Ordinary Differential Equations

lonelyneutrino

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1 First Order Differential Equations

Definition

First order ODEs are of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

A function $y = \phi(x)$ is a solution on an interval I if $\phi'(x) = f(x, \phi(x))$ for all $x \in I$.

Example

A simple example of a first order ODE is

$$\frac{dy}{dx} = 2x \tag{2}$$

A solution to this equation is $y = x^2 + C$, where C is a constant.

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y \tag{3}$$

A solution to this equation is $y = C \exp\left\{\int p(x)\, dx\right\}$, where C is a constant.

Definition

An initial value problem (IVP) is a differential equation along with a specified value, called the initial condition, that the solution must satisfy at a given point. For example, the IVP for the first order ODE

$$\frac{dy}{dx} = f(x, y) \tag{4}$$

might specify that $y(x_0) = y_0$ for some point x_0 in the interval of interest.

1.1 First Order Linear ODEs

Definition

A first order linear ODE is an equation of the form

$$\frac{dy}{dx} + p(x)y = g(x) \tag{5}$$

where $p(x)$ and $g(x)$ are continuous functions on an interval I.

Theorem 1.1: Integrating Factor Method

The general solution to the first order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \tag{6}$$

is given by

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)\,dx + C\right) \tag{7}$$

where $\mu(x) = \exp\{\int p(x)\,dx\}$ is the integrating factor and C is an arbitrary constant.

Proof. To prove this theorem, we start with the first order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \tag{8}$$

We multiply both sides by the integrating factor $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \tag{9}$$

If $\mu(x)$ is chosen such that $\mu'(x) = \mu(x)p(x)$, then the left-hand side becomes the derivative of the product $\mu(x)y$. The integrating factor is given by:

$$\mu(x) = \exp\left\{\int p(x)\,dx\right\} \tag{10}$$

The left-hand side can be rewritten as the derivative of a product:

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x) \tag{11}$$

Integrating both sides with respect to x gives:

$$\mu(x)y = \int \mu(x)g(x)\,dx + C \tag{12}$$

Finally, we solve for y:

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)\,dx + C\right) \tag{13}$$

Example

Consider the first order linear ODE

$$y' + \frac{1}{2}y = \frac{1}{2}\exp\{x/3\} \tag{14}$$

To solve this ODE, we first identify the integrating factor:

$$\mu(x) = \exp\left\{\int \frac{1}{2}\,dx\right\} = \exp\left\{\frac{1}{2}x\right\} \tag{15}$$

Multiplying both sides of the ODE by the integrating factor gives:

$$\exp\left\{\frac{1}{2}x\right\}y' + \frac{1}{2}\exp\left\{\frac{1}{2}x\right\}y = \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} \tag{16}$$

The left-hand side can be rewritten as:

$$\frac{d}{dx}\left(\exp\left\{\frac{1}{2}x\right\}y\right) = \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} \tag{17}$$

Integrating both sides with respect to x gives:

$$\exp\left\{\frac{1}{2}x\right\}y = \int \frac{1}{2}\exp\left\{\frac{5}{6}x\right\}dx + C \tag{18}$$

Finally, we solve for y:

$$y = \exp\left\{-\frac{1}{2}x\right\}\left(\int \frac{1}{2}\exp\left\{\frac{5}{6}x\right\} dx + C\right) \tag{19}$$

$$y = \frac{3}{5}\exp\left\{\frac{1}{2}x\right\} + C\exp\left\{-\frac{1}{2}x\right\} \tag{20}$$