Hypothesis Test on the Mean

Three forms of tests of hypothesis on the mean of a distribution:

I. $H_0\colon \mu \leq \mu_0$, $H_1\colon \mu > \mu_0$ right-tailed test (the rejection region is the right-tailed region).
II. $H_0\colon \mu \geq \mu_0$, $H_1\colon \mu < \mu_0$ left-tailed test (the rejection region is the left-tailed region).
III. $H_0\colon \mu = \mu_0$, $H_1\colon \mu \neq \mu_0$ two-tailed test (the rejection region consists of both lower and upper tail regions).

These three forms can also (commonly) be expressed as:

I. $H_0\colon \mu = \mu_0$, $H_1\colon \mu > \mu_0$
II. $H_0\colon \mu = \mu_0$, $H_1\colon \mu < \mu_0$
III. $H_0\colon \mu = \mu_0$, $H_1\colon \mu \neq \mu_0$

Why can we do this?

Values of the test statistic that lead us to reject μ_0 and to conclude that $\mu > \mu_0$ will also lead us to reject any value less than μ_0.
Values of the test statistic that lead us to reject μ_0 and to conclude that $\mu < \mu_0$ will also lead us to reject any value greater than μ_0.

Test on the Mean

Example: The maximum acceptable level for exposure to microwave radiation in the USA is an average of 10 microwatts per square centimeter. It is feared that a large television transmitter may be polluting the air nearby by pushing the level of microwave radiation above the safe limit. The microwave radiation level is measured in the area at randomly selected time over a one week period and obtained a sample of 25 readings with sample mean $\bar{X} = 10.3\ and\ sample\ standard\ deviation\ S = 2$. Design a test to find out if the microwave radiation level is above the average safety level at the significance level of 0.1.

Hypothesis test on the mean (critical region method):

We intend to test: $H_0\!: \ \mu = 10, \ \ H_1\!: \ \mu > 10$ (right-tailed test with a right-tailed rejection region)

The test statistic is : $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim T_{n-1}$. Assuming H_0 is true, the test statistic becomes $\frac{\bar{X}-10}{s/\sqrt{25}} \sim T_{24}$

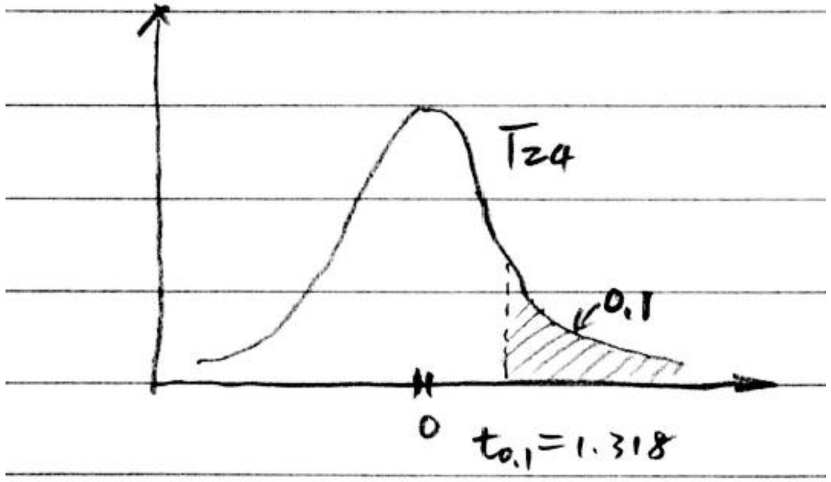
To find the rejection region, we need to select α value, the level of significance. α is also the maximum level of type I error can be tolerated.

If we make a type I error, the transmitter will be shut down unnecessarily;
If we make a type II error, we shall fail to detect a potential health hazard;
Hence, we want α to be small but not so small as to force β to be extremely large. We choose $\alpha = 0.1$

Test on the Mean

Example: (continued)

Hypothesis test on the mean (critical region method):



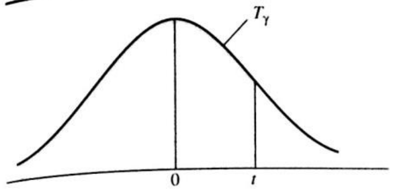
According to the definition of α: $P[type\ I\ error] = P[T_{24} \geq t_{0.1}] = 0.1$
From the T-table, we read out: $t_{0.1} = 1.318$ (critical point for right-tailed probability of 0.1)
Hence, we shall reject H_0 in favor of H_1 if the observed value of the test statistic is 1.318 or larger.

From the 25 sample data, we calculated that $\bar{x} = 10.3\ and\ s = 2$
Then,

$$\frac{\bar{X} - 10}{s/\sqrt{25}} = \frac{10.3 - 10}{2/5} = 0.75 < 1.318$$

We are unable to reject H_0. I.e., the sample data do not support the contention that the transmitter is lifting the average microwave radiation level above the safe limit.

TABLE VI
T distribution



Column heading = cumulative probability
Row heading = degrees of freedom
Row ∞ = standard normal values

	$P[T_\gamma \le t]$								
γ	.6	.75	.9	.95	.975	.99	.995	.999	.9995
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	318.317	636.607
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.611	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.385	3.646
31	0.256	0.682	1.309	1.696	2.040	2.453	2.744	3.375	3.633
32	0.255	0.682	1.309	1.694	2.037	2.449	2.738	3.365	3.622
33	0.255	0.682	1.308	1.692	2.035	2.445	2.733	3.356	3.611
34	0.255	0.682	1.307	1.691	2.032	2.441	2.728	3.348	3.601
35	0.255	0.682	1.306	1.690	2.030	2.438	2.724	3.340	3.591
36	0.255	0.681	1.306	1.688	2.028	2.434	2.719	3.333	3.582
37	0.255	0.681	1.305	1.687	2.026	2.431	2.715	3.326	3.574
38	0.255	0.681	1.304	1.686	2.024	2.429	2.712	3.319	3.566
39	0.255	0.681	1.304	1.685	2.023	2.426	2.708	3.313	3.558
40	0.255	0.681	1.303	1.684	2.021	2.423	2.704	3.307	3.551
41	0.255	0.681	1.303	1.683	2.020	2.421	2.701	3.301	3.544
42	0.255	0.680	1.302	1.682	2.018	2.418	2.698	3.296	3.538
43	0.255	0.680	1.302	1.681	2.017	2.416	2.695	3.291	3.532
44	0.255	0.680	1.301	1.680	2.015	2.414	2.692	3.286	3.526

Test on the Mean

Example: The maximum acceptable level for exposure to microwave radiation in the USA is an average of Of 10 microwatts per square centimeter. It is feared that a large television transmitter may be polluting the air nearby by pushing the level of microwave radiation above the safe limit. The microwave radiation level is measured in the area at randomly selected time over a one week period and obtained a sample of 25 readings with sample mean $\bar{X} = 10.3$ $and\ sample\ standard\ deviation\ S = 2.$. Design a test to find out if the microwave radiation level is above the average safety level.

Hypothesis test on the mean (p-value method) (p test):

We intend to test: H_0: $\mu = 10$, H_1: $\mu > 10$ (right-tailed test)

Assuming H_0 is true, the test statistic is $\frac{\bar{X}-10}{s/\sqrt{25}} \sim T_{24}$. The observed value of this statistic is: $\frac{10.3-10}{2/\sqrt{25}} = 0.75$

Since this is a right-tailed test, the p-value can be evaluated as:

$$p\_value = P[T_{24} \geq 0.75] = 1 - P[T_{24} < 0.75]$$

From the T-table, we have,

$$P[T_{24} < 0.685] = 0.75, P[T_{24} < 1.318] = 0.90$$

Hence,

$$0.75 < P[T_{24} < 0.75] < 0.90 \Rightarrow 0.1 < P[T_{24} \geq 0.75] < 0.25 \Rightarrow 0.1 < p < 0.25$$

Therefore, H_0 can not be rejected at 0.1 confidence level.

Test on the Variance

Three forms of test of hypothesis on the variance of a distribution:

I. $H_0\colon \sigma^2 = {\sigma_0}^2, \quad H_1\colon \sigma^2 > {\sigma_0}^2$ right-tailed test

II. $H_0\colon \sigma^2 = {\sigma_0}^2, \quad H_1\colon \sigma^2 < {\sigma_0}^2$ left-tailed test

III. $H_0\colon \sigma^2 = {\sigma_0}^2, \quad H_1\colon \sigma^2 \neq {\sigma_0}^2$ two-tailed test

The test statistic is $\frac{(n-1)S^2}{{\sigma_0}^2} \sim {\mathrm{X}^2}_{n-1}$ when H_0 is true.

Let x_{ob} be the observed value of the test statistic X, calculated from the sample. Then, to carry out the p-test, we have,

$p_{value} = P[X \geq x_{ob} | H_0 \; is \; true]$, for right-tailed test

$p_{value} = P[X \leq x_{ob} | H_0 \; is \; true]$, for left-tailed test

$p_{value} = 2\min\{P[X \leq x_{ob} | H_0 \; is \; true],\; P[X \geq x_{ob} | H_0 \; is \; true]\}$, for two-tailed test

Test on the Variance

Example: The following sample is drawn from a normal random variable. The researcher claim that the standard deviation of the random variable is less than 1.5. Do these data support the contention of the researcher (at the significance level of $\alpha = 0.1$)?

6.2	1.9	4.4	4.9	3.5
4.6	4.2	1.1	1.3	4.8
4.1	3.7	2.5	3.7	4.2
1.4	2.6	1.5	3.9	3.2

We test the following hypothesis: H_0: $\sigma = 1.5$; H_1: $\sigma < 1.5$

This is equivalent to test : H_0: $\sigma^2 = 2.25$; H_1: $\sigma^2 < 2.25$

The test statistic is: $\frac{(n-1)S^2}{\sigma^2} \sim \mathrm{X}^2_{n-1} \sim \mathrm{X}^2_{19}$

From the sample data, we calculated $\bar{x} = 3.39$, $S = 1.41$

Assuming H_0 is true, the observed value of the test statistic $\frac{(n-1)S^2}{{\sigma_0}^2}$ is 16.79

Since this is a left-tailed test, the p-value can be evaluated as:

$$p_{value} = P[\mathrm{X}^2_{19} \leq 16.79]$$

From the chi-squared probability table, we have,

$$P[\mathrm{X}^2_{19} \leq 14.6] = 0.25, P[\mathrm{X}^2_{19} \leq 18.3] = 0.50$$

Hence, $0.25 < p_{value} < 0.5$, i.e., $p_{value} > \alpha$

Therefore, H_0 can not be rejected. i.e., the sampled data do not support the claim of the researcher

Test on the Variance

Practice Example: The following sample is drawn from a normal random variable. The researcher claim that the standard deviation of the random variable is less than 1.5. Do these data support the contention of the researcher (at the significance level of $\alpha = 0.1$)? Use the rejection region method.

6.2	1.9	4.4	4.9	3.5
4.6	4.2	1.1	1.3	4.8
4.1	3.7	2.5	3.7	4.2
1.4	2.6	1.5	3.9	3.2

We test the following hypothesis: H_0: $\sigma = 1.5$; H_1: $\sigma < 1.5$

This is equivalent to test : H_0: $\sigma^2 = 2.25$; H_1: $\sigma^2 < 2.25$

The test statistic is: $\frac{(n-1)S^2}{\sigma^2} \sim \mathrm{X}^2_{n-1} \sim \mathrm{X}^2_{19}$

From the sample data, we calculated $\bar{x} = 3.39$, $S = 1.41$

Assuming H_0 is true, the observed value of the test statistic $\frac{(n-1)S^2}{{\sigma_0}^2}$ $is\ 16.79$

Since this is a left-tailed test, H_0 will be rejected when the observed value of the test statistics is less or equal to x_0. The value of x_0 is determined such that

$$P[type\ I\ error] = P[\mathrm{X}^2_{19} \leq x_0] = \alpha = 0.1$$

From the chi-squared probability table, we have,

$$x_0 = 11.7$$

Since the observed value of the test statistic is $16.79 > x_0 = 11.7$

Therefore, H_0 can not be rejected. i.e., the sampled data do not support the claim of the researcher

TABLE IV

Cumulative chi-squared distribution *(concluded)*

$P[\chi^2_\gamma \le t]$

γ \ F	0.005	0.010	0.025	0.050	0.100	0.250	0.500	0.750	0.900	0.950	0.975	0.990	0.995
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7