Joint Distribution

Independence:

Let (X, Y) be a two dimensional continuous random variable with joint density $f_{XY}(\mathrm{x}, \mathrm{y})$ and marginal density f_X and f_Y, respectively. X and Y are independent if and only if

$$f_{XY}(x, y) = f_X(x) f_Y(y), \qquad for\ all\ x\ and\ y$$

Practice Example: The joint density for (X, Y) is $f_{XY}(x, y) = c,\ 8.5 \le x \le 10.5, 120 \le y \le 240$. Are X and Y independent?

We already learned that the joint density function is $f_{XY}(x, y) = \frac{1}{240}$

The marginal density functions can be calculated as:

$$f_X(x) = \int_{120}^{240} \frac{1}{240} dy = \frac{1}{2}, 8.5 \le x \le 10.5$$

$$f_Y(y) = \int_{8.5}^{10.5} \frac{1}{240} dx = \frac{1}{120}, 120 \le y \le 240$$

It is true that, $f_{XY}(x, y) = f_X(x) f_Y(y),\ for\ all\ x\ and\ y$

Therefore, X and Y are independent.

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Independence:

Let (X, Y) be a two dimensional continuous random variable with joint density $f_{XY}(\mathrm{x}, \mathrm{y})$ and marginal density f_X *and* f_Y, respectively. X *and* Y are independent if and only if

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Practice Example: Are (X, Y) with the following joint density independent?

(1) Answer the question by inspecting the joint probability table
(2) Answer the question by checking against the independence condition

X\Y	0	1	2	3
0	0.840	0.030	0.020	0.010
1	0.060	0.010	0.008	0.002
2	0.010	0.005	0.004	0.001

From the table, it is easy to find out that:

$f_X(0) = 0.90; f_Y(0) = 0.91; f_{XY}(0{,}0) = 0.84$

$$f_{XY}(0{,}0) \neq f_X(0) \times f_Y\ (0)$$

Hence, they are not independent!

Joint Distribution

Expected Value:
Let (X, Y) be a two dimensional continuous random variable with joint density $f_{XY}(\mathrm{x}, \mathrm{y})$. Let $H(X, Y)$ be a random variable. The expected value of $H(X, Y)$ is given by:

Discrete case: $E[H(X,Y)] = \sum_{all\ x} \sum_{all\ y} H(x,y) f_{XY}(x,y)$; $E[XY] = \sum_{all\ x} \sum_{all\ y} xy f_{XY}(x,y)$
Continuous case:$E[H(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x,y) f_{XY}(x,y) dydx$; $E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y) dydx$

Practice: Evaluate **Univariate expectation via the joint density. I.e., Find** $E[H(X,Y)]$**, when** $\boldsymbol{H(X,Y) = X}$; $\boldsymbol{H(X,Y) = Y}$

Actually,

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x,y) dydx = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY}(x,y) dydx = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x,y) dydx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x,y) dxdy = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y) dxdy = \int_{-\infty}^{\infty} y f_Y(y) dy$$

Switched the order of integration

Joint Distribution

Expected Value:

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Continuous case:$E[H(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x,y) f_{XY}(x,y)dydx$; $E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y)dydx$

Practice Example: The joint density for (X, Y) is $f_{XY}(x,y) = \frac{1}{240}, 8.5 \le x \le 10.5, 120 \le y \le 240.$

Find $E[XY], E[X], and\ E[Y]$

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x,y)dydx = \int_{8.5}^{10.5} \int_{120}^{240} \frac{x}{240} dydx = \int_{8.5}^{10.5} \frac{x}{2} dx = 9.5$$

$$E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x,y)dxdy = \int_{120}^{240} \int_{8.5}^{10.5} \frac{y}{240} dxdy = \int_{120}^{240} \frac{y}{120} dy = 180$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y)dydx = \int_{120}^{240} \int_{8.5}^{10.5} \frac{xy}{240} dxdy = \int_{120}^{240} \frac{19y}{240} dy = 1710$$

Joint Distribution

Expected Value:

Let (X, Y) be a two dimensional continuous random variable with joint density $f_{XY}(\mathrm{x}, \mathrm{y})$. Let $H(X, Y)$ be a random variable. The expected value of $H(X, Y)$ is given by:

Discrete case: $E[H(X,Y)] = \sum_{all\ x} \sum_{all\ y} H(x,y) f_{XY}(x,y)$; $E[XY] = \sum_{all\ x} \sum_{all\ y} xy f_{XY}(x,y)$

Continuous case: $E[H(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x,y) f_{XY}(x,y) dydx$; $E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y) dydx$

Practice Example: Evaluate $E[X+Y]$ ***show that*** $E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) f_{XY}(x,y) dydx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x,y) dydx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x,y) dxdy$$

$$= \int_{-\infty}^{\infty} x f_X(x) dx + \int_{-\infty}^{\infty} y f_Y(y) dy = E[X] + E[Y]$$

Switched the order of integration

Joint Distribution

Covariance:

Let $X \text{ and } Y$ be random variables with means $\mu_X \text{ and } \mu_Y$ respectively. The covariance between $X \text{ and } Y$, denoted by $Cov(X,Y) \text{ or } \sigma_{XY}$ is given by:

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

Can this quantity represent some kind of correlation between $X \text{ and } Y$?

Prove that $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[XY - \mu_Y X - \mu_X Y + \mu_X\mu_Y] \\ &= E[XY] - \mu_Y E[X] - \mu_X E[Y] + \mu_X\mu_Y \\ &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

Evaluate $Cov(X,Y)$ when $X = Y$.

Can $Cov(X,Y) < 0$?

Joint Distribution

Prove that $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.
We already learned that When $X \ and \ Y \ are \ independent, Var(X+Y) = Var(X) + Var(Y)$

Proof:

$$\begin{aligned} Var(X+Y) &= E[(X+Y)^2] - (E[X+Y])^2 \\ &= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\ &= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2 \\ &= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2 + 2E[XY] - 2E[X]E[Y] \\ &= Var(X) + Var(Y) + 2Cov(X,Y) \end{aligned}$$

Joint Distribution

Theorem: Let (X, Y) be a two dimensional continuous random variable with joint density $f_{XY}(\mathrm{x}, \mathrm{y})$. If $X\ and\ Y$ are independent then $E[XY] = E[X]E[Y]$

Proof:

It is know that if $X\ and\ Y\ are\ independent,$ *we have* $f_{XY}(x, y) = f_X(x)f_Y(y)$

Then,

$$E[XY] = \iint_{-\infty}^{\infty} xyf_{XY}(x, y)dydx = \iint_{-\infty}^{\infty} xyf_X(x)f_Y(y)dydx$$

$$= \int_{-\infty}^{\infty} xf_X(x) \int_{-\infty}^{\infty} yf_Y(y)dy\, dx = \int_{-\infty}^{\infty} xf_X(x)\, E[Y]dx = E[Y] \int_{-\infty}^{\infty} xf_X(x)\, dx = E[Y]E[X]$$

Joint Distribution

Theorem: Let (X, Y) be a two dimensional continuous random variable with joint density $f_{XY}(\mathrm{x}, \mathrm{y})$. If $X\ and\ Y$ are independent then $E[XY] = E[X]E[Y]$

From this theorem, we have

$$X\ and\ Y\ are\ independent\ \Rightarrow Cov(X, Y) = 0$$

But, $\quad Cov(X, Y) = 0 \nRightarrow X\ and\ Y\ are\ independent$!!!

Here is an example:

Let $X \sim N(0,1)$. We have, $m_X(t) = e^{\frac{t^2}{2}}$
Then,

$$E[X] = \frac{d}{dt} m_X(t)_{t=0} = te^{\frac{t^2}{2}}\Big|_{t=0} = 0; E[X^2] = \frac{d^2}{dt^2} m_X(t)_{t=0} = \left[t^2 e^{\frac{t^2}{2}} + e^{\frac{t^2}{2}}\right]_{t=0} = 1;$$

$$E[X^3] = \frac{d^3}{dt^3} m_X(t)_{t=0} = \left[t^3 e^{\frac{t^2}{2}} + 3te^{\frac{t^2}{2}}\right]_{t=0} = 0$$

Let $Y = X^2$. $X\ and\ Y$ are related. But,

$$Cov(X, Y) = E[XY] - E[X]E[Y] = E[X^3] - E[X]E[X^2] = 0$$

Joint Distribution

Covariance:

Let X and Y be random variables with means μ_X and μ_Y respectively. The covariance between X and Y, denoted by $Cov(X, Y)$ or σ_{XY} is given by $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]$

Practice Example: Find $Cov(X_1, Y_1)$ and $Cov(X_2, Y_2)$. Can you tell the correlation of X and Y before the calculation?

X1\Y1	0	1	2
0	0.40	0.006	0.004
1	0.006	0.30	0.004
2	0.004	0.006	0.27

$$E[X_1Y_1] = 1.4$$
$$E[X_1] = 0.87$$
$$E[Y_1] = 0.868$$
$$Cov(X_1, Y_1) = 0.645$$

X2\Y2	0	1	2
0	0.1	0.1	0.1
1	0.13	0.13	0.14
2	0.1	0.1	0.1

$$E[X_2Y_2] = 1.01$$
$$E[X_2] = 1.0$$
$$E[Y_2] = 1.01$$
$$Cov(X_2, Y_2) = 0$$