Axioms of Probability (definition)

Let S be a sample space for an experiment, and A an event within S. A function P is called a probability function if:

(1) $0 \leq P[A] \leq 1\,, \quad for\ all\ A \subset S;$

(2) $P[S] = 1$

(3) If $A_1, A_2, \ldots A_n$ is a collection of *mutually exclusive* events in S, then $P[A_1 \cup A_2 \cup \cdots \cup A_n] = \sum_{i=1}^{n} P[A_i]$ (addition rule)

Important Properties of Probability

Let $A \; and \; B$ be events. Then,

$(1) \; P[\emptyset] = 0;$

(2) $A \subset B \Rightarrow P[A] \leq P[B]$

(3) $P[A] \leq 1$

(4) $P[A'] = 1 - P[A]$

(5) $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[A \cap C] + P[A \cap B \cap C]$$

Properties of Probability Derived from the Three Axioms

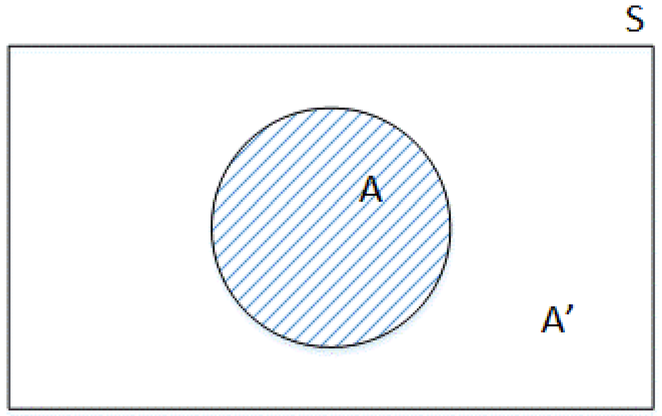
Complementarity rule: $P[A] + P[A'] = 1 \; or \; P[A'] = 1 - P[A]$, where, $A \subset S, A' is \; the \; complement \; of \; A$

Proof:

$S = A \cup A'$, A and A' are mutually exclusive.

Hence

$$P[S] = P[A \cup A'] = P[A] + P[A'] = 1$$



Venn Diagram

Now, can you prove that $P[\emptyset] = 0$?

Properties of Probability Derived from the Three Axioms

$$If\ A \subset B, then\ P[A] \leq P[B]$$

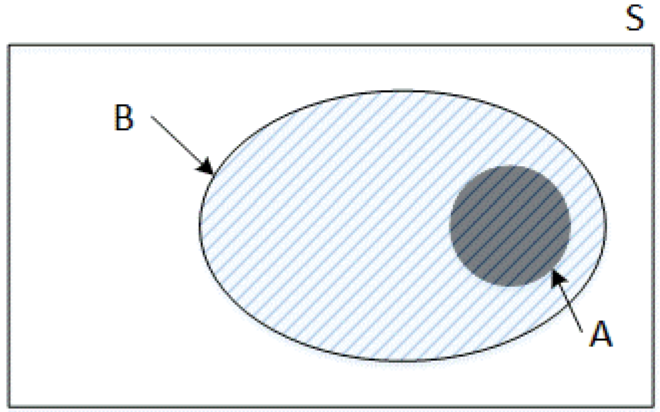
Proof:

$B = A \cup (B \cap A'),\ A$ and $B \cap A'$ are mutually exclusive (?). Hence

$$P[B] = P[A \cup (B \cap A')] = P[A] + P[B \cap A']$$

Since $P[B \cap A'] \geq 0$, (?)

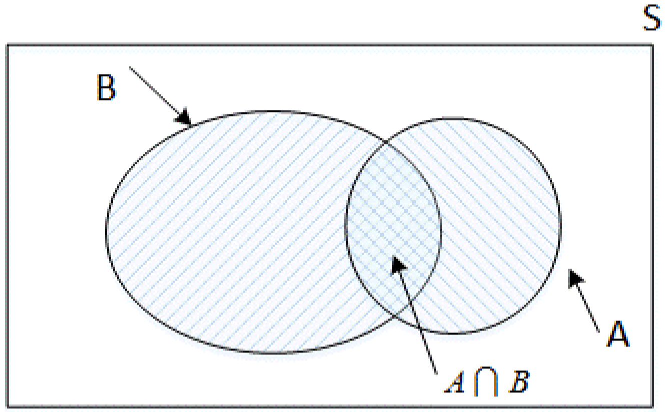
therefore $P[B] \geq P[A]$



Venn Diagram

Properties of Probability Derived from the Three Axioms

General addition rule: $If\ A, B \subset S, then\ P[A \cup B] = P[A] + P[B] - P[A \cap B]$



Venn Diagram

Proof: From the Venn diagram, we can see that,

$A \cup B = A \cup (B \cap A')$; $A\ and\ B \cap A'\ are\ mutually\ exclusive$;
$B = (A \cap B) \cup (A' \cap B)$; $A \cap B\ and\ A' \cap B\ are\ mutually\ exclusive$

Hence,

$P[A \cup B] = P[A] + P[A' \cap B]$ (1)
$P[B] = P[A \cap B] + P[A' \cap B] \Rightarrow P[A' \cap B] = P[B] - P[A \cap B]$ (2)

Substitute (2) into (1), we have

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Example on Using Properties of Probability

The following table presents probabilities for the number of times that a certain computer system will crash in the course of a week. Let A be the event that there are more than two crashes during the week, and let B be the event that the system crashes at least once. Find a sample space. Then find the subsets of the sample space correspond to the event A and event B. Then find $P[A] and\ P[B]$.

Solution:

The sample space $S = \{0,1,2,3,4\}$,

The events: $A = \{3,4\}, B = \{1,2,3,4\}$

Let x be the number of crashes happened during the week. Then,

$$P[A] = P[(x = 3) \cup (x = 4)] = P[x = 3] + P[x = 4] = 0.04 + 0.01 = 0.05$$

$$P[B] = P[(x = 1) \cup (x = 2) \cup (x = 3) \cup (x = 4)]$$

$$= P[x = 1] + P[x = 2] + P[x = 3] + P[x = 4]$$

$$= 0.30 + 0.05 + 0.04 + 0.01 = 0.40$$

Or, $P[B] = 1 - P[B'] = 1 - P[x = 0] = 1 - 0.60 = 0.40$

# of crashes	probability
0	0.60
1	0.30
2	0.05
3	0.04
4	0.01

Q: What are the properties used in calculating P[A] and P[B]?

Probability axiom (3)

Example on General addition rule

Example: A chemist analyzes seawater samples for two heavy metals: lead and mercury. Past experience indicates that 38% of the samples contain toxic level of lead or mercury; 32% contain toxic level of lead and 16% contain toxic level of mercury. What is the probability that a randomly selected sample will contain toxic level of lead only.

Solution:

Let $A_1 = \{the\ sample\ contains\ toxic\ level\ of\ lead\}$

Let $A_2 = \{the\ sample\ contains\ toxic\ level\ of\ mercury\}$

We are given, $P[A_1 \cup A_2] = 0.38, P[A_1] = 0.32, P[A_2] = 0.16$

By the addition rule: $P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$

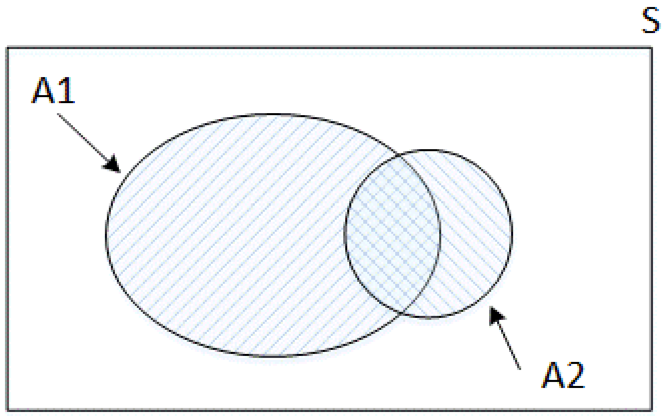
$$P[A_1 \cap A_2] = 0.32 + 0.16 - 0.38 = 0.10$$

Therefore,

$$P[sample\ contains\ toxic\ level\ of\ lead\ and\ mercury] = P[A_1 \cap A_2] = 0.10$$

$$P[sample\ contains\ toxic\ level\ of\ lead\ only] = P[A_1 \cap A_2']$$
$$= P[A_1] - P[A_1 \cap A_2] = 0.32 - 0.1 = 0.22$$

$$P[sample\ contains\ toxic\ level\ of\ mercury\ only] = P[A_2 \cap A_1']$$
$$= P[A_2] - P[A_1 \cap A_2] = 0.16 - 0.10 - 0.06$$



Venn Diagram