Matroids and combinatorial optimization

Combinatoria 2022/2023

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Table of Contents

Combinatorial optimization

Many combinatorial optimization problems can be formulated as follows:

- a set system $(E, \mathcal{F})$, i.e. a finite set E and some $\mathcal{F} \subseteq 2^E$;
- a cost function $c : \mathcal{F} \mapsto \mathbb{R}$;

Find an element of $\mathcal{F}$ whose cost is minimum or maximum.

Goal

We will introduce **independence systems** and **matroids** and show that many combinatorial optimization problems can be described in this context.
The main reason why matroids are important is that a **simple greedy algorithm can be used for optimization over matroids**.

Table of Contents

2 Indipendence systems and matroids

Definition (Indipendence system)

A set system $(E, \mathcal{F})$ is an **indipendence system** if and only if:

- $(M1)$ $\emptyset \in \mathcal{F}$;
- $(M2)$ if $X \subseteq Y$, $Y \in \mathcal{F}$ then $X \in \mathcal{F}$.

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The elements of $\mathcal{F}$ are called *independent*, the elements of $2^E \setminus \mathcal{F}$ *dependent*. Maximal independent sets are called **bases**. For $X \subseteq E$, the maximal independent subsets of X are called bases of X.

Example

Consider any finite subset E of a vector space V. We can define an independence system by choosing $\mathcal{F} = \{Y \in 2^E \ : \ Y \text{ are linearly independent } \}$.
This set system clearly satisfies $(M1)$ (there is no way to obatin $\vec{0}$ from a linear combination of an empty set of vectors) and $(M2)$.
Keeping in mind this example, we introduce some definitions.

Definition (Rank)

Let $(E, \mathcal{F})$ be an independence system. For $X \subseteq E$ we define the rank of X by:

$$r(X) := \max\{|Y| : Y \subseteq X,\ Y \in \mathcal{F}\}.$$

The rank of a finite collection of vectors X is just the size of any basis for X. Does this hold for any indipendence system? No.

Counter example

Consider the indipendence set system $(E, \mathcal{F})$ where $E = \{1, 2, 3\}$ and
$\mathcal{F} = \Big\{\emptyset, \{1, 2\}, \{1\}, \{2\}, \{3\}\Big\}$
We have two basis for E with different cardinality!
This suggests that subsets of linearly indipendent vectors have another property, in addition to $(M1)$ and $(M2)$. This is the *augmentation property*: Given two finite subsets of linearly indipendent vectors X and Y such that $|X| < |Y|$, we can find some vectors of Y to add to X such that they have the same rank.

Definition (Matroid)

An indipendence system is a **matroid** if and only if:

- $(M3)$ if $X, Y \in \mathcal{F}$, $|X| < |Y|$ then $\exists y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{F}$.

Introduced in a paper called *"On the Abstract Properties of Linear Dependence"* by Hassler Whitney in 1935.

Equivalent axiom

This new axiom captures the propriety we were looking for:

Theorem

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Proof

$\implies$ Suppose a subset $X \subseteq E$ has two bases of different size $|B_1| > |B_2|$, we can add elements of the bigger base to the small and still be an indipendent set, but B_2 was choosen to be maximal.

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$\impliedby$ Let $X, Y \in \mathcal{F}$ such that $|X| > |Y|$. This means Y can't be a basis for $Y \cup X$. This means there must be an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$. $\square$

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Theorem (Bases characterization)

Let E be a finite set and $\mathcal{B} \subseteq 2^E$. $\mathcal{B}$ is the set of bases of some matroid $(E, \mathcal{F})$ if and only if:

- $(B1)$ $\mathcal{B} \neq \emptyset$.
- $(B2)$ *For any* $A, B \in \mathcal{B}$ *and* $a \in A \setminus B$ *there exists* $b \in B \setminus A$ *such that* $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

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$(B2)$ is called the *base exchange property*.

Proof $\implies$ The set of bases of a matroid satisfies $(B1)$ by $(M1)$, and $(B2)$: the set $A \setminus \{a\}$ is still indipendent for $(M2)$, so by $(M3)$ we can find an element in $B \setminus A$ to create a new basis.
$\impliedby$ Combinatorial Optimization. Theory and Algorithms (Bernhard Korte, Jens Vygen), theorem 13.9, page 337. $\square$

Properties of the rank function

Theorem

Let $\mathcal{M}$ be a matroid on a finite set E. The rank function r of $\mathcal{M}$ has the following properties:

- $(R1)$ $0 \leq r(X) \leq |X|$ *for all* $X \subseteq E$.
- $(R2)$ *If* $X \subseteq Y \subseteq E$ *then* $r(X) \leq r(Y)$.
- $(R3)$ *For every* $X, Y \subseteq E$ $r(X \cup Y) + r(X \cap Y \leq r(X) + r(Y)$.

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Proof $(R1)$ and $(R2)$ follow from the definition, they are satisfied in any indipendence system.

Given a finite set E and a function $r : 2^E \mapsto \mathbb{N}$, we can define a matroid in the following way:

$$\mathcal{F} := \{F \subseteq E \ : \ r(F) = |F|\}$$

Corollary

Let E be a finite set. A function $r : 2^E \mapsto \mathbb{N}$ is the rank function of a matroid on E if and only if r satisfies properties $(R1)$, $(R2)$ and $(R3)$.

Proof Combinatorial Optimization. Theory and Algorithms (Bernhard Korte, Jens Vygen), Page 338. □

Rank quotient

We introduce another definition of rank of subsets for indipendent set system, the size of the smallest base contained:

Definition (Lower rank)

Let $(E, \mathcal{F})$ be an independence system. For $X \subseteq E$ we define the lower rank by:

$$\rho(X) := \min\{|Y| : Y \subseteq X,\ Y \in \mathcal{F} \text{ and } Y \cup \{x\} \notin \mathcal{F}\ \forall x \in X \setminus Y\}.$$

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Definition (Rank quotient)

The rank quotient of $(E, \mathcal{F})$ is defined by:

$$q(E, \mathcal{F}) := \min_{X \subseteq E} \frac{\rho(X)}{r(X)}$$

It follows from the definition that $q(E, \mathcal{F}) \leq 1$, and an indipendence system is a matroid if and only if $q(E, \mathcal{F}) = 1$, since this is equivalent to the statment that every bases for any $X \subseteq E$ has the same cardinality.

$(E, \mathcal{F})$ **is a Matroid** $\iff q(E, \mathcal{F}) = 1$

Table of Contents

Maximization and minimization problems for IS

Definition (Maximization problem)

Istance: A set system $(E, \mathcal{F})$ and a cost function $c : E \mapsto \mathbb{R}$.
Task: Find $X \in \mathcal{F}$ such that $c(X) := \sum_{e \in X} c(e)$ is maximum.

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Definition (Minimization problem)

Istance: A set system $(E, \mathcal{F})$ and a cost function $c : E \mapsto \mathbb{R}$.
Task: Find a basis B such that $c(B) := \sum_{e \in B} c(e)$ is minimum.

Note: $c(\emptyset) = 0$.

There is a natural duality between minimization and maximization problems on indipendence systems: finding a minimum cost basis B in $(E, \mathcal{F})$ is the same as finding the maximum cost complement $E \setminus B$. This suggests the following definition:

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Let $(E, \mathcal{F})$ be an indipendence system. We define its **dual** $(E, \mathcal{F}^*)$ so that the bases in $\mathcal{F}^*$ are exaclty the complements of the bases of $\mathcal{F}$.

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It's clear that $(E, \mathcal{F}^{**}) = (E, \mathcal{F})$.

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So which sets are in $\mathcal{F}^*$? Every subset of the dual basis $\mathcal{B}^*$, this means:

$$\mathcal{F}^* = \{F \subseteq E \ : \ \exists B \in \mathcal{B} \text{ s.t. } B \cap F = \emptyset\}$$

Properties of the dual

Theorem

Let $(E, \mathcal{F})$ be an indipendence system, $(E, \mathcal{F}^*)$ its dual, and let r and r^* be the corresponding rank functions.

1. *$(E, \mathcal{F})$ is a matroid if and only if $(E, \mathcal{F}^*)$ is a matroid.*
2. *If $(E, \mathcal{F})$ is a matroid, then $r^*(E) = |F| + r(E \setminus F) - r(E)$ for $F \subseteq E$.*

Proof We only give a proof of 2. The idea to prove 1. is via the rank function characterization: verify that r^* satisfies $(R1)$, $(R2)$ and $(R3)$.

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$$r^*(F) = \max_{I \subseteq F,\, I \in \mathcal{F}^*} |I|$$

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- *Maximum weight stable set problem:* given a graph G and weights $c : V(G) \mapsto \mathbb{R}$, find a stable set $X \subseteq V(G)$ of maximum weight. Here $E = V(G)$ and $\mathcal{F} = \{F \subseteq E : F \text{ is stable in } G\}$.

Examples

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- *TSP:* given a complete undirected graph G and weights $c : E(G) \mapsto \mathbb{R}^+$, find a minimum weight Hamiltonian circuit in G. Here $\mathcal{F} = \{F \subseteq E : F \text{ is a subset of edges in an Hamiltonian circuit in } G\}$.

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3 Maximization and minimization problems

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- *MST:* given a connected undirected graph G and weights $c : E(G) \mapsto \mathbb{R}$, find a minimum weight spanning tree in G. Here $E = E(G)$ and the elements of $\mathcal{F}$ are the edge sets of the forests in G.

The first two problems are NP-hard, while the last one is in P. The indipendence sytems of the first two problems are not matroids, while the last one is! This isn't a coincidence, we will show that simple (polynomial) algorithms solve optimization problems on matroids.

The IS of MST is a matroid

Proposition

The set system $(E, \mathcal{F})$, where E is the set of edges of some undirected graph G and $\mathcal{F} := \{F \subseteq E \ : \ (V(G), F) \text{ is a forest } \}$ is a matroid.

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Proof It is obvious that is an independence sytem, it remains to show that $(M3)$ holds. We will prove its contrapositive.Let $X, Y \in \mathcal{F}$, and suppose $Y \cup \{x\} \notin \mathcal{F}$ for all $x \in X \setminus Y$. We show $|X| \leq |Y|$. For each $x = \{u, v\} \in X$, u and v are in the same connected component of $(V(G), Y)$. Hence each connected component $Z \subseteq (V(G), X)$ is a subset of a connected component of $(V(G), Y)$. So the number of connected component p of the forest $(V(G), X)$ is greater or equal to the number q of connected components of the forest $(V(G), Y)$. But $|V(G)| - |X| = p \geq q = |V(G)| - |Y|$, implying $|X| \leq |Y|$. $\square$

Table of Contents

4 Greedy algorithms

Again, let $(E, \mathcal{F})$ be an independence system and $c : E \mapsto \mathbb{R}^+$ (non-negative weights). We consider the **Maximization problem** for $(E, \mathcal{F}, c)$ and formulate two “greedy” algorithms.

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The first one, called **Best-In** builds the solution choosing at each step the best possible element.

Greedy algorithms

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The second, called **Worst-Out** at each step removes the worst element, until a feasible solution is atteined.

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The first one, called **Best-In** builds the solution choosing at each step the best possible element.
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They boath employ the use of an oracle to decide if a given subset $X \subseteq E$ is feasible.

The best-in greedy algorithm for maximization

Best-In-Greedy algorithm

Input: An independence system $(E, \mathcal{F})$, given by an independence oracle, weights $c : E \mapsto \mathbb{R}^+$.

Output: A set $F \in \mathcal{F}$.

1. Sort E such that $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_n)$.
2. Set $F \leftarrow \emptyset$.
3. **For** i = 1 to n **do**:
   If $F \cup \{e_i\} \in \mathcal{F}$ **then** set $F \leftarrow F \cup \{e_i\}$

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The output F of Best-In-Greedy algorithm will be a basis for E.

The second algorithm requires a more complicated oracle. Given a set $F \subseteq E$, this oracle decides whether F contains a basis. Let us call such an oracle a basis-superset oracle.

Worst-out-Greedy algorithm

Input: An independence system $(E, \mathcal{F})$, given by a basis-superset oracle, weights $c : E \mapsto \mathbb{R}^+$.

Output: A basis $F \in \mathcal{F}$.

1. Sort E such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_n)$.
2. Set $F \leftarrow E$.
3. **For** i = 1 to n **do**:
   If $F \setminus \{e_i\}$ contains a basis **then** set $F \leftarrow F \setminus \{e_i\}$

Greedy algorithms for minimization

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Quality of the solution

Theorem (Jenkyns 1976, Korte and Hausmann 1978)

Let $(E, \mathcal{F})$ be an independence system. For $c : E \mapsto \mathbb{R}^+$ we denote by $G(E, \mathcal{F}, c)$ the cost of some solution found by the Best-In-Greedy for the Minimization problem, and by $OPT(E, \mathcal{F}, c)$ the cost of an optimum solution. Then

$$q(E, \mathcal{F}) \leq \frac{c(G)}{c(OPT)} \leq 1$$

for all $c : E \mapsto \mathbb{R}^+$. There is a cost function where the lower bound is attained.

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Proof Let $E = \{e_1, \ldots, e_n\}$ and $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_n)$. Let G_n be the solution found by Best-In-Greedy, while O_n is an optimum solution.

We define $E_j := \{e_1, \dots, e_j\}$, $G_j := G_n \cap E_j$ and $O_j := O_n \cap E_j$. Set $d_n := c(e_n)$ and $d_j := c(e_j) - c(e_{j+1})$ for $j = 1, \dots, n-1$.
Since $O_j \subseteq O_n$, $O_j \in \mathcal{F}$ by $(M2)$. We have $|O_j| \leq r(E_j)$.

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$$
\begin{aligned}
c(G_n) &= \sum_{j=1}^{n} (|G_j| - |G_{j-1}|) c(e_j) \\
&= \sum_{j=1}^{n} |G_j| d_j \\
&\geq \sum_{j=1}^{n} \rho(E_j) d_j
\end{aligned}
$$

$$
\begin{aligned}
&\geq q(E, \mathcal{F}) \sum_{j=1}^{n} r(E_j) d_j \\
&\geq q(E, \mathcal{F}) \sum_{j=1}^{n} |O_j| d_j \\
&= q(E, \mathcal{F}) \sum_{j=1}^{n} (|O_j| - |O_{j-1}|) c(e_j) \\
&= q(E, \mathcal{F}) c(O_n)
\end{aligned}
$$

this proves the inequality.

We now prove that the lower bound is tight. Choose $F \subseteq E$ and bases B_1, B_2 of F such that $\frac{|B_1|}{|B_2|} = q(E, \mathcal{F})$, define the cost function:

$$c(e) := \begin{cases} 1 \text{ for } e \in F \\ 0 \text{ for } e \in E \setminus F \end{cases}$$

If we sort $e_1, \ldots, e_n$ such that $B_1 = \{e_1, \ldots e_{|B_1|}\}$ the algorithm will find B_1 as a solution, and the lower bound is attained. $\square$

Rado-Edmonds theorem

Theorem (Rado 1957, Edmonds 1971)

An independence system $(E, \mathcal{F})$ is a matroid if and only if the BEST-IN-GREEDY finds an optimum solution for the MAXIMIZATION PROBLEM for $(E, \mathcal{F}, c)$ for all cost functions $c : E \mapsto \mathbb{R}^+$.

This is one of the rare cases where we can define a structure by its algorithmic behaviour.

Quality of the solution for the minimization problem

Theorem

Korte and Monma 1979 Let $(E, \mathcal{F})$ be an independence system. For $c : E \mapsto \mathbb{R}^+$ let $G(E, \mathcal{F}, c)$ denote a solution found by Worst-Out-Greedy for the Minimization problem. Then

$$1 \leq \frac{G(E, \mathcal{F}, c)}{OPT(E, \mathcal{F}, c)} \leq \max_{F \subseteq E} \frac{|F| - \rho^*(F)}{|F| - r^*(F)}$$

for all $c : E \mapsto \mathbb{R}^+$, where ρ^* and r^* are the rank functions of the dual indipendence system $(E, \mathcal{F}^*)$. There is a cost function where the upper bound is attained.

Proof We use the same notation as in the previous proof. At each step, the partial solution will is $(E \setminus E_j) \cup G_j$, by construction contains a basis of E. Its complement $E_j \setminus G_j$, like G_j in Best-In-Greedy is a basis for E_j in the dual $\mathcal{F}^*$, thus:

$$|E_j| - |G_J| \geq \rho^*(E_j)$$

Likewise, the complement of the solution $E_j \setminus O_j$ is indipendnent in $\mathcal{F}^*$, thus:

$$|E_J| - |O_J| \leq r^*(E_j)$$

We now use the same trick to compute the cost of the solution G_n:

$$
\begin{aligned}
c(G_n) &= \sum_{j=1}^{n} (|G_j| - |G_{j-1}|) c(e_j) \\
&= \sum_{j=1}^{n} |G_j| d_j \\
&\leq \sum_{j=1}^{n} |E_j| - \rho^*(E_j) d_j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(|E_j| - \rho^*(E_j)\right) \frac{|E_j| - r^*(E_j)}{|E_j| - r^*(E_j)} d_j \\
&\leq \max_{F \subseteq E} \frac{|F| - \rho^*(F)}{|F| - r^*(F)} \sum_{j=1}^{n} |O_j| d_j \qquad \square \\
&= \max_{F \subseteq E} \frac{|F| - \rho^*(F)}{|F| - r^*(F)} \sum_{j=1}^{n} (|O_j| - |O_{j-1}|) c(e_j) \\
&= \max_{F \subseteq E} \frac{|F| - \rho^*(F)}{|F| - r^*(F)} c(O_n)
\end{aligned}
$$

To prove that the upper bound is tight we can use the same strategy as in the Edmons-Rado theorem. $\square$

We have shown that using **Best-In-Greedy for maximization problems** on matroids will always find the best solution, likewise **Worst-Out-Greedy for minimization problems**.

Remarks

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If we apply the **Worst-Out-Greedy** to the maximization problem or the **Best-In-Greedy** to the minimization problems, **there is no positive lower/finite upper bound for** $\frac{G}{OPT}$.

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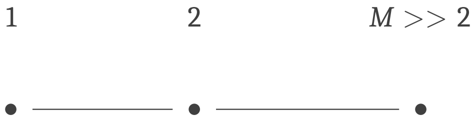
If we apply the **Worst-Out-Greedy** to the maximization problem or the **Best-In-Greedy** to the minimization problems, **there is no positive lower/finite upper bound for** $\frac{G}{OPT}$.
To see this try finding a minimal vertex cover of maximum weight or a maximal stable set of minimum weight in this simple graph:



However in the case of matroids, it does not matter whether we use the Best-In-Greedy or the Worst-Out-Greedy: since all bases have the same cardinality, the minimization problem for $(E, \mathcal{F}, c)$ is equivalent to the maximization problem for $(E, \mathcal{F}, c')$, where

$$
\begin{aligned}
c'(e) &:= M - c(e) \qquad \forall e \in E \\
M &:= 1 + \max\{c(e) \,:\, e \in E\}
\end{aligned}
$$

Therefore

Corollary

Kruskal's algorithm solves the MST problem optimally.

Matroids and combinatorial optimization

Thank you for listening!