Estimation

2. Maximum likelihood estimate:

Example 1 (discrete case): Let $x_1, x_2, \dots, x_n$ be a random sample from a Poisson distribution with parameter k. We want to find the value of k that given the maximum probability of observing this sample.

Solution:

Recall the Poisson density function: $P[X = x] = f(x) = \frac{e^{-k}k^x}{x!},\ x = 0,1,2,\dots$

The probability of obtaining the given sample is:

$$P[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n] = P[X_1 = x_1]P[X_2 = x_2]\cdots P[X_n = x_n]$$

$$= \prod_{i=1}^{n} P[X_i = x_i] = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-k}k^{x_i}}{x_i!} = L(k)$$

This probability is a function of k and is called the ***likelihood function***.

$$L(k) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-k}k^{x_i}}{x_i!} = \frac{e^{-nk}k^{\sum x_i}}{\prod x_i!}$$

We want to find the value of k that maximize this likelihood function!!!

Estimation

2. **Maximum likelihood estimate:**

Solution: (continue)

$$L(k) = \prod_{i=1}^{n} \frac{e^{-k} k^{x_i}}{x_i!} = \frac{e^{-nk} k^{\sum x_i}}{\prod x_i!}$$

We first take the natural logarithm of $L(k)$ to obtain the ***log likelihood function***:

$$\ln L(k) = -nk + \ln k \sum_{i=1}^{n} x_i - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

Notice that the value of k that maximizes $\ln L(k)$ also maximize $L(k)$ (***why?***)

$$\frac{d}{dk}\ln L(k) = -n + \frac{1}{k}\sum_{i=1}^{n} x_i = 0 \Rightarrow k = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} \Rightarrow \hat{k} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

How about $\frac{d}{dk}L(k) = 0$?

Estimation

2. **Maximum likelihood estimate:**

Steps:

1. Obtain a random sample $x_1, x_2, \dots, x_n$ from the distribution with density $f(x)$ and associated parameter θ.
2. Define the likelihood function $L(\theta) = \prod_{i=1}^{n} f(x_i)$ and the log likelihood function $\ln L(\theta)$
3. Find the expression for θ that maximizes $L(\theta)\ or \ln L(\theta)$
4. Replace θ by $\hat{\theta}$ to obtain the estimator

Estimation

2. **Maximum likelihood estimate:**

Practice Example 1: The following is a random sample from a Poisson distribution with parameter k. Estimate the value of k from the sample.

x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
9	10	8	12	12	10	8	8	8	9

$$\hat{k} = \bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 9.4$$

This sample was created by using Matlab
x = poissrnd(10,[1 10])

The true value of k is 10

Estimation

2. **Maximum likelihood estimate:**

Example 2 (continuous case): Let $x_1, x_2, \dots, x_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Find the maximum likelihood estimator for $\mu \ \ and \ \sigma^2$

Solution:

Recall the density function of a normal distribution is: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

We construct the ***likelihood function*** as:

$$L(\mu,\sigma) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

The ***log likelihood function*** is:

$$\ln L(\mu,\sigma) = -n\left(\ln\sqrt{2\pi} + \ln\sigma\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Estimation

2. **Maximum likelihood estimate:**

Example 2: Let $x_1, x_2, \dots, x_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Find the maximum likelihood estimator for $\mu \ and\ \sigma^2$

Solution: (continue)

$$\ln L(\mu,\sigma) = -n\left(\ln\sqrt{2\pi} + \ln\sigma\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Take the derivatives of the *log likelihood function* with respect to $\mu \ and\ \sigma$ and let them equal to zero, we have,

$$\frac{\partial \ln L(\mu,\sigma)}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0 \Rightarrow n\mu = \sum_{i=1}^{n}x_i \Rightarrow \mu = \frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\frac{\partial \ln L(\mu,\sigma)}{\partial\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

Hence, the maximum likelihood estimator for $\mu \ and\ \sigma^2$ are:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}x_i\,, \qquad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

Estimation

2. **Maximum likelihood estimate:**

Example 2: The following is a random sample from a normal distribution with parameter $\mu\ and\ \sigma^2$. Estimate the value of $\mu\ and\ \sigma^2$ from the sample.

x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
12.5	11.4	10.2	3.3	15.6	11.8	8.5	10.1	8.7	1.2

$$\hat{\mu} = \bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 9.33$$

$$\widehat{\sigma^2} = \frac{1}{10}\sum_{i=1}^{10} (x_i - \hat{\mu})^2 = 4.28$$

This sample was created by using Matlab
x = normrnd(10,5, [1 10])

The true values of $\mu\ and\ \sigma$ are 10 and 5

Estimation

Problem with point estimation:

No accuracy measurement is provided with the estimator

We want something like $P[L_1 \leq \mu \leq L_2] \approx 95\%$, where $L_1\ and\ L_2$ are statistics related to $\bar{X}$.

Confidence interval, interval estimate.

This need the distribution of $\bar{X}$, which is a function of random variable X

Find the distribution of functions of random variables - Moment generating function based method

Theorem 7.3.1: Let X and Y be random variables with moment generating functions $m_X(t)$ and $m_Y(t)$, respectively. If $m_X(t) = m_Y(t)$ for all t in some open interval about 0, then X and Y have the same distribution.

Theorem 7.3.2: Let X_1 and X_2 be independent random variables with moment generating functions $m_{X_1}(t)$ and $m_{X_2}(t)$, respectively. Let $Y = X_1 + X_2$. The moment generating function for Y is $m_Y(t) = m_{X_1}(t)\, m_{X_2}(t)$.

Proof:

$$m_Y(t) = E[e^{tY}] = E\left[e^{t(X_1+X_2)}\right] = E[e^{tX_1}e^{tX_2}]$$

Since X_1 and X_2 are independent, e^{tX_1} and e^{tX_2} are also independent.

Hence, $E[e^{tX_1}e^{tX_2}] = E[e^{tX_1}]E[e^{tX_2}]$. i.e., $m_Y(t) = m_{X_1}(t)\, m_{X_2}(t)$.

Example: Let $X_1, X_2, \dots, X_n$ be independent ***normal random variables*** with means $\mu_1, \mu_2, \dots, \mu_n$ and variances ${\sigma_1}^2, {\sigma_2}^2, \dots, {\sigma_n}^2$, respectively. Let $\mathrm{Y} = X_1 + X_2 + \cdots + X_n$. Find the m.g.f. of Y.

From the problem statement, we have,

$$m_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}, i = 1,2, \dots, n$$

Since $X_1, X_2, \dots, X_n$ are independent, we have,

$$m_Y(t) = m_{X_1}(t) m_{X_2}(t) \cdots m_{X_n}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} \dots e^{\mu_n t + \frac{1}{2}\sigma_n^2 t^2} = e^{\sum \mu_i t + \frac{1}{2}(\sum \sigma_i^2) t^2}$$

Hence, we have,

$$Y \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

i.e., Y is a normal random variable with mean $\mu = \sum_{i=1}^{n} \mu_i$ and variance $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$

Distribution of a linear function of X:

Let X be a random variable with moment generating function $m_X(t)$. Let $Y = \alpha + \beta X$. Find $m_Y(t)$.

$$m_Y(t) = E[e^{tY}] = E\left[e^{t(\alpha+\beta X)}\right] = E\left[e^{\alpha t}e^{\beta tX}\right] = e^{\alpha t}E\left[e^{(\beta t)X}\right] = e^{\alpha t}m_X(\beta t)$$

Distribution of a linear function of X:

Example: Let X be a normal random variables with mean $\mu = 10$ and variance$\sigma^2 = 4$. Find the distribution of $Y = 8 + 3X$.

Solution:

$$m_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = e^{10t + 2t^2}$$
$$m_Y(t) = e^{8t} e^{10\times(3t) + 2\times(3t)^2} = e^{8t} e^{30t + 18t^2} = e^{38t + 18t^2}$$

Hence, Y is a normal variable with $\mu_Y = 38 \;\; and \;\; \sigma_Y^2 = 36$, i.e., $Y \sim N(38{,}36)$

Note: To verify the mean and variance values of Y, we do the following calculation:

$$E[Y] = E[8 + 3X] = 8 + 3E[X] = 8 + 3 \times 10 = 38$$
$$VarY = Var(8 + 3X) = 0 + 3^2 VarX = 9 \times 4 = 36$$

However, it is not sufficient to find $E[Y] and\ VarY\ and\ then\ claim\ that\ Y \sim N(38{,}36)$ without finding the moment generating function of Y