ESE 326 Practice Exam for the Final F2017

1. Consider the random variables $(X, Y)$ with following joint density

$$f_{XY}(x,y) = cxy, \quad c\ is\ a\ constant, \quad 0 < y < x < 1$$

(1) Find the value of $c$ so that $f_{XY}$ is a valid density function
(2) Find the marginal density for $X$. Find $E[X]\ and\ E[X^2]$
(3) Find the marginal density for $Y$. Find $E[Y]\ \ and\ E[Y^2]$
(4) Are $X\ and\ Y$ independent? Why?
(5) Find the curve of regression of $X\ on\ Y$. Is the regression linear?

2. let $\bar{X}_1\ and\ \bar{X}_2$ be the sample means based on independent samples of size $n_1\ and\ n_2$ drawn from normal distributions $X_1\ and\ X_2$ with means $\mu_1\ and\ \mu_2$ and variances ${\sigma_1}^2\ and\ {\sigma_2}^2$ respectively.

(1) Show that $\bar{X}_1 - \bar{X}_2$ is an unbiased estimator of $\mu_1 - \mu_2$

(2) Show that $\bar{X}_1 - \bar{X}_2$ is a normal random variable with mean $\mu_1 - \mu_2$ and variance $\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}$

Let's assume that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ for the following questions.

(3) The pooled variance is defined as: $S_p^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$

Show that $\frac{(n_1+n_2-2)S_p^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}$

(4) Show that $\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{S_p^2(\frac{1}{n_1}+\frac{1}{n_2})}} \sim T_{n_1+n_2-2}$

3. The following are two random samples of sizes $n_1\ and\ n_2$ from two normal populations with means $\mu_1\ and\ \mu_2$ and variances $\sigma_1^2\ and\ \sigma_2^2$ respectively.

| Population 1 | Population 2 |
|---|---|
| $n_1 = 11$ | $n_2 = 9$ |
| $\bar{x}_1 = 18.55$ | $\bar{x}_2 = 14.67$ |
| $S_1^2 = 10.27$ | $S_2^2 = 7.75$ |

(1) Test the equality of the variances at the significance level of 0.2

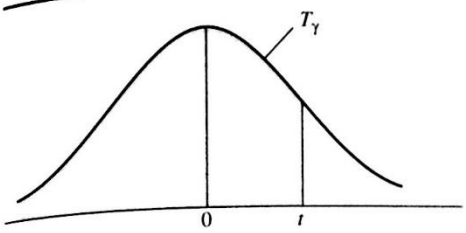
(2) Find the 95% confidence interval on $\mu_1 - \mu_2$. Based on this interval, is there evidence that a real difference exists?

(3) Test the hypothesis $H_0$: $\mu_1 = \mu_2$ $H_1$: $\mu_1 \neq \mu_2$ by using p test with significant level of $\alpha = 0.1$.

(4) Verify the result of (3) by finding the rejection region of the test statistics.

**The tables of T distribution and F distribution are attached.**

**TABLE VI**
**$T$ distribution**



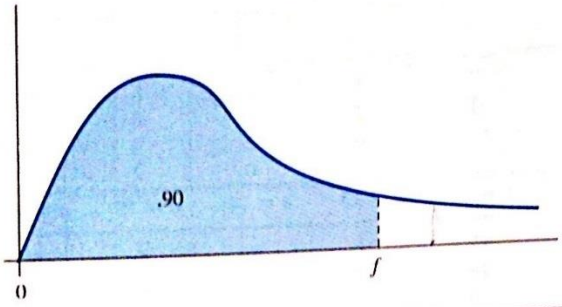

Column heading = cumulative probability

Row heading = degrees of freedom

Row $\infty$ = standard normal values

| | $P[T_\gamma \le t]$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | .6 | .75 | .9 | .95 | .975 | .99 | .995 | .999 | .9995 |
| 1 | 0.325 | 1.000 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.317 | 636.607 |
| 2 | 0.289 | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.327 | 31.598 |
| 3 | 0.277 | 0.765 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 10.215 | 12.924 |
| 4 | 0.271 | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 | 8.610 |
| 5 | 0.267 | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 | 6.869 |
| 6 | 0.265 | 0.718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| 7 | 0.263 | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| 8 | 0.262 | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| 9 | 0.261 | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| 10 | 0.260 | 0.700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| 11 | 0.260 | 0.697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
| 12 | 0.259 | 0.695 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 | 4.318 |
| 13 | 0.259 | 0.694 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 | 4.221 |
| 14 | 0.258 | 0.692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 | 4.140 |
| 15 | 0.258 | 0.691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 | 4.073 |
| 16 | 0.258 | 0.690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.686 | 4.015 |
| 17 | 0.257 | 0.689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.646 | 3.965 |
| 18 | 0.257 | 0.688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.611 | 3.922 |
| 19 | 0.257 | 0.688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.579 | 3.883 |
| 20 | 0.257 | 0.687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.552 | 3.850 |
| 21 | 0.257 | 0.686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.527 | 3.819 |
| 22 | 0.256 | 0.686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.505 | 3.792 |
| 23 | 0.256 | 0.685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.485 | 3.768 |
| 24 | 0.256 | 0.685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.467 | 3.745 |
| 25 | 0.256 | 0.684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.450 | 3.725 |
| 26 | 0.256 | 0.684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.435 | 3.707 |
| 27 | 0.256 | 0.684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.421 | 3.690 |
| 28 | 0.256 | 0.683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.408 | 3.674 |
| 29 | 0.256 | 0.683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.396 | 3.659 |
| 30 | 0.256 | 0.683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.385 | 3.646 |
| 31 | 0.256 | 0.682 | 1.309 | 1.696 | 2.040 | 2.453 | 2.744 | 3.375 | 3.633 |
| 32 | 0.255 | 0.682 | 1.309 | 1.694 | 2.037 | 2.449 | 2.738 | 3.365 | 3.622 |
| 33 | 0.255 | 0.682 | 1.308 | 1.692 | 2.035 | 2.445 | 2.733 | 3.356 | 3.611 |
| 34 | 0.255 | 0.682 | 1.307 | 1.691 | 2.032 | 2.441 | 2.728 | 3.348 | 3.601 |
| 35 | 0.255 | 0.682 | 1.306 | 1.690 | 2.030 | 2.438 | 2.724 | 3.340 | 3.591 |
| 36 | 0.255 | 0.681 | 1.306 | 1.688 | 2.028 | 2.434 | 2.719 | 3.333 | 3.582 |
| 37 | 0.255 | 0.681 | 1.305 | 1.687 | 2.026 | 2.431 | 2.715 | 3.326 | 3.574 |
| 38 | 0.255 | 0.681 | 1.304 | 1.686 | 2.024 | 2.429 | 2.712 | 3.319 | 3.566 |
| 39 | 0.255 | 0.681 | 1.304 | 1.685 | 2.023 | 2.426 | 2.708 | 3.313 | 3.558 |
| 40 | 0.255 | 0.681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 3.307 | 3.551 |
| 41 | 0.255 | 0.681 | 1.303 | 1.683 | 2.020 | 2.421 | 2.701 | 3.301 | 3.544 |
| 42 | 0.255 | 0.680 | 1.302 | 1.682 | 2.018 | 2.418 | 2.698 | 3.296 | 3.538 |
| 43 | 0.255 | 0.680 | 1.302 | 1.681 | 2.017 | 2.416 | 2.695 | 3.291 | 3.532 |
| 44 | 0.255 | 0.680 | 1.301 | 1.680 | 2.015 | 2.414 | 2.692 | 3.286 | 3.526 |

## TABLE IX
## *F* distribution




Column heading = numerator degrees of freedom

Row heading = denominator degrees of freedom

Points given are $f_{.10}$ points

For degrees of freedom > 120, use row or column 120

$P[F_{\gamma_1,\gamma_2} \leq f] = .90$

| $\gamma_2$ \ $\gamma_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 39.862 | 49.500 | 53.593 | 55.833 | 57.240 | 58.204 | 58.906 | 59.439 | 59.857 | 60.195 |
| 2 | 8.526 | 9.000 | 9.162 | 9.243 | 9.293 | 9.326 | 9.349 | 9.367 | 9.381 | 9.392 |
| 3 | 5.538 | 5.462 | 5.391 | 5.343 | 5.309 | 5.285 | 5.266 | 5.252 | 5.240 | 5.231 |
| 4 | 4.545 | 4.325 | 4.191 | 4.107 | 4.051 | 4.010 | 3.979 | 3.955 | 3.936 | 3.920 |
| 5 | 4.060 | 3.780 | 3.619 | 3.520 | 3.453 | 3.405 | 3.368 | 3.339 | 3.316 | 3.297 |
| 6 | 3.776 | 3.463 | 3.289 | 3.181 | 3.108 | 3.055 | 3.014 | 2.983 | 2.958 | 2.937 |
| 7 | 3.589 | 3.257 | 3.074 | 2.961 | 2.883 | 2.827 | 2.785 | 2.752 | 2.725 | 2.703 |
| 8 | 3.458 | 3.113 | 2.924 | 2.806 | 2.726 | 2.668 | 2.624 | 2.589 | 2.561 | 2.538 |
| 9 | 3.360 | 3.006 | 2.813 | 2.693 | 2.611 | 2.551 | 2.505 | 2.469 | 2.440 | 2.416 |
| 10 | 3.285 | 2.924 | 2.728 | 2.605 | 2.522 | 2.461 | 2.414 | 2.377 | 2.347 | 2.323 |
| 11 | 3.225 | 2.860 | 2.660 | 2.536 | 2.451 | 2.389 | 2.342 | 2.304 | 2.274 | 2.248 |
| 12 | 3.177 | 2.807 | 2.606 | 2.480 | 2.394 | 2.331 | 2.283 | 2.245 | 2.214 | 2.188 |
| 13 | 3.136 | 2.763 | 2.560 | 2.434 | 2.347 | 2.283 | 2.234 | 2.195 | 2.164 | 2.138 |
| 14 | 3.102 | 2.726 | 2.522 | 2.395 | 2.307 | 2.243 | 2.193 | 2.154 | 2.122 | 2.095 |
| 15 | 3.073 | 2.695 | 2.490 | 2.361 | 2.273 | 2.208 | 2.158 | 2.119 | 2.086 | 2.059 |
| 16 | 3.048 | 2.668 | 2.462 | 2.333 | 2.244 | 2.178 | 2.128 | 2.088 | 2.055 | 2.028 |
| 17 | 3.026 | 2.645 | 2.437 | 2.308 | 2.218 | 2.152 | 2.102 | 2.061 | 2.028 | 2.001 |
| 18 | 3.007 | 2.624 | 2.416 | 2.286 | 2.196 | 2.130 | 2.079 | 2.038 | 2.005 | 1.977 |
| 19 | 2.990 | 2.606 | 2.397 | 2.266 | 2.176 | 2.109 | 2.058 | 2.017 | 1.984 | 1.956 |
| 20 | 2.975 | 2.589 | 2.380 | 2.249 | 2.158 | 2.091 | 2.040 | 1.999 | 1.965 | 1.937 |
| 21 | 2.961 | 2.575 | 2.365 | 2.233 | 2.142 | 2.075 | 2.023 | 1.982 | 1.948 | 1.920 |
| 22 | 2.949 | 2.561 | 2.351 | 2.219 | 2.128 | 2.061 | 2.008 | 1.967 | 1.933 | 1.904 |
| 23 | 2.937 | 2.549 | 2.339 | 2.207 | 2.115 | 2.047 | 1.995 | 1.953 | 1.919 | 1.890 |
| 24 | 2.927 | 2.538 | 2.327 | 2.195 | 2.103 | 2.035 | 1.983 | 1.941 | 1.906 | 1.877 |
| 25 | 2.918 | 2.528 | 2.317 | 2.184 | 2.092 | 2.024 | 1.971 | 1.929 | 1.895 | 1.866 |
| 26 | 2.909 | 2.519 | 2.307 | 2.174 | 2.082 | 2.014 | 1.961 | 1.919 | 1.884 | 1.855 |
| 27 | 2.901 | 2.511 | 2.299 | 2.165 | 2.073 | 2.005 | 1.952 | 1.909 | 1.874 | 1.845 |
| 28 | 2.894 | 2.503 | 2.291 | 2.157 | 2.064 | 1.996 | 1.943 | 1.900 | 1.865 | 1.836 |
| 29 | 2.887 | 2.495 | 2.283 | 2.149 | 2.057 | 1.988 | 1.935 | 1.892 | 1.857 | 1.827 |
| 30 | 2.881 | 2.489 | 2.276 | 2.142 | 2.049 | 1.980 | 1.927 | 1.884 | 1.849 | 1.819 |
| 31 | 2.875 | 2.482 | 2.270 | 2.136 | 2.042 | 1.973 | 1.920 | 1.877 | 1.842 | 1.812 |
| 32 | 2.869 | 2.477 | 2.263 | 2.129 | 2.036 | 1.967 | 1.913 | 1.870 | 1.835 | 1.805 |
| 33 | 2.864 | 2.471 | 2.258 | 2.123 | 2.030 | 1.961 | 1.907 | 1.864 | 1.828 | 1.799 |
| 34 | 2.859 | 2.466 | 2.252 | 2.118 | 2.024 | 1.955 | 1.901 | 1.858 | 1.822 | 1.793 |
| 35 | 2.855 | 2.461 | 2.247 | 2.113 | 2.019 | 1.950 | 1.896 | 1.852 | 1.817 | 1.787 |
| 36 | 2.850 | 2.456 | 2.243 | 2.108 | 2.014 | 1.945 | 1.891 | 1.847 | 1.811 | 1.781 |
| 37 | 2.846 | 2.452 | 2.238 | 2.103 | 2.009 | 1.940 | 1.886 | 1.842 | 1.806 | 1.776 |
| 38 | 2.842 | 2.448 | 2.234 | 2.099 | 2.005 | 1.935 | 1.881 | 1.838 | 1.802 | 1.772 |
| 39 | 2.839 | 2.444 | 2.230 | 2.095 | 2.001 | 1.931 | 1.877 | 1.833 | 1.797 | 1.767 |
| 40 | 2.835 | 2.440 | 2.226 | 2.091 | 1.997 | 1.927 | 1.873 | 1.829 | 1.793 | 1.763 |
| 50 | 2.809 | 2.412 | 2.197 | 2.061 | 1.966 | 1.895 | 1.840 | 1.796 | 1.760 | 1.729 |
| 60 | 2.791 | 2.393 | 2.177 | 2.041 | 1.946 | 1.875 | 1.819 | 1.775 | 1.738 | 1.707 |
| 120 | 2.748 | 2.347 | 2.130 | 1.992 | 1.896 | 1.824 | 1.767 | 1.722 | 1.684 | 1.652 |