Solutions of HW9

Chapter 7

31. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from an exponential random variable with parameter β.

Let's find the moments based estimator:

$$E[X] = \beta$$

Replace both sides by its estimators, we have,

$$M_1 = \hat{\beta}$$

i.e., $\hat{\beta} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Let's find the maximum likelihood estimator:

The likelihood function is:

$$L(\beta) = \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-X_i/\beta} = \frac{1}{\beta^n} e^{-\frac{1}{\beta}\sum_{i=1}^{n} X_i}$$

The log likelihood function is:

$$\ln L(\beta) = -n \ln \beta - \frac{1}{\beta}\sum_{i=1}^{n} X_i$$

Take the derivative of the log likelihood function:

$$\frac{d}{d\beta}\ln L(\beta) = -\frac{n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} X_i$$

Let the derivative equal to zero. We have,

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

The maximum likelihood estimator and the moments-based estimator has the same form.

32. (a) $\hat{\lambda} = \bar{x} = 10.3$ calls

(b) The time of arrival of the first call of the day is exponential with parameter

$$\beta = \frac{1}{\lambda}.$$

$$\hat{\mu} = \hat{\beta} = \frac{1}{\hat{\lambda}} = \frac{1}{10.3} = .0971 \text{ hours}$$

Since .0971 hours = 5.83 minutes and the center opens at 9 a.m., the average time of arrival of the first call is estimated to be 9:058.

33. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a gamma random variable with parameter $\alpha = 2$ and β $unknown$.

We have, $f(x) = \frac{1}{\Gamma(2)\beta^2} x e^{-x/\beta}$

The likelihood function is:

$$L(\beta) == \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} \frac{1}{\Gamma(2)\beta^2} X_i e^{-X_i/\beta} = \frac{1}{\Gamma(2)\beta^{2^n}} \prod_{i=1}^{n} X_i\, e^{-\frac{1}{\beta}\sum_{i=1}^{n} X_i}$$

The log likelihood function is:

$$\ln L(\beta) = -n(\ln \Gamma(2) + 2\ln \beta) + \ln \prod_{i=1}^{n} X_i - \frac{1}{\beta}\sum_{i=1}^{n} X_i$$

Take the derivative of the log likelihood function, we have,

$$\frac{d}{d\beta} \ln L(\beta) = -\frac{2n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} X_i$$

Let this derivative equal to zero. We have,

$$\hat{\beta} = \frac{1}{2n}\sum_{i=1}^{n} X_i = \frac{1}{2}\bar{X}$$

(b) Since $\mu = E[X] = \alpha\beta = 2\beta$

We take the estimator of μ as $\hat{\mu} = 2\hat{\beta}$

$$E[\hat{\mu}] = E\left[2\hat{\beta}\right] = 2E\left[\hat{\beta}\right] = 2E\left[\tfrac{1}{2}\bar{X}\right] = E[\bar{X}] = \mu$$

Hence, this is an unbiased estimator.

(c) we calculated that $\bar{X} = 70.5$. Therefore, $\hat{\beta} = 35.25$

(d) $\bar{\mu} = 2\hat{\beta} = 70.5$

37. (a) Normal distribution: $\mu = 2, \sigma^2 = 9$
(b) Normal distribution: $\mu = 0, \sigma^2 = 4$
(c) Geometric distribution: $p = 0.25$
(d) Binomial distribution: $n = 5, p = 0.5$
(e) Poisson distribution: $k = 6$
(f) Gamma distribution: $\alpha = 5, \beta = 3$
(g) Chi-squared distribution: $\gamma = 16$
(h) Exponential distribution: $\beta = 0.5$

39. $m_Y(t) = E[e^{Yt}] = E\left[e^{(a_0+a_1X_1+a_2X_2+\cdots+a_nX_n)t}\right] = E[e^{a_0t}e^{a_1X_1t}e^{a_2X_2t} \ldots e^{a_nX_nt}]$

Since $X_1, X_2, \ldots, X_n$are independent. $e^{a_1X_1t}, e^{a_2X_2t}, \ldots, e^{a_nX_nt}$ are also independent. Therefore, we have,

$m_Y(t) = E[e^{a_0t}e^{a_1X_1t}e^{a_2X_2t} \ldots e^{a_nX_nt}] = E[e^{a_0t}]E[e^{a_1X_1t}] \ldots E[e^{a_nX_nt}]$

$$= e^{a_0t}m_{X_1}(a_1t)m_{X_2}(a_2t) \ldots m_{X_n}(a_nt) = e^{a_0t}\prod_{i=1}^{n} m_{X_i}(a_it)$$

41. We have that $m_{X_i}(t) = e^{\mu_i t+\sigma_i{}^2t^2/2}, i = 1,2, \ldots, n$. Then, applying the result from question 39 gives:

$m_Y(t) = e^{a_0t}\prod_{i=1}^{n} m_{X_i}(a_it) = e^{a_0t}e^{\mu_1a_1t+\sigma_1{}^2a_1{}^2t^2/2} \ldots e^{\mu_na_nt+\sigma_n{}^2a_n{}^2t^2/2}$

$= e^{(a_0+\mu_1a_1+\cdots+\mu_na_n)t+(\sigma_1{}^2a_1{}^2+\cdots+\sigma_n{}^2a_n{}^2)t^2/2}$

Hence, Y is following a normal distribution with

$\mu = a_0 + \sum_{i=1}^{n} a_i\mu_i\,, \sigma^2 = \sum_{i=1}^{n} a_i{}^2\sigma_i{}^2$

42. $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n$

In reference to Exercise 41, $a_0 = 0$ and $a_1 = \frac{1}{n}$ for i = 1, 2, …, n.

Thus, $\mu_{\bar{X}} = \sum_{i=1}^{n}\frac{1}{n}\mu = \frac{1}{n}\cdot n\mu = \mu$ and $\sigma_{\bar{X}}^2 = \sum_{i=1}^{n}\frac{1}{n^2}\sigma^2 = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$

Since $X_1, X_2, \ldots, X_n$ constitute a random sample from a normal population, and thus are independent normal random variables, $\bar{X}$ is normal.

43. We have, $m_{X_1}(t) = (1-2t)^{-5/2}, m_{X_2}(t) = (1-2t)^{-10/2}$

Since X_1 *and* X_2 are independent, we have

$m_{X_1+X_2}(t) = m_{X_1}(t)m_{X_2}(t) = (1-2t)^{-15/2}$

Hence, $X_1 + X_2$ is a chi-squared R.V. with degree of freedom of 15

44. Since X_i is a chi-squared random variable with parameter γ_i,

$$m_{X_i}(t) = (1-2t)^{\gamma_i/2}$$

Using Exercise 39, $m_Y(t) = \prod_{i=1}^{n} m_{X_i}(t)$

Substitution then yields $m_Y(t) = \prod_{i=1}^{n}(1-2t)^{\gamma_i/2} = (1-2t)^{\sum\gamma_i/2}$

which is the moment generating function of a chi-squared random variable with parameter $\gamma = \sum_{i=1}^{n}\gamma_i$.

45. We have, $X_i \sim N(\mu, \sigma^2),\ i = 1,2, \ldots, n$. Then, $\frac{X_i - \mu}{\sigma} \sim N(0,1)$. i.e., $\frac{X_i - \mu}{\sigma}$ is a standard normal R.V.

Hence, $\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_1^2$

Applying the results from Q44 gives,

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$$

i.e., $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$ has a chi-squared distribution with n degree of freedom.