How to assign a value to a probability?

- **The personal approach**

Example:

An oil spill has occurred. An environmental scientist asks, “What is the probability that this spill can be contained before it cause widespread damage to nearby beach?”
To answer this question, scientists are called upon to make value judgement, that is, to assign a probability to the event based on informed *personal opinion*.

Its accuracy depends on ?

The accuracy of the information available and the ability (experience) of the opinion provider. Sometime this maybe very expensive.

How to assign a value to a probability?

- **The personal approach**

Try to assign a probability to the following events:

“*the sun will rise tomorrow morning*”

“*I will live forever*”

“*I will win the lottery some day*”

“*A randomly selected student will get an A in this course*”

“*Mr. X is 25 years old and he will die within 20 years*”

How to assign a value to a probability?

- **The relative frequency approach (experiment method)**

Example:

An electrical engineer is studying the peak demand at a power plant. It is observed that on 80 of the 100 days randomly selected for study from past records, the peak demand occurred between 6 and 7 PM. It is natural to assume that the probability of this occurring on another day is approximately $\frac{80}{100} = 0.80$

This number is not simply a personal opinion. It is based on ***repeated experimentation and observation***. It is a *relative frequency*.

The relative frequency approach can be used whenever the experiment can be repeated many times and the results observed. In such cases, the probability of the occurrence of event A, denoted by $P[A]$, is approximated as follows:

$$P[A] \approx \frac{f}{n} = \frac{number\ of\ times\ event\ A\ occurred}{number\ of\ times\ experiment\ was\ run}$$

- The experiment must be repeatable.
- For a large number of trials, the approximate probability obtained is usually quite accurate

How to assign a value to a probability?

- **The relative frequency approach (experiment method)**

Practice example:

Some trees in a forest are showing signs of disease. A random sample of 200 trees of various sizes are examined yielding the following results:

Size	Disease free	Doubtful	Diseased	Total
Large	35	18	15	68
Medium	46	32	14	92
Small	24	8	8	40
Total	105	58	37	200

What is the probability that a randomly selected tree is diseased? 37/200

What is the probability that a randomly selected tree is large in size? 68/200

What is the probability that a randomly selected tree is small and diseased? 8/200

How to assign a value to a probability?

- **The classical approach**

Example: Suppose a “fun size” bag of M&Ms contains 9 brown candies, 6 yellow candies, 7 red candies, 4 orange candies, 2 blue candies, and 2 green candies. Suppose that a candy is randomly selected.

(1) What is the probability that the selected candy is brown?

$$\frac{9}{9+6+7+4+2+2} = \frac{3}{10}$$

(2) What is the probability that the selected candy is blue?

$$\frac{2}{9+6+7+4+2+2} = \frac{1}{15}$$

How to assign a value to a probability?

- **The classical approach**

This method can be used only when it is reasonable to assume that the possible outcomes of the experiment are equally likely. In this case, the probability of the occurrence of event A Is given by the following classical formula:

$$P[A] = \frac{n(A)}{n(S)} = \frac{number\ of\ ways\ A\ can\ occur}{number\ of\ ways\ the\ experiment\ can\ proceed}$$

- It does not require experimentation
- The probability is accurate through counting

How to assign a value to a probability?

- **The classical approach**

Practice Problem:

What is the probability that a child born to a couple heterozygous for eye color (each with genes for both brown and blue eyes) will be brown-eyed?

Note the following facts about the eye color of a child:

- A child receives one gene from each parent.
- Each parent is just as likely to contribute a gene for brown eyes as for blue eyes.
- The gene for brown eyes is dominant.

How to assign a value to a probability?

- **The classical approach**

Practice Problem:

What is the probability that a child born to a couple heterozygous for eye color (each with genes for both brown and blue eyes) will be brown-eyed?

Note the following facts about the eye color of a child:

- A child receives one gene from each parent. Hence the possible eye color gene combinations for the child are (brown, blue), (blue, brown), (blue, blue), and (brown, brown). Where the first member of each pair is the gene received from the father.
- Each parent is just as likely to contribute a gene for brown eyes as for blue eyes, all four possibilities are equally likely.
- The gene for brown eyes is dominant, i.e., three of the four possibilities lead to a brown-eyed child.

Hence, the probability that the child will be brown-eyed is $\frac{3}{4} = 0.75$

This figure is not a personal opinion, nor is it based on repeated experimentations.

Permutations and Combinations

- **Permutation:**

A permutation is an ***arrangement of objects in a definite order***

Example: Let's count the permutation from three letters $A, B, and\ C$
All the permutations: $ABC, ACB, BAC, BCA, CAB, CBA$ (total 6)

- **Combination:**

A combination is a ***selection of objects without regard to order***

Example: count the possible results to choose two from four letters A, B, C, D
All possible results: AB, AC, AD, BC, BD, CD (total 6)
Note, when counting combinations, AB and BA are counted as the same result, i.e., the order of the letter does not matter.

Counting Permutations

Example:

How many possible license plates could be stamped if each plate is required to have exactly 3 letters and 3 numbers?

Chain positions:	1st	2nd	3rd	4th	5th	6th
choices	10	10	10	26	26	26

\# of permutations: $10 \times 10 \times 10 \times 26 \times 26 \times 26 = 17{,}576{,}000$

How many possible license plates could be stamped if each plate is required to have exactly 3 unique letters and 3 unique numbers?

Chain positions:	1st	2nd	3rd	4th	5th	6th
choices	10	9	8	26	25	24

\# of permutations: $10 \times 9 \times 8 \times 26 \times 25 \times 24 = 11{,}232{,}000$

Counting Permutations

The multiplication principle:

Consider an experiment taking place in k stages. Let n_i denote the number of ways in which stage i can occur for $i = 1,2, \ldots, k$. Altogether the experiment can occur in $n_1 \times n_2 \times n_3 \times \cdots n_k = \prod_{i=1}^{k} n_i$ ways.

Questions when applying the multiplication principle:

1. Can objects be repeated at different stages?
2. Restriction for different stages

Counting Permutations

Example: A segment of RNA is composed of "words". Each word is composed of a chain of three ribonucleotides. Each of the ribonucleotide in the chain can be one of Adenine (A), Uracil (U), Guanine (G), or Cytosine (C).

(1) How many words can be formed?
(2) How many of these words involve some repetition?
(3) How many of the words end with Uracil or Cytosine and have no repetition?

(1) $4 \times 4 \times 4 = 64$, repetition allowed

(2) We first find words without repetition: $4 \times 3 \times 2 = 24$
The remaining $64 - 24 = 40$ must have some repetition

(3) The third position has restrictions. The number of this position can only be 2
Therefore, without repetition, this can be calculated as: $3 \times 2 \times 2 = 12$

Counting Permutations

Example: suppose we have n distinct objects but we are going to use only r of them in each arrangement. How many permutations are possible?
Let's apply the multiplication rule:

Positions:	1st	2nd	3rd	…	rth
Choices	n	n-1	n-2		n-r+1

The number of permutations, denoted by ${}_{r}^{n}P$, can be calculated as:

$${}_{r}^{n}P = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Where $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ is the factorial of n

Special case, when $r = n$, the result is $n!$, this is the permutations from n distinct objects.

Question: What are the assumptions in the above calculation?

Objects are distinct, no repetition; no restrictions on any position in the arrangement

Practice: ${}_{4}^{9}P = ?$ 3024 ${}_{7}^{7}P = ?$ 5040

Counting Combinations

Let's reconsider the problem of counting the number of permutations of r objects from n distinct ones.
This problem can be treated in two stages:
First, the r objects must be selected from n distinct ones. Without order, this is a combination problem. Let ${}_{r}^{n}C$ be the number of combinations of selecting r from n distinct objects.
Next, arrange the r selected objects in order. There are $r!$ Ways to do this.
Therefore, we have:

$${}_{r}^{n}P = {}_{r}^{n}C \times r!$$

Thus,

$${}_{r}^{n}C = \frac{{}_{r}^{n}P}{r!} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

Practice:

${}_{3}^{5}C = \binom{5}{3} = ?$ 10

${}_{0}^{5}C = \binom{5}{0} = ?$ 1

Counting Combinations

Example: A foundry ships a lot of 20 engine blocks of which 5 contain internal flaws. The purchaser will select three blocks at random and test them. The lot will be accepted if no flaws are found. What is the probability that this lot will be accepted?

Solution:
The number of ways to select 3 blocks from 20 is:

$${}^{20}_{3}C = \frac{20!}{3!\,17!} = 1140$$

In order to obtain no flaw engine for test, all 3 of the sampled engines must be selected from the 15 unflawed engines in the lot. This is counted as:

$${}^{15}_{3}C = \frac{15!}{3!\,12!} = 455$$

Therefore,

$$P[lot\ is\ accepted] = P[three\ selected\ engines\ are\ flawless] = \frac{455}{1140} \approx 40\%$$