Solution of HW4

Chapter 3

24. (a) (i) The drilling of a well (trial) results in a strike (success) or not a strike (failure)

(ii) Trials are identical and independent with $p = \frac{1}{13}$ for each well

(iii) X = the number of trials (wells drilled) before the first success (strike)

(b) $f(x) = \begin{cases} \left(\frac{12}{13}\right)^{x-1}\left(\frac{1}{13}\right), & \text{x=1,2,3,...} \\ 0 & , \textit{otherwise} \end{cases}$

(c) $m_X(t) = \dfrac{\frac{1}{13}e^t}{1-\frac{12}{13}e^t}, t < -\ln\frac{12}{13}$

(d) $E\left[X\right] = \frac{1}{p} = \frac{1}{1/13} = 13$ $\qquad E\left[X^2\right] = \frac{1+q}{p^2} = \dfrac{1+\frac{12}{13}}{\left(\frac{1}{13}\right)^2} = 325$

$\sigma^2 = \frac{q}{p^2} = \dfrac{\frac{12}{13}}{\left(\frac{1}{13}\right)^2} = 156 \qquad \sigma = \sqrt{156} = 12.49$

(e) $f(x) = q^{x-1}p,\ x = 1, 2, 3, \ldots$, then,

F (x_0) = $P\left[X \le x_0\right]$ = $\sum_{x=1}^{x_0} q^{x-1}p$

= $\dfrac{p(1-q^{x_0})}{1-q}$, the sum of the first x_0 terms of a geometric series

= $\dfrac{p(1-q^{x_0})}{p} = 1-q^{x_0}$

Therefore,

P[X ≥ 2] = 1 – F(1) = $1-\left(1-\left(\frac{12}{13}\right)\right) = \frac{12}{13}$

34. (a) $m_X(t) = E[e^{tX}] = \frac{1}{n}\sum_{i=1}^{n} e^{tx_i}$

(b) $\frac{dm_X(t)}{dt} = \frac{1}{n}(x_1 e^{tx_1} + x_2 e^{tx_2} + \cdots + x_n e^{tx_n})$

$$E[X] = \left.\frac{dm_X(t)}{dt}\right|_{t=0} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)$$

$$\frac{d^2 m_X(t)}{dt^2} = \frac{1}{n}\left(x_1^2 e^{tx_1} + x_2^2 e^{tx_2} + \cdots + x_n^2 e^{tx_n}\right)$$

$$E[X^2] = \left.\frac{d^2 m_X(t)}{dt^2}\right|_{t=0} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right)$$

$$\sigma^2 = E[X^2] - (E[X])^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right)$$

(c) $\mu_Y = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{10}\left(\frac{9\cdot 10}{2}\right) = 4.5$

$$E[Y^2] = \frac{1}{n}\sum_{i=1}^{n} y_i^2 = \frac{1}{10}\left(\frac{9\cdot 10\cdot 19}{6}\right) = 28.5$$

$$\sigma^2 = 28.5 - (4.5)^2 = 8.25$$

35. $f(x) = ce^{-x}, x = 1,2,3\ldots$

(a) $\sum_x f(x) = \sum_{x=1}^{\infty} ce^{-x} = c\sum_{x=1}^{\infty} e^{-x} = \frac{ce^{-1}}{1-e^{-1}} = \frac{c}{e-1} = 1$

This requires that $c = e - 1$.

Since $c = e - 1 > 0$, $f(x) = ce^{-x} > 0$

(b) $m_X(t) = E[e^{Xt}] = \sum_x e^{xt} f(x) = \sum_x e^{xt} ce^{-x} = c\sum_{x=1}^{\infty} e^{x(t-1)} = \frac{ce^{t-1}}{1-e^{t-1}}$

(c) $\frac{d}{dt} m_X(t) = \frac{d}{dt}\frac{ce^{t-1}}{1-e^{t-1}} = \frac{ce^{t-1}(1-e^{t-1}) + ce^{t-1}}{\left(1-e^{t-1}\right)^2}$

$$E[X] = \frac{d}{dt} m_X(t)\Big|_{t=0} = \frac{ce^{-1}(1-e^{-1}) + ce^{-1}}{\left(1-e^{-1}\right)^2} = \frac{e}{e-1}$$

36. (a) $f(x) = \begin{cases} \binom{15}{x}(.2)^x (.8)^{15-x} & , x = 0,1,\ldots,15 \\ 0 & , otherwise \end{cases}$

(b) $m_X(t) = \left(.8 + .2e^t\right)^{15}$

(c) $E\left[X\right]$ = np = (15)(.2) = 3

$VarX$ = npq = (15)(.2)(.8) = 2.4

(d) $\frac{dm_X(t)}{dt} = 15(.8+.2e^t)^{14}\left(.2e^t\right)$

$$E\left[X\right] = \frac{dm_X(t)}{dt}\Bigg|_{t=0} = 15\left(.8+.2\right)^{14}(.2) = 3$$

$$\frac{d^2 m_X(t)}{dt^2} = 15[\left(.8+.2e^t\right)\left(.2e^t\right) + \left(.2e^t\right)14\left(.8+.2e^t\right)^{13}\left(.2e^t\right)]$$

$$= 15[(.2e^t)\left(.8+.2e^t\right)^{13}\left(.8+3e^t\right)]$$

$$E\left[X\right] = \frac{d^2 m_X(t)}{dt^2}\Bigg|_{t=0} = 15[(.2)(.8+.2)^{13}(.8+3)] = 11.4$$

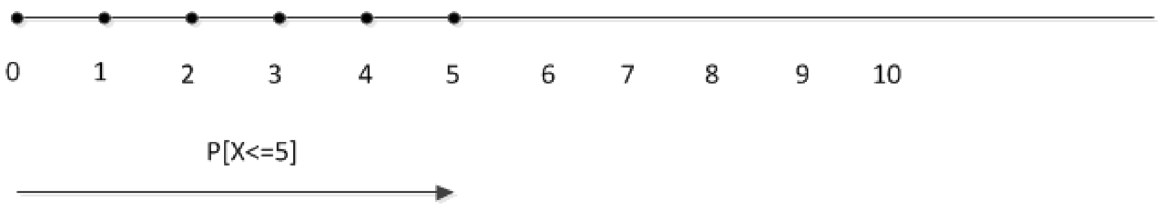
$VarX$ = 11.4 – 3^2 = 2.4

(e) P[X ≤ 1] = P[X = 0] + P[X = 1]

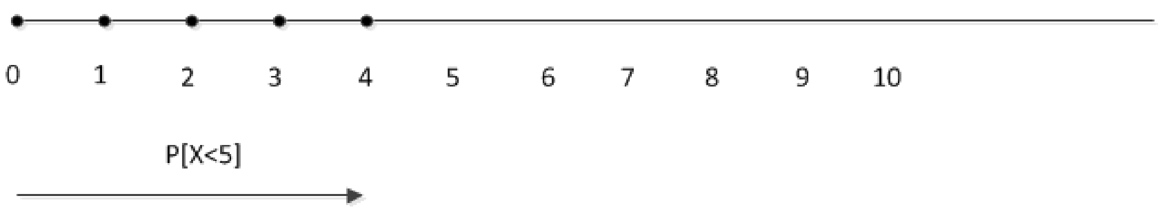
$$= \binom{15}{0}(.2)^0(.8)^{15} + \binom{15}{1}(.2)^1(.8)^{14}$$

= .0352 + .1319 = .1671

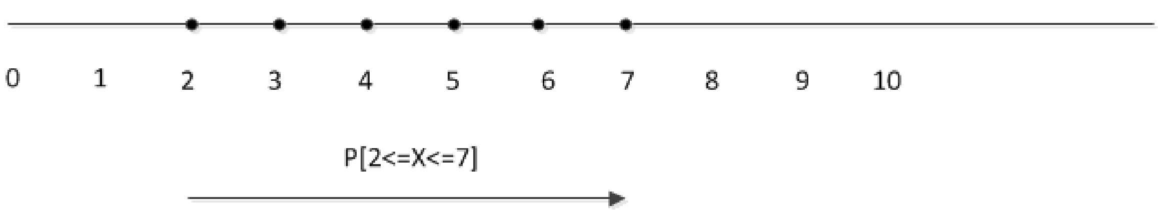
(a) $P[X \le 5] = F(5) = .9389$



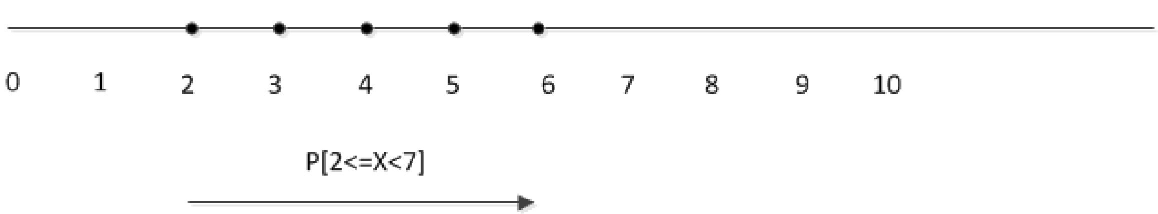
$P[X < 5] = P[X \le 4] = F(4) = .8358$



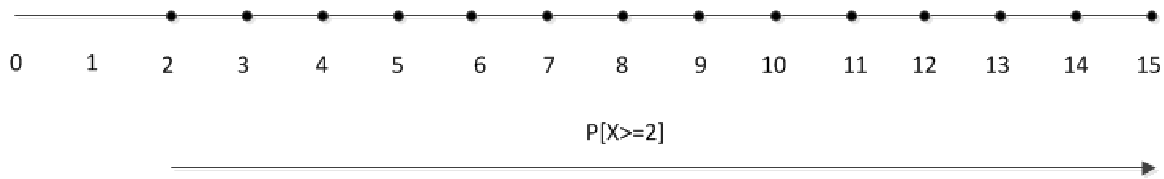
$P[2 \le X \le 7] = P[X \le 7] - P[X \le 1] = F(7) - F(1) = .9958 - .1671 = .8287$



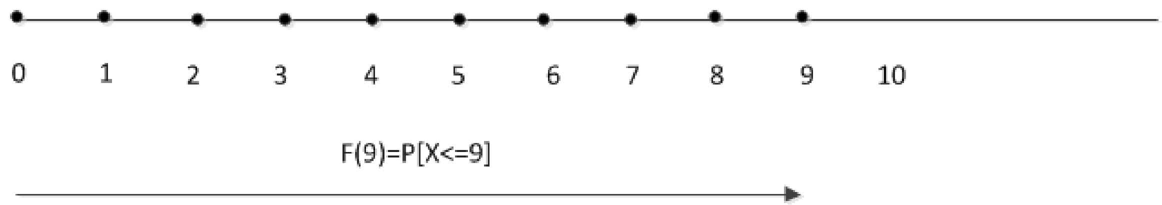
$P[2 \le X < 7] = P[2 \le X \le 6] = F(6) - F(1) = .9819 - .1671 = .8148$



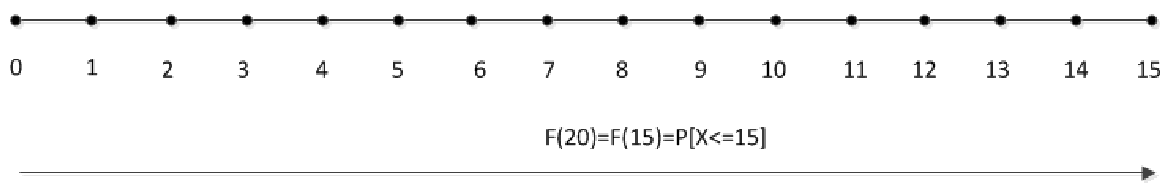
$P[X \ge 2] = 1 - F(2) = 1 - .3980 = .6020$



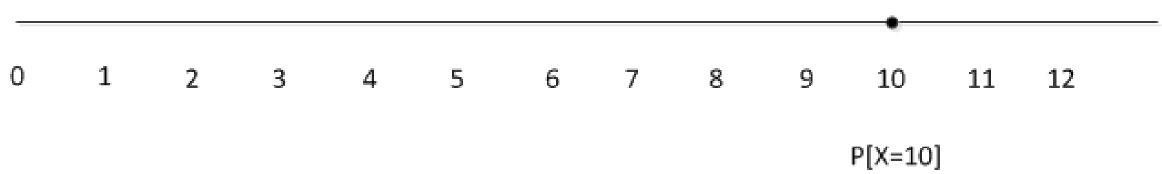
F(9) = .9999



F(20) = 1



P[X = 10] = F(10) – F(9) = 1 – .9999 = .0001



43. (a)

$$m_X(t) = E[e^{tx}] = \sum_x e^{tx}\binom{n}{x}p^x(1-p)^{n-x} = \sum_{x=0}^{n}\binom{n}{x}e^{tx}p^x(1-p)^{n-x} = \sum_{x=0}^{n}\binom{n}{x}\left(pe^t\right)^x(1-p)^{n-x}$$

$$= \left(1-p+pe^t\right)^n$$

(b) $\dfrac{d}{dt}m_X(t) = npe^t\left(1-p+pe^t\right)^{n-1}$

$$E[X] = \frac{d}{dt} m_X(t)|_{t=0} = np$$

(c) $\frac{d^2}{dt^2} m_X(t) = npe^t(1 - p + pe^t)^{n-1} + npe^t(n-1)pe^t(1 - p + pe^t)^{n-2}$

$$E[X^2] = \frac{d^2}{dt^2} m_X(t)|_{t=0} = np + n(n-1)p^2$$

(d) $VarX = E[X^2] - (E[X])^2 = np + n(n-1)p^2 - (np)^2 = np - np^2 = np(1-p) = npq$

49. $m_X(t) = (pe^t)^r(1 - qe^t)^{-r}$

$\frac{d}{dt} m_X(t) = r(pe^t)^r(1 - qe^t)^{-r-1}$

$E[X] = \frac{d}{dt} m_X(t)|_{t=0} = \frac{r}{p}$

50. $m_X(t) = (pe^t)^r \left(1 - qe^t\right)^{-r}$

$$\frac{dm_X(t)}{dt} = (pe^t)^r \left(-r\left(1-qe^t\right)^{-(r+1)}(-qe^t)\right) + \left(1-qe^t\right)^{-r} r\left(pe^t\right)^{r-1}(pe^t)$$

$$= rqe^t\left(pe^t\right)^r\left(1-qe^t\right)^{-(r+1)} + r\left(pe^t\right)^r\left(1-qe^t\right)^{-r}$$

$$= r(pe^t)^r\left(1-qe^t\right)^{-(r+1)}$$

$$\frac{d^2 m_X(t)}{d^2 t} = r\left(\left(pe^t\right)^r\left(-(r+1)\left(1-qe^t\right)^{-(r+2)}\left(-qe^t\right)\right) + \left(1-qe^t\right)^{-(r+1)} r\left(pe^t\right)^{r-1}\left(pe^t\right)\right)$$

$$= r\left(pe^t\right)^r\left(1-qe^t\right)^{-(r+2)}\left(qe^t + r\right)$$

$$\left.\frac{d^2 m_X(t)}{d^2 t}\right|_{t=0} = rp^r(1-q)^{-(r+2)}(q+r) = \frac{rq + r^2}{p^2}$$

$$VarX = \frac{r^2 + rq}{p^2} - \frac{r^2}{p^2} = \frac{rq}{p^2}$$

61. (a) $E[X] = k = 10$

(b) $VarX = k = 10$

(c) $\sigma_X = \sqrt{VarX} = \sqrt{10}$

(d) $f(x) = \frac{e^{-10}10^x}{x!}$

(e) $P[X \leq 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4]$

$$= \frac{e^{-10}10^0}{0!} + \frac{e^{-10}10^1}{1!} + \frac{e^{-10}10^2}{2!} + \frac{e^{-10}10^3}{3!} + \frac{e^{-10}10^4}{4!} = 0.0293$$

(f) $P[X < 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3]$

$$= \frac{e^{-10}10^0}{0!} + \frac{e^{-10}10^1}{1!} + \frac{e^{-10}10^2}{2!} + \frac{e^{-10}10^3}{3!} = 0.0103$$

(g) $P[X = 4] = \frac{e^{-10}10^4}{4!} = 0.0189$

(h) $P[X \geq 4] = 1 - P[X < 4] = 0.9897$

(i) $P[4 \leq X \leq 9] = P[X = 4] + P[X = 5] + P[X = 6] + P[X = 7]$

$$+ P[X = 8] + P[X = 9]$$

$$= \frac{e^{-10}10^4}{4!} + \frac{e^{-10}10^5}{5!} + \frac{e^{-10}10^6}{6!} + \frac{e^{-10}10^7}{7!} + \frac{e^{-10}10^8}{8!} + \frac{e^{-10}10^9}{9!} = 0.4476$$

64. Let X: the number of destructive earthquakes per year
X is Poisson with parameter k = λ = 1 destructive earthquake per year

Let Y: the number of destructive earthquakes in a six-month period

Y is Poisson with parameter λs = 1(.5) = .5

P[Y ≥ 1] = 1 – P[Y ≤ 0] = 1 – .607 = .393

Yes, P[Y ≥ 3] = 1 – P[Y ≤ 2] = 1 – .986 = .014, which indicates a small chance of this event occurring.

69. $m_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-k}k^x}{x!} = e^{-k} \sum_{x=0}^{\infty} \frac{(ke^t)^x}{x!} = e^{-k}e^{ke^t} = e^{k(1-e^t)}$

$E[X] = \frac{dm_X(t)}{dt}_{t=0} = e^{k(e^t-1)} \cdot ke^t \,|_{t=0} = k$

$E[X^2] = \frac{d^2m_X(t)}{dt}_{t=0} = e^{k(e^t-1)} \cdot (ke^t)^2 + e^{k(e^t-1)} \cdot ke^t|_{t=0} = k^2 + k$

$VarX = E[X^2] - (E[X])^2 = k^2 + k - k^2 = k$