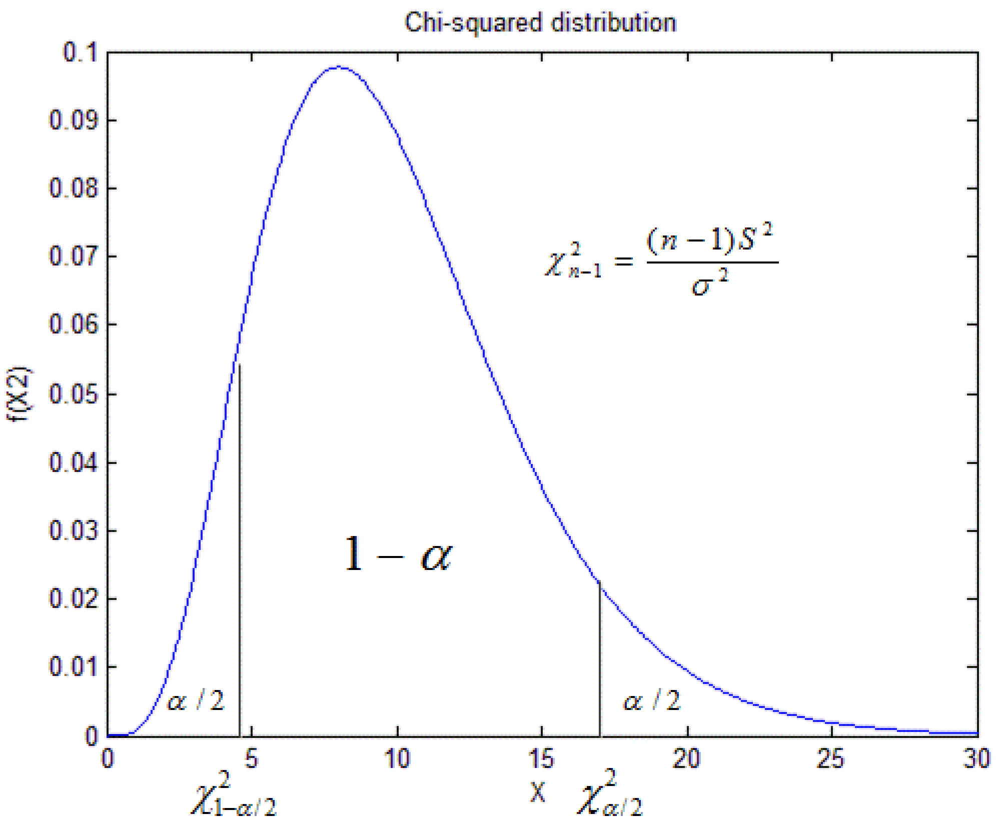
Interval Estimation of Variance

Now, we can use the R.V. $\frac{(n-1)S^2}{\sigma^2}$ to derive a $100(1-\alpha)\%$ confidence interval on σ^2.

Consider the X_{n-1}^2 graph. where, ***$\chi_{\frac{\alpha}{2}}^2$ is the critical point for right-tailed probability of $\frac{\alpha}{2}$, and $\chi_{1-\frac{\alpha}{2}}^2$ is the critical point for the right-tailed probability of $1-\frac{\alpha}{2}$.***



Then,

$$P\left[\chi_{1-\frac{\alpha}{2}}^2 \le \frac{(n-1)S^2}{\sigma^2} \le \chi_{\frac{\alpha}{2}}^2\right] = 1-\alpha$$

$$\Rightarrow P\left[\frac{1}{\chi_{\frac{\alpha}{2}}^2} \le \frac{\sigma^2}{(n-1)S^2} \le \frac{1}{\chi_{1-\frac{\alpha}{2}}^2}\right] = 1-\alpha$$

$$\Rightarrow P\left[\frac{(n-1)S^2}{\chi_{\frac{\alpha}{2}}^2} \le \sigma^2 \le \frac{(n-1)S^2}{\chi_{1-\frac{\alpha}{2}}^2}\right] = 1-\alpha$$

i.e., the $100(1-\alpha)\%$ confidence interval on σ^2 is $[L_1, L_2]$, where $L_1 = \frac{(n-1)S^2}{\chi_{\frac{\alpha}{2}}^2}$, and $L_2 = \frac{(n-1)S^2}{\chi_{1-\frac{\alpha}{2}}^2}$

Interval Estimation of Variance

Example: The following 25 samples are drawn from a normal distribution:

3.4	3.6	4.0	0.4	2.0
3.0	3.1	4.1	1.4	2.5
1.4	2.0	3.1	1.8	1.6
3.5	2.5	1.7	5.1	0.7
4.2	1.5	3.0	3.9	3.0

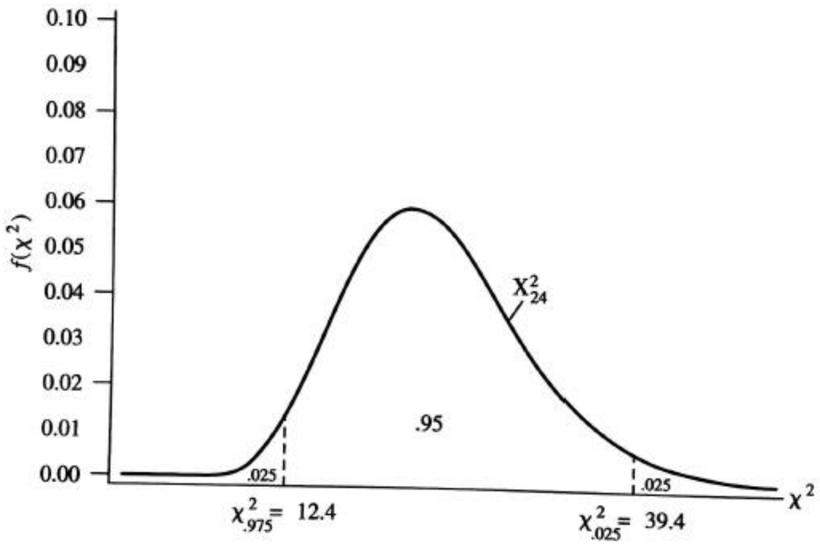


FIGURE 8.3
Partition of the X^2_{24} curve needed to construct a 95% confidence interval on the variance in relative I/O content of the consulting firm of Example 8.1.1.

Find the 95% confidence interval of the standard deviation σ of the distribution.

the $100(1-\alpha)\%$ confidence interval on σ^2 is $[L_1, L_2]$, where $L_1 = \frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}}$, and $L_2 = \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}}$

From the sampled data, it can be calculated that $S^2 = 1.408$
Then, $L_1 = \frac{24\times1.4}{39.4} = 0.858; L_2 = \frac{24\times1.4}{12.4} = 2.725$
Hence, the 95% confidence interval of σ^2 is $[0.858{,}2.725]$. The 95% confidence interval of σ is $[0.926{,}1.65]$

TABLE IV

Cumulative chi-squared distribution ***(concluded)***

$P[\chi^2_\gamma \le t]$

γ \ F	0.005	0.010	0.025	0.050	0.100	0.250	0.500	0.750	0.900	0.950	0.975	0.990	0.995
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

Interval Estimation of mean: with unknown variance

We already learned that when variance σ^2 is known, $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is a standard normal random variable.

However, when σ is replaced by its estimator s (sample standard deviation), the random variable $\frac{\bar{X}-\mu}{s/\sqrt{n}}$ follows a student-t (or T) distribution.

The T distribution:

Let Z be a standard normal random variable and let ${\rm X}^2{}_\gamma$ be an independent chi-squared random variable with γ degree of freedom. The random variable $\frac{Z}{\sqrt{\frac{{\rm X}^2{}_\gamma}{\gamma}}}$ is said to follow a T distribution with γ degree of freedom. i.e., $\frac{Z}{\sqrt{\frac{{\rm X}^2{}_\gamma}{\gamma}}} \sim T_\gamma$

The density function of a T_γ random variable is: $f(t) = \frac{\Gamma(\gamma+1)/2}{\Gamma(\frac{\gamma}{2})\sqrt{\pi\gamma}}(1+\frac{t^2}{\gamma})^{-(\gamma+1)/2}, -\infty < t < \infty, \gamma > 0$

$f(t) = f(-t)$? Find the value of t that maximize $f(t)$. Can you image the shape of $f(t)$?

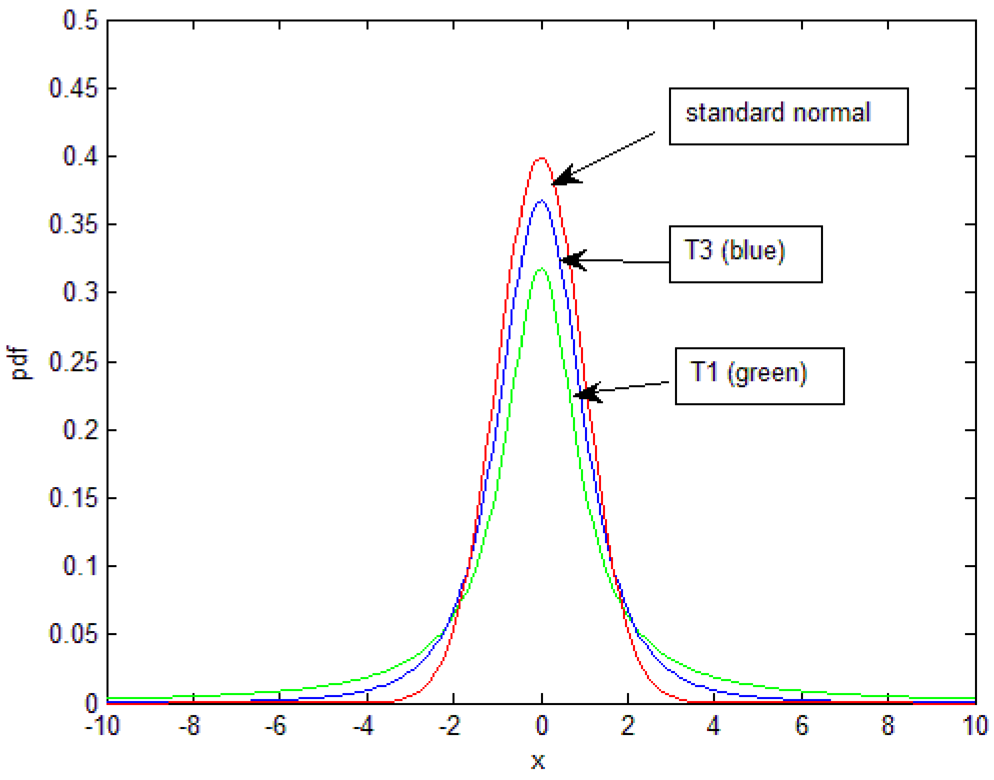
Interval Estimation of mean: with unknown variance

The T distribution:

The density function of a T_γ random variable is:

$$f(t) = \frac{\Gamma(\gamma+1)/2}{\Gamma(\frac{\gamma}{2})\sqrt{\pi\gamma}}(1+\frac{t^2}{\gamma})^{-(\gamma+1)/2}, -\infty < t < \infty$$

- $T_\gamma, \gamma = 1,2,3,\ldots$, denotes a T random variable with γ degree of freedom.
- The density is bell-shaped and symmetric about $t = 0$
- As γ increases, the variance of the T random variable decrease.
- As γ increases, the curve of the T random variable approaches the standard normal curve.



The T distribution table:

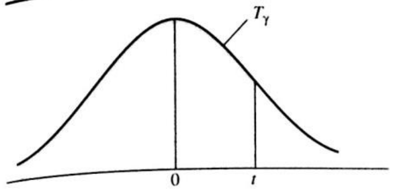
Define t_r as $P\left[T_\gamma \geq t_r\right] = r \Rightarrow P\left[T_\gamma \leq t_r\right] = 1 - r$, i.e., **$t_r$ is the critical point for right-tailed probability r.**

Then, $P\left[-t_{\alpha/2} \leq T_\gamma \leq t_{\alpha/2}\right] = 1 - \alpha$

Example: read the values of t_α using the T-table

$P[T_{10} \leq t_{0.1}] = 0.90$ $\quad t_{0.1} = 1.372$
$P[T_{10} \leq t_{0.9}] = 0.10$ $\quad t_{0.9} = -1.372$
$P[-t \leq T_{10} \leq t] = 0.95$ $\quad t = 2.228$

TABLE VI
T distribution



Column heading = cumulative probability
Row heading = degrees of freedom
Row ∞ = standard normal values

	$P[T_\gamma \leq t]$								
γ	.6	.75	.9	.95	.975	.99	.995	.999	.9995
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	318.317	636.607
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.611	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.385	3.646
31	0.256	0.682	1.309	1.696	2.040	2.453	2.744	3.375	3.633
32	0.255	0.682	1.309	1.694	2.037	2.449	2.738	3.365	3.622
33	0.255	0.682	1.308	1.692	2.035	2.445	2.733	3.356	3.611
34	0.255	0.682	1.307	1.691	2.032	2.441	2.728	3.348	3.601
35	0.255	0.682	1.306	1.690	2.030	2.438	2.724	3.340	3.591
36	0.255	0.681	1.306	1.688	2.028	2.434	2.719	3.333	3.582
37	0.255	0.681	1.305	1.687	2.026	2.431	2.715	3.326	3.574
38	0.255	0.681	1.304	1.686	2.024	2.429	2.712	3.319	3.566
39	0.255	0.681	1.304	1.685	2.023	2.426	2.708	3.313	3.558
40	0.255	0.681	1.303	1.684	2.021	2.423	2.704	3.307	3.551
41	0.255	0.681	1.303	1.683	2.020	2.421	2.701	3.301	3.544
42	0.255	0.680	1.302	1.682	2.018	2.418	2.698	3.296	3.538
43	0.255	0.680	1.302	1.681	2.017	2.416	2.695	3.291	3.532
44	0.255	0.680	1.301	1.680	2.015	2.414	2.692	3.286	3.526