Continuous Distributions

Continuous random variable: A random variable is continuous if it can assume any value in some interval or intervals of real numbers and *the probability that it assumes any specific value is 0*.

In discrete value case, $f(x) = P[X = x], x\ real$, i.e., the density function can be assigned as the probability that the RV X is taking a specific value x. This definition of density function can not be used in continuous case.

Continuous density: Let X be a continuous random variable. A function f such that

(1) $f(x) \geq 0$
(2) $\int_{-\infty}^{\infty} f(x)dx = 1$
(3) $P[a \leq x \leq b] = \int_a^b f(x)dx\ , for\ a\ and\ b\ real$

is called a density for X.

Can the value of $f(x) > 1$?

Continuous Distributions

Example: The density function of the lead concentration in gasoline is:

$$f(x) = \begin{cases} 12.5x - 1.25, & 0.1 \leq x \leq 0.5 \; grams \\ 0, & otherwise \end{cases}$$

Verify $f(x)$ satisfies the definition of density function and find the probability that the lead concentration in a randomly selected liter of gasoline will lie between 0.2 and 0.3 grams inclusive.

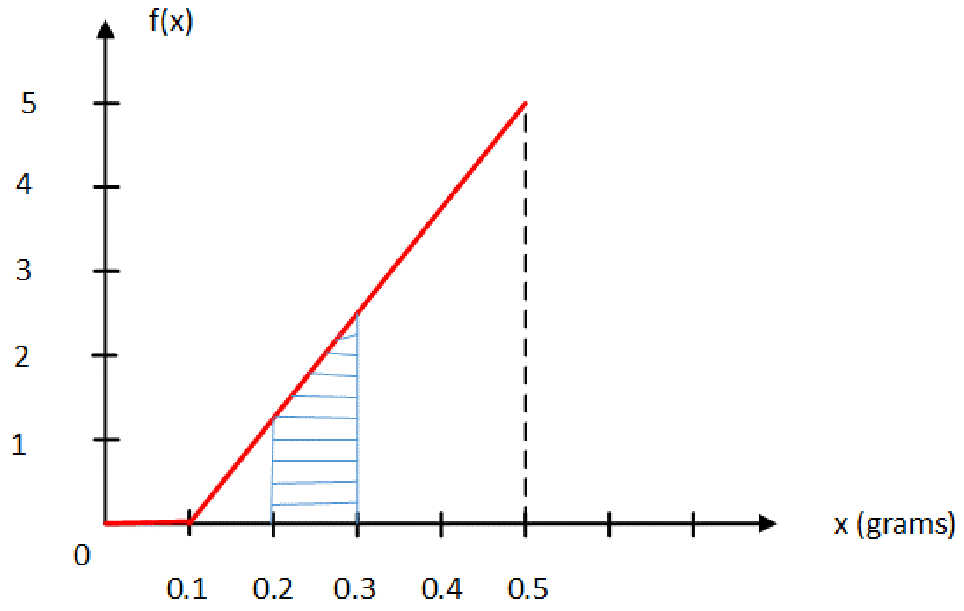
(1) I can be clearly seen from the graph that $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)dx = \int_{0.1}^{0.5}(12.5x - 1.25)dx = \left[\frac{12.5x^2}{2} - 1.25x\right]_{0.1}^{0.5} = 1$

Hence, $f(x)$ is a density function

$$P[0.2 \leq x \leq 0.3] = \int_{0.2}^{0.3} f(x)dx = \int_{0.2}^{0.3} (12.5x - 1.25)dx$$

$$= \left[\frac{12.5x^2}{2} - 1.25x\right]_{0.2}^{0.3} = 0.1875$$



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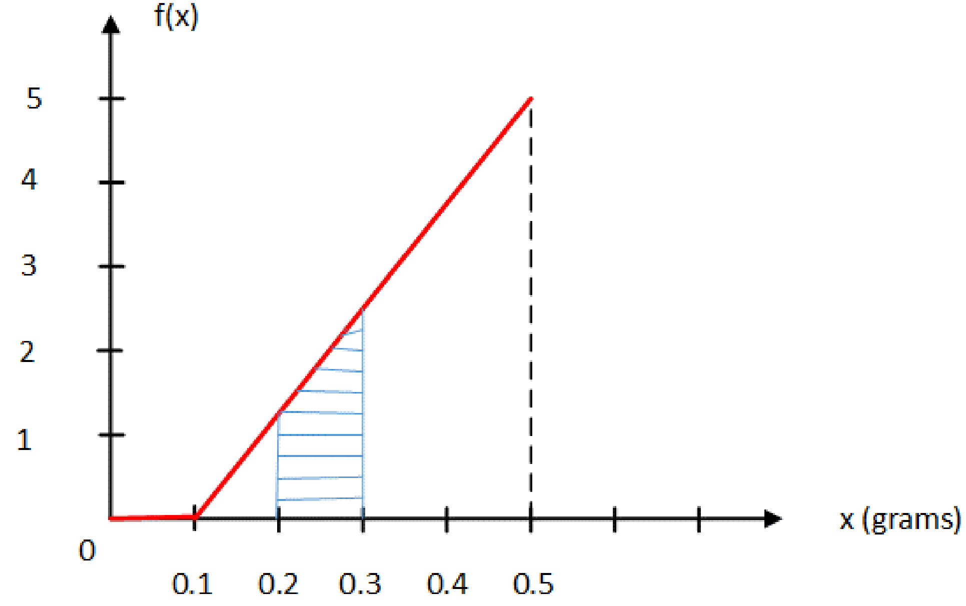
(1) In continuous case, we have,

$$P[a \leq x \leq b] = P[a \leq x < b] = P[a < x \leq b] = P[a < x < b]$$

But in discrete case, this is not true!!!

(2) The probability $P[a \leq x \leq b]$ is the area under the graph of density function f between $x = a \; and \; x = b$



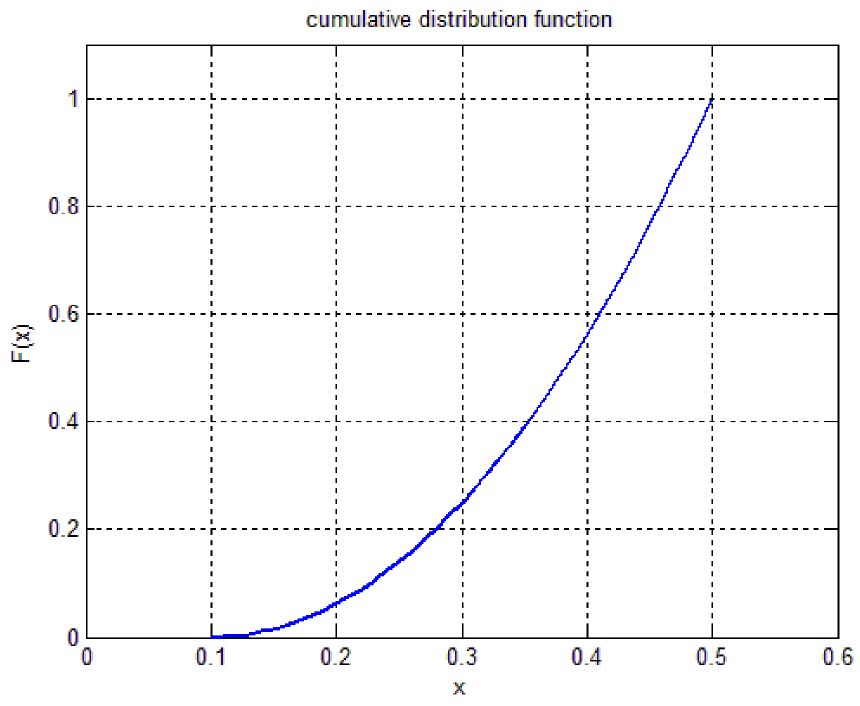
Continuous Cumulative Distribution Function

Definition: Let X be continuous with density f. The cumulative distribution function for X, denoted by F, is defined by: $F(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt\,,\ x\ real$.

Example: Find the cumulative distribution function of the gasoline example.

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt = \int_{0.1}^{x} (12.5t - 1.25)dt$$
$$= 6.25x^2 - 1.25x + 0.0625$$

$$F(x) = \begin{cases} 0, & x < 0.1 \\ 6.25x^2 - 1.25x + 0.0625, & 0.1 \leq x \leq 0.5 \\ 1, & x > 0.5 \end{cases}$$

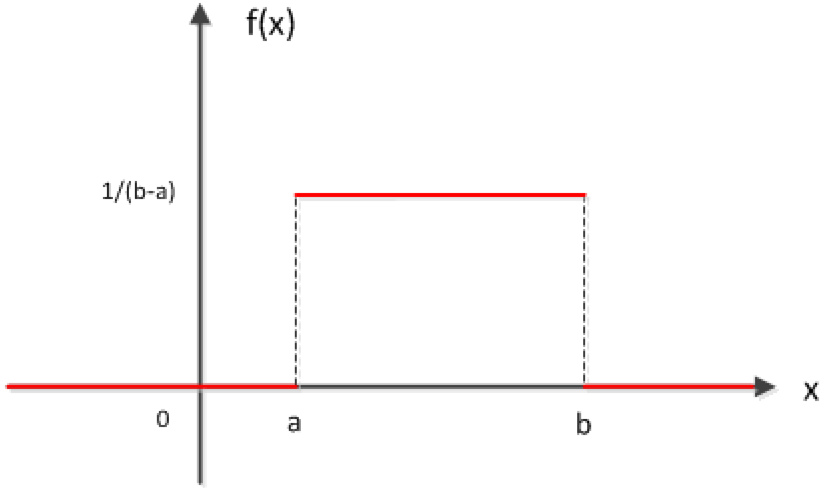


Continuous Cumulative Distribution Function

Practice problem: A random variable follows an uniform distribution if its density function is

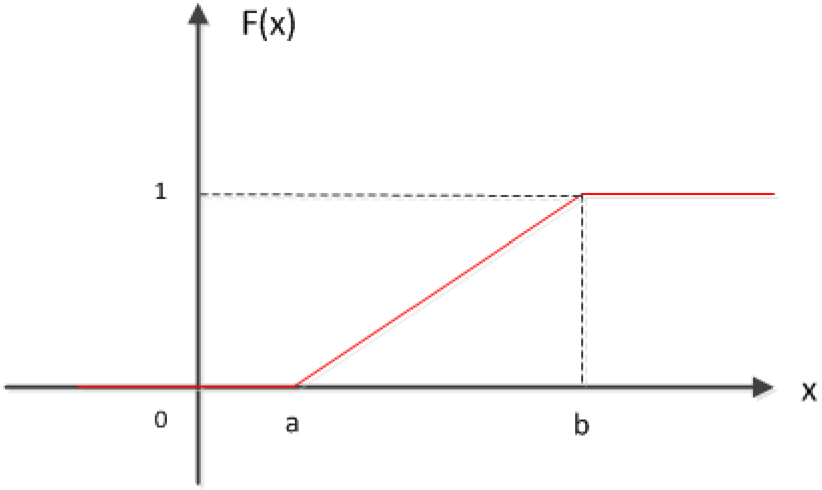
$$f(x) = \frac{1}{b-a}, a \leq x \leq b$$

Find the cumulative distribution function $F(x)$ of an uniform distribution.



$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} \frac{1}{b-a}dt = \frac{x-a}{b-a}$$

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$



Continuous Cumulative Distribution Function

Obtaining $f(x)$ from $F(x)$ in the continuous case:

$$f(x) = F'(x)$$

Example: verify this using the result from the previous example.

Parameters for Continuous Distributions

Expected value: Let X be a continuous RV with density function $f(x)$. Let $H(X)$ be a random variable. The expected value of $H(X)$, denoted by $E[H(X)]$, is given by:

$$E[H(X)] = \int_{-\infty}^{\infty} H(x)f(x)dx$$

Provided that the integration is finite.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \mu_x$$

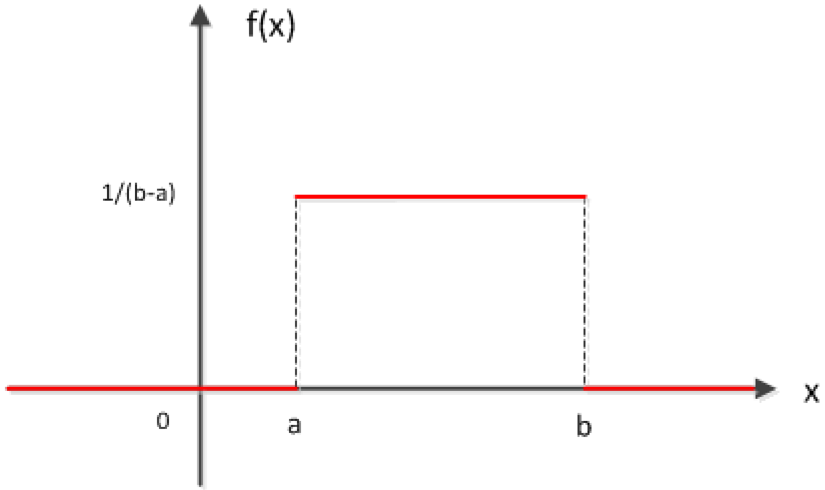
Variance: $\sigma^2 = VarX = E[(X-\mu_x)^2] = E[X^2] - (E[X])^2$

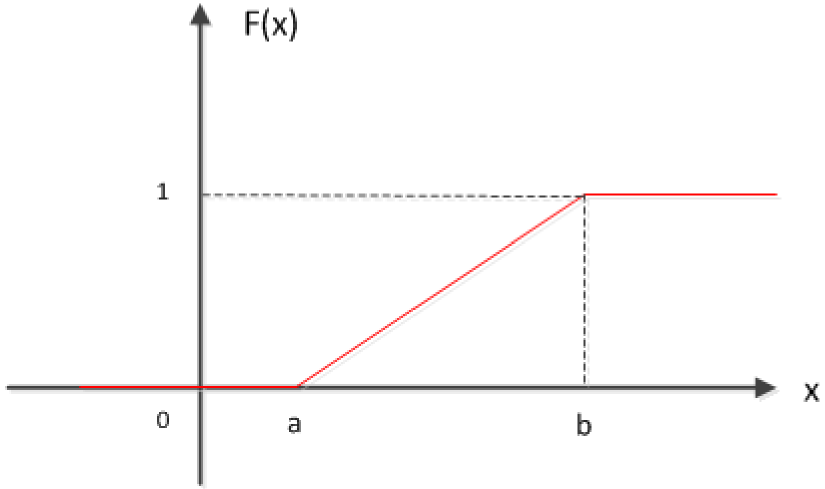
Parameters for Continuous Distributions

Practice Example: Consider an uniform random variable X with density function

$$f(x) = \frac{1}{b-a}, a \leq x \leq b$$

Find $E[X]$ and $VarX$





Solution:

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{a}^{b} \frac{x}{b-a}dx = \frac{1}{b-a}\int_{a}^{b} xdx = \frac{1}{b-a}\frac{x^2}{2}\Big|_a^b = \frac{a+b}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{a}^{b} \frac{x^2}{b-a}dx = \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b = \frac{b^2+ab+a^2}{3}$$

$$VarX = E[X^2] - (E[X])^2 = \frac{(a-b)^2}{12}$$

Gamma Distribution

Gamma function: $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz\ ,\ \alpha > 0$

Properties of gamma function:
(1) $\Gamma(1) = 1$
(2) $For\ \alpha > 1, \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

It is easy to verify (1). Actually, $\Gamma(1) = \int_0^\infty e^{-z}dz = [e^{-z}]_\infty^0 = 1$
To verify (2), we have,

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz = -\int_0^\infty z^{\alpha-1}de^{-z} = -\left([z^{\alpha-1}e^{-z}]_0^\infty - \int_0^\infty e^{-z}dz^{\alpha-1}\right) = \int_0^\infty e^{-z}dz^{\alpha-1} - [z^{\alpha-1}e^{-z}]_0^\infty$$

$$= \int_0^\infty (\alpha-1)z^{\alpha-2}e^{-z}dz = (\alpha-1)\Gamma(\alpha-1)$$

Where, we uses integration by parts and the result that $[z^{\alpha-1}e^{-z}]_0^\infty = 0$.

Actually, $\left[\frac{z^{\alpha-1}}{e^z}\right]_{z=\infty} = \left[\frac{(z^{\alpha-1})'}{(e^z)'}\right]_{z=\infty} = \left[\frac{(\alpha-1)z^{\alpha-2}}{e^z}\right]_{z=\infty} = \cdots = \left[\frac{(\alpha-1)(\alpha-2)\ldots\times2\times1}{e^z}\right]_{z=\infty} = 0$