Comparing two means: variances equal ($\sigma_1 = \sigma_2$)

let $\bar{X}_1$ and $\bar{X}_2$ be the sample means based on independent samples of size n_1 and n_2 drawn from normal distributions with means μ_1 and μ_2 and variances ${\sigma_1}^2$ $and {\sigma_2}^2$ respectively. We already learned that:

$\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2})$. Hence, we have,

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}}} \sim Z \sim N(0,1)$$

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim Z \sim N(0,1), when\ \sigma_1^2 = \sigma_2^2 = \sigma^2$$

Let's define the "***pooled variance***": ${S_p}^2 = \frac{(n_1-1){S_1}^2 + (n_2-1){S_2}^2}{n_1+n_2-2}$.

Then

$$\frac{(n_1 + n_2 - 2){S_p}^2}{\sigma^2} \sim {\chi^2}_{n_1+n_2-2}$$

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{{S_p}^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim T_{n_1+n_2-2}$$

Comparing two means: variances equal ($\boldsymbol{\sigma_1 = \sigma_2}$)

let S_1^2 and S_2^2 be the sample variances based on independent samples of size n_1 and n_2 drawn from normal distributions with means μ_1 and μ_2 and variances ${\sigma_1}^2$ $and {\sigma_2}^2$ respectively. When $\sigma_1 = \sigma_2 = \sigma$, then,

$$\frac{(n_1 + n_2 - 2){S_p}^2}{\sigma^2} \sim {\chi^2}_{n_1+n_2-2}$$

Where, ${S_p}^2 = \frac{(n_1-1){S_1}^2+(n_2-1){S_2}^2}{n_1+n_2-2}$.

Proof:

According to the problem statement, we have,

$$\frac{(n_1 - 1)S_1^2}{\sigma^2} \sim \mathrm{X}^2_{n_1-1}; \frac{(n_2 - 1)S_2^2}{\sigma^2} \sim \mathrm{X}^2_{n_2-1}$$

Then,

$$\frac{(n_1 + n_2 - 2)S_p^2}{\sigma^2} = \frac{(n_1 - 1)S_1^2}{\sigma^2} + \frac{(n_2 - 1)S_2^2}{\sigma^2} = S_{p_1}^2 + S_{p_2}^2$$

And

$$m_{S_p^2}(t) = m_{S_{p_1}^2}(t) \cdot m_{S_{p_2}^2}(t) = (1 - 2t)^{-(n_1-1)/2} \cdot (1 - 2t)^{-(n_2-1)/2} = (1 - 2t)^{-(n_1+n_2-2)/2}$$

Hence, we have,

$$\frac{(n_1 + n_2 - 2)S_p^2}{\sigma^2} \sim \mathrm{X}^2_{n_1+n_2-2}$$

Comparing two means: variances equal ($\sigma_1 = \sigma_2$)

let $\bar{X}_1$ *and* $\bar{X}_2$, and S_1^2 *and* S_2^2 are the sample means and sample variances based on independent samples of size n_1 *and* n_2 drawn from normal distributions with means μ_1 *and* μ_2 and variances ${\sigma_1}^2$ *and*${\sigma_2}^2$ respectively. When $\sigma_1 = \sigma_2 = \sigma$, then,

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{{S_p}^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim T_{n_1+n_2-2}$$

Where, ${S_p}^2 = \frac{(n_1-1){S_1}^2+(n_2-1){S_2}^2}{n_1+n_2-2}$.

Proof:

We already proved that $\frac{(n_1+n_2-2)S_p^2}{\sigma^2} \sim \mathrm{X}^2_{n_1+n_2-2}$. Also, $\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{\sigma^2(\frac{1}{n_1}+\frac{1}{n_2})}} \sim Z \sim N(0,1)$

Following to the definition of T distribution,

$$\frac{\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma^2(\frac{1}{n_1} + \frac{1}{n_2})}}}{\sqrt{\frac{\frac{(n_1 + n_2 - 2)S_p^2}{\sigma^2}}{n_1 + n_2 - 2}}} \sim T_{n_1+n_2-2} \Rightarrow \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{S_p^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim T_{n_1+n_2-2}$$

Comparing two means: variances equal ($\boldsymbol{\sigma_1 = \sigma_2}$)

Find the confidence interval of $\boldsymbol{\mu_1 - \mu_2}$:

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{{S_p}^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim T_{n_1+n_2-2}$$

Then, the bounds of a $100(1-\alpha)\%$ confidence interval on $\boldsymbol{\mu_1 - \mu_2}$ are:

$$(\bar{X}_1 - \bar{X}_2) \pm t_{\alpha/2}\sqrt{{S_p}^2(\frac{1}{n_1} + \frac{1}{n_2})}$$

$t_{\alpha/2}$ can be found from the $T_{n_1+n_2-2}$ probability table.

Comparing two means: variances equal ($\sigma_1 = \sigma_2$)

Example: A study is conducted to estimate the difference in the mean occupational exposure to radioactivity in utility workers in the years 1973 and 1979. These data based on independent samples of workers for the 2 years are obtained as:

1973	1979
$n_1 = 16$	$n_2 = 16$
$\bar{x}_1 = 0.94$	$\bar{x}_2 = 0.62$
${s_1}^2 = 0.040$	${s_2}^2 = 0.028$

Assuming the two populations are normal with means μ_1 and μ_2 and variances ${\sigma_1}^2$ and ${\sigma_2}^2$ respectively. Find the 95% confidence interval of $\mu_1 - \mu_2$.

Solution: We first check for equality of variances by testing: $H_0: {\sigma_1}^2 = {\sigma_2}^2$; $H_1: {\sigma_1}^2 \neq {\sigma_2}^2$ at the $\alpha = 0.2$ level.

Assuming H_0 is true, the test statistic $\frac{{S_1}^2}{{S_2}^2} \sim F_{15,15}$. The observed value of the test statistic is $\frac{{s_1}^2}{{s_2}^2} = 0.04/0.028 \approx 1.43$

Since $\frac{{s_1}^2}{{s_2}^2} = \frac{0.04}{0.028} \approx 1.43 > 1$ and the test is two-tailed,

$$p_{value} = 2P\left[F_{15,15} \geq 1.43\right]$$

Solution: (continued)

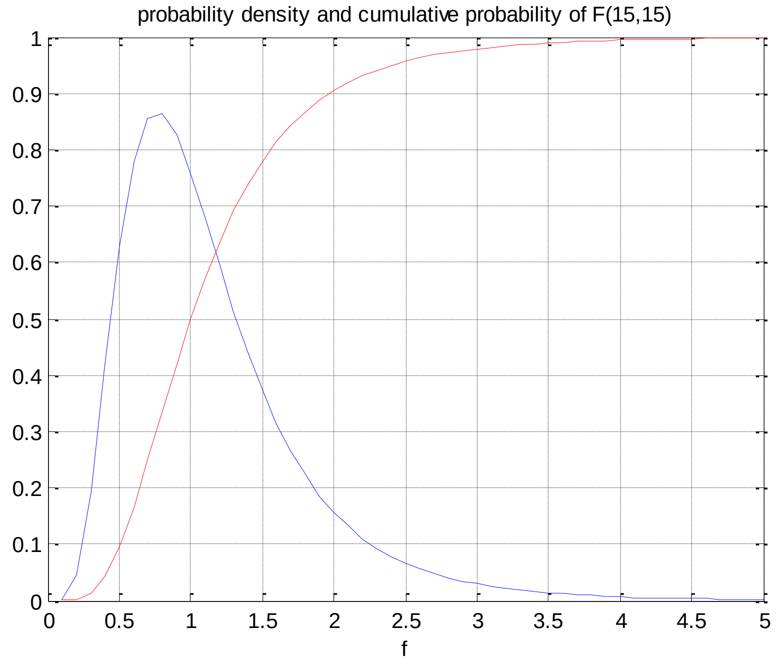
From the F probability table, we find out $P\left[F_{15,15} \geq 1.972\right] = 0.1$, $P\left[F_{15,15} \leq 1.972\right] = 0.9$. Hence, we have,

$$P\left[F_{15,15} \geq 1.43\right] > 0.1$$

Therefore, we have,

$$p_{value} > 0.2$$

We are unable to reject H_0 at the significance level $\alpha = 0.2$.
Hence, $H_1: {\sigma_1}^2 = {\sigma_2}^2$ still holds.



Solution: (continued)

Since ${\sigma_1}^2 = {\sigma_2}^2$, we can use the pooled variance for the confidence interval estimate:

$${S_p}^2 = \frac{(n_1 - 1){s_1}^2 + (n_2 - 1){s_2}^2}{n_1 + n_2 - 2} = 0.034$$

From the T probability table, we find out that for the degree of freedom of 30, we have, $t_{0.025} = 2.042$
The bounds for the 95% confidence interval are:

$$(\bar{X}_1 - \bar{X}_2) \pm t_{\frac{\alpha}{2}}\sqrt{{S_p}^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = (0.94 - 0.62) \pm 2.042 \times \sqrt{0.034 \times \left(\frac{1}{16} + \frac{1}{16}\right)} = 0.32 \pm 0.13 \text{ , i.e., } [0.19, 0.45]$$

What does this confidence interval tell you?

We can be 95% confident that the difference in mean exposure to radioactivity for the 2 years is between 0.19 and 0.45. This interval does not contain the number 0 and is positive valued throughout. This indicates that the mean exposure in 1973 was higher than that in 1979

Comparing two means: variances equal ($\sigma_1 = \sigma_2$)

Hypothesis test on mean comparison:

I. $H_0\colon \mu_1 = \mu_2, \quad H_1\colon \mu_1 > \mu_2$
II. $H_0\colon \mu_1 = \mu_2, \quad H_1\colon \mu_1 < \mu_2$
III. $H_0\colon \mu_1 = \mu_2, \quad H_1\colon \mu_1 \neq \mu_2$

Under the assumption that H_0 is true, the test statistic becomes:

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)_0}{\sqrt{{S_p}^2(\frac{1}{n_1} + \frac{1}{n_2})}} \sim T_{n_1+n_2-2}$$

Where $(\mu_1 - \mu_2)_0$ denotes the difference in population means assuming H_0 is true.

This test statistic is used only when the variances of the two populations equal, i.e.,$\sigma_1 = \sigma_2$!!!
You need to carry out test on comparison of variances before carrying out test on comparison of means!!!

Comparing two means: variances equal ($\sigma_1 = \sigma_2$)

Example: A study of the tensile strength of ductile iron annealed or strengthened at two different temperatures is conducted. It is thought that the lower temperature will yield the higher mean tensile strength. These data results:

1450 °F	1650 °F
$n_1 = 10$	$n_2 = 16$
$\bar{x}_1 = 18{,}900 psi$	$\bar{x}_2 = 17{,}500 psi$
${S_1}^2 = 1600$	${S_2}^2 = 2500$

Solution:

First, let's test if we have $\sigma_1 = \sigma_2$. i.e., we want to test H_0: $\sigma_1^2 = \sigma_2^2$; H_1: $\sigma_1^2 \neq \sigma_2^2$

We choose $\frac{S_2^2}{S_1^2}$ as the test statistic. Assuming H_0 is true, $\frac{S_2^2}{S_1^2} \sim F_{15,9}$ and the observed value of this statistic is $\frac{2500}{1600} = 1.56$

Since the observed value is $1.56 > 1$, and this is a two-tailed test, we have,

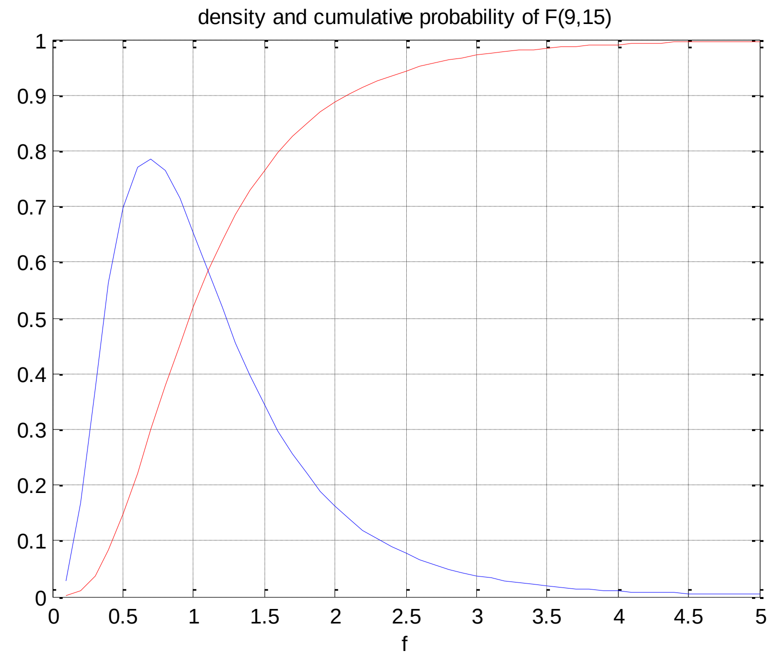
$$p_{value} = 2P\left[F_{15,9} \geq 1.56\right]$$

From the F-table, we have $P\left[F_{15,9} \geq 2.34\right] = 0.10 \Rightarrow P\left[F_{15,9} \geq 1.56\right] > 0.10$

Hence,

$$p_{value} > 0.20$$

Therefore, H_0 should not be rejected at the significance level of 0.20. i.e., $\sigma_1^2 = \sigma_2^2$ holds.



95% table

TABLE IX
***F* distribution (continued)**

γ_2 \ γ_1	9	10	11	12	13	14	15	16
1	240.543	241.881	242.983	243.905	244.689	245.363	245.949	246.462
2	19.385	19.396	19.405	19.412	19.419	19.424	19.429	19.433
3	8.812	8.786	8.763	8.745	8.729	8.715	8.703	8.692
4	5.999	5.964	5.936	5.912	5.891	5.873	5.858	5.844
5	4.772	4.735	4.704	4.678	4.655	4.636	4.619	4.604
6	4.099	4.060	4.027	4.000	3.976	3.956	3.938	3.922
7	3.677	3.637	3.603	3.575	3.550	3.529	3.511	3.494
8	3.388	3.347	3.313	3.284	3.259	3.237	3.218	3.202
9	3.179	3.137	3.102	3.073	3.048	3.025	3.006	2.989
10	3.020	2.978	2.943	2.913	2.887	2.865	2.845	2.828
11	2.896	2.854	2.818	2.788	2.761	2.739	2.719	2.701
12	2.796	2.753	2.717	2.687	2.660	2.637	2.617	2.599
13	2.714	2.671	2.635	2.604	2.577	2.554	2.533	2.515
14	2.646	2.602	2.566	2.534	2.507	2.484	2.463	2.445
15	2.588	2.544	2.507	2.475	2.448	2.424	2.403	2.385
16	2.538	2.494	2.456	2.425	2.397	2.373	2.352	2.333
17	2.494	2.450	2.413	2.381	2.353	2.329	2.308	2.289
18	2.456	2.412	2.374	2.342	2.314	2.290	2.269	2.250
19	2.423	2.378	2.340	2.308	2.280	2.256	2.234	2.215
20	2.393	2.348	2.310	2.278	2.250	2.225	2.203	2.184
21	2.366	2.321	2.283	2.250	2.222	2.197	2.176	2.156
22	2.342	2.297	2.259	2.226	2.198	2.173	2.151	2.131
23	2.320	2.275	2.236	2.204	2.175	2.150	2.128	2.109
24	2.300	2.255	2.216	2.183	2.155	2.130	2.108	2.088
25	2.282	2.236	2.198	2.165	2.136	2.111	2.089	2.069
26	2.265	2.220	2.181	2.148	2.119	2.094	2.072	2.052
27	2.250	2.204	2.166	2.132	2.103	2.078	2.056	2.036
28	2.236	2.190	2.151	2.118	2.089	2.064	2.041	2.021
29	2.223	2.177	2.138	2.105	2.075	2.050	2.027	2.007
30	2.211	2.165	2.126	2.092	2.063	2.037	2.015	1.995
31	2.199	2.153	2.114	2.080	2.051	2.026	2.003	1.983
32	2.189	2.142	2.103	2.070	2.040	2.015	1.992	1.972
33	2.179	2.133	2.093	2.060	2.030	2.004	1.982	1.961
34	2.170	2.123	2.084	2.050	2.021	1.995	1.972	1.952
35	2.161	2.114	2.075	2.041	2.012	1.986	1.963	1.942
36	2.153	2.106	2.067	2.033	2.003	1.977	1.954	1.934
37	2.145	2.098	2.059	2.025	1.995	1.969	1.946	1.926
38	2.138	2.091	2.051	2.017	1.988	1.962	1.939	1.918
39	2.131	2.084	2.044	2.010	1.981	1.954	1.931	1.911
40	2.124	2.077	2.038	2.003	1.974	1.948	1.924	1.904
50	2.073	2.026	1.986	1.952	1.921	1.895	1.871	1.850
60	2.040	1.993	1.952	1.917	1.887	1.860	1.836	1.815
120	1.959	1.910	1.869	1.834	1.803	1.775	1.750	1.728

90% table

TABLE IX
***F* distribution (continued)**

γ_2 \ γ_1	11	12	13	14	15	16	17	18	19	20
1	60.473	60.705	60.903	61.072	61.220	61.350	61.464	61.566	61.658	61.740
2	9.401	9.408	9.414	9.420	9.425	9.429	9.432	9.435	9.438	9.441
3	5.223	5.216	5.210	5.205	5.200	5.196	5.193	5.190	5.187	5.185
4	3.907	3.896	3.886	3.878	3.870	3.864	3.858	3.853	3.849	3.844
5	3.282	3.268	3.257	3.247	3.238	3.230	3.223	3.217	3.212	3.207
6	2.920	2.905	2.892	2.881	2.871	2.863	2.855	2.848	2.842	2.836
7	2.684	2.668	2.654	2.643	2.632	2.623	2.615	2.607	2.601	2.595
8	2.519	2.502	2.488	2.475	2.464	2.455	2.446	2.438	2.431	2.425
9	2.396	2.379	2.364	2.351	2.340	2.329	2.320	2.312	2.305	2.298
10	2.302	2.284	2.269	2.255	2.244	2.233	2.224	2.215	2.208	2.201
11	2.227	2.209	2.193	2.179	2.167	2.156	2.147	2.138	2.130	2.123
12	2.166	2.147	2.131	2.117	2.105	2.094	2.084	2.075	2.067	2.060
13	2.116	2.097	2.080	2.066	2.053	2.042	2.032	2.023	2.014	2.007
14	2.073	2.054	2.037	2.022	2.010	1.998	1.988	1.979	1.970	1.962
15	2.037	2.017	2.000	1.985	1.972	1.961	1.950	1.941	1.932	1.924
16	2.005	1.985	1.968	1.953	1.940	1.928	1.917	1.908	1.899	1.891
17	1.978	1.958	1.940	1.925	1.912	1.900	1.889	1.879	1.870	1.862
18	1.954	1.933	1.916	1.900	1.887	1.875	1.864	1.854	1.845	1.837
19	1.932	1.912	1.894	1.878	1.865	1.852	1.841	1.831	1.822	1.814
20	1.913	1.892	1.875	1.859	1.845	1.833	1.821	1.811	1.802	1.794
21	1.896	1.875	1.857	1.841	1.827	1.815	1.803	1.793	1.784	1.776
22	1.880	1.859	1.841	1.825	1.811	1.798	1.787	1.777	1.768	1.759
23	1.866	1.845	1.827	1.811	1.796	1.784	1.772	1.762	1.753	1.744
24	1.853	1.832	1.814	1.797	1.783	1.770	1.759	1.748	1.739	1.730
25	1.841	1.820	1.802	1.785	1.771	1.758	1.746	1.736	1.726	1.718
26	1.830	1.809	1.790	1.774	1.760	1.747	1.735	1.724	1.715	1.706
27	1.820	1.799	1.780	1.764	1.749	1.736	1.724	1.714	1.704	1.695
28	1.811	1.790	1.771	1.754	1.740	1.726	1.715	1.704	1.694	1.685
29	1.802	1.781	1.762	1.745	1.731	1.717	1.705	1.695	1.685	1.676
30	1.794	1.773	1.754	1.737	1.722	1.709	1.697	1.686	1.676	1.667
31	1.787	1.765	1.746	7.729	1.714	1.701	1.689	1.678	1.668	1.659
32	1.780	1.758	1.739	1.722	1.707	1.694	1.682	1.671	1.661	1.652
33	1.773	1.751	1.732	1.715	1.700	1.687	1.675	1.664	1.654	1.645
34	1.767	1.745	1.726	1.709	1.694	1.680	1.668	1.657	1.647	1.638
35	1.761	1.739	1.720	1.703	1.688	1.674	1.662	1.651	1.641	1.632
36	1.756	1.734	1.715	1.697	1.682	1.669	1.656	1.645	1.635	1.626
37	1.751	1.729	1.709	1.692	1.677	1.663	1.651	1.640	1.630	1.620
38	1.746	1.724	1.704	1.687	1.672	1.658	1.646	1.635	1.624	1.615
39	1.741	1.719	1.700	1.682	1.667	1.653	1.641	1.630	1.619	1.610
40	1.737	1.715	1.695	1.678	1.662	1.649	1.636	1.625	1.615	1.605
50	1.703	1.680	1.660	1.643	1.627	1.613	1.600	1.588	1.578	1.568
60	1.680	1.657	1.637	1.619	1.603	1.589	1.576	1.564	1.553	1.543
120	1.625	1.601	1.580	1.562	1.545	1.530	1.516	1.504	1.493	1.482

Comparing two means: variances equal ($\sigma_1 = \sigma_2$)

Example: A study of the tensile strength of ductile iron annealed or strengthened at two different temperatures is conducted. It is thought that the lower temperature will yield the higher mean tensile strength. These data results:

1450 °F	1650 °F
$n_1 = 10$	$n_2 = 16$
$\bar{x}_1 = 18{,}900psi$	$\bar{x}_2 = 17{,}500psi$
${S_1}^2 = 1600$	${S_2}^2 = 2500$

Solution:

Next, let's test if we have $\mu_1 = \mu_2$. i.e., we want to test H_0: $\mu_1 = \mu_2; H_1: \mu_1 > \mu_2$

Since $\sigma_1^2 = \sigma_2^2$, the pooled variance can be used. i.e., ${S_p}^2 = \frac{(n_1-1){S_1}^2+(n_2-1){S_2}^2}{n_1+n_2-2} = 2162.5$

Assuming H_0 is true, the test statistic $\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)_0}{\sqrt{{S_p}^2(\frac{1}{n_1}+\frac{1}{n_2})}} \sim T_{24}$ and the observed value is $\frac{18900-17500}{\sqrt{2162.5\times(\frac{1}{10}+\frac{1}{16})}} = 74.68$

Since this is a right-tailed test, $p_{value} = P[T_{24} \geq 74.68]$

From the T-table, we have, $P[T_{24} \geq 3.745] = 0.0005$

Hence, $p_{value} = P[T_{24} \geq 74.68] < 0.0005$

Therefore, H_0 should be rejected. i.e., the evidence strongly support that $\mu_1 > \mu_2$.

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

We already know that:

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}}} \sim Z$$

When the variances of the two populations unequal, we replace ${\sigma_1}^2$ and ${\sigma_2}^2$ by ${S_1}^2$ and ${S_2}^2$. Then,

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{{S_1}^2}{n_1} + \frac{{S_2}^2}{n_2}}} \sim T_\gamma$$

Where the degree of freedom γ has to be estimated from the samples as follows:

$$\gamma = \frac{\left(\frac{{S_1}^2}{n_1} + \frac{{S_2}^2}{n_2}\right)^2}{\frac{\left(\frac{{S_1}^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{{S_2}^2}{n_2}\right)^2}{n_2 - 1}}$$

γ will be rounded ***down*** to the nearest integer.

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

Example: The following are the information on two independent samples from two normal distributions with means μ_A *and* μ_B and variances ${\sigma_A}^2$ *and* ${\sigma_B}^2$. Test the following hypothesis: H_0: $\mu_A = \mu_B$, $H_1: \mu_A > \mu_B$

Population A	Population B
$n_A = 25$	$n_B = 16$
$\bar{x}_A = 380$	$\bar{x}_B = 370$
$S_A^2 = 100$	$S_B^2 = 400$

In order to carry out the test on μ_A *and* μ_B, we have to find out if $\sigma_A^2 = \sigma_B^2$. Hence,
First, we want to do the test on $H_0: \sigma_A^2 = \sigma_B^2; H_1: \sigma_A^2 \neq \sigma_B^2$

Assuming H_0 is true, the test statistic $\frac{S_B^2}{S_A^2} \sim F_{15,24}$ and the observed value is $\frac{S_B^2}{S_A^2} = 4$

Since $\frac{S_B^2}{S_A^2} = 4 > 1$ and this is a two tailed test, we have,

$$p_{value} = 2P[F_{15,24} \geq 4]$$

From the F-table, we have, $P\left[F_{15,24} \geq 2.108\right] = 0.05$
Hence, $P\left[F_{15,24} \geq 4\right] < P\left[F_{15,24} \geq 2.108\right] = 0.05$
And $p_{value} = 2P\left[F_{15,24} \geq 4\right] < 2 \times 0.05 = 0.10$
Therefore, H_0 should be rejected at the significance level of 0.1. i.e., we have $\sigma_A^2 \neq \sigma_B^2$

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

Example: The following are the information on two independent samples from two normal distributions with means μ_A and μ_B and variances $\sigma_A{}^2$ and $\sigma_B{}^2$. Test the following hypothesis: H_0: $\mu_A = \mu_B$, $\quad H_1: \mu_A > \mu_B$

Population A	Population B
$n_A = 25$	$n_B = 16$
$\bar{x}_A = 380$	$\bar{x}_B = 370$
$S_A^2 = 100$	$S_B^2 = 400$

Secondly, let's carry out the test on the means: H_0: $\mu_A = \mu_B$, $\quad H_1: \mu_A > \mu_B$

Since $\sigma_A^2 \neq \sigma_B^2$, we choose $\frac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{\sqrt{\frac{S_A{}^2}{n_A} + \frac{S_B{}^2}{n_B}}}$ as the test statistic. Assuming H_0 is true, we have,

$$\frac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{\sqrt{\frac{S_A{}^2}{n_A} + \frac{S_B{}^2}{n_B}}} \sim T_\gamma, where\ \ \gamma = \frac{\left(\frac{S_A{}^2}{n_A} + \frac{S_B{}^2}{n_B}\right)^2}{\frac{\left(\frac{S_A{}^2}{n_A}\right)^2}{n_A - 1} + \frac{\left(\frac{S_B{}^2}{n_B}\right)^2}{n_B - 1}} \approx 19$$

The observed value of this statistic can be calculated as 1.86

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

Example: The following are the information on two independent samples from two normal distributions with means μ_A and μ_B and variances ${\sigma_A}^2$ and ${\sigma_B}^2$. Test the following hypothesis: H_0: $\mu_A = \mu_B$, H_1: $\mu_A > \mu_B$

Population A	Population B
$n_A = 25$	$n_B = 16$
$\bar{x}_A = 380$	$\bar{x}_B = 370$
$S_A^2 = 100$	$S_B^2 = 400$

Since this is a right-tailed test, we have,

$$p_{value} = P[T_{19} \geq 1.86]$$

From T-table, we have,

$$P[T_{19} \geq 1.729] = 0.05; P[T_{19} \geq 2.093] = 0.025$$

Hence,

$$0.025 < p_{value} < 0.05$$

Since $p_{value} < \alpha = 0.1$, H_0 should be rejected at the significance level of 0.1.
i.e., the sampled data supports H_1: $\mu_A > \mu_B$

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

Example: The following are the information on two independent samples from two normal distributions with means μ_A *and* μ_B and variances ${\sigma_A}^2$ *and* ${\sigma_B}^2$. Test the following hypothesis: H_0: $\mu_A = \mu_B$, H_1: $\mu_A > \mu_B$
(Notice that this example is different from the previous example. The sample variances in this example are ten times of those in previous example) .

Population A	Population B
$n_A = 25$	$n_B = 16$
$\bar{x}_A = 380$	$\bar{x}_B = 370$
$S_A^2 = 1000$	$S_B^2 = 4000$

In order to carry out the test on μ_A *and* μ_B, we have to find out if $\sigma_A^2 = \sigma_B^2$. Hence,
First, we want to do the test on $H_0: \sigma_A^2 = \sigma_B^2; H_1: \sigma_A^2 \neq \sigma_B^2$

Assuming H_0 is true, the test statistic $\frac{S_B^2}{S_A^2} \sim F_{15,24}$ and the observed value is $\frac{S_B^2}{S_A^2} = 4$

Since $\frac{S_B^2}{S_A^2} = 4 > 1$ and this is a two tailed test, we have,

$$p_{value} = 2P[F_{15,24} \geq 4]$$

From the F-table, we have, $P\left[F_{15,24} \geq 2.108\right] = 0.05$
Hence, $P\left[F_{15,24} \geq 4\right] < P\left[F_{15,24} \geq 2.108\right] = 0.05$
And $p_{value} = 2P\left[F_{15,24} \geq 4\right] < 2 \times 0.05 = 0.10$
Therefore, H_0 should be rejected at the significance level of 0.1. i.e., we have $\sigma_A^2 \neq \sigma_B^2$

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

Example: The following are the information on two independent samples from two normal distributions with means μ_A *and* μ_B and variances $\sigma_A{}^2$ *and* $\sigma_B{}^2$. Test the following hypothesis: H_0: $\mu_A = \mu_B$, H_1: $\mu_A > \mu_B$
(Notice that this example is different from the previous example. The sample variances in this example are ten times of those in previous example) .

Population A	Population B
$n_A = 25$	$n_B = 16$
$\bar{x}_A = 380$	$\bar{x}_B = 370$
$S_A^2 = 1000$	$S_B^2 = 4000$

Secondly, let's carry out the test on the means: H_0: $\mu_A = \mu_B$, H_1: $\mu_A > \mu_B$

Since $\sigma_A^2 \neq \sigma_B^2$, we choose $\frac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{\sqrt{\frac{S_A{}^2}{n_A} + \frac{S_B{}^2}{n_B}}}$ as the test statistic. Assuming H_0 is true, we have,

$$\frac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{\sqrt{\frac{S_A{}^2}{n_A} + \frac{S_B{}^2}{n_B}}} \sim T_\gamma, where\ \gamma = \frac{\left(\frac{S_A{}^2}{n_A} + \frac{S_B{}^2}{n_B}\right)^2}{\frac{\left(\frac{S_A{}^2}{n_A}\right)^2}{n_A - 1} + \frac{\left(\frac{S_B{}^2}{n_B}\right)^2}{n_B - 1}} \approx 19$$

The observed value of this statistic can be calculated as 0.588

Comparing two means: variances unequal ($\sigma_1 \neq \sigma_2$)

Example: The following are the information on two independent samples from two normal distributions with means μ_A and μ_B and variances ${\sigma_A}^2$ and ${\sigma_B}^2$. Test the following hypothesis: H_0: $\mu_A = \mu_B$, H_1: $\mu_A > \mu_B$
(Notice that this example is different from the previous example. The sample variances in this example are ten times of those in previous example) .

Population A	Population B
$n_A = 25$	$n_B = 16$
$\bar{x}_A = 380$	$\bar{x}_B = 370$
$S_A^2 = 1000$	$S_B^2 = 4000$

Since this is a right-tailed test, we have,

$$p_{value} = P[T_{19} \geq 0.588]$$

From T-table, we have,

$$P[T_{19} \leq 0.257] = 0.6; P[T_{19} \leq 0.688] = 0.75$$

Hence,

$$0.25 < p_{value} < 0.4$$

Since $p_{value} > \alpha = 0.1$, H_0 should not be rejected at the significance level of 0.1.