习题课 5

1. $$D_n = \begin{vmatrix} b & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & b+a_1 \end{vmatrix}$$

沿第1行展开

$$D_n = bD_{n-1} + M_{12} \quad (n \geq 2)$$

$$M_{12} = \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & b & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & 0 & \cdots & b & -1 \\ a_n & a_{n-2} & a_{n-3} & \cdots & a_2 & b+a_1 \end{vmatrix}$$

$= (-1)^{n-1+1} \cdot a_n \cdot (-1)^{n-2}$ (沿第1列展开)

$$\Rightarrow \quad D_n - bD_{n-1} = a_n$$
$$D_{n-1} - bD_{n-2} = a_{n-1}$$
$$\cdots$$
$$D_2 - bD_1 = a_2 \qquad D_1 = b+a_1$$

$$\Rightarrow \quad D_n - b^{n-1}D_1 = a_n + b a_{n-1} + \cdots + b^{n-2}a_2$$
$$D_n = b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-1}b + a_n$$

$n=1$，经验证，上式同样成立

2. $$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ \vdots & & & \ddots & & \vdots \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}$$

$n \geq 3$

$$D_n = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & & & \ddots & & \vdots \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1-n & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1-n & \cdots & 1 & 1 \\ \vdots & & \vdots & \ddots & & \vdots \\ \frac{1}{2}n(n+1) & 3 & 4 & \cdots & n & 1 \end{vmatrix}$$

$$= \frac{1}{2}n(n+1)\cdot(-1)^{n+1}\cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1-n & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1-n & 1 \end{vmatrix}$$

$$= \frac{1}{2}(-1)^{n+1}\cdot n(n+1) \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -n & 0 & \cdots & 0 & 0 \\ 0 & -n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n & 0 \end{vmatrix}$$

$$= \frac{1}{2}(-1)^{n+1}\cdot n(n+1)\cdot(-1)^{1+n-1}\cdot(-n)^{n-2}$$
$$= \frac{1}{2}(-1)^{n-1}(n+1)\,n^{n-1}$$

$n=1,2$，经过验证，上式同样成立

3. $$D_n = \begin{vmatrix} a+b & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

$$D_n=(a+b)D_{n-1}-a\begin{vmatrix} b & a & 0 & \cdots & 0 \\ 0 & a+b & 0 & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

$$=(a+b)D_{n-1}-abD_{n-2}$$

$D_n-aD_{n-1}=b(D_{n-1}-aD_{n-2})$ $\quad D_n-bD_{n-1}=a(D_{n-1}-bD_{n-2})$ $\quad D_1=a+b$ $\quad D_2=a^2+ab+b^2$

$n\geq 3$ $\quad D_n-aD_{n-1}=b^{n-2}(D_2-aD_1)=b^n$ $\quad D_n-bD_{n-1}=a^{n-2}(D_2-bD_1)=a^n$

$\Rightarrow$ 若 $a\neq b$，$D_n=\frac{a^{n+1}-b^{n+1}}{a-b}$ ；若 $a=b$ ，$D_n=(n+1)a^n$

$n=1,2$ 上式同样成立

4.

$$D_n=\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix}$$

若 a_i 中至少有两个为零，则 $D_n=0$（有两行相同）

若 a_i 中至多有一个为零，不妨设 $a_1,\cdots,a_{n-1}$ 均不为零

$$D_n=\begin{vmatrix} a_1 & 0 & \cdots & 0 & -a_n \\ 0 & a_2 & \cdots & 0 & -a_n \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & a_{n-1} & -a_n \\ 1 & 1 & \cdots & 1 & 1+a_n \end{vmatrix}$$

$$=a_1a_2\cdots a_{n-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & -\frac{a_n}{a_1} \\ 0 & 1 & \cdots & 0 & -\frac{a_n}{a_2} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -\frac{a_n}{a_{n-1}} \\ 1 & 1 & \cdots & 1 & 1+a_n \end{vmatrix}$$

$$=a_1a_2\cdots a_{n-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & -\frac{a_n}{a_1} \\ 0 & 1 & \cdots & 0 & -\frac{a_n}{a_2} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -\frac{a_n}{a_{n-1}} \\ 0 & 0 & \cdots & 0 & 1+a_n+\frac{a_n}{a_1}+\frac{a_n}{a_2}+\cdots+\frac{a_n}{a_{n-1}} \end{vmatrix}$$

$$=a_1a_2\cdots a_{n-1}\cdot(-1)^{n+n}\left(1+a_n+\frac{a_n}{a_1}+\cdots+\frac{a_n}{a_{n-1}}\right)$$

$$=a_1\cdots a_{n-1}\left(1+a_n+\frac{a_n}{a_1}+\cdots+\frac{a_n}{a_{n-1}}\right)$$

若 $a_n\neq 0$ $\quad D_n=a_1\cdots a_n\left(1+\sum_{i=1}^{n}\frac{1}{a_i}\right)$

5. $$D_n=\begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & & & \ddots & & \vdots \\ z & z & z & \cdots & z & x \end{vmatrix} \quad (y\neq z)$$

法一 当 $n\geq 2$

$$D_n=\begin{vmatrix} x & y & y & \cdots & y & 0+y \\ z & x & y & \cdots & y & 0+y \\ z & z & x & \cdots & y & 0+y \\ \vdots & & & \ddots & & \vdots \\ z & z & z & \cdots & z & (x-y)+y \end{vmatrix}$$

$$=\begin{vmatrix} x & y & y & \cdots & y & 0 \\ z & x & y & \cdots & y & 0 \\ z & z & x & \cdots & y & 0 \\ \vdots & & & \ddots & & \vdots \\ z & z & z & \cdots & z & x-y \end{vmatrix}+\begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & & & \ddots & & \vdots \\ z & z & z & \cdots & z & y \end{vmatrix}$$

$$=(x-y)D_{n-1}+\begin{vmatrix} x-z & y-x & 0 & \cdots & 0 & 0 \\ 0 & x-z & y-x & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & x-z & 0 \\ z & z & z & \cdots & z & y \end{vmatrix}$$

$$=(x-y)D_{n-1}+(-1)^{n+n}y\cdot(x-z)^{n-1}$$

$$=(x-y)D_{n-1}+y(x-z)^{n-1}$$

由 $|A|=|A^T|$，对 A^T 用上述结论，得 $D_n=(x-z)D_{n-1}+z(x-y)^{n-1}$

联立解得 $D_n=\dfrac{y(x-z)^n-z(x-y)^n}{y-z}$

$n=1$，经过验证，上式同样成立

法二（感谢万骐滔同学提供的解答）

$$D_n=\begin{vmatrix} x-y & 0 & 0 & \cdots & 0 & y \\ z-x & x-y & 0 & \cdots & 0 & y \\ 0 & z-x & x-y & \cdots & 0 & y \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & z-x & x \end{vmatrix}$$

$$=(x-y)D_{n-1}+(x-z)\cdot\begin{vmatrix} 0 & 0 & \cdots & 0 & y \\ z-x & x-y & \cdots & 0 & y \\ 0 & z-x & \cdots & 0 & y \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & z-x & x \end{vmatrix}$$

$$=(x-y)D_{n-1}+(x-z)\cdot(-1)^{1+n-1}\cdot y\cdot(z-x)^{n-2}$$

$$=(x-y)D_{n-1}+y\cdot(x-z)^{n-1}$$ 后同法一

法三 （感谢李一轲同学提供的解答）

（前同法一） 硬解数列通项：$D_n = (x-y)D_{n-1} + y(x-z)^{n-1}$ $D_1 = x$

这是一个一阶非齐次线性递推

齐次递推（$D_n = (x-y)D_{n-1}$）通解形式为 $D_n^{(h)} = C_{(x,y,z)}(x-y)^{n-1}$

非齐次递推（$D_n = (x-y)D_{n-1} + y(x-z)^{n-1}$）有特解形如 $D_n^{(p)} = K_{(x,y,z)}(x-z)^{n-1}$

代回得 $K(x-z)^{n-1} = (x-y)K(x-z)^{n-2} + y(x-z)^{n-1}$

$$K(x-z) = (x-y)K + y(x-z) \quad (\text{不妨设 } x \neq z)$$

$$K = \frac{y(x-z)}{y-z}$$

$$\Rightarrow D_n = D_n^{(p)} + D_n^{(h)} = C(x-y)^{n-1} + \frac{y(x-z)}{y-z}\cdot(x-z)^{n-1}$$

$D_1 = x \Rightarrow C = \dfrac{z(y-x)}{y-z}$

故 $D_n = \dfrac{y(x-z)^n - z(x-y)^n}{y-z}$