Solution of HW5

Chapter 4

3. $f(x) = \frac{1}{10}e^{-x/10},\ x > 0$

(a) verify that $f(x)$ is a valid density function for a continuous random variable

For $x > 0, f(x) > 0$. Also,

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{10}e^{-x/10}dx = -e^{-\frac{x}{10}}\Big|_0^\infty = 1$$

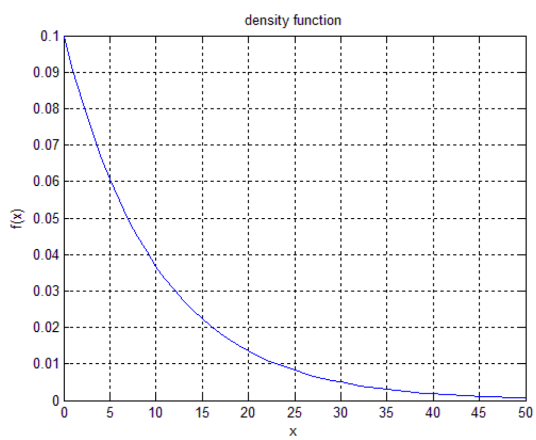
(b) $P[X \le 7] = \int_0^7 \frac{1}{10}e^{-x/10}dx = 0.5034$

$$P[X \ge 7] = 1 - P[X < 7] = 1 - \int_0^7 \frac{1}{10}e^{-\frac{x}{10}}dx = 1 - 0.5034 = 0.4966$$

$$P[X = 7] = 0$$

(c) $P[1 < X < 2] = \int_1^2 \frac{1}{10}e^{-x/10}dx = 0.0861$. Hence, it is unusual that a phone call lasts between 1 and 2 minutes.

(d)



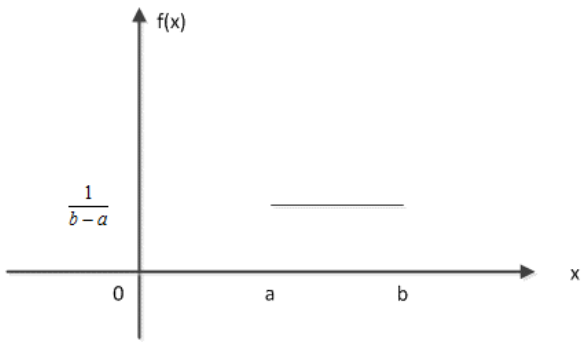
5. $f(x) = \frac{1}{b-a}, a < x < b$

(a) $f(x) \ge 0, for\ a < x < b$

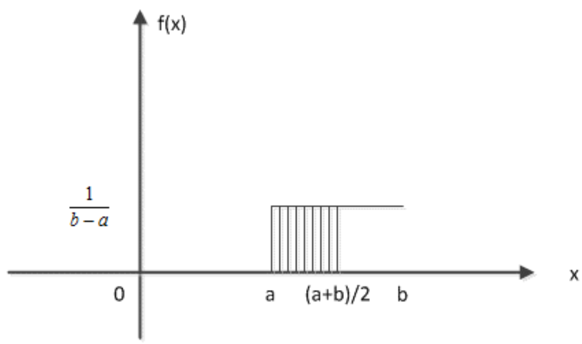
$$\int_{-\infty}^\infty f(x)dx = \int_a^b \frac{1}{b-a}dx = \frac{1}{b-a}x\Big|_a^b = 1$$

Hence, $f(x)$ is a density function.

(b)



(c)



(d) $P\left[X \le \frac{a+b}{2}\right] = 0.5$

(e) $P[c \le X \le d] = P[e \le X \le f]$. Probabilities are equal over intervals of equal length.

10. $F(x) = P[X \le x] = \int_a^x \frac{1}{b-a}\,dt = \frac{t}{b-a}\Big|_a^x = \frac{x-a}{b-a}$, for a < x < b

Thus, $F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x \ge b \end{cases}$

16. $\mu = E[X] = \int_{25}^{50} x \frac{1}{\ln 2} \frac{1}{x}\,dx = \frac{1}{\ln 2} x \Big|_{25}^{50} = \frac{25}{\ln 2} = 36.23$ pounds

$$E[X^2] = \int_{25}^{50} x^2 \frac{1}{\ln 2} \frac{1}{x}\,dx = \frac{1}{\ln 2} \cdot \frac{x^2}{2}\Big|_{25}^{50} = \frac{1875}{2\ln 2} = 1358.6956$$

$$\sigma^2 = 1358.6956 - (36.23)^2 = 51.67$$

$\sigma = \sqrt{51.67} = 7.188$ pounds

17. $f(x) = \frac{1}{10}e^{-x/10},\ x > 0$

(a) $m_X(t) = E[e^{tx}] = \int_0^\infty e^{tx}\frac{1}{10}e^{-x/10}dx = \frac{1}{10}\int_0^\infty e^{(t-0.1)x}dx$

$= \frac{1}{10}\int_0^\infty \left(e^{(t-0.1)}\right)^x dx = \frac{1}{10(t-0.1)}e^{(t-0.1)}\Big|_0^\infty$

This integration is finite $m_X(t) = (1-10t)^{-1}$ when $t < 0.1$

(b) $\frac{d}{dt}m_X(t) = 10(1-10t)^{-2}$

$E[X] = \frac{d}{dt}m_X(t)|t = 0 = 10$ minutes

(c) $E[X^2] = \frac{d^2}{dt^2}m_X(t)|t = 0 = 200(1-10t)^{-3}|t = 0 = 200$

$VarX = E[X^2] - (E[X])^2 = 200 - 100 = 100$

$\sigma_X = \sqrt{VarX} = 10$

23. $f(x) = \left(\frac{50}{6}\right)x^{-3}, 2 < x < 10$

(a) $f(x) \geq 0, 2 < x < 10$

$\int_{-\infty}^\infty f(x)dx = \int_2^{10}\left(\frac{50}{6}\right)x^{-3}dx = -\frac{25}{6x^2}\Big|_2^{10} = 1$

Hence, $f(x)$ is a density function

(b) $F(x) = P[X \leq x] = \int_{-\infty}^x f(t)dt = \int_2^x \left(\frac{50}{6}\right)t^{-3}dt = \frac{25}{24} - \frac{25}{6x^2}$

$P[X \leq 4] = F(4) = \frac{25}{24} - \frac{25}{96} = \frac{75}{96} = 0.78125$

(c) $E[X] = \int_{-\infty}^\infty xf(x)dx = \int_2^{10}\left(\frac{50}{6}\right)x^{-2}dx = \frac{10}{3}$

(d) $E[X^2] = \int_{-\infty}^\infty x^2 f(x)dx = \int_2^{10}\left(\frac{50}{6}\right)x^{-1}dx = 13.412$

$VarX = E[X^2] - (E[X])^2 = 2.3009$

25. (a) $\int_0^\infty z^2 e^{-z}dz = \Gamma(3) = 2\Gamma(2) = 2 \times 1 \times \Gamma(1) = 2$

(b) $\int_0^\infty z^7 e^{-z}dz = \Gamma(8) = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

(c) $\int_0^\infty x^3 e^{-x/2}dx$

Let $z = \frac{x}{2} \Longrightarrow x = 2z$. Then $dz = \left(\frac{1}{2}\right) dx \Longrightarrow dx = 2dz$

Substitute these into the expression, we have,

$\int_0^\infty x^3 e^{-x/2} dx = \int_0^\infty 8z^3 e^{-z} 2dz = 16 \int_0^\infty z^3 e^{-z} dz = 16 \times \Gamma(4) = 96$

(d) $\int_0^\infty \left(\frac{1}{16}\right) x e^{-x/4} dx$

Let $z = \frac{x}{4} \Longrightarrow x = 4z$. Then $dz = \left(\frac{1}{4}\right) dx \Longrightarrow dx = 4dz$

Substitute these into the expression, we have,

$\int_0^\infty \left(\frac{1}{16}\right) x e^{-x/4} dx = \int_0^\infty \left(\frac{1}{16}\right) 4z e^{-z} 4dz = \int_0^\infty z e^{-z} dz = \Gamma(2) = 1$

28. Let $z = \dfrac{x}{\beta}$. Then $x = \beta z$ and $dx = \beta dz$.

Substitution yields

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \beta^{\alpha-1} z^{\alpha-1} e^{-z} \beta dz$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \beta^\alpha \int_0^\infty z^{\alpha-1} e^{-z} dz = \frac{1}{\Gamma(\alpha)} \cdot \Gamma(\alpha) = 1$$

30. $$\frac{dm_X(t)}{dt} = -\alpha(1-\beta t)^{-(\alpha+1)}(-\beta) = \alpha\beta(1-\beta t)^{-(\alpha+1)}$$

$$E[X] = \left.\frac{dm_X(t)}{dt}\right|_{t=0} = \alpha\beta$$

$$\frac{d^2 m_X(t)}{dt^2} = -(\alpha+1)\alpha\beta(1-\beta t)^{-(\alpha+2)}(-\beta)$$

$$= \alpha(\alpha+1)\beta^2(1-\beta t)^{-(\alpha+2)}$$

$$E[X^2] = \left.\frac{d^2 m_X(t)}{dt^2}\right|_{t=0} = \alpha(\alpha+1)\beta^2$$

$$VarX = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$