Gamma Distribution

Gamma function: $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz \ , \ \alpha > 0$

Properties of gamma function:

(1) $\Gamma(1) = 1$

(2) $For\ \alpha > 1, \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

Examples: evaluate the following expressions

(1)$\int_0^\infty z^3 e^{-z} dz$ (2)$\int_0^\infty \frac{1}{54} x^2 e^{-\frac{x}{3}} dx$

$$\int_0^\infty z^3 e^{-z} dz = \Gamma(4) = 3 \times \Gamma(3) = 3 \times 2 \times \Gamma(2) = 3 \times 2 \times 1 \times \Gamma(1) = 6$$

To evaluate $\int_0^\infty \frac{1}{54} x^2 e^{-\frac{x}{3}} dx$, let $z = \frac{x}{3}$. Then, $dz = \frac{1}{3} dx$, and

$$\int_0^\infty \frac{1}{54} x^2 e^{-\frac{x}{3}} dx = \int_0^\infty \frac{1}{54} (3z)^2 e^{-z} 3dz = \frac{27}{54} \int_0^\infty z^2 e^{-z} dz = \frac{1}{2} \Gamma(3) = \frac{1}{2} \times 2 \times \Gamma(2) = 1$$

Gamma Distribution

Gamma distribution: A random variable X with density

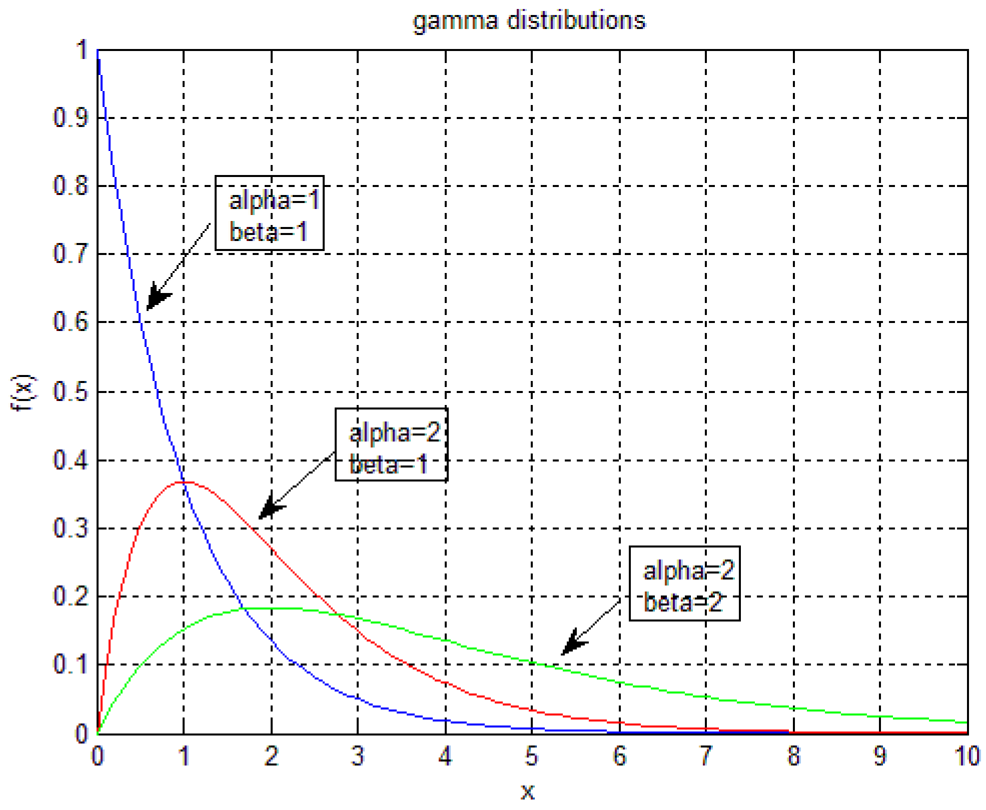
$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad x > 0, \alpha > 0, \beta > 0$$

Is said to have a gamma distribution with parameter $\alpha\ and\ \beta$

$$\alpha: shape\ parameter;\ \ \beta: rate\ parameter$$

Exponential and ***Chi-squared*** distributions are special cases of gamma distribution.

Prove that $f(x)$ is a density function !



$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} d(\frac{x}{\beta}) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (z)^{\alpha-1} e^{-z} dz = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

Gamma Distribution

Gamma distribution: A random variable X with density $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad x > 0, \alpha > 0, \beta > 0$

Is said to have a gamma distribution with parameter $\alpha\ and\ \beta$

Prove that:

The *moment generating function* for a Gamma R.V. X is: $m_X(t) = (1 - \beta t)^{-\alpha}$

$$m_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-\frac{(1-\beta t)x}{\beta}} dx$$

Let $z = \frac{(1-\beta t)x}{\beta} \Rightarrow x = \frac{\beta z}{1-\beta t}$. Then, $\frac{dx}{dz} = \frac{\beta}{1-\beta t} \Rightarrow dx = \frac{\beta dz}{1-\beta t}$

Substitute $x\ and\ dx$ into the integration, we have,

$$m_X(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \left(\frac{\beta z}{1-\beta t}\right)^{\alpha-1} e^{-z} \frac{\beta dz}{1-\beta t} = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{\beta^\alpha}{(1-\beta t)^\alpha} \int_0^\infty z^{\alpha-1} e^{-z} dz$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{\beta^\alpha}{(1-\beta t)^\alpha} \Gamma(\alpha) = (1 - \beta t)^{-\alpha}$$

Gamma Distribution

Practice Problem:

The *moment generating function* for a Gamma R.V. X is: $m_X(t) = (1 - \beta t)^{-\alpha}$

Use moment generating function to prove that $E[X] = \alpha\beta$ and $VarX = \alpha\beta^2$

$$\frac{d}{dt} m_X(t) = (-\alpha)(-\beta)(1 - \beta t)^{-(\alpha+1)} \Rightarrow E[X] = \frac{d}{dt} m_X(t)_{t=0} = \alpha\beta$$

$$\frac{d^2}{dt^2} m_X(t) = (\alpha + 1)\alpha\beta^2(1 - \beta t)^{-(\alpha+2)} \Rightarrow E[X^2] = \frac{d^2}{dt^2} m_X(t)_{t=0} = \alpha^2\beta^2 + \alpha\beta^2$$

$$VarX = E[X^2] - (E[X])^2 = \alpha\beta^2$$