eta) = \begin{cases} 1 & \text{if } \eta = \zeta \text{ on } T, \\ 0 & \text{otherwise.} \end{cases}$$

If ν is a probability measure on X, define

$$\nu(\zeta) = \int f_\zeta\,d\nu.$$

For $x \in T$ and $\zeta \in \{0, 1\}^T$, put

$$\Gamma(x, \zeta) = \int c(x, \eta) f_\zeta(\eta) \nu(d\eta).$$

An important consequence of Proposition 2.7 and Theorem 2.14 which shows how we will prove that certain measures are Gibbs states is the following: $\nu \in \mathscr{G}$ if and only if

$$\Gamma(x, \zeta) = \Gamma(x, \zeta_x)$$

for all T and all $x \in T$ and $\zeta \in \{0, 1\}^T$.

The entropy of ν on T relative to the potential $\{J_R\}$ is defined as

$$H_T(\nu) = \sum_\zeta \nu(\zeta) \log \nu(\zeta) - \int \left[\sum_{R \subset T} J_R \chi_R \right] d\nu,$$

where the first sum is over all $\zeta \in \{0, 1\}^T$. Note that this depends only on ν through its finite-dimensional distribution corresponding to the coordinates in T. The definition of $H_T(\nu)$ is best motivated by considering the case of a finite S with $T = S$. Then this coincides with Definition 4.1 of Chapter II if the π there is taken to be the Gibbs state given in (1.4). The first step is to obtain a useful expression for the time derivative of the relative entropy of $\nu S(t)$ on T. Note that the terms in the first sum of (5.2) below are all nonnegative.

Lemma 5.1. *Suppose that $\nu \in \mathscr{P}$ has the property that $\nu(\zeta) > 0$ for all $\zeta \in \{0, 1\}^T$. Then*

$$\begin{aligned} 2 \frac{d}{dt} H_T(\nu S(t)) \Big|_{t=0} &= - \sum_{x \in T} \sum_\zeta [\Gamma(x, \zeta) - \Gamma(x, \zeta_x)] \log \frac{\Gamma(x, \zeta)}{\Gamma(x, \zeta_x)} \\ &\quad + \sum_{x \in T} \sum_\zeta [\Gamma(x, \zeta) - \Gamma(x, \zeta_x)] \\ &\quad \times \left\{ V(x, \zeta) + \log \frac{\Gamma(x, \zeta)}{\nu(\zeta)} - \log \frac{\Gamma(x, \zeta_x)}{\nu(\zeta_x)} \right\}, \end{aligned} \tag{5.2}$$

where

$$V(x, \zeta) = 2 \sum_{x \in R \subset T} J_R \chi_R(\zeta).$$

Proof. Set $\nu_t = \nu S(t)$. We will compute the derivative of the two terms in $H_T(\nu_t)$ separately. Sums on ζ will always be over $\zeta \in \{0, 1\}^T$, and sums on

x will be over $x \in T$. Since $\nu(\zeta) > 0$ for all $\zeta \in \{0, 1\}^T$ and since

$$\sum_\zeta \int \Omega f_\zeta \, d\nu = \int \Omega 1 \, d\nu = 0,$$

$$\begin{aligned}
\frac{d}{dt} \sum_\zeta \nu_t(\zeta) \log \nu_t(\zeta) \bigg|_{t=0} &= \sum_\zeta [1 + \log \nu(\zeta)] \frac{d}{dt} \nu_t(\zeta) \bigg|_{t=0} \\
&= \sum_\zeta [1 + \log \nu(\zeta)] \int \Omega f_\zeta \, d\nu \\
&= \sum_\zeta \log \nu(\zeta) \int \sum_x c(x, \eta)[f_\zeta(\eta_x) - f_\zeta(\eta)] \, d\nu \\
&= \sum_{x,\zeta} \log \nu(\zeta)[\Gamma(x, \zeta_x) - \Gamma(x, \zeta)].
\end{aligned}$$

Making the change of variables $\zeta \to \zeta_x$ in the sum on ζ of the first term, we obtain

$$\frac{d}{dt} \sum_\zeta \nu_t(\zeta) \log \nu_t(\zeta) \bigg|_{t=0} = \sum_{x,\zeta} \Gamma(x, \zeta) \log \frac{\nu(\zeta_x)}{\nu(\zeta)}. \tag{5.3}$$

For the second term in $H_T(\nu_t)$,

$$\begin{aligned}
\frac{d}{dt} \int \left[\sum_{R \subset T} J_R \chi_R \right] d\nu_t \bigg|_{t=0} &= \int \left[\sum_{R \subset T} J_R \Omega \chi_R \right] d\nu \\
&= -2 \sum_x \sum_{x \in R \subset T} J_R \int c(x, \eta) \chi_R(\eta) \, d\nu
\end{aligned}$$

since

$$\chi_R(\eta_x) - \chi_R(\eta) = \begin{cases} -2\chi_R(\eta) & \text{if } x \in R, \\ 0 & \text{if } x \notin R. \end{cases}$$

But for $R \subset T$,

$$\begin{aligned}
\int c(x, \eta) \chi_R(\eta) \, d\nu &= \sum_\zeta \int c(x, \eta) f_\zeta(\eta) \chi_R(\eta) \, d\nu \\
&= \sum_\zeta \chi_R(\zeta) \Gamma(x, \zeta),
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt} \int \left[\sum_{R \subset T} J_R \chi_R \right] d\nu_t \bigg|_{t=0} &= -2 \sum_{x, \zeta} \sum_{x \in R \subset T} J_R \chi_R(\zeta) \Gamma(x, \zeta) \\
&= -\sum_{x,\zeta} \Gamma(x, \zeta) V(x, \zeta).
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) gives

$$\frac{d}{dt} H_T(\nu_t)\bigg|_{t=0} = \sum_{x,\zeta} \Gamma(x, \zeta)\left\{\log \frac{\nu(\zeta_x)}{\nu(\zeta)} + V(x, \zeta)\right\}.$$

Since $V(x, \zeta_x) = -V(x, \zeta)$ and

$$\log \frac{\nu(\zeta_x)}{\nu(\zeta)} = -\log \frac{\nu(\zeta)}{\nu(\zeta_x)},$$

this can be rewritten as

$$2\frac{d}{dt} H_T(\nu_t)\bigg|_{t=0} = \sum_{x,\zeta} [\Gamma(x, \zeta) - \Gamma(x, \zeta_x)]\left\{ V(x, \zeta) - \log \frac{\nu(\zeta)}{\nu(\zeta_x)}\right\},$$

from which (5.2) follows by adding and subtracting appropriate terms. □

Lemma 5.5. *Suppose that $\nu \in \mathscr{I}$. Then $\nu(\zeta) > 0$ for all $\zeta \in \{0, 1\}^T$.*

Proof. Since $\nu \in \mathscr{I}$ and $f_\zeta \in \mathscr{D}$,

$$\begin{aligned} 0 = \int \Omega f_\zeta \, d\nu &= \int \sum_{x \in T} c(x, \eta)[f_\zeta(\eta_x) - f_\zeta(\eta)] \, d\nu \\ &= \sum_{x \in T} \int c(x, \eta)[f_{\zeta_x}(\eta) - f_\zeta(\eta)] \, d\nu. \end{aligned}$$

Therefore, since $c(x, \eta) > 0$ for all x and η, $\nu(\zeta) = 0$ implies that $\nu(\zeta_x) = 0$ for all $x \in T$. Iterating this statement, we see that if $\nu(\zeta) = 0$ for some $\zeta \in \{0, 1\}^T$, then $\nu(\zeta) = 0$ for all $\zeta \in \{0, 1\}^T$. Since $\sum_\zeta \nu(\zeta) = 1$, this cannot occur. □

Theorem 5.6. *Suppose that $\nu \in \mathscr{I}$. Then*

$$\begin{aligned} &\sum_{x \in T} \sum_{\zeta \in \{0, 1\}^T} [\Gamma(x, \zeta) - \Gamma(x, \zeta_x)] \log \frac{\Gamma(x, \zeta)}{\Gamma(x, \zeta_x)} \\ &\quad = \sum_{x \in T} \sum_{\zeta \in \{0, 1\}^T} [\Gamma(x, \zeta) - \Gamma(x, \zeta_x)] \\ &\qquad \times \left\{ V(x, \zeta) + \log \frac{\Gamma(x, \zeta)}{\nu(\zeta)} - \log \frac{\Gamma(x, \zeta_x)}{\nu(\zeta_x)} \right\}. \end{aligned}$$

Proof. By Lemma 5.5, $\nu(\zeta) > 0$ for all $\zeta \in \{0, 1\}^T$. Since $\nu \in \mathscr{I}$, $\nu S(t) = \nu$ for all $t \geq 0$, so that $(d/dt)H_T(\nu S(t)) = 0$ for all $t \geq 0$. The result then follows from Lemma 5.1. □

For $x \in T$, define

$$\alpha_T(x) = \sum_{\zeta} [\Gamma(x, \zeta) - \Gamma(x, \zeta_x)] \log \frac{\Gamma(x, \zeta)}{\Gamma(x, \zeta_x)},$$

$$\beta_T(x) = \sum_{\zeta} |\Gamma(x, \zeta) - \Gamma(x, \zeta_x)|, \quad \text{and}$$

$$\rho_T(x) = 2 \sum_{x \in R \not\subset T} |J_R| + 2 \sum_{u \notin T} \sup_{\eta} \frac{|c(x, \eta_u) - c(x, \eta)|}{c(x, \eta)}.$$

If $x \notin T$, they are defined to be zero.

Corollary 5.7. *Suppose that $\nu \in \mathscr{I}$. Then*

$$\sum_{x \in T} \alpha_T(x) \le \sum_{x \in T} \rho_T(x) \beta_T(x).$$

Proof. By Theorem 5.6, it suffices to show that

$$\left| V(x, \zeta) + \log \frac{\Gamma(x, \zeta)}{\nu(\zeta)} - \log \frac{\Gamma(x, \zeta_x)}{\nu(\zeta_x)} \right| \le \rho_T(x)$$

for all $\zeta \in \{0, 1\}^T$. In order to do this, let

$$c_T(x, \zeta) = c(x, \zeta_T),$$

where for $\zeta \in \{0, 1\}^T$, $\zeta_T \in X$ is defined by

$$\zeta_T(u) = \begin{cases} \zeta(u) & \text{if } u \in T, \\ 1 & \text{if } u \notin T. \end{cases}$$

Since the process is a stochastic Ising model, the function

$$c_T(x, \zeta) \exp\left[\sum_{R \ni x} J_R \chi_{R \cap T}(\zeta) \right] = c(x, \zeta_T) \exp\left[\sum_{R \ni x} J_R \chi_R(\zeta_T) \right]$$

does not depend on the coordinate $\zeta(x)$. Therefore by adding and subtracting the log of this function and using the definitions of $V(x, \zeta)$ and $\Gamma(x, \zeta)$, we see that

$$V(x,\zeta)+\log\frac{\Gamma(x,\zeta)}{\nu(\zeta)}-\log\frac{\Gamma(x,\zeta_x)}{\nu(\zeta_x)}$$

$$=\log\frac{\int\frac{c(x,\eta)}{c_T(x,\zeta)}f_\zeta(\eta)\exp\left[-\sum_{x\in R\not\subset T}J_R\chi_{R\cap T}(\zeta)\right]d\nu}{\nu(\zeta)}$$

$$-\log\frac{\int\frac{c(x,\eta)}{c_T(x,\zeta_x)}f_{\zeta_x}(\eta)\exp\left[-\sum_{x\in R\not\subset T}J_R\chi_{R\cap T}(\zeta_x)\right]d\nu}{\nu(\zeta_x)}.$$

Since $\nu(\zeta)=\int f_\zeta\,d\nu$ and $\nu(\zeta_x)=\int f_{\zeta_x}\,d\nu$, the absolute value of this expression is bounded by

$$2\sum_{x\in R\not\subset T}|J_R|+2\sup\left\{\log\frac{c(x,\eta_1)}{c(x,\eta_2)}:\ \eta_1=\eta_2\text{ on }T\right\}.$$

The desired bound then follows by using the inequality $\log a\le a-1$. $\square$

In order to keep in mind where we are headed, a few observations should be made at this point. First, $\nu\in\mathscr{G}$ if and only if $\alpha_T(x)=0$ for all T and all $x\in T$. Secondly, $\alpha_T(x)=0$ implies that $\beta_T(x)=0$, since all the summands in the definition of $\alpha_T(x)$ are automatically nonnegative. Therefore one should be able to control the size of $\beta_T(x)$ in terms of that of $\alpha_T(x)$. Once that is done, it should be possible to use Corollary 5.7 to conclude that $\alpha_T(x)=0$ provided that $\rho_T(x)$ is small. But we automatically have

$$\lim_{T\uparrow S}\rho_T(x)=0$$

for all $x\in S$ by the definition of the potential and by the assumption that the rates satisfy (0.3) of Chapter III. In order to make effective use of these ideas, we need the following inequalities.

Lemma 5.8. (a) $\alpha_{T_1}(x)\le\alpha_{T_2}(x)$ *if* $x\in T_1\subset T_2$.

(b) $$\beta_T^2(x)\le 2[\sup_\eta c(x,\eta)]\alpha_T(x).$$

Proof. For part (a), take $x\in T_1\subset T_2$. Let $\Gamma_1(x,\zeta^1)$ and $\Gamma_2(x,\zeta^2)$ be defined relative to T_1 and T_2 respectively. Then

$$\alpha_{T_1}(x)=\sum_{\zeta_1}\phi[\Gamma_1(x,\zeta^1),\Gamma_1(x,\zeta^1_x)],\quad\text{and}$$

$$\alpha_{T_2}(x)=\sum_{\zeta_2}\phi[\Gamma_2(x,\zeta^2),\Gamma_2(x,\zeta_x^2)],$$

where the sums are over $\zeta_1\in\{0,1\}^{T_1}$ and $\zeta_2\in\{0,1\}^{T_2}$ respectively and

$$\phi(u,v)=(u-v)\log\frac{u}{v}$$

for $u, v>0$. Since ϕ is convex and homogeneous of degree one, it is subadditive. Therefore since

$$\Gamma_1(x,\zeta_1)=\sum_{\zeta_2=\zeta_1\text{ on }T_1}\Gamma_2(x,\zeta_2),$$

it follows that $\alpha_{T_1}(x)\leq\alpha_{T_2}(x)$. For part (b), use the symmetry and subadditivity of $\phi(u,v)$ to show that

$$\begin{aligned}\alpha_T(x)&=\sum_{\zeta}\phi[\Gamma(x,\zeta),\Gamma(x,\zeta_x)]\\&\geq\phi(M,m),\end{aligned}$$

where

$$M=\sum_{\zeta}\max\{\Gamma(x,\zeta),\Gamma(x,\zeta_x)\}$$

and

$$m=\sum_{\zeta}\min\{\Gamma(x,\zeta),\Gamma(x,\zeta_x)\}.$$

Since

$$\beta_T(x)=M-m$$

and

$$M\leq 2\sup_{\eta}c(x,\eta),$$

the desired result follows from the elementary inequality

$$M-m\leq M\log\frac{M}{m}\quad\text{for }0<m\leq M.\quad\square$$

For $x, u\in S$, define

$$\gamma(x,u)=\sum_{R\ni x,u}|J_R|+\sup_{\eta}|c(x,\eta_u)-c(x,\eta)|. \tag{5.9}$$

In view of (0.3) of Chapter III,

$$\sup_x \sum_u \gamma(x, u) < \infty \tag{5.10}$$

is equivalent to (2.6). Of course (5.10) is automatic if both the potential and flip rates are translation invariant on Z^d and of finite range. We will assume from now on that

$$\inf_{x,\eta} c(x, \eta) > 0.$$

This is automatic in the translation invariant case. By the definition of $\rho_T(x)$, we will then have the bound

$$\rho_T(x) \leq L \sum_{u \notin T} \gamma(x, u) \tag{5.11}$$

for some constant L.

Theorem 5.12. *Suppose that $S = Z^d$, that both the potential and the flip rates are translation invariant, and that (5.10) is satisfied. Then $\mathcal{I} \cap \mathcal{S} \subset \mathcal{G}$.*

Proof. Take $\nu \in \mathcal{I} \cap \mathcal{S}$. If T_n is a cube in Z^d of side length n, then T_{kn} is the union of k^d translates of T_n. Since $\nu \in \mathcal{S}$, it follows from part (a) of Lemma 5.8 that

$$\frac{1}{(kn)^d} \sum_{x \in T_{kn}} \alpha_{T_{kn}}(x) \geq \frac{1}{n^d} \sum_{x \in T_n} \alpha_{T_n}(x). \tag{5.13}$$

On the other hand, by (5.11),

$$\begin{aligned}\frac{1}{n^d} \sum_{x \in T_n} \rho_{T_n}(x) &\leq \frac{L}{n^d} \sum_{x \in T_n} \sum_{u \notin T_n} \gamma(0, u - x)\\ &= L \sum_v \gamma(0, v) \frac{|\{x \in T_n : x + v \notin T_n\}|}{n^d}.\end{aligned}$$

This tends to zero as $n \to \infty$ by (5.10) and the Dominated Convergence Theorem. By Corollary 5.7 and the fact that $\beta_T(x)$ is uniformly bounded, it follows that

$$\lim_{n \to \infty} \frac{1}{n^d} \sum_{x \in T_n} \alpha_{Tn}(x) = 0.$$

Therefore by (5.13), $\alpha_{T_n}(x) = 0$ for all $x \in T_n$, and hence by part (a) of Lemma 5.8, $\alpha_T(x) = 0$ for all T and all $x \in T$. Therefore $\nu \in \mathcal{G}$. $\square$

Holley (1971) in the finite range case, and Higuchi and Shiga (1975) under assumption (2.6), proved somewhat more than the conclusion of Theorem 5.12. (They considered only the choice (2.4) of flip rates.) They showed for $\nu \in \mathscr{S}$ that any limit of $\nu S(t)$ along a sequence of t's tending to ∞ must be in $\mathscr{G}$.

Theorem 5.14. *Suppose that $S = Z^d$ for $d = 1$ or $d = 2$, and assume that*

$$\sum_v |v| \sup_x \gamma(x, v+x) < \infty \tag{5.15}$$

where $|\cdot|$ is the l_∞ norm on Z^d. Then $\mathscr{I} = \mathscr{G}$.

Proof. Suppose $\nu \in \mathscr{I}$. By Corollary 5.7 and part (b) of Lemma 5.8,

$$\sum_{x \in T} \alpha_T(x) \le M \sum_{x \in T} \rho_T(x)\sqrt{\alpha_T(x)}$$

for some constant M. By the Schwarz inequality, this implies that

$$\left[\sum_{x \in T} \alpha_T(x)\right]^2 \le M^2\left[\sum_{x \in T} \rho_T(x)\right]\left[\sum_{x \in T} \rho_T(x)\alpha_T(x)\right]. \tag{5.16}$$

Let T_n be the cube $[-n, n]^d$ in Z^d. By (5.11),

$$\begin{aligned}\sum_{n=1}^{\infty} \rho_{T_n}(x) &\le L \sum_{n:\, T_n \ni x} \sum_{u \notin T_n} \gamma(x, u)\\ &\le L \sum_u \gamma(x, u)|\{n \ge 1 : x \in T_n, u \notin T_n\}|\\ &\le L \sum_u \gamma(x, u)|u - x|, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\sum_{x \in T_n} \rho_{T_n}(x) &\le L \sum_{x \in T_n} \sum_{u \notin T_n} \gamma(x, u)\\ &\le L \sum_v \sum_{\substack{x \in T_n\\ v+x \notin T_n}} \gamma(x, v+x)\\ &\le L\, d(2n+1)^{d-1} \sum_v |v| \sup_x \gamma(x, v+x).\end{aligned}$$

Therefore by (5.15),

$$\sup_x \sum_{n=1}^{\infty} \rho_{T_n}(x) < \infty, \quad \text{and} \tag{5.17}$$

$$\sup_n \frac{1}{n^{d-1}} \sum_{x \in T_n} \rho_{T_n}(x) < \infty. \tag{5.18}$$

It remains to show that if $d=1$ or $d=2$, (5.16), (5.17), and (5.18) imply that $\alpha_{T_n}(x)=0$ for all $x\in T_n$ and all n. In order to do this, use part (a) of Lemma 5.8 and (5.17) to show that

$$\begin{aligned}\sum_{x\in T_n}\alpha_{T_n}(x)&\geq\varepsilon\sum_{x\in T_n}\alpha_{T_n}(x)\sum_{k=1}^{n}\rho_{T_k}(x)\\ &\geq\varepsilon\sum_{k=1}^{n}\sum_{x\in T_k}\alpha_{T_k}(x)\rho_{T_k}(x),\end{aligned}\tag{5.19}$$

where ε is the reciprocal of the quantity in (5.17). Let

$$\delta_k=\sum_{x\in T_k}\alpha_{T_k}(x)\rho_{T_k}(x).$$

By (5.16) and (5.19),

$$\varepsilon^2\left(\sum_{k=1}^{n}\delta_k\right)^2\leq M^2\delta_n\sum_{x\in T_n}\rho_{T_n}(x),$$

so that by (5.18),

$$\left(\sum_{k=1}^{n}\delta_k\right)^2\leq N\delta_n n^{d-1}$$

for some constant N. If for some n, $\delta_n>0$, then for all larger n it follows that

$$\frac{1}{n^{d-1}}\leq N\left\{\frac{1}{\sum_{k=1}^{n-1}\delta_k}-\frac{1}{\sum_{k=1}^{n}\delta_k}\right\}.$$

The series on the right converges, so that if $d\leq 2$, we have reached a contradiction. Therefore in this case, $\delta_n=0$ for all n, so $\alpha_{T_n}(x)=0$ for all $x\in T_n$ and all n by (5.16). □

6. Notes and References

Section 1. Definitions of infinite Gibbs states were given at about the same time by Minlos (1967), Dobrushin (1968a), and Lanford and Ruelle (1969). Definition 1.5 is Dobrushin's version, while Minlos' definition is the equivalent form given in Theorem 1.16(c). The equivalence was proved by Dobrushin. Proposition 1.10 and its proof are in the same spirit as those results which state that every Markov random field is a Gibbs state. For more on this, see Chapter 1 of Preston (1974b) and the references given there. The conclusions of Theorem 1.21 are known as the Griffiths inequalities. They were proved by Griffiths (1967) for pair potentials and by Kelly and Sherman (1968) for many body potentials. Ginibre (1970) gave a further generalization of these inequalities. The proof of (1.23) given here is due to Ginibre (1969). The treatment of Theorem 1.27 and its corollaries follows Holley (1974a).

Section 2. Stochastic Ising models were first proposed in a special case by Glauber (1963), and then generalized and studied by Dobrushin (1971a, b). Versions of Theorem 2.13 and 2.14 were proved by Spitzer (1971b) in the case of a finite S, and by Dobrushin (1971b) and Logan (1974) for general S. Other papers containing results of this type are Higuchi and Shiga (1975), Glötzl (1981), and Ding and Chen (1981). Corresponding results for exclusion processes with speed change were proved by Logan (1974), Georgii (1979), Glötzl (1982), and Yan, Chen, and Ding (1982a, b). Theorem 2.16 is due to Holley (1974a). That paper is a survey of the stochastic Ising model as of 1974. A survey of interacting particle systems which emphasizes the role of the stochastic Ising model is given by Durrett (1981).

Section 3. Theorem 3.1 with sufficient condition (3.2) is due to Dobrushin (1968b). Although his proof did not use the connection with the stochastic Ising model, it was based on a similar type of contraction principle. The general technique of using the stochastic Ising model to prove results about the Gibbs states such as absence of phase transition, exponential decay of correlations, and analyticity of correlation functions is due to Holley and Stroock (1976b). Corollary 3.8 is due to Lanford and Ruelle (1969). Theorem 3.9 was proved independently by Ruelle (1968) and by Dobrushin (1969). The proof given here is due independently to Lanford and Preston, and comes via Spitzer. The very precise necessary condition (3.11) and sufficient condition (3.12) for the occurrence of phase transition in one-dimensional systems were only obtained after a number of weaker results had been proved. Most important was the work of Dyson (1969a, b). His necessary condition was

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{N} nJ(n)}{\log(\log N)}=0$$

and his sufficient condition was

$$\sum_{n}\frac{\log(\log n)}{n^3J(n)}<\infty.$$

The proof of Theorem 3.14 is based on the well-known Peierls (1936) argument. This approach was further developed by Griffiths (1964) and Dobrushin (1965).

Section 4. This section is based on Holley and Stroock (1976c). Other papers which contain similar results are Higuchi and Shiga (1975) and Sullivan (1975c). One difference in the treatment given here is that Lemma 4.3 is proved without appealing to the spectral theorem.

Section 5. The results in this section are taken from Holley and Stroock (1977a). This paper is the culmination of the development of the relative entropy technique in this context which was initiated by Holley (1971) and was continued by Higuchi and Shiga (1975) and Moulin Ollagnier and Pinchon (1977). Holley and Stroock proved Theorem 5.14 in the finite range case only, but they indicated that generalizations of the type given here were possible. The relative entropy technique has been used in other contexts by Holley (1972c), Holley and Stroock (1981), Fritz (1982), and Liggett (1983a).

7. Open Problems

1. Is the conclusion of Theorem 5.14 true in dimensions greater than two?

2. Is the final statement in Theorem 2.16 true without the attractiveness assumption?

3. Is it true that any two uniformly positive stochastic Ising models with respect to the same potential are either both or neither exponentially ergodic in the uniform norm? The L_2 version of this statement is proved in Corollary 4.18. Holley (1985) has made some progress on this problem.

4. A spin system is a stochastic Ising model with respect to the potential $J_R \equiv 0$ if and only if $c(x, \eta_x) = c(x, \eta) > 0$ for all $x \in S$ and $\eta \in X$. The unique Gibbs state corresponding to this potential is the product measure ν with density $\frac{1}{2}$. Note that if $c(x, \eta_x) = c(x, \eta)$ and the process is attractive, then $c(x, \eta)$ is independent of η, so there are no interesting attractive examples in this class. In how great a generality can it be proved that $\mathscr{I} = \{\nu\}$? Some results in this direction are Corollary 4.21 and Theorem 5.14, as well as Theorem 1.12 of Chapter III.

5. Is it true that every translation invariant strictly positive spin system on Z^d with finite range has an invariant measure which is a Gibbs state? This is plausible in view of Proposition 1.10 and the fact that the strict positivity of the rates should imply that an invariant measure is somewhat smooth. Using techniques similar to those of Section 5, Künsch (1984b) has shown that if a translation invariant spin system on Z^d has an invariant measure which is translation invariant and Gibbs with respect to some potential, then every invariant measure which is translation invariant is again Gibbs with respect to the same potential. Thus an affirmative solution to this problem, together with some estimate on the size of the potential which would permit the application of Theorem 3.9, would give an affirmative solution to the positive rates conjecture for attractive spin systems (Problem 6 in Chapter III).

6. The proof of nonergodicity for appropriate stochastic Ising models is based on Theorem 3.14. Can a proof of nonergodicity be constructed which is based on a direct analysis of the time evolution itself rather than on its invariant measures? If so, that would provide an alternative proof of Theorem 3.14. More importantly, it would potentially generalize to spin systems which are not reversible.

7. In the context of Corollary 2.18, is it the case that

$$\lim_{t\to\infty} \nu_\rho S(t) = \bar{\nu} \quad \text{if } \rho > \tfrac{1}{2}, \quad \text{and}$$

$$\lim_{t\to\infty} \nu_\rho S(t) = \underline{\nu} \quad \text{if } \rho < \tfrac{1}{2},$$

where ν_ρ is the product measure on X with

$$\nu_\rho\{\eta\colon \eta(x) = 1\} = \rho \quad \text{for all } x \in Z^d?$$

What can be said about convergence for more general initial distributions?