Theorem 8.2.1:
Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. The random variable,

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim T_{n-1}$$

i.e., $\frac{\bar{X}-\mu}{s/\sqrt{n}}$ follows a T distribution with n-1 degree of freedom.

Proof: we already know that $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ and $\frac{(n-1)S^2}{\sigma^2} \sim \mathrm{X}_{n-1}^2$

Then, following the definition of a T random variable,

$$\frac{Z}{\sqrt{\mathrm{X}_\gamma^2/\gamma}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\frac{S}{\sigma}} = \frac{\bar{X}-\mu}{S/\sqrt{n}}$$

Since $\bar{X}$ and S are independent if the samples are from a normal population, we have,

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim T_{n-1}$$

Theorem 8.2.2:
Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. A $100(1-\alpha)\%$ confidence interval on μ is given by:

$$\bar{X} \mp t_{\alpha/2} S/\sqrt{n}$$

Proof: we use the statistic $\frac{\bar{X}-\mu}{S/\sqrt{n}}$ to find the confidence interval of the mean. Since $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim T_{n-1}$

We have

$$P\left[-t_{\alpha/2} \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq t_{\alpha/2}\right] = 1-\alpha \Rightarrow P\left[(-t_{\alpha/2})S/\sqrt{n} \leq \bar{X}-\mu \leq (t_{\alpha/2})S/\sqrt{n}\right] = 1-\alpha$$

$$\Rightarrow P\left[(-t_{\alpha/2})\frac{S}{\sqrt{n}} - \bar{X} \leq -\mu \leq \left(t_{\alpha/2}\right)\frac{S}{\sqrt{n}} - \bar{X}\right] = 1-\alpha$$

$$\Rightarrow P\left[\bar{X} - \left(t_{\alpha/2}\right)\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + (t_{\alpha/2})\frac{S}{\sqrt{n}}\right] = 1-\alpha$$

Therefore, the $100(1-\alpha)\%$ confidence interval of μ is: $[\bar{X} - \left(t_{\alpha/2}\right)\frac{S}{\sqrt{n}}, \bar{X} + \left(t_{\alpha/2}\right)\frac{S}{\sqrt{n}}]$

Theorem 8.2.2:
Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. A $100(1-\alpha)\%$ confidence interval on μ is given by:

$$\bar{X} \mp t_{\alpha/2} S/\sqrt{n}$$

Example: The following random samples are drawn from a normal distribution with mean μ and variance σ^2. Find the 95% confidence interval on μ.

52.7	43.9	41.7	71.5	47.6	55.1
62.2	56.5	33.4	61.8	54.3	50.1
45.3	63.4	53.9	65.5	66.6	70.0
52.4	38.6	46.1	44.4	60.7	56.4

From this sample, we can calculate $\bar{X} = 53.92, S^2 = 101.48, S \approx 10.07$. From the T cumulative probability table, we have, $t_{0.025} = 2.069$. Hence,

$$L_1 = \bar{X} - \frac{t_{0.025} S}{\sqrt{n}} = 53.92 - \frac{2.069 \times 10.07}{\sqrt{24}} = 49.67; L_2 = \bar{X} + \frac{t_{0.025} S}{\sqrt{n}} = 53.92 + \frac{2.069 \times 10.07}{\sqrt{24}} = 58.17$$

The 95% confidence interval on μ is $[49.67, 58.17]$

Hypothesis Test

A pharmaceutical company developed a new lung cancer treatment medicine. They claim that the cure rate of this new medicine is more than 50%. To verify this claim, a clinical experiment was conducted. A sample of 100 lung cancer patients were given the new medicine.

(1) After one month, it was reported that only 5 patients were cured. Does this result reject the claim of the company?

(2) If 72 patients were reported cured, does this result support (fail to reject) the claim of the company?

(3) What is the critical value N_0, if the number of patients cured is more than N_0, the claim will be supported?

(4) If the sample result reject the claim of the company, what is the risk of rejecting the claim?

These questions can be answered by Hypothesis test design

Hypothesis Test

- A ***hypothesis*** is a premise or claim that we want to test (investigate)

- A ***hypothesis test*** is a statistical test that is used to determine whether there is enough evidence in a sample of data to infer that a certain conclusion (in the form of a hypothesis) is true for the entire population

- A hypothesis test examines two *opposing hypotheses* about a population:
  - The ***null hypothesis*** is the currently accepted statement (value of a parameter)(H_0).
  - The ***alternative hypothesis*** is the research hypothesis (the statement the researcher want to be able to conclude is true) (H_1)

- H_0 and H_1 are always mathematically complement each other, i.e., H_0 is the negation of H_1

- A test will remain with the null hypothesis until there is enough evidence (sample data) to support the alternative hypothesis. i.e., H_0 will remain true until it is rejected by the evidence.

Hypothesis Test

Example: Highway engineers have found that the proper alignment of the vehicle's headlights is a major factor affecting the performance of reflective highway signs. It is thought that more than 50% of the vehicle on the road have misaimed headlights. If this contention can be supported statistically, then a new tougher inspection program will be put into operation.

Null hypothesis and alternative hypothesis:

Let p denote the proportion of vehicle in operation that have misaimed headlights. Since we wish to support the statement that $p > 0.5$, we have the two hypothesis:

$$H_0: p \leq 0.5$$
$$H_1: p > 0.5$$

There are two hypothesis involved in the hypothesis test on population parameter θ, (p in the example):

The hypothesis proposed by the researcher (called research hypothesis), denoted by H_1, is called the *alternative* (or research) hypothesis.

The negation of H_1, denoted by H_0, is called the *null* hypothesis. ***The equality is always in the null hypothesis***!!!

Hypothesis Test

How to choose H_0 and H_1

Example 1: Suppose that a doctor claims that 17 year olds have an average body temperature that is higher than the commonly accepted average human temperature of 98.6 degrees Fahrenheit. A simple random statistical sample of 25 people, each of age 17, is selected. The average temperature of the 17 year olds is found to be 98.9 degrees, with standard deviation of 0.6 degrees.

H_0: $\mu_T \leq 98.6°$ $\quad\quad$ H_1: $\mu_T > 98.6°$

Example 2: Specifications for a water pipe call for a mean breaking strength μ of more than 2000lb per linear foot. Engineer will perform a hypothesis test to decide whether or not to use a certain kind of pipe. They will select a random sample of 1 ft sections of pipe, measure their break strengths, and perform a hypothesis test.

H_0: $\mu \leq 2000$ $\quad\quad$ H_1: $\mu > 2000$