Discretized Quantum Adiabatic Computation

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1 Introduction

This project leverages the Trotter-Suzuki Approximation to discretize Hamiltonian time evolution. By doing so, it proposes a classical algorithm that simulations quantum adiabatic computation. QUBO problems, which are formulated as Ising models are solved through the adiabatic evolution of a Hamiltonian with a trivial groundstate.

1.1 Limitations

Quantum adiabatic algorithms scale rather poorly because Pauli matrices for N particles are order 2^N matrices. There must be reasonable gaps between the energy values of various eigenstates. Time interpolation must be carefully controlled to prevent eigenvalue quasi-crossings.

2 Algorithm

2.1 Notation

The standard computational basis is used for this implementation, which is as follows: $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. The corresponding eigenvalues subject to the Pauli-Z quantum gate yield the eigenvalues +1 and -1 respectively-

$$\sigma_z |0\rangle = |0\rangle$$

$$\sigma_z |1\rangle = -|1\rangle$$

The application of a Hadamard gate transforms the qubits to a Hadamard basis-

$$\mathcal{H}|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\mathcal{H}|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

2.2 The Problem

$$f(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij} x_i x_j \tag{1}$$

Let $\sigma_i = 2\mathbf{x}_i - 1 \implies \mathbf{x}_i = \frac{\sigma_i+1}{2}$

$$(2)$$

$$f(\sigma) = \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}\left(\frac{\sigma_i+1}{2}\right)\left(\frac{\sigma_j+1}{2}\right) \tag{3}$$

$$= \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(\sigma_i\sigma_j + \sigma_i + \sigma_j + 1) \tag{4}$$

The constant can be neglected throughout the optimization process, but must be added at the end to ensure an accurate result.

$$= \frac{1}{4}\sum_{ij} Q_{ij}\sigma_i\sigma_j + \frac{1}{4}\sum_{i}(Q_{ij}+Q_{ji})\sigma_i + \frac{1}{4}\sum_{ij} Q_{ij} \tag{5}$$

$$= -\sum_{i} J_{ij}\sigma_i\sigma_j + \sum_{i} h_i\sigma_i + C \tag{6}$$

where

$$J = -\frac{Q}{4}$$

$$h_i = \frac{1}{4}\sum_{i} Q_{ij} + Q_{ji}$$

$$C = \frac{1}{4}\sum_{ij} Q_{ij}$$

Note that for an N-order QUBO, N qubits are required, which results in a system size of 2^N.

2.3 The initial Hamiltonian

The initial Hamiltonian must not commute with the final Hamiltonian to ensure a proper evolution. It must also have a trivial groundstate. For this implementation, a tranverse field Hamiltonian is chosen-

$$H = -\sum_{i} \sigma_i^X$$

The groundstate eigenvector is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)^{\otimes N}$.

$$H\,|+\rangle^{\otimes N} = -N$$

The groundstate can be prepared easily using the Hadamard gate-

$$\mathcal{H}^{\otimes N}\,|0\rangle^{\otimes N} = |+\rangle^{\otimes N}$$

2.4 The final Hamiltonian

The final Hamiltonian is the quantum version of the Ising model represented above.

$$H = -\sum_{ij} J_{ij}\sigma_i^Z\sigma_j^Z + \sum_{i} h_i\sigma_i^X$$

2.5 The adiabatic Hamiltonian

$$H(t) = (1 - s(t))H_i + s(t)H_f$$

where $s : [t, T] \mapsto [0, 1]$

2.6 The Gap Function

$$g(s) = E_1(s) - E_0(s)$$

It defines the lowest excitation energy at each step of the evolution. It is significant because the total evolution time has the condition

$$T >> \frac{1}{\min\{g(s)\}}$$

2.7 Time Evolution

The quantum state and groundstate are evolved adiabatically through the discretized time evolution operator. For small timesteps, the time evolution operator is defined as

$$U(t, t + \Delta t) \approx \exp\{-i\Delta t H(t)\} \tag{7}$$

$$U(t, t + \Delta t) \approx \exp\left\{-i\frac{\Delta t}{2}\frac{t}{T}H_f\right\} \exp\left\{-i\Delta t\left(1 - \frac{t}{T}\right)\frac{t}{T}H_i\right\} \exp\left\{-i\frac{\Delta}{2}\frac{t}{T}H_f\right\} \tag{8}$$

The above equation is a second order Trotter decomposition. The quantum state is evolved as shown below.

$$|\psi\rangle_{t+\Delta t} = U(t, t + \Delta t)\,|\psi\rangle \tag{9}$$