

Ordinary Differential Equations

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1 First Order Differential Equations

Definition

First order ODEs are of the form

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

A function $y = \phi(x)$ is a solution on an interval I if $\phi'(x) = f(x, \phi(x))$ for all $x \in I$.

Example

A simple example of a first order ODE is

$$\frac{dy}{dx} = 2x \quad (2)$$

A solution to this equation is $y = x^2 + C$, where C is a constant.

Example

$$\frac{dy}{dx} = p(x)y \quad (3)$$

A solution to this equation is $y = C \exp\{\int p(x) dx\}$, where C is a constant.

Definition

An initial value problem (IVP) is a differential equation along with a specified value, called the initial condition, that the solution must satisfy at a given point. For example, the IVP for the first order ODE

$$\frac{dy}{dx} = f(x, y) \quad (4)$$

might specify that $y(x_0) = y_0$ for some point x_0 in the interval of interest.

1.1 First Order Linear ODEs

Definition

A first order linear ODE is an equation of the form

$$\frac{dy}{dx} + p(x)y = g(x) \quad (5)$$

where $p(x)$ and $g(x)$ are continuous functions on an interval I .

Theorem 1.1: Integrating Factor Method

The general solution to the first order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \quad (6)$$

is given by

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)g(x) dx + C \right) \quad (7)$$

where $\mu(x) = \exp\left\{\int p(x) dx\right\}$ is the integrating factor and C is an arbitrary constant.

Proof. To prove this theorem, we start with the first order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \quad (8)$$

We multiply both sides by the integrating factor $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \quad (9)$$

If $\mu(x)$ is chosen such that $\mu'(x) = \mu(x)p(x)$, then the left-hand side becomes the derivative of the product $\mu(x)y$. The integrating factor is given by:

$$\mu(x) = \exp\left\{\int p(x) dx\right\} \quad (10)$$

The left-hand side can be rewritten as the derivative of a product:

$$\frac{d}{dx} (\mu(x)y) = \mu(x)g(x) \quad (11)$$

Integrating both sides with respect to x gives:

$$\mu(x)y = \int \mu(x)g(x) dx + C \quad (12)$$

Finally, we solve for y :

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)g(x) dx + C \right) \quad (13)$$

Example

Consider the first order linear ODE

$$y' + \frac{1}{2}y = \frac{1}{2}\exp\{x/3\} \quad (14)$$

To solve this ODE, we first identify the integrating factor:

$$\mu(x) = \exp\left\{\int \frac{1}{2} dx\right\} = \exp\left\{\frac{1}{2}x\right\} \quad (15)$$

Multiplying both sides of the ODE by the integrating factor gives:

$$\exp\left\{\frac{1}{2}x\right\}y' + \frac{1}{2}\exp\left\{\frac{1}{2}x\right\}y = \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} \quad (16)$$

The left-hand side can be rewritten as:

$$\frac{d}{dx} \left(\exp\left\{\frac{1}{2}x\right\}y \right) = \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} \quad (17)$$

Integrating both sides with respect to x gives:

$$\exp\left\{\frac{1}{2}x\right\}y = \int \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} dx + C \quad (18)$$

Finally, we solve for y :

$$y = \exp\left\{-\frac{1}{2}x\right\} \left(\int \frac{1}{2} \exp\left\{\frac{5}{6}x\right\} dx + C \right) \quad (19)$$

$$y = \frac{3}{5} \exp\left\{\frac{1}{2}x\right\} + C \exp\left\{-\frac{1}{2}x\right\} \quad (20)$$

1.2 Separable ODEs

Definition

A separable first order differential equation is of the form

$$\frac{dy}{dx} = f(x)g(y)$$

where $f(x)$ and $g(y)$ are functions of x and y , respectively.

Definition

A homogeneous first order differential equation is of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

where f is a function of the single variable $\frac{y}{x}$.

A homogeneous equation is solved by making the substitution $v = \frac{y}{x}$, which transforms the equation into a separable form.

Example

$$\frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy}$$

Let $v = \frac{y}{x}$, then $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting these into the original equation gives:

$$v + x\frac{dv}{dx} = \frac{x^2 - 3(vx)^2}{2x(vx)}$$