Solution of HW6

Chapter 4

38. (a) mean = γ = 15

    variance = 2γ = 2(15) = 30

    (b) Since X_γ^2 is a gamma random variable with $\beta = 2$ and $\alpha = \frac{\gamma}{2}$,

    $$f(x) = \frac{1}{\Gamma\left(\frac{\gamma}{2}\right) 2^{\gamma/2}} x^{\gamma/2 - 1} e^{-x/2}, \quad x > 0$$

    So, obviously, when γ = 15,

    $$f(x) = \frac{1}{\Gamma\left(\frac{15}{2}\right) 2^{15/2}} x^{15/2 - 1} e^{-x/2}, \quad x > 0$$

    (c) $m_X(t) = (1 - \beta t)^{-\alpha} = (1 - 2t)^{-15/2}$

    (d) P[X_{15}^2 $\leq$ 5.23] = .01

    P[X_{15}^2 $\geq$ 22.3] = 1 – .9 = .10

    P[6.26 $\leq$ X_{15}^2 $\leq$ 27.5] = F(27.5) – F(6.26) = .975 – .025 = .95

    $\chi^2_{.01}$ = 30.6

    $\chi^2_{.05}$ = 25.0

    $\chi^2_{.95}$ = 7.26

39. (a) 0.9418

    (b) 0.9418

    (c) 0

    (d) 0.0582

    (e) 0.8543

    (f) 1.28

(g) -1.28

(h) 1.96

(i) 1.645

42. (a) P[90 < X < 122] = $P\left[\frac{90-106}{8} < \frac{X-106}{8} < \frac{122-106}{8}\right]$

= P[-2 < Z < 2] = .9772 – .0228 = .9544

(b) P[X ≤ 120] = $P\left[\frac{X-106}{8} < \frac{120-106}{8}\right]$ = P[Z ≤ 1.75] = .9599

(c) Need to find x_0 such that P[X ≤ x_0] = .25.

P[X ≤ x_0] = $P\left[Z \le \frac{x_0-106}{8}\right]$ = .25 $\Rightarrow$ $\frac{x_0-106}{8}$ = $z_{.75}$ = -.675

$\Rightarrow$ x_0 = 106 – (.675)(8) = 100.6 mg/100 ml

(d) yes, P[X > 130] = $P\left[\frac{\text{X-100}}{8} > \frac{130-106}{8}\right]$ = P[Z > 3]

= 1 – F(3) = 1 - .9987 = .0013,

which indicates that a fasting blood glucose level greater than 130 is quite abnormal

45. $X \sim N(\mu, \sigma^2), f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{1}{2}\frac{(\text{x}-\mu)^2}{\sigma^2}\right]$

(a) $G(y) = P[Y \le y] = P[e^X \le y] = P[X \le \ln(y)] = F(\ln(y))$

(b) $G'(y) = \frac{d}{dy}G(y) = \frac{d}{dy}F(\ln(y)) = \frac{dF(\ln(y))}{d(\ln(y))} \times \frac{d(\ln(y))}{dy} = F'(\ln(y))\frac{1}{y}$

(c) $g(y) = G'(y) = F'(\ln(y))\frac{1}{y} = \frac{f(\ln(y))}{y} = \frac{1}{\sqrt{2\pi}\sigma y} exp\left[-\frac{1}{2}\frac{(\ln(y)-\mu)^2}{\sigma^2}\right]$

48. P[128 < X < 178] = P[-25 < X – 153 < 25] = P[-σ < X – μ < σ]
= P[-1 < Z < 1] = .68 or 68%

P[X > 228] = P[(X – 153) > 3(25)] = P[(X – μ) > 3 σ]

= P[Z > 3] = .00135 or .135%

50. Chebyshev's guarantees that P[|X – μ | < 3σ] $\geq$ 1 – $\frac{1}{3^2}$ = .89

yes, both the normal probability rule and Chebyshev's inequality assign a high probability to a normal random variable being within 3 σ of its mean.

The normal probability rule yields a stronger statement.

Chapter 5

4. (a) (i) Since n is a positive integer, $\frac{2}{n(n+1)} > 0$

(ii) $\sum_{y=1}^{n}\sum_{x=y}^{n}\frac{2}{n(n+1)} = \frac{2}{n(n+1)}\sum_{y=1}^{n}(n-y+1) = \frac{2}{n(n+1)}\left(\frac{n(n+1)}{2}\right) = 1$

There are 1+2+⋯+n = $\frac{n(n+1)}{2}$ points each with probability $\frac{2}{n(n+1)}$.

(b) $f_X(x) = \sum_{y=1}^{x}\frac{2}{n(n+1)} = \frac{2x}{n(n+1)}, \quad x = 1, 2, 3, \ldots, n$

$f_Y(y) = \sum_{x=y}^{n}\frac{2}{n(n+1)} = \frac{2(n-y+1)}{n(n+1)}, \quad y = 1, 2, 3, \ldots, n$

(c) X and Y are not independent since

$$f_X(x)\cdot f_Y(y) = \frac{2x}{n(n+1)}\cdot\frac{2(n-y+1)}{n(n+1)} \neq \frac{2}{n(n+1)} = f_{XY}(x, y)$$

(d) When n=5 the region over which (X, Y) is defined is

P{X $\leq$ 3 and Y $\leq$ 2] = $\sum_{y=1}^{2}\sum_{x=y}^{3}\frac{2}{5(6)} = \frac{2}{30}\sum_{y=1}^{2}(3-y+1) = \frac{2}{30}(3+2) = \frac{10}{30}$

P[X $\leq$ 3] = $\sum_{x=1}^{3}\frac{2x}{5(6)} = \frac{2}{30}\left(\frac{3(4)}{2}\right) = \frac{12}{30}$

P[Y $\leq$ 2] = $\sum_{y=1}^{2}\frac{2(5-y+1)}{5(6)} = \frac{2}{30}(5+4) = \frac{18}{30}$

The simplest way to find the above probabilities is to count the number of points in the region that are in the event and multiply by $\frac{2}{5(6)}$.

5. (a) $P[X = 0\ and\ Y = 0] = 0.4$

   (b) $P[X \geq 1\ and\ Y \leq 1] = 0.300 + 0.04 + 0.009 + 0.008 + 0.005 + 0.04$

   $$+0.01 + 0.008 + 0.007 + 0.002 = 0.429$$

   (c) $f_X(0) = 0.525;\ f_X(1) = 0.354;\ f_X(2) = 0.062;\ f_X(3) = 0.027;$

   $f_X(4) = 0.024;\ f_X(5) = 0.010$

   $f_Y(0) = 0.762;\ f_Y(1) = 0.167;\ f_Y(2) = 0.053;\ f_Y(3) = 0.018;$

   (d) $P[X \geq 2] = 1 - f_X(0) - f_X(1) = 0.121$

   (e) $P[Y = 1\ or\ Y = 2] = f_Y(1) + f_Y(2) = 0.22$

   (f) $f_{XY}(X = 0, Y = 0) = 0.400 \neq f_X(0) \times f_Y(0) = 0.4001$. X and Y are not independent.

14. $\int_0^1 \int_0^1 \int_0^1 c\left(x_1 x_2 x_3\right) dx_1 dx_2 dx_3 = c\left(\int_0^1 x dx\right)^3 = c \cdot \frac{1}{8} = 1 \ \Rightarrow\ c = 8$