

习题课 5

$$1. \quad D_n = \begin{vmatrix} b & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & b+a_1 \end{vmatrix} \quad (n \geq 2)$$

沿第1行展开

$$D_n = b D_{n-1} + M_{12}$$

$$M_{12} = \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-2} & a_{n-3} & \cdots & a_2 & b+a_1 \end{vmatrix}$$

$$= (-1)^{n+1} \cdot a_n \cdot (-1)^{n-2} \quad (\text{沿第1列展开})$$

$$\Rightarrow D_n - b D_{n-1} = a_n$$

$$D_{n-1} - b D_{n-2} = a_{n-1}$$

...

$$D_2 - b D_1 = a_2 \quad D_1 = b + a_1$$

$$\Rightarrow D_n - b^{n-1} D_1 = a_n + b a_{n-1} + \cdots + b^{n-2} a_2$$

$$D_n = b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-1} b + a_n \quad (n \geq 2)$$

$$2. \quad D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}$$

$$n \geq 3 \quad D_n = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1-n & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}n(n+1) & 3 & 4 & \cdots & n & 1 \end{vmatrix}$$

$$= \frac{1}{2}n(n+1) \cdot (-1)^{n+1} \cdot \begin{vmatrix} 1-n & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1-n & \cdots & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(-1)^{n+1} \cdot n(n+1) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -n & 0 & \cdots & 0 & 0 \\ 0 & -n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n & 0 \end{vmatrix}$$

$$= \frac{1}{2}(-1)^{n+1} \cdot n(n+1) \cdot (-1)^{n+1} \cdot (-n)^{n-2}$$

$$= \frac{1}{2}(-1)^{n+1} (n+1) n^{n-1}$$

$n=1, 2$, 经过验证, 上式同样成立

$$3. \quad D_n = \begin{vmatrix} a+b & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

$$D_n = (a+b)D_{n-1} - a \begin{vmatrix} b & a & 0 & \dots & 0 \\ b & a+b & 0 & \dots & 0 \\ 0 & b & a+b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a+b \end{vmatrix}$$

$$= (a+b)D_{n-1} - abD_{n-2}$$

$$D_n - aD_{n-1} = b(D_{n-1} - aD_{n-2}) \quad D_n - bD_{n-1} = a(D_{n-1} - bD_{n-2}) \quad D_1 = a+b \quad D_2 = a^2+ab+b^2$$

$$n \geq 3 \quad D_n - aD_{n-1} = b^{n-2}(D_2 - aD_1) = b^n \quad D_n - bD_{n-1} = a^{n-2}(D_2 - bD_1) = a^n$$

$$\Rightarrow \text{若 } a \neq b, D_n = \frac{a^{n+1} - b^{n+1}}{a-b}; \text{ 若 } a=b, D_n = (n+1)a^n$$

$n=1,2$ 上式同样成立

4.

$$D_n = \begin{vmatrix} 1+a_1 & 1 & \dots & 1 \\ 1 & 1+a_2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1+a_n \end{vmatrix}$$

若 a_i 中至少有两个为零, 则 $D_n = 0$ (有两行相同)

若 a_i 中至多有一个为零, 不妨设 a_1, \dots, a_{n-1} 均不为零

$$D_n = \begin{vmatrix} a_1 & 0 & \dots & 0 & -a_n \\ 0 & a_2 & \dots & 0 & -a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & -a_n \\ 1 & 1 & \dots & 1 & 1+a_n \end{vmatrix}$$

$$= a_1 a_2 \dots a_{n-1} \begin{vmatrix} 1 & 0 & \dots & 0 & -\frac{a_n}{a_1} \\ 0 & 1 & \dots & 0 & -\frac{a_n}{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{a_n}{a_{n-1}} \\ 1 & 1 & \dots & 1 & 1+a_n \end{vmatrix}$$

$$= a_1 a_2 \dots a_{n-1} \begin{vmatrix} 1 & 0 & \dots & 0 & -\frac{a_n}{a_1} \\ 0 & 1 & \dots & 0 & -\frac{a_n}{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{a_n}{a_{n-1}} \\ 0 & 0 & \dots & 0 & 1+a_n + \frac{a_n}{a_1} + \frac{a_n}{a_2} + \dots + \frac{a_n}{a_{n-1}} \end{vmatrix}$$

$$= a_1 a_2 \dots a_{n-1} \cdot (-1)^{n+n} \left(1+a_n + \frac{a_n}{a_1} + \dots + \frac{a_n}{a_{n-1}} \right)$$

$$= a_1 \dots a_{n-1} \left(1+a_n + \frac{a_n}{a_1} + \dots + \frac{a_n}{a_{n-1}} \right)$$

$$\text{若 } a_n \neq 0 \quad D_n = a_1 \dots a_n \left(1 + \sum_{i=1}^n \frac{1}{a_i} \right)$$

$$5. D_n = \begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & \cdots & z & x \end{vmatrix} \quad (y \neq z)$$

法一 当 $n \geq 2$

$$D_n = \begin{vmatrix} x & y & y & \cdots & y & 0+y \\ z & x & y & \cdots & y & 0+y \\ z & z & x & \cdots & y & 0+y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & \cdots & z & (x-y)+y \end{vmatrix}$$

$$= \begin{vmatrix} x & y & y & \cdots & y & 0 \\ z & x & y & \cdots & y & 0 \\ z & z & x & \cdots & y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & \cdots & z & x-y \end{vmatrix} + \begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & \cdots & z & y \end{vmatrix}$$

$$= (x-y)D_{n-1} + \begin{vmatrix} x-z & y-x & 0 & \cdots & 0 & 0 \\ 0 & x-z & y-x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x-z & 0 \\ z & z & z & \cdots & z & y \end{vmatrix}$$

$$= (x-y)D_{n-1} + (-1)^{n+1} y \cdot (x-z)^{n-1}$$

$$= (x-y)D_{n-1} + y(x-z)^{n-1}$$

由 $|A| = |A^T|$, 对 A^T 用上述结论, 得 $D_n = (x-z)D_{n-1} + z(x-y)^{n-1}$

$$\text{联立解得 } D_n = \frac{y(x-z)^n - z(x-y)^n}{y-z}$$

$n=1$, 经过验证, 上式同样成立

法二 (感谢万骐涵同学提供的解答)

$$D_n = \begin{vmatrix} x-y & 0 & 0 & \cdots & 0 & y \\ z-x & x-y & 0 & \cdots & 0 & y \\ 0 & z-x & x-y & \cdots & 0 & y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z-x & x \end{vmatrix}$$

$$= (x-y)D_{n-1} + (x-z) \cdot \begin{vmatrix} 0 & 0 & \cdots & 0 & y \\ z-x & x-y & \cdots & 0 & y \\ 0 & 0 & \cdots & 0 & y \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z-x & x \end{vmatrix}$$

$$= (x-y)D_{n-1} + (x-z) \cdot (-1)^{1+n-1} \cdot y \cdot (z-x)^{n-2}$$

$$= (x-y)D_{n-1} + y \cdot (x-z)^{n-1} \quad \text{后同法一}$$

法三 (感谢李一轩同学提供的解答)

(前同法一) 硬解数列通项: $D_n = (x-y)D_{n-1} + y(x-z)^{n-1}$ $D_1 = x$

这是一个一阶非齐次线性递推

齐次递推 ($D_n = (x-y)D_{n-1}$) 通解形式为 $D_n^{(h)} = C_{(x,y,z)} (x-y)^{n-1}$

非齐次递推 ($D_n = (x-y)D_{n-1} + y(x-z)^{n-1}$) 有特解形如 $D_n^{(p)} = K_{(x,y,z)} (x-z)^{n-1}$

代入得 $K(x-z)^{n-1} = (x-y)K(x-z)^{n-2} + y(x-z)^{n-1}$

$K(x-z) = (x-y)K + y$ (不妨设 $x \neq z$)

$$K = \frac{y(x-z)}{x-z}$$

$$\Rightarrow D_n = D_n^{(p)} + D_n^{(h)} = C(x-y)^{n-1} + \frac{y(x-z)}{x-z} \cdot (x-z)^{n-1}$$

$$D_1 = x \Rightarrow C = \frac{z(y-x)}{x-y}$$

$$\text{故 } D_n = \frac{y(x-z)^n - z(x-y)^n}{y-z}$$