Linear Algebra II

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1 Basics

Example

For a set $\mathcal{S}$, let $\mathbb{F}^{\mathcal{S}}$ be the set of all functions from $\mathcal{S}$ to $\mathbb{F}$. Then, defined over canonical addition and scalar multiplications, $\mathbb{F}^{\mathcal{S}}$ is a vector space. The additive identity is the zero function 0, defined as $0(x) = 0$. The additive inverse can be defined as $-f : \mathcal{S} \to \mathbb{F}$ defined as $-f(x) = -(f(x)) \forall x \in \mathcal{S}$.

Note that $\mathbb{F}^n$ and $\mathbb{F}^\infty$ are special cases of $\mathbb{F}^{\mathcal{S}}$, where $\mathcal{S}$ is a finite set of size n or an infinite set, respectively.

Note that the empty set ϕ is not a vector space, nor is it a subspace of any vector space.

Example

The set of differentiable real-valued functions is a subspace of $\mathbb{R}^{\mathbb{R}}$. Note that in calculus, the sum of two continuous functions is continuous, and the sum of two differentiable functions is differentiable. Also, scalar multiples of continuous and differentiable functions are continuous and differentiable, respectively.

Definition

Let $V_1, \cdots V_n$ be subspaces of a vector space $\mathbf{V}$. Then, the sum of these subspaces is defined as

$$V_1 + V_2 + \cdots + V_n = \{v_1 + v_2 + \cdots + v_n \mid v_i \in V_i \text{ for all } i\}$$

Example

Let

$$V_1 = \left\{(w, w, x, x) \in \mathbb{F}^4 | w, x \in \mathbb{F}\right\}$$

$$V_2 = \left\{(y, y, y, z) \in \mathbb{F}^4 | y, z \in \mathbb{F}\right\}$$

Now, let $v_1 \in V_1$ and $v_2 \in V_2$. Then, we can write

$$v_1 = (w_1, w_1, x_1, x_1)$$

$$v_2 = (y_2, y_2, y_2, z_2)$$

for some $w_1, x_1, y_2, z_2 \in \mathbb{F}$. Then, we have

$$v_1 + v_2 = (w_1 + y_2, w_1 + y_2, x_1 + y_2, x_1 + z_2) \in V_1 + V_2$$

Let W be defined as

$$W = \left\{(x, x, y, z) \in \mathbb{F}^4 | x, y, z, \in \mathbb{F}\right\}$$

Then, $v_1 + v_2 \in W$ so $V_1 + V_2 \subseteq W$.
Let $w \in W$. Then, we can write

$$w = (x_w, x_w, y_w, z_w)$$

for some $x_w, y_w, z_w \in \mathbb{F}$. Then, we have

$$w = (x_w, x_w, y_w, z_w) = (x_w, x_w, y_w, y_w) + (0, 0, 0, z_w - y_w) \in V_1 + V_2$$

$\therefore W = V_1 + V_2$

Lemma 1.1

For any subspaces $V_1, \cdots V_n$ of a vector space $\mathbf{V}$, $V_1 + \cdots + V_n$ is a subspace of $\mathbf{V}$. It is also the smallest subspace of V that contains all elements of the form $v_1 + \cdots + v_n$ where $v_i \in V_i$ for all i.

Proof. From the definition and that $V_1, \cdots V_n$ are subspaces, Since the subspaces themselves are closed under addition and scalar multiplication, $V_1 + \cdots + V_n$ is also closed under addition and scalar multiplication. Also, the zero vector $\mathbf{0}$ is in each of the subspaces, so $\mathbf{0} \in V_1 + \cdots + V_n$. Thus, $V_1 + \cdots + V_n$ is a subspace of $\mathbf{V}$.

Note: Generally, the set theoretic union is rarely a subspace, except for trivial cases where one space is a subspace of the other. However, intersections of subspaces are generally subspaces.

Definition: Direct sum

Let $V_1, \cdots V_n$ be subspaces of a vector space $\mathbf{V}$. Then, the sum $V_1 + \cdots + V_n$ is called a direct sum if each element of $V_1 + \cdots + V_n$ can be written in one and only one way as $v_1 + \cdots + v_n$ where $v_i \in V_i$ for all i. In this case, we say that the sum is a direct sum, denoted by

$$W = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

Example

Let

$$U = \left\{(x, x, y) \in \mathbb{F}^3 | x, y \in \mathbb{F}\right\}$$

Let

$$W = \left\{(x, 0, 0) \in \mathbb{F}^3 | x \in \mathbb{F}\right\}$$

Then, U and W are subspaces of $\mathbb{F}^3$. Any arbitrary vector in $\mathbb{F}^3$ can be written as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ b \\ c \end{pmatrix} + \begin{pmatrix} a-b \\ 0 \\ 0 \end{pmatrix}$$

Since this is a unique representation, $U \oplus W = \mathbb{F}^3$.

Theorem 1.2

Let $V_1, \cdots V_n$ be subspaces of a vector space $\mathbf{V}$. Then, $V_1 + \cdots V_n = V_1 \oplus \cdots \oplus V_n$ if and only if the only way to write $\mathbf{0}$ as $v_1 + \cdots + v_n$ where $v_i \in V_i$ for all i is to take each $v_i = \mathbf{0}$. In other words, if $v_1 + \cdots + v_n = \mathbf{0}$ implies that each $v_i = \mathbf{0}$, then the sum is a direct sum.

Proof. Suppose that $V_1 + \cdots V_n$ is a direct sum. Then, the additive identity can be written as the sum of additive identities from each subspace. By definition of a direct sum, this is the *only* way to write the additive identity as a sum.
Suppose that the only way to write zero is as the sum of additive identities from each subspace. Consider an arbitrary vector $v \in V$. Suppose that there are two different ways of writing the sum,

$$v = u_1 + \cdots u_n; u_k \in V_k$$

$$v = v_1 + \cdots v_n; v_k \in V_k$$

Then, we can subtract these two equations

$$0 = (u_1 - v_1) + \cdots + (u_n - v_n); (u_k - v_k) \in V_k$$

Since the only way to write zero is as the sum of additive identities from each subspace, we must have $u_k - v_k = 0$ for all k. Thus, $u_k = v_k$ for all k, and the representation is unique. Therefore, by definition, the sum $V_1 + \cdots V_n$ is a direct sum.

Theorem 1.3

Let U and W be subspaces of a vector space $\mathbf{V}$. Then, the sum $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof. Suppose that $U + W$ is a direct sum. Let $v \in U \cap W$. Then, $v \in U$, and $-v \in W$

$$0 = v + (-v)$$

Since the representation is unique, we must have $v = 0$. Thus, $U \cap W = \{0\}$.

Conversely, suppose that $U \cap W = \{0\}$. Let $u \in U$ and $w \in W$. Then, we can write

$$u + w = 0$$

From the previous result, it suffices to show that $u = w = 0$. This implies that w is the additive inverse of u, meaning $u, w \in U \cap W = \{0\}$. Therefore, $u = w = 0$ and $U + W = U \oplus W$.