LECTURE 3

Announcements:

1. Homework 1 Out TODAY.
2. Register on Gradescope.
3. Quiz 2 Out today evening

   (Quiz 1 Average ~ 0.7)

Divide & Conquer "Methodology" for designing algorithms

→ Mergesort

→ Fast multiplication

→ Exponentiation

Analysis of D & C algorithms:
Master theorem.

Today: Fast Fourier Transform

→ Very simple and revolutionizing idea

Applications:

Audio: MP3's, Synthesizers, Shazam, Voice recognition
Noise Cancelling headphones

Image/video: DVD, JPEG, MRI Scan,
Ultrasound

Physics: Optics, Spectrometry, ...

Math: PDEs, ...

Algorithms: Integer multiplication, ...

Discrete Fourier Transform (DFT).

Fourier Transform: "Represent periodic functions" in a nice way.

Light waves, Sound waves, Orbits

1806: Gauss discovered FFT to compute the orbit of an asteroid.

1963: Cooley & Tucker published a paper on FFT

DFT: Need some complex arithmetic first

Review of Complex numbers:

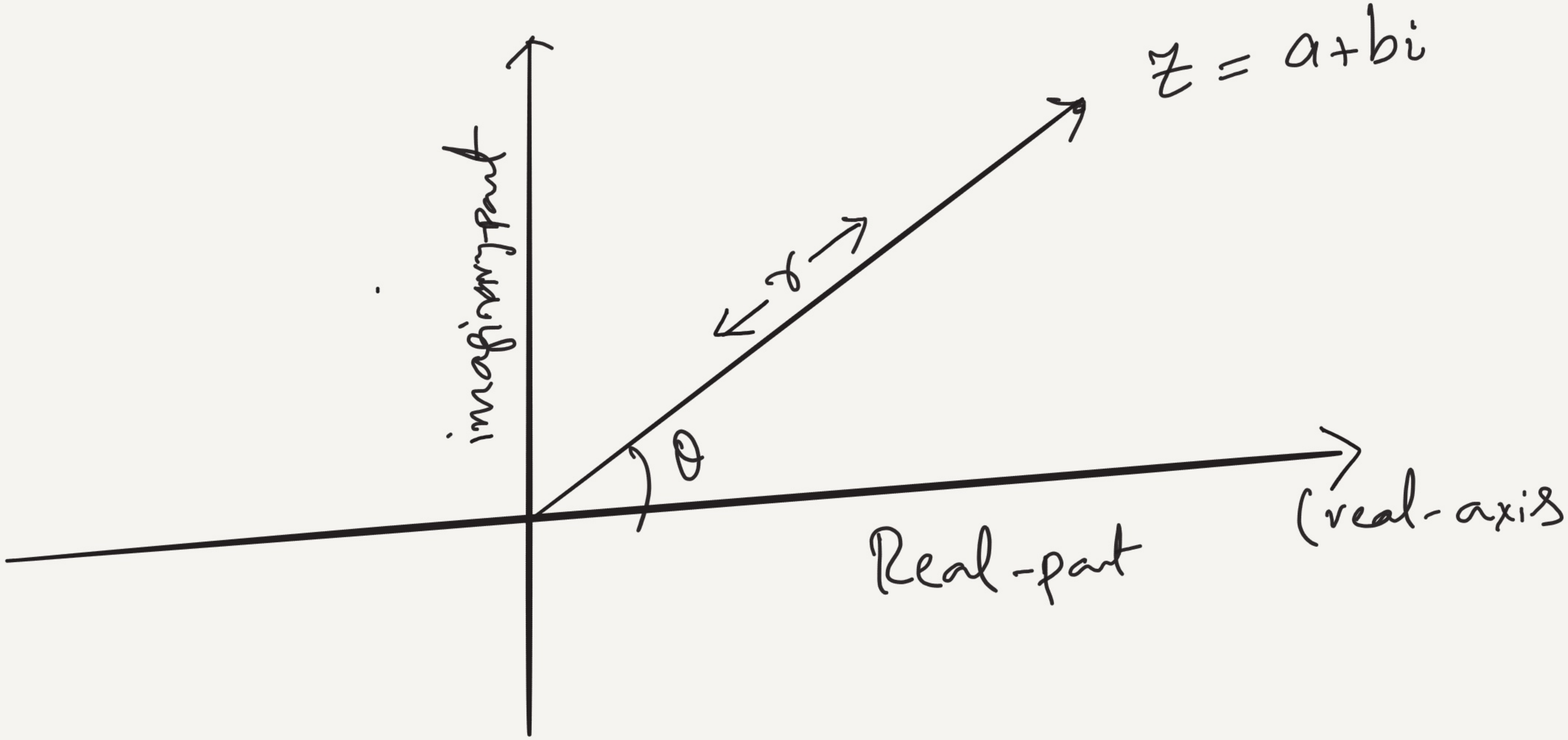
$$z = \underbrace{a}_{\text{real part}} + b\,i$$

imaginary part.

"Polar Coordinates"

$$z = \underbrace{r}_{\text{positive real number}} \cdot e^{i\theta}$$

"phase of z".



$$z_1 = a_1 + b_1 i$$

$$z_2 = a_2 + b_2 i$$

$$z_1 \cdot z_2 = (a_1 + b_1 i) \cdot (a_2 + b_2 i) = (\quad) + (\quad) i$$

$$z_1 = r_1 \cdot e^{i\theta_1}$$

$$z_2 = r_2 \cdot e^{i\theta_2}$$

$$z_1 \cdot z_2 = (r_1 \cdot r_2) \cdot e^{i(\theta_1 + \theta_2)}$$

Roots of Unity:

Given a natural number n,

n^{th} roots of unity are solutions to

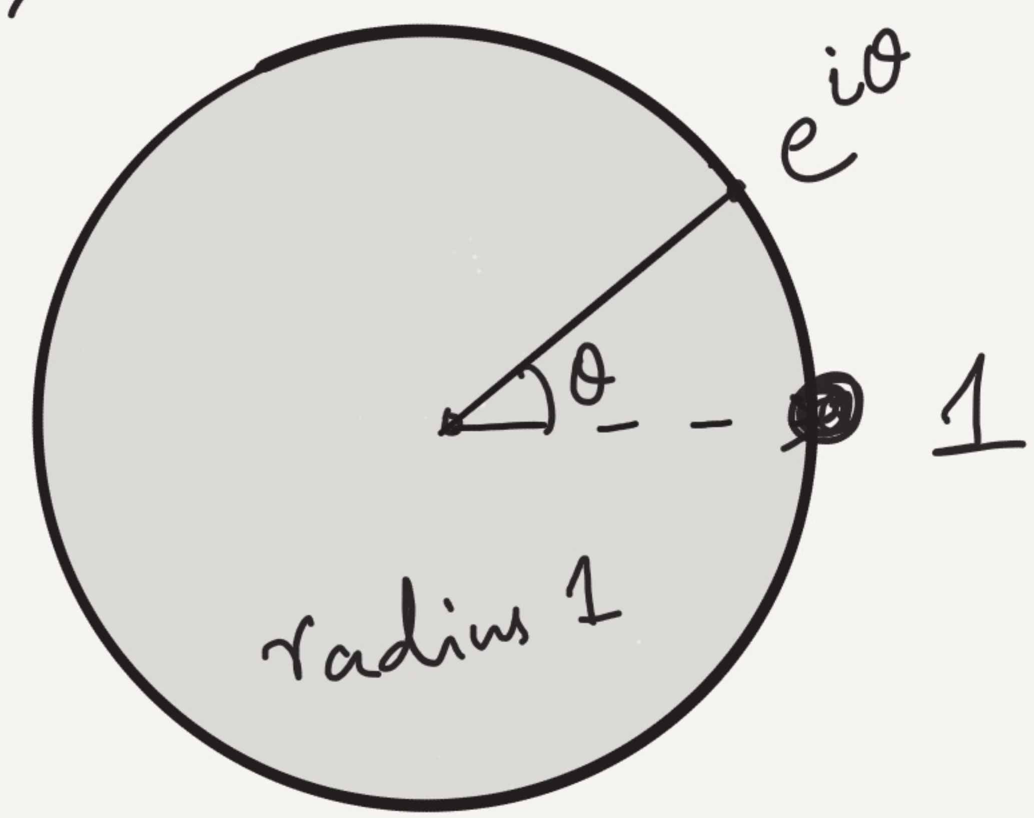
$$\boxed{z^n - 1 = 0}$$

$$z^n = 1$$

$$z = r \cdot e^{i\theta}$$

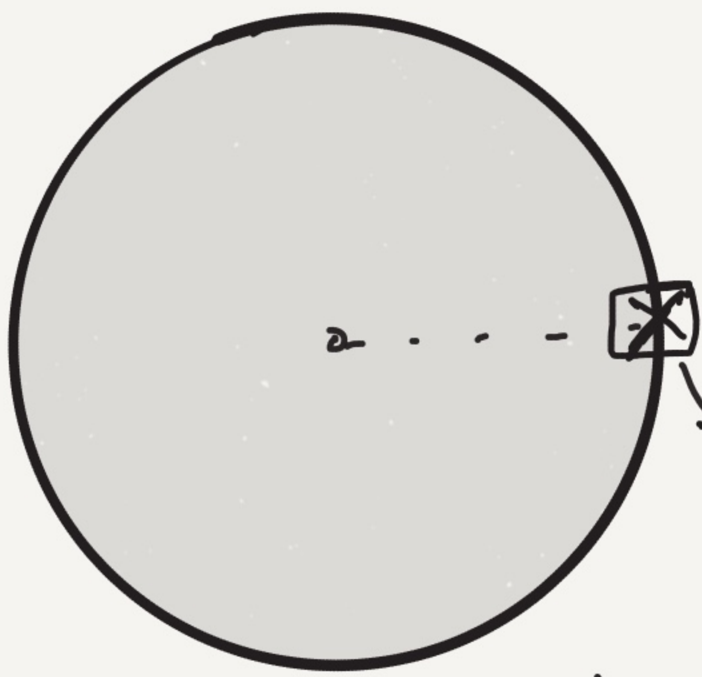
$$z^n = r^n \cdot e^{in\theta}$$

$$z^n = 1 \iff r = 1 \quad \text{and} \quad \boxed{e^{i \cdot n\theta} = 1}$$



$n = 1$

$e^{i \cdot \theta} = 1$

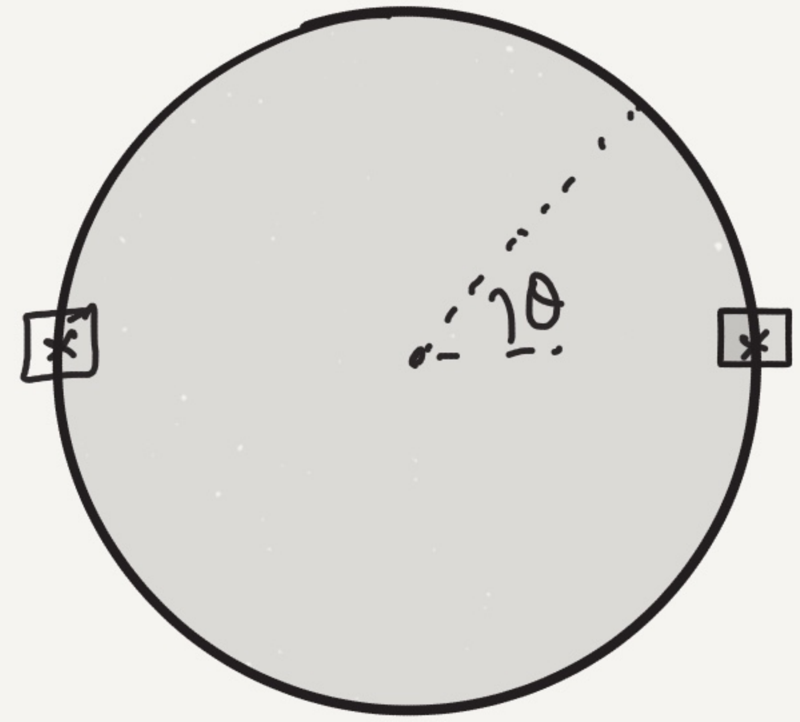


1st root of unity.

$n = 2$

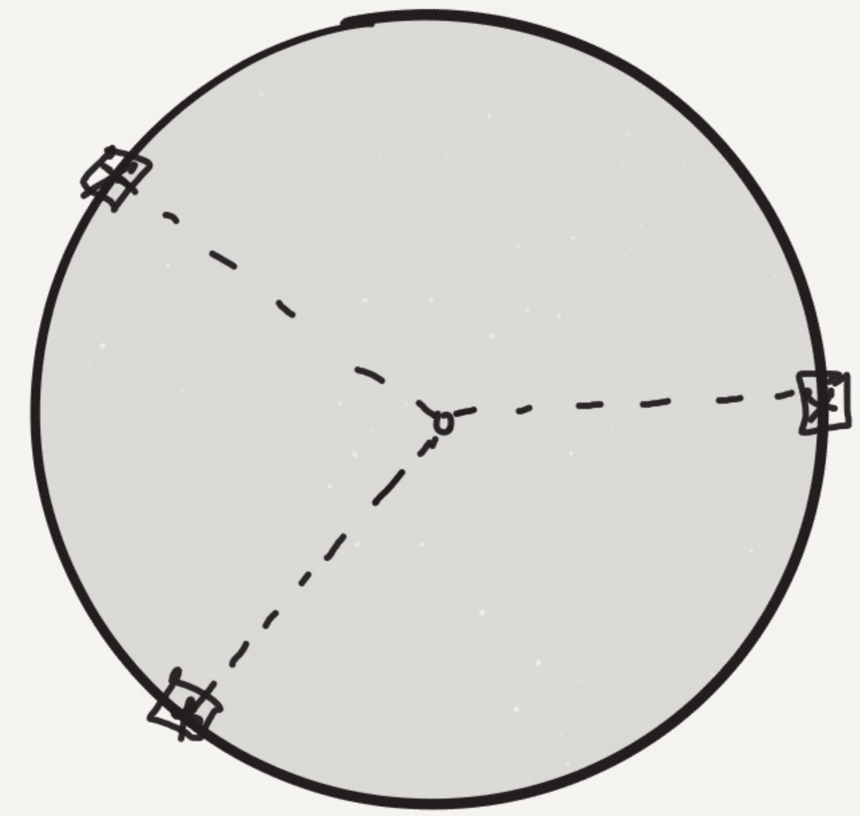
$\underbrace{z^2 = 1.}$

$e^{i \cdot 2\theta} = 1$



$e^{i \cdot 2\theta} = 1$

$n = 3$



$\underbrace{e^{i \cdot 3\theta} = 1}$

$\iff$ $3.\theta$ is an integer multiple of 2π.

Satisfied e.g.; $\boxed{\theta = 0, \ \theta = \frac{2\pi}{3}, \ \theta = \frac{4\pi}{3},}$ $\theta = \frac{6\pi}{3}, \ldots$

$n = 4$

$z^4 = 1$

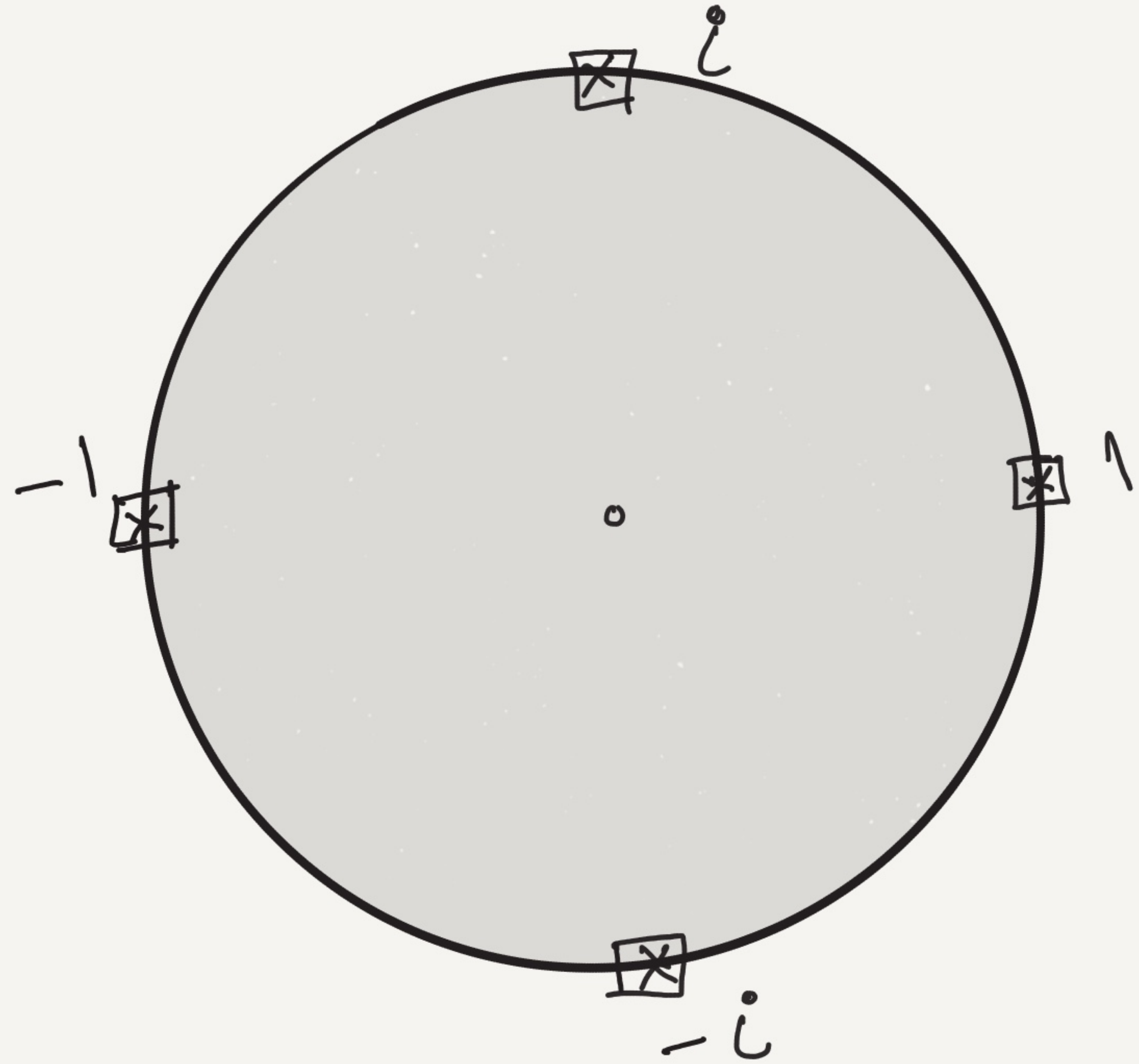
$e^{i \cdot 4\theta} = 1$

$\Leftrightarrow$ $4 \cdot \theta$ is an integer multiple of 2π

$\Leftrightarrow$ $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$,

$\theta = \frac{3\pi}{2}$, $(\theta = 2\pi, \dots)$

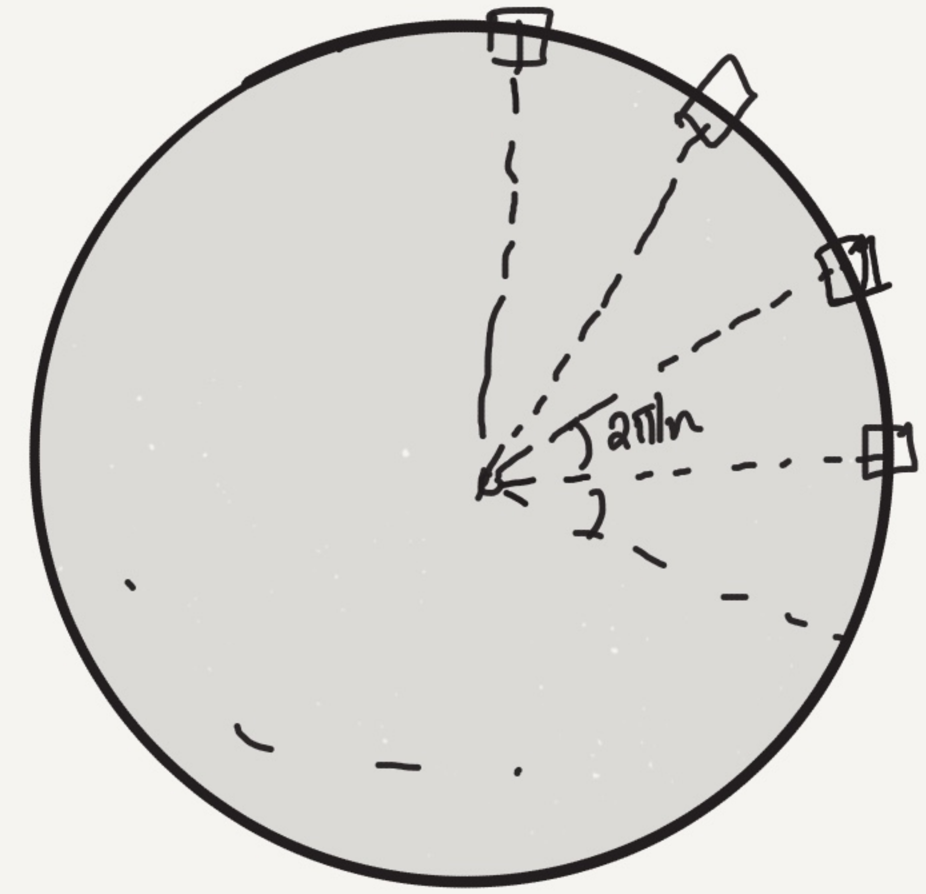
repeating



There are exactly 4 4^{th} roots of unity which are $1, i, -1, -i$.

General:

There are exactly n n^{th} roots of unity.



Roots are

$$1, e^{\left(i\cdot\frac{2\pi}{n}\right)}, e^{\left(\frac{i\cdot 2\pi}{n}\right)\cdot 2}, e^{\left(\frac{2\pi i}{n}\right)\cdot 3}, \ldots, e^{\left(\frac{2\pi i}{n}\right)(n-1)}$$

"Very Special numbers"

$$\boxed{\omega = e^{\frac{i\cdot 2\pi}{n}}}$$

DFT_n:

$$\bar{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} \xrightarrow{DFT(\bar{a})} \bar{s} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{bmatrix}$$

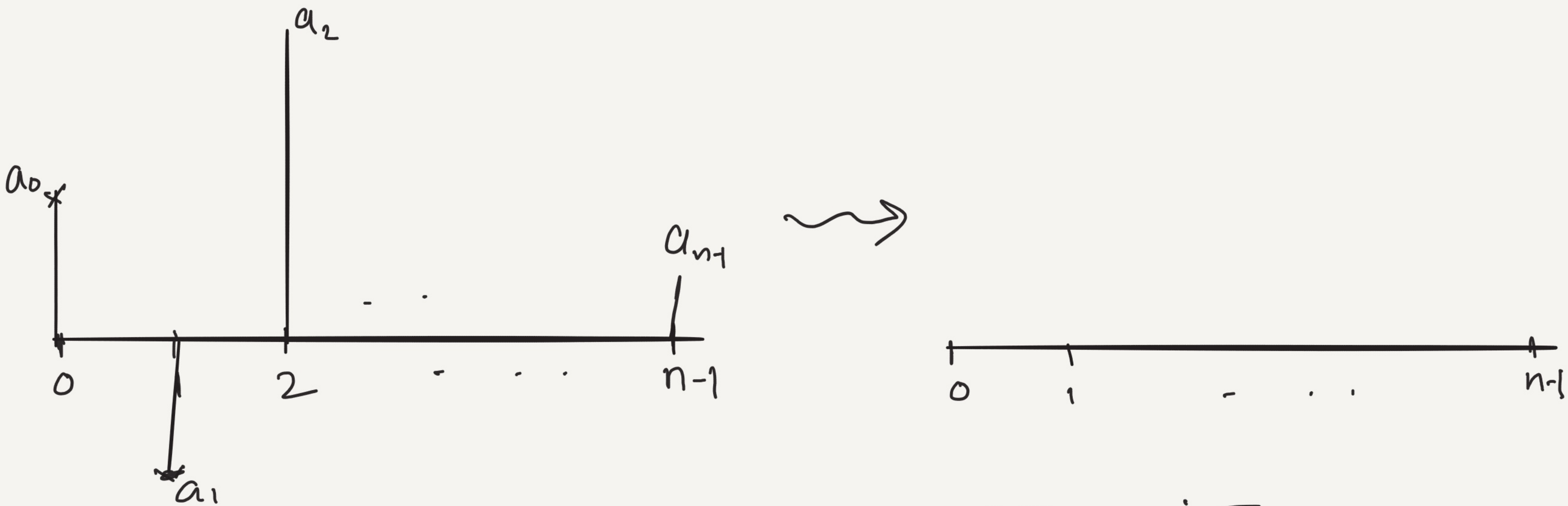
n dimensional vector

$$s_j = a_0 + \omega^{j} \cdot a_1 + \omega^{j \cdot 2} \cdot a_2 + \cdots + \cdots + \omega^{j \cdot (n-1)} \cdot a_{n-1}$$

j^{th} coordinate

$$s_j = \sum_{k=0}^{n-1} (\omega^j)^k \cdot a_k$$

$$= \sum_{k=0}^{n-1} e^{\left(\frac{2\pi i}{n}\right) \cdot j \cdot k} \cdot a_k$$



Examples: $n=2$; $\omega = e^{\frac{i\cdot 2\pi}{n}} = e^{i\cdot\pi} = -1$

(Euler's equation)

$$\bar{a} = \boxed{\begin{matrix} a_0 \\ a_1 \end{matrix}}$$

$$s_0 = \omega^{0\cdot 0} a_0 + \omega^{0\cdot 1}\cdot a_1$$
$$= a_0 + a_1$$

$$s_1 = \omega^{1\cdot 0} \cdot a_0 + \omega^{1\cdot 1} a_1$$

$$= a_0 + \omega \cdot a_1 = a_0 - a_1$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_0 + a_1 \\ a_0 - a_1 \end{bmatrix}$$

$$n = 3 \quad , \quad \omega = e^{\frac{2\pi i}{3}}.$$

$$= \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) \cdot i$$

$$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\bar{a} = \boxed{\begin{matrix} a_0 \\ a_1 \\ a_2 \end{matrix}}$$

$$s_0 = a_0 + a_1 + a_2 \quad \left(\omega^{0.0} \cdot a_0 + \omega^{0.1} \cdot a_1 + \omega^{0.2} \cdot a_2\right)$$

$$s_1 = \omega^{1.0} a_0 + \omega^{1.1} \cdot a_1 + \omega^{1.2} \cdot a_2$$

$$= a_0 + \omega \cdot a_1 + \omega^2 \cdot a_2$$

$$s_2 = \omega^{2.0} \cdot a_0 + \omega^{2.1} \cdot a_1 + \omega^4 \cdot a_2$$

$$= a_0 + \omega^2 \cdot a_1 + \omega \cdot a_2$$

$$\boxed{\omega^4 = \omega^3 \omega = \omega.}$$

Goal:

INPUT : $n, \quad \bar{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$

OUTPUT : $\bar{s} = \mathrm{DFT}_n(\bar{a})$.

Complex numbers on Computers??

We will assume:

→ Can store & "work" with Complex numbers

→ Multiplying or adding two Complex numbers is a "simple operation".

Evaluate algorithms in terms of these simple operations.

First attempt:

$$\boxed{s_j = \sum_{k=0}^{n-1} \omega^{j \cdot k} \cdot a_k}$$

For $j = 0, 1, \dots, n-1$

$s_j \leftarrow 0$

For $k = 0, \dots, n-1$

$s_j \leftarrow s_j + \omega^{j \cdot k} \cdot a_k$

There are n^2 iterations: Each involves one addition & one multiplication

$$\boxed{\text{So Complexity is } \Theta(n^2).}$$

Can we do better??

YES WE CAN!

Gauss 1806: $\Theta(n \log n)$

Cooley - Tucker 1963:

Identifying "structure" in the numbers

$$\boxed{\omega^{j \cdot k}}$$

INPUT: $\bar{a} := \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$

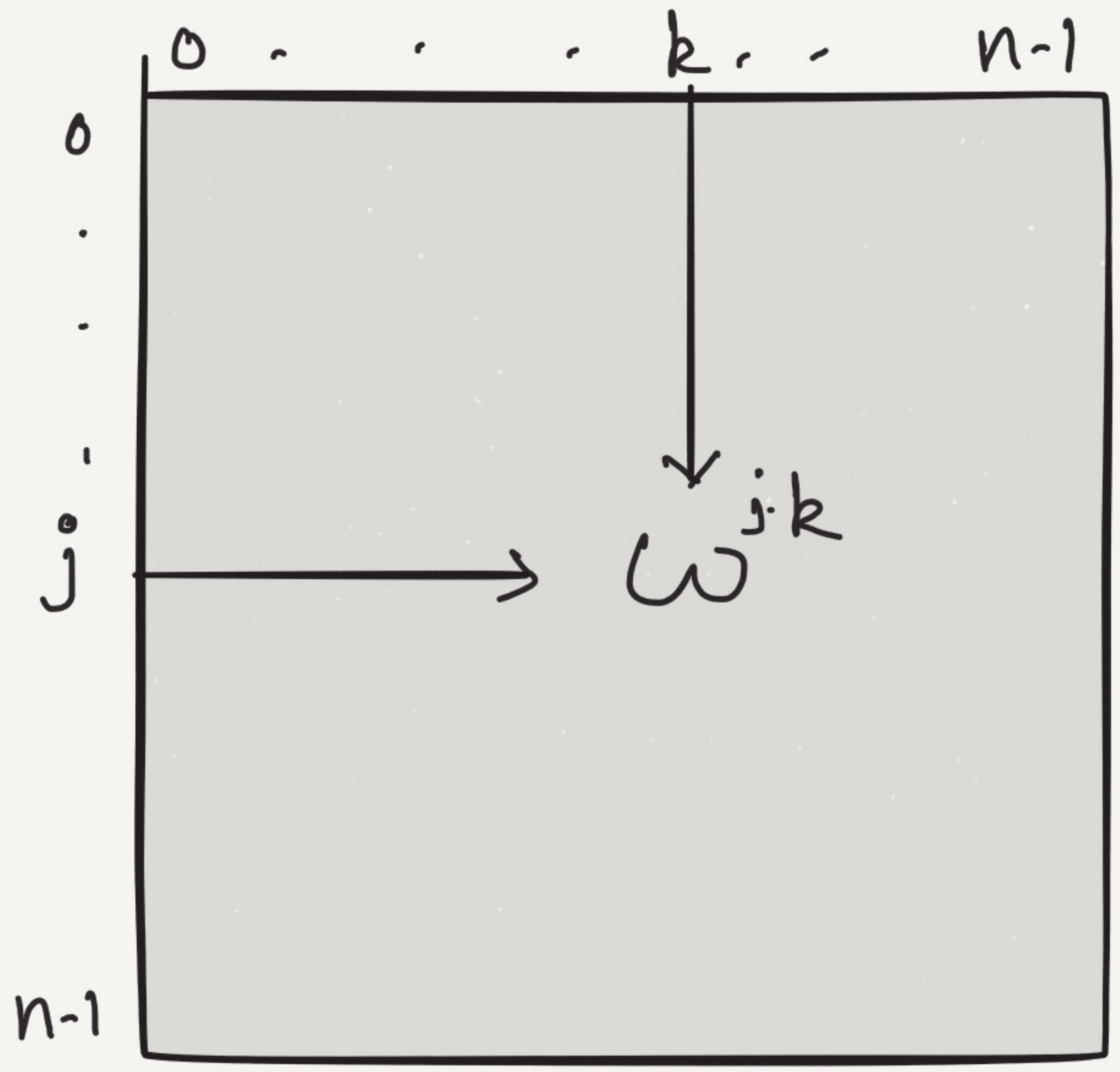
OUTPUT: $\bar{y}$; $y_j = \omega^{j \cdot 0} \cdot a_0 + \omega^{j \cdot 1} \cdot a_1 + \cdots + \omega^{j \cdot (n-1)} \cdot a_{n-1}$

$$\bar{y} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \\ \omega^{j \cdot 0} & \omega^{j \cdot 1} & \omega^{j \cdot 2} & \cdots & & \omega^{j \cdot (n-1)} \\ 1 & & & & & \\ \vdots & & & & & \\ 1 & & & & & \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

Rephrasing the problem:

INPUT: $\bar{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$

OUTPUT: $M_n \cdot \bar{a}$



$\rightarrow \mathrm{DFT}_n$

$$n=2: \quad M_2 \longrightarrow \begin{bmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 1} \\ \omega^{1\cdot 0} & \omega^{1\cdot 1} \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\omega = e^{\frac{2\pi i}{n}} = -1$$

$$n=3: \quad M_3 \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

$$\omega = e^{\frac{2\pi i}{3}}$$

$n = 4$; $\omega = e^{2\pi i/4}$

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega^5 \end{bmatrix}$$

$n = 4 \; ; \quad \omega = e^{2\pi i/4}$

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega^5 \end{bmatrix}$$

YELLOW = RED

ORANGE = $\omega^2 \cdot$ BLUE

If we move even columns (0^{th}, 2^{nd}) together

$$
\begin{array}{c|cccc}
 & 0^{th} & 2^{nd} & 1^{st} & 3^{rd} \\
\hline
0 & 1 & 1 & 1 & 1 \\
1 & 1 & \omega^2 & \omega & \omega^3 \\
2 & 1 & 1 & \omega^2 & \omega^2 \\
3 & 1 & \omega^2 & \omega^3 & \omega^5
\end{array}
\equiv
\begin{array}{|c|c|}
\hline
A & B \\
\hline
A & \omega^2 \cdot B \\
\hline
\end{array}
$$