Random Variables

A ***random variable*** assigns a numerical value to each outcome in a sample space with associated probabilities.

Example: Suppose that an electrical engineer has on hand six resistors. Three of them are labeled 10Ω and the other three are labeled 20Ω. The engineer wants to connect a 10Ω resistor and a 20Ω resistor in series to create a resistance of 30Ω. Now suppose that in fact the three resistors labeled 10Ω have actual resistance of $9, 10, and\ 11\Omega$, and that the three resistors labeled 20Ω have actual resistances of $19, 20, and\ 21\Omega$. The process of selecting one resistor of each type is an experiment whose sample space is presented as in the table:

Outcomes	X	Probability
(9,19)	28	1/9
(9,20)	29	1/9
(9,21)	30	1/9
(10,19)	29	1/9
(10,20)	30	1/9
(10,21)	31	1/9
(11,19)	30	1/9
(11,20)	31	1/9
(11,21)	32	1/9

X	P(X=x)
28	1/9
29	2/9
30	3/9
31	2/9
32	1/9

X is a random variable.

Discrete Random Variable (Discrete R.V.)

Definition: A random variable is discrete if it can assume at most a finite or a countably infinite number of possible values.

Notation: X , the variable; x, observed value of X

Examples:
(1) X: the number on a dice $\{1,2,3,4,5,6\}$

(2) In an immunology study, malignant plasmacytoma cells are exposed to the lymphocytes one at a time in the presence of a fusion-promoting agent and hope the cell will fuse. It is known that the probability that such a cell will fuse is $1/2$. ***Let $\boldsymbol{Y}$ denote the number of cells exposed to obtain the first fusion***.

Y can assume any value in the set of $\{1,2,3,\ldots\}$

Discrete Probability Density Function (probability mass function)

Definition: Let X be a discrete random variable. The function f given by: $f(x) = P[X = x]$ for x real, is called the ***density function*** of X iff

$$f(x) \geq 0 \;\; and \;\; \sum_{all\ x} f(x) = 1$$

Examples:

Let X denote the number from a thrown dice. X takes value from $\{1,2,3,4,5,6\}$.
Then, the probability density function of X is

$$f(x) = \frac{1}{6}, \qquad x = 1,2,3,4,5,6$$

To verify this, we have

$$f(x) = \frac{1}{6} > 0$$

$$\sum_{all\ x} f(x) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Discrete Probability Density Function (probability mass function)

Examples: In an immunology study, malignant plasmacytoma cells are exposed to the lymphocytes one at a time in the presence of a fusion-promoting agent and hope the cell will fuse. It is known that the probability that such a cell will fuse is $1/2$. Let Y denote the number of cells exposed to obtain the first fusion. Y can assume any value in the set of $\{1,2,3, \dots\}$.

Find the probability mass function of Y.

Solution:

$f(1) = P[Y = 1]$

$= P[the\ first\ cell\ fuses] = \frac{1}{2}$

$f(2) = P[Y = 2]$

$= P[(the\ first\ cell\ does\ not\ fuse)\ \cap (the\ second\ cell\ fused)]$

$= P[the\ first\ cell\ does\ not\ fuse]P[the\ second\ cell\ fuses] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$f(3) = P[Y = 3]$

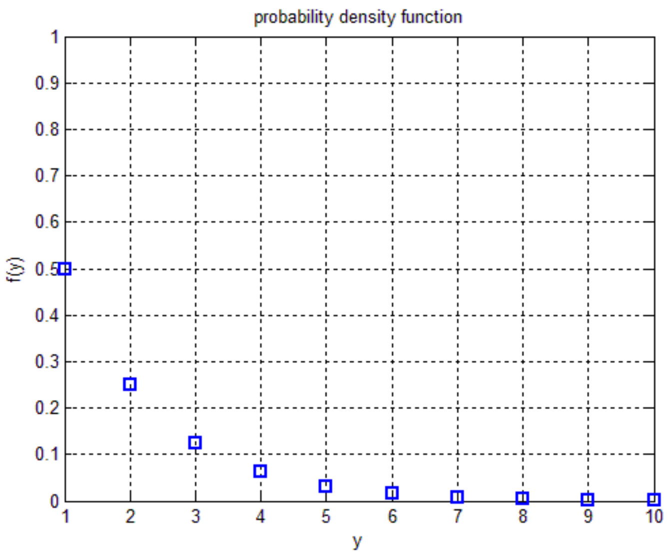
$= P[(the\ first\ cell\ does\ not\ fuse) \cap (the\ second\ cell\ dose\ not\ fuze) \cap (the\ third\ cell\ fuses)]$

$= P[the\ first\ cell\ does\ not\ fuse]P[the\ second\ cell\ dose\ not\ fuze]P[the\ third\ cell\ fuses]$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = (\frac{1}{2})^3 = \frac{1}{8}$

$\vdots$

In general, $$f(y) = \left(\frac{1}{2}\right)^y,\ y = 1,2,3, \dots$$



Discrete Probability Density Function (probability mass function)

Examples: In an immunology study, malignant plasmacytoma cells are exposed to the lymphocytes one at a time in the presence of a fusion-promoting agent and hope the cell will fuse. It is known that the probability that such a cell will fuse is $1/2$. Let Y denote the number of cells exposed to obtain the first fusion. Y can assume any value in the set of $\{1,2,3, \dots\}$. The probability mass function of Y is found as $f(y) = \left(\frac{1}{2}\right)^y$, $y = 1,2,3, \dots$.

Verify that this is a probability mass function.

Solution:

(1) $f(y) \geq 0, for\ y = 1,2,3, \dots$

(2) $\sum_{all\ y} f(y) = \sum_{y=1}^{\infty} \left(\frac{1}{2}\right)^y$

Recall a result from geometric series that $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r},\ |r| < 1$

Apply this result to (2), we have,

$$\sum_{y=1}^{\infty} \left(\frac{1}{2}\right)^y = \sum_{y=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{y-1} = \frac{1/2}{1 - 1/2} = 1$$

Therefore, $f(y)$ is a probability mass function

Examples: In an immunology study, malignant plasmacytoma cells are exposed to the lymphocytes one at a time in the presence of a fusion-promoting agent and hope the cell will fuse. It is known that the probability that such a cell will fuse is $1/2$. Let Y denote the number of cells exposed to obtain the first fusion. Y can assume any value in the set of $\{1,2,3, \ldots\}$. The probability mass function of Y is found as $f(y) = \left(\frac{1}{2}\right)^y$, $y = 1,2,3, \ldots$.
What is the probability that we need to expose four or more cells to obtain the first fusion?

Solution:

We need to find out $\mathrm{P}[\mathrm{Y} \geq 4]$.

$$\begin{aligned} P[Y \geq 4] &= P[Y = 4] + P[Y = 5] + P[Y = 6] + \cdots \\ &= 1 - P[Y \leq 3] = 1 - P[Y = 3] - P[Y = 2] - P[Y = 1] \\ &= 1 - f(3) - f(2) - f(1) \\ &= 1 - \frac{1}{8} - \frac{1}{4} - \frac{1}{2} = \frac{1}{8} \end{aligned}$$

Cumulative Distribution

Definition: Let X be a discrete random variable (RV) with density $f(x)$**.** The *cumulative distribution function (cdf)* for X**,** denoted by F, is defined by:

$$F(x) = P[X \leq x], \qquad for\ x\ real$$

Consider a specific real number x_0, according to the definition:

$$F(x_0) = P[X \leq x_0] = \sum_{x \leq x_0} f(x)$$

Example: Consider the immunology study example. Find the cumulative distribution function of the random variable Y

Solution:

$F(y_0) = \sum_{y \leq y_0} f(y) = \sum_{y=1}^{y_0} \left(\frac{1}{2}\right)^y = \sum_{y=1}^{y_0} \frac{1}{2}\left(\frac{1}{2}\right)^{y-1}$. Recall the following result: $\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}, r \neq 1$

Apply this result with $a = \frac{1}{2}\ and\ r = \frac{1}{2}$, we have,

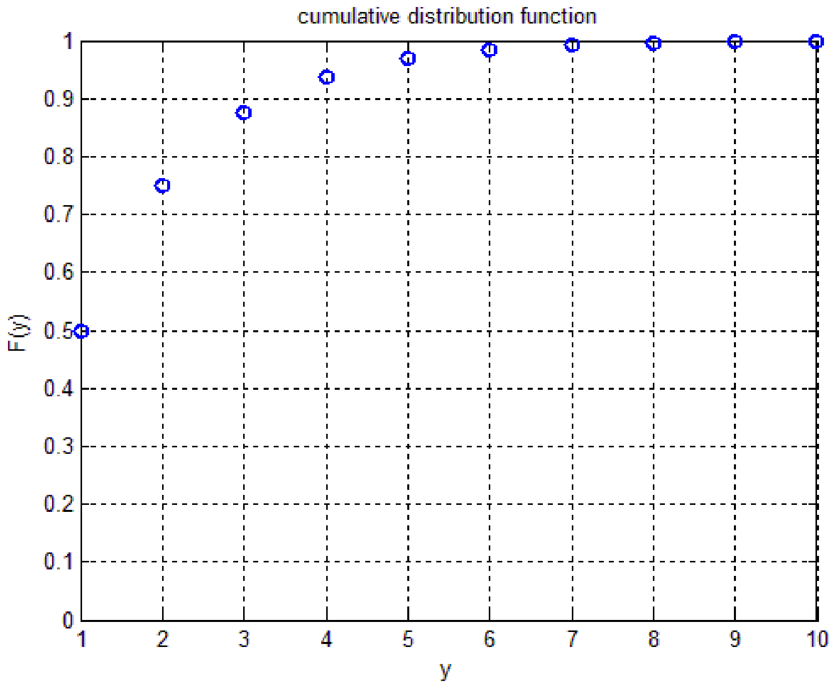
$F(y_0) = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{y_0}\right)}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^{y_0}$. When $y_0 = 7, F(7) = 1 - \left(\frac{1}{2}\right)^7 = \frac{127}{128}$

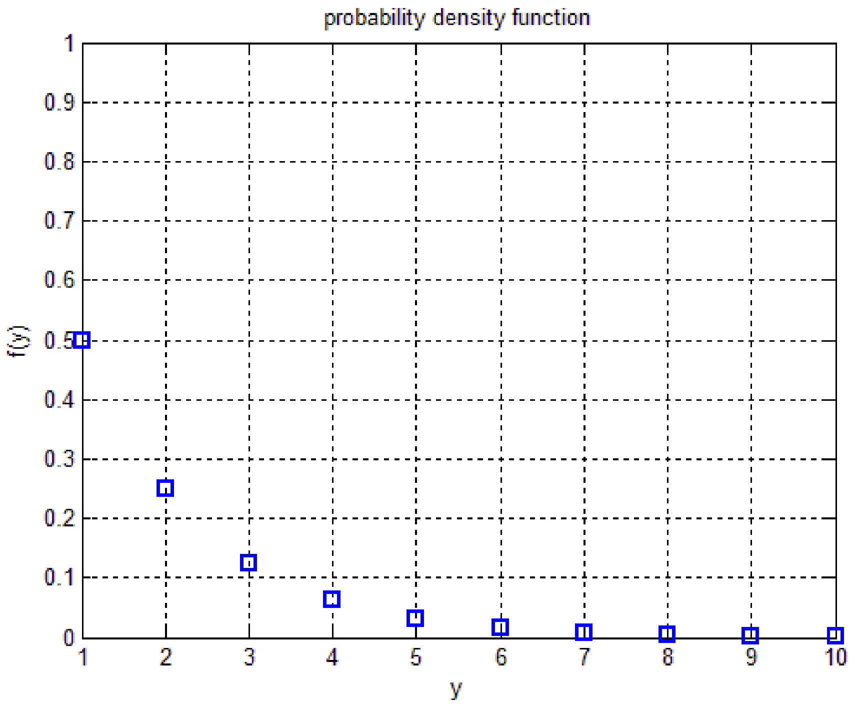
Cumulative Distribution

Example: Consider the immunology study example. Find the cumulative distribution function of the random variable Y

We can construct the following cumulative function table for the example:

y	1	2	3	4	5	6	7	…
F(y)	1/2	3/4	7/8	15/16	31/32	63/64	127/128	…



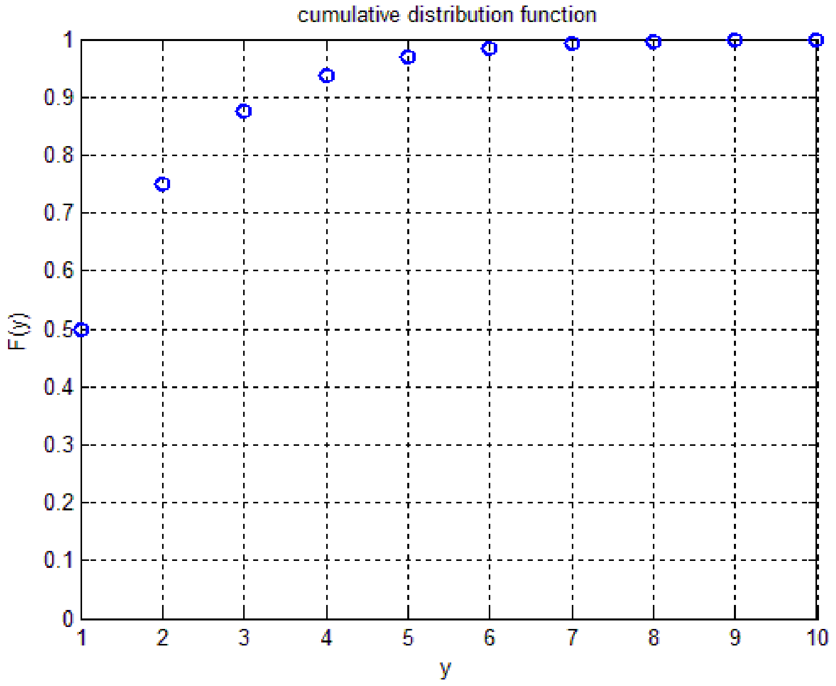


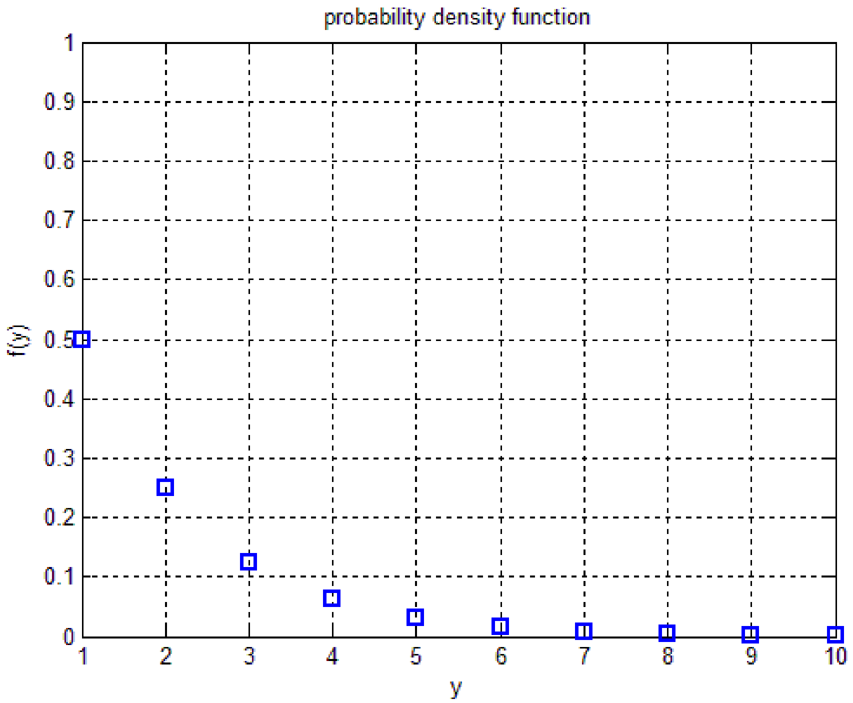
Cumulative Distribution Function vs Density Function

Let's assume a discrete random variable X can take values of $x_0, x_1, x_2, \ldots, x_n, \ldots$, then,

$$F(x_n) = \sum_{i=0}^{n} f(x_i)$$

$$f(x_n) = F(x_n) - F(x_{n-1})$$





Expected Value

Parameters to describe a RV

Definition: Let X be a discrete random variable (R.V.) with density $f(x)$.Let $H(X)$ be a function of random variable X. The expected value of $H(X)$, denoted by $E[H(X)]$, is given by:

$$E[H(X)] = \sum_{all\ x} H(x)f(x)$$

A special case of this definition, $H(X) = X,\ E[X] = \sum_{all\ x} xf(x)$

Example: Given the density of a discrete random variable X and Y as in the table. Calculate $E[X]\ and\ E[Y]$

x	1	2	3	4	5
f(x)	0.2	0.2	0.2	0.2	0.2

y	1	2	3	4	5
f(y)	0.9	0.05	0.02	0.02	0.01

Solution: $\mathrm{E}[\mathrm{X}] = 1 \times 0.2 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.2 + 5 \times 0.2 = 3$

$E[Y] = 1 \times 0.9 + 2 \times 0.05 + 3 \times 0.02 + 4 \times 0.02 + 5 \times 0.01 = 1.19$

Expected Value

Parameters to describe a RV

Example: Given the density of a discrete random variable X and Y as in the table. Calculate $E[X]$ and $E[Y]$

x	1	2	3	4	5
f(x)	0.2	0.2	0.2	0.2	0.2

y	1	2	3	4	5
f(y)	0.9	0.05	0.02	0.02	0.01