Chapter 1

Definition of the Markov Chain

In this chapter, we begin by defining the state space and subsequently provide a characterization of this space as the union of edge-disjoint cycles, making use of Euler's theorem for multigraphs. Following this, we introduce two distinct types of simple square transformations and demonstrate their capability to systematically construct any state, starting from the empty state. Together with their inverses these transformations are then employed to define an irreducible and aperiodic Markov Chain, converging to our chosen invariant distribution.

1.1 The space of link configurations and pairings

Definition 1.1.1. Let $G = (V, E)$ be a simple graph. A **link configuration** is an edge function $m : E \to \mathbb{N}$ such that $\forall x \in V$ we have $\sum_{y \sim x} m_{x,y} \in 2\mathbb{N}$.

Given a graph G, we denote the set of all its link configurations as Σ_G. Our focus will be specifically on instances where G is a grid subset of $\mathbb{Z}^2$ with periodic or free boundary conditions. In both cases, the vertex set is defined as

$$V_n := \{(i, j) \mid i, j \in \{0, \dots, n-1\}\}$$

while the edge sets for the free and the periodic boundary condition are, respectively:

$$E_n^f := \Big\{\{x, y\} \mid x, y \in V_n,\ \|x - y\| = 1\Big\}$$
$$E_n^p := E_n^f \cup \Big\{\{(i, 0), (i, n-1)\},\ \{(0, j), (n-1, j)\} \mid i, j \in \{0, \dots, n-1\}\Big\}$$

It's natural to define a *square* s to be an ordered tuple $s = (e_1, e_2, e_3, e_4)$ where the $e_i s$ are edges in a cycle of length four. We will use the convention that e_1 is the top edge, e_2 the right, e_3 the bottom and e_4 the left. We will define s_v to be the set of its vertices.

We can identify any square s with one of its vertices, for example the top-right one, this implies that the number of squares is $|V_n| = n^2$.

We will call $B \subset V_n$ the set of vertices of degree less than 4 in (V_n, E_n^f).

Definition 1.1.2. A **multicolor link configuration** of N colors is a tuple $(m^1, \dots, m^N)$ of link configurations.

Note that the parity constraints are independent for each color. We will now simply refer to them as link configurations and call single-color configuration as monochromatic.

Given a link configuration m, we will define a binary relation on each vertex which connects links of the same color. TODO rewrite, less notation, simpler, we don't have links but only natural numbers on each edge!

Definition 1.1.3. A **pairing** for a link configuration m is a collection of symmetric binary relations at each vertex on its incident links, $\pi = (\pi_x)_{x\in V}$ such that for all l_i, l_j, l_k incident links of vertex x:

1. $\neg(l_i \pi_x l_i)$,
2. if $l_i \pi_x l_j$ then they are the same color,
3. for any l_i there exists an l_j such that $l_i \pi_x l_j$,
4. if $l_i \pi_x l_j$ then for any $l_k \neq l_j$ $\neg(l_i \pi_x l_k)$.

In words, there are no self-pairings; only pairings between links of the same color are permitted. Every link is paired (this is possible since we always have an even number of links of the same color incident at each vertex), and each link can be paired only with one other link.

We denote the set of all possible pairing configurations of a link configuration m as $\mathcal{P}(m)$. Our state space will be the set $\Omega := \{(m, \pi) \,:\, m \in \Sigma, \pi \in \mathcal{P}(m)\}$.

1.2 Edge-disjoint cycle representation

Consider monochromatic states. We can view them as a multigraph with even degree on all vertices. By the Euler's theorem adapted for multigraphs, we can represent G as a union of edged disjoint cycles.

Lemma 1.2.1. Let $G = (V, E)$ be a multigraph with non-empty edge set, such that $\forall x \in V$ $d(x) \in 2\mathbb{N}$. Then G contains a cycle.

Proof. Since $|E| > 0$ there exsits a vertex $v \in V$ such that $d(v) \geq 2$. Starting from this vertex choose any edge, call it v, x_1, since $d(x_1)$ is even, there exists at least another edge unvisited edge. Keep going, since V is a finite set after enough steps we encounter an already visited vertex: we have found a cycle. ☐

Proposition 1.2.1. If $G = (V, E)$ is a multigraph in which $\forall v \in V$ $\deg(v) \in 2\mathbb{N}$ then the edge set E can be partitioned into edge-disjoint cycles.

Proof. We prove it by strong induction on the number of cycles. The base case is a multigraph with $|E| = 0$. Such a graph consists of one or more isolated vertices and the (empty) edge set can clearly be partitioned into a union of zero cycles.

Now suppose the result is true for every multigraph $G = (V, E)$ with $|E| \leq m$ edges whose vertices all have even degree. Consider a multigraph with $|E| = m + 1$: . From Lemma 1.2.1 we know that there is at least one cycle $C = (\mathcal{V}, \mathcal{E})$ contained in G. Then we can form a new graph $G' = (V, E')$ by removing the edges that appear in the cycle $E' := E \setminus \mathcal{E}$. Every vertex in the cycle has its degree reduced by two, vertices that didn't appear in the cycle maintain the same degree: parity is preserved. Since we removed at least two edges (the shortest cycle possible in a multigraph) $|E'| \leq m$ so that by the induction hypothesis we can partition the edge set of G' as disjoint cycles $E' = \bigcup_{i=1}^{N} C_i$. Then adding C to the partition of E' we obtain a partition for the original edge set E. $\square$

1.3 Square transformations

We define a class of transformations that map a state m to a new state m' by modifying only the links within a designated square on the grid. We limit ourselves to these transformations due to their local nature. Since our goal is to simulate a Markov chain on Σ, the locality of square transformations is crucial to make the exploration of the state space computationally tractable.

Definition 1.3.1. Given a square $s = (e_1, e_2, e_3, e_4)$ and four integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with the same parity, we define the *square transformation*

$$X_{s,(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} : S \subseteq \Sigma \to \Sigma$$

that sends a state m to a new state m' defined by

$$m'_e = \begin{cases} m_{e_i} + \alpha_i & \text{if } e = e_i \\ m_e & \text{otherwise.} \end{cases}$$

The domain S of the square transformation is defined to ensure the absence of negative links after applying the transformation:

$$S := \{m \in \Sigma \mid m_{e_i} \geq \alpha_i, \text{ for } i = 1, 2, 3, 4\}$$

Each square transformation is characterized by the square it acts on and the four integers associated with it. We will ofter represent them using the square diagram in figure 1.1:

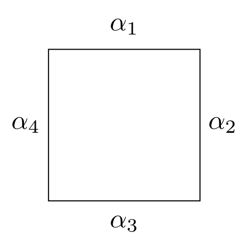


Figure 1.1: The diagram of the square transformation $X_{s,(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$.

The parity constraints on these integers ensure that the parity number of links incident at each vertex is preserved since in accordance with definition 1.1.1.

Given that our state is generally a multicolor link configuration, we require a distinct set of square transformations for each of the possible N colors, each acting only on a single color. When needed, we will explicitly indicate the color on which a transformation acts as X_s^c with $c = 1, \ldots, N$. A general square transformation will be denoted simply as X_s.

We define two special types of square transformations. The first type consists of a pair called *uniform* transformations $U_s^{\pm 1}$. The second type, called *single* transformations, includes a pair for each side of the square: $R_s^{\pm 2}$, $L_s^{\pm 2}$, $T_s^{\pm 2}$, and $B_s^{\pm 2}$, corresponding to *right,* *left, top,* and *bottom*, respectively. The associated integers for these transformations are presented in Table 1.1. In figure 1.8, the diagrams of uniform and single right transformations are drawn.

$U_s^{\pm 1}$	$(\pm 1, \pm 1, \pm 1, \pm 1)$
$T_s^{\pm 1}$	$(\pm 2, 0, 0, 0)$
$R_s^{\pm 1}$	$(0, \pm 2, 0, 0)$
$B_s^{\pm 1}$	$(0, 0, \pm 2, 0)$
$L_s^{\pm 1}$	$(0, 0, 0, \pm 2)$

Table 1.1: Associated integers of uniform and single square transformations.

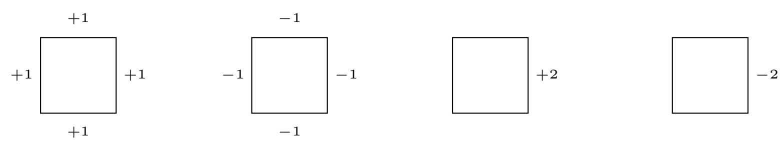


Figure 1.2: Diagrams of square transformations U^{+1}, U^{-1}, R^{+2} and R^{-2}, from left to right.

The importance of these simple transformations resides in the following proposition:

Proposition 1.3.1. All square transformations can be obtained by composing single and uniform transformations.

Proof. Let X_s denote any square transformation, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be its associated integers.

Choose the biggest integer, say α_1. Begin with the uniform transformation $U_s^{\alpha_1}$, obtained by applying the uniform transformation with the sign of α_1 $|\alpha_1|$ times. Since from the parity constraints follows that $\alpha_2 - \alpha_1 \equiv 0 \mod 2$, it makes sense to apply the right single -2 transformation a number $|\alpha_2 - \alpha_1|$ of times, denote this by $R_s^{(\alpha_2 - \alpha_1)}$. Similarly, apply left and bottom single transformations $L_s^{(\alpha_4 - \alpha_1)}$ and $B_s^{(\alpha_3 - \alpha_1)}$ respectively. We found the decomposition:

$$X_s = R_s^{(\alpha_2 - \alpha_1)} \circ B_s^{(\alpha_3 - \alpha_1)} \circ L_s^{(\alpha_4 - \alpha_1)} \circ U_s^{\alpha_1}$$

since the associated integers match. $\square$

1.4 State construction

We will prove that it is possible to construct any link configuration starting from the empty state (zero links on all edges) applying only a sequence of uniform $+1$ and the four single -2 transformations, in the case of free boundary conditions.

We will also prove that this task is impossible if we consider periodic boundary conditions: this is obvious when we our grid has odd length sides, since we can find states with an odd number of links, but square transformation preserves the parity of the total number of links in a state. The general case will follow from the topological properties of the torus.

Proposition 1.4.1. Any state $m \in \Sigma_G$, where $G = (V_n, E_n^f)$ is a box subset of $\mathbb{Z}^2$ with free boundary conditions, can be constructed from the empty state $0 \in \Sigma_G$ just by using the uniform $+1$ and the four single -2 trasformations.

Proof. Since we can work with any color independently, we consider a monochromatic state without loss of generality.

Since our state has even degree on all vertices of the lattice, we can apply proposition 1.2.1 on the multigraph obtained by identifying m_e as the number of edges of a multigraph. Call the edge-disjoint cycle partition of the edges $\{C_i\}_{i\in[N]}$, where N is the number of cycles. Since the cycles are edge disjoint, we can construct them separately, thus we only need to prove that we can build a single cycle[1].

Constructing a cycle starting from the empty state is straightforward: the *interior*[2] $\mathrm{Int}(C)$ of a cycle C is defined as a subset of squares such that:

$$\mathrm{Int}(C) := \{s \mid \text{ any path from } s_v \text{ to } B \text{ contains a vertex of } C\}$$

Recall that s_v is the set of vertices of square s and B is the set of boundary vertices of the graph.

If the cycle has more than 2 edges (the cycle is a simple graph) apply the uniform $+1$ to every square in $\mathrm{Int}(C)$ (which is at least one square). Subsequently, eliminate the undesired double edges with the single link -2 transformations.

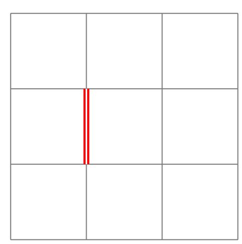


Figure 1.3: A two edge cycle with empty interior.

If the cycle has 2 edges (in this case the interior is empty), we can construct it by applying the uniform $+1$ transformation twice and then applying single -2 to eliminate the three undesired edges. □

Remark 1.4.1. Clearly, with the inverse transformations uniform -1 and single link $+2$, we can reach the empty state starting from any state.

Example 1.4.1. Suppose we want to build the red cycle in the left picture of figure 1.4.

1. Apply the uniform $+1$ mask on all squares on the interior of the cycle.

2. Use the -2 single link to remove the double edges inside.

[1]This argument can be made precise through induction on N.
[2]Here the assumption of free boundary conditions is necessary.

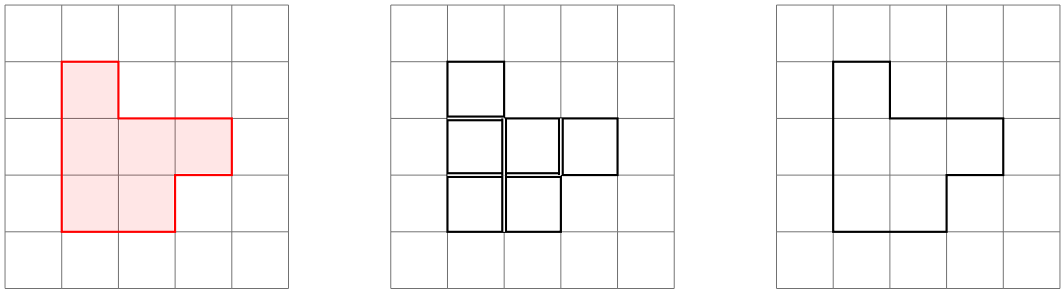


Figure 1.4: How to build any cycle with a well defined interior.

This procedure works on the plane, but not on the torus! In this case, the boundary B is empty, and we can't define the interior/exterior of a closed curve as in the proof of Proposition 1.4.1. Still, there are some simple closed curves that separate the torus into two regions, both of which have the curve as a boundary. Clearly, we can build these curves using the same procedure applied to any of these two regions.

However, this does not work in general. In fact, there exist closed curves that don't separate the torus into two regions, hence we can't apply the same procedure. We call these curves *non-separating*, see Figure 1.5.

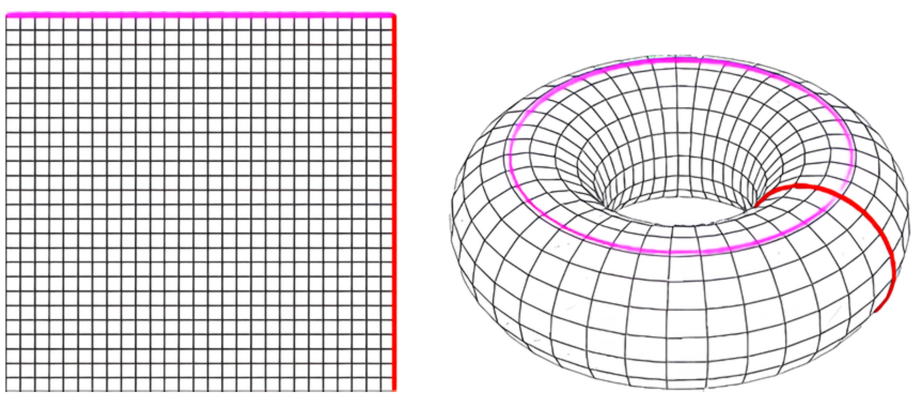


Figure 1.5: In pink an *horizontal non-separating* closed curve on the torus, in red a *vertical* one.

Being non-separating is a topological property, we will work with a characterization more suited for our setting.

We call an *horizontal loop* a simple closed curve on the torus that is parallel to horizontal links and does not intersect any vertex, like the curve in orange in figure 1.6. Their vertical counterpart is called *vertical loop*.

The characterization will exploit the fact that if a closed curve γ is separating, then the number of intersections with any horizontal and vertical loops l is even. We denote this number as $\mathrm{Cross}_l(\gamma)$.

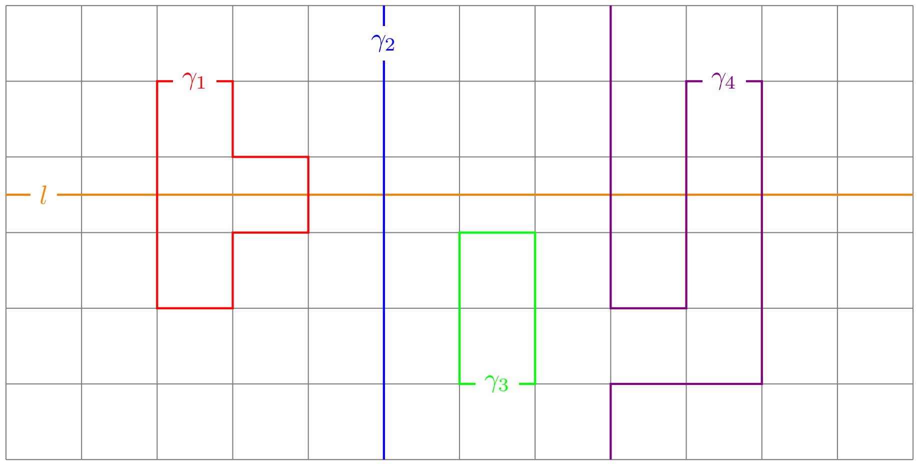


Figure 1.6: Four closed simple curves on the torus and an horizontal loop l in orange. Here $\text{Cross}_l(\gamma_1) = 2$, $\text{Cross}_l(\gamma_2) = 1$, $\text{Cross}_l(\gamma_3) = 0$ and $\text{Cross}_l(\gamma_4) = 3$.

Clearly if γ_1 and γ_2 are two edge-disjoint curves, then $\text{Cross}_l(\gamma \cup \eta) = \text{Cross}_l(\gamma) + \text{Cross}_l(\eta)$, so that any separating circuit, which can be written as the union of edge-disjoint cycles (which are closed simple curves), has even cross number.

We call a non-separating closed curve γ *vertical non-separating* if there exists an horizontal loop l such that $\text{Cross}_l(\gamma)$ is odd; we call it *horizontal non-separtating* if the same is true for a vertical loop l. The curves γ_2 and γ_4 in figure 1.6 are both vertical non-separating.

Proposition 1.4.2. A closed curve γ on the torus is non-separating if and only if we can find an horizontal/vertical loop l such that $\text{Cross}_l(\gamma)$ is odd.

Proof. We will prove that γ is separating if and only if $\text{Cross}_l(\gamma)$ is even for every horizontal/vertical loop l.

This can be proven using an argument similar to the one used in proving Jordan's curve theorem for polygons.

Assume γ is separating, call A and B the two disjoint regions of the torus such that $\partial A = \partial B = \gamma$. Choose a point x on any given any vertical/horizontal loop l which does not lie on γ. Then either $x \in A$ or $X \in B$. Suppose wlog that $x \in A$, if $\text{Cross}_l(\gamma)$ were odd, then also $x \in B$, but $A \cap B = \emptyset$.

Suppose now that for any vertical/horizontal loop l $\text{Cross}_l(\gamma)$ is even. We can then define two distinct subsets as follows:

1. Choose a point x at the center of a square s and an horizontal loop l that intersect x. Assign all points within square s to the set A.

2. Consider another square s' intersecting with l, and denote its center as x'. Divide loop l into two curves l_1 from x to x' and l_2 from x' to x. Since $\text{Cross}_l(\gamma)$ is even, the intersection of curves l_1 and l_2 with γ has the same parity. If it's even, assign all points in square s' to set A, otherwise to B.

3. Apply this procedure to the centers of the squares already assigned, using vertical loops to assign the remaining squares.

It's easy to see that the sets $A \setminus \gamma$ and $B \setminus \gamma$ have the desired properties, hence γ is separating. $\square$

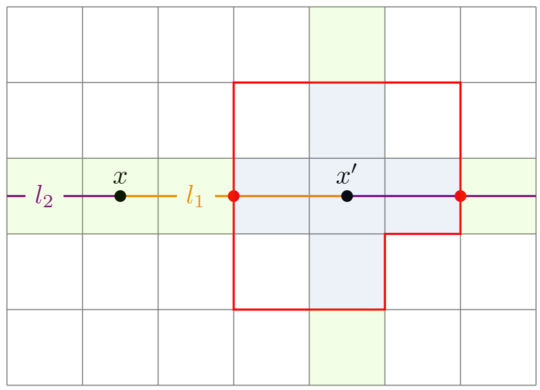


Figure 1.7: The procedure to define the subsets A and B discussed in the proof of proposition 1.4.2. The number of intersection of both l_1 and l_2 is odd.

Proposition 1.4.3. Let $m \in \Sigma_G$ with $G = (V_n, E_n^p)$ be a state consisting of only separating curves. If given a square transformation X_s the new state $m' = X_s(m)$ contains one non-separating curve γ, then it contains two.

Proof. We need to consider only transformation that add more links, and since by proposition 1.3.1 any square transformations can be written using uniform ± 1 and single ± 2, we are left with only uniform $+1$ and single $+2$.

If the new state $m' = X_s(m)$ contains a non-separating curve γ, then one or two of its edges was introduced by the transformation X_s. This means that in the original state m, all the vertices in γ were already in the same connected component H (a circuit is at least 2-edge-connected), then by Euler's theorem for multigraphs we can find in H a circuit C which visits every vertex γ. If we remove the edges of any path P joining any two vertices in H, then $H \setminus P$ contains a path visiting all edges (an eulerian path) joining the same vertices. We will employ this fact to build another non-separating curve.

We need to check two cases:

- *X_s is a single $+2$.* Call e_1 and e_2 the two new added links with endpoints x and y. Call γ the non-separating curve, which we can write as the union of a path P_{xy} from x to y contained in the circuit H and one of the new edges e_1:

  $$\gamma = P_{xy} \cup \{e_1\}$$

  Call $\tilde{P}_{xy}$ the eulerian path joining x to y contained in $H \setminus P_{xy}$, then the closed curve $\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_2\}$ is also non-separating, since given an horizontal/vertical loop l:

  $$\mathrm{Cross}_l(\tilde{\gamma}) = \mathrm{Cross}_l(C) - \mathrm{Cross}_l(\gamma)$$

  where $\mathrm{Cross}_l(C)$ is even, since it's a circuit contained in the original state m, and $\mathrm{Cross}_l(\gamma)$ is odd since γ is non-separating, this implies that $Cross(\tilde{\gamma})$ is also odd.

- X_s *is a uniform* $+1$. Call e_1, e_2, e_3 and e_4 the new added edges, x and y the two vertices on γ that also are affected by the transformation. Again we can write:

$$\gamma = P_{xy} \cup \{e_1, e_2\}$$
$$\tilde{\gamma} = \tilde{P}_{xy} \cup \{e_3, e_4\}$$

the same parity of crossings argument applies here for $\tilde{\gamma}$.

□

A straightforward consequence of this proposition is that we cannot reach a state with an odd number of non-separating curves starting from the empty state, as they can only be produced in pairs. More generally:

Corollary 1.4.1. Each parity number of non-separating curves (vertical and horizontal) is invariant with respect to square transformations.

To overcame this problem we are forced to introduce new non square transformations that don't preserve the parity of non-separating curves, such as:

$$V_k^{+1}(m)_e := \begin{cases} m_e + 1 & \text{if } e \text{ connects two vertices on the vertical line } x = k \\ m_e & \text{otherwise} \end{cases}$$

$$H_k^{+1}(m)_e := \begin{cases} m_e + 1 & \text{if } e \text{ connects two vertices on the horizontal line } y = k \\ m_e & \text{otherwise} \end{cases}$$

Notice how we don't need their inverses, since we can apply them twice and then use the single -2 to remove all links. The downside is that they affect all the edges on a given vertical/horizontal line: they are not local, in the sense that the number of affected edges depends linearly on the size of the grid.

1.5 Transition probabilities

We define a probability measure (1.1) on the set Ω, where each element is a link configuration with a compatible pairing, $\omega = (m, \pi)$.

$$\mathbb{P}(\omega) = \frac{1}{Z} \prod_{e \in E} \prod_{i=1}^{N} \frac{\beta^{m_e^i}}{m_e^i!} \prod_{x \in V} \frac{\Gamma(\frac{N}{2})}{2^{n_x(m)} \Gamma(\frac{N}{2} + n_x(m))} = \frac{1}{Z} \mu(m) \tag{1.1}$$

Where Z is a normalization constant, $\beta \in \mathbb{R}_{\geq 0}$, N is the number of colors and $n_x(m) := \frac{1}{2} \sum_{i=1}^{N} \sum_{y \sim x} m_{x,y}^i$ which we refer to as the *local time.*

Since it is independent of the pairing configuration, the marginal probability of the link configuration is simply

$$\rho(m) = \sum_{\pi \in \mathcal{P}(m)} \mathbb{P}(m, \pi) = \frac{1}{Z} \mu(m) |\mathcal{P}(m)| \tag{1.2}$$

where the number of possible pairings configuration is

$$|\mathcal{P}(m)| = \prod_{x\in V}\prod_{i=1}^{N}(2n_x(m)-1)!!$$

To sample from the probability measure (1.1) we first sample from the marginal measure (1.2) and then uniformly sample a pairing configuration.

Definition 1.5.1. We say that states $m, m' \in \Sigma$ are neighbours and write $m \sim m'$ if we can find a square transformation X_s^c such that $X_s^c(m) = m'$.

From a starting state $m \in \Sigma$, we attempt to transition to a neighboring state m' by applying one of the possible square transformations for a randomly chosen color c, which acts on a square s, both chosen uniformly at random.

The jump probability from state m to state m' is defined as

$$q(m' \mid m) = \begin{cases} \frac{1}{N}\frac{1}{|V|}\frac{1}{\mathcal{M}_{s,c}(m)} & \text{if } m' = X_s^c(m) \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{M}_{s,c}(m)$ is the number of possibile transformation of color $c = 1, \ldots, N$ that can be applied to square s with link configuration m, and $|V|$ is the total number of squares in the grid (which is equal to the number of vertices).

We define the transition probability

$$P(m \to m') = q(m' \mid m)A(m', m)$$

where $A(m', m)$ is called the *acceptance probability* from state m to state m'. Imposing the detailed balance equations, we find the condition

$$\frac{A(m', m)}{A(m, m')} = \frac{q(m \mid m')}{q(m' \mid m)}\frac{\rho(m')}{\rho(m)} = \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')}\frac{\rho(m')}{\rho(m)} \tag{1.3}$$

A common choice which satisfies condition (1.3) is the Metropolis-Hastings acceptance probability:

$$A(m', m) = \min\left(1, \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')}\frac{\rho(m')}{\rho(m)}\right) \tag{1.4}$$

Another possible choice is the Glauber acceptance probability:

$$A(m', m) = \left(1 + \frac{\mathcal{M}_{s,c}(m')}{\mathcal{M}_{s,c}(m)}\frac{\rho(m)}{\rho(m')}\right)^{-1}$$

Since m and m' only differ by a transformation in one square s, we can compute efficiently the ratios $\rho(m')/\rho(m)$.

Putting it all together, in the case of Metropolis acceptance probability the transition probabilities are:

$$P(m \to m') = \begin{cases} \frac{1}{N}\frac{1}{|V|}\frac{1}{\mathcal{M}_{s,c}(m)}\min\left(1, \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')}\frac{\rho(m')}{\rho(m)}\right) & \text{if } m' \sim m \\ \sum_{m''\sim m}\frac{1}{N}\frac{1}{|V|}\frac{1}{\mathcal{M}_{s,c}(m)}\left[1 - \min\left(1, \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m'')}\frac{\rho(m'')}{\rho(m)}\right)\right] & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

The probability $P(m \to m)$ is defined to ensure the transition matrix is stochastic:

$$\sum_{m' \in \Sigma} P(m \to m') = \frac{1}{N}\frac{1}{|V|}\sum_{m' \sim m}\frac{1}{\mathcal{M}_{s,c}(m)} \tag{1.6}$$

$$= \frac{1}{N}\frac{1}{|V|}\sum_{s}\sum_{m'=X_s(m)}\frac{1}{\mathcal{M}_{s,c}(m)} \tag{1.7}$$

$$= \frac{1}{N}\frac{1}{|V|}\sum_{s} N = 1 \tag{1.8}$$

We can compute the explicit the probability rations used in the acceptance probability for each of our possibile neighbours.

Transition	**Probability Ratio** $\rho(m')/\rho(m)$
$m' = U_s^{+1}(m)$	$\frac{\beta^4}{16}\prod_{e \in s}(m_e^i + 1)^{-1}\prod_{x \in s_v}\left(\frac{N}{2} + n_x\right)^{-1}$
$m' = U_s^{-1}(m)$	$\frac{16}{\beta^4}\prod_{e \in s} m_e^i \prod_{x \in s_v}\left(\frac{N}{2} + n_x - 1\right)$
$m' = R_s^{+2}(m)$	$\frac{\beta^2}{4(m^i_{\{x,y\}} + 2)(m^i_{\{x,y\}} + 1)}\left(\frac{N}{2} + n_x\right)^{-1}\left(\frac{N}{2} + n_y\right)^{-1}$
$m' = R_s^{-2}(m)$	$\frac{4m^i_{\{x,y\}}(m^i_{\{x,y\}} - 1)}{\beta^2}\left(\frac{N}{2} + n_x - 1\right)\left(\frac{N}{2} + n_y - 1\right)$

Table 1.2: Probability ratios for various transitions.

To improve convergence to equilibrium, in particular for low β, we introduce a set of additional square transformation. Note that all of these can be obtained by using the previous ones, we do it only for convergence purposes.

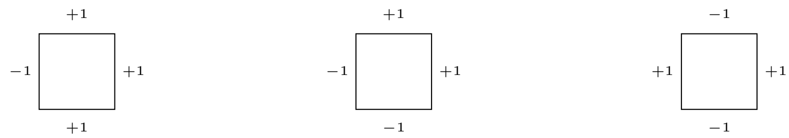


Figure 1.8: The square transformations *triple*, *swap-a* and *swap-o* from left to right.

1.6 Irreducibiliy

The key ingredient is Proposition 1.4.1, which asserts that with free boundary conditions any link configuration can be constructed starting from the empty configuration by applying only the uniform $+1$ and single link -2 transformations. If we also include their inverses, uniform -1 and single link $+2$, we can reach the empty configuration starting from any state.

Remark 1.6.1. If m and m' are neighbours, then the transition probabilities $P(m \to m')$ and $P(m' \to m)$ are strictly positive. This follows from the definition of the acceptance probability and the fact that the probability measure (1.2) is striclty positive on all states.

Proposition 1.6.1. The Markov Chain with state space Σ_G with $G = (V_n, E_n^f)$ and transition probabilities (1.5) is irreducibile.

Proof. Let $m, m' \in \Sigma$ be any states. According to Proposition 1.4.1, there exists a sequence of transformations allowing us to construct state m' from the empty state 0. This sequence has positive probability since it is the product of strictly positive terms, as stated in Remark 1.6.1. Consequently, we have proven that $P(0 \to m') > 0$.
Moreover, by Remark 1.4.1 and the same reasoning, we also have $P(m \to 0) > 0$, which implies that $P(m \to m') > 0$.

□

1.7 Aperiodicity

Since the chain is irreducible, we only need to prove that a state has period one. By our choice, if we can always find a state m and one of its neighbors m' such that the acceptance probability $A(m, m')$ is strictly less than one, then one can deduce from (1.5) that $P(m \to m) > 0$, ensuring that the state m has period one.

Proposition 1.7.1. For any fixed $\beta > 0$ there exists a state $m \in \Sigma$ such that $A(m', m) < 1$ where m' is obtained by applying a single link $+2$ square trasnformation.

Proof. By the definition of acceptance probability (1.4), we need to check

$$\frac{\rho(m')}{\rho(m)} \frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')} < 1$$

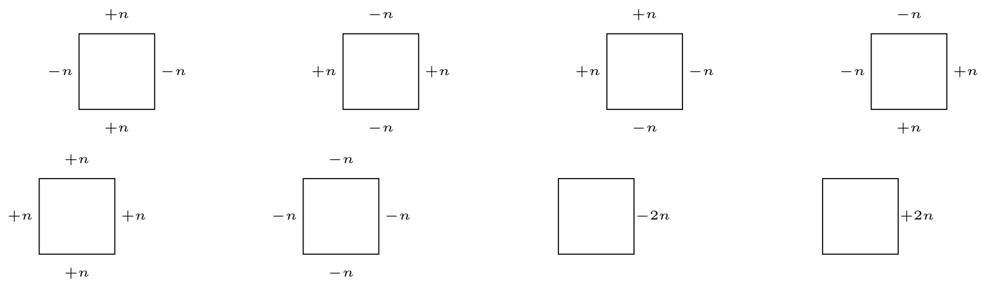
Since every transformation that we can apply at square s to state m can also be applied on state m', we have $\frac{\mathcal{M}_{s,c}(m)}{\mathcal{M}_{s,c}(m')} \leq 1$.

We can compute the probability ratio:

$$\frac{\rho(m')}{\rho(m)} = \frac{1}{4} \frac{\beta^2}{(m^i_{\{x,y\}}+2)(m^i_{\{x,y\}}+1)} \left(\frac{N}{2}+n_x\right)^{-1} \left(\frac{N}{2}+n_y\right)^{-1} \xrightarrow{m_{\{x,y\}}\to\infty} 0$$

so we can make the acceptance probability arbitrary small if we choose a state with enough links on a single edge.

□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1.8 Simulation

Euristic for exapling the $\beta^{1/2}$ dependence of average number of link vs β.

Try to maximize the probabilty, assume uniform number of link x.

$$\approx \frac{\beta^x}{x!}\Gamma \approx \left(\frac{\beta e^3}{2x^2}\right)^x$$

log is a concave function, thus is the same to maximize

$$x \log \beta e^3 - x \log 2x^2$$

which has stationary points

$$2x^2 = \beta e^3$$
$$x \sim \beta^{1/2}$$

1.9 useless stuff

Any cycle can be obtained as the boundary of the unions of square boxes:

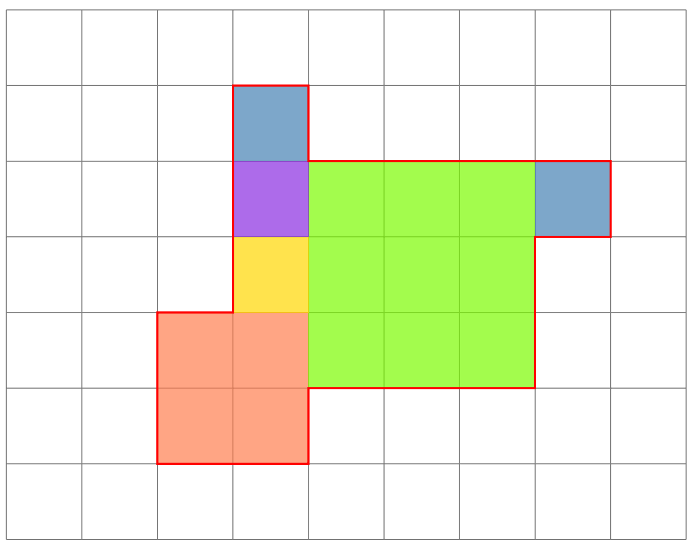

Figure 1.9: A cycle as the boundary of the union of squares.

Since any cycle Thus if we can build any $n \times n$ square we are done (there is the case with links on the same edge! easy just use mask $+2$, need to encompass this case better) Since evert internal double edge can be removes using the single link -2 mask.

We prove it using induction: The base case with 1×1 squares are trivial, just use the uniform $+1$ mask. Suppose we have a $(n-1) \times (n-1)$ square, then we can add to a side all $+1$ uniform masks, then -2 to each internal boundary and obtain a $n \times n$ square. $\square$

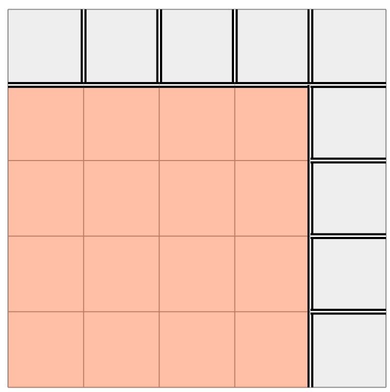

Figure 1.10: Add the missing part with the uniform $+1$ mask, then remove the double internal edges with the single link -2.