

# Ordinary Differential Equations

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## 1 First Order Differential Equations

### Definition

First order ODEs are of the form

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

A function  $y = \phi(x)$  is a solution on an interval  $I$  if  $\phi'(x) = f(x, \phi(x))$  for all  $x \in I$ .

### Example

A simple example of a first order ODE is

$$\frac{dy}{dx} = 2x \quad (2)$$

A solution to this equation is  $y = x^2 + C$ , where  $C$  is a constant.

### Example

$$\frac{dy}{dx} = p(x)y \quad (3)$$

A solution to this equation is  $y = C \exp\{\int p(x) dx\}$ , where  $C$  is a constant.

### Definition

An initial value problem (IVP) is a differential equation along with a specified value, called the initial condition, that the solution must satisfy at a given point. For example, the IVP for the first order ODE

$$\frac{dy}{dx} = f(x, y) \quad (4)$$

might specify that  $y(x_0) = y_0$  for some point  $x_0$  in the interval of interest.

### 1.1 First Order Linear ODEs

#### Definition

A first order linear ODE is an equation of the form

$$\frac{dy}{dx} + p(x)y = g(x) \quad (5)$$

where  $p(x)$  and  $g(x)$  are continuous functions on an interval  $I$ .

#### Theorem 1.1: Integrating Factor Method

The general solution to the first order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \quad (6)$$

is given by

$$y = \frac{1}{\mu(x)} \left( \int \mu(x)g(x) dx + C \right) \quad (7)$$

where  $\mu(x) = \exp\left\{\int p(x) dx\right\}$  is the integrating factor and  $C$  is an arbitrary constant.

*Proof.* To prove this theorem, we start with the first order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \quad (8)$$

We multiply both sides by the integrating factor  $\mu(x)$ :

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \quad (9)$$

If  $\mu(x)$  is chosen such that  $\mu'(x) = \mu(x)p(x)$ , then the left-hand side becomes the derivative of the product  $\mu(x)y$ . The integrating factor is given by:

$$\mu(x) = \exp\left\{\int p(x) dx\right\} \quad (10)$$

The left-hand side can be rewritten as the derivative of a product:

$$\frac{d}{dx} (\mu(x)y) = \mu(x)g(x) \quad (11)$$

Integrating both sides with respect to  $x$  gives:

$$\mu(x)y = \int \mu(x)g(x) dx + C \quad (12)$$

Finally, we solve for  $y$ :

$$y = \frac{1}{\mu(x)} \left( \int \mu(x)g(x) dx + C \right) \quad (13)$$

### Example

Consider the first order linear ODE

$$y' + \frac{1}{2}y = \frac{1}{2}\exp\{x/3\} \quad (14)$$

To solve this ODE, we first identify the integrating factor:

$$\mu(x) = \exp\left\{\int \frac{1}{2} dx\right\} = \exp\left\{\frac{1}{2}x\right\} \quad (15)$$

Multiplying both sides of the ODE by the integrating factor gives:

$$\exp\left\{\frac{1}{2}x\right\}y' + \frac{1}{2}\exp\left\{\frac{1}{2}x\right\}y = \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} \quad (16)$$

The left-hand side can be rewritten as:

$$\frac{d}{dx} \left( \exp\left\{\frac{1}{2}x\right\}y \right) = \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} \quad (17)$$

Integrating both sides with respect to  $x$  gives:

$$\exp\left\{\frac{1}{2}x\right\}y = \int \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} dx + C \quad (18)$$

Finally, we solve for  $y$ :

$$y = \exp\left\{-\frac{1}{2}x\right\} \left( \int \frac{1}{2} \exp\left\{\frac{5}{6}x\right\} dx + C \right) \quad (19)$$

$$y = \frac{3}{5} \exp\left\{\frac{1}{2}x\right\} + C \exp\left\{-\frac{1}{2}x\right\} \quad (20)$$