# Exponential Distribution

***Exponential distribution:*** An *exponential* random variable is a gamma random variable with $\alpha = 1$. The density for an exponential random variable is in the form:

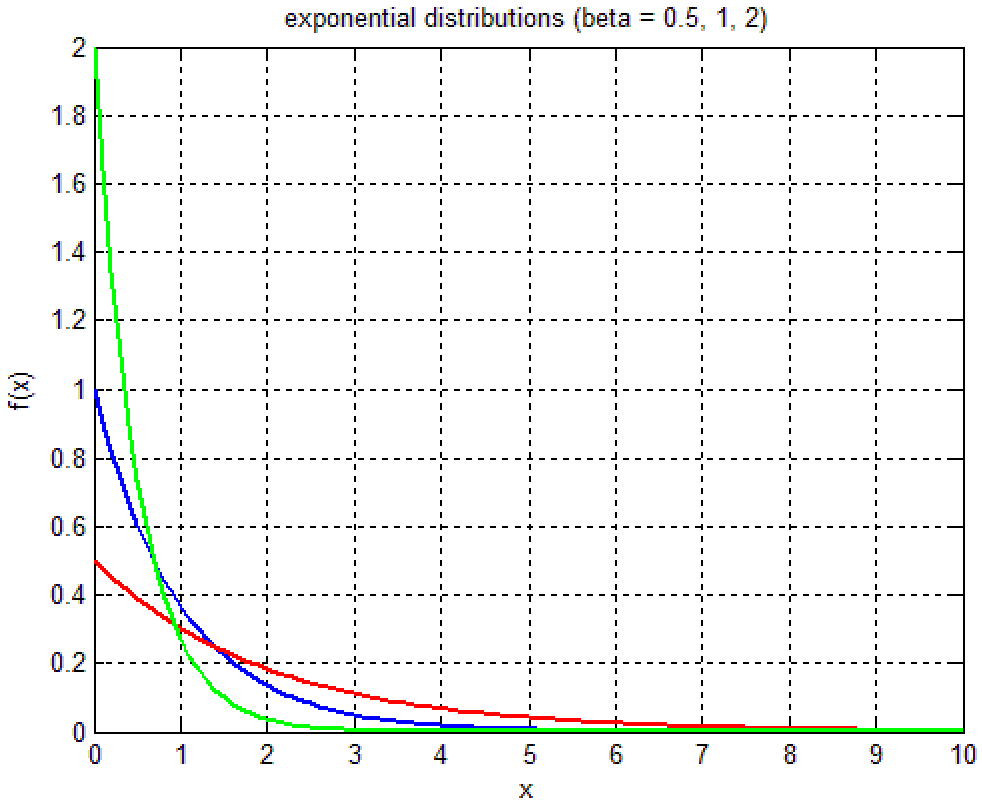
$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x > 0, \beta > 0$$

The density function of a Gamma R.V. is

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}, \qquad x > 0, \alpha > 0, \beta > 0$$

When $\alpha = 1$, we have,

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \qquad x > 0, \beta > 0$$




Q: in the graph, which curve has $\beta = 0.5$?

# Exponential Distribution

***Exponential distribution:*** An *exponential* random variable is a gamma random variable with $\alpha = 1$. The density for an exponential random variable is in the form:

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \qquad x > 0, \beta > 0$$

What is the moment generating function of an exponential R.V.? What are $E[X]$ and $VarX$?

The moment generating function of a Gamma R.V. $X$ is

$$m_X(t) = (1 - \beta t)^{-\alpha}$$

When $\alpha = 1,$ we have, the moment generating function of an exponential random variable $X$ with parameter $\beta$ is

$$m_X(t) = (1 - \beta t)^{-1}$$

$$E[X] = \beta; VarX = \beta^2$$

# Exponential Distribution

***Exponential distribution:*** An *exponential* random variable is a gamma random variable with $\alpha = 1$. The density for an exponential random variable is in the form:

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x > 0, \beta > 0$$

**Example:** Consider a Poisson process with parameter $\lambda$ representing the rate of occurrence of an event within a time interval. Let $Y$ denote the time of the occurrence of the first event. $Y$ has an exponential distribution with $\beta = 1/\lambda$.

The cumulative distribution function of $Y$ is: $F(y) = P[Y \le y] = 1 - P[Y > y]$
$P[Y > y]$ is the probability that the first occurrence of the event takes place after time $y$. The event $Y > y$ happens only if no occurrence of the event are recorded in the time interval $[0, y]$.

Let $X$ denote the number of occurrence of the event in time interval $[0, y]$. Then $X$ is a Poisson random variable with parameter $\lambda y$. Thus,

$$P[Y > y] = P[X = 0] = \frac{e^{-\lambda y}(\lambda y)^0}{0!} = e^{-\lambda y}$$

Then,

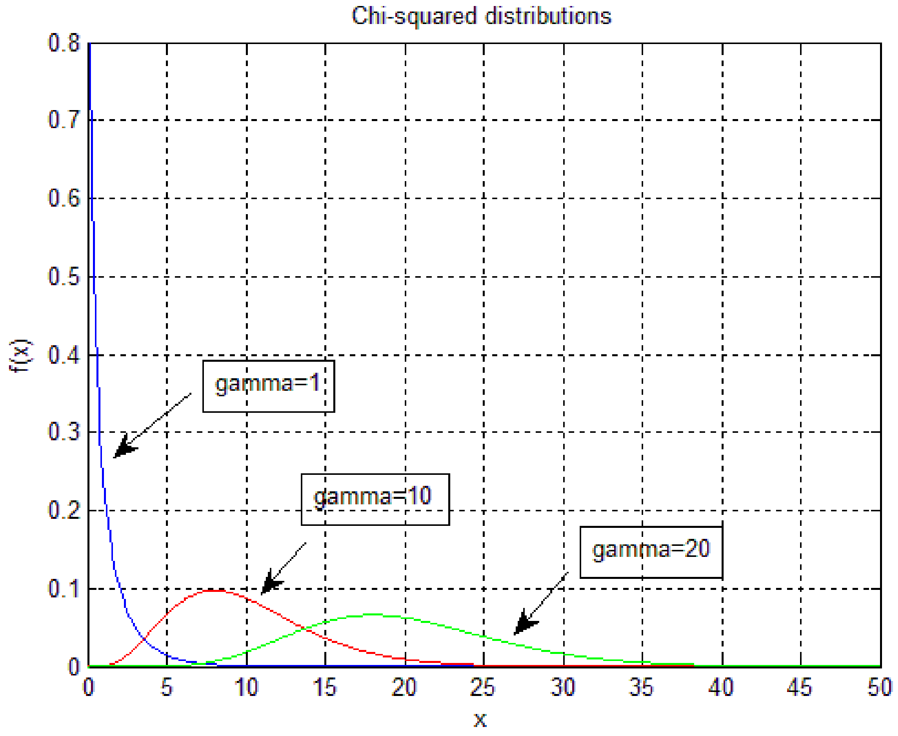
$$F(y) = 1 - P[Y > y] = 1 - e^{-\lambda y} \Rightarrow f(y) = F'(y) = \lambda e^{-\lambda y} = \frac{1}{\beta} e^{-\frac{y}{\beta}}, \qquad \beta = \frac{1}{\lambda}$$

i.e., $Y$ has an exponential distribution with parameter $\beta = \frac{1}{\lambda}$

# Chi-squared Distribution

***Chi-squared distribution:*** Let $X$ be a gamma random variable with $\beta = 2$ $and$ $\alpha = \frac{\gamma}{2}$, $for$ $\gamma$ a positive integer. $X$ is said to have a chi-squared distribution with $\gamma$ *degree of freedom*. We denote this variable by $X_\gamma^2$. The density function of a chi-squared RV is in the form:

$$f(x,\gamma) = \frac{x^{(\frac{\gamma}{2}-1)}e^{-x/2}}{2^{\frac{\gamma}{2}}\,\Gamma(\frac{\gamma}{2})}, x > 0$$




What is the m.g.f of a Chi-squared random variable $X$ ? What are $E[X]$ $and$ $VarX$ ?

The moment generating function of a Gamma R.V. $X$ is $m_X(t) = (1 - \beta t)^{-\alpha}$

When $\beta = 2$ $and$ $\alpha = \frac{\gamma}{2}$, we have, the moment generating function of an exponential random variable $X$ with parameter $\gamma$ is

$$m_X(t) = (1 - 2t)^{-\gamma/2}$$

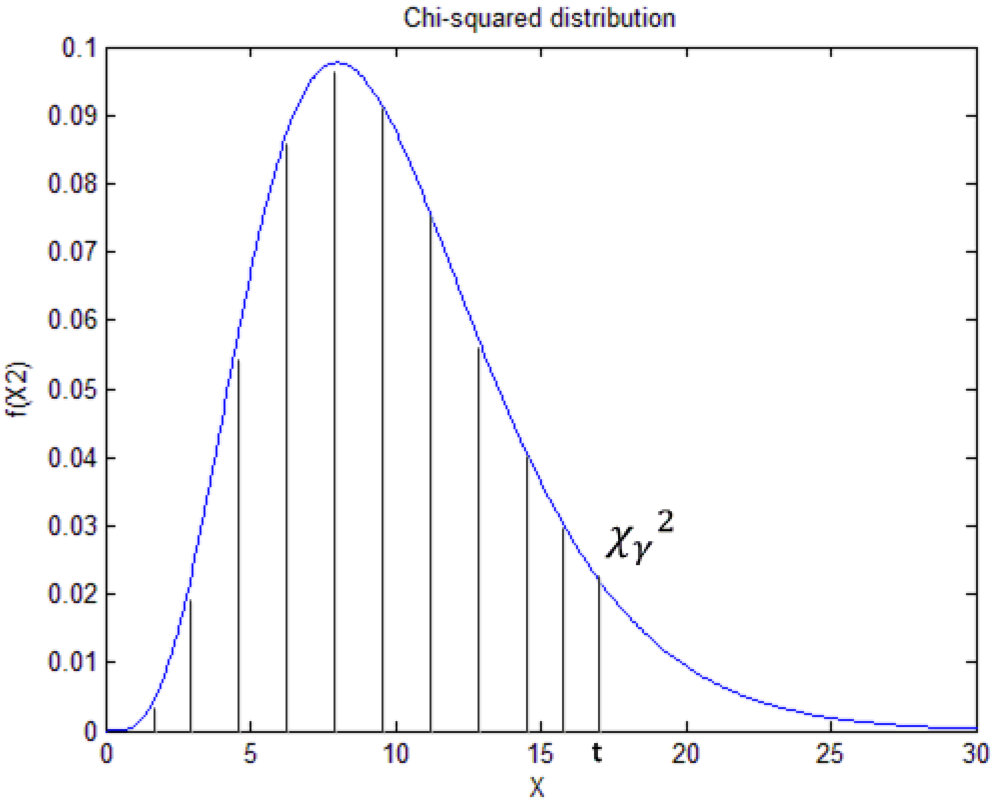
$E[X] = \gamma$; $VarX = 2\gamma$

# Chi-squared Distribution

***How to read Table IV:*** *cumulative distribution function*

$$P\left[X_{\gamma}{}^{2} \le t\right] = \alpha$$

$t$ is called the ***critical point*** for ***left-tailed probability*** of $\alpha$

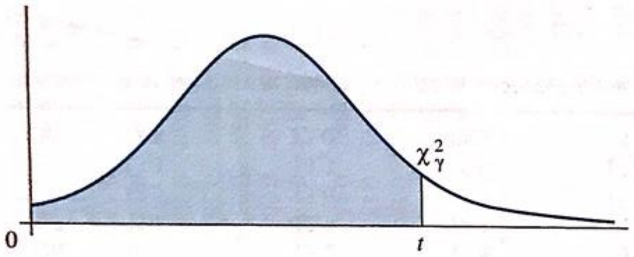


# Chi-squared Distribution

***How to read Table IV:***
cumulative distribution function

$$P[X_\gamma{}^2 \leq t]$$

**TABLE IV**
**Cumulative chi-squared distribution**




$P[\chi^2_\gamma \leq t]$

| $\gamma$ \ $F$ | 0.005 | 0.010 | 0.025 | 0.050 | 0.100 | 0.250 | 0.500 | 0.750 | 0.900 | 0.950 | 0.975 | 0.990 | 0.995 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0000393 | 0.000157 | 0.000982 | 0.00393 | 0.0158 | 0.102 | 0.455 | 1.32 | 2.71 | 3.84 | 5.02 | 6.63 | 7.88 |
| 2 | 0.0100 | 0.0201 | 0.0506 | 0.103 | 0.211 | 0.575 | 1.39 | 2.77 | 4.61 | 5.99 | 7.38 | 9.21 | 10.6 |
| 3 | 0.0717 | 0.115 | 0.216 | 0.352 | 0.584 | 1.21 | 2.37 | 4.11 | 6.25 | 7.81 | 9.35 | 11.3 | 12.8 |
| 4 | 0.207 | 0.297 | 0.484 | 0.711 | 1.06 | 1.92 | 3.36 | 5.39 | 7.78 | 9.49 | 11.1 | 13.3 | 14.9 |
| 5 | 0.412 | 0.554 | 0.831 | 1.15 | 1.61 | 2.67 | 4.35 | 6.63 | 9.24 | 11.1 | 12.8 | 15.1 | 16.7 |
| 6 | 0.676 | 0.872 | 1.24 | 1.64 | 2.20 | 3.45 | 5.35 | 7.84 | 10.6 | 12.6 | 14.4 | 16.8 | 18.5 |
| 7 | 0.989 | 1.24 | 1.69 | 2.17 | 2.83 | 4.25 | 6.35 | 9.04 | 12.0 | 14.1 | 16.0 | 18.5 | 20.3 |
| 8 | 1.34 | 1.65 | 2.18 | 2.73 | 3.49 | 5.07 | 7.34 | 10.2 | 13.4 | 15.5 | 17.5 | 20.1 | 22.0 |
| 9 | 1.73 | 2.09 | 2.70 | 3.33 | 4.17 | 5.90 | 8.34 | 11.4 | 14.7 | 16.9 | 19.0 | 21.7 | 23.6 |
| 10 | 2.16 | 2.56 | 3.25 | 3.94 | 4.87 | 6.74 | 9.34 | 12.5 | 16.0 | 18.3 | 20.5 | 23.2 | 25.2 |
| 11 | 2.60 | 3.05 | 3.82 | 4.57 | 5.58 | 7.58 | 10.3 | 13.7 | 17.3 | 19.7 | 21.9 | 24.7 | 26.8 |
| 12 | 3.07 | 3.57 | 4.40 | 5.23 | 6.30 | 8.44 | 11.3 | 14.8 | 18.5 | 21.0 | 23.3 | 26.2 | 28.3 |

***Practice***: Find the critical values using the table

$P[X^2_5 \leq t] = 0.90$
$P[X^2_{10} \leq t] = 0.95$
$P[X^2_5 \leq t] = 0.05$
$P[X^2_{10} \leq t] = 0.005$

9.24
18.3
1.15
2.16

# Chi-squared Distribution

*Complement cumulative distribution function*

$$P[{X_\gamma}^2 \geq t]$$

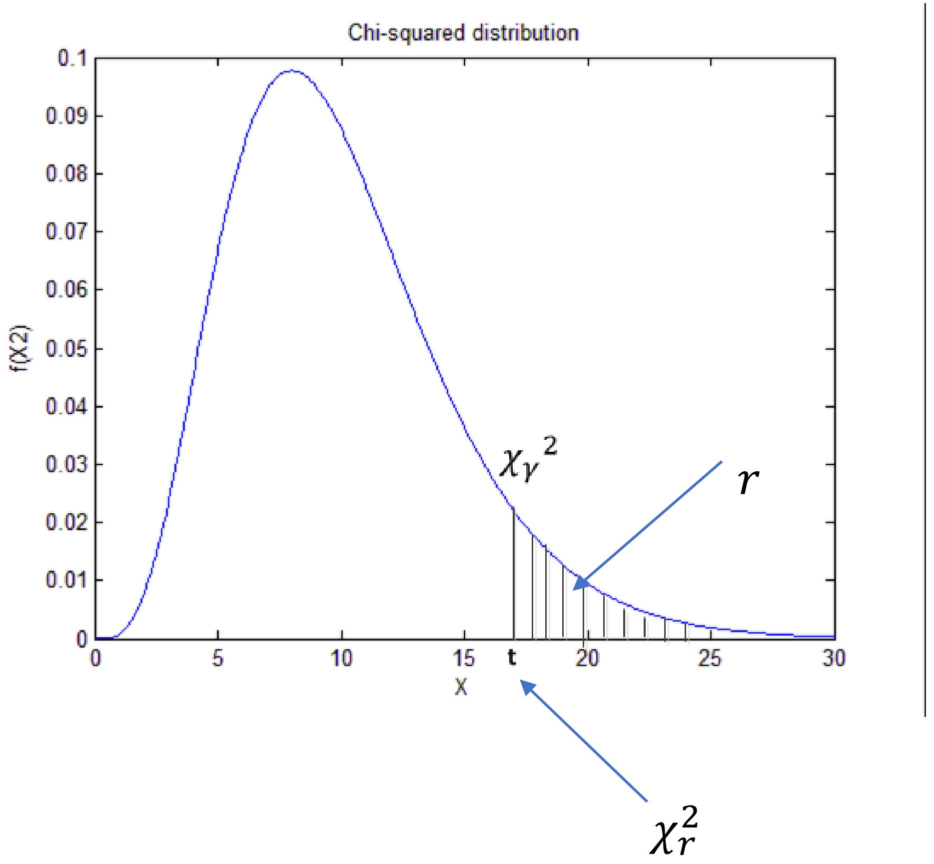
In practice, we use $\chi_r^2$ to denote the point associated with a chi-squared random variable such that

$$P\left[X_\gamma^2 \geq \chi_r^2\right] = r$$

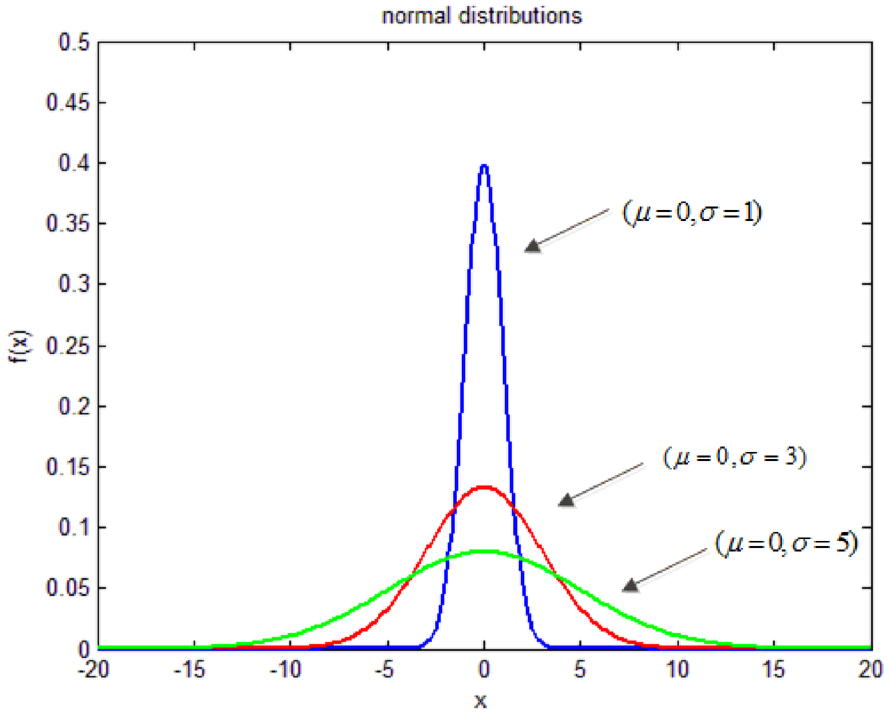
$\chi_r^2$ is called the ***critical point*** for the ***right-tailed probability*** of $r$

For example, for $\gamma = 10$, find the value of $\chi_{0.05}^2$ using table IV.

$\chi_{0.05}^2 = 18.3$



# Normal Distribution




A random variable $X$ with density

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \qquad -\infty \le x \le \infty, -\infty \le \mu \le \infty, \sigma > 0$$

Is said to have a ***normal distribution*** with parameter $\mu\ and\ \sigma$.

$$X \sim N(\mu, \sigma^2)$$

**The moment generating function** of a normal random variable with parameter $\mu\ and\ \sigma$ is

$$m_X(t) = e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$$

**Practice:** use $m_X(t)$ of a normal random variable to find $E[X]\ and\ VarX$

$$\frac{d}{dt} m_X(t) = (\mu + \sigma^2 t) e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \Rightarrow E[X] = \frac{d}{dt} m_X(t)_{t=0} = \mu$$

$$\frac{d^2}{dt^2} m_X(t) = \sigma^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} + (\mu + \sigma^2 t)^2 e^{(\mu t + \frac{1}{2}\sigma^2 t^2)} \Rightarrow E[X^2] = \frac{d^2}{dt^2} m_X(t)_{t=0} = \sigma^2 + \mu^2$$

$$VarX = E[X^2] - (E[X])^2 = \sigma^2$$

# Normal Distribution

A random variable $X$ with density

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \qquad -\infty \le x \le \infty, -\infty \le \mu \le \infty, \sigma > 0$$

Is said to have a ***normal distribution*** with parameter $\mu$ $and$ $\sigma$. Commonly this is represented as $X \sim N(\mu, \sigma^2)$

**Practice:** Let $X$ be a normal random variable with $\mu = 1, \sigma = 0.25,$
Find:

$f(x = 0)$

$f(x = 2)$

$f(x = \mu)$

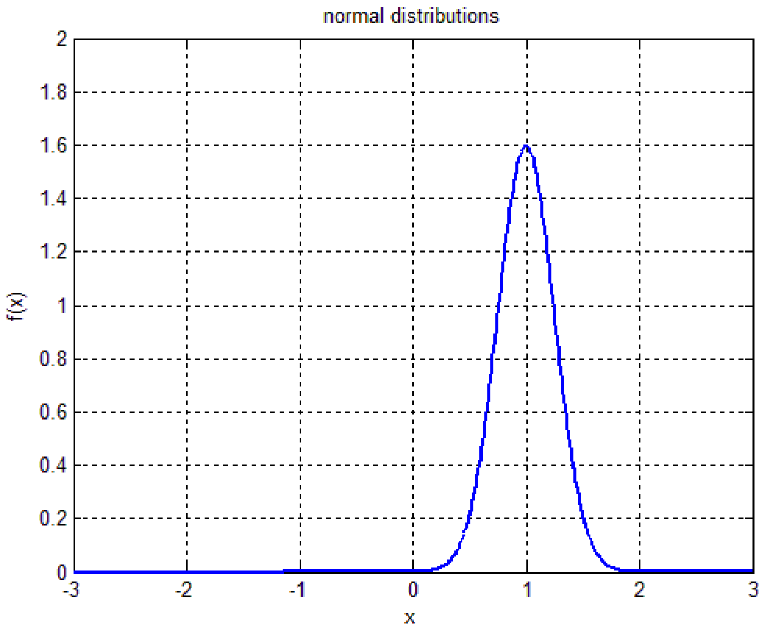
$f(x = \mu + \sigma)$

$$f(x = 0) = \frac{1}{\sqrt{2\pi} \times 0.25} e^{-8} \approx 10^{-4}$$

$$f(x = 2) = \frac{1}{\sqrt{2\pi} \times 0.25} e^{-8} \approx 10^{-4}$$

$$f(x = \mu) = \frac{1}{\sqrt{2\pi} \times 0.25} e^{0} \approx 1.6$$

$$f(x = \mu + \sigma) = \frac{1}{\sqrt{2\pi} \times 0.25} e^{-0.5} \approx 0.96$$



# Standard Normal Distribution

***Standardization theorem:***

Let $X$ be a normal random variable with $\mu$ $and$ $\sigma$. The variable $\frac{X-\mu}{\sigma}$ is standard normal with mean 0 and standard deviation 1. The density function of a standard normal variable is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty \leq x \leq \infty$$

A standard normal RV is usually denoted using $Z$.

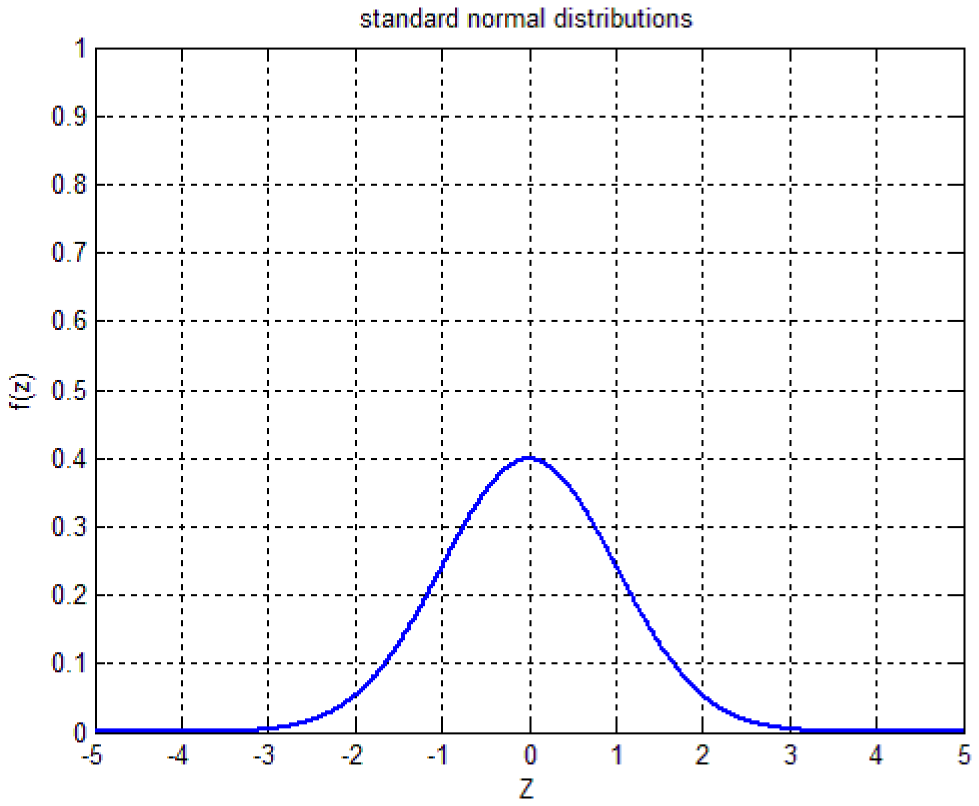
$$Z \sim N(0,1)$$

Show that $E[Z] = 0$ $and$ $VarZ = 1$

Actually, given that $E[X] = \mu$ $and$ $VarX = \sigma^2$, we have,

$$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = E\left[\frac{X}{\sigma}\right] - \frac{\mu}{\sigma} = 0$$

$$VarZ = Var\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2} VarX = 1$$



# Standard Normal Distribution

***Standardization theorem:***

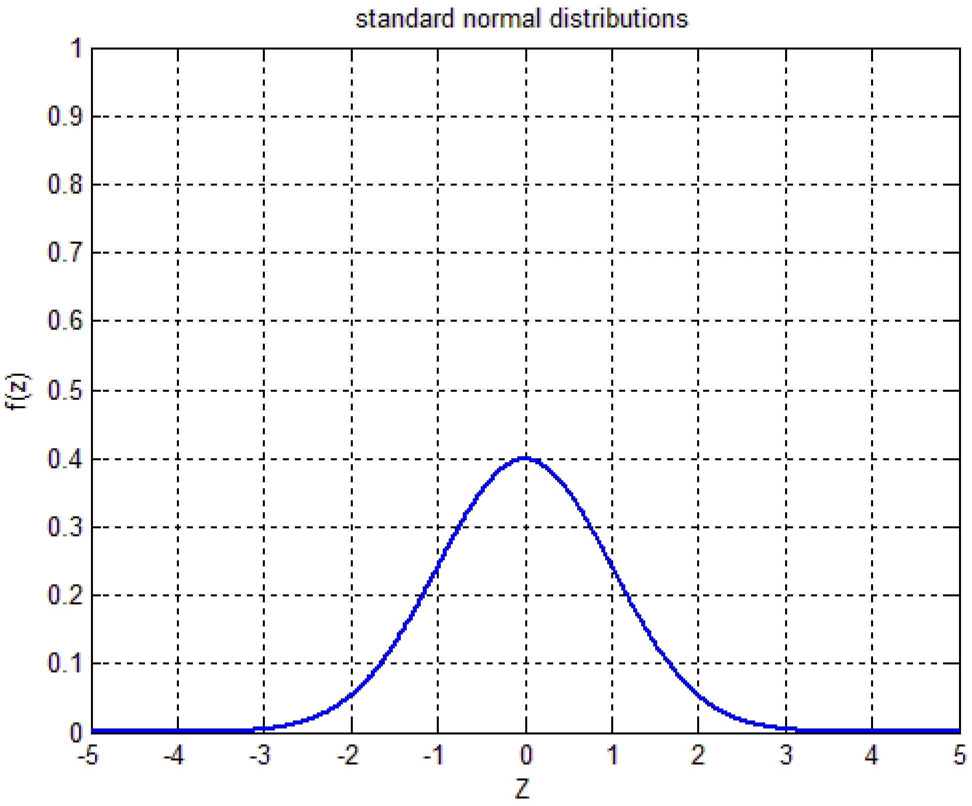
Let $X$ be a normal random variable with $\mu\ and\ \sigma$. The variable $\frac{x-\mu}{\sigma}$ is standard normal with mean 0 and standard deviation 1. The density function of a standard normal variable is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty \leq x \leq \infty$$

A standard normal RV is usually denoted using $Z$.

$$Z \sim N(0,1)$$

***This density function is symmetric with respect to $Z = 0$!!!***




What is $P[Z \geq 0]$?
Is $P[Z \geq 2] = P[Z \leq -2]$ true?

# The Standard Normal cumulative distribution function table

The integral of the normal density function has no close-form expression. Hence the calculation of probabilities from normal densities requires a numerical method. To solve this problem, a standard normal distribution table is created.

This table represents the following cumulative probability of a standard normal random variable $Z$: (*left-tailed probability*)

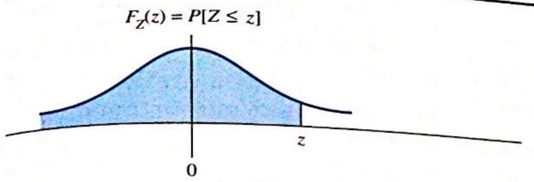
$$F_Z(z) = P[Z \leq z]$$

**Practice problem:** Find the probabilities using the standard normal probability table.

| | |
|---|---|
| $P[Z \leq -2.11]$ | 0.0174 |
| $P[Z < -2.11]$ | 0.0174 |
| $P[Z \geq -2.11]$ | 0.9826 |
| $P[Z \leq 2.11]$ | 0.9826 |
| $P[Z > 2.11]$ | 0.0174 |



TABLE V
**Cumulative distribution: Standard normal**

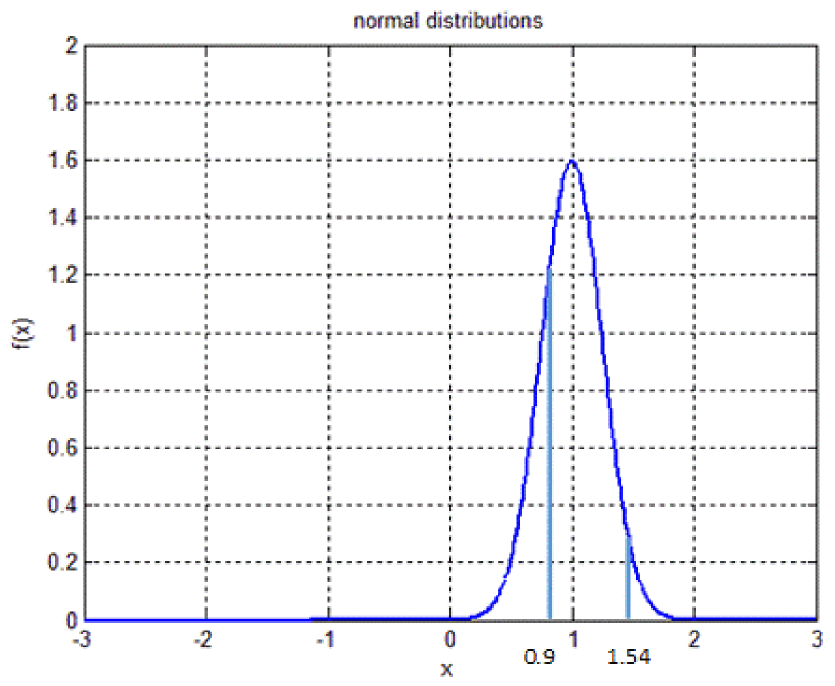



$F_Z(z) = P[Z \leq z]$

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| −3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| −3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| −3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| −3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| −2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| −2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| −2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| −2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| −2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| −2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| −2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| −2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| −2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| −2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| −1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| −1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| −1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| −1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| −1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| −1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| −1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| −1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| −1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| −1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| −0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| −0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1921 | 0.1894 | 0.1867 |
| −0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| −0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| −0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| −0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| −0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| −0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| −0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| −0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

# The Standard Normal cumulative distribution function table

**Example:** Let $X$ denote the number of grams of hydrocarbons emitted by an automobile per mile. Assuming that $X$ is normal with $\mu = 1\ gram$ and $\sigma = 0.25\ gram$, find the probability that a randomly selected automobile will emit between 0.9 and 1.54 grams of hydrocarbons per mile.




**Solution:** To find $P[0.9 \leq X \leq 1.54]$, we first ***standardize*** $X$

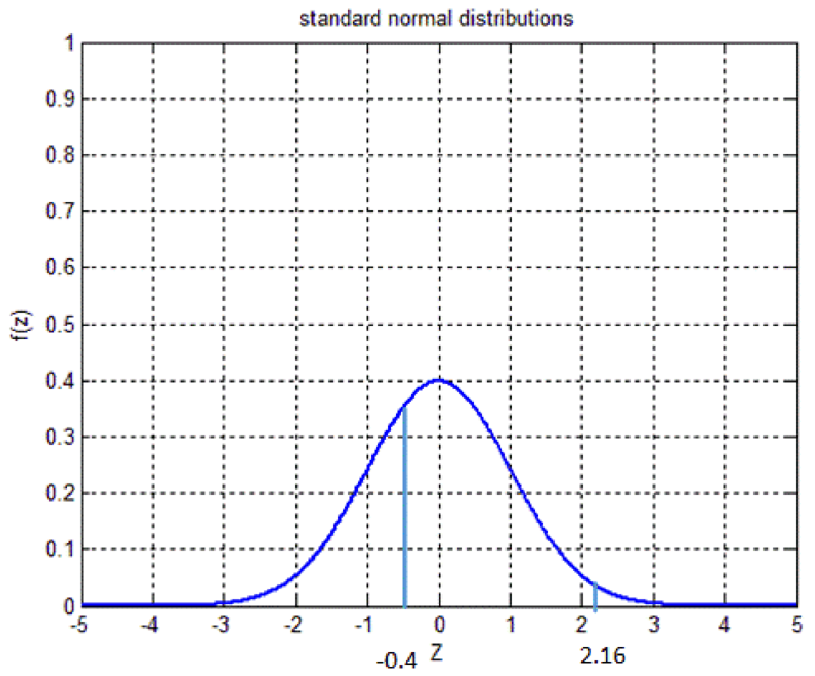
$$P[0.9 \leq X \leq 1.54] = P[\frac{0.9-1}{0.25} \leq \frac{X-1}{0.25} \leq \frac{1.54-1}{0.25}]$$

Notice that the random variable $\frac{X-1}{0.25} = Z$ is a standard normal RV. Therefore, the problem becomes finding

$$P[-0.4 \leq Z \leq 2.16]$$

This can be done using the standard normal cumulative distribution table:

$$P[-0.4 \leq Z \leq 2.16] = P[Z \leq 2.16] - P[Z \leq -0.4] = F(2.16) - F(-0.4)$$
$$= 0.9846 - 0.3446 = 0.64$$



# Normal Probability Rules

Let $X$ be a normal random variable with parameters $\mu$ $and$ $\sigma$. Then

$$P[-\sigma < X - \mu < \sigma] \approx 0.68$$
$$P[-2\sigma < X - \mu < 2\sigma] \approx 0.95$$
$$P[-3\sigma < X - \mu < 3\sigma] \approx 0.997$$

$$P[-\sigma < X - \mu < \sigma] = P\left[-1 < \frac{X-\mu}{\sigma} < 1\right] = P[-1 < Z < 1] \approx 0.68$$
$$P[-2\sigma < X - \mu < 2\sigma] = P\left[-2 < \frac{X-\mu}{\sigma} < 2\right] = P[-2 < Z < 2] \approx 0.95$$
$$P[-3\sigma < X - \mu < 3\sigma] = P\left[-3 < \frac{X-\mu}{\sigma} < 3\right] = P[-3 < Z < 3] \approx 0.997$$

If X is a normal variable, we can be 99.7% sure that a value x will be within $3\sigma$ distance from the center $\mu$

It is 0.3% rare that a value of a normal variable can be $3\sigma$ away from the center $\mu$.