Direct Methods of Dynamic Optimization without Constraints

1 Introduction

The aim of this exercise is to introduce some methods of direct dynamical optimization without additional constraints imposed on control or state variables, except for state equation (which can be treated as a form of constraint). They involve a method of a direct gradient in the control space, a method of adjoint directions, a method of the second variation and a method of variable metric. Among those, the first and the second method will be presented in details.

2 Problem statement

Let us suppose that a process is described by the following difference state equation:

$$x_{n+1} = f_n(x_n, u_n), \tag{1}$$

and a performance index to be minimized is given by

$$J = \sum_{n=0}^{N-1} L_n(x_n, u_n), \tag{2}$$

where

x_n – k–dimensional state vector in a time moment n,
u_n – r–dimensional control vector in a time moment n,
n – present time,
N – optimization horizon,
L_n – cost function.

It is assumed that the initial state x_0 and final time N are given, the final state x_N is free. In that case the performance index may have more complex form and contain an element dependent on the final state x_N, i.e.:

$$J = h_N(x_N) + \sum_{n=0}^{N-1} L_n(x_n, u_n).$$

It is also assumed that all functions involved in the problem are differentiable with respect to the state x_n and the control u_n. The goal is to find a control sequence u_n^o, $n = 0, 1, \ldots, N-1$, which minimizes the performance index (2), the optimal trajectory x_n^o, $n = 0, 1, \ldots, N$, corresponding to this control and optimal value of the performance index.

3 Necessary conditions for optimal control

The optimal control u_n^o and trajectory x_n^o should satisfy the state equation (1). Therefore

$$\sum_{n=0}^{N-1} L_n(x_n^o, u_n^o) \leq \sum_{n=0}^{N-1} L_n(x_n, u_n)$$

and

$$x_{n+1}^o = f_n(x_n^o, u_n^o).$$

In order to solve this problem the method of Lagrange multipliers will be applied, since it makes possible to transform the problem of minimizing the performance index (2) under constraint resulting from the state equation (1) into a problem without constraints. After introduction of Lagrange multipliers p_{n+1}, the modified performance index takes the following form:

$$\overline{J} = \sum_{n=0}^{N-1} \left\{ L_n(x_n, u_n) + p_{n+1}^T \left[f_n(x_n, u_n) - x_{n+1} \right] \right\}. \tag{3}$$

From the calculus of variations it stems that the necessary condition of extremum of a functional under no constraints is that the first variation of this functional is equal to zero. Hence $\delta\overline{J} = 0$ shoul be satisfied. Before proceeding with calculation of the first variation of the modified performance index (3), it is convenient to rewrite it in the following form:

$$\overline{J} = p_0^T x_0 + \sum_{n=0}^{N-1} \left\{ L_n(x_n, u_n) + p_{n+1}^T f_n(x_n, u_n) - p_n^T x_n \right\} - p_N^T x_N.$$

Let

$$\overline{L}_n(x_n, u_n, p_n, p_{n+1}) = H_n(x_n, u_n, p_{n+1}) - p_n^T x_n, \tag{4}$$

where the function $H_n(...)$, defined as

$$H_n(x_n, u_n, p_{n+1}) = L_n(x_n, u_n) + p_{n+1}^T f_n(x_n, u_n) \tag{5}$$

is called a hamiltonian.

Then the first variation of the functional (3) can be described in the following way:

$$\delta\overline{J} = p_0^T \delta x_0 + \sum_{n=0}^{N-1} \left[\frac{\partial \overline{L_n}}{\partial x_n} \delta x_n + \frac{\partial \overline{L_n}}{\partial u_n} \delta u_n \right] - p_N^T \delta x_N. \tag{6}$$

Taking into account (4) we obtain

$$\delta\overline{J} = p_0^T \delta x_0 + \sum_{n=0}^{N-1} \left[\left(\frac{\partial H_n}{\partial x_n} - p_n^T \right) \delta x_n + \frac{\partial H_n}{\partial u_n} \delta u_n \right] - p_N^T \delta x_N. \tag{7}$$

The first component in the formula (7) is equal to zero due to the assumption that the initial state is given – then $\delta x_0 = 0$. To make the variation of the functional dependent only on variation of control δu_n, the Lagrange multiplier should make the second element in (7) equal to zero. Since $\delta x_n \neq 0$, that is satisfied if

$$p_n = \left(\frac{\partial H_n}{\partial x_n} \right)^T, \quad n = N-1, N-2, \ldots, 0, \tag{8}$$

$$p_N = 0. \tag{9}$$

The relation (8) is called an adjoint equation.

All considerations presented above lead to the following relation:

$$\delta\overline{J} = \sum_{n=0}^{N-1} \frac{\partial H_n}{\partial u_n} \delta u_n = \sum_{n=0}^{N-1} b_n^T \delta u_n = 0. \tag{10}$$

where

$$b_n = \left[\frac{\partial H_n}{\partial u_n} \right]^T. \tag{11}$$

The vector function b_n is called the functional gradient reduced to the control space. If the control is optimal, it cannot be improved by a variation of any form. Therefore the functional gradient computed along the optimal trajectory is to be equal to zero (if such gradient exists and no constraints on control are given). Hence the necessary condition of optimality is given by

$$b_n^o = \frac{\partial H_n(x_n^o, u_n^o, p_{n+1}^o)}{\partial u_n^o} = 0. \tag{12}$$

4 A method of direct gradient in the control space

The reasoning presented in the preceeding section leads to the conclusion that the solution of the problem consists in finding a sequence of control u_n^o, $n = 0, 1, \ldots, N-1$, satisfying the following conditions:

$$x_{n+1} = f_n(x_n, u_n), \qquad x_0 \text{ – given}, \tag{13}$$

$$p_n = \left(\frac{\partial H_n}{\partial x_n} \right)^T, \quad p_N = 0, \tag{14}$$

$$b_n = \left[\frac{\partial H_n}{\partial u_n}\right]^T = 0 \tag{15}$$

where

$$H_n = L_n + p_{n+1}^T f_n. \tag{16}$$

The norm of the reduced gradient can be computed according to the following relation:

$$\|b_n\| = \left(\sum_{n=0}^{N-1} b_n^T b_n\right)^{1/2}. \tag{17}$$

The method of direct gradient consists in development of trajectories and control improved iteratively, starting from certain, arbitrarily assumed initial sequence of control $u_n^{(1)}$, where the improvement is made along the direction of the functional gradient. The raw version of the algorithm can be described in the following way:

1. Assume the number of iterations K, precision ε and the first approximation of the control sequence $u_n^{(1)}$, $n = 0, 1, \ldots, N-1$.
2. Calculate values of x_n, $n = 1, \ldots, N-1, N$, following the state equation (13).
3. Determine the form of hamiltonian from (16).
4. Calculate values of p_n, $n = N-1, \ldots, 1, 0$ using the adjoint equation (14)
5. Find the reduced gradient b_n, $n = 0, 1, \ldots, N-1$ using (11).
6. Calculate the gradient norm (17).
7. If the computed norms are smaller than the precision ε then stop the algorithm. The optimal solution has been found.
8. Otherwise, if the gradient norm is greater than the precision ε, check if the number of iterations passed the given value K. If so, stop the algorithm. The solution has been found after K iterations.
9. Otherwise find improved sequence of control.
10. Repeat the preceeding steps starting from point 2.

It should be stressed that although the optimal control u_n^o, trajectory x_n^o and Lagrange multiplier p_{n+1}^o, $n = 0, 1, \ldots, N-1$ should satisfy (15), meaning that the functional gradient is equal to zero, in practice it is impossible to achieve this due to numerical application of this method. Instead, it is assumed that the value of the gradient norm should be smaller than some threshold, defined by ε. If this condition is satisfied, then the obtained control will be assumed to be optimal. However, very small ε leads to great number of iterations needed to find the solution. Hence the algorithm requires other stop criterion which is defined by maximum allowable number of iterations. If the algorithm stop is caused by this criterion, it means that the solution found differs somehow from the optimal. Then it is necessary to analyse the results of single iterations and, possibly, increase the maximum number of iterations. If none of the stop criteria is met, then the control found in previous iteration can be improved according to the following relation:

$$u_n^{(i+1)} = u_n^{(i)} - t b_n^{(i)}, \quad n = 0, 1, \ldots, N-1, \quad t > 0 \tag{18}$$

and used as a basis for next iteration. A trajectory $x_n^{(i+1)}$, $n = 0, 1, \ldots, N-1$, corresponds to this control, and can be calculated from (13). Substituting $u_n^{(i+1)}$ and $x_n^{(i+1)}$ into the performance index (2) we obtain

$$J(t) = \sum_{n=0}^{N-1} \left[L_n(x_n^{(i+1)}(t), u_n^{(i+1)}(t))\right]. \tag{19}$$

The calculations of the value of the performance index should be made taking into account different values of t in order to find a best way to utilize the improvement direction. Eventually, a value t^o is found, minimizing the performance index.

In some cases, involving e.g. linear state equations and the cost function in the form $L_n(x_n, u_n) = L(x_n, u_n)$, it is possible to find analytical form of the performance index $J(t)$. Applying one of the variety of methods for finding minimum for a function of one variable, an optimal t^o, minimizing the performance index, is determined. Having found t^o a sequence of control $u_n^{(i+1)}$, $n = 0, 1, \ldots, N-1$ is calculated from the following relation:

$$u_n^{(i+1)} = u_n^{(i)} - t^o b_n^{(i)}.$$

The calculation is repeated until the norm of the gradient $b_n^{(i)}$ is small enough or the final number of iteration K is reached.

5 The conjugated gradient method in the control space

The method described in the previous section, though easy to implement, has several drawbacks. Even in the case of multivariable quadratic functions the gradient direction is not the best direction for searching function minimum. One of the most crucial drawbacks is the increase of the base dimension. Moreover, every iteration is initiated with no information about the area being investigated. Hence, other methods have been developed basing on the direct gradient algorithm. One of such approaches, similarly as in the static optimization, is the method of conjugated gradients which takes into account coupled directions.

Two directions $d^{(i)}$ and $d^{(j)}$ are said to be coupled with respect to some positive defined matrix A if

$$d^{(i)} A d^{(j)} = 0 \quad \text{for} \quad i \neq j \tag{20}$$

In this aproach, the improvement directions are chosen in a way that each of them is coupled with the preceeding ones. Coupled drections are linear independent, therefore the dimension of the base is never decreased.

The main difference between this method and direct gradient one lays in the way of improvement of direction. In the method of conjugated gradient it is calculated in the i-th iteration according to the following rule:

$$d_n^{(i)} = -b_n^{(i)} + c^{(i)} d_n^{(i-1)}, \quad c^{(1)} = 0, \quad n = 0, 1, \ldots, N-1 \tag{21}$$

Values of $c^{(i)}$ should be chosen to make $d_n^{(i)}$ and $d_n^{(i-1)}$ coupled with respect to the Hessian of the performance index determined with respect to control variable. It can be reached e.g. by applying Fletcher–Reeves version of the algorithm:

$$c^{(i)} = \frac{\left\| b^{(i)} \right\|^2}{\left\| b^{(i-1)} \right\|^2}. \tag{22}$$

It is assumed in this case that the optimization along the gradient has been done precisely. Although the convergence of this version has been proved for quadratic functionals, it can also be used in problems involving other performance indices.

There exists a number of other methods for determining value of the coefficient $c^{(i)}$ [1]. Having calculated the coupled direction $d_n^{(i)}$, the control $u_n^{(i+1)}$ can be determined from the following relation:

$$u_n^{(i+1)} = u_n^{(i)} + t^o d_n^{(i)}, \quad n = 0, 1, \ldots, N-1 \tag{23}$$

Value of t^o is determined as in the method of direct gradient. The form of the algorithm is the same as in that case.

In [2] the following properties of the conjugated gradient method have been proved:

1. The coupled direction always determines the direction of decrease of the functional value.
2. If the problem under consideration is an LQ problem, then, beginning from the second iteration, the value of the performance index $J^{\{i)}$ resulting from the application of conjugated gradient method is always smaller than the one stemming from the application of the direct gradient method, assuming that the first control sequence $u_n^{(1)}$ is the same for both methods.

The latter property leads to a conclusion that the method of conjugated gradient converges to the problem solution much faster compared to the direct gradient method in the broad class of problems which can be approximated by an LQ problem in the neighbourhood of the optimal solution.

6 An Example

To make the method of direct gradient more understandable, let us consider the exemplary problem.

A process is described by the following state equation:

$$x_{n+1} = x_n + u_n, \quad x_0 = 1.$$

A performance index to be minimized is given by:

$$J = \frac{1}{2} \sum_{n=0}^{3} \left(x_n^2 + u_n^2\right).$$

Having applied formulae (16), (13), (14), (11) we obtain, respectively:

$$H_n = \frac{1}{2} \left(x_n^2 + u_n^2\right) + p_{n+1} \left(x_n + u_n\right),$$

$$p_n = x_n + p_{n+1}, \quad n = 3, 2, 1, 0,$$

$$p_4 = 0,$$

$$b_n = u_n + p_{n+1}.$$

Let us suppose that in the first iteration we arbitrarily assume the following control vector: $u_0^{(1)} = 1$, $u_1^{(1)} = 9$, $u_2^{(1)} = 8$, $u_3^{(1)} = 1$. Then, in next iterations we obtain results shown in the Table 4.1.

Table 4.1

Iteration number	Control vector u_0	u_1	u_2	u_3	State vector x_0	x_1	x_2	x_3	Value of the performance index
1	1;	9;	8;	1	1;	2;	11;	19	317
2	−4, 67;	2, 30;	3, 36;	0, 83	1;	−3, 67;	−1, 36;	2, 0	26, 69
3	−0, 49;	0, 70;	0, 46;	0, 38	1;	0, 50;	1, 21;	1, 67	3, 30
4	0, 99;	0, 09;	0, 09;	0, 31	1;	0, 0;	0, 09;	0, 19	1, 07
5	−0, 59;	−0, 13;	−0, 07;	0, 13	1;	0, 41;	0, 28;	0, 21	0, 84
⋮		⋮				⋮			⋮
15	−0, 61;	−0, 23;	−0, 08;	0, 0	1;	0, 38;	0, 15;	0, 07	0, 81

7 Problems

1. Determine necessary conditions for extremum of the functional

$$J = h(x_N) + \sum_{n=0}^{N-1} L_n(x_n, u_n),$$

for a process described by the state equation (1) when x_n is given.

2. Develop an algorithm for searching minimum along the gradient direction for the functional $J(t)$ for methods of direct and conjugate gradients.

3. Present a precise block diagram of algorithms for direct and conjugate gradient methods in the control space.

4. Determine optimal control and value of the performance index for a problem defined by linear state equation

$$x_{n+1} = A_n x_n + B_n u_n, \quad x_0 \text{ is given},$$

and the quadratic performance index to be minimized

$$J = x_n^T Q x_n + \sum_{n=0}^{N-1} \left(x_n^T Q x_n + 2 x_n^T G u_n + u_n^T H u_n\right).$$

by means of the direct and conjugate gradient methods.

5. Solve the problem 4 using analytical form of the step t, minimizing the performance index $J(t)$.
6. Using direct and conjugated gradient methods determine optimal control for a plant described by state equation

$$x_{n+1} = Ax_n^2 + Bu_n,$$

given x_0 and performance index

$$J = \sum_{n=0}^{N-1} \left(Qx_n^2 + Hu_n^2\right) + \frac{1}{2}Fx_n^4.$$

7. Using direct and conjugated gradient methods determine optimal control for a plant described by state equation

$$x_{n+1} = A \left[\begin{array}{c} \cos x_{1n} \\ \sin x_{2n} \end{array} \right] + Bu_n,$$

given x_0 and performance index

$$J = \sum_{n=0}^{N-1} \left(x_n^T Q x_n + Hu_n^2\right),$$

where $x_n = [x_{1n}, x_{2n}]^T$.

References

[1] W. Findeisen, J. Szymanowski, A. Wierzbicki, *Teoria i metody obliczeniowe optymalizacji*, PWN, Warszawa, 1997 (*in Polish*).

[2] Lasdon, Mitter, Waren, *The conjugate gradient method for optimal control problems*, IEEE Trans. on Automatic Control, 1967, No.2.