习题课 5

1.
$$D_n = \begin{vmatrix} b & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & b+a_1 \end{vmatrix}$$

沿第 1 行展开
$$D_n = b D_{n-1} + M_{12} \quad (n \geq 2)$$
$$M_{12} = \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-2} & a_{n-3} & \cdots & a_2 & b+a_1 \end{vmatrix}$$
$$= (-1)^{n+1} \cdot a_n \cdot (-1)^{n-2} \quad (沿第 1 列展开)$$

$$\Rightarrow \quad D_n - b D_{n-1} = a_n$$
$$D_{n-1} - b D_{n-2} = a_{n-1}$$
$$\cdots$$
$$D_2 - b D_1 = a_2 \quad D_1 = b + a_1$$

$$\Rightarrow \quad D_n - b^{n-1} D_1 = a_n + b a_{n-1} + \cdots + b^{n-2} a_2$$
$$D_n = b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-1} b + a_n$$
$n=1$，经验证，上式同样成立

2.
$$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ \vdots & & & & & \vdots \\ 2 & 3 & 4 & \cdots & & n \end{vmatrix}$$

$n \geq 3$
$$D_n = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & & & & & \vdots \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1-n & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & & & & & & \vdots \\ \frac{1}{2}n(n+1) & 3 & 4 & \cdots & & n & 1 \end{vmatrix}$$

$$= \frac{1}{2} n(n+1) \cdot (-1)^{n+1} \cdot \begin{vmatrix} 1-n & 1 & \cdots & 1 & 1 \\ 1 & 1-n & \cdots & 1 & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & 1-n \end{vmatrix}$$

$$= \frac{1}{2} (-1)^{n+1} \cdot n(n+1) \begin{vmatrix} -n & 0 & \cdots & 1 & 1 \\ 0 & -n & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & -n & 0 \end{vmatrix}$$

$$= \frac{1}{2} (-1)^{n+1} \cdot n(n+1) \cdot (-1)^{1+n-1} \cdot (-n)^{n-2}$$
$$= \frac{1}{2} (-1)^{n-1} (n+1) n^{n-1}$$
$n=1, 2$，经过验证，上式同样成立

3.
$$D_n = \begin{vmatrix} a+b & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

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$$D_n = (a+b)D_{n-1} - a \begin{vmatrix} b & a & 0 & \cdots & 0 \\ 0 & a+b & 0 & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

$$= (a+b)D_{n-1} - ab\, D_{n-2}$$

$D_n - aD_{n-1} = b(D_{n-1} - aD_{n-2})$   $D_n - bD_{n-1} = a(D_{n-1} - bD_{n-2})$   $D_1 = a+b$   $D_2 = a^2+ab+b^2$

$n \geq 3$   $D_n - aD_{n-1} = b^{n-2}(D_2 - aD_1) = b^n$   $D_n - bD_{n-1} = a^{n-2}(D_2 - bD_1) = a^n$

$\Rightarrow$ 若 $a \neq b$, $D_n = \dfrac{a^{n+1} - b^{n+1}}{a-b}$ ; 若 $a = b$, $D_n = (n+1)a^n$

$n = 1, 2$ 上式同样成立

4.

$$D_n = \begin{vmatrix} 1+a_1 & 1 & \cdots & & 1 \\ 1 & 1+a_2 & & & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & & \cdots & 1+a_n \end{vmatrix}$$

若 $a_i$ 中至少有两个为零，则 $D_n = 0$（有两行相同）

若 $a_i$ 中至多有一个为零，不妨设 $a_1, \cdots, a_{n-1}$ 均不为零

$$D_n = \begin{vmatrix} a_1 & 0 & \cdots & 0 & -a_n \\ 0 & a_2 & \cdots & 0 & -a_n \\ & & \ddots & & \\ 0 & 0 & \cdots & a_{n-1} & -a_n \\ 1 & 1 & \cdots & 1 & 1+a_n \end{vmatrix}$$

$$= a_1 a_2 \cdots a_{n-1} \begin{vmatrix} 1 & 0 & \cdots & 0 & -\frac{a_n}{a_1} \\ 0 & 1 & \cdots & 0 & -\frac{a_n}{a_2} \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -\frac{a_n}{a_{n-1}} \\ 1 & 1 & \cdots & 1 & 1+a_n \end{vmatrix}$$

$$= a_1 a_2 \cdots a_{n-1} \begin{vmatrix} 1 & 0 & \cdots & 0 & -\frac{a_n}{a_1} \\ 0 & 1 & \cdots & 0 & -\frac{a_n}{a_2} \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -\frac{a_n}{a_{n-1}} \\ 0 & 0 & \cdots & 0 & 1+a_n+\frac{a_n}{a_1}+\frac{a_n}{a_2}+\cdots+\frac{a_n}{a_{n-1}} \end{vmatrix}$$

$$= a_1 a_2 \cdots a_{n-1} \cdot (-1)^{n+n}\left(1+a_n+\frac{a_n}{a_1}+\cdots+\frac{a_n}{a_{n-1}}\right)$$

$$= a_1 \cdots a_{n-1}\left(1+a_n+\frac{a_n}{a_1}+\cdots+\frac{a_n}{a_{n-1}}\right)$$

若 $a_n \neq 0$   $D_n = a_1 \cdots a_n\left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)$

5. $D_n = \begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & & & & & \vdots \\ z & z & z & \cdots & z & x \end{vmatrix}$ $(y \neq z)$

法一 当 $n \geq 2$

$D_n = \begin{vmatrix} x & y & y & \cdots & y & 0+y \\ z & x & y & \cdots & y & 0+y \\ z & z & x & \cdots & y & 0+y \\ \vdots & & & & & \vdots \\ z & z & z & \cdots & z & (x-y)+y \end{vmatrix}$

$= \begin{vmatrix} x & y & y & \cdots & y & 0 \\ z & x & y & \cdots & y & 0 \\ z & z & x & \cdots & y & 0 \\ \vdots & & & & & \vdots \\ z & z & z & \cdots & z & x-y \end{vmatrix} + \begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & & & & & \vdots \\ z & z & z & \cdots & z & y \end{vmatrix}$

$= (x-y)D_{n-1} + \begin{vmatrix} x-z & y-x & 0 & \cdots & 0 & 0 \\ 0 & x-z & y-x & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdots & x-z & 0 \\ z & z & z & \cdots & z & y \end{vmatrix}$

$= (x-y)D_{n-1} + (-1)^{n+n} y \cdot (x-z)^{n-1}$

$= (x-y)D_{n-1} + y(x-z)^{n-1}$

由 $|A| = |A^T|$ ，对 $A^T$ 用上述结论，得 $D_n = (x-z)D_{n-1} + z(x-y)^{n+1}$

联立解得 $D_n = \dfrac{y(x-z)^n - z(x-y)^n}{y-z}$

$n=1$ ，经过验证，上式同样成立

法二 (感谢万骐洛同学提供的解答)

$D_n = \begin{vmatrix} x-y & 0 & 0 & \cdots & 0 & y \\ z-x & x-y & 0 & \cdots & 0 & y \\ 0 & z-x & x-y & \cdots & 0 & y \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & z-x & x \end{vmatrix}$

$= (x-y)D_{n-1} + (x-z) \cdot \begin{vmatrix} 0 & 0 & \cdots & 0 & y \\ z-x & x-y & \cdots & 0 & y \\ 0 & 0 & \cdots & 0 & y \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & z-x & x \end{vmatrix}$

$= (x-y)D_{n-1} + (x-z) \cdot (-1)^{1+n-1} \cdot y \cdot (z-x)^{n-2}$

$= (x-y)D_{n-1} + y \cdot (x-z)^{n-1}$ 后同法一

法三 (感谢李一轲同学提供的解答)

(前同法一) 硬解数列通项：$D_n = (X-Y)D_{n-1} + Y(X-Z)^{n-1}$    $D_1 = X$

这是一个一阶非齐次线性递推

齐次递推 $(D_n = (X-Y)D_{n-1})$ 通解形式为 $D_n^{(h)} = C_{(x,y,z)}(X-Y)^{n-1}$

非齐次递推 $(D_n = (X-Y)D_{n-1} + Y(X-Z)^{n-1})$ 有特解形如 $D_n^{(P)} = K_{(x,y,z)}(X-Z)^{n-1}$

代回得  $K(X-Z)^{n-1} = (X-Y)K(X-Z)^{n-2} + Y(X-Z)^{n-1}$

$\qquad K(X-Z) = (X-Y)K + Y$  (不妨设 $X \ne Z$)

$\qquad K = \dfrac{Y(X-Z)}{Y-Z}$

$\Rightarrow D_n = D_n^{(P)} + D_n^{(h)} = C(X-Y)^{n-1} + \dfrac{Y(X-Z)}{Y-Z}\cdot(X-Z)^{n-1}$

$\qquad D_1 = X \Rightarrow C = \dfrac{Z(Y-X)}{X-Y}$

故  $D_n = \dfrac{Y(X-Z)^n - Z(X-Y)^n}{Y-Z}$