Diffusions and orthogonal polynomials.

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Diffusion semigroups are described through their generators, which are in general in $\mathbb{R}^n$ or an open set in it second order differential operators of the form

$$L(f)(x) = \sum_{ij} a^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial f}{\partial x_i}.$$

The easiest cases are when one is able to diagonalize this operator in an basis of orthogonal polynomials, since then one is able to have a quite explicit description of the associated law of the underlying process. In dimension 1, there are not many examples of such a situation. It reduces to the family of Jacobi, Laguerre and Hermite polynomials. In higher dimension, many examples come from Lie group actions of homogeneous spaces, random matrices, root systems, isoparametric surfaces or other algebraic constructions.

We shall give a complete characterization of the problem : on which open sets in $\mathbb{R}^n$ one may expect to find a probability measure for which the associated orthogonal polynomials are eigenvectors of diffusion operators.

Provided we rank polynomials according to their natural degree, we provide a complete description of all the models in dimension 2, where we are able to completely solve this problem. There are exactly 11 compact sets (up to affine transformations), and 7 non compact ones, on which there exist such a measure. We shall also describe all the associated measures and operators.