LINEAR QUADRATIC PROBLEM

1 Introduction

Linear quadratic problem plays a special role in the optimization theory and its applications. It is one of the few cases where the control law can be determined in analytical form and does not require development of any additional methods to utilize the necessary conditions of optimal control. It can be easily applied to a variety of real-life problems. The choice of the performance index influences dynamical properties of the system it has been developed for. Furthermore, many problems that are not explicitly described in LQ form can be, under certain conditions, transformed in a way that allows using solutions of the LQ problem. That, in turn, may serve as a basis for further work in a variety of optimization problems both theoretical and their applications, involving tracking problem, robust and adaptive control and many others.

2 Time-invariant LQ problem.

The system under consideration is described by the following state equation:

$$x_{i+1} = Ax_i + Bu_i, \tag{1}$$

where x_i is n – dimensional state vector, u_i – r – dimensional control vector $(i = 0, 1, \dots, N-1)$, x_0 – a given initial state, A – $n \times n$ matrix, B – $n \times r$ matrix.

The aim is to find control u_i minimizing the performance index defined as

$$J = \frac{1}{2}\sum_{i=0}^{N-1}\left(x_i^T Q x_i + u_i^T R u_i\right) + \frac{1}{2}x_N^T F x_N. \tag{2}$$

where matrices Q and F are nonnegatively definite, R – positively definite, and all of them are symmetrical.

Using the principle of optimality and the method of dynamic programming (or the method of Lagrange multipliers), it is possible to prove that the optimal control u_i satisfies the following relation [1]

$$u_i = -\left(R + B^T K_{i+1} B\right)^{-1} B^T K_{i+1} A x_i \tag{3}$$

and the optimal value of the performance index is

$$J^o = \frac{1}{2}x_0^T K_0 x_0 \tag{4}$$

where K_i is a symmetrical, non-negatively definite solution of the Riccati equation:

$$K_i = A^T\left(K_{i+1} - K_{i+1}B\left(R + B^T K_{i+1} B\right)^{-1} B^T K_{i+1}\right)A + Q, \tag{5}$$

$i = 0, 1, \dots, N-1$, $K_N = F$.

Moreover, the optimal cost-to-go is given by

$$J_i^o = \underset{\substack{\{u_k\} \\ k=i,\ldots,N-1}}{Min} \frac{1}{2} \left\{ \sum_{k=i}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) + x_N^T F x_N \right\} = \frac{1}{2} x_i^T K_i x_i \qquad (6)$$

One of the basic variation of the problem stated above is a problem with infinite horizon, i.e. $N \to \infty$. In this case it is justifiable to assume that $F = 0$. Then, the final condition for the equation (5) takes the form $K_N = 0$. The solution of (5) leads to the following relation:

$$\lim_{i \to \infty} K_{N-i} = \hat{K} \qquad (7)$$

which means that K_i becomes a matrix $\hat{K}$ of constant elements. This conclusion forms the basis for the most frequently used algorithm for finding $\hat{K}$, consisting in solving (5) in consecutive steps, until $K_i = K_{i+1}$. Since such procedure may result in subsequently calculated K_i that are not symmetric, an often applied approach is to compute all of its elements k_i^{st} $(s = 1, 2, \ldots, n,\ t = 1, 2, \ldots, n)$ first, and modify their values next, according to the following relation:

$$\overline{k_i^{st}} = \overline{k_i^{ts}} = \frac{k_i^{st} + k_i^{ts}}{2}.$$

However, this makes the problem more complex in numerical terms because it is required to calculate n^2 elements in each step, instead of $\frac{1}{2}(n+1)n$. On the other hand, it ensures against error propagation.

Another method of calculating $\hat{K}$ stems directly from (5). It is clear that $\hat{K}$ is a solution of the algebraic nonlinear equation:

$$\hat{K} = A^T \left(\hat{K} - \hat{K} B \left(R + B^T \hat{K} B \right)^{-1} B^T \hat{K} \right) A + Q \qquad (8)$$

It should be stressed, however, that in the case of infinite horizon, an additional condition

$$rank \left[B : AB : \ldots : A^{n-1} B \right] = n$$

must be satisfied to guarantee that the solution of the problem exists. This condition determines absolute controllability of the system (1).

The algorithm of determining the optimal control for finite horizon N consists of the following points:

1. Assume $i = N$, $K_i = F$.
2. Determine and store values of K_i from (5) for $i = N-1, N-2, \ldots, 0$.
3. Calculate value of the performance index from (4).
4. Determine u_i and x_{i+1} from (3) and (1), respectively, for $i = 0, 1, \ldots, N-1$.

It is worth noticing that x_i can be calculated without explicitly computing u_i. Substituting u_i determined by (3) into (1) we obtain:

$$x_{i+1} = \left(I - \left(R + B^T K_{i+1} B \right)^{-1} B^T K_{i+1} \right) A x_i,$$

where I is a $n \times n$ identity matrix.

It should be also noticed that all numerical algorithms developed for solving LQ problem require computing elements of K_i in the opposite direction to x_i. However, in contrast to two point boundary value problems, often appearing in other dynamic optimization problems, K_i can be calculated separately, since it depends neither on x_i nor on u_i.

As indicated in the introduction to this exercise, analytical results of the LQ problem are very often used to find solutions of other problems. Below, some of these problems are presented.

3 Examples of problem modification

3.1 Time varying LQ problem

If the parameters in the state equation or in the performance index are not constant, but changing from one stage to another according to some specified rule, the problem is called time varying. In that case the state equation takes the following form:

$$x_{i+1} = A_i x_i + B_i u_i$$

and the performance index is given by

$$J = \frac{1}{2} \sum_{i=0}^{N-1} \left(x_i^T Q_i x_i + u_i^T R_i u_i \right) + \frac{1}{2} x_N^T F x_N.$$

Though the problem may seem much more complex at the first glance, it appears that the formula determining the optimal control is almost the same as in time-invariant problem. Simply substituting A_i, B_i, Q_i, R_i for A, B, Q, R, respectively, in (3) we obtain the solution of this problem. However, in this case, it does not make sense to consider the problem of $N \to \infty$, but in very special cases.

3.2 LQ problem with disturbance

One of the possible modifications of standard LQ problem takes into account different external disturbances that can appear in the control process. They can be introduced into the model by an additive element in the state equation:

$$x_{i+1} = Ax_i + Bu_i + w_i \tag{9}$$

It is proven that the control minimizing the performance index (2) is given by

$$u_i = -\left(R + B^T K_{i+1} B \right)^{-1} B^T \left(K_{i+1} A x_i - g_{N-i-1} \right), \tag{10}$$

where g_i is a solution of the following equation:

$$g_{i+1} = A^T \left(I - B \left(R + B^T K_{N-i+1} B \right)^{-1} B^T K_{N-i+1} \right) g_i - K_{N-i-1} w_{N-i-2}, \tag{11}$$

$$i = 0, 1, \ldots, N-1,$$

$$g_0 = 0.$$

It should be stressed that the above relations imply that control at the i–th stage depends not only on the state x_i, as in the standard LQ problem, but on all future disturbances (till the $N-2$ step) as well.

3.3 Tracking problem

Another important problem concerns the task of tracking given trajectory. It is defined by the state equation (1) and the performance index

$$J = \frac{1}{2}\sum_{i=0}^{N-1}\left[(x_i - s_i)^T Q (x_i - s_i) + u_i^T R u_i\right] + \frac{1}{2}(x_N - s_N)^T F (x_N - s_N)\,. \tag{12}$$

where values of s_i determine given trajectory.

It can be easily transformed into the LQ problem with disturbances and the optimal control can be found using (10) without any additional assumptions.

4 Problems.

1. Prove the control law (3).
2. Develop detailed algorithm for finding the optimal control in scalar ($n = r = 1$) and two-dimensional case ($n = 2$, $r = 1$), for finite and infinite horizon.
3. Using algorithms developed in point 2. calculate values of control for given x_0 and parameters A, B, Q, R, F, N. Analyse the solution of Riccati equation both for finite and infinite horizon, the form of the optimal control and the shape of the otpimal trajectory. Compute the eigenvalues of the matrix $\left(I - \left(R + B^T K_{i+1} B\right)^{-1} B^T K_{i+1}\right) A$.
4. Prove the control law (10) in the scalar case. Analyse the case of constant disturbance.
5. Derive the control law for the tracking problem, transforming it to the problem with disturbance. Analyse the case of constant reference trajectory.
6. In the problem with disturbances the control law (10) can be applied if future disturbances are known. Analyse how the form of the disturbance affects the optimal control and the optimal trajectory.
7. Find the solution of the LQ problem with performance index including cross products $2x_i^T S u_i$. In order to derive the control law transform the problem into standard LQ by proper change of variables.

References

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[2] M. Athans, P.L.Falb, Optimal Control, McGraw-Hill, New York, 1966.