On The Metric Nature
of (Differential) Logical Relations

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───── **Abstract** ─────────────────────────────────

Differential logical relations are a method to measure distances between higher-order programs. They differ from standard methods based on program metrics in that differences between functional programs are themselves functions, relating errors in input with errors in output, this way providing a more fine grained, contextual, information. The aim of this paper is to clarify the metric nature of differential logical relations. While previous work has shown that these do not give rise, in general, to (quasi-)metric spaces nor to partial metric spaces, we show that the distance functions arising from such relations, that we call quasi-quasi-metrics, can be related to both quasi-metrics and partial metrics, the latter being also captured by suitable relational definitions. Moreover, we exploit such connections to deduce some new compositional reasoning principles for program differences.

1 Introduction

Program equivalence is a crucial concept in program semantics, and ensures that different implementations of a program produce *exactly* the same results under the same conditions, i.e., in any environment. This concept is fundamental in program verification, code optimization, and for enabling reliable refactoring: by proving that two programs are equivalent, developers and compiler designers can confidently replace one with the other, knowing that the behavior and outcomes will remain consistent. In this respect, guaranteeing that the underlying notion of program equality is a congruence is of paramount importance.

In the research communities mentioned above, however, it is known that comparing programs through a notion of equivalence without providing the possibility of measuring the distance between non-equivalent programs makes it impossible to validate many interesting and useful program transformations [29]. All this has generated interest around the concepts of program metrics and more generally around the study of techniques through which to quantitatively compare non-equivalent programs, so as, e.g., to validate those program transformations which do not introduce too much of an error [32, 28].

What corresponds, in a quantitative context, to the concept of congruence? Once differences are measured by some (pseudo-)metric, a natural answer to this question is to require that any language construct does not increase distances, that is, that they are *non-expansive*. Along with this, the standard properties of (pseudo-)metrics, like the triangle inequality $d(x,z) + d(z,y) \geqslant d(x,y)$, provide general principles that are very useful in metric reasoning, replacing standard qualitative principles (e.g., in this case, transitivity $\mathrm{eq}(x,z) \wedge \mathrm{eq}(z,y) \vdash \mathrm{eq}(x,y)$).

Still, as already observed in many occasions [11, 9], the restriction to language constructs that are non-expansive with respect to some purely numerical metric turns out too severe in practice. On the one hand, the literature focusing on higher-order languages has mostly restricted its attention to linear or graded languages [32, 2], due to well-known difficulties in constructing metric models for full "simply-typed" languages [12]. On the other hand, even if one restricts to a linear language, the usual metrics defined over functional types are hardly useful in practice, as they assign distances to functions f, g via a comparison of their values in the worst case: for instance, as shown in [11], the two maps $\lambda x.x, \lambda x. \sin(x) : \mathsf{Real} \to \mathsf{Real}$, although behaving very closely around 0, are typically assigned the distance ∞, since their values grow arbitrarily far from each other in the worst case.

The *differential logical relations* [11, 9, 30, 10] have been introduced as a solution to the aforementioned problems. In this setting, which natively works for unrestricted higher-order languages, the distance between two programs is not necessarily given as a single number: for instance, two programs of functional type are far apart according to a function itself, which measures how the error in the output depends on the *error* in the input, but also on the *value* of the input itself. This way the notion of distance becomes sufficiently expressive, at the same time guaranteeing the possibility of compositional reasoning. This paradigm also scales to languages with duplication, recursion [9] and works even in presence of effects [10].

In the literature on program metrics, it has become common to consider metrics valued on arbitrary *quantales* [23, 37]. This means that, as for the differential logical relations, the distance between two points needs not be a non-negative real, but can belong to any suitable algebra of "quantities". This has led to the study of different classes of quantale-valued metrics, each characterized by a particular formulation of the triangular law. Among this, *quasi-metrics* [20] and *partial metrics* [4, 24] have been explored for the study of domains, even for higher-order languages [18, 27]. While the first obey the usual triangular inequality, or transitivity, the second obey a *stronger* transitivity condition, also taking into account the replacement of standard reflexivity $d(x, x) = 0$ by a weaker *quasi-reflexivity* condition $d(x, x) \leqslant d(x, y)$, implying that a point need not be a distance zero from itself.

A natural question is thus: do the distances between programs that are obtained via differential logical relations constitute some form of (quantale-valued) metric? In particular, what forms do transitivity and reflexivity do these relations support? The original paper [11] defined symmetric differential logical relations and gave a very weak form of triangle inequality. Subsequent works, relating to the more natural asymmetric case, have either ignored the metric question [9, 10] or shown that the distances produced must violate *both* the reflexivity of quasi-metric and the strong transitivity of partial metrics [18, 30].

This paper aims at providing a bridge between current methods for higher-order program differences and the well-established literature on quantale-valued metrics. More specifically, we show that the distances produced by differential logical relations, that we call *quasi-quasi-metrics* (or *quasi²-metrics*), satisfy the *quasi*-reflexivity of partial metrics and the standard transitivity of quasi-metrics. Such metrics thus sit somehow *in between* quasi-metrics and partial metrics. We will establish precise connections between all those. We also exploit these results to deduce some new principles of compositional reasoning about program differences arising from the different forms of transitivity at play. Finally, we introduce a deductive system, inspired from the quantitative equational theories of Mardare et al. [28], to deduce differences between programs.

Contributions   Our contributions can be summarized as follows:

- We introduce a new class of quantale-valued metrics, called quasi²-metrics. We show that

92 each such metric gives rise to two *observational quasi-metrics* over programs, and can be
93 seen as a relaxation of partial *quasi*-metrics [25]. This is in Section 3;

94 ▪ we establish the equivalence of the cartesian closed structure of quasi^2-metrics with
95 the standard definition of differential logical relations. We also show that observational
96 quasi-metrics as well as partial quasi-metrics can be captured by suitable families of logical
97 relations. We exploit all such definitions to deduce some new compositional reasoning
98 principles for program differences. This spans through Sections 4-7;

99 ▪ finally, we introduce an equational theory for program differences via a syntactic pre-
100 sentation of differential logical relations and we formulate two conjectures about the
101 comparison of the different notions of program distances introduced. This is in Sections 8
102 and 9.

2   From Logical Relations to Differential Logical Relations

104 In this section we recall how differential logical relations can be seen as a quantitative
105 generalization of standard logical relations, at the same time highlighting the metric coun-
106 terparts of qualitative notions like equivalences and preorders. Moreover, we introduce
107 *quasi-quasi-metrics* as the metric counterpart of *quasi*-reflexive and transitive relations.

108 **Logical Relations**   The theory of logical relations is well-known and has been exploited in
109 various directions to establish *qualitative* properties of type systems, like e.g. termination
110 [19], bisimulation [34] or parametricity [31, 22]. The idea is to start from some basic binary
111 relation $\rho_o \subseteq o \times o$ over the terms of some ground type o. The relation ρ_o can then be *lifted*
112 to a family of binary relations $\rho_A \subseteq A \times A$, where A varies over all simple types constructed
113 starting from o (indeed, one may consider recursive [15], polymorphic [33, 31] or monadic
114 [21] types as well, but we here limit our discussion to simple types). The lifting is defined
115 inductively by:

116 $$(t, t') \in \rho_{A \times B} \iff (\mathsf{fst}(t), \mathsf{fst}(t')) \in \rho_A \text{ and } (\mathsf{snd}(t), \mathsf{snd}(t')) \in \rho_B, \qquad (\wedge)$$

117 $$(t, t') \in \rho_{A \Rightarrow B} \iff (\forall s, s' \in A) \ (s, s') \in \rho_A \implies (t\,s, t'\,s') \in \rho_B. \qquad (\Rightarrow)$$

118 Typically, one wishes to establish a so-called *fundamental lemma*, stating that well-typed
119 programs $x : A \vdash t : B$ *preserve relations*. This means that, for *any* choice of a family of
120 logical relations ρ_A defined as above, one can prove

121 $$(\forall s, s' \in A) \ (s, s') \in \rho_A \implies (t[s/x], t[s'/x]) \in \rho_B. \qquad \text{(Fundamental Lemma)}$$

122 Notice that this is equivalent to the instance of reflexivity $(\lambda x.t, \lambda x.t) \in \rho_{A \Rightarrow B}$.

123 Of particular interest are the *equivalence* relations (that is, those which are reflexive,
124 symmetric and transitive) and the *preorders* (that is, the reflexive and transitive ones). We
125 here focus on the latter, as we will not consider symmetry in this paper (see Remark 3). A
126 fundamental observation is that the logical relation lifting preserves preorders (and indeed,
127 equivalences): if ρ_A and ρ_B are reflexive and transitive, then $\rho_{A \times B}$ and $\rho_{A \Rightarrow B}$ are reflexive and
128 transitive as well, *provided that the fundamental lemma holds*. The case of the function space
129 is the most interesting one: as we observed above, the reflexivity condition $(t, t) \in \rho_{A \Rightarrow B}$
130 coincides with the fact that the function t is relation-preserving; transitivity, instead, can be
131 proved by combining relation-preservation, the reflexivity of ρ_A and the transitivity of ρ_B.

132 Any logical relation $\rho \subseteq A \times A$ induces an equivalence $\equiv_\rho$, called the *observational*
133 *equivalence*, where $t \equiv_\rho u$ iff for all $s \in A$, $(s, t) \in \rho$ iff $(s, u) \in \rho$. Intuitively, two terms t, u are

equivalent if the relation ρ cannot distinguish them. For example, if the definition of ρ_o on basic types only depends on the values $\mathsf{t} \Rightarrow^* \mathsf{v}$ produced by terms, one can usually deduce that terms are indistinguishable from their associated values, that is $\mathsf{t} \equiv_\rho \mathsf{v}$. In the absence of symmetry, one obtains two *observational preorders* $\sqsubseteq_\rho^l, \sqsubseteq_\rho^r \subseteq \mathsf{A} \times \mathsf{A}$ defined by:

$$\mathsf{s} \sqsubseteq_\rho^l \mathsf{t} \iff (\forall \mathsf{u} \in \mathsf{A})\ (\mathsf{t}, \mathsf{u}) \in \rho \implies (\mathsf{s}, \mathsf{u}) \in \rho,$$

$$\mathsf{s} \sqsubseteq_\rho^r \mathsf{t} \iff (\forall \mathsf{u} \in \mathsf{A})\ (\mathsf{u}, \mathsf{s}) \in \rho \implies (\mathsf{u}, \mathsf{t}) \in \rho.$$

These preorders satisfy the following useful and easily provable properties:

▸ **Proposition 1.** *For any binary relation $\rho \subseteq \mathsf{A} \times \mathsf{A}$ and $c \in \{l, r\}$,*

(i.) $\sqsubseteq_\rho^c \supseteq \rho$ *iff ρ is transitive;*

(ii.) $\sqsubseteq_\rho^c \subseteq \rho$ *iff ρ is reflexive;*

(iii.) $\sqsubseteq_\rho^c = \rho$ *iff ρ is a preorder;*

(iv) The following hold:

$$(\forall \mathsf{s}, \mathsf{t}, \mathsf{u} \in \mathsf{A})\ \mathsf{s} \sqsubseteq_\rho^l \mathsf{t}\ \wedge\ (\mathsf{t}, \mathsf{u}) \in \rho \implies (\mathsf{s}, \mathsf{u}) \in \rho, \qquad \text{(left transitivity)}$$

$$(\forall \mathsf{s}, \mathsf{t}, \mathsf{u} \in \mathsf{A})\ (\mathsf{s}, \mathsf{t}) \in \rho\ \wedge\ \mathsf{t} \sqsubseteq_\rho^r \mathsf{u} \implies (\mathsf{s}, \mathsf{u}) \in \rho. \qquad \text{(right transitivity)}$$

The reason why we delve into these basic properties of preorders is that we will soon explore their (less trivial!) quantitative counterparts, that arise naturally in the theory of differential logical relations. In particular, the left and right transitivity conditions will correspond to *stronger* variants of the triangular inequality for metric spaces.

Beyond preorders, we are interested in the following weaker notion:

▸ **Definition 2** (quasi-preorder). *A relation $\leq\ \subseteq \mathsf{A} \times \mathsf{A}$ is called a* quasi-preorder *if it is transitive and (left-)quasi-reflexive, that is,* $\mathsf{t} \leq \mathsf{u} \implies \mathsf{t} \leq \mathsf{t}$.

Quasi-preorders are obtained by weakening the reflexivity condition of preorders: intuitively, only the points which are smaller than someone are smaller than themselves. One can easily develop a theory of logical relations for quasi-preorders. The sole delicate point is that, in order to let such relations lift to function spaces, one has to slightly modify the relation lifting as follows:

$$(\mathsf{t}, \mathsf{t}') \in \rho_{\mathsf{A}\Rightarrow\mathsf{B}} \iff (\forall \mathsf{s}, \mathsf{s}' \in \mathsf{A})\ (\mathsf{s}, \mathsf{s}') \in \rho_\mathsf{A} \implies (\mathsf{t}\mathsf{s}, \mathsf{t}'\mathsf{s}') \in \rho_\mathsf{B}\ \wedge\ (\mathsf{t}\mathsf{s}, \mathsf{t}\mathsf{s}') \in \rho_\mathsf{B}. \qquad (\Rightarrow^*)$$

Compared to $(\Rightarrow)$, $(\Rightarrow^*)$ includes a second clause $(\mathsf{t}\mathsf{s}, \mathsf{t}\mathsf{s}') \in \rho_\mathsf{B}$ relating the action of t on both s and s'. With this definition, one can easily check that if $\rho_\mathsf{A}, \rho_\mathsf{B}$ are quasi-preorders, and the fundamental lemma holds, then $\rho_{\mathsf{A}\times\mathsf{B}}$ and $\rho_{\mathsf{A}\Rightarrow\mathsf{B}}$ are quasi-preorders as well.

Differential Logical Relations   We now have all elements to discuss what happens when extending logical relations to a quantitative setting. Rather than considering binary relations $\rho \subseteq \mathsf{A} \times \mathsf{A}$ expressing that a certain property holds for two terms s, t or not, we will consider *ternary* relations $\rho \subseteq \mathsf{A} \times \mathcal{Q}_\mathsf{A} \times \mathsf{A}$, where $(\mathsf{s}, \mathsf{a}, \mathsf{t}) \in \rho$ indicates that a certain relation holds of s, t to *a certain extent*, quantified via $\mathsf{a} \in \mathcal{Q}_\mathsf{A}$. Here $\mathcal{Q}_\mathsf{A}$ is a *quantale*, an algebraic structure (recalled in the next section) that captures several properties of quantities as expressed by e.g. non-negative real numbers.

In fact, just like for standard logical relations, a differential logical relation $\rho_o \subseteq o \times \mathcal{Q}_o \times o$ on a ground type can be *lifted* to a family of binary relations $\rho_\mathsf{A} \subseteq \mathsf{A} \times \mathcal{Q}_\mathsf{A} \times \mathsf{A}$ over simple types.

First, we define, by induction, the quantales $\mathcal{Q}_{A \times B} = \mathcal{Q}_A \times \mathcal{Q}_B$ and $\mathcal{Q}_{A \Rightarrow B} = A \Rightarrow (\mathcal{Q}_A \rightharpoonup \mathcal{Q}_B)$, where $\mathcal{Q}_A \rightharpoonup \mathcal{Q}_B$ is the quantale of monotone functions. We then define the lifting of ρ_o by:

$$((\mathsf{t}, \mathsf{u}), (a, b), (\mathsf{t}', \mathsf{u}')) \in \rho_{A \times B} \iff (\mathsf{t}, a, \mathsf{t}') \in \rho_A \text{ and } (\mathsf{u}, b, \mathsf{u}') \in \rho_A,$$

$$(\mathsf{t}, f, \mathsf{t}') \in \rho_{A \Rightarrow B} \iff (\forall \mathsf{s}, \mathsf{s}' \in A, \forall a \in \mathcal{Q}_A) \text{ if } (\mathsf{s}, a, \mathsf{s}') \in \rho_A, \text{ then}$$

$$(\mathsf{t}\,\mathsf{s}, f(\mathsf{t})(a), \mathsf{t}\,\mathsf{s}') \in \rho_B \text{ and } (\mathsf{t}\,\mathsf{s}, f(\mathsf{t})(a), \mathsf{t}'\,\mathsf{s}') \in \rho_B.$$

Notice that the definition of $\rho_{A \Rightarrow B}$ closely imitates the clause $(\Rightarrow^*)$ for quasi-preorders. Also observe that the quantale $\mathcal{Q}_{A \Rightarrow B}$ for the function type is itself a set of functions relating terms of type A and quantities in $\mathcal{Q}_A$ with quantities in $\mathcal{Q}_B$. As we show in Section 5, this definition gives rise to an interpretation of the simply typed λ-calculus where a fundamental lemma holds under the following form: for all terms $x : A \vdash \mathsf{t} : B$ and choice of a family of differential logical relations ρ_A as above, there exists a map $\mathsf{t}^\bullet : A \Rightarrow (\mathcal{Q}_A \rightharpoonup \mathcal{Q}_B)$ such that

$$(\forall \mathsf{s}, \mathsf{s}' \in A, \forall a \in \mathcal{Q}_A) \ (\mathsf{s}, a, \mathsf{s}') \in \rho_A \ \Rightarrow \ (\mathsf{t}\,\mathsf{s}, \mathsf{t}^\bullet(\mathsf{s})(a), \mathsf{t}\,\mathsf{s}') \in \rho_B. \qquad \text{(fundamental lemma)}$$

The function $\mathsf{t}^\bullet$ behaves like some sort of *derivative* of t: it relates errors in input with errors in output. This connection is investigated in more detail in [9, 30].

So far, everything works just as in the standard, qualitative, case. However, the quantitative setting is well visible when we consider the corresponding notions of equivalences and preorders. Recall that an (integral) quantale is, in particular, an ordered monoid $(\mathcal{Q}, +, 0, \leqslant)$ of which 0 is the minimum element. For a differential logical relation $\rho \subseteq A \times \mathcal{Q}_A \times A$, reflexivity, symmetry and transitivity translate into the following conditions:

$$(\forall \mathsf{t} \in A) \ (\mathsf{t}, 0, \mathsf{t}) \in \rho, \qquad \qquad \text{(reflexivity)}$$

$$(\forall \mathsf{t}, \mathsf{u} \in A, \forall a \in \mathcal{Q}_A) \ (\mathsf{t}, a, \mathsf{u}) \in \rho \ \Rightarrow \ (\mathsf{u}, a, \mathsf{t}) \in \rho, \qquad \qquad \text{(symmetry)}$$

$$(\forall \mathsf{s}, \mathsf{t}, \mathsf{u} \in A, \forall a, b \in \mathcal{Q}_A) \ (\mathsf{s}, a, \mathsf{t}) \in \rho \ \wedge \ (\mathsf{t}, b, \mathsf{u}) \in \rho \ \Rightarrow \ (\mathsf{s}, a + b, \mathsf{u}) \in \rho. \qquad \text{(transitivity)}$$

It is clear then that equivalence relations translate, in the quantitative setting, into some kind of metric space. Similarly, the quantitative counterpart of preorders are the so-called *quasi-metric spaces* [20], essentially, metrics without a symmetry condition, indeed a very well-studied class of metrics. In particular, we will show that, similarly to preorders, any ternary relation $\rho \subseteq A \times \mathcal{Q}_A \times A$ gives rise to left and right *observational quasi-metrics* $q_\rho^l, q_\rho^r : A \times A \to \mathcal{Q}_A$ satisfying properties analogous to those of Proposition 1.

▶ **Remark 3.** While in the original definition [11] differential logical relations were symmetric, symmetry was abandoned in all subsequent works. The first reason is that several interesting notions of program difference, like e.g. those arising from *incremental computing* [9, 6, 1], are not symmetric. A second reason is that the cartesian closure is problematic in presence of both quasi-reflexivity and symmetry [30].

There is, however, an important point on which differential logical relations differ from standard logical relations: while the former lift preorders well to all simple types, their quantitative counterpart, the quasi-metrics, are *not* preserved by the higher-order lifting of differential logical relations. Indeed, we observed that an essential ingredient in the lifting of the reflexivity property is the fundamental lemma; yet, in the framework of differential logical relations, the fundamental lemma produces, for any term $\mathsf{t} : A \Rightarrow B$, the "reflexivity" condition $(\mathsf{t}, \mathsf{t}^\bullet, \mathsf{t}) \in \rho_{A \Rightarrow B}$, which differs from standard reflexivity in that the distance is $\mathsf{t}^\bullet$ and *not* the minimum element 0. This means that the metric structure arising from differential logical relation cannot be that of standard (quasi-)metric spaces. Rather, it must be something close to the *partial* metric spaces [4, 24], that is, metric spaces in which the

condition $d(x,x) = 0$ is replaced by the quasi-reflexivity condition $d(x,x) \leqslant d(x,y)$. We will discuss the connections with partial metric spaces in the next sections.

By replacing reflexivity with quasi-reflexivity, we obtain the quantitative counterpart of quasi-preorders, that we call *quasi^2-metrics* (being "quasi" both in the sense of quasi-metrics, i.e. the rejection of symmetry, and of quasi-preorders, i.e. the weakening of reflexivity).

▸ **Definition 4.** *For a set X and a quantale $\mathcal{Q}$, a relation $\rho \subseteq X \times \mathcal{Q} \times X$ is called* quasi-quasi-metric *(or more concisely* quasi^2-metric*) if it is transitive and satisfies the condition*

$$(\forall x, y \in X, \forall a \in \mathcal{Q}) \ (x, a, y) \in \rho \ \Rightarrow \ (x, a, x) \in \rho. \qquad \text{(quasi-reflexivity)}$$

As shown in Section 4, the quasi^2-metrics capture the properties of distances which are preserved by differential logical relations: indeed, the argument showing that the quasi-preorders lift to all simple types scales well to the quantitative setting, showing that a quasi^2-metric on the base types gives rise to quasi^2-metrics on all simple types.

The obvious question, however, is: what are these quasi^2-metrics? How are they related to the more standard quasi-metrics and partial metrics? This is what we are going to do in the following section.

3   Quasi^2-Metric Spaces

In this section we use the language of quantale-valued relations to explore the connections between the quasi^2-metrics introduced in the previous section and the more well-established notions of quasi-metric and partial quasi-metric spaces.

Quantale-Valued Relations   Let us recall that a quantale $\mathcal{Q}$ is a complete lattice $(\mathcal{Q}, \sqsubseteq)$ endowed with a continuous monoidal operation $\otimes$, with unit 1. A quantale $\mathcal{Q}$ is *unital* when $1 = \top$ and *commutative* when $\otimes$ is commutative. Suppose $\mathcal{Q}$ is commutative. Given $x, y \in \mathcal{Q}$, their *residual* is defined as $x \multimap y := \bigvee \{z \in \mathcal{Q} \mid z \otimes x \sqsubseteq y\}$ where $\sqsubseteq$ is the partial order of $\mathcal{Q}$. Notice that $z \sqsubseteq x \multimap y$ iff $z \otimes x \sqsubseteq y$, and that $(x \multimap y) \otimes x \sqsubseteq y \sqsubseteq x \multimap (y \otimes x)$. A commutative quantale $\mathcal{Q}$ is *divisible* [24] if for all $x, y \in \mathcal{Q}$, $x \sqsubseteq y$ holds iff $y \otimes (y \multimap x) = x$. Equivalently, $\mathcal{Q}$ is divisible iff, whenever $x \sqsubseteq y$, there exists z such that $x = y \otimes z$. In the following we will use $\mathcal{Q}$ to refer to a commutative, unital and divisible quantales.

▸ **Example 5.** The *Lawvere quantale* is formed by the non-negative extended reals $[0, +\infty]$ with the *reversed* order $x \sqsubseteq y := x \geqslant y$, and with addition as monoidal operation. Notice that the ordering of quantales is *reversed* with respect to usual metric intuitions: the "0" element is the $\top$, joins correspond to taking infs, etc.

Given a quantale $\mathcal{Q}$ and sets X, Y, a *$\mathcal{Q}$-relation over X, Y* is a map $s\colon X \times Y \to \mathcal{Q}$, which can be visualized as a matrix with values in $\mathcal{Q}$. For $\mathcal{Q}$-relations $s, t\colon X \times Y \to \mathcal{Q}$, we write $s \sqsubseteq t$ when $s(x,y) \sqsubseteq t(x,y)$ for all $x \in X$ and $y \in Y$. Given $\mathcal{Q}$-relations $s\colon X \times Y \to \mathcal{Q}$, $t\colon Y \times Z \to \mathcal{Q}$ and $u\colon X \times Z \to \mathcal{Q}$, $w\colon Z \times Y \to \mathcal{Q}$, we define the $\mathcal{Q}$-relations $s \otimes t\colon X \times Z \to \mathcal{Q}$ and $u \multimap s\colon Z \times Y \to \mathcal{Q}$ and $s \circ\!\!-\, w\colon X \times Z \to \mathcal{Q}$ via the two operations:

$$(s \otimes t)(x,z) = \bigvee_{y \in Y} s(x,y) \otimes t(y,z),$$

$$(u \multimap s)(z,y) = \bigwedge_{x \in X} u(x,z) \multimap s(x,y), \qquad (s \circ\!\!-\, w)(x,z) = \bigwedge_{y \in Y} w(z,y) \multimap s(x,y).$$

The monoidal product $\otimes$ and the residuals $\multimap$, $\circ\!\!-$ of $\mathcal{Q}$-relations satisfy properties analogous to residuals in $\mathcal{Q}$, e.g. $s \otimes (s \multimap t) \sqsubseteq t$, $(t \circ\!\!-\, s) \otimes s \sqsubseteq t$. It is well-known that $\mathcal{Q}$-relations

form a category $\mathcal{Q}\mathbf{Rel}$ whose objects are sets and such that $\mathcal{Q}\mathbf{Rel}(X, Y)$ are the $\mathcal{Q}$-relations from X to Y. The operation $s \otimes t$ is the composition operator of this category, while the identities are the relations defined as $\mathbf{1}_X(x, x) = 1 = \top$ and $\mathbf{1}_X(x, y \neq x) = \bot$.

Finally, for any relation $s \in \mathcal{Q}\mathbf{Rel}(X, X)$, define the relations $\Delta_1 s, \Delta_2 s \in \mathcal{Q}\mathbf{Rel}(X, X)$ by $\Delta_1 s = s \circ \Delta \circ \pi_1$ and $\Delta_2 s = s \circ \Delta \circ \pi_2$, that is, $\Delta_1 s(x, y) := s(x, x)$, $\Delta_2 s(x, y) = s(y, y)$.

Quasi^2- and Quasi-Metric Spaces For a relation $s \in \mathcal{Q}\mathbf{Rel}(X, X)$, reflexivity $s(x, x) = 1$ and transitivity $s(x, z) \otimes s(z, y) \sqsubseteq s(x, y)$ can be written more concisely as $s \sqsupseteq \mathbf{1}_X$ and $s \otimes s \sqsubseteq s$. A relation s satisfying both such properties is called a *quasi-metric over X* (with values in $\mathcal{Q}$). The following construction generalizes the observational preorders to $\mathcal{Q}$-relations:

▸ **Proposition 6.** *For all $s \in \mathcal{Q}\mathbf{Rel}(X, X)$, the relations $q_s^l := s \multimapinv s, q_s^r := s \multimap s \in \mathcal{Q}\mathbf{Rel}(X, X)$ are quasi-metrics and, for $c \in \{l, r\}$, the following hold:*

(i.) $q_s^c \sqsupseteq s$ iff s is transitive;

(ii.) $q_s^c \sqsubseteq s$ iff s is reflexive;

(iii.) $q_s^c = s$ iff s is a quasi-metric;

(iv) $q_s^l \otimes s \sqsubseteq s$ and $s \otimes q_s^r \sqsubseteq s$, that is, the following hold:

$$(\forall x, y, z \in X) \quad q_s^l(x, z) \otimes s(z, y) \sqsubseteq s(x, y), \qquad\qquad \text{(left transitivity)}$$

$$(\forall x, y, z \in X) \quad s(x, z) \otimes q_s^r(z, y) \sqsubseteq s(x, y). \qquad\qquad \text{(right transitivity)}$$

We call the quasi-metrics q_s^l, q_s^r the *left and right observational quasi-metric of s.*

Quasi^2-metrics correspond to $\mathcal{Q}$-relations $s \in \mathcal{Q}\mathbf{Rel}(X, X)$ satisfying transitivity $s \otimes s \sqsubseteq s$ and quasi-reflexivity $s \sqsubseteq \Delta_1 s$ (i.e. $s(x, y) \sqsubseteq s(x, x)$). From transitivity, we deduce that, for a quasi^2-metric s, both $q_s^l, q_s^r \sqsupseteq s$ hold, that is, the observational quasi-metrics yield *tighter* distances than s. This implies that left and right transitivity read as *stronger* forms of the triangular inequality. In particular, the following alternative characterization of quasi^2-metrics holds:

▸ **Proposition 7.** *For any quasi-reflexive $s \in \mathcal{Q}\mathbf{Rel}(X, X)$, s is a quasi^2-metric iff there exists a quasi-metric $q \sqsupseteq s$ such that either $s \otimes q \sqsubseteq s$ or $q \otimes s \sqsubseteq s$ holds.*

Proof. Suppose there exists a quasi-metric $q \sqsupseteq s$ such that $s \otimes q \sqsubseteq s$ holds. Then $s \otimes s \sqsubseteq s \otimes q \sqsubseteq s$, so s is transitive. A similar argument works if q is such that $q \otimes s \sqsubseteq s$. Conversely, if s is a quasi^2-metric it is enough to let $q := q_s^r$ and use Proposition 6 (iv). ◂

Partial Metric Spaces Let us now discuss the connection with partial metric spaces. We here consider the non-symmetric variant of the partial metric spaces from [4], called partial *quasi*-metric spaces (PQM) [25]. As we anticipated, these are metrics p for which the usual reflexivity condition $p(x, x) = 1$ is replaced by the weaker quasi-reflexivity condition $p(x, x) \sqsupseteq p(x, y)$. However, unlike the quasi^2-metrics just discussed, PQMs satisfy a *stronger* transitivity condition. When $\mathcal{Q} = [0, +\infty]$ is the Lawvere quantale, this condition reads as

$$p(x, z) + p(z, y) - p(z, z) \sqsupseteq p(x, y). \qquad\qquad \text{(strong transitivity in } [0, +\infty])$$

The idea is that the self-distance of the central term z is "subtracted". For a general quantale $\mathcal{Q}$, this becomes:

$$p(x, z) \otimes (p(z, z) \multimap p(z, y)) \sqsubseteq p(x, y). \qquad\qquad \text{(strong transitivity)}$$

Define the relations $\Theta_s^l, \Theta_s^r \in \mathcal{Q}\mathbf{Rel}(X, X)$ by $\Theta_s^l(x, y) = s(y, y) \multimap s(x, y)$ and $\Theta_s^r(x, y) = s(x, x) \multimap s(x, y)$. A PQM can be thus more concisely be defined as a relation $s \in \mathcal{Q}\mathbf{Rel}(X, X)$ satisfying $s \sqsubseteq \Delta_1 s$ and $s \otimes \Theta_s^r \sqsubseteq s$. Notice that strong transitivity $s \otimes \Theta_s^r \sqsubseteq s$ looks similar to the right transitivity $s \otimes q_s^r \sqsubseteq s$. Indeed, the following result relates the relations Θ_s^c and q_s^c:

▸ **Proposition 8.** *For all $s \in \mathcal{Q}\mathbf{Rel}(X, X)$ and $c \in \{l, r\}$, $q_s^c \sqsubseteq \Theta_s^c$. Moreover, if s is quasi-reflexive, $\Theta_s^c \sqsubseteq q_s^c$ holds iff Θ_s^c is a quasi-metric iff s is a partial quasi-metric.*

Proof. We only argue for $c = r$, the other case being similar. From $q_s^r(x, y) = \bigwedge_z s(z, x) \multimap s(z, y) \sqsubseteq s(x, x) \multimap s(x, y) = (\Theta_s^r)(x, y)$ we deduce that $q_s^r \sqsubseteq \Theta_s^r$. The converse direction $q_s^r \sqsupseteq \Theta_s^r$ corresponds to showing that $s(z, x) \otimes (s(x, x) \multimap s(x, y)) \sqsubseteq s(z, y)$, which holds iff s is a partial quasi-metric. We have thus shown that s is a PQM iff $\Theta_s^r = q_s^r$. This also implies that, if s is a PWM, Θ_s^r is a quasi-metric. Finally, suppose Θ_s^r is a quasi-metric. By quasi-reflexivity, and the divisibility of $\mathcal{Q}$, we have that $s(x, z) = s(x, x) \otimes (s(x, x) \multimap s(x, z))$. We then have $s(x, z) \otimes (s(z, z) \multimap s(z, y)) = s(x, x) \otimes (s(x, x) \multimap s(x, z)) \otimes (s(z, z) \multimap s(z, y)) \sqsubseteq s(x, y)$, so s is a partial quasi-metric. ◂

The result above suggests that the partial quasi-metrics can be seen as limit cases of the quasi^2-metrics, namely those for which the quasi-metric $q_s^r(x, y)$ can be written under the simpler form $\Theta_s^r(x, y) = s(x, x) \multimap s(x, y)$.

Unfortunately, while the standard definition of differential logical relations preserves quasi^2-metrics, it does *not* preserve partial quasi-metrics: [18, 30] show that the function space constructions lifts PQMs into PQMs only when the monoidal product of the underlying quantales is *idempotent* (one talks in this case of a partial *ultra*-metric, since strong transitivity becomes $p(x, z) \wedge p(z, y) \sqsubseteq p(x, y)$). Nevertheless, we will show in Section 6 how one can capture PQMs via a suitable family of logical relations.

4 Differential Logical Relations as Quasi^2-Metrics

In this section we provide a semantic presentation of differential logical relations by defining a cartesian closed category of quasi^2-metrics, this way highlighting the close correspondence between these two notions.

From $\mathcal{Q}$-Relations to Ternary Relations   While in the previous section we discussed $\mathcal{Q}$-relations, that is, *binary* relations valued in a quantale $\mathcal{Q}$, the theory of differential logical relations is expressed in terms of *ternary* relations $\rho \subseteq X \times \mathcal{Q} \times X$. In fact, any such relation $\rho \subseteq X \times \mathcal{Q} \times X$ induces a $\mathcal{Q}$-relation $\widehat{\rho} \in \mathcal{Q}\mathbf{Rel}(X, X)$ defined by

$$\widehat{\rho}(x, y) = \bigvee \{a \in \mathcal{Q} \mid (x, a, y) \in \rho\}.$$

Intuitively, $\widehat{\rho}(x, y)$ is the *smallest* (recall the inversion of the order) distance between x and y. This correspondence can be made more precise as follows: let a ternary relation $\rho \subseteq X \times \mathcal{Q} \times X$ be said $\mathcal{Q}$-*closed* when the following hold:

- $(x, a, y) \in \rho$ and $a' \sqsubseteq a$ implies $(x, a', y) \in \rho$;
- if $(x, a_i, y) \in \rho$, for all $i \in I$, then $(x, \bigvee_{i \in I} a_i, y) \in \rho$.

We have the following correspondence:

▸ **Lemma 9.** *The map $\rho \mapsto \widehat{\rho}$ defines a bijection between the $\mathcal{Q}$-closed relations $\rho \subseteq X \times \mathcal{Q} \times X$ and $\mathcal{Q}\mathbf{Rel}(X, X)$.*

Proof. Let ρ, τ be closed and let $\widehat{\rho}(x, y) = \widehat{\tau}(x, y)$. Observe that, for all $x, y \in X$, by $\mathcal{Q}$-closure we have $(x, \widehat{\rho}(x, y), y) \in \rho$. Suppose now that $(x, a, y) \in \tau$, then $a \sqsubseteq \widehat{\tau}(x, y) = \widehat{\rho}(x, y)$, and from $(x, \widehat{\rho}(x, y), y) \in \rho$ and $a \sqsubseteq \widehat{\rho}(x, y)$ we deduce $(x, a, y) \in \rho$. By a similar argument we can also prove that $(x, a, y) \in \rho$ implies $(x, a, y) \in \tau$, so in the end $\rho = \tau$. We conclude then that the map $\rho \mapsto \widehat{\rho}$ is injective. For surjectivity, observe that any $s \in \mathcal{Q}\mathbf{Rel}(X, X)$ induces a relation $(x, a, y) \in \rho^s$ iff $a \sqsubseteq s(x, y)$, so that $s = \widehat{\rho^s}$. ◄

In the sequel, we will identify metrics with their corresponding $\mathcal{Q}$-closed relations.

A Cartesian Closed Category of Quasi^2-Metrics We now define a category of quasi^2-metrics. Let us recall notations from Section 2. For sets A and B, we denote the set of functions from A to B by $A \Rightarrow B$; for quantales $\mathcal{Q}$ and $\mathcal{R}$, we denote the set of monotone functions from $\mathcal{Q}$ to $\mathcal{R}$ by $\mathcal{Q} \rightharpoonup \mathcal{R}$. Below, we write $f \cdot x$ for the application of $f : A \to B$ to $x \in A$, and we suppose that $(-) \cdot (-)$ is left-associative, i.e., $f \cdot x \cdot y$ is an abbreviation of $(f \cdot x) \cdot y$.

The category **Qqm** of quasi^2-metrics is defined as follows:

- objects are triples $X = (\mathcal{Q}_X, |X|, \rho_X)$ consisting of a quantale $\mathcal{Q}_X$, a set $|X|$ and a quasi^2-metric $\rho_X \subseteq |X| \times \mathcal{Q}_X \times |X|$;
- morphisms from X to Y are triples (f, a, f') consisting of functions $f, f' : |X| \to |Y|$ and $a : |X| \to (\mathcal{Q}_X \rightharpoonup \mathcal{Q}_Y)$ such that for all $(x, b, x') \in \rho_X$, we have $(f \cdot x, a \cdot x \cdot b, f' \cdot x') \in \rho_Y$ and $(f \cdot x, a \cdot x \cdot b, f \cdot x') \in \rho_Y$.

The identity morphism on an object X is $(\mathrm{id}_X, i_X, \mathrm{id}_X)$ consisting of the identity function id_X on $|X|$ and a function $i_X : |X| \to (\mathcal{Q}_X \rightharpoonup \mathcal{Q}_X)$ given by $i_X \cdot x \cdot a = a$. The composition of $(f, a, f') : X \to Y$ and $(g, b, g') : Y \to Z$ is $(g \circ f, c, g' \circ f')$ where $c : |X| \to (\mathcal{Q}_X \rightharpoonup \mathcal{Q}_Z)$ is given by $c \cdot x = (b \cdot (f \cdot x)) \circ (a \cdot x)$.

▶ **Proposition 10.** *The category* **Qqm** *is cartesian closed.*

The cartesian closed structure corresponds to the construction of differential logical relations in Section 2. The terminal object $\top$ is $(\{*\}, \{*\}, \rho_\top)$ where $\rho_\top = \{(*, *, *)\}$, and the product of X and Y is $X \times Y = (\mathcal{Q}_X \times \mathcal{Q}_Y, |X| \times |Y|, \rho_{X \times Y})$, where $\rho_{X \times Y}$ is given by

$$((x, y), (a, b), (x', y')) \in \rho_{X \times Y} \iff (x, a, x') \in \rho_X \text{ and } (y, b, y') \in \rho_Y.$$

The exponential $X \Rightarrow Y$ is given by $(|X| \Rightarrow (\mathcal{Q}_X \rightharpoonup \mathcal{Q}_Y), |X| \Rightarrow |Y|, \rho_{X \Rightarrow Y})$ where

$$(f, a, f') \in \rho_{X \Rightarrow Y} \iff \text{for all } (x, b, x') \in \rho_X \text{ and } g \in \{f, f'\}, (f \cdot x, a \cdot x \cdot b, g \cdot x') \in \rho_Y.$$

Here, the quantale structure of $\mathcal{Q}_{X \times Y}$ and $\mathcal{Q}_{X \Rightarrow Y}$ are given by the pointwise manner. The first projection from $X \times Y$ to Y is given by $(\mathrm{proj}_{X,Y}, \varpi_{X,Y}, \mathrm{proj}_{X,Y})$ where $\mathrm{proj}_{X,Y} : |X| \times |Y| \to |X|$ is the first projection, and

$$\varpi_{X,Y} : |X| \times |Y| \to (\mathcal{Q}_X \times \mathcal{Q}_Y \rightharpoonup \mathcal{Q}_X)$$

is given by $\varpi_{X,Y} \cdot (x, y) \cdot (a, b) = a$. The second projection is given in the same manner. The tupling of $(f, a, f') : Z \to X$ and $(g, b, g') : Z \to Y$ is $(\langle f, g \rangle, \langle a, b \rangle, \langle f', g' \rangle)$ where $\langle f, g \rangle : |Z| \to |X| \times |Y|$ and $\langle a, b \rangle : |Z| \to (\mathcal{Q}_Z \rightharpoonup \mathcal{Q}_X \times \mathcal{Q}_Y)$ are the tupling of f, g and a, b respectively:

$$\langle f, g \rangle \cdot z = (f \cdot z, g \cdot z), \qquad \langle a, b \rangle \cdot z \cdot c = (a \cdot z \cdot c, a \cdot z \cdot c).$$

The currying of $(f, a, f') : Z \times X \to Y$ is $(f^\wedge, a^\wedge, f'^\wedge)$ where $f^\wedge : |Z| \to (|X| \Rightarrow |Y|)$ and $f'^\wedge : |Z| \to (|X| \Rightarrow |Y|)$ are the currying of the following functions

$$f : |Z| \times |X| \to |Y|, \qquad f' : |Z| \times |X| \to |Y|,$$

$$
\frac{x : A \in \Gamma}{\Gamma \vdash x : A}
\qquad
\frac{r \in \mathbb{R}}{\Gamma \vdash \underline{r} : \mathsf{Real}}
\qquad
\frac{\Gamma \vdash t_1 : \mathsf{Real} \quad \dots \quad \Gamma \vdash t_{\mathrm{ar}(\phi)} : \mathsf{Real}}{\Gamma \vdash \phi(t_1, \dots, t_{\mathrm{ar}(\phi)}) : \mathsf{Real}}
\qquad
\frac{\Gamma \vdash t : A \Rightarrow B \quad \Gamma \vdash s : A}{\Gamma \vdash t\,s : B}
$$

$$
\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\,t : A \Rightarrow B}
\qquad
\frac{\Gamma \vdash t : A \quad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : A \times B}
\qquad
\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{fst}(t) : A}
\qquad
\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathsf{snd}(t) : B}
$$

🟨 **Figure 1** Typing Rules

and $a^\wedge : |Z| \to (\mathcal{Q}_Z \multimap \mathcal{Q}_{X \Rightarrow Y})$ is the currying of

$$a \colon |Z| \times |X| \to (\mathcal{Q}_{Z \times X} \multimap \mathcal{Q}_Y),$$

namely, $a^\wedge \cdot z \cdot a \cdot x \cdot b$ is defined to be $a \cdot (z, x) \cdot (a, b)$. The evaluation morphism

$$(\mathrm{eval}_{X,Y}, \varepsilon_{X,Y}, \mathrm{eval}_{X,Y}) \colon (X \Rightarrow Y) \times X \to Y$$

consists of the evaluation function $\mathrm{eval}_{X,Y} \colon (|X| \Rightarrow |Y|) \times |X| \to |Y|$ and $\varepsilon_{X,Y}$ is given by

$$\varepsilon_{X,Y} \cdot (a, b) \cdot (f, x) = a \cdot x \cdot b.$$

▸ **Example 11.** We define an object $R \in \mathbf{Qqm}$ to be $(\mathbb{R}, [0, +\infty], \rho_R)$, where

$$(x, a, x') \in \rho_R \iff |x - x'| \sqsupseteq a.$$

Observe that the distance $\widehat{\rho}_R : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ is just the Euclidean distance $\widehat{\rho}_R(x, y) = |y - x|$. For functions $f, g \colon \mathbb{R} \to \mathbb{R}$, an element $a \in \mathcal{Q}_{R \Rightarrow R}$ satisfies $(f, a, g) \in \rho_{R \Rightarrow R}$ if and only if we have $a \cdot x \cdot b \sqsubseteq \bigwedge_{|x-y| \sqsupseteq b} |f \cdot x - g \cdot y|$, i.e., a bounds gaps between outputs of f and g. In particular, we have $(f, \top, f) \in \rho_{R \Rightarrow R}$ if and only if f is a constant function. We note that the largest element $\top \in \mathcal{Q}_{R \Rightarrow R}$ is given by $\top \cdot x \cdot b = 0$.

🟨 5 The Fundamental Lemma

In this section we establish the fundamental lemma of differential logical relations for a simply typed lambda calculus Λ_{Real}, by relying on the cartesian closed category $\mathbf{Qqm}$ of quasi^2-metrics. We then apply this result to measure differences between functions.

Syntax and Set-theoretic Semantics   Our language Λ_{Real} comprises a type of real numbers and first order functions on $\mathbb{R}$. Let Var be a countably infinite set of variables. We define *type*s and *term*s as follows:

(type) $A, B \coloneqq \mathsf{Real} \mid A \times B \mid A \Rightarrow B,$

(term) $t, s \coloneqq x \in \mathsf{Var} \mid \underline{r} \mid \phi(t_1, \dots, t_n) \mid t\,s \mid \lambda x : A.\,t \mid \langle t, s \rangle \mid \mathsf{fst}(t) \mid \mathsf{snd}(s).$

Here, r varies over $\mathbb{R}$, and ϕ varies over the set of multi-arity functions on $\mathbb{R}$, namely, ϕ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ for some $n \in \mathbb{N}$. We call n the *arity* of ϕ, and we denote the arity of ϕ by $\mathrm{ar}(\phi)$. We adopt the standard typing rules given in Figure 1. Below, we denote the set of types by $\mathbf{Type}$ and the set of closed terms of type A by $\mathbf{T}_A$.

We denote the standard set theoretic interpretation of Λ_{Real} by $(\!|-|\!)$. (See [26] for example.) To be concrete, the interpretation $(\!|A|\!)$ of a type A is a set inductively defined by

$$(\!|\mathsf{Real}|\!) = \mathbb{R}, \qquad (\!|A \times B|\!) = (\!|A|\!) \times (\!|B|\!), \qquad (\!|A \Rightarrow B|\!) = (\!|A|\!) \Rightarrow (\!|B|\!);$$

and we interpret a term $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ as a function $(\!|t|\!)$ from $(\!|A_1|\!) \times \cdots \times (\!|A_n|\!)$ to $(\!|B|\!)$. The function $(\!|t|\!)$ is given by induction on the derivation of $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ as follows:

$$(\!|x_i|\!) \cdot (x_1, \ldots, x_n) = x_i,$$

$$(\!|r|\!) \cdot (x_1, \ldots, x_n) = r,$$

$$(\!|\phi(t_1, \ldots, t_{\mathrm{ar}(\phi)})|\!) \cdot (x_1, \ldots, x_n) = \phi((\!|t_1|\!) \cdot (x_1, \ldots, x_n), \ldots, (\!|t_{\mathrm{ar}(\phi)}|\!) \cdot (x_1, \ldots, x_n)),$$

$$(\!|t\,s|\!) \cdot (x_1, \ldots, x_n) = (\!|t|\!) \cdot (x_1, \ldots, x_n) \cdot ((\!|s|\!) \cdot (x_1, \ldots, x_n)),$$

$$(\!|\lambda x : A.\, t|\!) \cdot (x_1, \ldots, x_n) \cdot y = (\!|t|\!) \cdot (x_1, \ldots, x_n, y),$$

$$(\!|\mathsf{fst}(t)|\!) \cdot (x_1, \ldots, x_n) = \text{the first component of } (\!|t|\!) \cdot (x_1, \ldots, x_n),$$

$$(\!|\mathsf{snd}(t)|\!) \cdot (x_1, \ldots, x_n) = \text{the second component of } (\!|t|\!) \cdot (x_1, \ldots, x_n),$$

$$(\!|\langle t, s \rangle|\!) \cdot (x_1, \ldots, x_n) = ((\!|t|\!) \cdot (x_1, \ldots, x_n), (\!|s|\!) \cdot (x_1, \ldots, x_n)).$$

The Fundamental Lemma We inductively define a quasi[2]-metric space $[\![A]\!]$ by

$$[\![\mathsf{Real}]\!] = R, \qquad [\![A \times B]\!] = [\![A]\!] \times [\![B]\!], \qquad [\![A \Rightarrow B]\!] = [\![A]\!] \Rightarrow [\![B]\!],$$

and we simply denote the structure of an object $[\![A]\!]$ by $(|A|, \mathcal{Q}_A, \rho_A)$. It is straightforward to check that for every type A, we have $|A| = (\!|A|\!)$. The quasi[2]-metrics $[\![A]\!]$ are the categorical interpretation of types A, and the following fundamental lemma is derived from the categorical interpretation of Λ_{Real}-terms in $\mathbf{Qqm}$.

▸ **Theorem 12** (Fundamental Lemma). *Let* $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ *be a typing context. For every term* $\Gamma \vdash t : A$, *and for every* $(x, a, x') \in \rho_{A_1 \times \cdots \times A_n}$, *we have*

$$((\!|t|\!) \cdot x, \{\!|t|\!\} \cdot x \cdot a, (\!|t|\!) \cdot x') \in \rho_A$$

where we inductively define $\{\!|t|\!\} \in \mathcal{Q}_{A_1 \times \cdots \times A_n \Rightarrow B}$ *as follows:*
- *We define* $\{\!|x_i|\!\} \cdot (x_1, \ldots, x_n) \cdot (a_1, \ldots, a_n)$ *to be* a_i.
- *We define* $\{\!|r|\!\} \cdot x \cdot a$ *to be* $\underline{0}$.
- *We define* $\{\!|\phi(t_1, \ldots, t_n)|\!\} \cdot x \cdot a$ *to be* $\phi^{\mathrm{d}}((\!|t_1|\!) \cdot x, \ldots, (\!|t_n|\!) \cdot x, \{\!|t_1|\!\} \cdot x \cdot a, \ldots, \{\!|t_n|\!\} \cdot x \cdot a)$
  where we define $\phi^{\mathrm{d}} : \mathbb{R}^n \times [0, +\infty]^n \to [0, +\infty]$ *by*

$$\phi^{\mathrm{d}}(y_1, \ldots, y_n, b_1, \ldots, b_n) = \bigwedge_{|y_1 - z_1| \sqsupseteq b_1} \cdots \bigwedge_{|y_n - z_n| \sqsupseteq b_n} |\phi(y_1, \ldots, y_n) - \phi(z_1, \ldots, z_n)|.$$

- *We define* $\{\!|t\,s|\!\} \cdot x \cdot a$ *to be* $\{\!|t|\!\} \cdot x \cdot a \cdot ((\!|s|\!) \cdot x) \cdot (\{\!|s|\!\} \cdot x \cdot a)$.
- *We define* $\{\!|\lambda x : A.\, t|\!\} \cdot (x_1, \ldots, x_n) \cdot (a_1, \ldots, a_n) \cdot y \cdot b$ *to be* $\{\!|t|\!\} \cdot (x_1, \ldots, x_n, y) \cdot (a_1, \ldots, a_n, b)$.
- *We define* $\{\!|\langle t, s \rangle|\!\} \cdot x \cdot a$ *to be* $(\{\!|t|\!\} \cdot x \cdot a, \{\!|s|\!\} \cdot x \cdot a)$.
- *We define* $\{\!|\mathsf{fst}(t)|\!\} \cdot x \cdot a$ *to be the first component of* $\{\!|t|\!\} \cdot x \cdot a$.
- *We define* $\{\!|\mathsf{snd}(t)|\!\} \cdot x \cdot a$ *to be the second component of* $\{\!|t|\!\} \cdot x \cdot a$.

Proof. The triple $((\!|t|\!), \{\!|t|\!\}, (\!|t|\!))$ is the interpretation of $\Gamma \vdash t : A$ in the cartesian closed category $\mathbf{Qqm}$ where we interpret $\Gamma \vdash \phi(t_1, \ldots, t_n)$ by

$$[\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{\langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle} R \times \cdots \times R \xrightarrow{(\phi, \phi^{\mathrm{d}}, \phi)} R.$$

The statement follows from that $((\!|t|\!), \{\!|t|\!\}, (\!|t|\!))$ is a morphism from $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ to $[\![A]\!]$ in $\mathbf{Qqm}$. ◂

The fundamental lemma is a way to compositionally reason about distances.

443   ▸ **Example 13.** Let us fix a positive real number ϵ. We define $D_\epsilon \colon (\mathbb{R} \Rightarrow \mathbb{R}) \to (\mathbb{R} \Rightarrow \mathbb{R})$ by

444    $$D_\epsilon \cdot f = \lambda x : \mathbb{R}. \, \frac{f(x + \epsilon) - f(x)}{\epsilon}.$$

445   For $f \colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, $D_\epsilon \cdot f \cdot x$ calculates an approximation of the derivative of f at x.
446   By the fundamental lemma, we obtain $(D_\epsilon, E_\epsilon, D_\epsilon) \in \rho_{(R \Rightarrow R) \Rightarrow (R \Rightarrow R)}$ where E_ϵ is a function
447   from $|R \Rightarrow R|$ to $\mathcal{Q}_{R \Rightarrow R} \rightharpoonup \mathcal{Q}_{R \Rightarrow R}$ given by

448    $$E_\epsilon \cdot f \cdot a = \lambda x : \mathbb{R}. \, \lambda b : [0, +\infty]. \, \frac{a \cdot (x + \epsilon) \cdot b + a \cdot x \cdot b}{\epsilon}.$$

449   In Example 19, we will observe that $(\mathrm{id}_\mathbb{R}, a, \sin)$ is an element of $\rho_{R \Rightarrow R}$ where $\mathrm{id}_\mathbb{R}$ is the
450   identity function on $\mathbb{R}$, and $a \in \mathcal{Q}_{R \Rightarrow R}$ is given by $a \cdot x \cdot b = |x - \sin(x)| + b$. By applying
451    $(D_\epsilon, E_\epsilon, D_\epsilon)$ to $(\mathrm{id}_\mathbb{R}, a, \sin)$, we obtain $(D_\epsilon \cdot \mathrm{id}_\mathbb{R}, a', D_\epsilon \cdot \sin) \in \rho_R$ where

452    $$a' \cdot x \cdot b = \frac{|x + \epsilon - \sin(x + \epsilon)| + |x - \sin(x)| + 2b}{\epsilon}.$$

453   From this, we see that the distance between $D_\epsilon \cdot \mathrm{id}_\mathbb{R} \cdot 0$ and $D_\epsilon \cdot \sin \cdot 0$ is bounded by $\frac{|\epsilon - \sin(\epsilon)|}{\epsilon}$.
454   We note that a' is not the exact distance between $D_\epsilon \cdot \mathrm{id}_\mathbb{R}$ and $D_\epsilon \cdot \sin$. For example, while
455    $|D_{0.1} \cdot \mathrm{id}_\mathbb{R} \cdot 0 - D_{0.1} \cdot \sin \cdot 0.1| \approx 0.01$, we have $a' \cdot 0 \cdot 0.1 \approx 2$. This gap stems in the fact
456   that $(D_\epsilon, E_\epsilon, D_\epsilon)$ takes all functions into account and can not exploit continuity of specific
457   functions.

6   Quasi-Metric Logical Relations

459   As described in Section 3, any quasi^2-metric gives rise to left and right observational quasi-
460   metrics. In this section, we introduce a class of logical relations γ_A that capture the left
461   observational quasi-metric associated to ρ_A. We will then show how such relations can be
462   used to derive over-approximations of distances between functions.
463   For a type A, we define $\gamma_\mathsf{A} \subseteq |\mathsf{A}| \times \mathcal{Q}_\mathsf{A} \times |\mathsf{A}|$ by induction on A as follows:

464    $$(x, a, x') \in \gamma_\mathsf{Real} \iff |x - x'| \sqsupseteq a,$$
465    $$(f, a, f') \in \gamma_{\mathsf{A} \Rightarrow \mathsf{B}} \iff \text{for all } (x, b, x') \in \rho_\mathsf{A}, \, (f \cdot x, a \cdot x \cdot b, f' \cdot x) \in \gamma_\mathsf{B}, \text{ and}$$
466    $$\text{for all } (f', b, f') \in \rho_{\mathsf{A} \Rightarrow \mathsf{B}}, \, (f, a \otimes b, f) \in \rho_{\mathsf{A} \Rightarrow \mathsf{B}},$$
467    $$((x, y), (a, b), (x', y')) \in \gamma_{\mathsf{A} \times \mathsf{B}} \iff (x, a, x') \in \gamma_\mathsf{A} \text{ and } (y, b, y') \in \gamma_\mathsf{B}.$$

468   We give some explanation on the definition of $\gamma_{\mathsf{A} \Rightarrow \mathsf{B}}$. The definition consists of two conditions.
469   The first condition means that, if (f, a, f') is an element of $\gamma_{\mathsf{A} \Rightarrow \mathsf{B}}$, then the distance $a \cdot x \cdot b$
470   over-approximates the distance between f and f' at *the same* point x (rather than on
471   distinct points, as is the case for the relation $\rho_{\mathsf{A} \Rightarrow \mathsf{B}}$). The second condition means that a
472   also over-approximates the *gap* between the self-distance of f' and the self-distance of f.
473   Let us introduce a notation. For a type A and $x \in |\mathsf{A}|$, we write $\lceil x \rceil \in \mathcal{Q}_\mathsf{A}$ for $\hat{\rho}_\mathsf{A}(x, x)$, i.e.,

474    $$\lceil x \rceil = \sup\{a \in \mathcal{Q}_\mathsf{A} \mid (x, a, x) \in \rho_\mathsf{A}\}.$$

475   Since ρ_A is closed under supremum, for any $x \in |\mathsf{A}|$, we have $(x, \lceil x \rceil, x) \in \rho_\mathsf{A}$.

476   ▸ **Lemma 14.** *For every type A and for every $x, x' \in |\mathsf{A}|$, if $(x, \lceil x \rceil, x') \in \rho_\mathsf{A}$, then $x = x'$.*

Proof. By induction on A. It is straightforward to check the case Real and the case $A \times B$. For the case $A \Rightarrow B$, if $(f, \lceil f \rceil, f') \in \rho_{A \Rightarrow B}$, then for any $x \in |A|$, we have

$$(f \cdot x, \lceil f \rceil \cdot x \cdot \lceil x \rceil, f' \cdot x) \in \rho_A.$$

Here, by the induction hypothesis,

$$\lceil f \rceil \cdot x \cdot \lceil x \rceil = \sup\{a \in \mathcal{Q}_B \mid \text{for all } (x, \lceil x \rceil, x') \in \rho_A, (f \cdot x, a, f \cdot x') \in \rho_B\}$$
$$= \sup\{a \in \mathcal{Q}_B \mid (f \cdot x, a, f \cdot x) \in \rho_B\}$$
$$= \lceil f \cdot x \rceil.$$

Hence, $f' \cdot x = f \cdot x$.   ◂

▸ **Lemma 15.** *For any type* A *and* B*, if* $f \in |A \Rightarrow B|$ *and* $x \in |A|$*, then* $\lceil f \rceil \cdot x \cdot \lceil x \rceil = \lceil f \cdot x \rceil$*.*

Proof. This is shown in the proof of Lemma 14.   ◂

▸ **Lemma 16.** *For every type* A*, if* $(x, a \otimes \lceil x' \rceil, x') \in \rho_A$*, then* $(x, a, x') \in \gamma_A$*.*

Proof. By induction on A. The only non-trivial case is $A \Rightarrow B$. If $(f, a \otimes \lceil f' \rceil, f') \in \rho_{A \Rightarrow B}$, then for any $(x, b, x') \in \rho_A$, since $(x, b \vee \lceil x \rceil, x) \in \rho_A$, we obtain

$$(f \cdot x, (a \cdot x \cdot (b \vee \lceil x \rceil)) \otimes (\lceil f' \rceil \cdot x \cdot (b \vee \lceil x \rceil)), f' \cdot x) \in \rho_B.$$

It follows from monotonicity of a and Lemma 15 that we have

$$(f \cdot x, (a \cdot x \cdot b) \otimes (\lceil f' \rceil \cdot x \cdot \lceil x \rceil), f' \cdot x) = (f \cdot x, (a \cdot x \cdot b) \otimes \lceil f' \cdot x \rceil, f' \cdot x) \in \rho_B.$$

By the induction hypothesis, we conclude $(f \cdot x, a \cdot x \cdot b, f' \cdot x) \in \gamma_B$. For any $(f', b, f') \in \rho_{A \Rightarrow B}$, since $b \sqsubseteq \lceil f' \rceil$, it follows from $(f, a \otimes \lceil f' \rceil, f') \in \rho_{A \Rightarrow B}$ that $(f, a \otimes b, f') \in \rho_{A \Rightarrow B}$. By left-quasi-reflexivity, we obtain $(f, a \otimes b, f) \in \rho_{A \Rightarrow B}$.   ◂

Let q_A^l indicate the quasi-metric representing the left observational quasi-metrics associated with the quasi^2-metric ρ_A.

▸ **Proposition 17.** *For every type* A*, we have* $q_A^l = \gamma_A$*.*

Proof. We first show that γ_A is a subset of q_A^l by induction on A. It is straightforward to check the case Real and the case $A \times B$. We check the case $A \Rightarrow B$. Let (f, a, f') be an element of $\gamma_{A \Rightarrow B}$, and let (f', a', f'') be an element of $\rho_{A \Rightarrow B}$. We show that $(f, a \otimes a', f'')$ is an element of $\rho_{A \Rightarrow B}$. For any $(x, b, x') \in \rho_A$, since $(x, b, x) \in \rho_A$, we have

$$(f \cdot x, a \cdot x \cdot b, f' \cdot x) \in \gamma_B,$$
$$(f' \cdot x, a' \cdot x \cdot b, f'' \cdot x') \in \rho_B.$$

Hence, by the induction hypothesis, we see that $(f \cdot x, (a \otimes a') \cdot x \cdot b, f'' \cdot x')$ is an element of ρ_B. It remains to check that $(f \cdot x, (a \otimes a') \cdot x \cdot b, f \cdot x')$ is an element of ρ_B. Since $(f', a', f'') \in \rho_{A \Rightarrow B}$, we have $(f', a', f') \in \rho_{A \Rightarrow B}$. Then, by the definition of $\gamma_{A \Rightarrow B}$, we obtain $(f, a \otimes a', f) \in \rho_{A \Rightarrow B}$. Hence, $(f \cdot x, (a \otimes a') \cdot x \cdot b, f \cdot x')$ is an element of ρ_B. We next show that q_A^l is a subset of γ_A. Again, it is straightforward to check the case Real and the case $A \times B$. We check the case $A \Rightarrow B$. Let (f, a, f') be an element of $q_{A \Rightarrow B}^l$, and let (x, b, x') be an element of ρ_A. Since $(f, a \otimes \lceil f' \rceil, f') \in \rho_{A \Rightarrow B}$ and $(x, \lceil x \rceil, x) \in \rho_A$, we obtain

$$(f \cdot x, (a \cdot x \cdot \lceil x \rceil) \otimes (\lceil f' \rceil \cdot x \cdot \lceil x \rceil), f' \cdot x) = (f \cdot x, (a \cdot x \cdot \lceil x \rceil) \otimes \lceil f' \cdot x \rceil, f' \cdot x) \in \rho_B$$

Hence, by Lemma 16, $(f \cdot x, a \cdot x \cdot \lceil x \rceil, f' \cdot x)$ is an element of ρ_B. Since a is monotone, for any $(x, b, x') \in \rho_A$, we have $(f \cdot x, a \cdot x \cdot b, f' \cdot x) \in \rho_B$. If $(f', b, f') \in \rho_{A \Rightarrow B}$, then by the definition of $q_{A \Rightarrow B}^l$, we obtain $(f, a \otimes b, f) \in \rho_{A \Rightarrow B}$.   ◂

We can use Proposition 17 to over-approximate ρ-distances in terms of γ-distances and the left observational quasi-metric. Let us sketch our idea. First, thanks to Proposition 17 and Proposition 6, we can exploit left-transitivity to pass from a γ-distance between t and s and a self-ρ-distance of s to a ρ-distance between t and s:

$$(\langle\!|\mathsf{t}|\!\rangle, a, \langle\!|\mathsf{s}|\!\rangle) \in \gamma_\mathsf{A} \text{ and } (\langle\!|\mathsf{s}|\!\rangle, b, \langle\!|\mathsf{s}|\!\rangle) \in \rho_\mathsf{A} \implies (\langle\!|\mathsf{t}|\!\rangle, a \otimes b, \langle\!|\mathsf{s}|\!\rangle) \in \rho_\mathsf{A}. \qquad \text{(left transitivity)}$$

Second, thanks to the fundamental lemma, we can always obtain a ρ-distance by summing a γ-distance with the self-distance $\{\!|\mathsf{s}|\!\}$:

$$(\langle\!|\mathsf{t}|\!\rangle, a, \langle\!|\mathsf{s}|\!\rangle) \in \gamma_\mathsf{A} \implies (\langle\!|\mathsf{t}|\!\rangle, a \otimes \{\!|\mathsf{s}|\!\}, \langle\!|\mathsf{s}|\!\rangle) \in \rho_\mathsf{A}. \qquad (\rho \sqsupseteq \gamma \otimes \text{self-}\rho)$$

The following result exploits this last idea to bound the distance between two functions f and g by summing the "vertical distance" between f and g (that is, the distance of $f(x)$ and $g(x)$ for some fixed x) with an approximation of the self-distances of f and g:

▸ **Theorem 18.** *Let* A *be a type. For any* $f, f' \in |\mathsf{A} \Rightarrow \mathsf{Real}|$ *and any* $a, a' \in \mathcal{Q}_{\mathsf{A} \Rightarrow \mathsf{Real}}$, *if*

$\quad$■$\quad$ $|f \cdot x - f' \cdot x| \sqsupseteq a \cdot x \cdot b$ *for all* $(x, b, x') \in \rho_\mathsf{A}$; *and*

$\quad$■$\quad$ $(f, a', f) \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}$ *and* $(f', a', f') \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}$,

then $(f, a \otimes a', f') \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}$.

Proof. By the definition of $\gamma_{\mathsf{A} \Rightarrow \mathsf{Real}}$, we obtain

$$(f, a \otimes (\lceil f' \rceil \multimap \lceil f \rceil), f') \in \gamma_{\mathsf{A} \Rightarrow \mathsf{Real}}.$$

Therefore, it follows from Proposition 17 that

$$(f, a \otimes (\lceil f' \rceil \multimap \lceil f \rceil) \otimes \lceil f' \rceil, f') \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}.$$

Since $(f, a', f) \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}$ and $(f', a', f') \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}$, we have $a' \sqsubseteq (\lceil f' \rceil \multimap \lceil f \rceil) \otimes \lceil f' \rceil$. Hence, $(f, a \otimes a', f') \in \rho_{\mathsf{A} \Rightarrow \mathsf{Real}}$. ◄

▸ **Example 19.** Let $\mathrm{id}_\mathbb{R}$ be the identity function on $\mathbb{R}$. By the fundamental lemma with a simple calculation, we obtain $(\mathrm{id}_\mathbb{R}, a', \mathrm{id}_\mathbb{R}) \in \rho_{R \Rightarrow R}$ and $(\sin, a', \sin) \in \rho_{R \Rightarrow R}$ where $a' \cdot x \cdot b = b$. By Theorem 18, $a \in \mathcal{Q}_{R \Rightarrow R}$ given by $a \cdot x \cdot b = |x - \sin(x)|$ satisfies $(\mathrm{id}_\mathbb{R}, a \otimes a', \sin) \in \rho_{R \Rightarrow R}$. To be concrete, $(a \otimes a') \cdot x \cdot b = |x - \sin(x)| + b$, which means that the distance between x and $\sin(y)$ is small when x and y are close to 0.

▸ Remark 20. Due to asymmetry in the definition of the exponential $X \Rightarrow Y$ in **Qqm**, it is not clear how to capture the *right* observational quasi-metrics in a similar manner. However, we will see that right observational quasi-metrics can be captured by partial metric logical relations introduced in the next section.

7 Partial Metric Logical Relations

As discussed in Section 3, the quasi²-metrics ρ_A are not, in general, partial metrics. In this section we introduce a family of differential logical relations $(\eta_\mathsf{A})_{\mathsf{A} \in \mathsf{Types}}$ that defines a class of partial quasi-metrics over Λ_{Real}. The fundamental (indeed, the only) difference with respect to the family ρ_A is, as it may be expected, in the case of the function type.

For any type A, we define $\eta_\mathsf{A} \subseteq |\mathsf{A}| \times \mathcal{Q}_\mathsf{A} \times |\mathsf{A}|$ by induction on A as follows:

$$(x, a, x') \in \eta_\mathsf{Real} \iff |x - x'| \sqsupseteq a,$$

$$(f, a, f') \in \eta_{\mathsf{A} \Rightarrow \mathsf{B}} \iff \text{there are } a_1, a_2 \in \mathcal{Q}_{\mathsf{A} \Rightarrow \mathsf{B}} \text{ such that } a_1 \otimes a_2 \sqsupseteq a \text{ and}$$
$$\text{for all } (x, b, x') \in \eta_\mathsf{A}, \ (f \cdot x, a_1 \cdot x \cdot b, f \cdot x') \in \eta_\mathsf{B} \text{ and}$$
$$(f \cdot x', a_2 \cdot x \cdot b, f' \cdot x') \in \eta_\mathsf{B},$$

$$((x, y), (a, b), (x', y')) \in \eta_{\mathsf{A} \times \mathsf{B}} \iff (x, a, x') \in \eta_\mathsf{A} \text{ and } (y, b, y') \in \eta_\mathsf{B}.$$

The idea of the definition of $\eta_{\mathsf{A} \Rightarrow \mathsf{B}}$ is that if $(f, a, f') \in \eta_{\mathsf{A} \Rightarrow \mathsf{B}}$, then a must be larger than or equal to the sum of the self-distance of f and of the "vertical" distances between f and f'.

For $(x, a, x') \in \eta_{\mathsf{A} \Rightarrow \mathsf{B}}$, we call a pair $a_1, a_2 \in \mathcal{Q}_{\mathsf{A} \Rightarrow \mathsf{B}}$ satisfying the condition in the definition of $(x, a, x') \in \eta_{\mathsf{A} \Rightarrow \mathsf{B}}$ a *decomposition* of $(x, a, x') \in \eta_{\mathsf{A} \Rightarrow \mathsf{B}}$.

The following result shows that the relations η_A are $\mathcal{Q}$-closed.

▸ **Lemma 21.** *For any type A, the relation ρ_A is $\mathcal{Q}$-closed.*

An immediate consequence of the lemma is the following:

▸ **Corollary 22.** *For all $(x, a, x') \in \eta_\mathsf{A}$, the set of decompositions of $(x, a, x') \in \eta_\mathsf{A}$ is a complete lattice.*

The following result shows that the relations η_A define partial quasi-metrics on all types.

▸ **Proposition 23.** *For all types A:*
- *If $(x, a, x') \in \eta_\mathsf{A}$, then $(x, a, x) \in \eta_\mathsf{A}$.*
- *If $(x, a, z) \in \eta_\mathsf{A}$ and $(z, b, y) \in \eta_\mathsf{A}$, then there exists $c_1, c_2 \in \mathcal{Q}_\mathsf{A}$ such that $a \otimes b \sqsubseteq c_1 \otimes c_2$, $(z, c_1, z) \in \eta_\mathsf{A}$ and $(x, c_2, y) \in \eta_\mathsf{A}$. In particular, $(x, a \otimes (c_1 \multimap b), y) \in \eta_\mathsf{A}$.*

Proof. We only prove the second, more delicate, statement. By induction on A, we show that there is a map $\varphi_\mathsf{A} \colon \mathcal{Q}_\mathsf{A} \times \mathcal{Q}_\mathsf{A} \Rightarrow \mathcal{Q}_\mathsf{A} \times \mathcal{Q}_\mathsf{A}$ such that if $(x, a, z) \in \eta_\mathsf{A}$ and $(z, b, y) \in \eta_\mathsf{A}$, then $(c_1, c_2) = \varphi_\mathsf{A}(a, b)$ satisfies the required conditions. For the base case, we define $\varphi_\mathsf{Real}(a, b) = (0, a + b)$. For the case $\mathsf{A} = (\mathsf{B} \Rightarrow \mathsf{C})$, let $(a_1, a_2) \in \mathcal{Q}_{\mathsf{B} \Rightarrow \mathsf{C}} \times \mathcal{Q}_{\mathsf{B} \Rightarrow \mathsf{C}}$ and $(b_1, b_2) \in \mathcal{Q}_{\mathsf{B} \Rightarrow \mathsf{C}} \times \mathcal{Q}_{\mathsf{B} \Rightarrow \mathsf{C}}$ be the greatest decompositions of a and b, respectively. We define $\varphi_{\mathsf{B} \Rightarrow \mathsf{C}}(a, b)$ by

$$\varphi_{\mathsf{B} \Rightarrow \mathsf{C}}(a, b) = (a_1 \otimes k, b_1 \otimes l)$$

where $(k \cdot w \cdot d, l \cdot w \cdot d) = \varphi_\mathsf{C}(a_2 \cdot w \cdot d, b_2 \cdot w \cdot d)$. Below, we write (c_1, c_2) for $\varphi_{\mathsf{B} \Rightarrow \mathsf{C}}(a, b)$. Let us check that $\varphi_{\mathsf{B} \Rightarrow \mathsf{C}}$ is a witness.
- We first show that $(z, c_1, z) \in \eta_{\mathsf{B} \Rightarrow \mathsf{C}}$. For any $(w, d, w') \in \eta_\mathsf{B}$, we have

$$(x \cdot w', a_2 \cdot w \cdot d, z \cdot w') \in \eta_\mathsf{C}, \tag{1}$$
$$(z \cdot w, b_1 \cdot w \cdot d, z \cdot w') \in \eta_\mathsf{C}, \tag{2}$$
$$(z \cdot w', b_2 \cdot w \cdot d, y \cdot w') \in \eta_\mathsf{C}. \tag{3}$$

Then, by applying the induction hypothesis to (1) and (3), we obtain

$$(z \cdot w', k \cdot w \cdot d, z \cdot w') \in \eta_\mathsf{C}. \tag{4}$$

By (2) and (4), we obtain $(z, c_1, z) \in \eta_{\mathsf{B} \Rightarrow \mathsf{C}}$.

587 ▪ We next show that $(x, c_2, y) \in \eta_{B \Rightarrow C}$. For any $(w, d, w') \in \eta_B$, we have

588
$$(x \cdot w, a_1 \cdot w \cdot d, x \cdot w') \in \eta_C, \tag{5}$$

589
$$(x \cdot w', a_2 \cdot w \cdot d, z \cdot w') \in \eta_C, \tag{6}$$

590
$$(z \cdot w', b_2 \cdot w \cdot d, y \cdot w') \in \eta_C. \tag{7}$$

591 By applying the induction hypothesis to (6) and (7), we obtain

592
$$(x \cdot w', l \cdot w \cdot d, y \cdot w') \in \eta_C. \tag{8}$$

593 By (5) and (8), we obtain $(x, c_2, y) \in \eta_{B \Rightarrow C}$.

594 ▪ Finally, we have

595
$$(c_1 \cdot w \cdot d) \otimes (c_2 \cdot w \cdot d) = (a_1 \cdot w \cdot d) \otimes (k \cdot w \cdot d) \otimes (b_1 \cdot w \cdot d) \otimes (l \cdot w \cdot d)$$

596
$$\sqsupseteq (a_1 \cdot w \cdot d) \otimes (a_2 \cdot w \cdot d) \otimes (b_1 \cdot w \cdot d) \otimes (b_2 \cdot w \cdot e)$$

597
$$\sqsupseteq (a \cdot w \cdot d) \otimes (b \cdot w \cdot d).$$

598 ◄

599 By adapting the definition of γ_A from Section 6, we can capture the left observational
600 quasi-metrics q_A^l associated with the partial quasi-metrics η_A. Moreover, by Proposition 8,
601 the *right* observational quasi-metric q_A^r satisfies $(x, \hat{\eta}_A(x, x) \multimap a, y) \in q_A^r \iff (x, a, y) \in \eta_A$.
602 Thanks to this, we can capture this quasi-metrics via the logical relations $\delta_A \subseteq |A| \times \mathcal{Q}_A \times |A|$
603 defined by induction on A, letting the base and product case being defined as for γ_A, and the
604 function case being as follows:

605
$$(f, a, f') \in \delta_{A \Rightarrow B} \iff \text{for all } (f, b, f) \in \eta_{A \Rightarrow B}, \ (f, a \otimes b, f') \in \eta_{A \Rightarrow B}.$$

606 ▸ **Proposition 24.** *For every type* A, *we have* $q_A^r = \delta_A$.

607 **Proof.** The only interesting case is that of a function type $A = B \Rightarrow C$. By Proposition 8,
608 $(f, a, f') \in q_A^r$ holds iff for all $a \sqsubseteq \hat{q}_A^r(f, f') = \hat{q}_A^r(f, f) \multimap \hat{q}_A^r(f, f')$, which is in turn equivalent
609 to $a \otimes \hat{q}_A^r(f, f) \sqsubseteq \hat{q}_A^r(f, f')$. This implies then that $(f, a, f') \in q_A^r$ iff for all $(f, b, f) \in q_A^r$ (i.e. for
610 all $b \sqsubseteq \hat{q}_A^r(f, f)$), $(f, a \otimes b, f') \in q_A^r$ (i.e. $a \otimes b \sqsubseteq \hat{q}_A^r(f, f')$), that is, iff $(f, a, f') \in \delta_A$. ◄

611 **8** **A Quantitative Equational Theory**

612 The goal of this section is to introduce an equational theory to formally deduce differences
613 between programs. To this end, we first give a syntactic presentation of differential logical
614 relations internally to the language of Λ_{Real}, and then introduce a deductive system to deduce
615 program differences.

616 While our idea is inspired by the quantitative equational theories of Mardare et al. [28], it
617 differs in two respects: first, distances need not be real numbers, but are presented as arbitrary
618 Λ_{Real}-programs; second, non-expansiveness is replaced by the condition corresponding to the
619 fundamental lemma of differential logical relations.

620 **Preparation** Before we go into construction, we prepare some syntactic counter parts of
621 constructions in the fundamental lemma for **Qqm**. We first inductively define a type $A^\bullet$ by

622
$$\mathsf{Real}^\bullet = \mathsf{Real}, \qquad (A \Rightarrow B)^\bullet = A \Rightarrow A^\bullet \Rightarrow B^\bullet, \qquad (A \times B)^\bullet = A^\bullet \times B^\bullet.$$

$$x^\bullet = \dot{x} \qquad \underline{r}^\bullet = \underline{0} \qquad (t\,s)^\bullet = t^\bullet\,s\,s^\bullet \qquad (\lambda x : A.\, t)^\bullet = \lambda x : A.\, \lambda \dot{x} : A^\bullet.\, t^\bullet \qquad \langle t, s \rangle^\bullet = \langle t^\bullet, s^\bullet \rangle$$

$$(\mathsf{fst}(t))^\bullet = \mathsf{fst}(t^\bullet) \qquad (\mathsf{snd}(t))^\bullet = \mathsf{snd}(t^\bullet) \qquad (\phi(t_1, \dots, t_n))^\bullet = \phi^{\mathrm{d}}(t_1, \dots, t_n, t_1^\bullet, \dots, t_n^\bullet)$$

■ **Figure 2** Derivative of Term

This is a syntactic counter part of quantales $\mathcal{Q}_A$. The reason that we define $\mathsf{Real}^\bullet$ to be Real even though $\mathsf{Real}^\bullet$ should be a type of non-negative extended real numbers is to keep the syntax of Λ_{Real} simple. It is possible to extend Λ_{Real} with a type $\mathsf{Real}_{\geqslant 0}^\infty$ of non-negative extended real numbers and types $A \rightarrow B$ of monotone functions. We next give syntactic counter part of $\{t\}$. For this purpose, we suppose that there is a partition $\mathsf{Var} = \mathsf{Var}_0 \cup \mathsf{Var}_1$, i.e., there are mutually disjoint subsets $\mathsf{Var}_0, \mathsf{Var}_1 \subseteq \mathsf{Var}$ such that Var is equal to $\mathsf{Var}_0 \cup \mathsf{Var}_1$. Furthermore, we suppose that there is a bijection $\dot{(-)} \colon \mathsf{Var}_0 \to \mathsf{Var}_1$. In the sequel, we denote variables in Var_1 by dotted symbols $\dot{x}, \dot{y}, \dot{z}, \dots$, and we denote variables in Var_0 by $x, y, z, \dots$. Based on this convention, for a typing context $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$, we define a typing context $\Gamma^\bullet$ by $\Gamma^\bullet = (\dot{x}_1 : A_1^\bullet, \dots, \dot{x}_n : A_n^\bullet)$. Now, for a term $\Gamma \vdash t : A$, we define a term $\Gamma, \Gamma^\bullet \vdash t^\bullet : A$, which we call the *derivative* of t, in Figure 2. The definition of $t^\bullet$ corresponds to the definition of $\{t\}$, and we can find the same construction in [9].

Syntactic Differential Logical Relations By adopting the structure of **Qqm**, we define a type-indexed family $\{\delta_A^{\mathrm{log}} \subseteq \mathbf{T}_A \times \mathbf{T}_{A^\bullet} \times \mathbf{T}_A\}_{A \in \mathbf{Type}}$ of ternary predicates as follows:

$$(t, a, t') \in \delta_{\mathsf{Real}}^{\mathrm{log}} \iff \text{there are } r, r' \in \mathbb{R} \text{ and } s \in [0, +\infty] \text{ such that } |r - r'| \sqsupseteq s \text{ and}$$
$$\vdash t = \underline{r} : \mathsf{Real} \text{ and } \vdash a = \underline{s} : \mathsf{Real} \text{ and } \vdash t' = \underline{r'} : \mathsf{Real},$$

$$(t, a, t') \in \delta_{A \Rightarrow B}^{\mathrm{log}} \iff \text{for any } (s, b, s') \in \delta_A^{\mathrm{log}}, (t\,s, a\,s\,b, t'\,s') \in \delta_B^{\mathrm{log}} \text{ and } (t\,s, a\,s\,b, t\,s') \in \delta_B^{\mathrm{log}},$$

$$(t, a, t') \in \delta_{A \times B}^{\mathrm{log}} \iff (\mathsf{fst}(t), \mathsf{fst}(a), \mathsf{fst}(t')) \in \delta_A^{\mathrm{log}} \text{ and } (\mathsf{snd}(t), \mathsf{snd}(a), \mathsf{snd}(t')) \in \delta_B^{\mathrm{log}}$$

where we write $\Gamma \vdash t = s : A$ when the equality between $\Gamma \vdash t : A$ and $\Gamma \vdash s : A$ is derivable from the standard equational theory consisting of $\beta\eta$-equalities extended with the following axiom for every multi-arity function ϕ:

$$\Gamma \vdash \phi(\underline{r_1}, \dots, \underline{r_{\mathrm{ar}(\phi)}}) = \underline{\phi(r_1, \dots, r_{\mathrm{ar}(\phi)})} : \mathsf{Real}.$$

Although $\mathbf{T}_A$ is not a quantale in general, we can show that δ_A^{log} satisfies "left quasi-reflexivity", "transitivity" and a fundamental lemma in the following form.

▶ **Proposition 25.** *Let* A *be a type.*
- *If* $(t, a, t') \in \delta_A^{\mathrm{log}}$, *then* $(t, a, t) \in \delta_A^{\mathrm{log}}$.
- *If* $(t, a, t') \in \delta_A^{\mathrm{log}}$ *and* $(t', a', t'') \in \delta_A^{\mathrm{log}}$, *then* $(t, \mathsf{add}_A\,a\,a', t'') \in \delta_A^{\mathrm{log}}$ *where* $\mathsf{add}_A \in \mathbf{T}_{A \Rightarrow A \Rightarrow A}$ *is given by*

$$\mathsf{add}_{\mathsf{Real}} = \lambda xy : \mathsf{Real}.\, \underline{\mathsf{add}}(x, y),$$
$$\mathsf{add}_{A \Rightarrow B} = \lambda xy : A \Rightarrow B.\, \lambda z : A.\, \mathsf{add}_B\,(x\,z)\,(y\,z),$$
$$\mathsf{add}_{A \times B} = \lambda xy : A \times B.\, \langle \mathsf{add}_A\,\mathsf{fst}(x)\,\mathsf{fst}(y), \mathsf{add}_B\,\mathsf{snd}(x)\,\mathsf{snd}(y) \rangle.$$

- *For any term* $x_1 : A_1, \dots, x_n : A_n \vdash t : A$, *and for any family* $\{(s_i, a_i, s_i') \in \delta_{A_i}^{\mathrm{log}}\}_{1 \leqslant i \leqslant n}$,

$$(t[s_1/x_1, \dots, s_n/x_n], t^\bullet[s_1/x_1, \dots, s_n/x_n, a_1/\dot{x}_1, \dots, a_n/\dot{x}_n], t'[s_1'/x_1, \dots, s_n'/x_n]) \in \delta_A^{\mathrm{log}}.$$

$$\frac{|r - r'| \leqslant s}{\Gamma \vdash (\underline{r}, \underline{s}, \underline{r'}) : \mathsf{Real}} \qquad \frac{\Gamma \vdash (\mathsf{t}_1, \mathsf{a}_1, \mathsf{t}'_1) : \mathsf{Real} \quad \ldots \quad \Gamma \vdash (\mathsf{t}_n, \mathsf{a}_n, \mathsf{t}'_n) : \mathsf{Real}}{\Gamma \vdash (\phi(\mathsf{t}_1, \ldots, \mathsf{t}_n), \phi^{\mathrm{d}}(\mathsf{t}_1, \ldots, \mathsf{t}_n, \mathsf{a}_1, \ldots, \mathsf{a}_n), \phi(\mathsf{t}'_1, \ldots, \mathsf{t}'_n)) : \mathsf{Real}}$$

$$\frac{\mathsf{x} : \mathsf{A} \in \Gamma}{\Gamma \vdash (\mathsf{x}, \dot{\mathsf{x}}, \mathsf{x}) : \mathsf{A}} \qquad \frac{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{Real} \quad \Gamma \vdash (\mathsf{t}', \mathsf{a}', \mathsf{t}'') : \mathsf{Real}}{\Gamma \vdash (\mathsf{t}, \mathsf{a} + \mathsf{a}', \mathsf{t}'') : \mathsf{Real}} \qquad \frac{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{Real}}{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}) : \mathsf{Real}}$$

$$\frac{\Gamma, \mathsf{x} : \mathsf{A}, \dot{\mathsf{x}} : \mathsf{A}^{\bullet} \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{B}}{\Gamma \vdash (\lambda \mathsf{x} : \mathsf{A}. \mathsf{t}, \lambda \dot{\mathsf{x}} : \mathsf{A}^{\bullet}. \lambda \mathsf{x} : \mathsf{A}. \mathsf{a}, \lambda \mathsf{x} : \mathsf{A}. \mathsf{t}') : \mathsf{A} \Rightarrow \mathsf{B}} \qquad \frac{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A} \Rightarrow \mathsf{B} \quad \Gamma \vdash (\mathsf{s}, \mathsf{b}, \mathsf{s}') : \mathsf{A}}{\Gamma \vdash (\mathsf{t}\,\mathsf{s}, \mathsf{a}\,\mathsf{s}\,\mathsf{b}, \mathsf{t}'\,\mathsf{s}') : \mathsf{B}}$$

$$\frac{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A} \times \mathsf{B}}{\Gamma \vdash (\mathsf{fst}(\mathsf{t}), \mathsf{fst}(\mathsf{a}), \mathsf{fst}(\mathsf{t}')) : \mathsf{A}} \qquad \frac{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A} \times \mathsf{B}}{\Gamma \vdash (\mathsf{snd}(\mathsf{t}), \mathsf{snd}(\mathsf{a}), \mathsf{snd}(\mathsf{t}')) : \mathsf{B}} \qquad \frac{\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A} \quad \Gamma \vdash (\mathsf{s}, \mathsf{b}, \mathsf{s}') : \mathsf{B}}{\Gamma \vdash (\langle \mathsf{t}, \mathsf{s} \rangle, \langle \mathsf{a}, \mathsf{b} \rangle, \langle \mathsf{t}', \mathsf{s}' \rangle) : \mathsf{A} \times \mathsf{B}}$$

$$\frac{\Gamma \vdash \mathsf{t} = \mathsf{s} : \mathsf{A} \quad \Gamma' \vdash \mathsf{t}' = \mathsf{s}' : \mathsf{A} \quad \Gamma, \Gamma^{\bullet} \vdash \mathsf{a} = \mathsf{b} : \mathsf{A}^{\bullet} \quad \Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A}}{\Gamma \vdash (\mathsf{s}, \mathsf{b}, \mathsf{s}') : \mathsf{A}}$$

🟨 **Figure 3** Derivation Rules

656 ▪ *If* $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\log}$ *and* $\vdash \mathsf{t} = \mathsf{s} : \mathsf{A}$ *and* $\vdash \mathsf{a} = \mathsf{b} : \mathsf{A}^{\bullet}$ *and* $\vdash \mathsf{t}' = \mathsf{s}' : \mathsf{A}$, *then* $(\mathsf{s}, \mathsf{b}, \mathsf{s}') \in \delta_{\mathsf{A}}^{\log}$.

657 **Proof Sketch.** We prove the statement by induction on A. We only check the case $\mathsf{A} \Rightarrow \mathsf{B}$.
658 It is straightforward to derive "left-quasi-reflexivity" from the definition of $\delta_{\mathsf{A} \Rightarrow \mathsf{B}}^{\log}$. This is
659 why we modify the definition of differential logical relation given in [11]. For transitivity,
660 we shall show that for any $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\log}$, $(\mathsf{t}', \mathsf{a}', \mathsf{t}'') \in \delta_{\mathsf{A}}^{\log}$ and $(\mathsf{s}, \mathsf{b}, \mathsf{s}') \in \delta_{\mathsf{A}}^{\log}$, we have
661 $(\mathsf{t}\,\mathsf{s}, (\mathsf{a} + \mathsf{a}')\,\mathsf{b}\,\mathsf{s}, \mathsf{t}''\,\mathsf{s}') \in \delta_{\mathsf{B}}^{\log}$. By the induction hypothesis, we obtain $(\mathsf{s}, \mathsf{b}, \mathsf{s}) \in \delta_{\mathsf{A}}^{\log}$. Therefore,

662 $$(\mathsf{t}\,\mathsf{s}, \mathsf{a}\,\mathsf{b}\,\mathsf{s}, \mathsf{t}'\,\mathsf{s}) \in \delta_{\mathsf{B}}^{\log}, \qquad (\mathsf{t}'\,\mathsf{s}, \mathsf{a}'\,\mathsf{b}\,\mathsf{s}, \mathsf{t}''\,\mathsf{s}') \in \delta_{\mathsf{B}}^{\log}.$$

663 Then, by transitivity of $\delta_{\mathsf{B}}^{\log}$, we obtain $(\mathsf{t}\,\mathsf{s}, (\mathsf{a} + \mathsf{a}')\,\mathsf{b}\,\mathsf{s}, \mathsf{t}''\,\mathsf{s}') \in \delta_{\mathsf{B}}^{\log}$. We can prove the
664 fundamental lemma by induction on the derivation of $\Gamma \vdash \mathsf{t} : \mathsf{B}$. ◀

665 **Equational Metric** We introduce a formal system to infer $\delta^{\log}$-distances between terms.
666 For terms $\Gamma \vdash \mathsf{t} : \mathsf{A}$ and $\Gamma, \Gamma^{\bullet} \vdash \mathsf{a} : \mathsf{A}$ and $\Gamma \vdash \mathsf{t}' : \mathsf{A}$, we write $\Gamma \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A}$ when we
667 can derive this judgment from the rules given in Figure 3. Then, we define a type-indexed
668 ternary predicates $\{\delta_{\mathsf{A}}^{\mathrm{eq}} \subseteq \mathbf{T}_{\mathsf{A}} \times \mathbf{T}_{\mathsf{A}^{\bullet}} \times \mathbf{T}_{\mathsf{A}}\}_{\mathsf{A} \in \mathbf{Type}}$ by

669 $$(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\mathrm{eq}} \iff \vdash (\mathsf{t}, \mathsf{a}, \mathsf{t}') : \mathsf{A}.$$

670 We note that quasi-reflexivity and transitivity for arbitrary A follows from left quasi-reflexivity
671 and transitivity for Real. We can also show that δ^{eq} is subsumed by $\delta^{\log}$.

672 ▸ **Proposition 26.** *Let* A *be a type.*
673 ▪ *If* $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\mathrm{eq}}$, *then* $(\mathsf{t}, \mathsf{a}, \mathsf{t}) \in \delta_{\mathsf{A}}^{\mathrm{eq}}$.
674 ▪ *If* $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\mathrm{eq}}$ *and* $(\mathsf{t}', \mathsf{a}', \mathsf{t}'') \in \delta_{\mathsf{A}}^{\mathrm{eq}}$, *then* $(\mathsf{t}, \mathsf{add}_{\mathsf{A}}\,\mathsf{a}\,\mathsf{a}', \mathsf{t}'') \in \delta_{\mathsf{A}}^{\mathrm{eq}}$.
675 ▪ *For any term* $\mathsf{x}_1 : \mathsf{A}_1, \ldots, \mathsf{x}_n : \mathsf{A}_n \vdash \mathsf{t} : \mathsf{A}$, *and for any family* $\{(\mathsf{s}_i, \mathsf{a}_i, \mathsf{s}'_i) \in \delta_{\mathsf{A}_i}^{\mathrm{eq}}\}_{1 \leqslant i \leqslant n}$,

676 $$(\mathsf{t}[\mathsf{s}_1/\mathsf{x}_1, \ldots, \mathsf{s}_n/\mathsf{x}_n], \mathsf{t}^{\bullet}[\mathsf{s}_1/\mathsf{x}_1, \ldots, \mathsf{s}_n/\mathsf{x}_n, \mathsf{a}_1/\dot{\mathsf{x}}_1, \ldots, \mathsf{a}_n/\dot{\mathsf{x}}_n], \mathsf{t}'[\mathsf{s}'_1/\mathsf{x}_1, \ldots, \mathsf{s}'_n/\mathsf{x}_n]) \in \delta_{\mathsf{A}}^{\mathrm{eq}}.$$

677 ▪ *If* $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\mathrm{eq}}$, *then* $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta_{\mathsf{A}}^{\log}$.

678 **Proof.** By induction on A. ◀

9 A Lattice of Quasi^2-Metrics?

We conclude our presentation with a few open questions about the relations holding between the different notions of program difference introduced in this paper. When considering program equivalence, various non-equivalent notions have been introduced, such as observational equivalences, equivalences derived from denotational semantics or equational theories. Since observational equivalences are the coarsest equivalences and equational theories are the finest equivalences in many situations, denotational semantics gives various mathematical reasoning principles for observational equivalences as well as equational theories.

Similarly, in the last sections we have introduced various notions of program differences, all defined in terms of some form of differential logical relations. Therefore, it is reasonable to expect that a similar comparison should be possible for quasi^2-metrics. In particular, this suggests the following two questions:

- Does the type indexed family $\delta^{\log}$ give rise to the "coarsest family of quasi^2-metrics"?
- Does the type indexed family δ^{eq} give rise to the "finest family of quasi^2-metrics"?

We note that, although such differences are defined over $\mathbf{T}_{A^\bullet}$, which is not a quantale, we can easily associate $\delta^{\log}$ and δ^{eq} with quasi^2-metrics valued on the quantale $\mathcal{P}\mathbf{T}_{A^\bullet}$ of subsets of $\mathbf{T}_{A^\bullet}$, letting e.g. $(\mathsf{t}, a, \mathsf{t}') \in \tilde{\delta}^{\log}_A \iff$ for all $\mathsf{a} \in a$, $(\mathsf{t}, \mathsf{a}, \mathsf{t}') \in \delta^{\log}_A$, and similarly for δ^{eq}_A.

However, unfortunately, it is not straightforward to tackle these questions because of the two main obstacles. First, while two quasi^2-metrics valued over the same quantale can be easily compared, it is not clear how to compare two quasi^2-metrics defined over *different* quantales. Second, while in the case of logical relations, the argument that logical equivalence is the coarsest one relies on a notion of *observational equivalence*, it is not clear how to define a similar notion of *observational quasi^2-metric* for Λ_{Real}: since differences between programs describe relationship between differences of inputs and differences of outputs, when we are to measure differences between programs, we should observe differences between outputs of programs with respect to different contexts. Therefore, we should define a notion of differences between contexts *before* we define observational quasi^2-metric for Λ_{Real}. How can we define differences between contexts?

10 Related Work

Differential logical relations for a simply typed language were introduced in [11], and later extended to languages with monads [10], and related to incremental computing [9]. Moreover, a unified framework for operationally-based logical relations, subsuming differential logical relations, was introduced in [7]. The connections with metric spaces and partial metric spaces have been explored already in [18, 30], on the one hand providing a series of negative results that motivate the present work, and on the other hand producing a class of metric and partial metric models based on a different relational construction.

The literature on the interpretation of linear or graded lambda-calculi in the category of metric spaces and non-expansive functions is ample [32, 16, 2, 17, 13]. A related approach is that of quantitative algebraic theories [28], which aims at capturing metrics over algebras via an equational presentation. These have been extended both to quantale-valued metrics [8] and to the simply typed (i.e. non graded) languages [12], although in the last case the non-expansivity condition makes the construction of interesting algebras rather challenging.

The literature on partial metric spaces is vast, as well. Introduced by Matthews [4], they have been largely explored for the metrization of domain theory [5, 35, 36] and, more recently, of λ-theories [27]. An elegant categorical description of partial metrics via the quantaloid of *diagonals* is introduced in [24]. As this construction is obviously related to the notion of

quasi-reflexivity here considered, it would be interesting to look for analogous categorical descriptions of the quasi^2-metrics here introduced.

11   Conclusion

In this paper we have explored the connections between the notions of program distance arising from differential logical relations and those defined via quasi-metrics and partial quasi-metrics. As discussed in Section 9, our results suggest natural and important questions concerning the comparison of all the notions of distance considered in this paper. At the same time, our results provide a conceptual bridge that could be used to exploit methods and results from the vast area of research on quantale-valued relations [23, 37] for the study of program distances in higher-order programming languages. For instance, natural directions are the characterization of limits and, more generally, of topological properties via logical relations, as suggested by recent work [3], although in a qualitative setting.

While we here focused on non-symmetric differential logical relations, understanding the metric structure of the symmetric case, as in [11], would be interesting as well. Notice that this would require to abandon quasi-reflexivity, cf. Remark 3.

Finally, while in this paper we only considered simple types, the notion of quasi^2-metric is robust enough to account for other constructions like e.g. monadic types as in [10]. It is thus natural to explore the application of methods arising from quasi-metric or partial quasi-metrics for the study of languages with effects like e.g. probabilistic choice.

References

1   Mario Alvarez-Picallo, Alex Eyers-Taylor, Michael Peyton Jones, and C.-H. Luke Ong. Fixing incremental computation. In Luís Caires, editor, *Programming Languages and Systems*, pages 525–552, Cham, 2019. Springer International Publishing.

2   Arthur Azevedo de Amorim, Marco Gaboardi, Justin Hsu, Shin-ya Katsumata, and Ikram Cherigui. A semantic account of metric preservation. In *Proceedings POPL 2017*, pages 545–556, New York, NY, USA, 2017. Association for Computing Machinery. `doi:10.1145/3009837.3009890`.

3   Gilles Barthe, Raphaëlle Crubillé, Ugo Dal Lago, and Francesco Gavazzo. On feller continuity and full abstraction. *Proc. ACM Program. Lang.*, 6(ICFP), August 2022. `doi:10.1145/3547651`.

4   Michael Bukatin, Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh. Partial metric spaces. *American Mathematical Monthly*, 116:708–718, 10 2009. `doi:10.4169/193009709X460831`.

5   Michael A. Bukatin and Joshua S. Scott. Towards computing distances between programs via Scott domains. In Sergei Adian and Anil Nerode, editors, *Logical Foundations of Computer Science*, pages 33–43, Berlin, Heidelberg, 1997. Springer Berlin Heidelberg. `doi:10.1007/3-540-63045-7_4`.

6   Yufei Cai, Paolo G. Giarrusso, Tillmann Rendel, and Klaus Ostermann. A theory of changes for higher-order languages: incrementalizing λ-calculi by static differentiation. In *Proceedings of the 35th ACM SIGPLAN Conference on Programming Language Design and Implementation*, PLDI '14, pages 145–155, New York, NY, USA, 2014. Association for Computing Machinery. `doi:10.1145/2594291.2594304`.

7   Francesco Dagnino and Francesco Gavazzo. A fibrational tale of operational logical relations: Pure, effectful and differential. *Logical Methods in Computer Science*, Volume 20, Issue 2, Apr 2024. URL: `https://lmcs.episciences.org/11041`, `doi:10.46298/lmcs-20(2:1)2024`.

8   Fredrik Dahlqvist and Renato Neves. A complete v-equational system for graded lambda-calculus. In Marie Kerjean and Paul Blain Levy, editors, *Proceedings of the 39th Conference on*

772   *the Mathematical Foundations of Programming Semantics, MFPS XXXIX, Indiana University,*
773   *Bloomington, IN, USA, June 21-23, 2023*, volume 3 of *EPTICS*. EpiSciences, 2023. URL:
774   `https://doi.org/10.46298/entics.12299`, `doi:10.46298/ENTICS.12299`.

9   Ugo Dal Lago and Francesco Gavazzo. Differential logical relations, Part II increments and
775   derivatives. *Theoretical Computer Science*, 895:34–47, 2021.

10   Ugo Dal Lago and Francesco Gavazzo. Effectful program distancing. *Proc. ACM Program.*
       Lang., 6(POPL), January 2022. `doi:10.1145/3498680`.

11   Ugo Dal Lago, Francesco Gavazzo, and Akira Yoshimizu. Differential logical relations, part I:
       the simply-typed case. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano
       Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming,*
       ICALP 2019, July 9-12, 2019, Patras, Greece, volume 132 of *LIPIcs*, pages 111:1–111:14.
       Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. URL: `https://doi.org/10.4230/`
       `LIPIcs.ICALP.2019.111`, `doi:10.4230/LIPICS.ICALP.2019.111`.

12   Ugo Dal Lago, Furio Honsell, Marina Lenisa, and Paolo Pistone. On Quantitative Algebraic
       Higher-Order Theories. In Amy P. Felty, editor, *7th International Conference on Formal*
       Structures for Computation and Deduction (FSCD 2022), volume 228 of *Leibniz International*
       Proceedings in Informatics (LIPIcs), pages 4:1–4:18, Dagstuhl, Germany, 2022. Schloss Dagstuhl
       – Leibniz-Zentrum für Informatik. URL: `https://drops.dagstuhl.de/entities/document/`
       `10.4230/LIPIcs.FSCD.2022.4`, `doi:10.4230/LIPIcs.FSCD.2022.4`.

13   Ugo Dal Lago, Naohiko Hoshino, and Paolo Pistone. On the Lattice of Program Metrics. In
       Marco Gaboardi and Femke van Raamsdonk, editors, *8th International Conference on Formal*
       Structures for Computation and Deduction (FSCD 2023), volume 260 of *Leibniz International*
       Proceedings in Informatics (LIPIcs), pages 20:1–20:19, Dagstuhl, Germany, 2023. Schloss
       Dagstuhl – Leibniz-Zentrum für Informatik. URL: `https://drops.dagstuhl.de/entities/`
       `document/10.4230/LIPIcs.FSCD.2023.20`, `doi:10.4230/LIPIcs.FSCD.2023.20`.

14   Derek Dreyer, Amal Ahmed, and Lars Birkedal. Logical step-indexed logical relations. In
       Proceedings of the 2009 24th Annual IEEE Symposium on Logic In Computer Science, LICS
       '09, pages 71–80, USA, 2009. IEEE Computer Society. `doi:10.1109/LICS.2009.34`.

15   Marco Gaboardi, Andreas Haeberlen, Justin Hsu, Arjun Narayan, and Benjamin C. Pierce.
       Linear dependent types for differential privacy. *SIGPLAN Not.*, 48(1):357–370, jan 2013.
       `doi:10.1145/2480359.2429113`.

16   Francesco Gavazzo. Quantitative behavioural reasoning for higher-order effectful programs:
       Applicative distances. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic*
       in Computer Science, LICS '18, pages 452–461, New York, NY, USA, 2018. `doi:10.1145/`
       `3209108.3209149`.

17   Guillaume Geoffroy and Paolo Pistone. A partial metric semantics of higher-order types
       and approximate program transformations. In *Computer Science Logic 2021 (CSL 2021)*,
       volume 183 of *LIPIcs–Leibniz International Proceedings in Informatics*, pages 35:1–35:18, 2021.
       `doi:10.4230/LIPIcs.CSL.2021.23`.

18   Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types*, volume 7 of *Cambridge*
       Tracts in Theoretical Computer Science. Cambridge University Press, 1989.

19   Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory: Selected Topics in*
       Point-Set Topology. New Mathematical Monographs. Cambridge University Press, 2013.

20   Jean Goubault-Larrecq, Slawomir Lasota, and David Nowak. Logical relations for monadic
       types. In Julian Bradfield, editor, *Computer Science Logic*, pages 553–568, Berlin, Heidelberg,
       2002. Springer Berlin Heidelberg.

21   Claudio Hermida, Uday S. Reddy, and Edmund P. Robinson. Logical relations and parametricity
       - A Reynolds programme for category theory and programming languages. In *Proceedings of*
       the Workshop on Algebra, Coalgebra and Topology (WACT 2013), volume 303 of *Electronic*
       Notes in Theoretical Computer Science, pages 149–180. Elsevier, 2014.

22   Dirk Hofmann, Gavin J Seal, and W Tholen. *Monoidal Topology: a Categorical Approach to*
       Order, Metric and Topology. Cambridge University Press, New York, 2014.

23 Dirk Hofmann and Isar Stubbe. Topology from enrichment: the curious case of partial metrics. *Cahiers de Topologie et Géométrie Différentielle Catégorique, LIX*, 4:307–353, 2018.

24 H.-P.A. Künzi, H. Pajoohesh, and M.P. Schellekens. Partial quasi-metrics. *Theoretical Computer Science*, 365(3):237–246, 2006. Spatial Representation: Discrete vs. Continuous Computational Models. URL: `https://www.sciencedirect.com/science/article/pii/S0304397506004993`, `doi:10.1016/j.tcs.2006.07.050`.

25 Roy L.Crole. *Categories for Types*. Cambridge University Press, 1993.

26 Valentin Maestracci and Paolo Pistone. The Lambda Calculus Is Quantifiable. In Jörg Endrullis and Sylvain Schmitz, editors, *33rd EACSL Annual Conference on Computer Science Logic (CSL 2025)*, volume 326 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 34:1–34:23, Dagstuhl, Germany, 2025. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: `https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2025.34`, `doi:10.4230/LIPIcs.CSL.2025.34`.

27 Radu Mardare, Prakash Panangaden, and Gordon Plotkin. Quantitative algebraic reasoning. In *Proceedings LICS 2016*. IEEE Computer Society, 2016.

28 Sparsh Mittal. A survey of techniques for approximate computing. *ACM Comput. Surv.*, 48(4), March 2016. `doi:10.1145/2893356`.

29 Paolo Pistone. On generalized metric spaces for the simply typed lambda-calculus. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, pages 1–14. IEEE, 2021. `doi:10.1109/LICS52264.2021.9470696`.

30 Gordon Plotkin and Martin Abadi. A logic for parametric polymorphism. In *TLCA '93, International Conference on Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 361–375. Springer Berlin Heidelberg, 1993.

31 Jason Reed and Benjamin C. Pierce. Distance makes the types grow stronger. *Proceedings ICFP 2010*, pages 157–168, 2010.

32 John C. Reynolds. Types, abstraction and parametric polymorphism. In R.E.A. Mason, editor, *Information Processing '83*, pages 513–523. North-Holland, 1983.

33 Davide Sangiorgi, Naoki Kobayashi, and Eijiro Sumii. Logical bisimulations and functional languages. In Farhad Arbab and Marjan Sirjani, editors, *International Symposium on Fundamentals of Software Engineering*, pages 364–379, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg.

34 Michel P. Schellekens. A characterization of partial metrizability: domains are quantifiable. *Theor. Comput. Sci.*, 305(1-3):409–432, 2003. Topology in Computer Science. `doi:10.1016/S0304-3975(02)00705-3`.

35 Michael B. Smyth. The constructive maximal point space and partial metrizability. *Ann. Pure Appl. Log.*, 137(1-3):360–379, 2006. `doi:10.1016/j.apal.2005.05.032`.

36 Isar Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems*, 256:95 – 116, 2014. Special Issue on Enriched Category Theory and Related Topics (Selected papers from the 33rd Linz Seminar on Fuzzy Set Theory, 2012). `doi:10.1016/j.fss.2013.08.009`.