ESE 351 **3:25pm - 4:00pm, Tuesday, February 24, 2015**

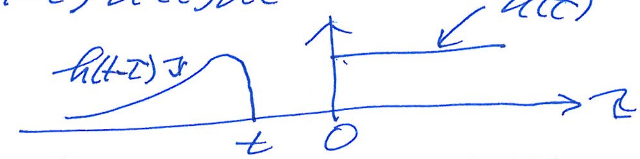
Test 3 (Chapters 4 and 5) Part 1 **Your Name:** ______________________

Closed book except for one personally made letter-size ($8\frac{1}{2} \times 11$) crib sheet with both sides allowed. No calculators are allowed. Write your name on the second line of every page now. Time is of the essence. If you get stuck, or the problem seems too intractable or the numbers seem unwieldy, you have probably gone off course; please move on to the next problem and come back to it later. If you use transform methods, you will get only a half of the credit. If you need extra sheets of paper, just ask for them.

1. [easy, 8 points] Consider a linear time-invariant system with the following impulse response: $h(t) = 4te^{-t}1(t)$. Find the system response $y(t)$ to the unit step input, $u(t) = 1(t)$, using the convolution method with graphs. Be sure to find $y(t)$ from $t = -\infty$ to $t = \infty$ and evaluate integrals. Hint: $\lim_{t\to\infty} h(t) = 0$.

$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau$

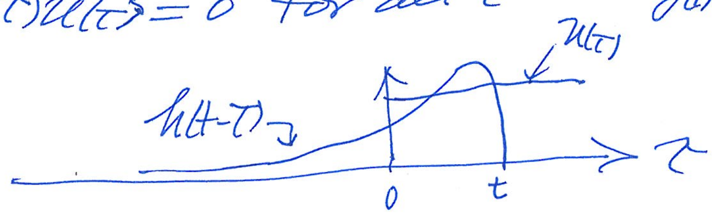
Case (1) $t<0$



No overlap between $h(t-\tau)$ and $u(\tau)$

$\Rightarrow h(t-\tau)u(\tau) = 0$ for all τ $\quad y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$

Case (2) $t \geq 0$



$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t 4(t-\tau)e^{-(t-\tau)}d\tau$

change of variable $\sigma \triangleq t-\tau \Rightarrow d\sigma = -d\tau$

$\tau = 0 \Rightarrow \sigma = t \qquad \tau = t \Rightarrow \sigma = 0$

$y(t) = \int_t^0 4\sigma e^{-\sigma}(-d\sigma) = \int_0^t 4\sigma e^{-\sigma} d\sigma$

Note $\frac{d(-e^{-\sigma})}{d\sigma} = e^{-\sigma}$ or $d(-e^{-\sigma}) = e^{-\sigma}d\sigma$

$y(t) = \int_0^t 4\sigma\, d(-e^{-\sigma}) = [4\sigma(-e^{-\sigma})]_0^t - \int_0^t 4(-e^{-\sigma})d\sigma$

$= -4[\sigma e^{-\sigma}]_0^t + 4\int_0^t e^{-\sigma}d\sigma$

$= -4te^{-t} + 4[-e^{-\sigma}]_0^t = -4te^{-t} - 4e^{-t} + 4$

Answer: $y(t) =$ $[4 - 4e^{-t} - 4te^{-t}]1(t)$ or $y(t) = \begin{cases} 4 - 4e^{-t} - 4te^{-t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$

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Test 3 (Chapters 4 and 5) Part 2 **Your Name:** ______________________

2. [harder, 14 points] Consider the system with the following input-output difference equation:

$$\mathcal{S}^2 y - 5\mathcal{S}y + 6y = u, \quad y(-1) = 0,\ y(0) = 0,$$

where the input is $u(k) = 2^k 1(k)$, and the initial conditions are as given above. Find first the value $y(1)$ at time $k = 1$. Find then the output $y(k)$ for $k > 1$. If you use the z transforms, you will get only half the credit.

$y(k+2) - 5y(k+1) + 6y(k) = u(k) = 2^k 1(k)$ ---(*)

$y(k+2) = 5y(k+1) - 6y(k) + u(k)$

$y(1) = 5\cancel{y(0)}^{0} - 6\cancel{y(-1)}^{0} + \cancel{u(-1)}^{0} = 0$

$y(2) = 5\cancel{y(1)}^{0} - 6\cancel{y(0)}^{0} + \cancel{u(0)}^{1} = 1$

We note that after this $u(k) = 2^k$ instead of $2^k 1(k)$

In other words, the above equation is the first time

$y(k+2) - 5y(k+1) + 6y(k) = 2^k$ works. ---(**)

Hence we are going to use $y(1) = 0$ and $y(0) = 0$ as the initial conditions for (**).

$(\mathcal{S}^2 - 5\mathcal{S} + 6)y = (\mathcal{S}-2)(\mathcal{S}-3)y = 2^k$ ---(***)

The annihilator for 2^k is $a = (\mathcal{S}-2)$

$(\mathcal{S}-2)^2(\mathcal{S}-2)y = 0$

$y(k) = c_1 2^k + c_2 3^k + c_3 k 2^{k-1}$

The particular solution is $y_p(k) = c_3 k 2^{k-1}$

$(\mathcal{S}-3)(\mathcal{S}-2)\, c_3 k 2^{k-1} = (\mathcal{S}-3)[c_3\{(k+1)2^k - k2^k\}]$

$= (\mathcal{S}-3)[c_3 2^k] = c_3(\mathcal{S}-3)2^k = c_3[2^{k+1} - 3 \cdot 2^k]$

$= c_3(-1)2^k \Longleftrightarrow 2^k$ {the righthand of (***)}

Hence $c_3 = -1$. Thus $y(k) = c_1 2^k + c_2 3^k - k 2^{k-1}$

$y(0) = c_1 + c_2 = 0$; $c_2 = -c_1$

$y(1) = 2c_1 + 3c_1 - 1 = 0$; $2c_1 - 3c_1 = -c_1 = 1$

$\Rightarrow c_2 = +1$

Answer: $y(1) =$ 0 ; $y(k) =$ $3^k - 2^k - k2^{k-1}$

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Test 3 (Chapters 4 and 5) Part 3 **Your Name:** ______________________

3. [medium, 8 points] Consider the system with the following input-output differential equation:

$$\mathcal{D}^2 y + 4\mathcal{D}y + 5y = \mathcal{D}u, \quad y(0-) = -5, \quad \mathcal{D}y(0-) = 8.$$

Suppose that the initial conditions at time $t = 0-$ are as given above and that the input is a unit impulse, $u(t) = \delta(t)$. Find first the output $y(0+)$ and its derivative $\mathcal{D}y(0+)$ at time $t = 0+$. Find then the general solution for the output $y(t)$ for the positive time $t > 0$. Namely you need not compute the values of the arbitrary constants such as C_1, C_2, A and ϕ. If you use the Laplace transforms, you will get only half the credit.

$\mathcal{D}^2 y \sim \mathcal{D}u = \delta^{(1)}$

$\mathcal{D}y \sim u = \delta$

$y \sim \mathcal{D}^{-1}u = 1$

$\mathcal{D}^{-1}y \sim \mathcal{D}^{-2}u = r \leftarrow$ continuous $\Rightarrow \mathcal{D}^{-1}y(0+) = \mathcal{D}^{-1}y(0-)$

$\int_{0-}^{0+} \rightarrow [\mathcal{D}y(0+) - \underbrace{\mathcal{D}y(0-)}_{8}] + 4[y(0+) - \underbrace{y(0-)}_{-5}] + 5\underbrace{[\mathcal{D}^{-1}y(0+) - \mathcal{D}^{-1}y(0-)]}_{\to 0}$ $= \delta(0+) - \delta(0-) = 0$

$\mathcal{D}y(0+) - 8 + 4y(0+) + 20 = 0 \Rightarrow \mathcal{D}y(0+) + 4y(0+) = -12$

$\int \rightarrow \mathcal{D}y + 4y + 5\mathcal{D}^{-1}y = u = \delta$

$\int_{0-}^{0+} \rightarrow [y(0+) - \underbrace{y(0-)}_{-5}] + 4\underbrace{[\mathcal{D}^{-1}y(0+) - \mathcal{D}^{-1}y(0-)]}_{\to 0} + 5[0] = 1$

$y(0+) + 5 = 1 \Rightarrow y(0+) = -4 \Rightarrow \mathcal{D}y(0+) = 4$

For $t > 0$, $(\mathcal{D}^2 + 4\mathcal{D} + 5)y = [(\mathcal{D}+2)^2 + 1^2]y = 0$

$\mathcal{D} = -2 \pm i \Rightarrow y(t) = Ae^{-2t}\cos(t + \phi)$

Although the following is not required, we will find the values of A and ϕ.

$y(0+) = A\cos\phi = -4$

$\mathcal{D}y(t) = -2Ae^{-2t}\cos(t+\phi) - Ae^{-2t}\sin(t+\phi)$

$\mathcal{D}y(0+) = -2A\cos\phi - A\sin\phi = 4$

$\alpha = A\cos\phi = -4$

$\beta = A\sin\phi = -4 - 2A\cos\phi = -4 - 2\times(-4) = 8 - 4 = 4$

$A = \sqrt{4^2 + 4^2} = 4\sqrt{2}$; $\phi = \frac{3}{4}\pi$

Answer: $y(0+) =$ -4 ; $\mathcal{D}y(0+) =$ 4 ; $y(t) =$ $Ae^{-2t}\cos(t+\phi)$