CS180: Algorithms and Complexity

Professor: Raghu Meka (raghum@cs)

Plan for Today

Master theorem

Integer multiplication

Exponentiation

Asymptotic analysis

Methodology for comparing run-times

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Given two functions $f, g : \mathbb{N} \to \mathbb{R}^+$

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f(n) = Θ(g(n)): iff both hold - there are constants c_1, c_2>0 so that eventually always $c_1 g(n) < f(n) < c_2 g(n)$

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Divide-and-conquer paradigm

Divide-and-conquer.

- Divide problem into several subproblems.
- Solve each subproblem recursively.
- Combine solutions to subproblems into overall solution.

Last class:

- Example for Mergesort
- Recursion analysis for mergesort

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Solution: O(n log n)

Proof by recursion tree

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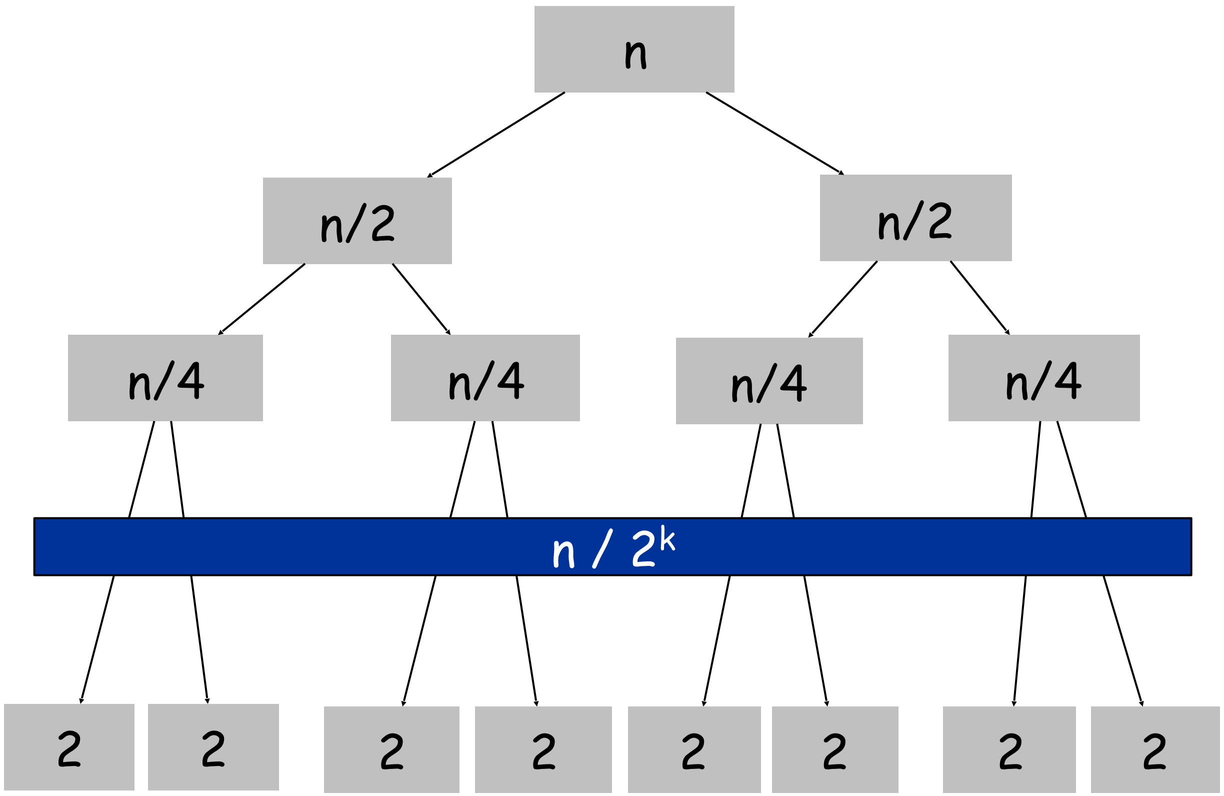

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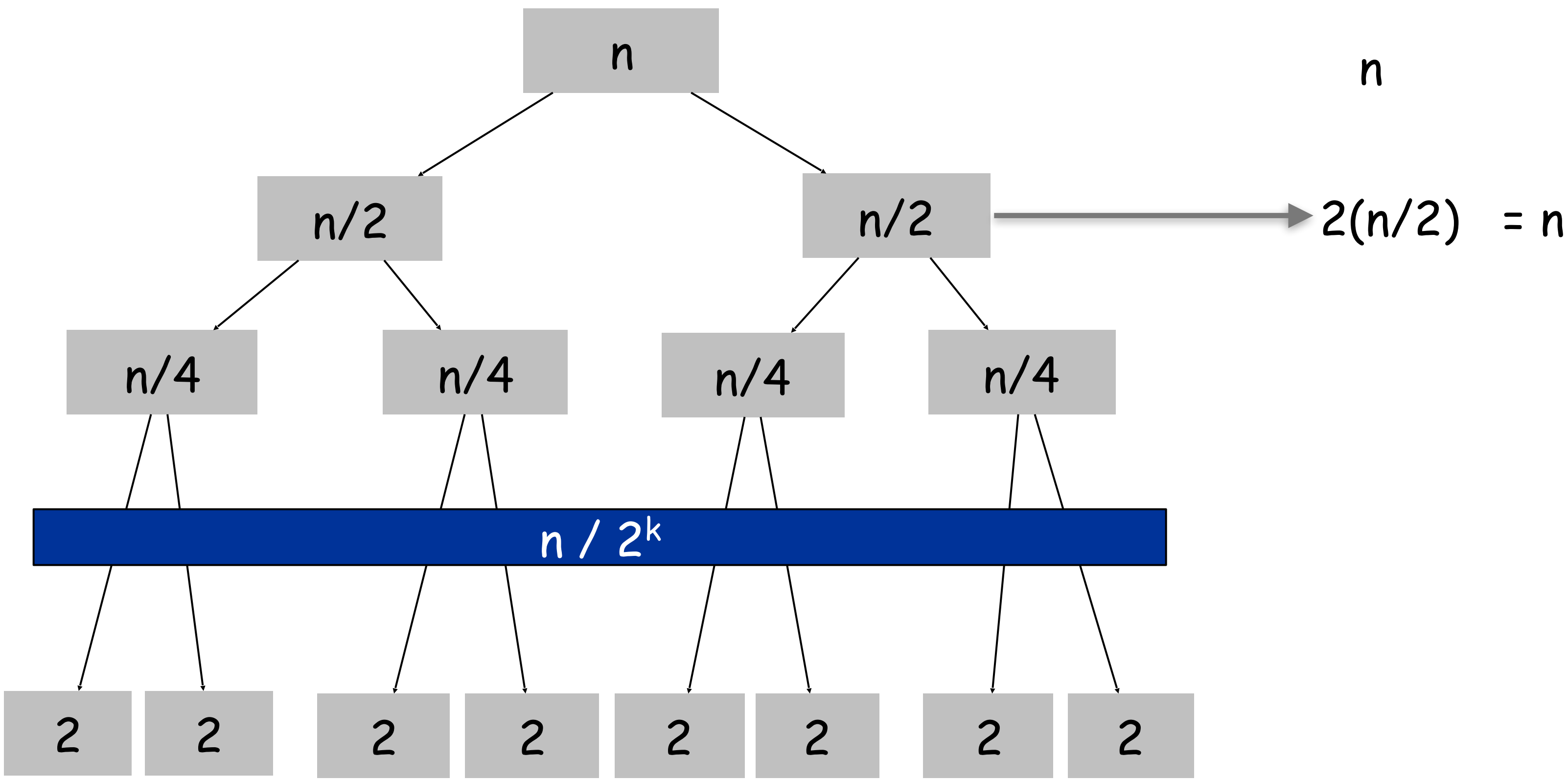

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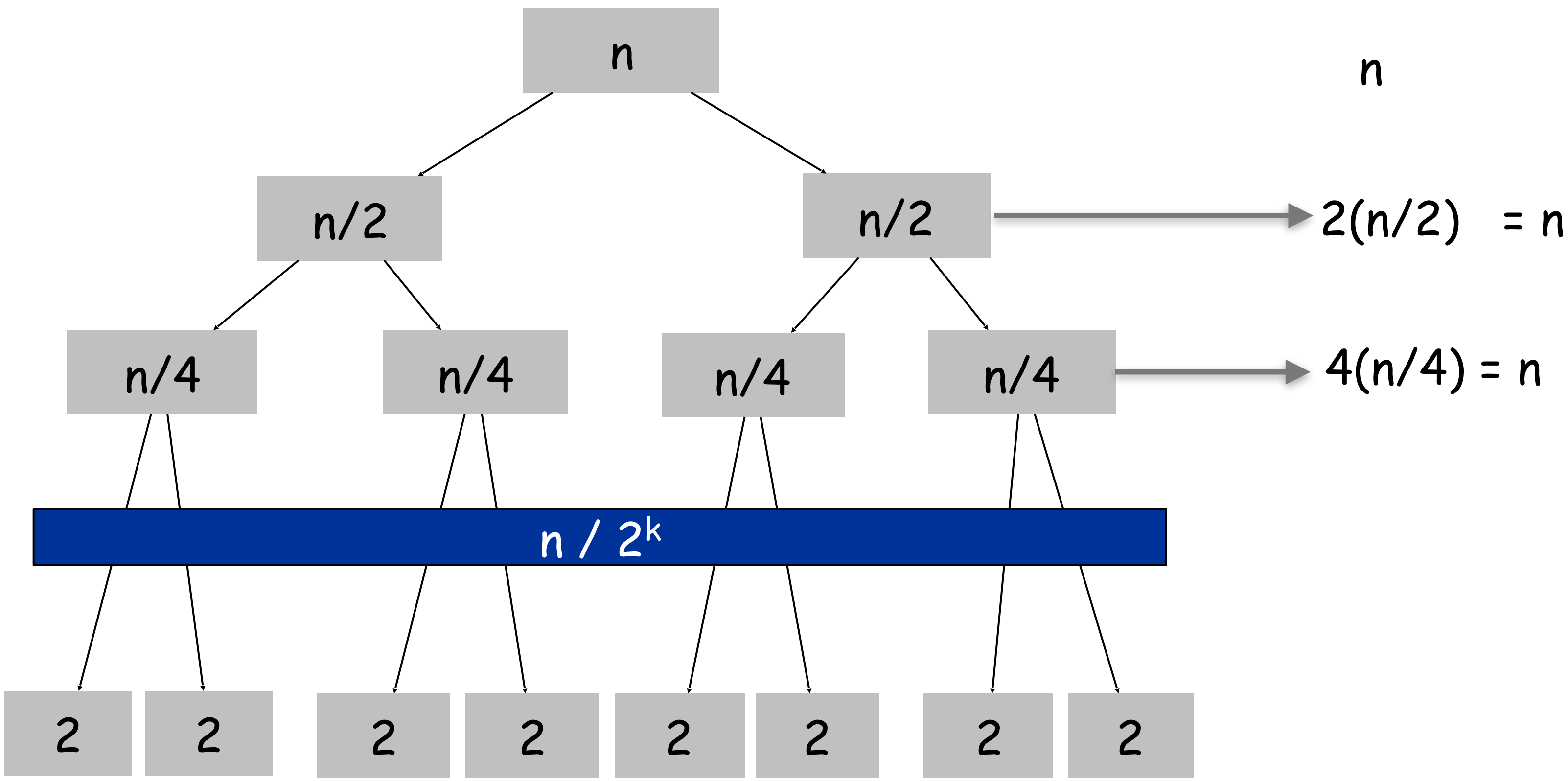

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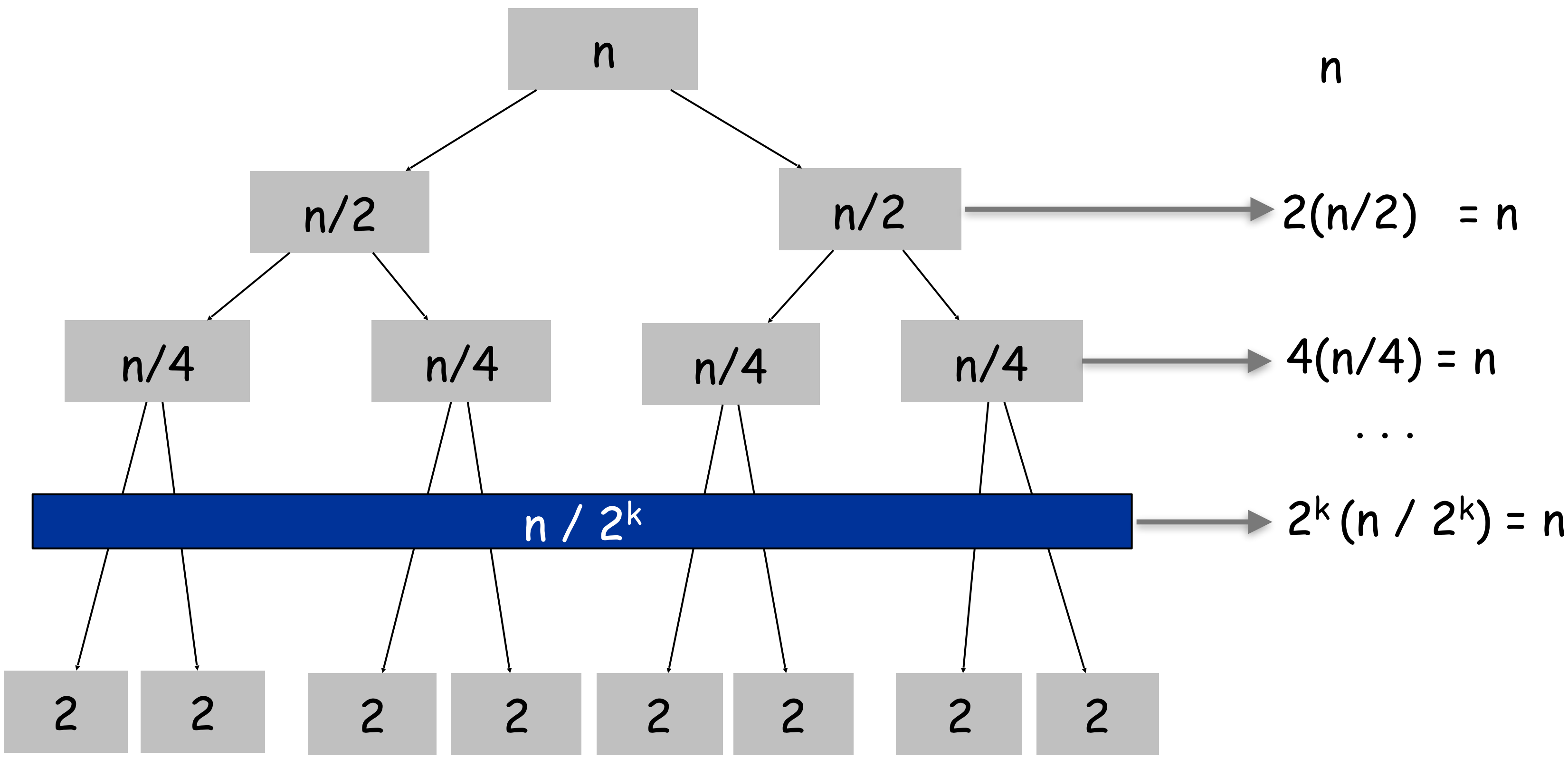

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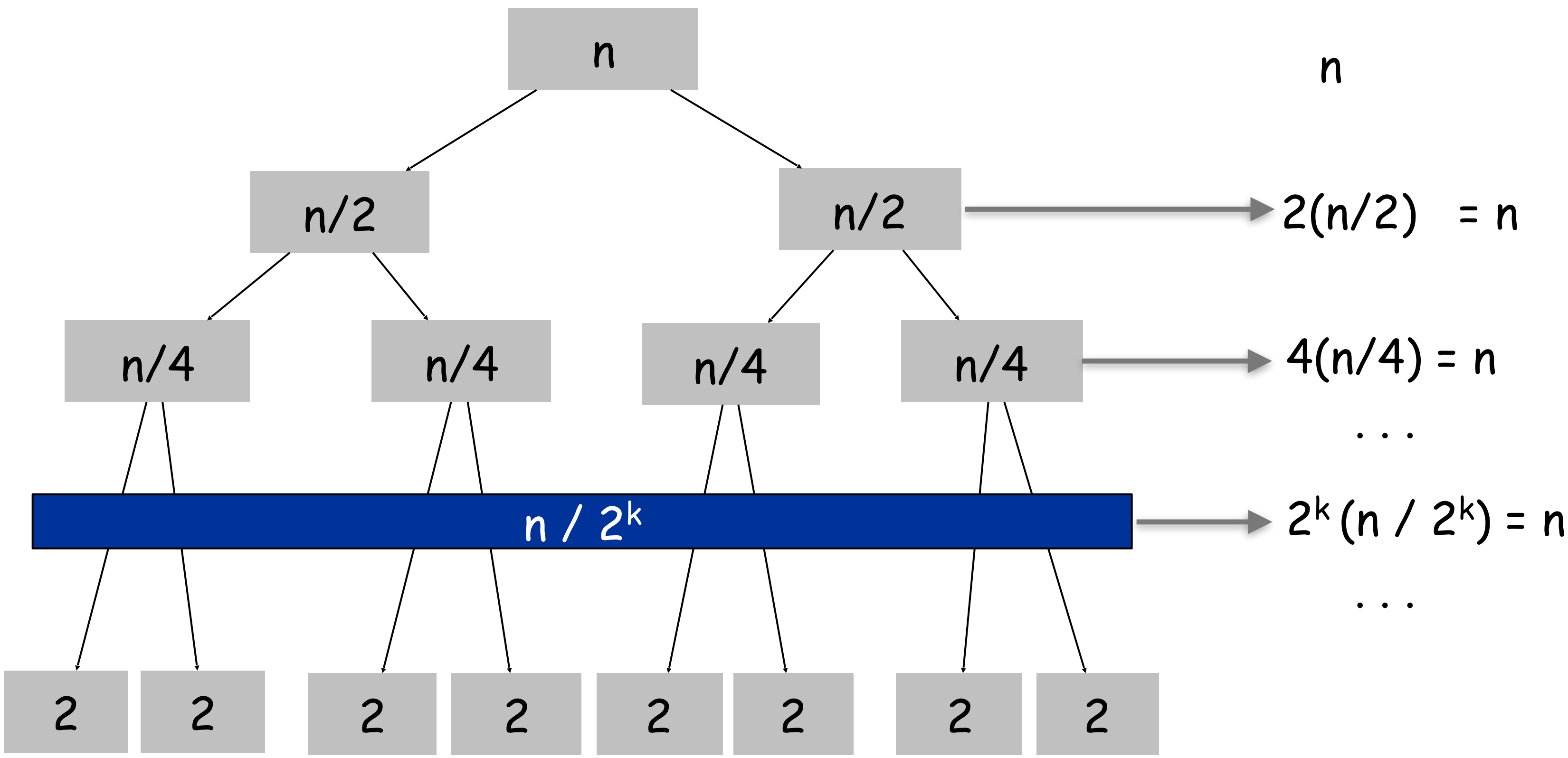

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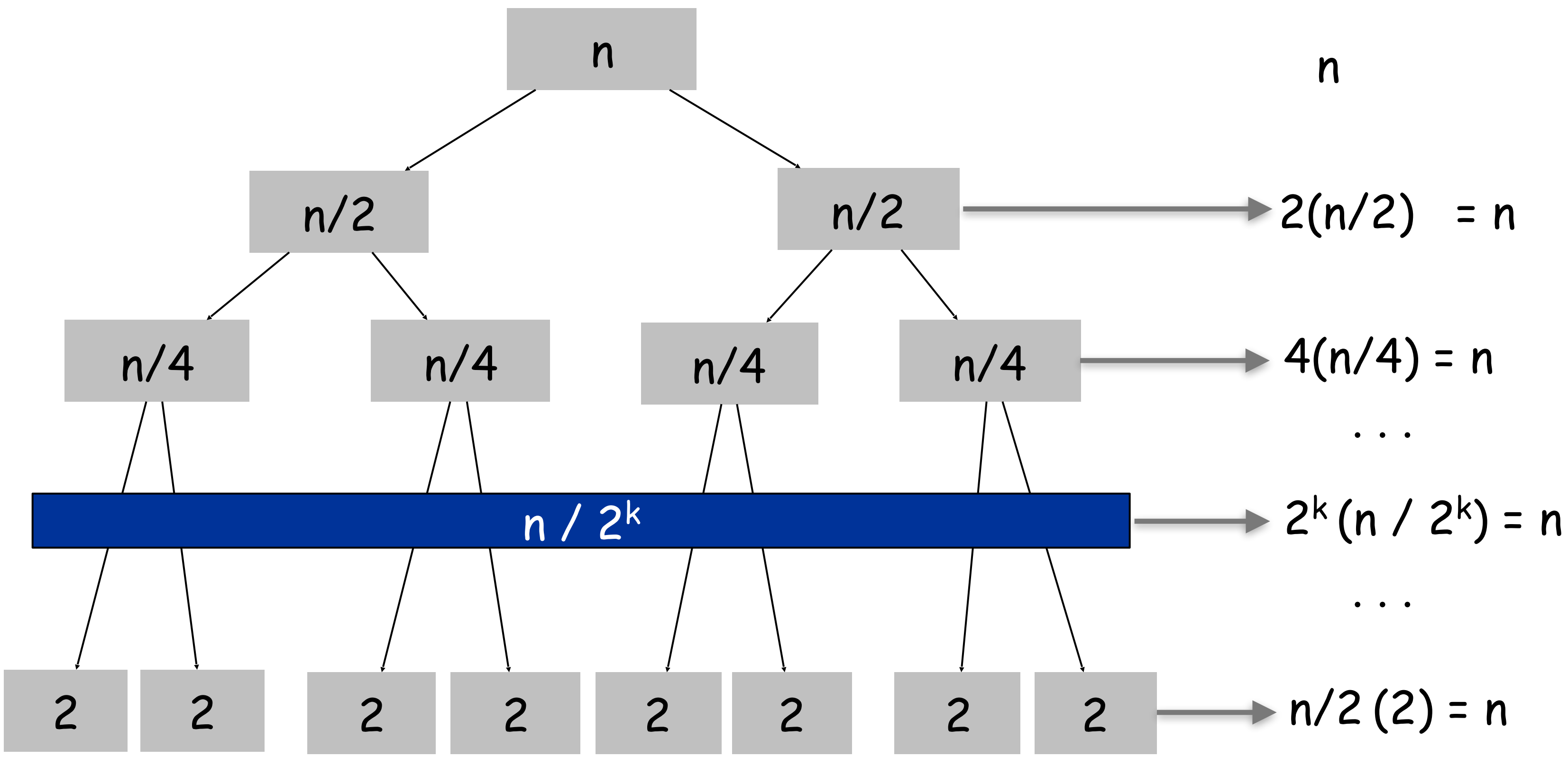

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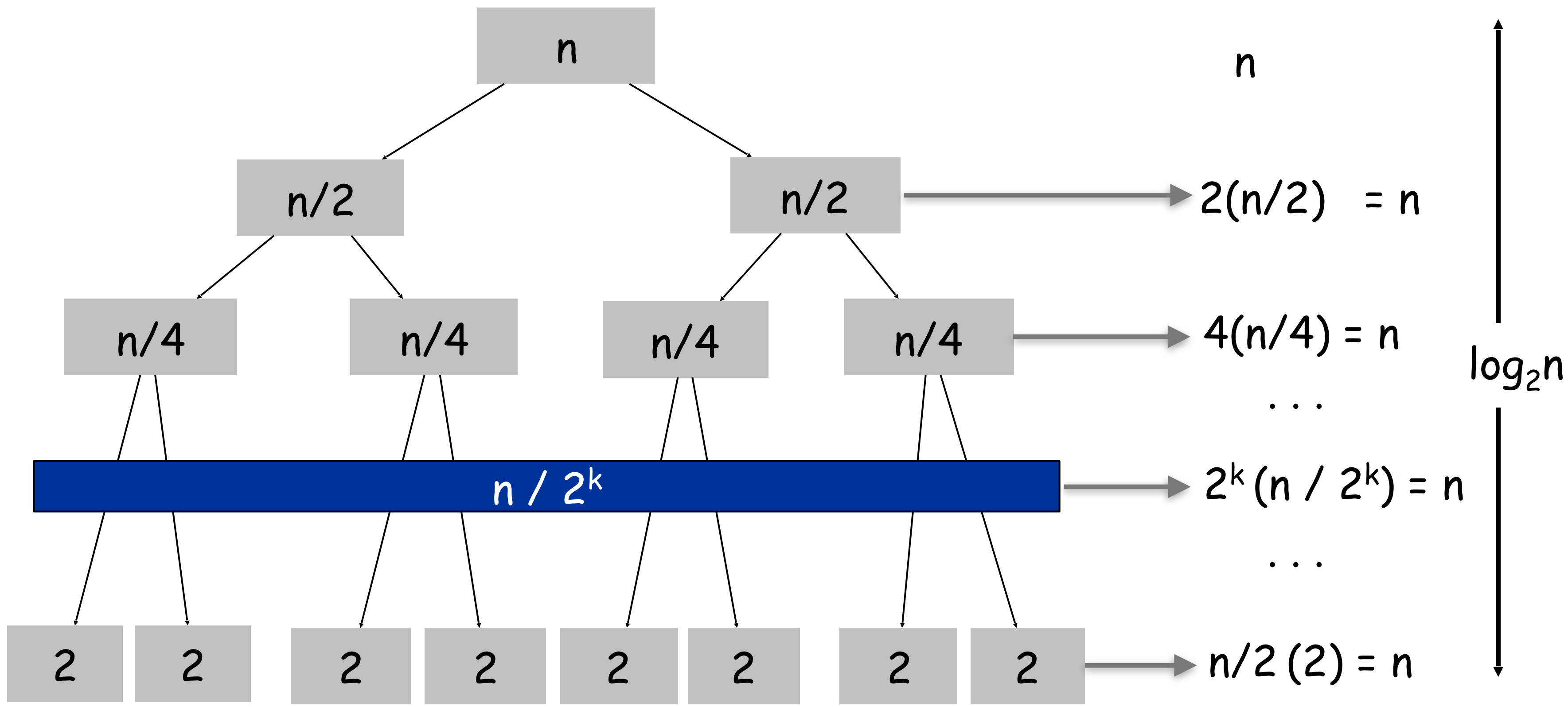

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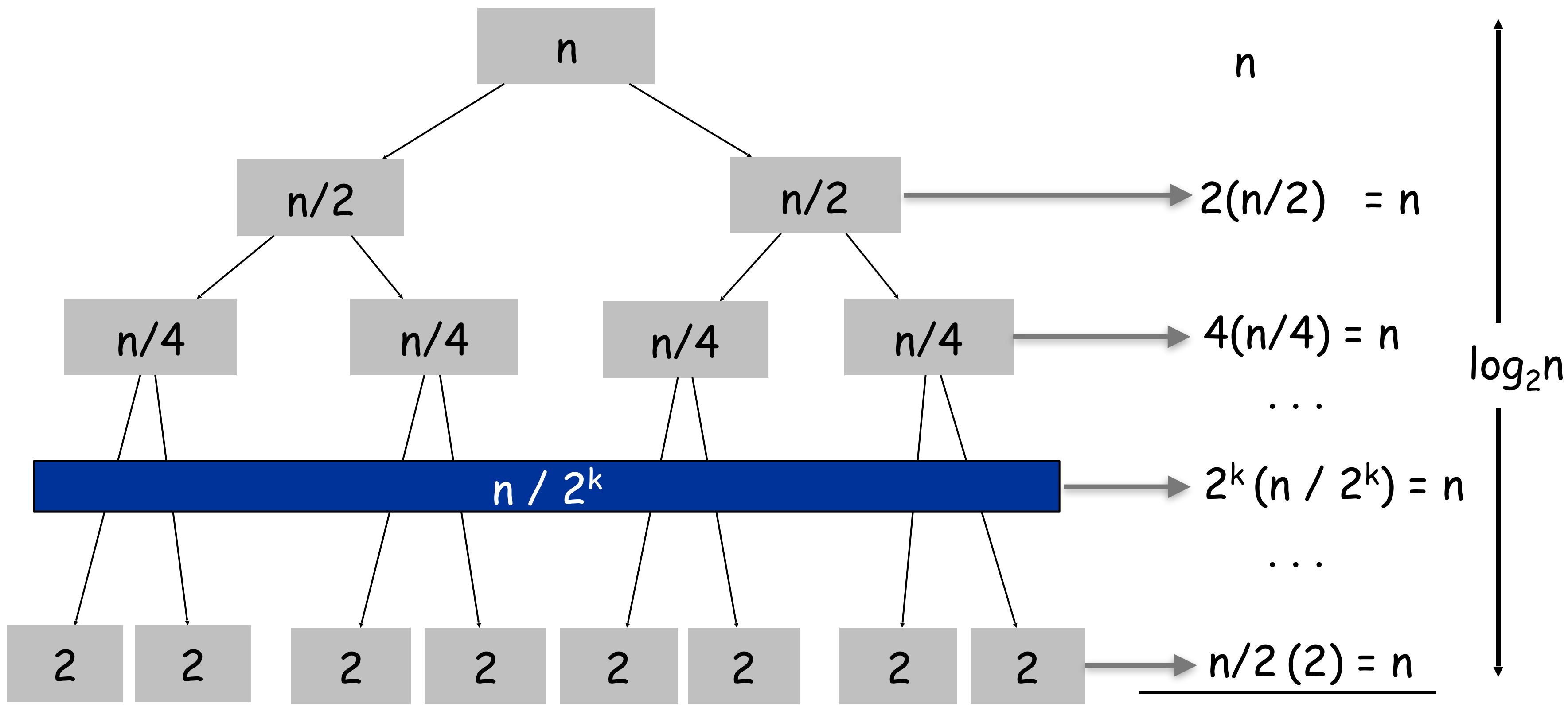

TOTAL = $n \log_2 n$

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- n / b^i = size of subproblem at level i.

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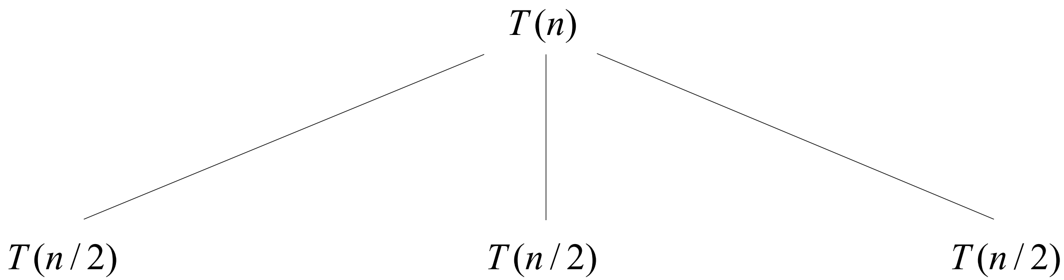

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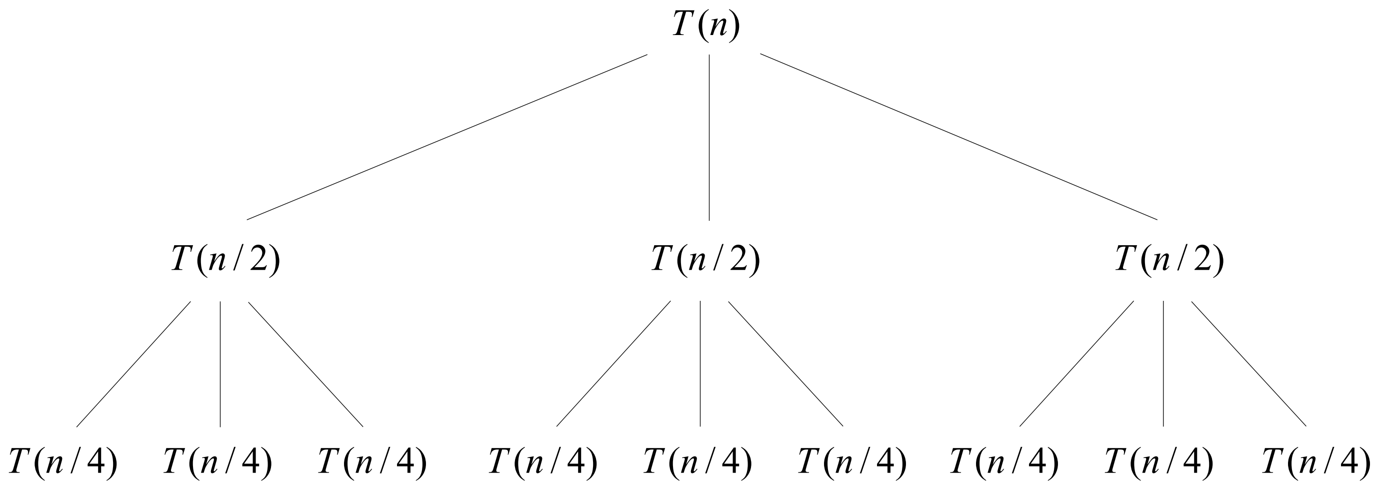

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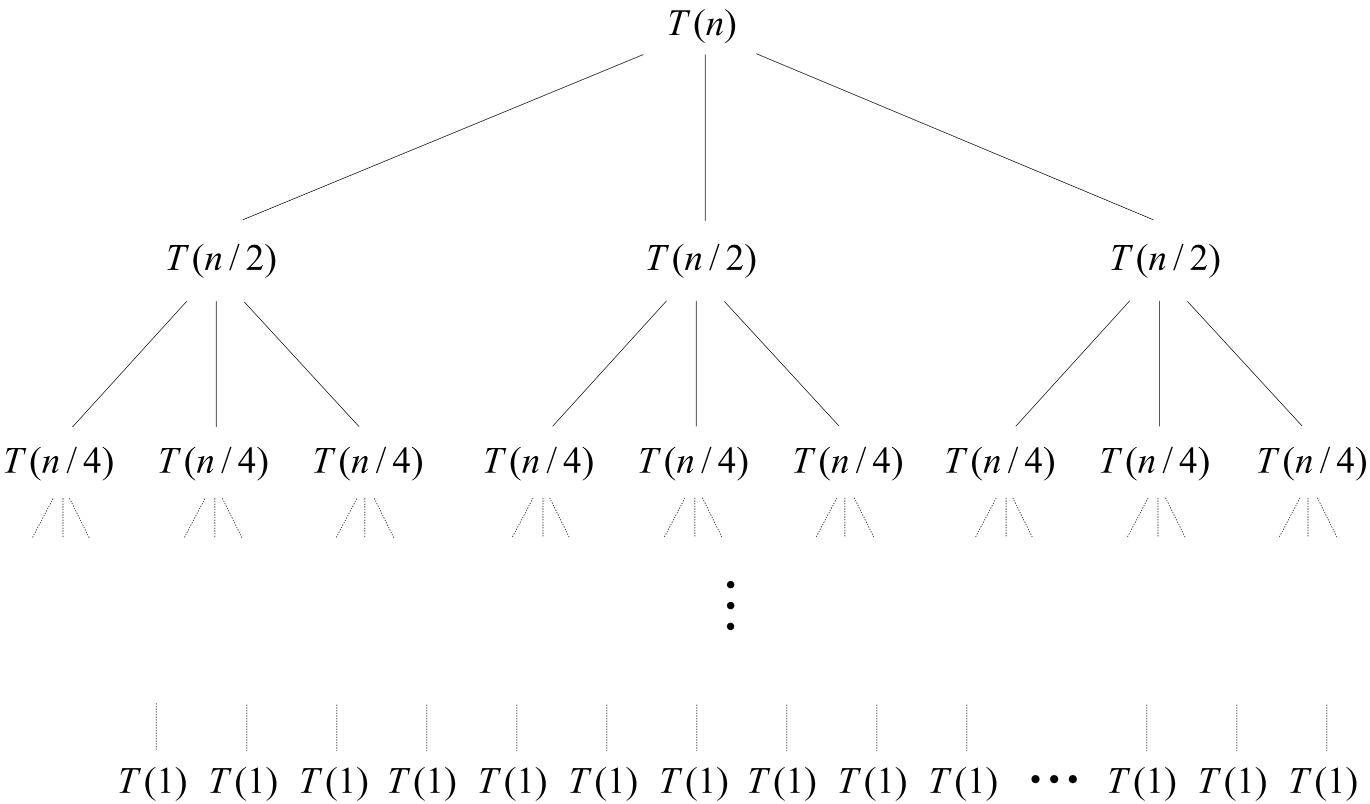

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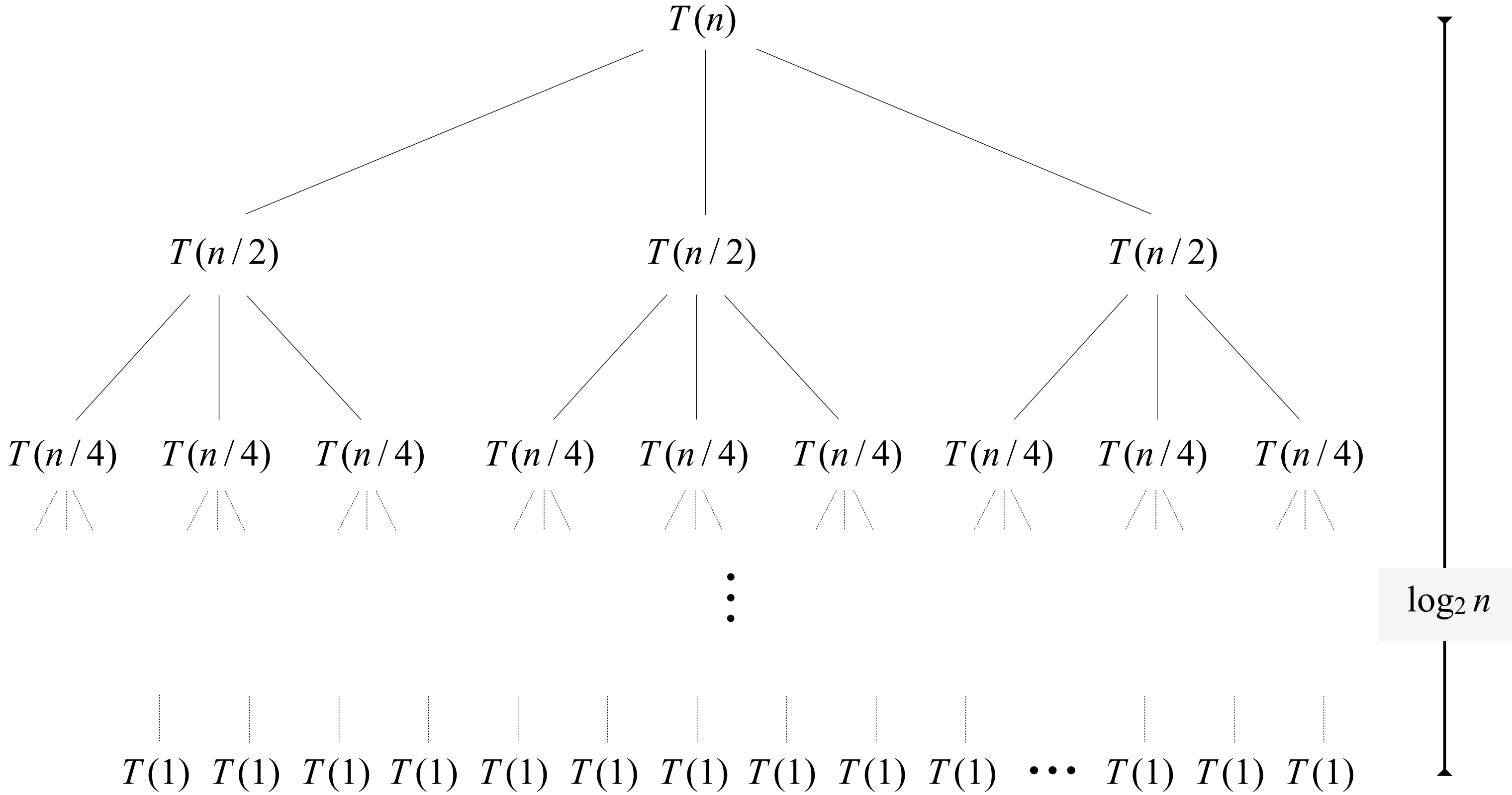

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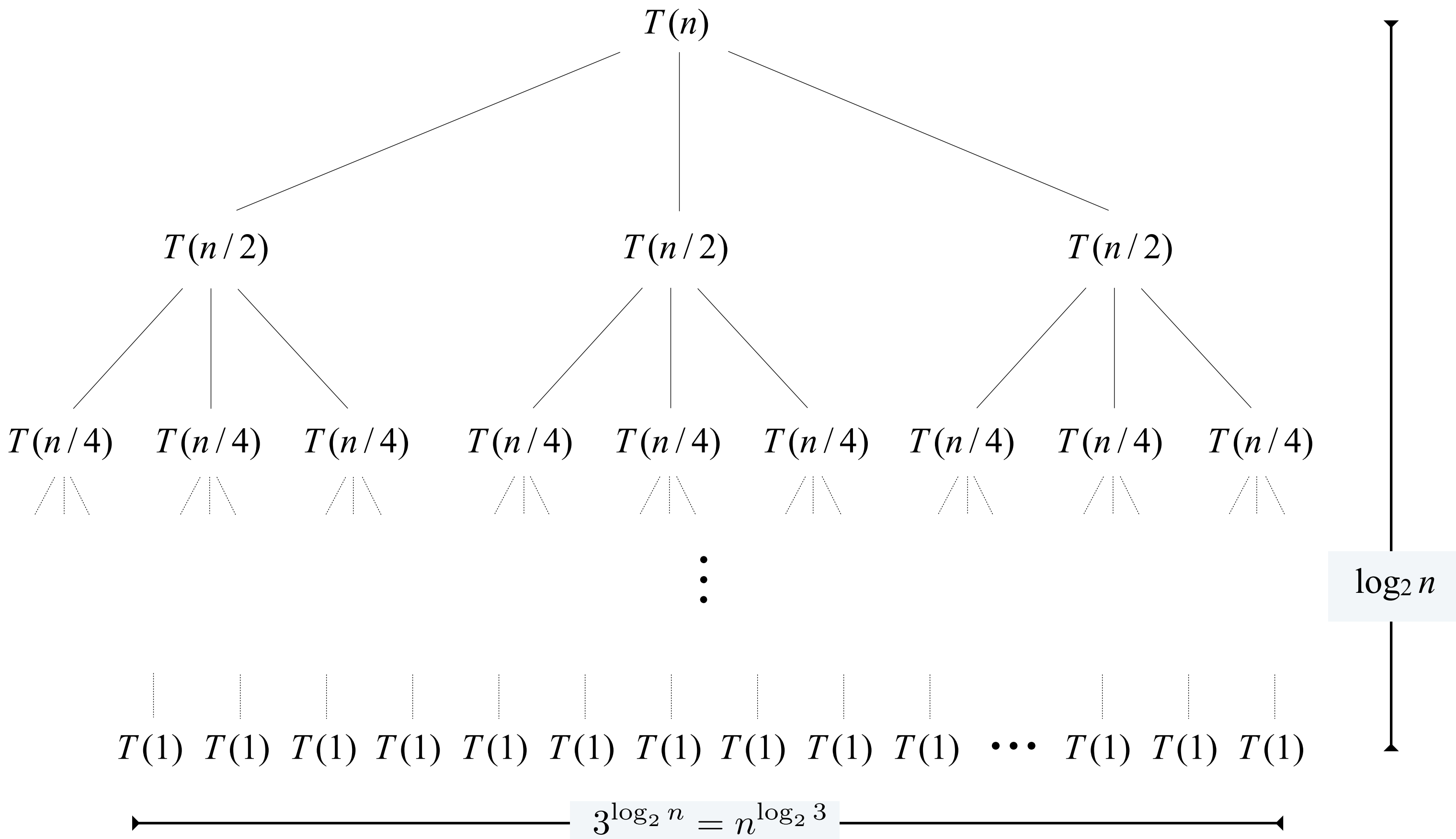

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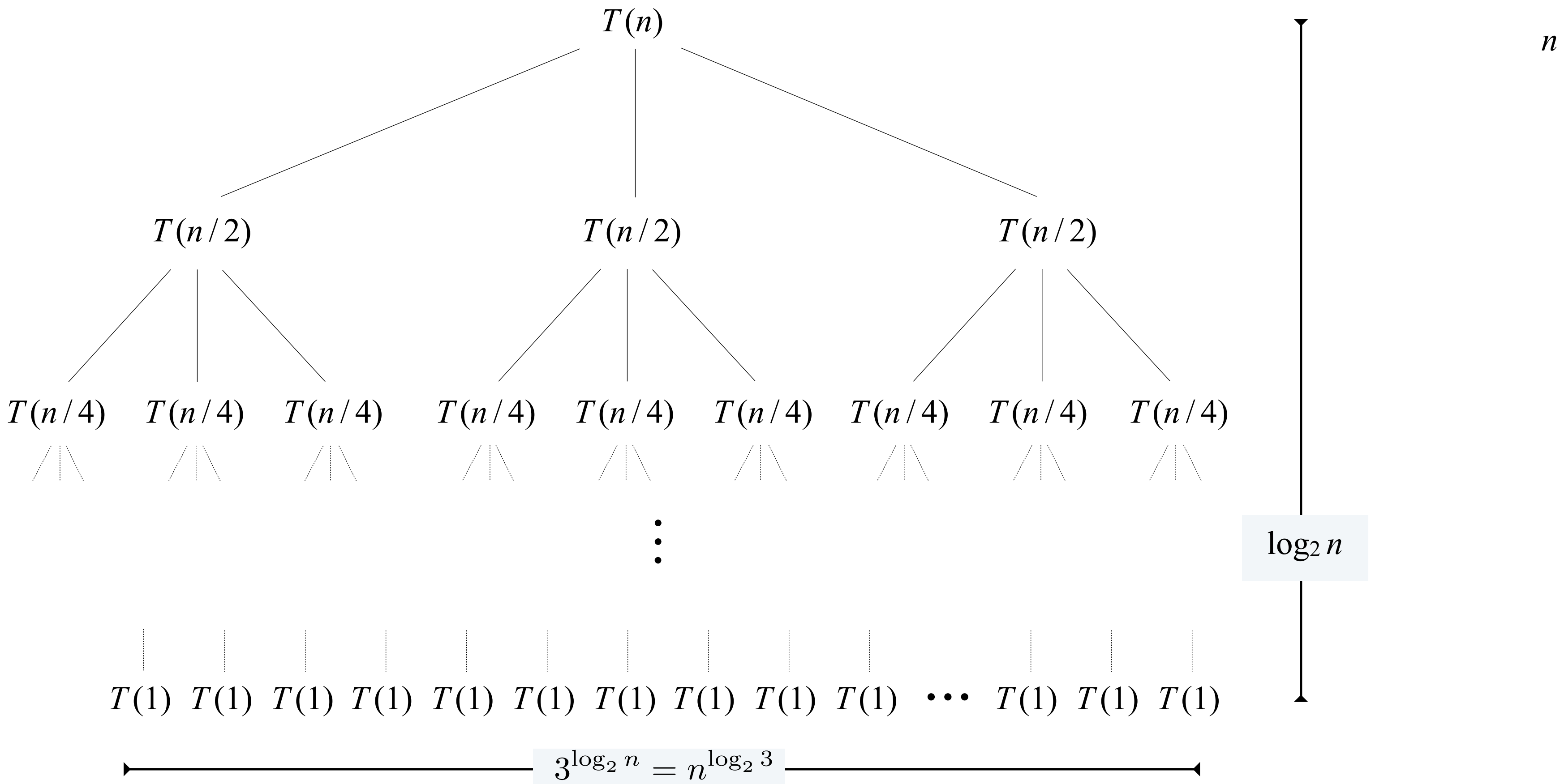

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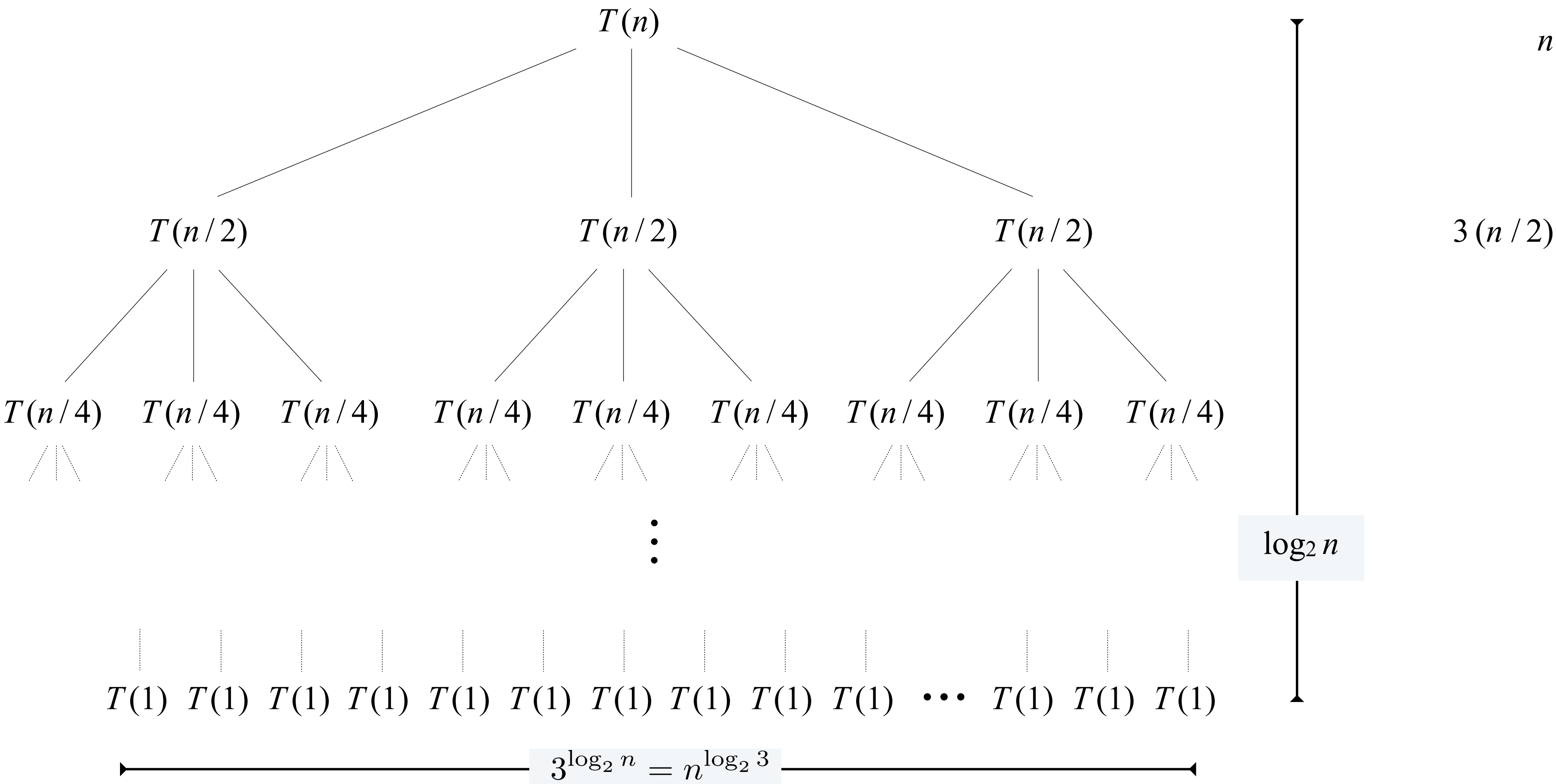

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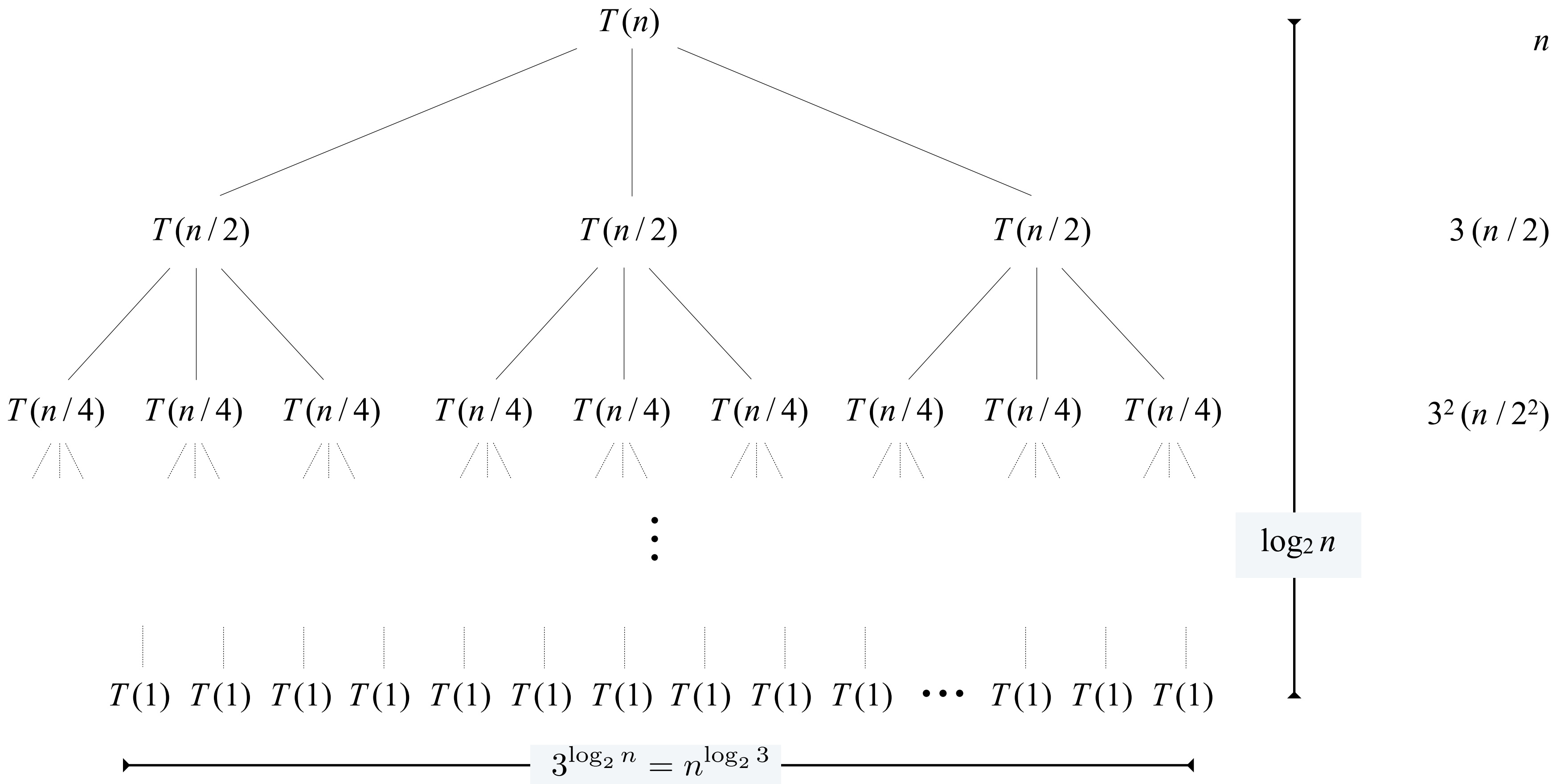

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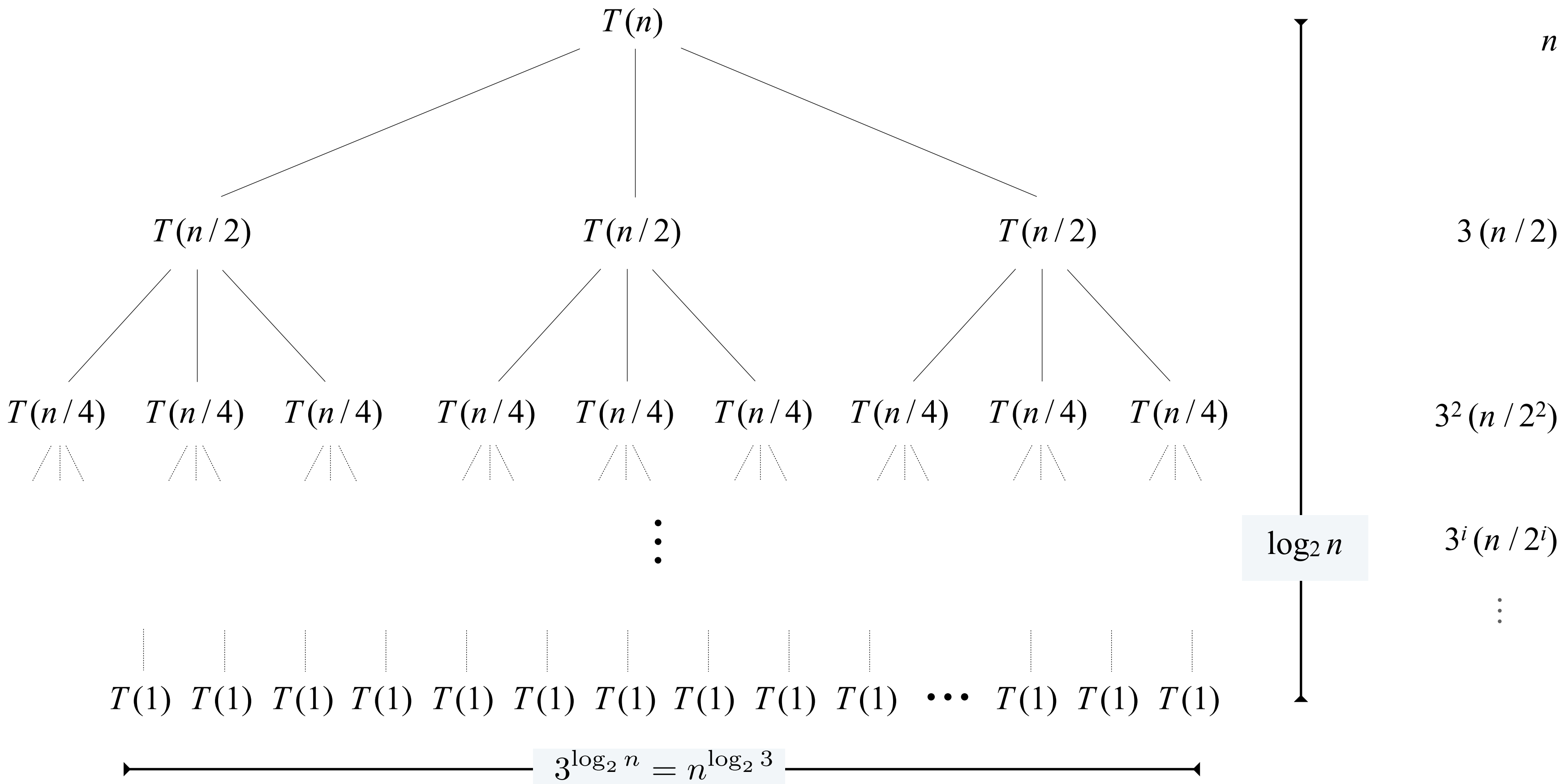

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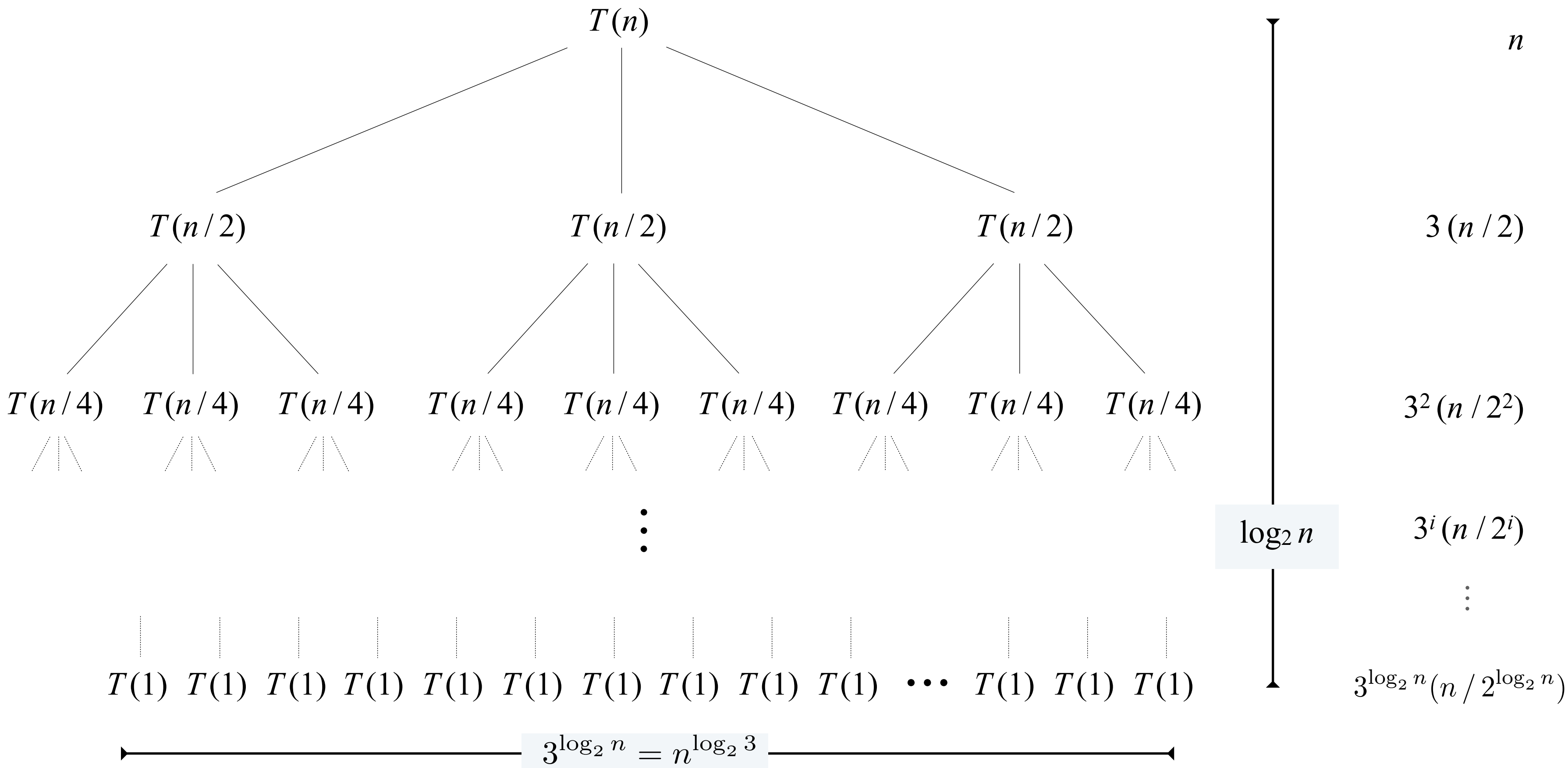

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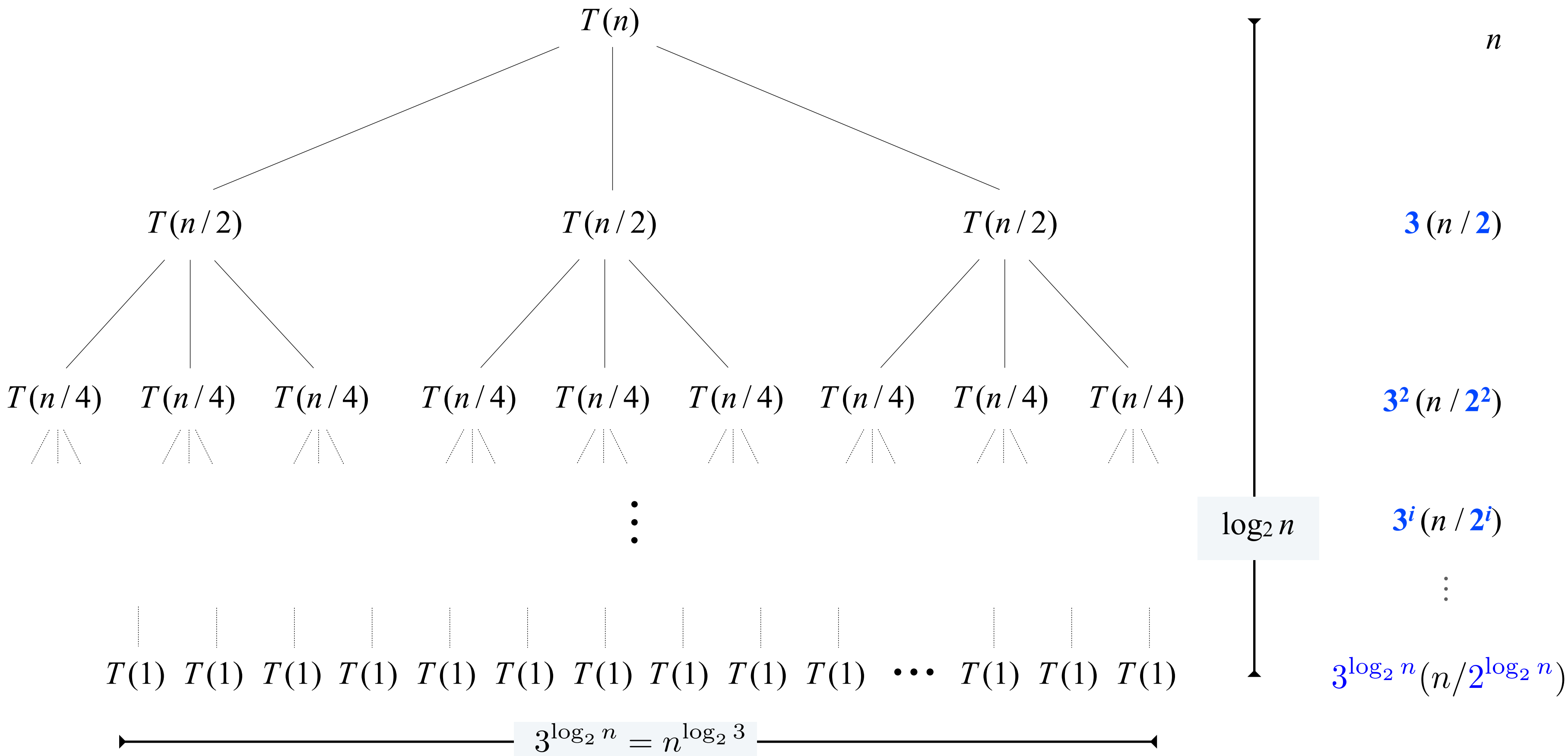

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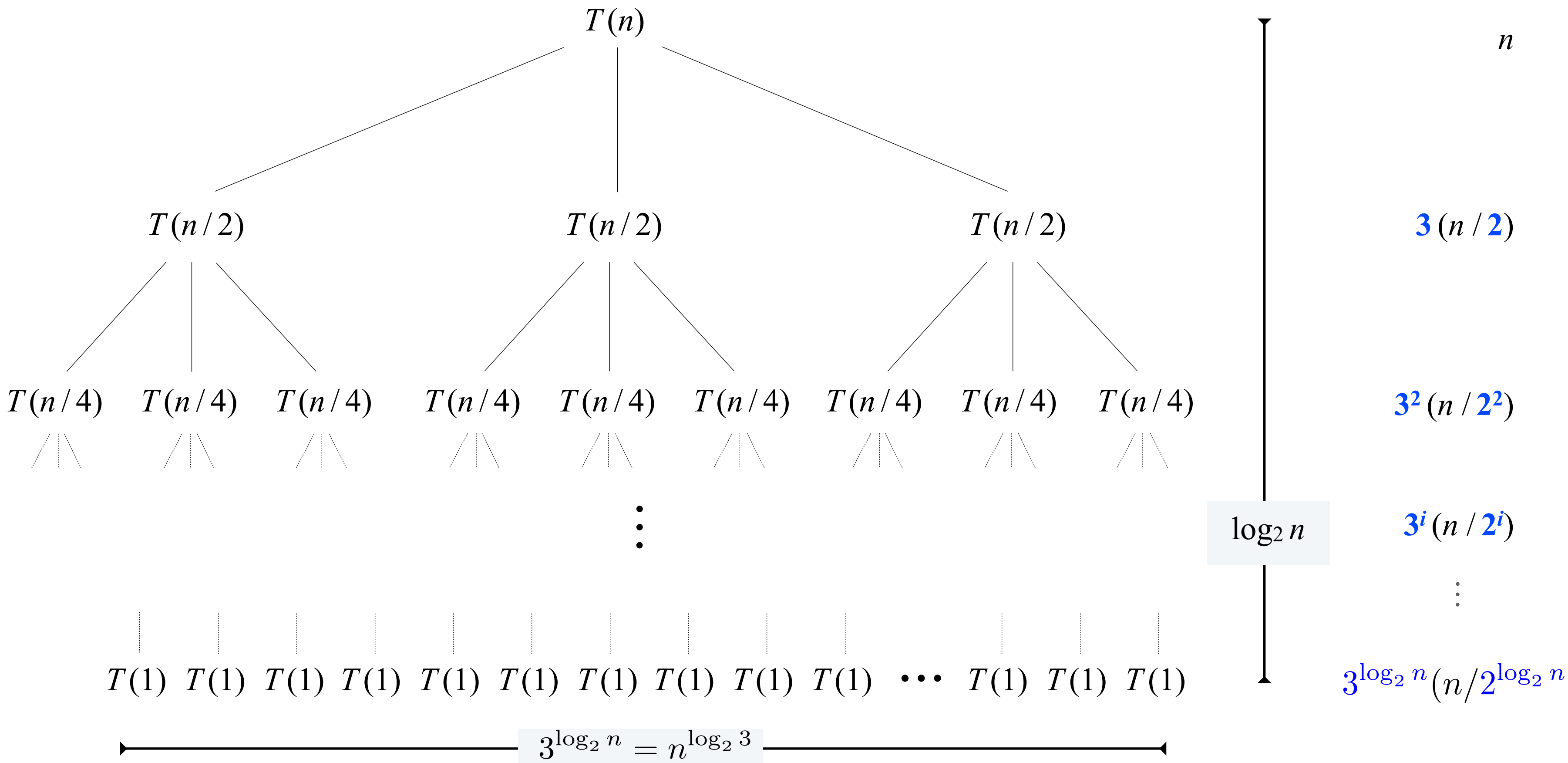

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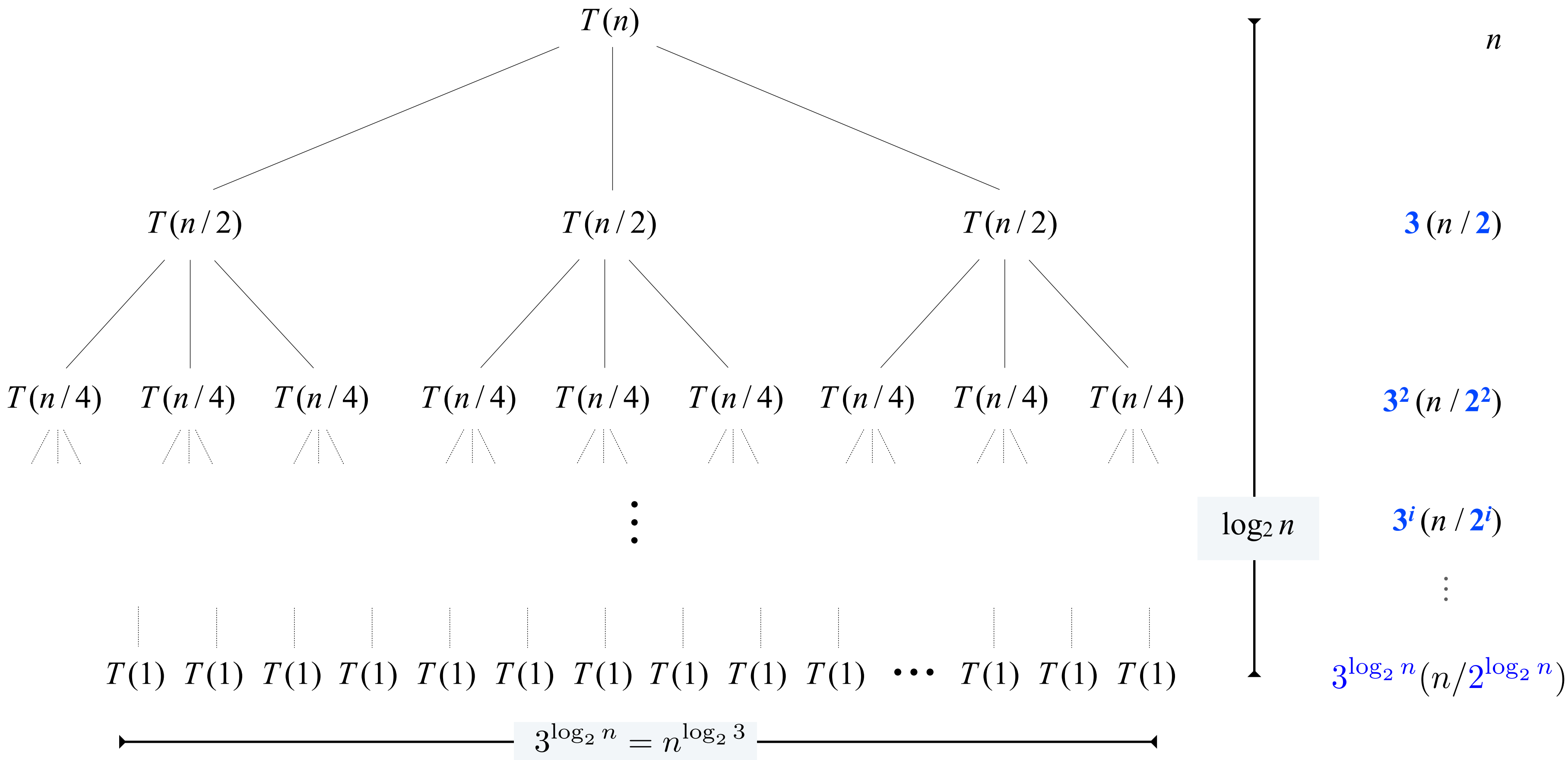

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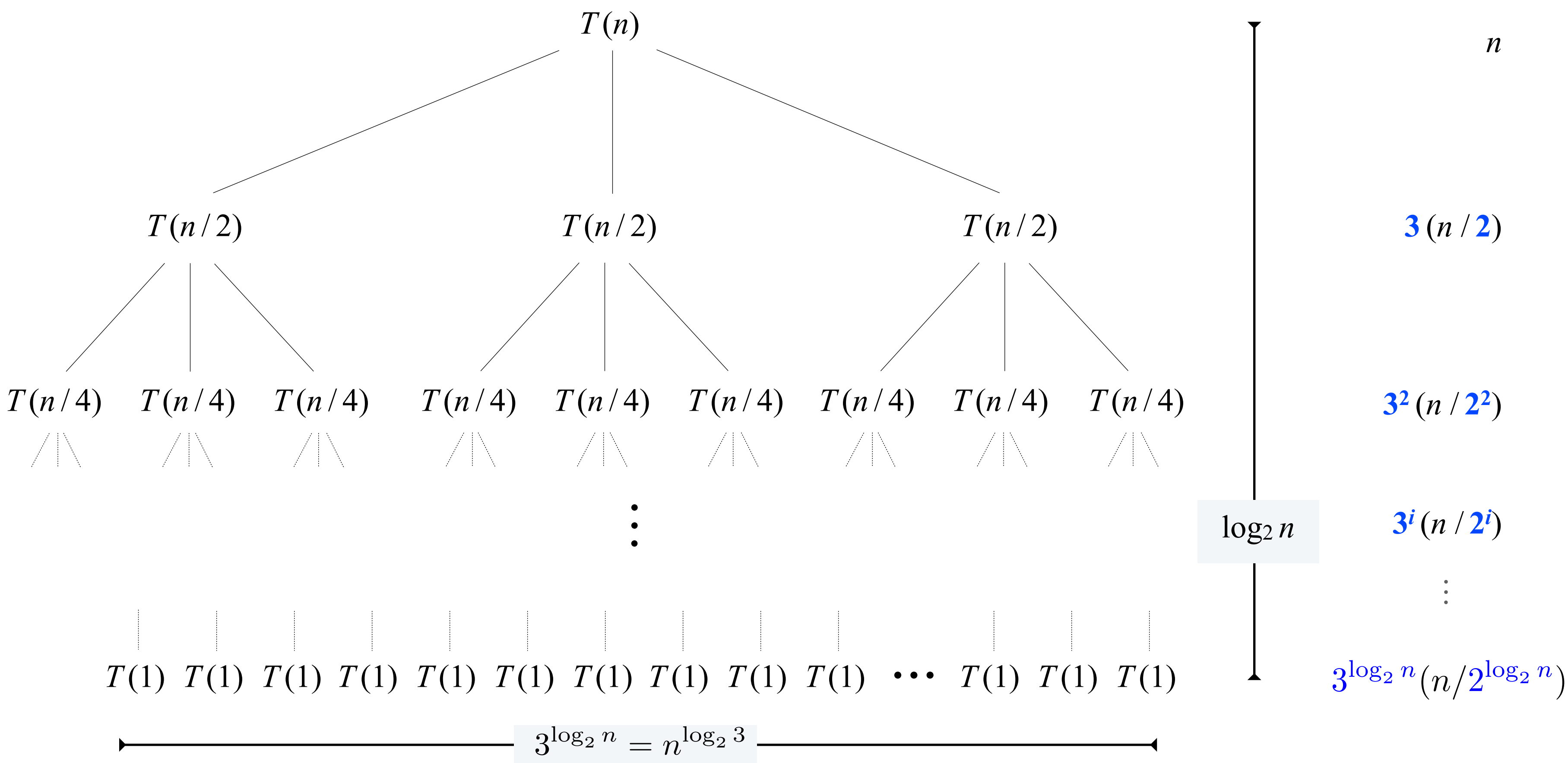

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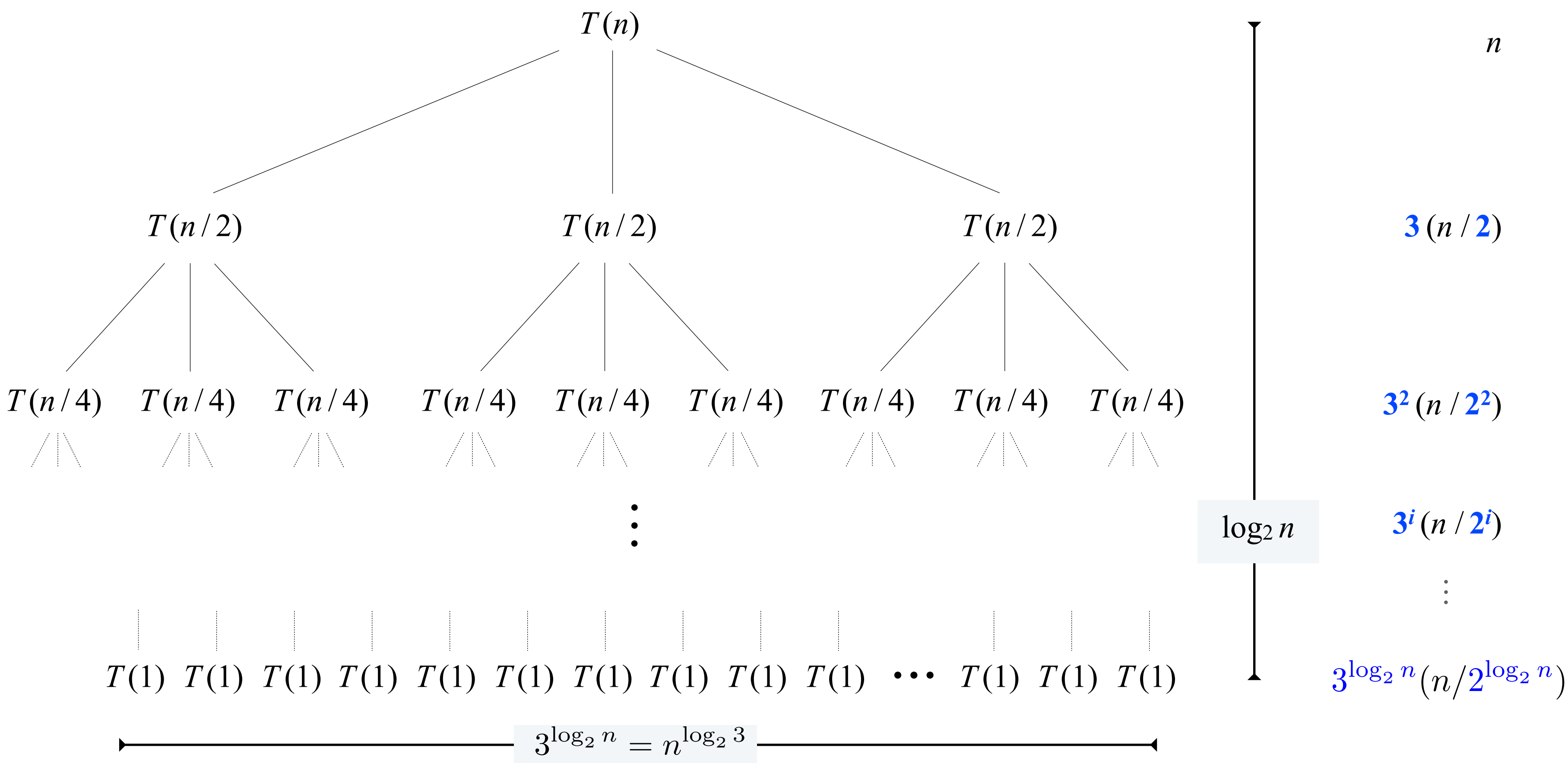

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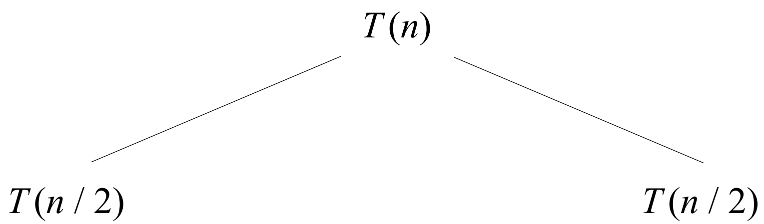

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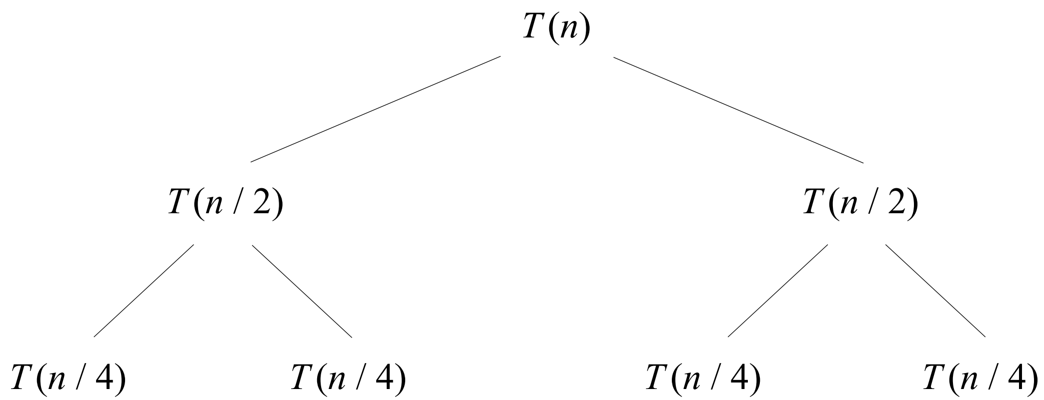

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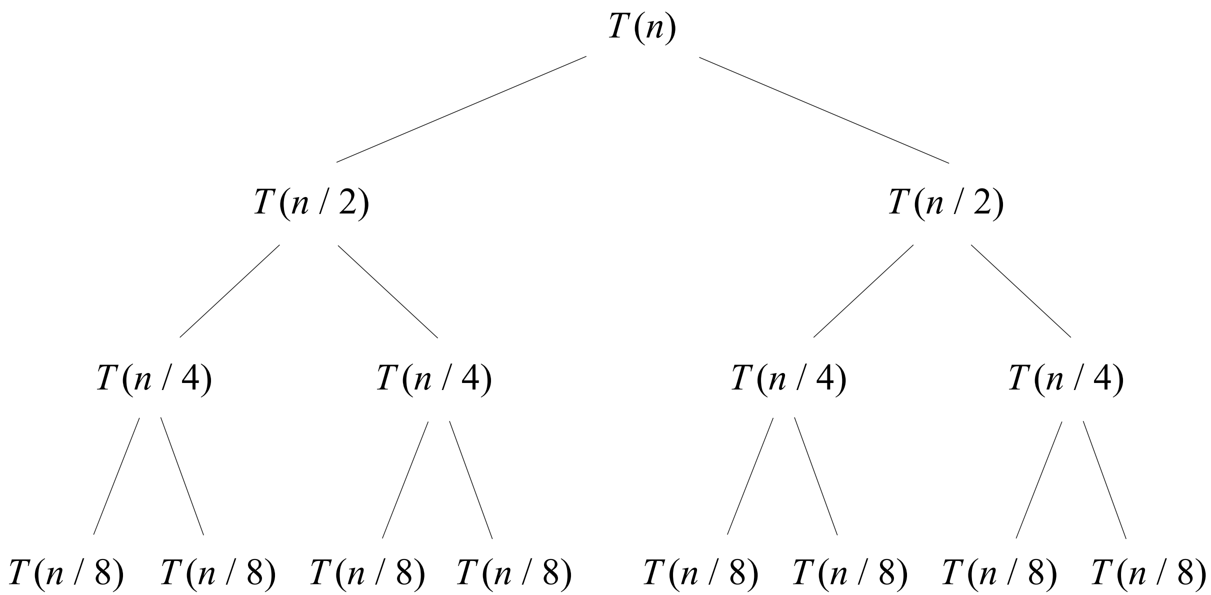

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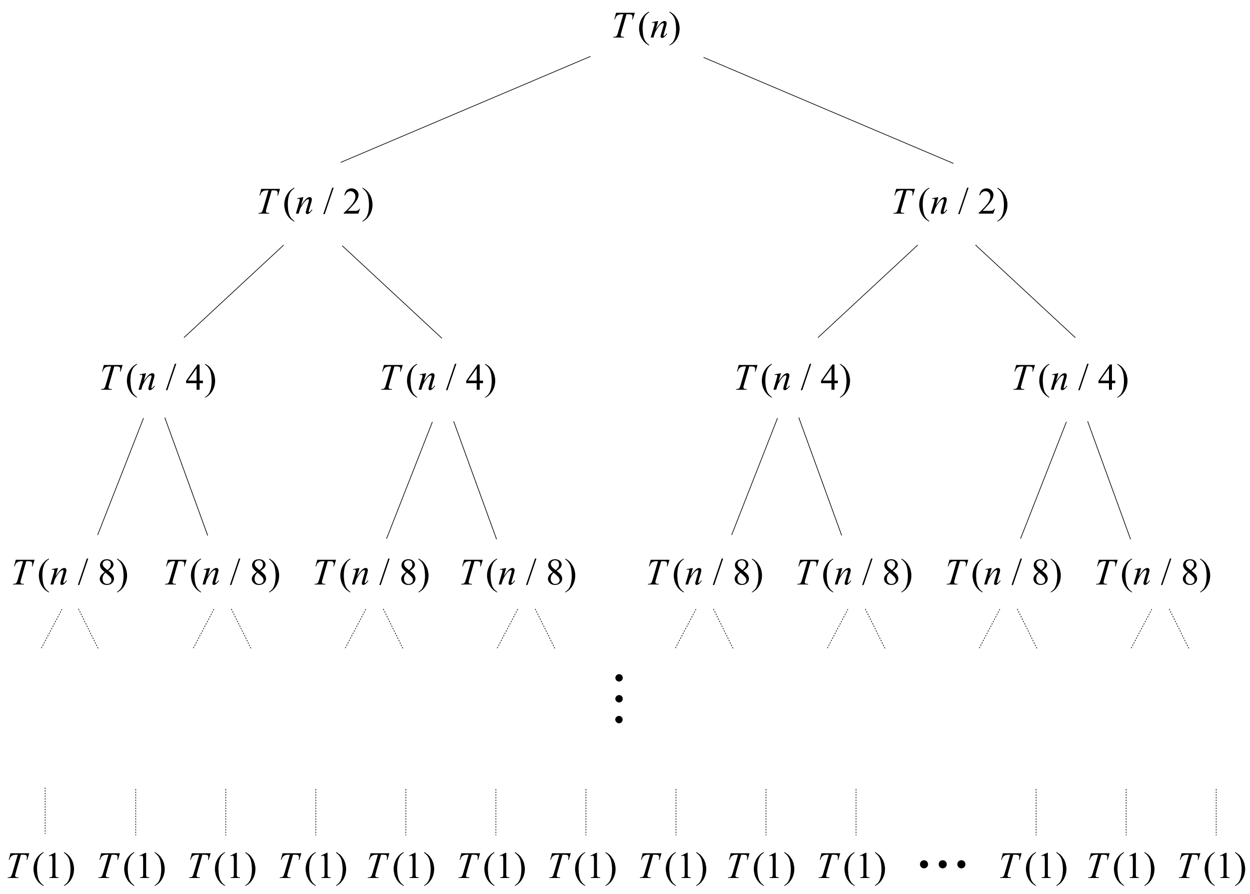

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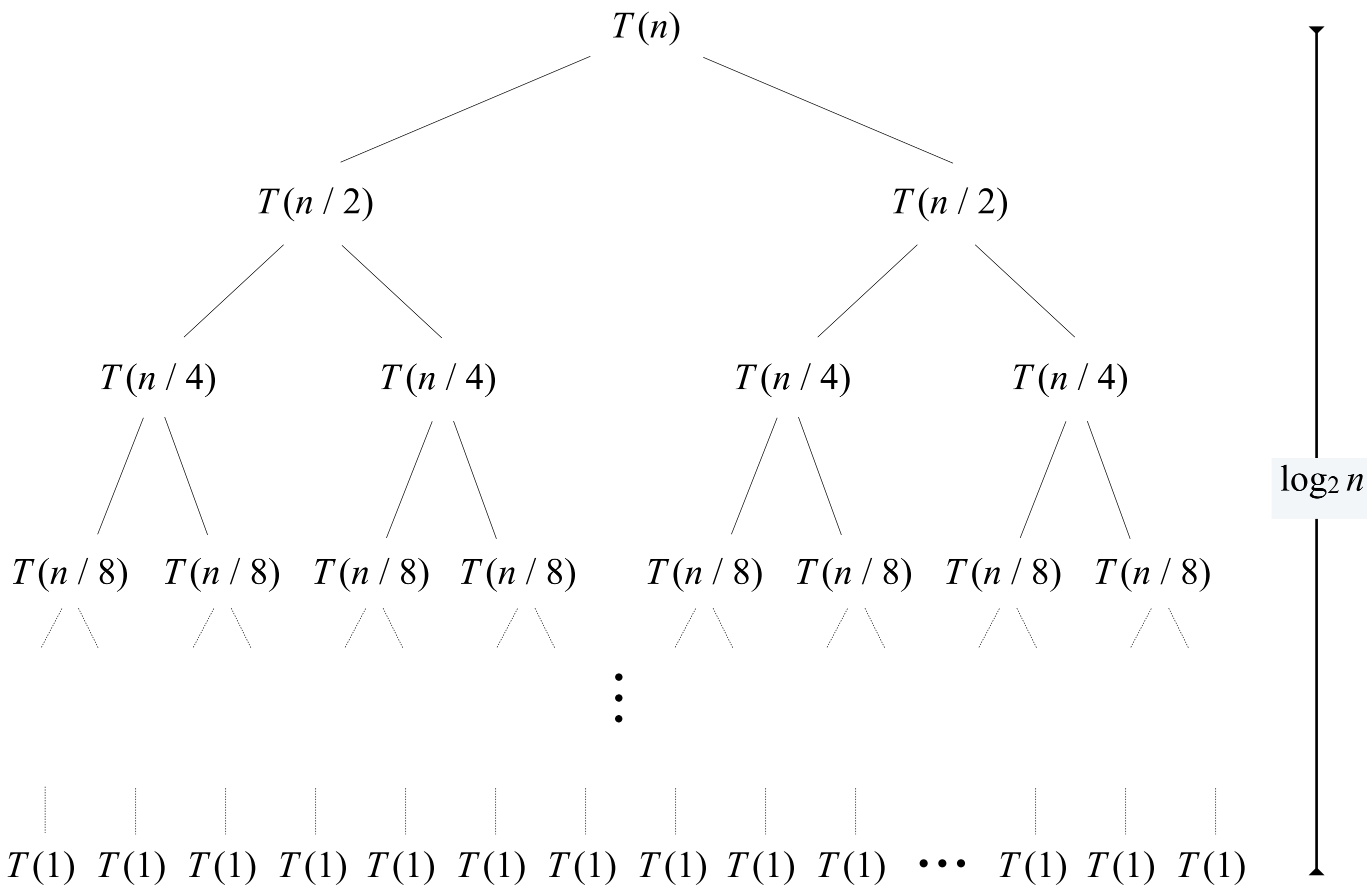

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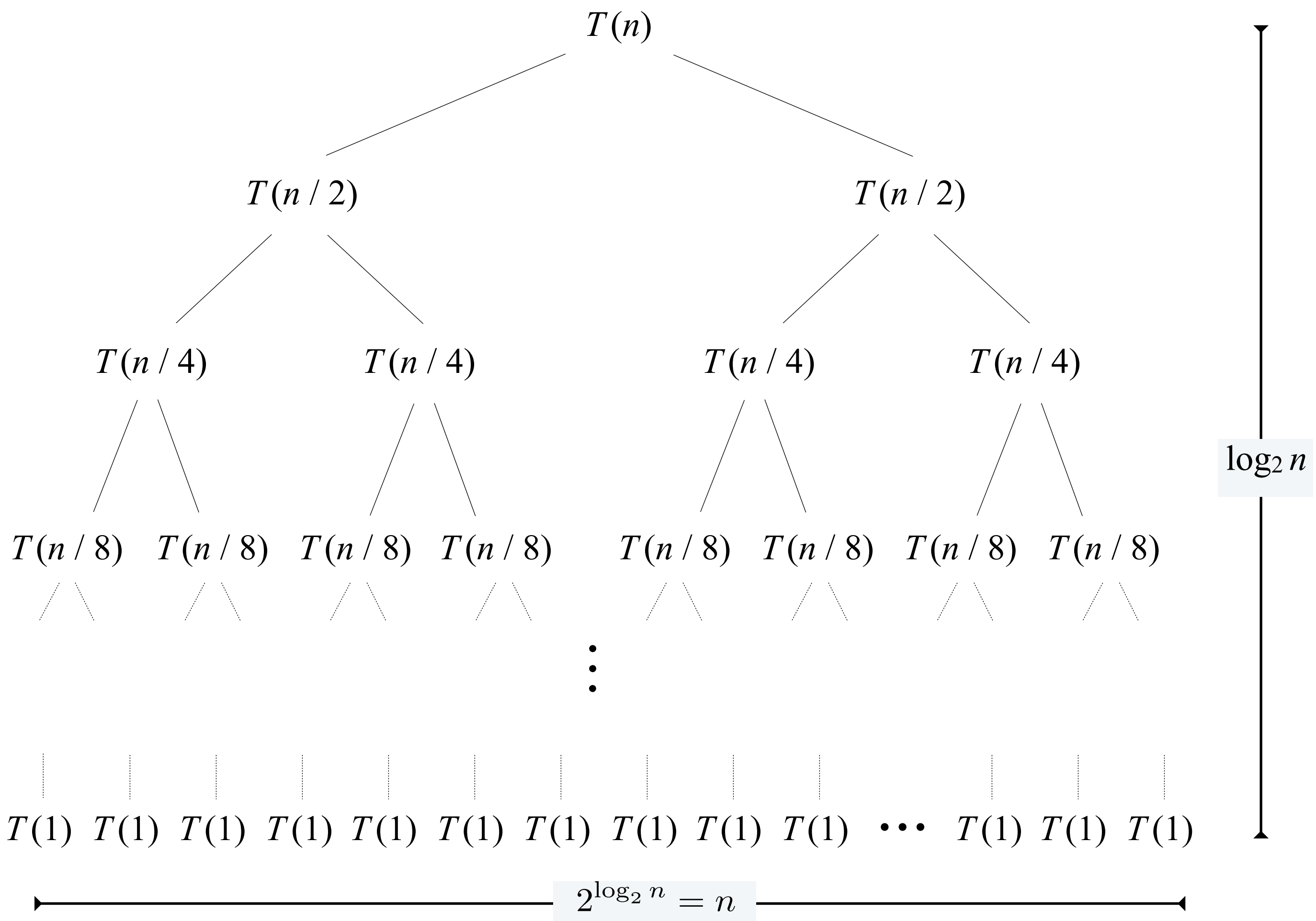

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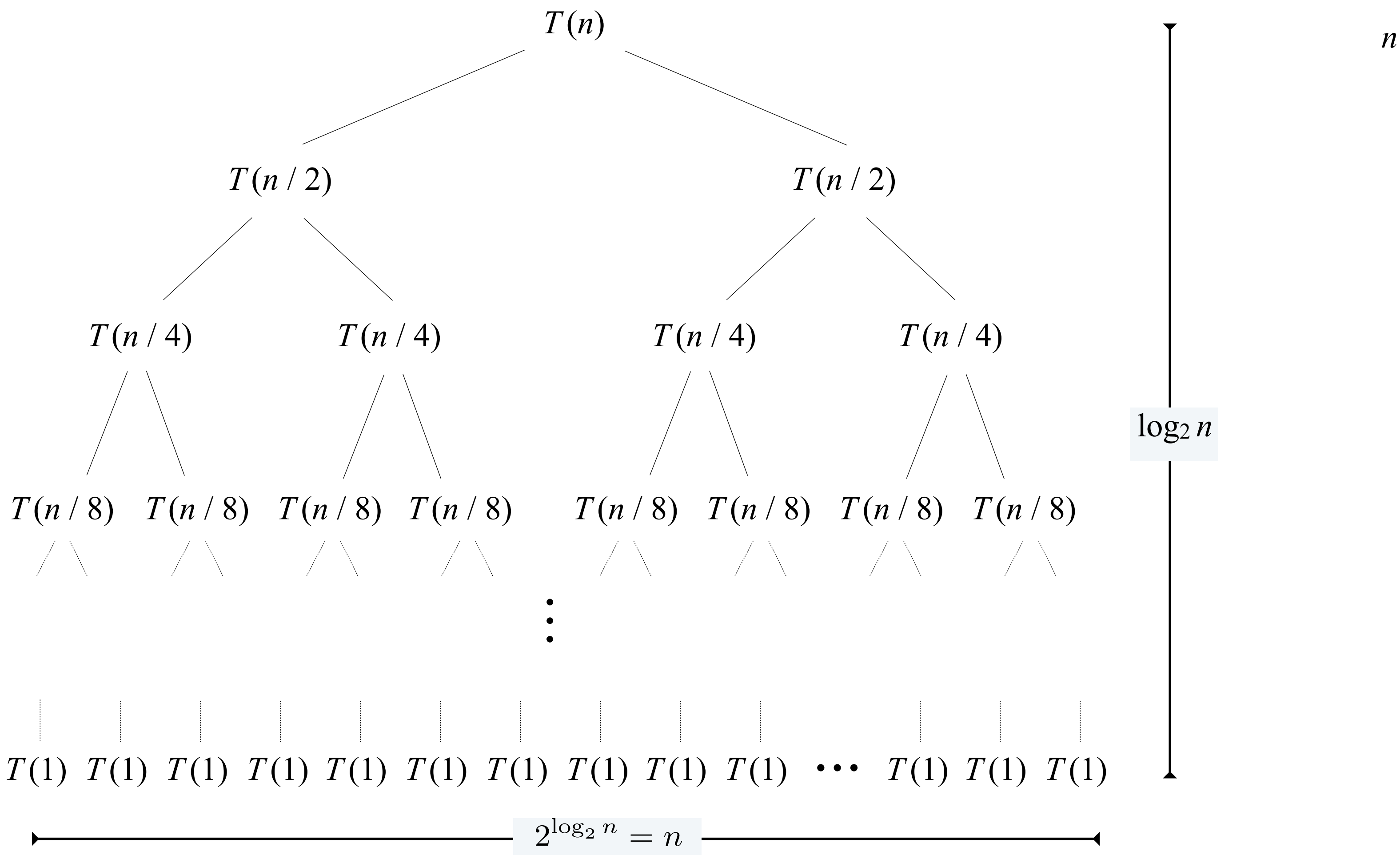

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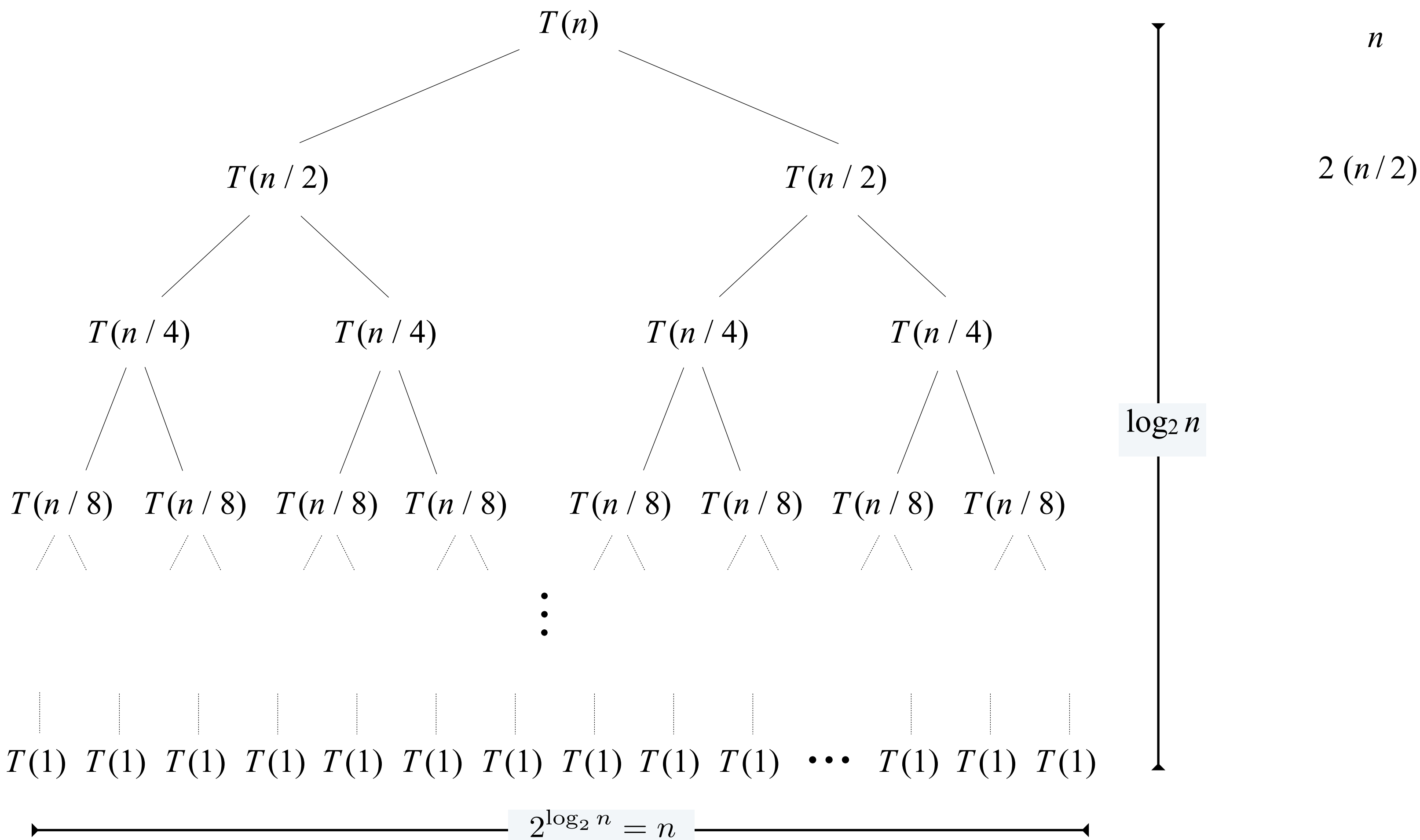

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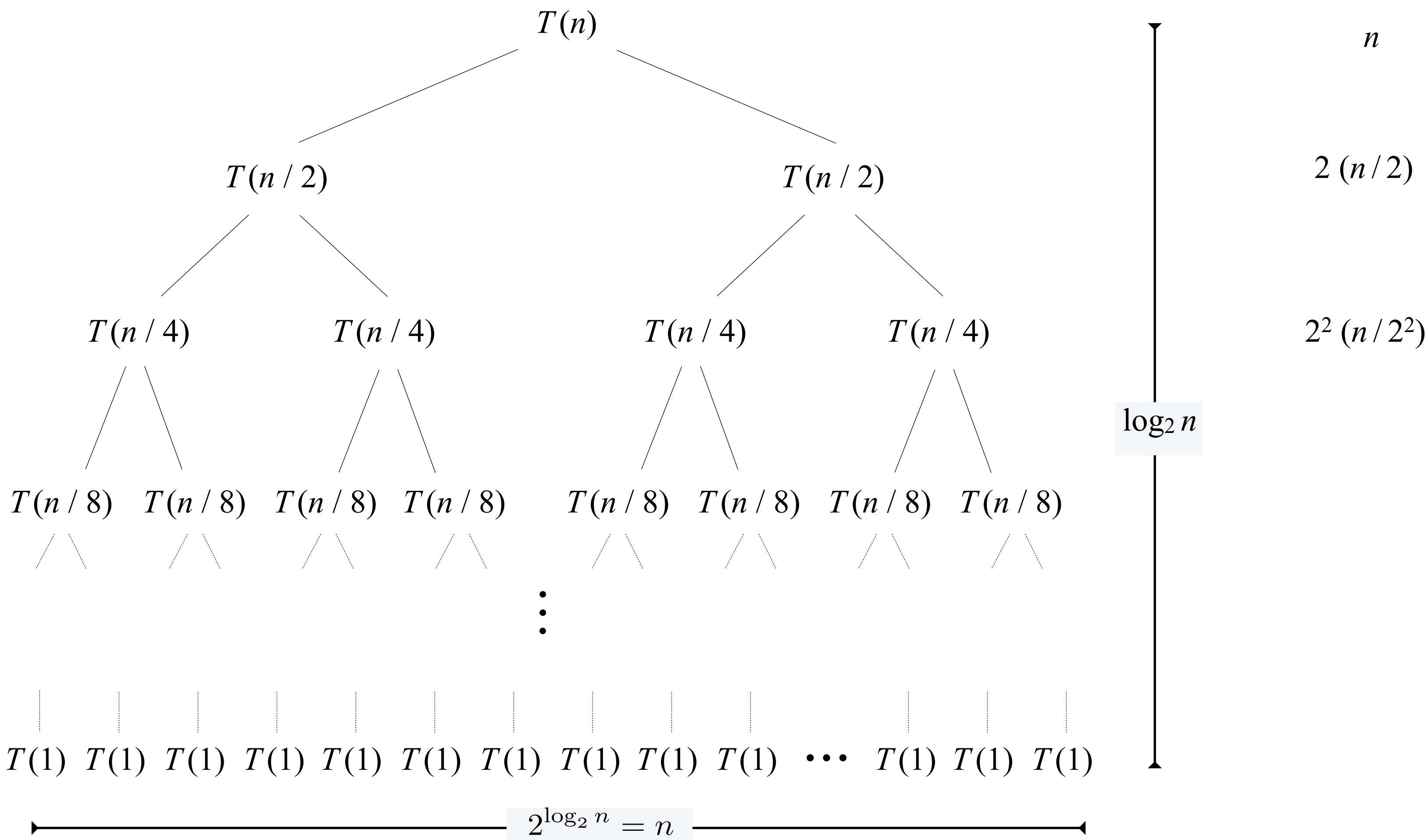

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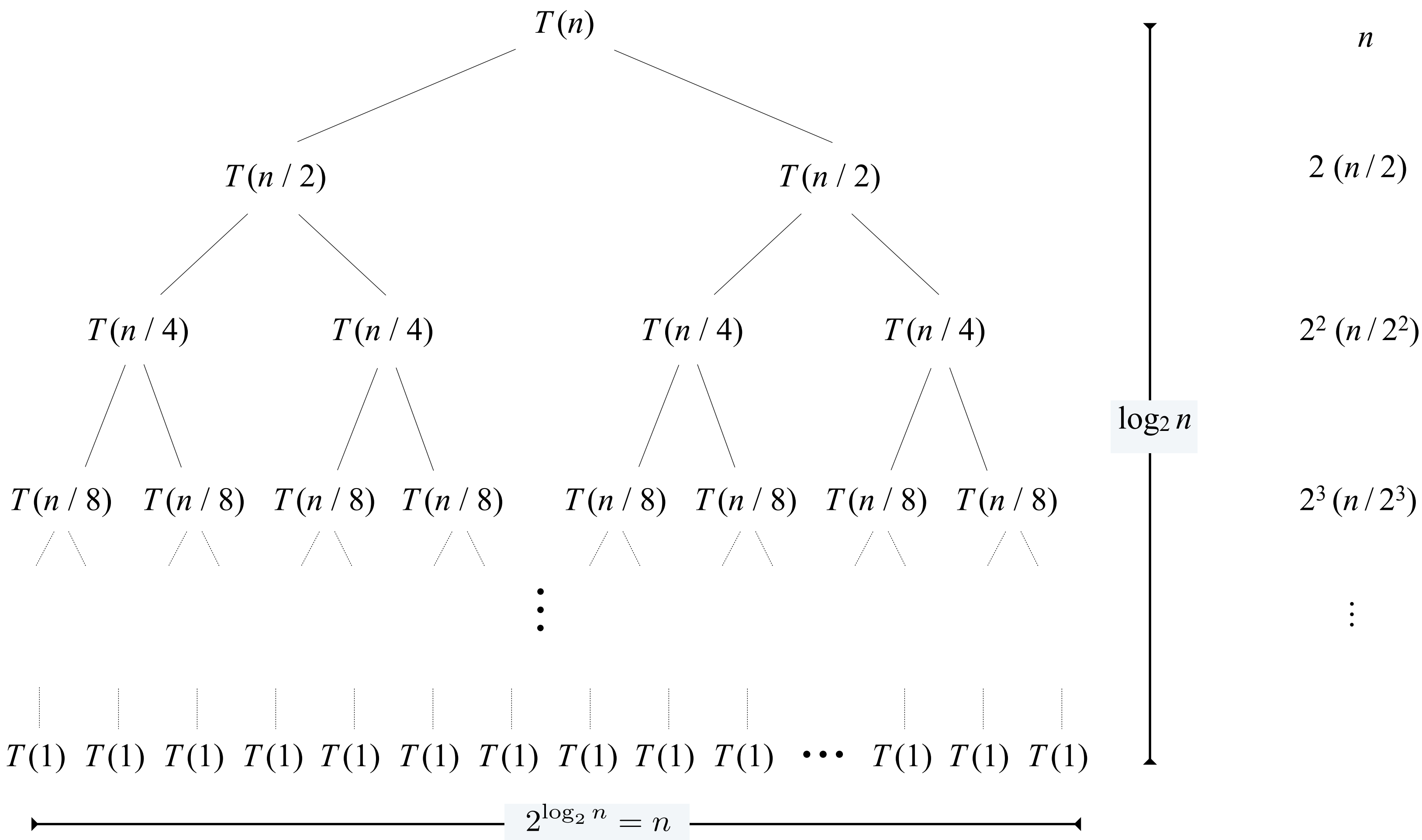

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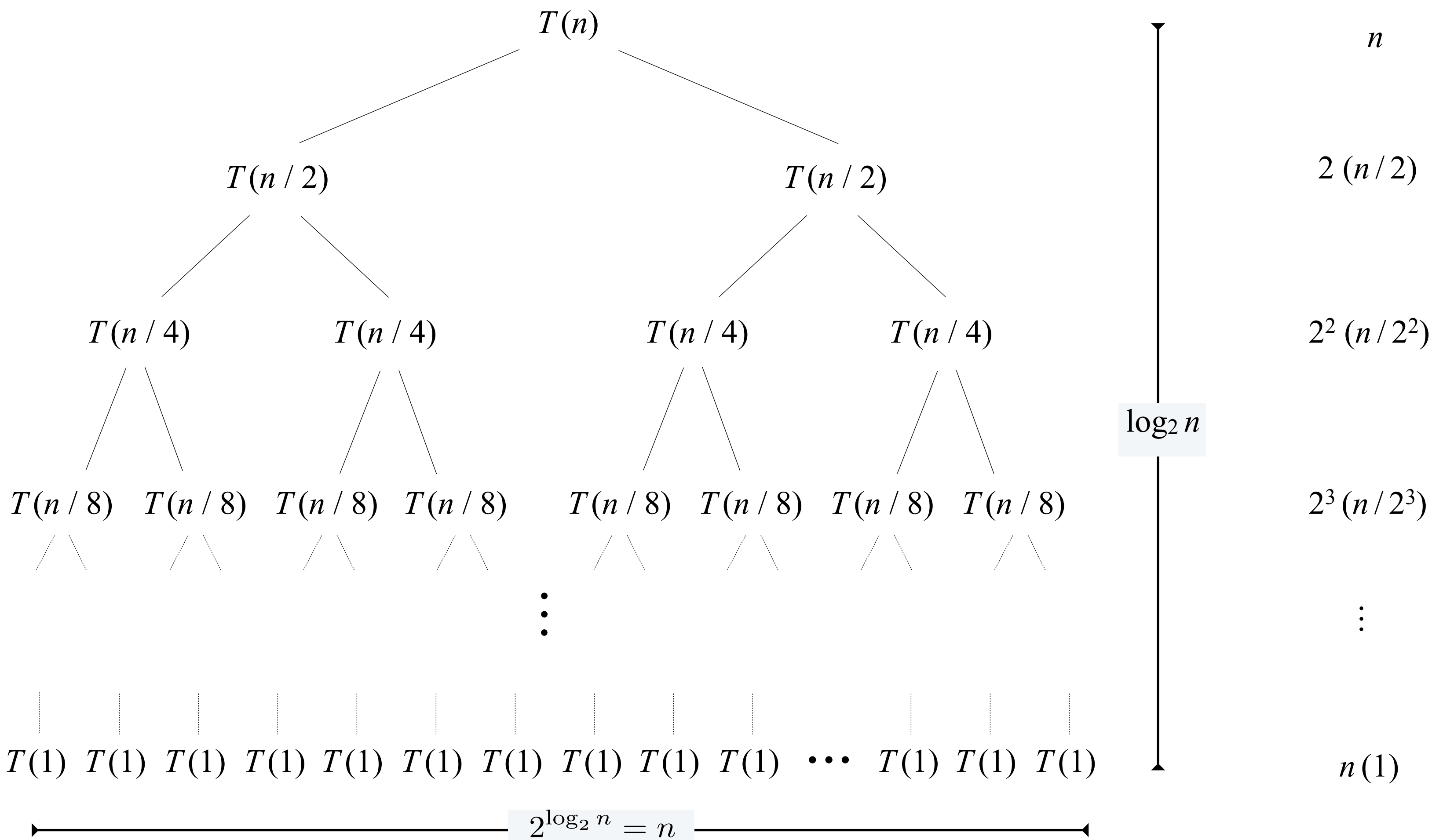

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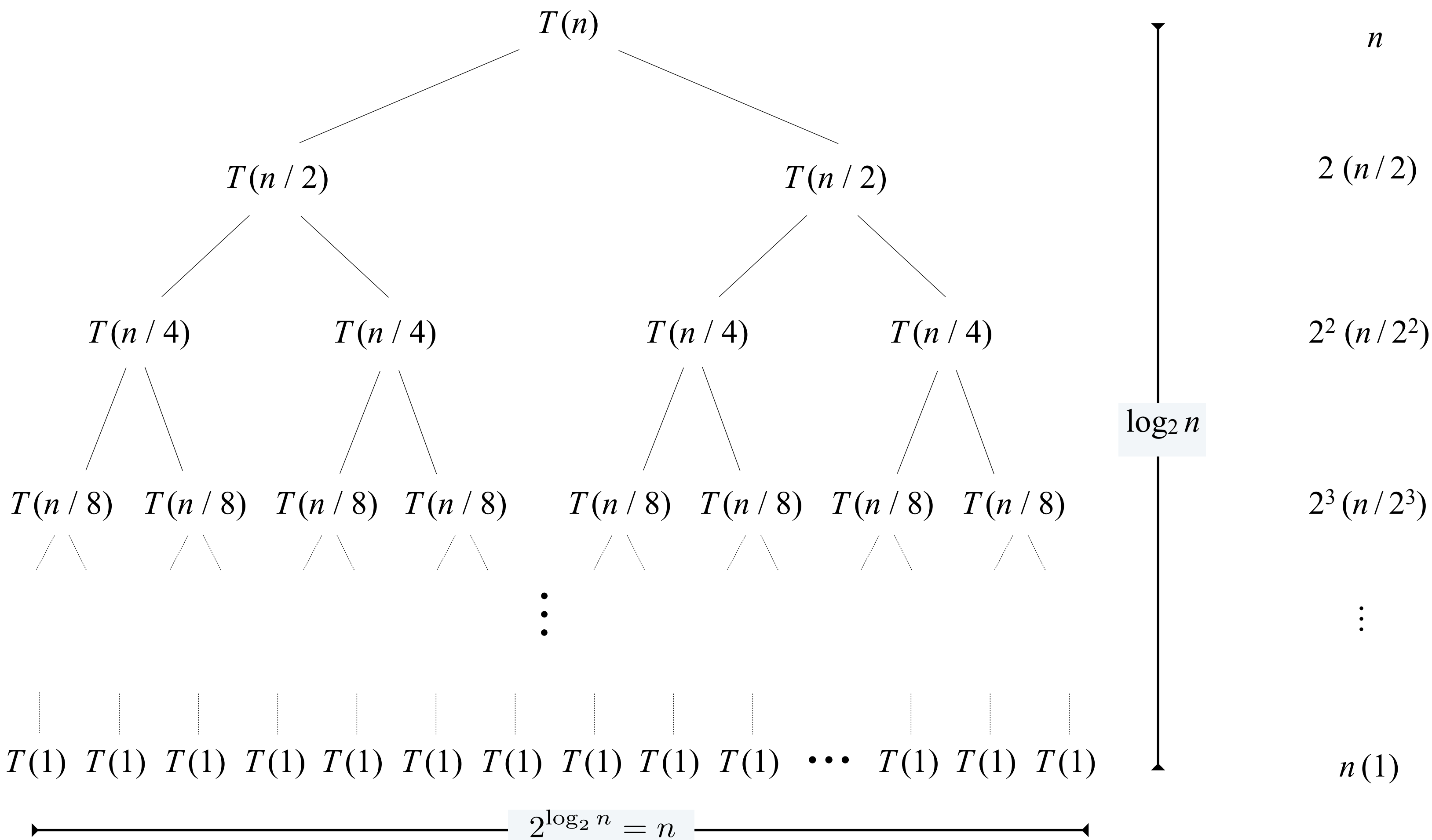

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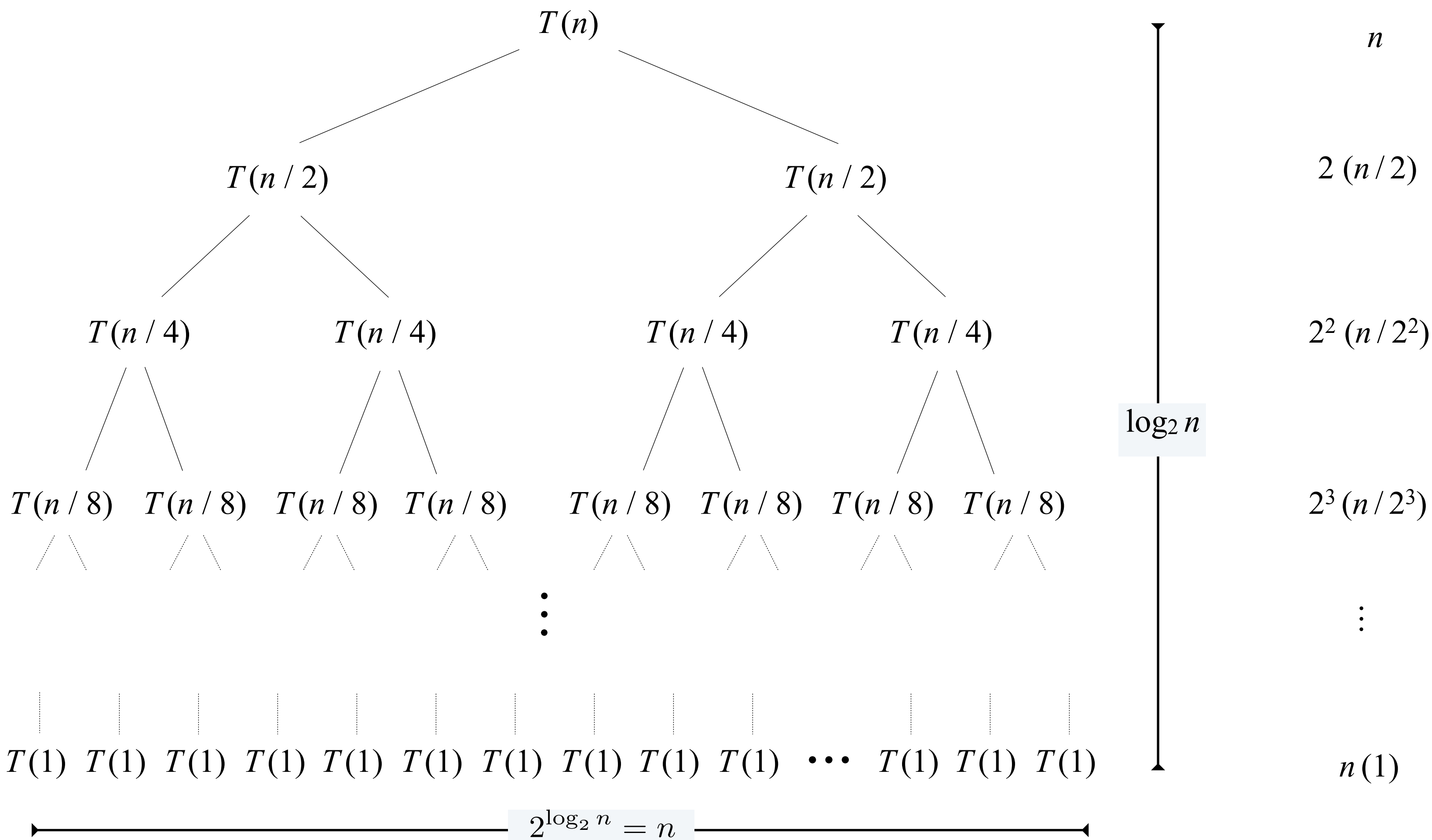

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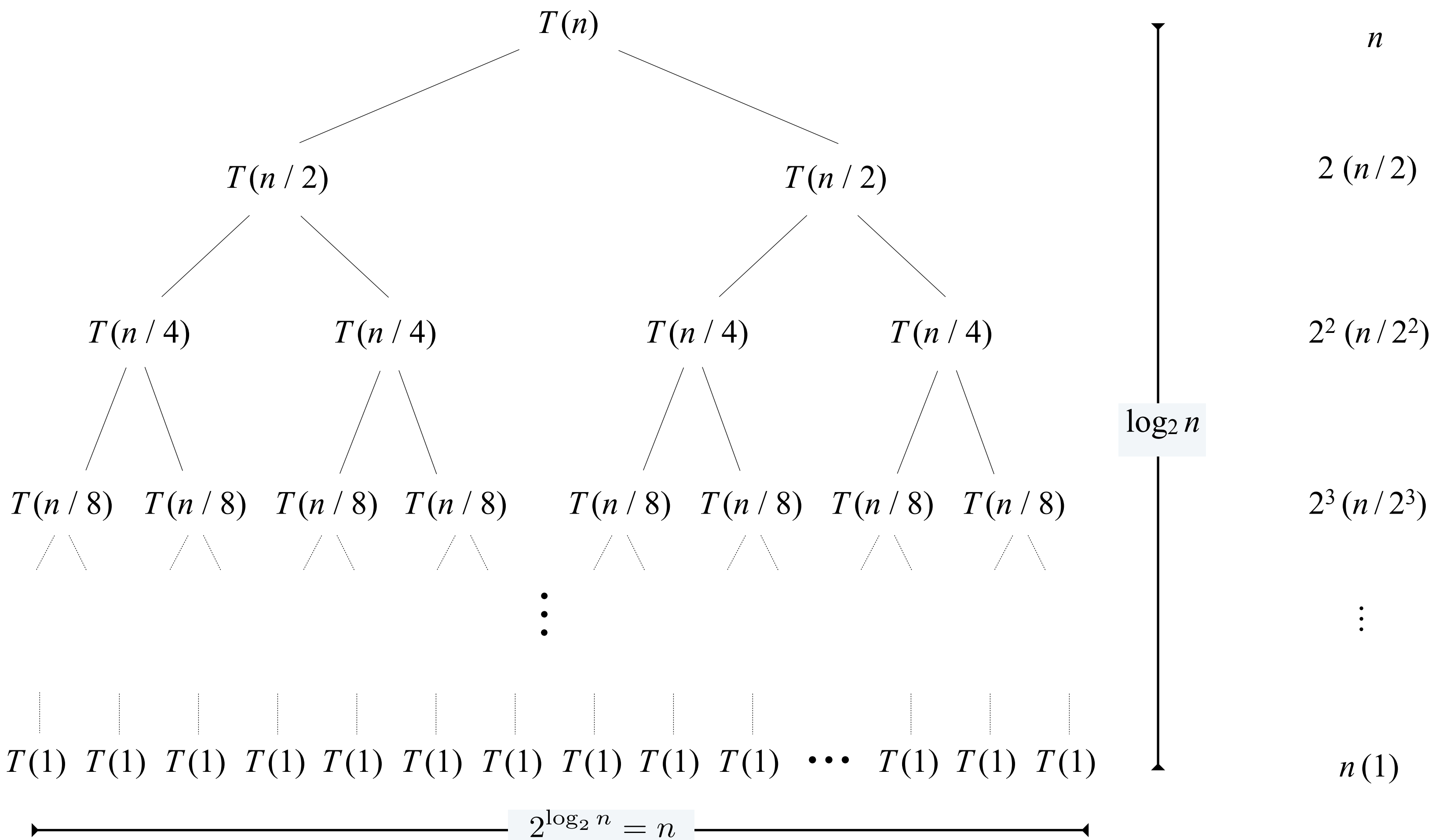

$r = 1$ $$T(n) = (1 + r + r^2 + r^3 + \ldots + r^{\log_2 n})\, n = n\,(\log_2 n + 1)$$

Case 3: Total cost dominated by cost at root

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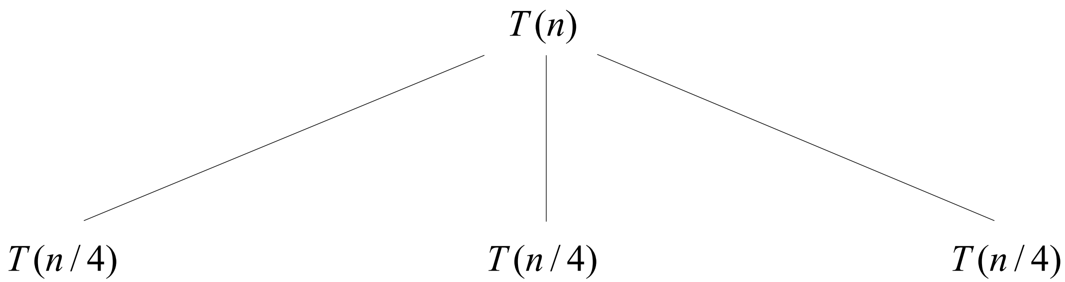

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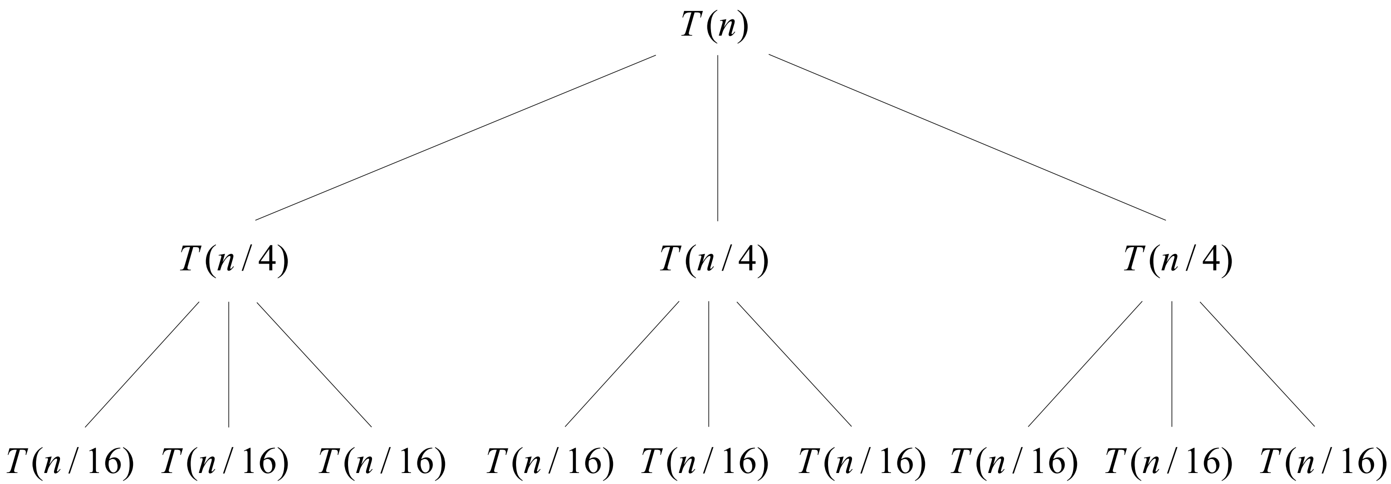

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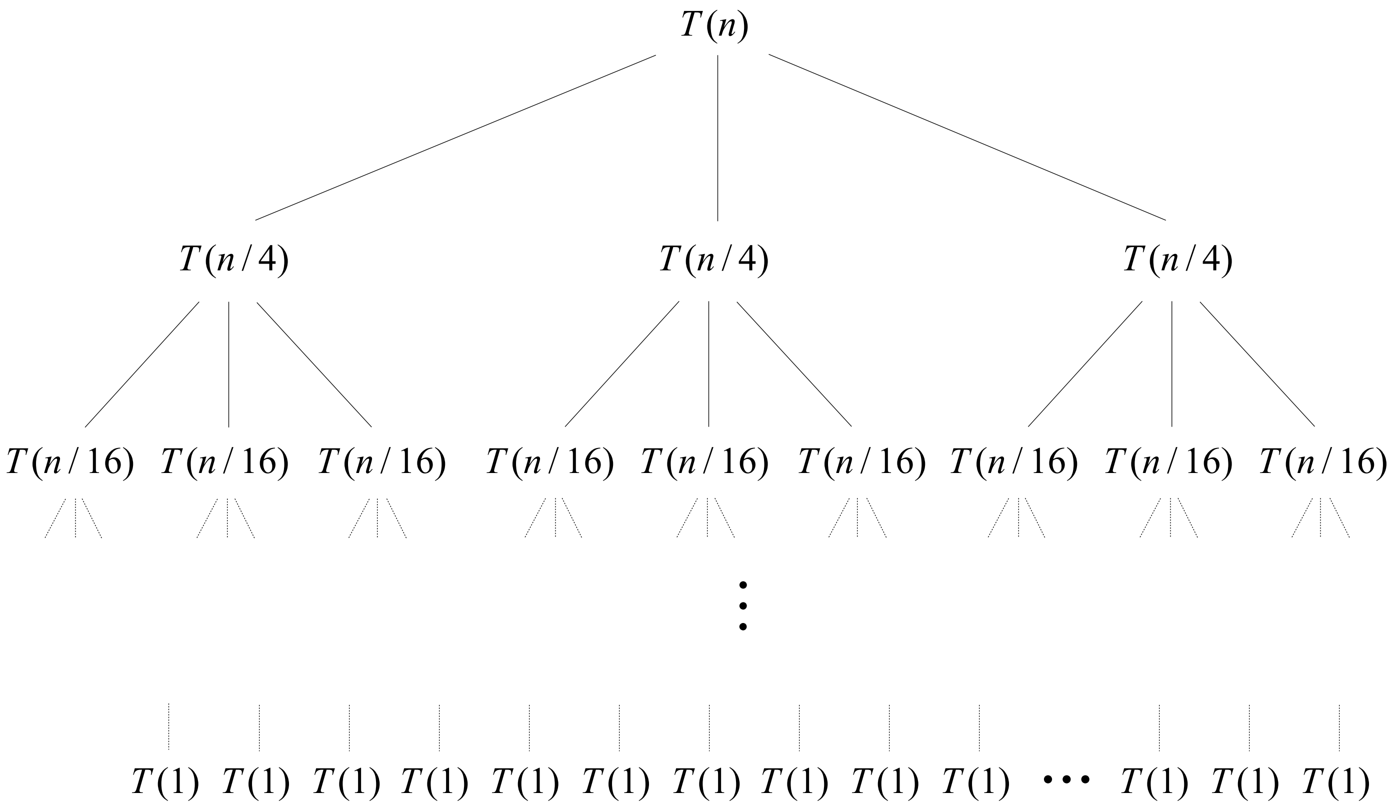

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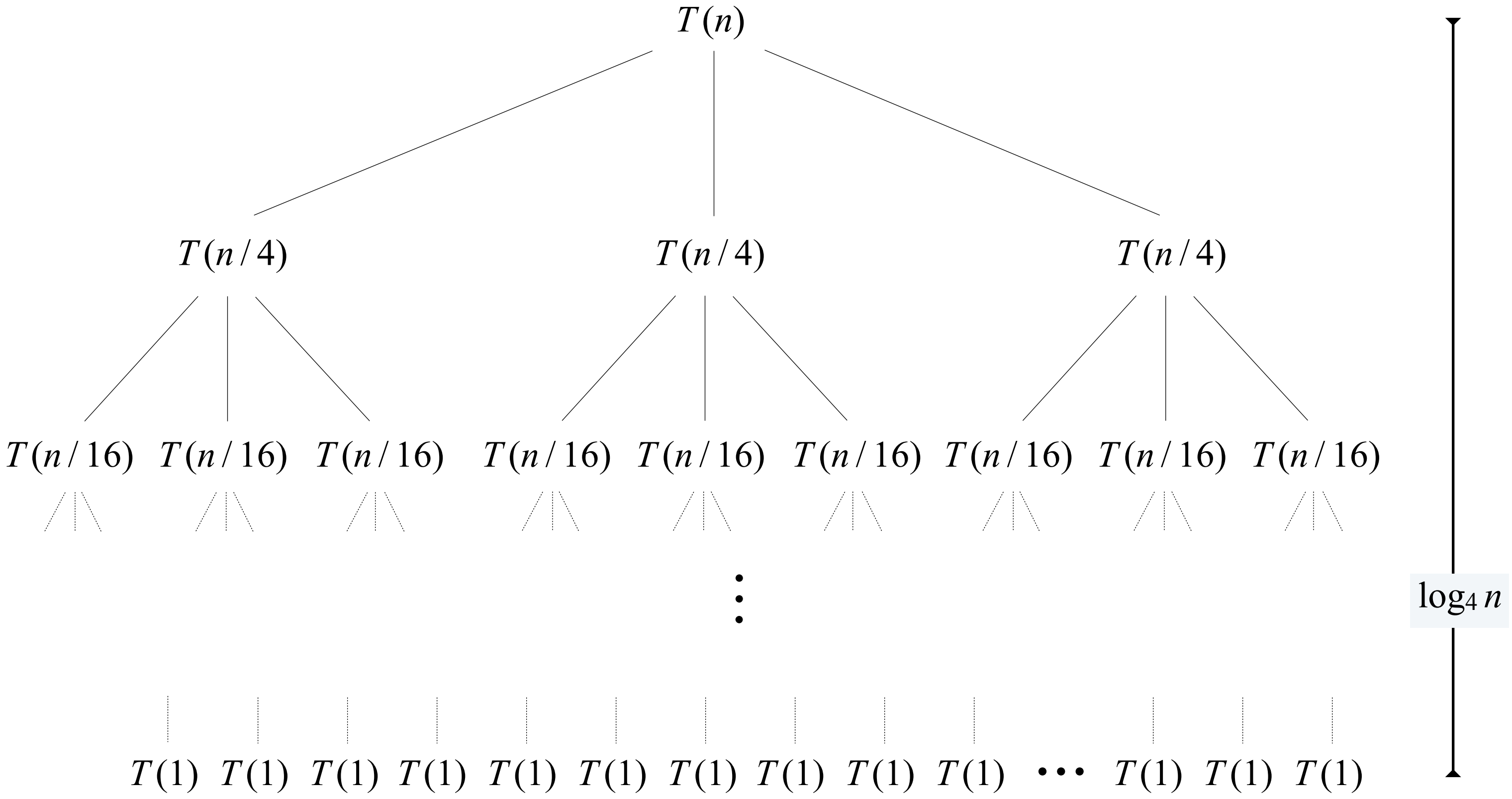

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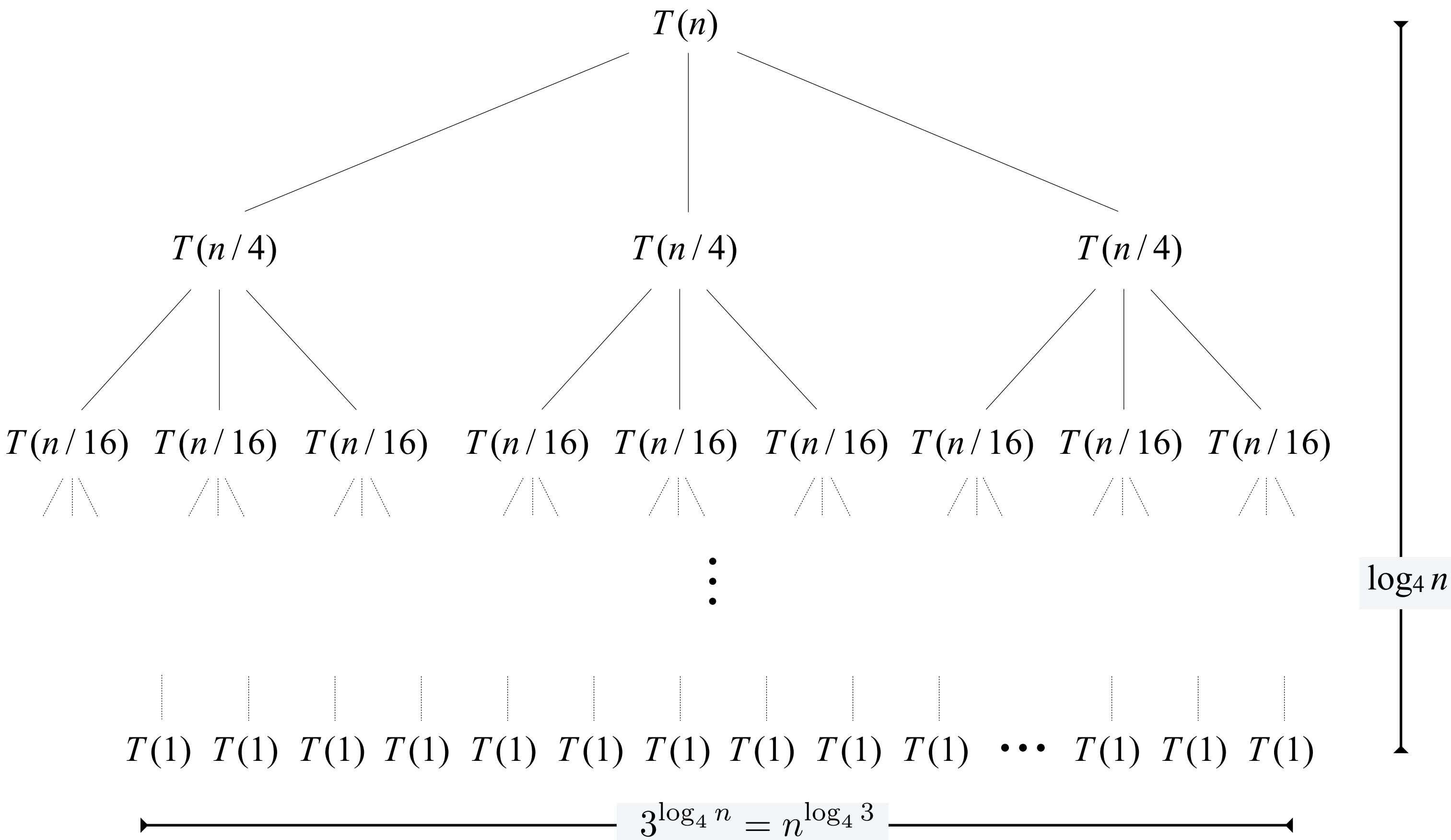

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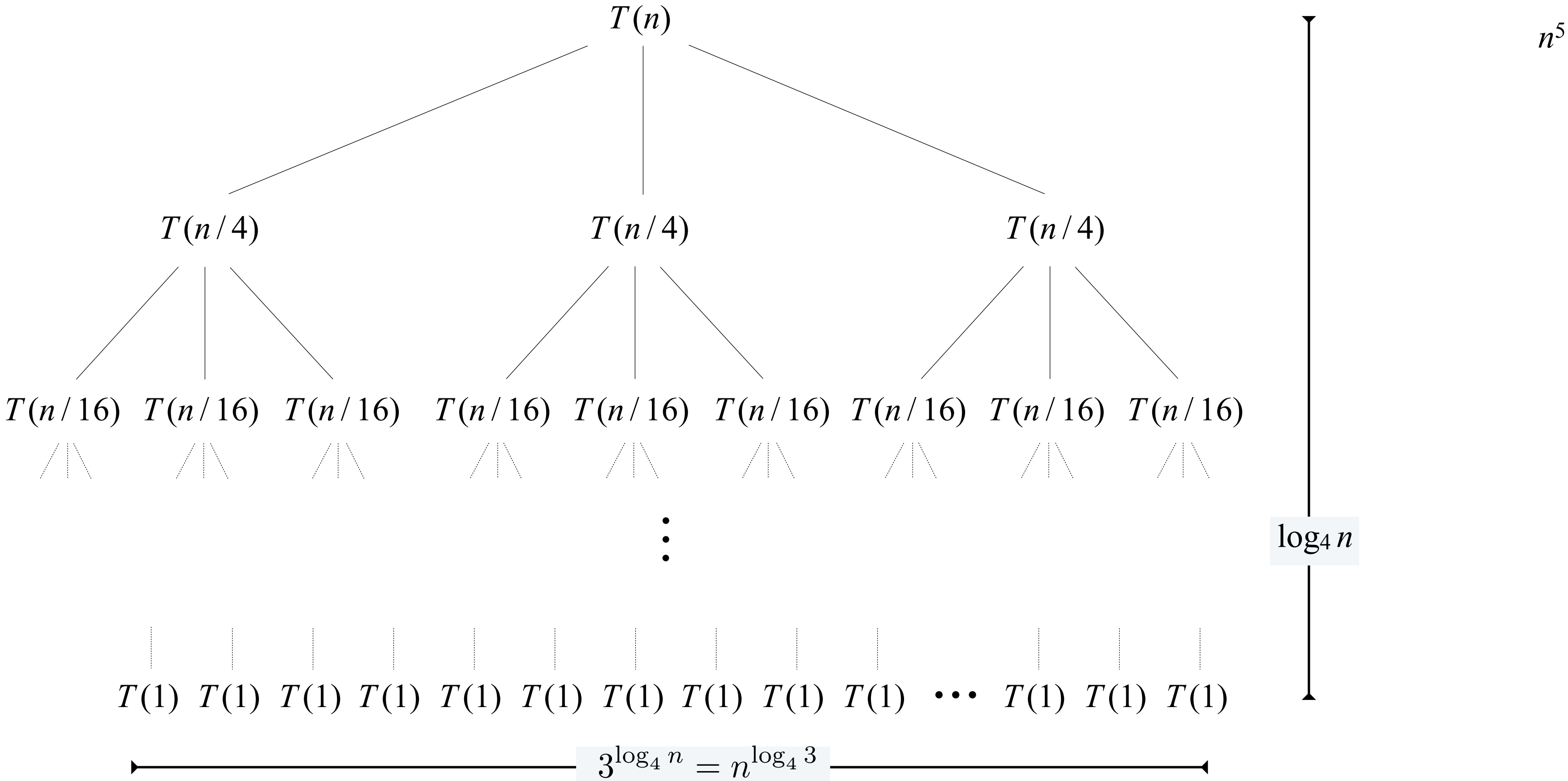

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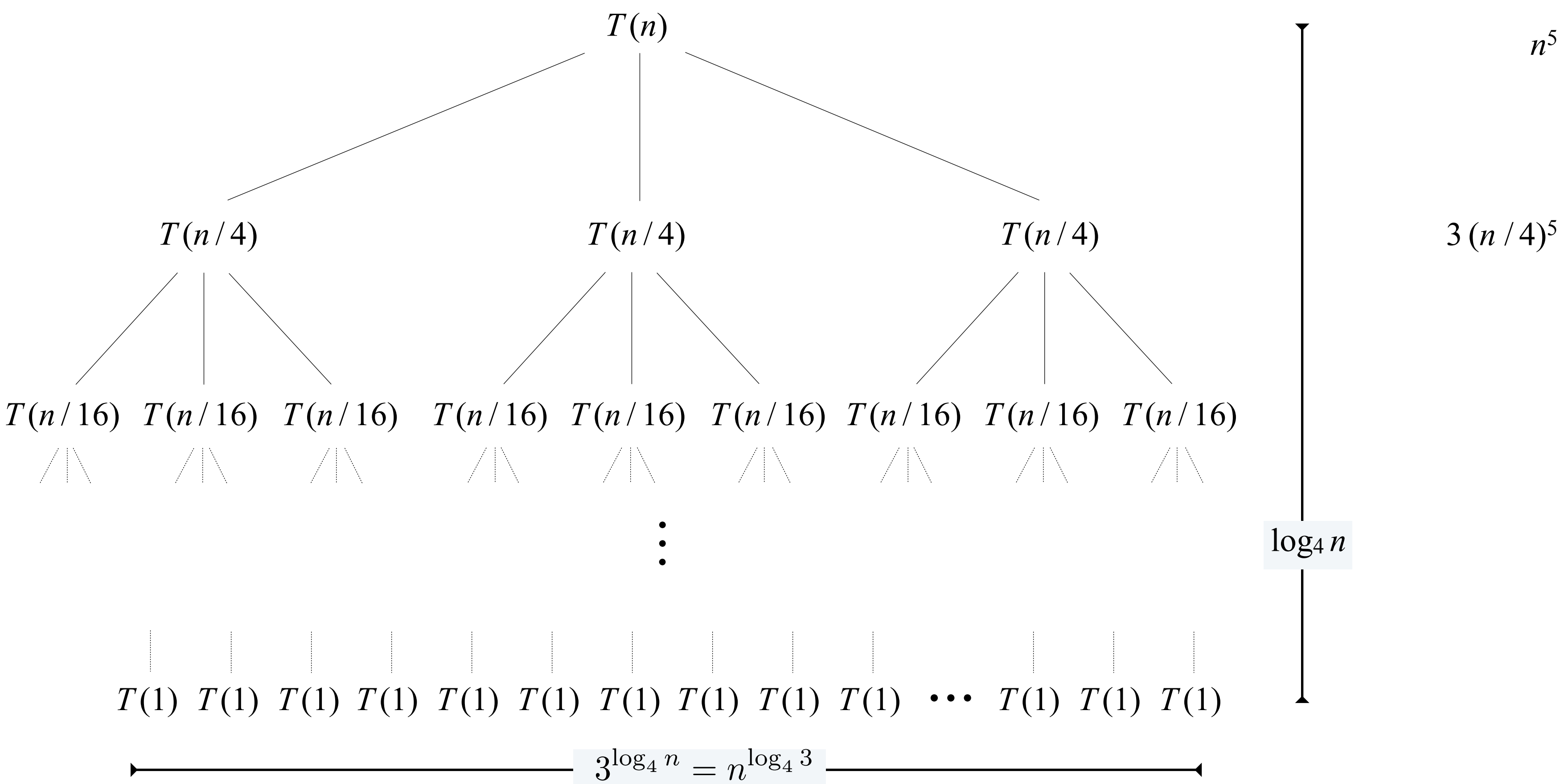

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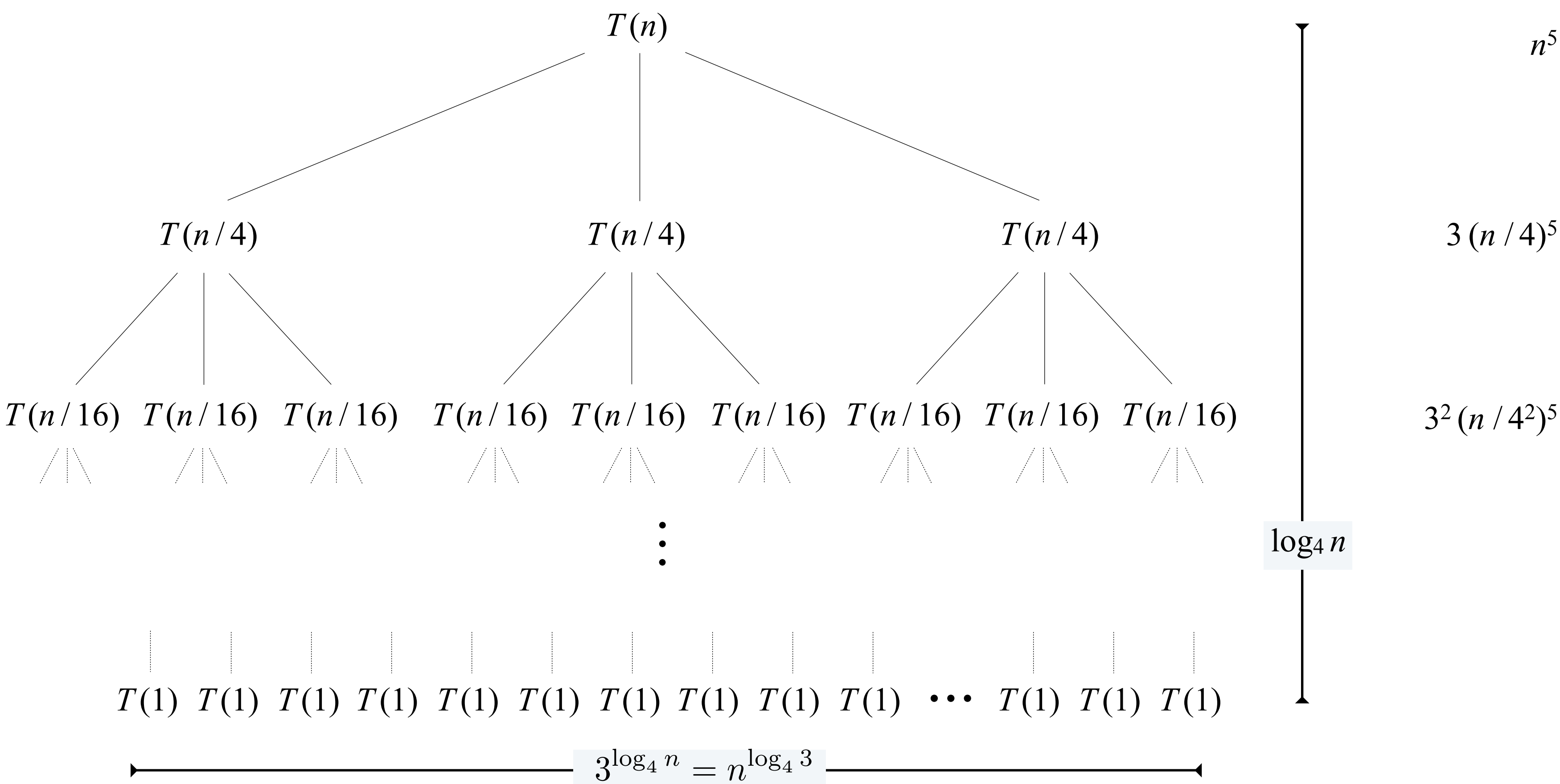

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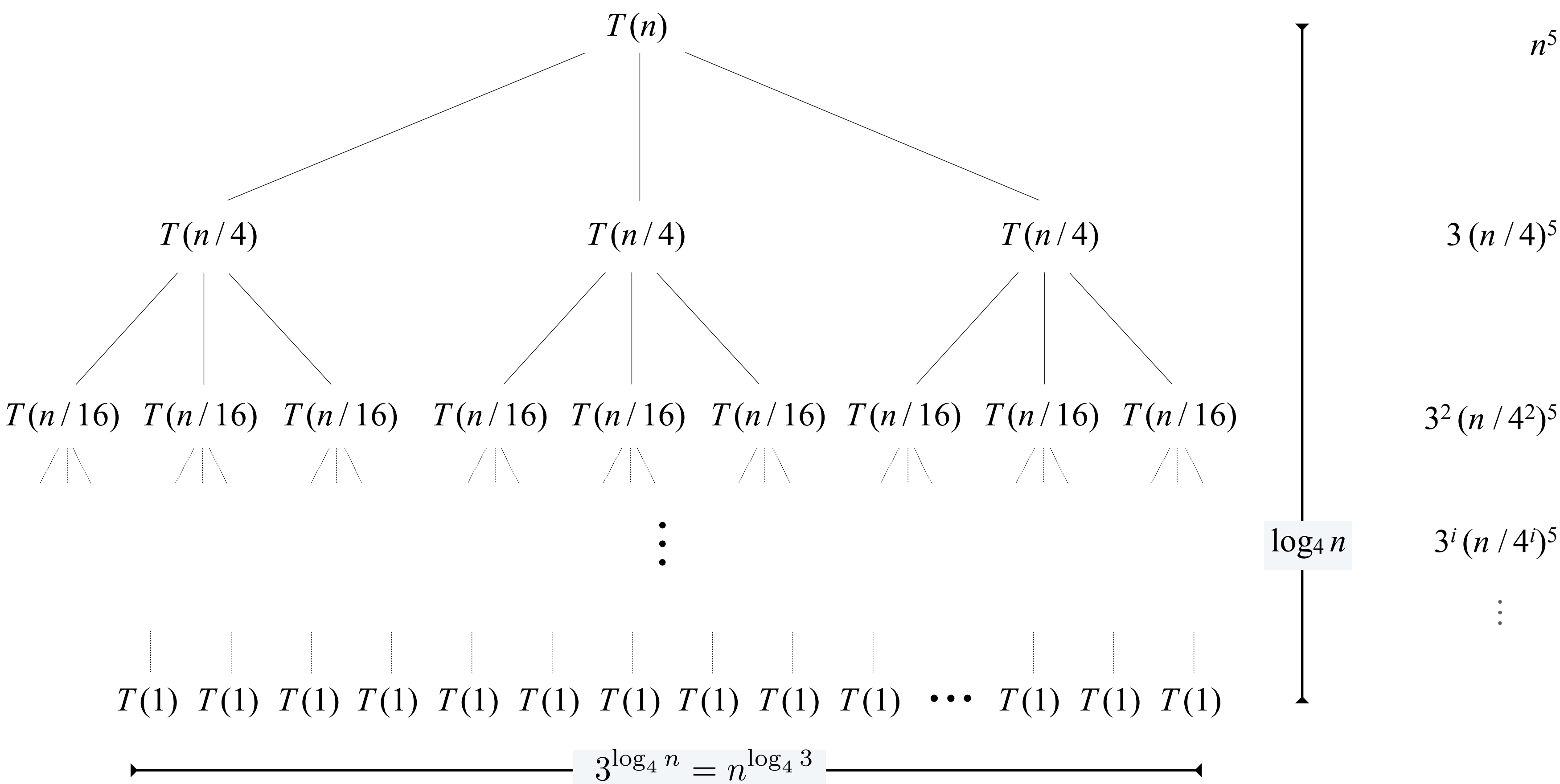

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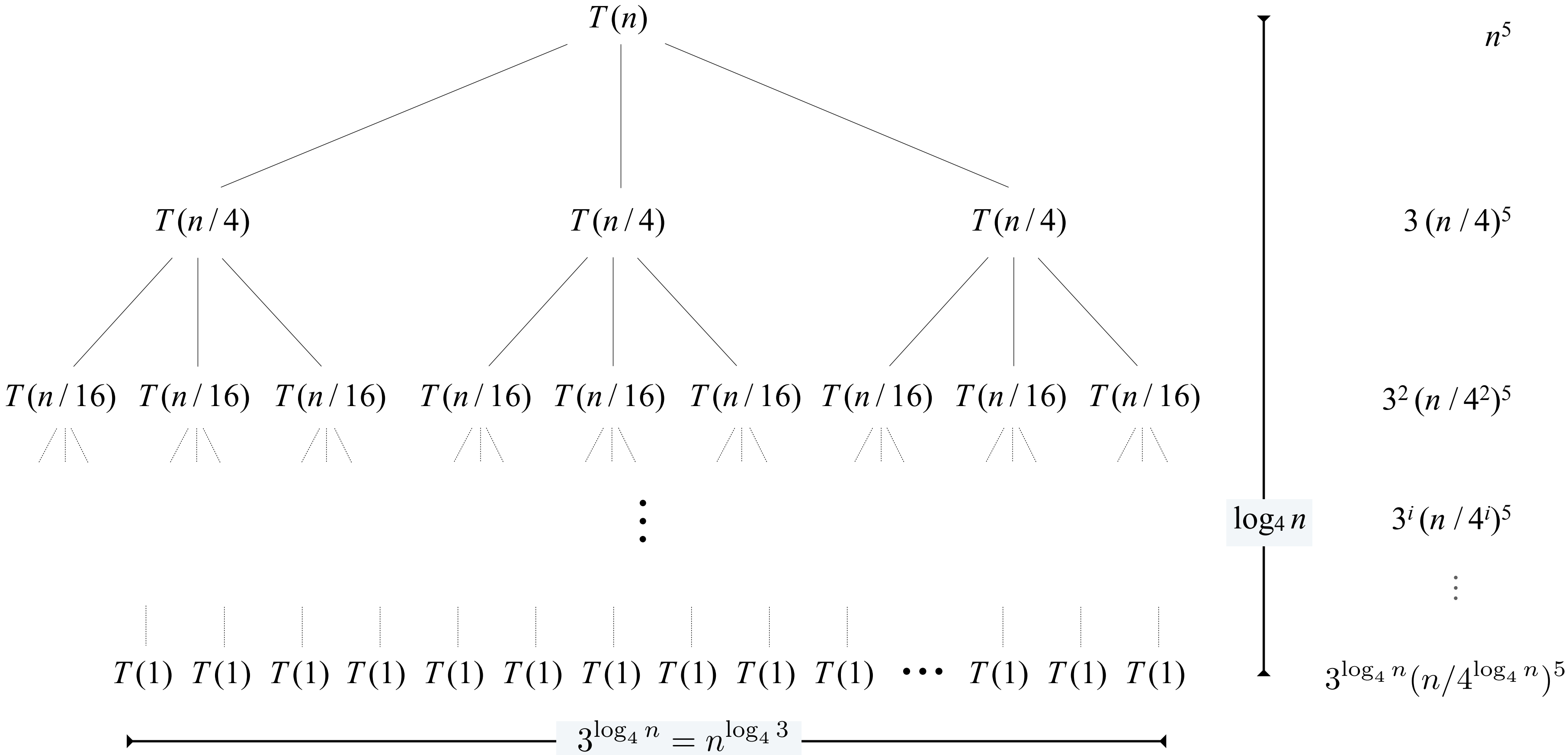

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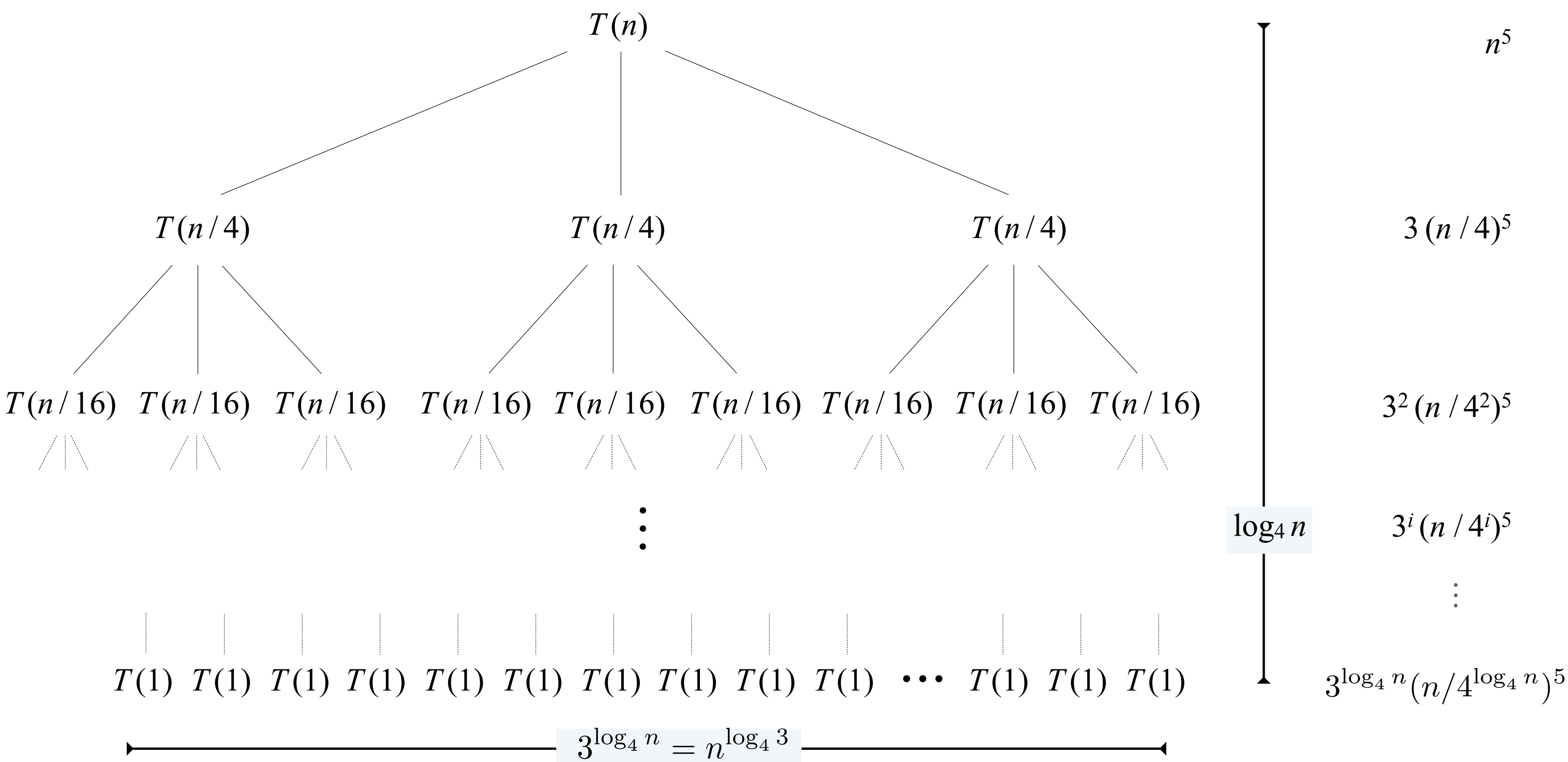

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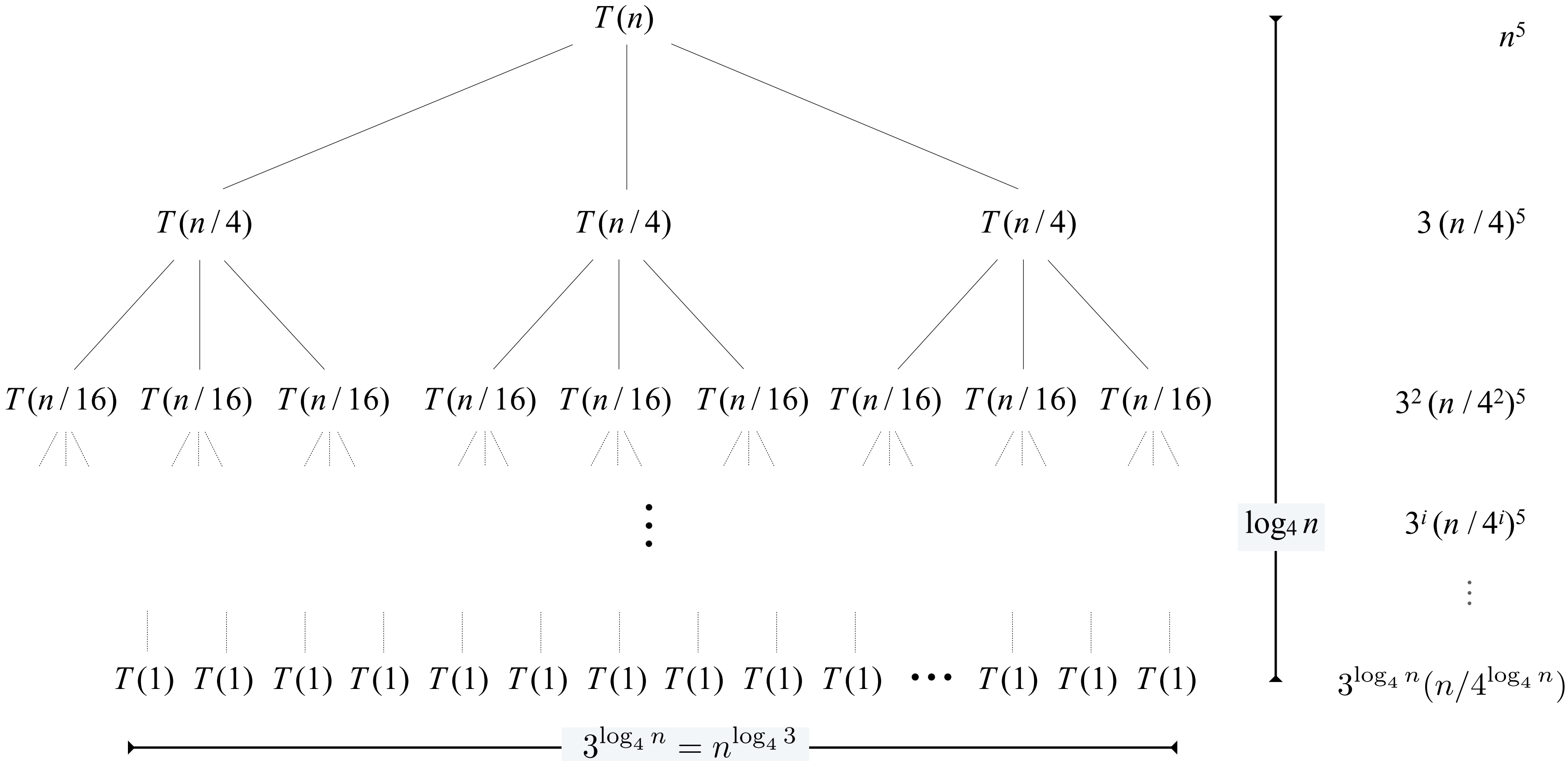

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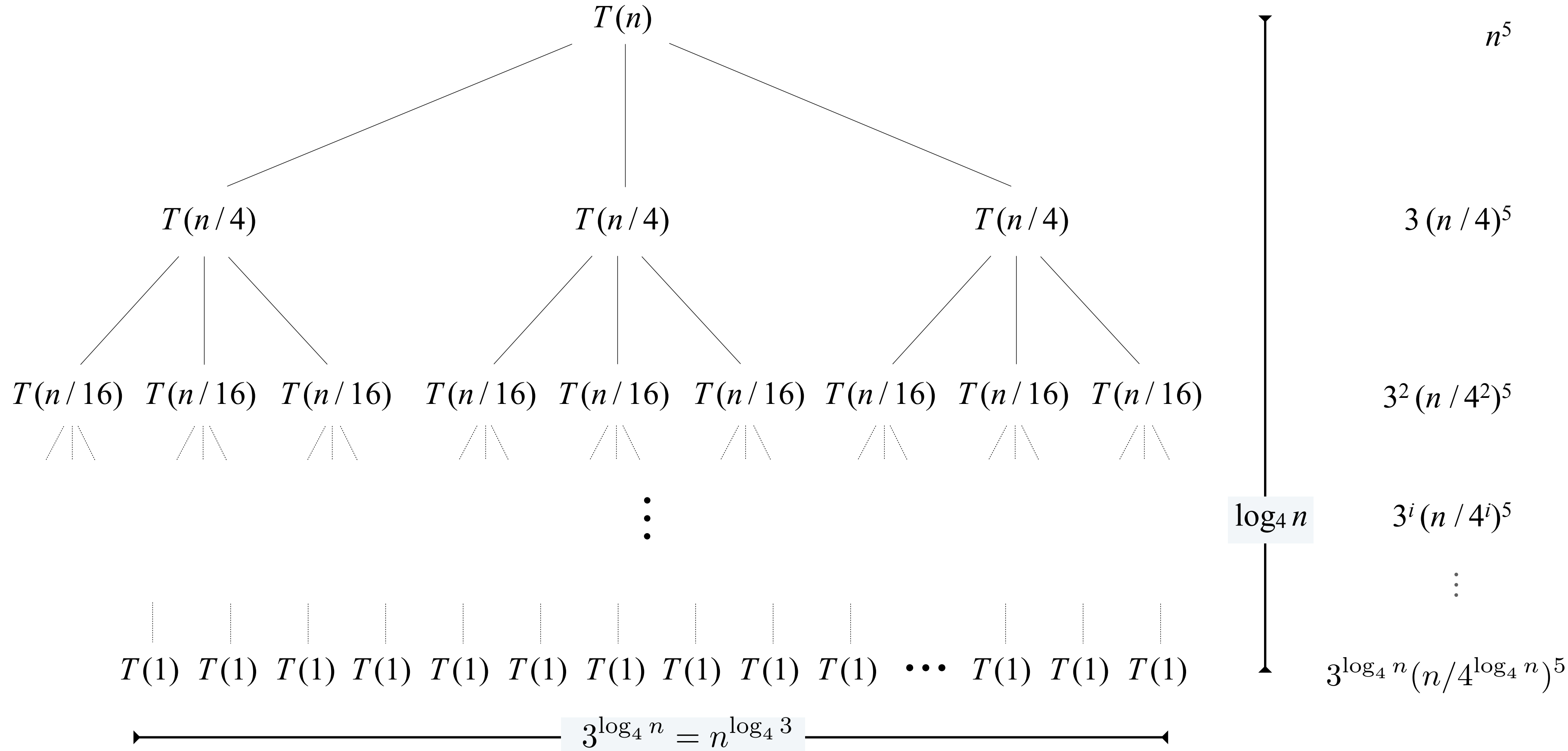

$r = 3 / 4^5 < 1$ $\quad n^5 \le T(n) = (1 + r + r^2 + r^3 + \ldots)\, n^5 \le \dfrac{1}{1-r}\, n^5$

Master theorem

Master theorem. Suppose that $T(n)$ is a function on the nonnegative integers satisfying the recurrence

$$\boxed{T(n) = a\,T\left(\frac{n}{b}\right) \;+\; f(n)}$$

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Proof sketch: Recursion tree, case analysis.

Plan for Today

Master theorem

Integer multiplication

Exponentiation

Basic arithmetic ops

Addition
INPUT: Two n-bit numbers a, b in binary
OUTPUT: (a + b) in binary format.

	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1

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Grade-school algorithm: O(n) operations.

Basic arithmetic ops

Subtraction
INPUT: Two n-bit numbers a, b in binary
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						0		
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
						0	0	0

Grade-school algorithm:

Basic arithmetic ops

Subtraction
INPUT: Two n-bit numbers a, b in binary
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+	0	1	1	1	1	1	0	1
		1	0	1	1	0	0	0

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	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	1	0	0	0

Grade-school algorithm: O(n) operations.

Basic arithmetic ops

Addition
INPUT: Two n-bit numbers a, b in binary
OUTPUT: (a + b) in binary format.

Subtraction
INPUT: Two n-bit numbers a, b in binary
OUTPUT: (a - b) in binary format.

Grade-school algorithms: $\Theta(n)$ operations.

Asymptotically optimal.

Basic arithmetic ops

Multiplication
INPUT: Two n-bit numbers a, b in binary
OUTPUT: (a x b) in binary format.

	1	1	0	1	0	1	0	1
x	0	1	1	1	1	1	0	1
?	?	?	?	?	?	?	?	?

Grade-school algorithm:

Basic arithmetic ops

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?	?	?	?	?	?	?	?	?

Grade-school algorithm: $O(n^2)$ operations.

Conjecture [Kolmogorov 1952]: This is optimal!

Basic arithmetic ops

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	1	1	0	1	0	1	0	1
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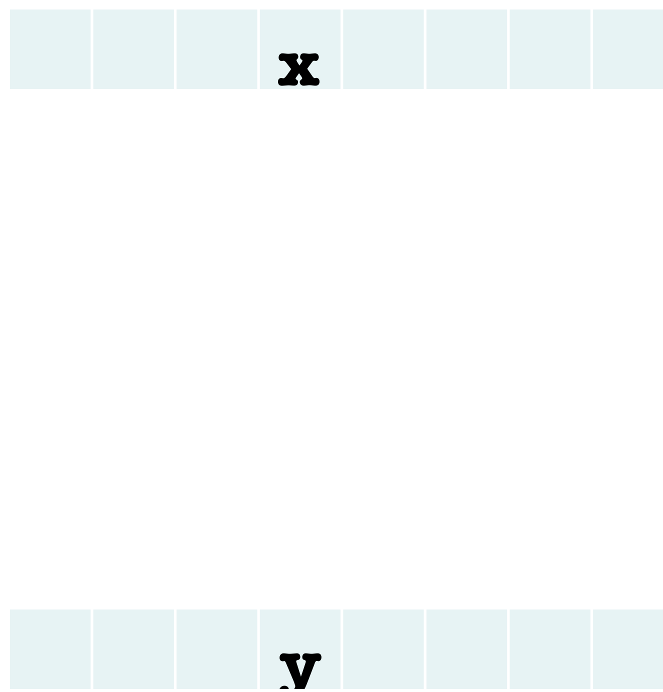
?	?	?	?	?	?	?	?	?

Grade-school algorithm: $O(n^2)$ operations.

Theorem [Karatsuba 1960]: Conjecture false!

Divide-and-conquer multiplication



Divide-and-conquer multiplication

1. Divide **x** and **y** into high and low-order bits

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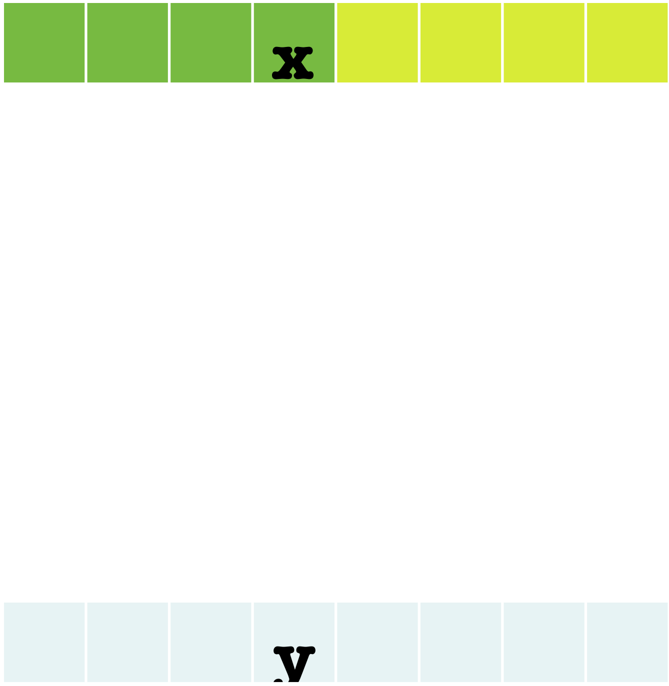

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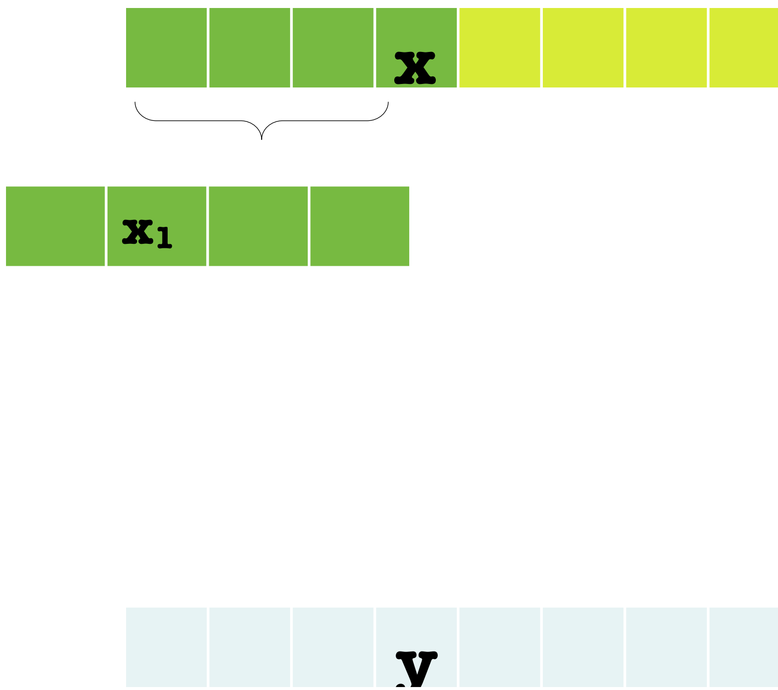

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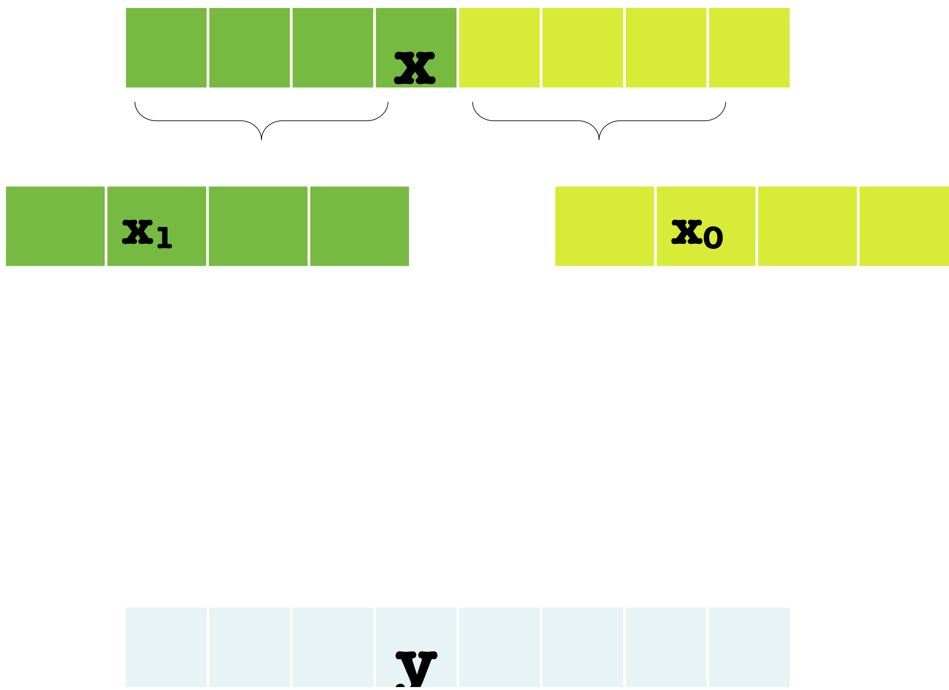

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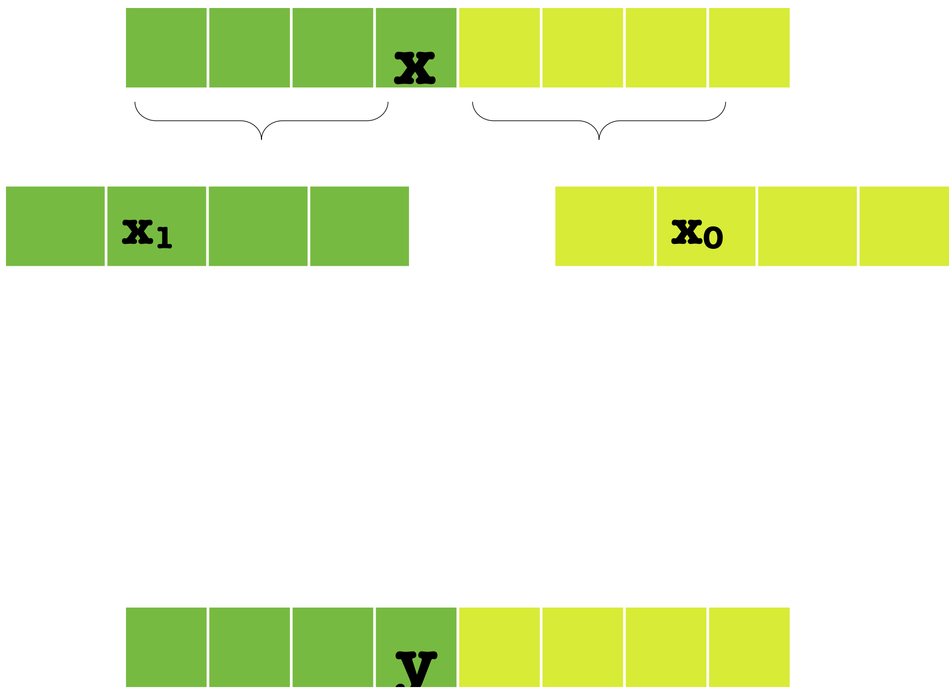

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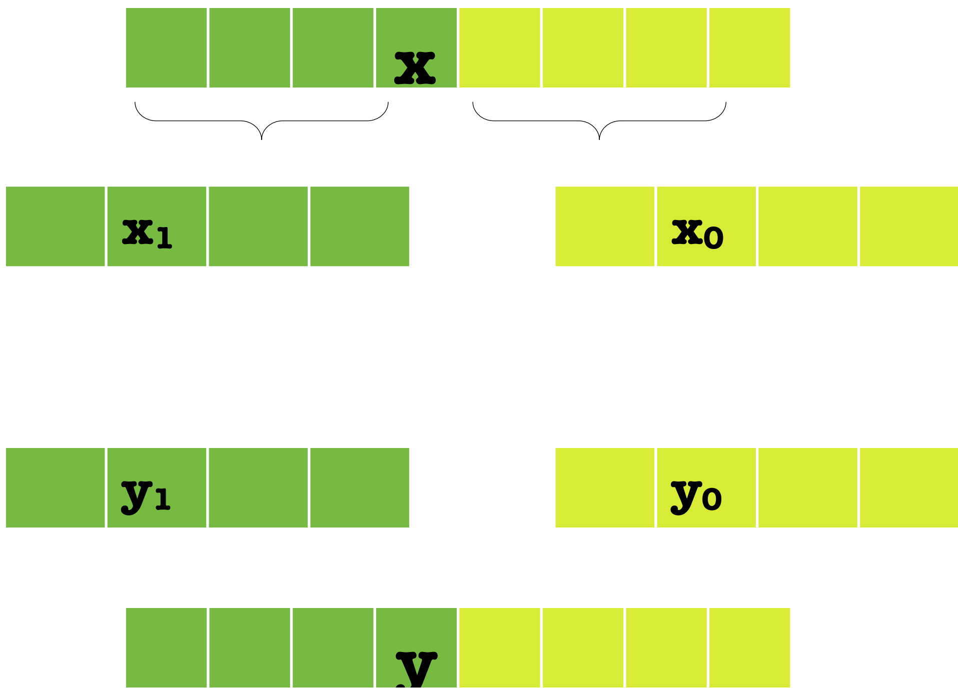
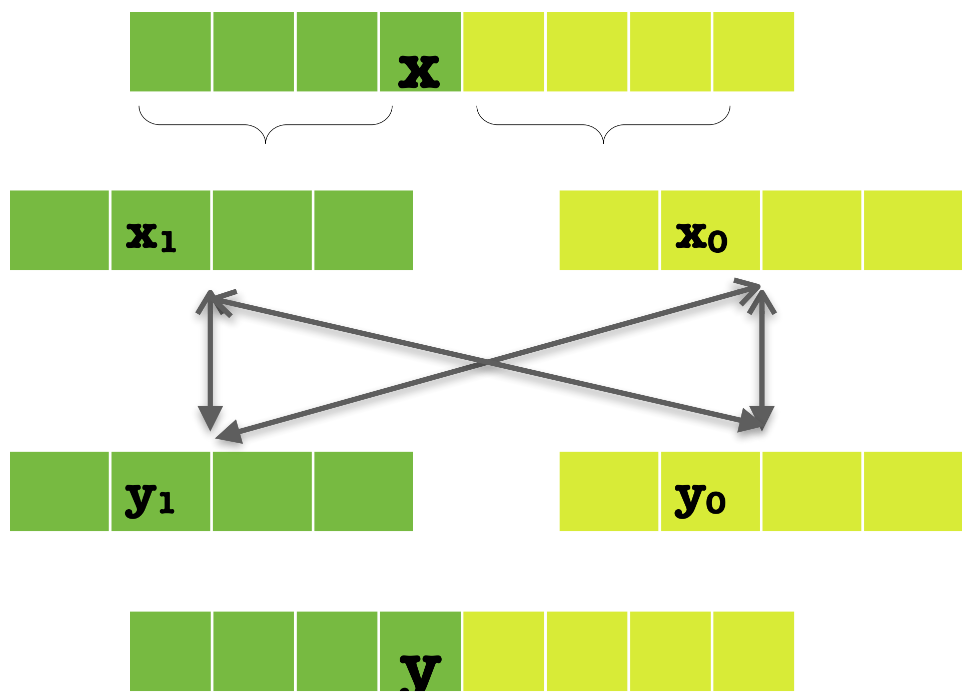


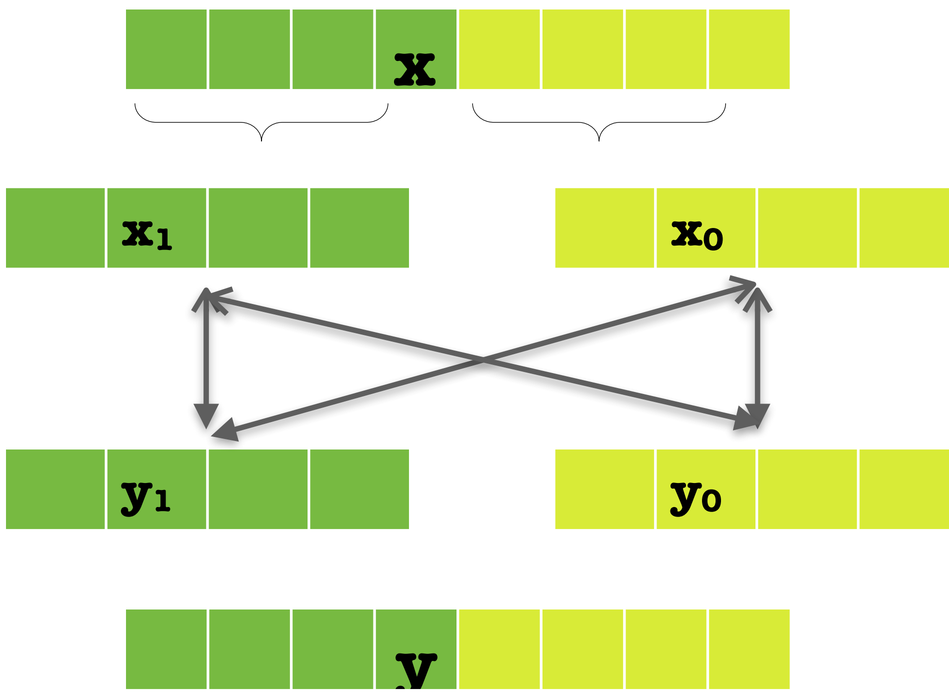
Divide-and-conquer multiplication

1. Divide **x** and **y** into high and low-order bits
2. Multiply four (n/2)-bit integers recursively.



Divide-and-conquer multiplication

1. Divide **x** and **y** into high and low-order bits
2. Multiply four (n/2)-bit integers recursively.
3. Shift and add to obtain **x*y**.



Divide-and-conquer multiplication

1. **Divide `x` and `y` into high and low-order bits**
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$$x = 2^{n/2}x_1 + x_0$$

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$$
\boxed{
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x &= 2^{n/2} x_1 + x_0 \\
y &= 2^{n/2} y_1 + y_0 \\
x \cdot y &= (2^{n/2} x_1 + x_0) \cdot (2^{n/2} y_1 + y_0) \\
&= 2^n (x_1 \cdot y_1) + 2^{n/2}((x_0 \cdot y_1) + (x_1 \cdot y_0)) + (x_0 \cdot y_0)
\end{aligned}
}
$$

Divide-and-conquer multiplication

1. **Divide $\mathbf{x}$ and $\mathbf{y}$ into high and low-order bits**
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x \cdot y &= (2^{n/2}x_1 + x_0) \cdot (2^{n/2}y_1 + y_0) \\
&= 2^n(x_1 \cdot y_1) + 2^{n/2}((x_0 \cdot y_1) + (x_1 \cdot y_0)) + (x_0 \cdot y_0)
\end{aligned}
$$

Ex. $x = \underbrace{1\ 0\ 0\ 0}_{x1}\ \underbrace{1\ 1\ 0\ 1}_{x0} \qquad y = \underbrace{1\ 1\ 1\ 0}_{y1}\ \underbrace{0\ 0\ 0\ 1}_{y0}$

Divide-and-conquer multiplication

Proposition. The divide-and-conquer multiplication algorithm requires $\Theta(n^2)$ bit operations to multiply two n-bit integers.

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Pf. Apply first-case of the master theorem to the recurrence: a = 4, b = 2, f(n) = O(n).

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Much ado about nothing??

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&= 2^n(x_1 \cdot y_1) + 2^{n/2}((x_0 \cdot y_1) + (x_1 \cdot y_0)) + (x_0 \cdot y_0)
\end{aligned}
$$

Karatsuba's trick: Four for the product of three!

$$
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$$= 2^n (x_1 \cdot y_1) + 2^{n/2} ((x_0 \cdot y_1) + (x_1 \cdot y_0)) + (x_0 \cdot y_0)$$

$$t_1 = x_1 y_1, \quad t_0 = x_0 y_0, \quad t_{10} = (x_1 + x_0) \cdot (y_1 + y_0)$$

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$$x_0y_1 + x_1y_0 = (x_1 + x_0)(y_1 + y_0) - x_1y_1 - x_0y_0$$

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$$\begin{aligned} x_0y_1 + x_1y_0 &= (x_1 + x_0)(y_1 + y_0) - x_1y_1 - x_0y_0 \\ &= t_{10} - t_1 - t_0 \end{aligned}$$

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$$= t_{10} - t_1 - t_0$$

All four multiplications, for only three multiplications!

Karatsuba's algorithm

KARATSUBA-MULTIPLY(x, y, n)

1. IF(n=1): RETURN $x \times y$.
2. ELSE:
   (a) $m = \lceil n/2 \rceil$. Set $x = 2^m x_1 + x_0$.
   (b) Set $y = 2^m y_1 + y_0$.

Karatsuba's algorithm

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number of bits

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Left-half of x

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Karatsuba's algorithm

Karatsuba-Multiply(x, y, n)

1. If(n=1): Return $x \times y$.
2. Else:
   (a) $m = \lceil n/2 \rceil$. Set $x = 2^m x_1 + x_0$.
   (b) Set $y = 2^m y_1 + y_0$.
   (c) $t_1 =$ Karatsuba-Multiply(x_1, y_1, m).
   (d) $t_0 =$ Karatsuba-Multiply(x_0, y_0, m).
   (e) $t_{10} =$ Karatsuba-Multiply$(x_1 + x_0, y_1 + y_0, m)$.
   (f) Return $2^{2m} t_1 + 2^m (t_{10} - t_1 - t_0) + t_0$

Karatsuba multiplication analysis

Proposition. Karatsuba's algorithm requires $O(n^{1.585})$ bit operations to multiply two n-bit integers.

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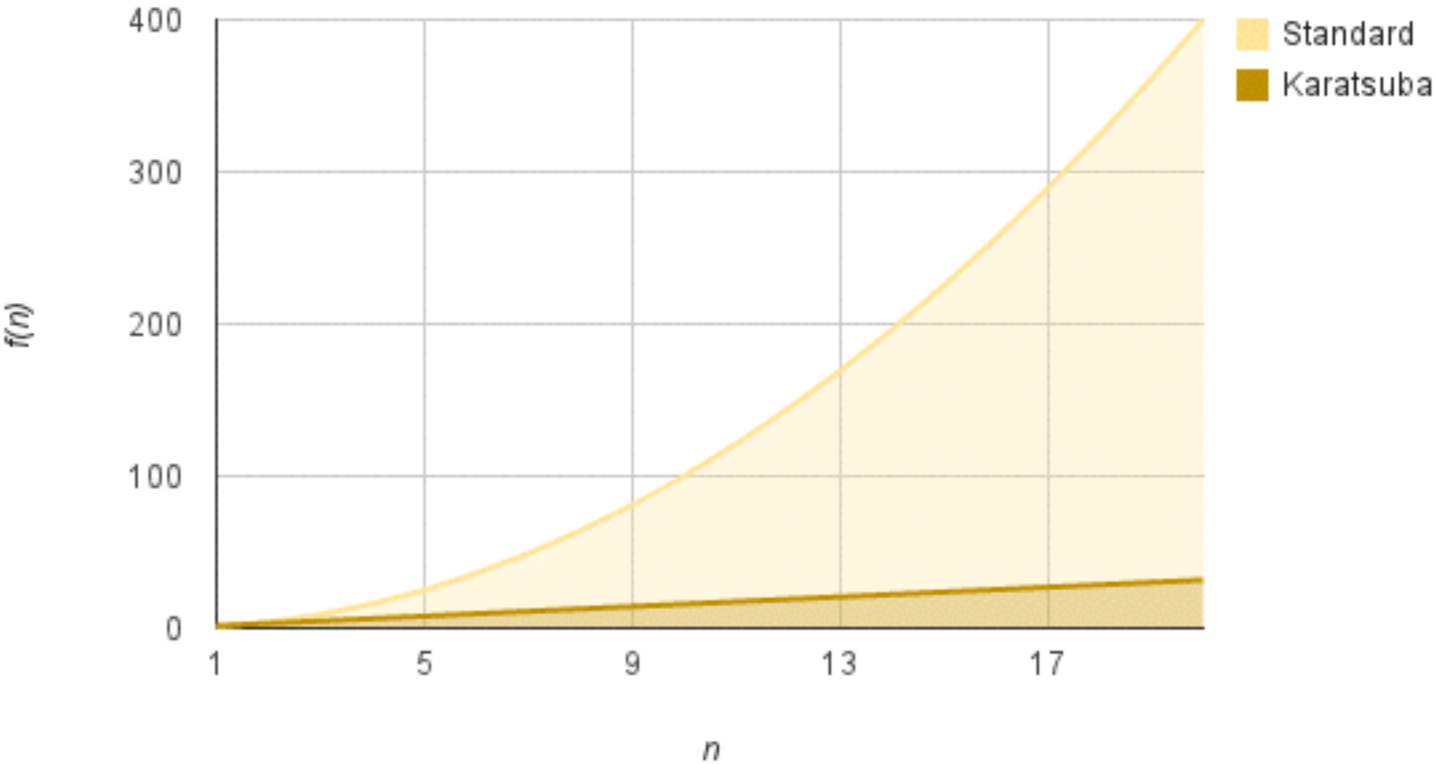
Practice. Faster than grade-school algorithm for about 320-640 bits.

Karatsuba multiplication analysis



n^2 grows much quicker than $n^{1.585}$!

History of integer multiplication

year	algorithm	order of growth
?	brute force	$\Theta(n^2)$
1962	Karatsuba-Ofman	$\Theta(n^{1.585})$

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GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»

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?	?	$\Theta(n \log n)$

GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

Used in maple, mathematica, matlab, crypto, …

GMP
«Arithmetic without limitations»

Plan for Today

Master theorem

Integer multiplication

Exponentiation

Plan for Today

Master theorem

Integer multiplication

Exponentiation : Very useful in number theory,
Cryptography

Basic arithmetic ops

Exponentiation
INPUT: Given two numbers a, n
OUTPUT: a^n in binary format.

Example: a = 11(3), n = 10.
1110011010101001

Absolutely critical in cryptography.

Basic arithmetic ops

Exponentiation
INPUT: Given two numbers a, n
OUTPUT: a^n in binary format.

NAIVE-EXPONENTIATE(a,n)

Basic arithmetic ops

Exponentiation
INPUT: Given two numbers a, n
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NAIVE-EXPONENTIATE(a,n)

1. SET A = 1.
2. FOR $i = 1 : n$,
   SET $A = a \cdot A$.
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What is the running-time?

Basic arithmetic ops

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1. SET A = 1.
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   SET $A = a \cdot A$.
3. RETURN A.

What is the running-time?

Let us specialize to a = 3.

Exponentiation

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

Exponentiation

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

n=1: 3^1 = 11
n=2: 3^2 = 1001
n=3: 3^3 = 11011
n=4: 3^4 = 1010001
n=5: 3^5 = 11110011
n=6: 3^6 = ...

Exponentiation

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
2. FOR $i = 1 : n$,
   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

What is the running-time?

Exponentiation

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
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   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

O(1)

What is the running-time?

Exponentiation: Running-time

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INPUT: Given n
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NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
2. FOR $i = 1 : n$,
   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

Time in i'th iteration?

What is the running-time?

Exponentiation: Running-time

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
2. FOR $i = 1 : n$,
   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

O(# bits in A_{i-1})

What is the running-time?

Exponentiation: Running-time

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
2. FOR $i = 1 : n$,
   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

O(# bits in A_{i-1}) = O(i).

What is the running-time?

Exponentiation: Running-time

Running time = $$\begin{aligned} &O(1) + O(1 + 2 + \cdots + n) \\ &= O(1) + O(n(n+1)/2) \\ &= O(n^2). \end{aligned}$$

$\textsc{Naive-Exponentiate}(3,\textsc{n})$

1. $\textsc{Set}\ A_0 = 1.$
2. $\textsc{For}\ i = 1 : n,$
   $\textsc{Set}\ A_i = 3 \cdot A_{i-1}.$
3. $\textsc{Return}\ A_n.$

O(# bits in A_{i-1})
= O(i).

What is the running-time?

Exponentiation: Running-time

Running time = $$\begin{aligned} &O(1) + O(1 + 2 + \cdots + n) \\ &= O(1) + O(n(n+1)/2) \\ &= O(n^2). \end{aligned}$$

Proposition: Naive-Exponentiate runs in $O(n^2)$ time.

NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
2. FOR $i = 1 : n$,
   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

Exponentiation: Running-time

Running time = $$\begin{aligned} & O(1) + O(1 + 2 + \cdots + n) \\ &= O(1) + O(n(n+1)/2) \\ &= O(n^2). \end{aligned}$$

Proposition: Naive-Exponentiate runs in $O(n^2)$ time.

Can we do better?

NAIVE-EXPONENTIATE(3,N)

1. SET $A_0 = 1$.
2. FOR $i = 1 : n$,
   SET $A_i = 3 \cdot A_{i-1}$.
3. RETURN A_n.

Exponentiation: Running-time

Running time = $$\begin{aligned} &O(1) + O(1 + 2 + \cdots + n) \\ &= O(1) + O(n(n+1)/2) \\ &= O(n^2). \end{aligned}$$

Proposition: Naive-Exponentiate runs in $O(n^2)$ time.

Can we do better?

YES WE CAN!

Fast exponentiation

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

Recursive view of algorithm:

NAIVE-EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   $A_{n-1} =$ NAIVE-EXPONENTIATE(3,N-1)
   RETURN $3 \cdot A_{n-1}$.

Fast exponentiation

Exponentiation
INPUT: Given n
OUTPUT: 3^n in binary format.

Recursive view of algorithm:

NAIVE-EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   $A_{n-1} =$ NAIVE-EXPONENTIATE(3,N-1)
   RETURN $3 \cdot A_{n-1}$.

Not dividing enough!

Fast exponentiation

Divide and conquer algorithm:

EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Exponentiate}(3,\textsc{n})$

1. $\textsc{If}$ $n = 1$, $\textsc{Return}$ 3.
2. $\textsc{Else}$
   (a) $\textsc{Set}$ $A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor)$.
   (b) $\textsc{Set}$ $A_r = \textsc{Exponentiate}(3, \lceil n/2 \rceil)$.

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Exponentiate}(3,\textsc{n})$

1. $\textsc{If}\ n = 1,\ \textsc{Return}\ 3.$
2. $\textsc{Else}$
   (a) $\textsc{Set}\ A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor).$

   (b) $\textsc{Set}\ A_r = \textsc{Exponentiate}(3, \lceil n/2 \rceil).$

   (c) $\textsc{Return}\ A_\ell \cdot A_r.$

Fast exponentiation

Divide and conquer algorithm:

EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   (a) SET $A_\ell =$ EXPONENTIATE$(3, \lfloor n/2 \rfloor)$.
   (b) SET $A_r =$ EXPONEN
   (c) RETURN $A_\ell \cdot A_r$.

How to multiply?

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Exponentiate}(3,\textsc{n})$

1. $\textsc{If}$ $n = 1$, $\textsc{Return}$ 3.
2. $\textsc{Else}$

   (a) $\textsc{Set}$ $A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor)$.

   (b) $\textsc{Set}$ $A_r = \textsc{Exponentiate}(3, \lceil n/2 \rceil)$.

   (c) $\textsc{Return}$ $\textsc{Karatsuba-Multiply}(A_\ell, A_r)$.

Fast exponentiation

Divide and conquer algorithm:

EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   (a) SET A_ℓ = EXPONENTIATE($3, \lfloor n/2 \rfloor$).
   (b) SET A_r = EXPONENTIATE($3, \lceil n/2 \rceil$).
   (c) RETURN KARATSUBA-MULTIPLY(A_ℓ, A_r).

What is the running time?

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Exponentiate}(3,\textsc{n})$

1. $\textsc{If}$ $n = 1$, $\textsc{Return}$ 3.
2. $\textsc{Else}$
   (a) $\textsc{Set}$ $A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor)$.
   (b) $\textsc{Set}$ $A_r = \textsc{Exponentiate}(3, \lceil n/2 \rceil)$.
   (c) $\textsc{Return}$ $\textsc{Karatsuba-Multiply}(A_\ell, A_r)$.

Is this the best or can we do better?

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Exponentiate}(3,\textsc{n})$

1. $\textsc{If}\ n = 1,\ \textsc{Return}\ 3.$
2. $\textsc{Else}$
   (a) $\textsc{Set}\ A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor).$
   (b) $\textsc{Set}\ A_r = \textsc{Exponentiate}(3, \lceil n/2 \rceil).$
   (c) $\textsc{Return Karatsuba-Multiply}(A_\ell, A_r).$

Do you really need two calls?

Is this the best or can we do better?

Fast exponentiation

Divide and conquer algorithm:

EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   (a) SET $A_\ell =$ EXPONENTIATE$(3, \lfloor n/2 \rfloor)$.
   (b) SET $A_r =$ EXPONENTIATE$(3, \lceil n/2 \rceil)$.
   (c) RETURN KARATSUBA-MULTIPLY(A_ℓ, A_r).

Ex: If n even? NO!

Is this the best or can we do better?

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Exponentiate}(3,\textsc{n})$

1. $\textsc{If}\ n = 1,\ \textsc{Return}\ 3.$
2. $\textsc{Else}$
   (a) $\textsc{Set}\ A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor).$
   (b) $\textsc{Set}\ A_r = \textsc{Exponentiate}(3, \lceil n/2 \rceil).$
   (c) $\textsc{Return}\ \textsc{Karatsuba-Multiply}(A_\ell, A_r).$

Ex: If n even? NO!
If n odd? No!

Is this the best or can we do better?

Fast exponentiation

Divide and conquer algorithm:

Fast-Exponentiate(3,n)

1. If $n = 1$, Return 3.
2. Else
   (a) Set A_ℓ = Exponentiate$(3, \lfloor n/2 \rfloor)$.
   (b) If n Even
   (c)

Fast exponentiation

Divide and conquer algorithm:

FAST-EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   (a) SET A_ℓ = EXPONENTIATE($3, \lfloor n/2 \rfloor$).
   (b) IF n EVEN
   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD

Fast exponentiation

Divide and conquer algorithm:

$\textsc{Fast-Exponentiate}(3,\textsc{n})$

1. $\textsc{If}\ n = 1,\ \textsc{Return}\ 3.$
2. $\textsc{Else}$
   (a) $\textsc{Set}\ A_\ell = \textsc{Exponentiate}(3, \lfloor n/2 \rfloor).$
   (b) $\textsc{If}\ n\ \textsc{Even}$
   $\textsc{Return}\quad \textsc{Karatsuba-Multiply}(A_\ell, A_\ell).$
   (c) $\textsc{If}\ n\ \textsc{Odd}$
   $\textsc{Return}\quad 3 \cdot \textsc{Karatsuba-Multiply}(A_\ell, A_\ell).$

What is the running time?

Fast exponentiation: Running-time

T(n) = Time taken on input n.

FAST-EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   (a) SET A_ℓ = EXPONENTIATE($3, \lfloor n/2 \rfloor$).
   (b) IF n EVEN
   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD
   RETURN $3\cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

O(1)

Fast exponentiation: Running-time

$$T(n) = O(1) +$$

FAST-EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE

T(n/2)

   (a) SET A_ℓ = EXPONENTIATE$(3, \lfloor n/2 \rfloor)$.
   (b) IF n EVEN
       RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD
       RETURN $3\cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

Fast exponentiation: Running-time

$$T(n) = O(1) + T(n/2) +$$

FAST-EXPONENTIATE(3,N)

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   (a) SET $A_\ell =$ EXPONENTIATE(3, $\lfloor n/2 \rfloor$).
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   RETURN $3\cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

$O((\#\text{ bits in } A_\ell)^{(\log_2 3)})$

Fast exponentiation: Running-time

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   RETURN $3\cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

$$O(n^{(\log_2 3)})$$

Fast exponentiation: Running-time

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   RETURN 3·KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

O(n)

Fast exponentiation: Running-time

$$T(n) = O(1) + T(n/2) + O(n^{\log_2 3}) + O(n)$$

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O(n)

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$$T(n) = T(n/2) + O(n^{\log_2 3})$$

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Fast exponentiation: Running-time

$$T(n) = T(n/2) + O(n^{\log_2 3})$$

Proposition: Fast-Exponentiate runs in $O(n^{1.585})$ time.

FAST-EXPONENTIATE(3,N)

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Fast exponentiation: Running-time

$$T(n) = T(n/2) + O(n^{\log_2 3})$$

Proposition: Fast-Exponentiate runs in $O(n^{1.585})$ time.

Proof: Apply Master theorem,

FAST-EXPONENTIATE(3,N)

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Fast exponentiation: Running-time

$$T(n) = T(n/2) + O(n^{\log_2 3})$$

Proposition: Fast-Exponentiate runs in $O(n^{1.585})$ time.

Proof: Apply Master theorem, case 3 to T.
a = 1, b = 2, f(n) = $O(n^{1.585})$.
Big improvement over quadratic!

FAST-EXPONENTIATE(3,N)

1. IF $n = 1$, RETURN 3.
2. ELSE
   (a) SET A_ℓ = EXPONENTIATE(3, $\lfloor n/2 \rfloor$).
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   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
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   RETURN 3·KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

Summary for today

Integer multiplication

Fast exponentiation

Summary for today

Integer multiplication

Fast exponentiation

When in doubt,
Divide it up!

General case?

Exponentiation
INPUT: Given two numbers a, n
OUTPUT: a^n in binary format.

Example: a = 101011, n = 10?

Absolutely critical in cryptography.

General case?

Exponentiation
INPUT: Given two numbers a, n
OUTPUT: a^n in binary format.

Example: a = 101011, n = 10?

Absolutely critical in cryptography.

Same algorithm!

Fast exponentiation

Divide and conquer algorithm:

FAST-EXPONENTIATE(a, n)

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2. ELSE
   (a) SET $A_\ell = $ EXPONENTIATE$(a, \lfloor n/2 \rfloor)$.
   (b) IF n EVEN
   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD
   RETURN $a \cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

What is the running time?

Fast exponentiation

T(n) = Run-time on n.
m = # bits in a.

$\textsc{Fast-Exponentiate}(a, n)$

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2. $\textsc{Else}$
   (a) $\textsc{Set}$ $A_\ell = \textsc{Exponentiate}(a, \lfloor n/2 \rfloor)$.
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What is the running time?

Fast exponentiation

T(n) = Run-time on n.
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$\textsc{Fast-Exponentiate}(a, n)$

1. $\textsc{If}\ n = 1,\ \textsc{Return}\ a.$

O(1)

2. $\textsc{Else}$
   (a) $\textsc{Set}\ A_\ell = \textsc{Exponentiate}(a, \lfloor n/2 \rfloor).$
   (b) $\textsc{If}\ n\ \textsc{Even}$
       $\textsc{Return}\quad \textsc{Karatsuba-Multiply}(A_\ell, A_\ell).$
   (c) $\textsc{If}\ n\ \textsc{Odd}$
       $\textsc{Return}\quad a \cdot \textsc{Karatsuba-Multiply}(A_\ell, A_\ell).$

What is the running time?

Fast exponentiation

$$T(n) = O(1) +$$

$\textsc{Fast-Exponentiate}(a, n)$

1. $\textsc{If}$ $n = 1$, $\textsc{Return}$ a.
2. $\textsc{Else}$

T(n/2)

   (a) $\textsc{Set}$ $A_\ell = \textsc{Exponentiate}(a, \lfloor n/2 \rfloor)$.

   (b) $\textsc{If}$ n $\textsc{Even}$
   $\textsc{Return}$ $\textsc{Karatsuba-Multiply}(A_\ell, A_\ell)$.

   (c) $\textsc{If}$ n $\textsc{Odd}$
   $\textsc{Return}$ $a \cdot \textsc{Karatsuba-Multiply}(A_\ell, A_\ell)$.

What is the running time?

Fast exponentiation

$$T(n) = O(1) + T(n/2) +$$

FAST-EXPONENTIATE(a, n)

1. IF $n = 1$, RETURN a.
2. ELSE
   (a) SET $A_\ell =$ EXPONENTIATE$(a, \lfloor n/2 \rfloor)$.
   (b) IF n EVEN
   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD
   RETURN $a \cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

$O((\#\text{ bits in } A_\ell)^{(\log_2 3)})$

What is the running time?

Fast exponentiation

$$T(n) = O(1) + T(n/2) +$$

FAST-EXPONENTIATE(a, n)

1. IF $n = 1$, RETURN a.

\# bits < (# bits in a)·n

$O((\#\text{ bits in } A_\ell)^{(\log_2 3)})$

(a) SET $A_\ell =$ EXPONENTIATE(, $\lfloor n/2 \rfloor$).

(b) IF n EVEN
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What is the running time?

Fast exponentiation

$$T(n) = O(1) + T(n/2) +$$

FAST-EXPONENTIATE(a, n)

1. IF $n = 1$, RETURN a.

\# bits < m*n

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   (a) SET A_ℓ = EXPONENTIATE$(a, \lfloor n/2 \rfloor)$.

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What is the running time?

Fast exponentiation

$$T(n) = O(1) + T(n/2) +$$

FAST-EXPONENTIATE(a, n)

1. IF $n = 1$, RETURN a.

$$O((mn)^{(\log_2 3)})$$

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   (a) SET $A_\ell =$ EXPONENTIATE$(a, \lfloor n/2 \rfloor)$.

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   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

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   RETURN $a \cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

$O((mn)^{(\log_2 3)})$

Fast exponentiation

$$T(n) = O(1) + T(n/2) + O((mn)^{\log_2 3})$$

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Fast exponentiation: Running-time

$$T(n) = T(n/2) + O((mn)^{\log_2 3})$$

Proposition: Fast-Exponentiate runs in $O((mn)^{1.585})$ time.

FAST-EXPONENTIATE(a, n)

1. IF $n = 1$, RETURN a.
2. ELSE
   (a) SET A_ℓ = EXPONENTIATE$(a, \lfloor n/2 \rfloor)$.
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Fast exponentiation: Running-time

$$T(n) = T(n/2) + O((mn)^{\log_2 3})$$

Proposition: Fast-Exponentiate runs in $O((mn)^{1.585})$ time.

Proof: Apply Master theorem,

FAST-EXPONENTIATE(a, n)

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   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD
   RETURN $a\cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).

Fast exponentiation: Running-time

$$T(n) = T(n/2) + O((mn)^{\log_2 3})$$

Proposition: Fast-Exponentiate runs in $O((mn)^{1.585})$ time.

Proof: Apply Master theorem, case 3 to T.
$a = 1, b = 2, f(n) = O((mn)^{1.585})$.
Big improvement over quadratic!

FAST-EXPONENTIATE(a, n)

1. IF $n = 1$, RETURN a.
2. ELSE
   (a) SET $A_\ell =$ EXPONENTIATE$(a, \lfloor n/2 \rfloor)$.
   (b) IF n EVEN
   RETURN KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).
   (c) IF n ODD
   RETURN $a\cdot$KARATSUBA-MULTIPLY(A_ℓ, A_ℓ).