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EE 381 HW #3

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1. In how many ways can 10 people be seated on a bench if only 4 seats are avaliable?

Given:

order does matter b/c there is every time that someone gets seated, we no longer consider the seats that were taken for the next avaliable person. You are arranging seats.

a1 a2 a3 a4

10 9 8 7

Permutation:

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$10P4 = P(\overset{n}{10}, \overset{r}{4}) = \frac{10!}{(10-4!)} = \frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}}$$

$= \boxed{\text{5,040 ways to arrange seats}}$

2) Evaluate ...

Permutation: $P(n,r) = \dfrac{n!}{(n-r)!}$ NOTE: $0! = 1$

Combination: $C(n,r) = \dfrac{n!}{(n-r)!\,r!}$

a) $P(\overset{p}{8},\overset{r}{3}) = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = \dfrac{8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}} = \boxed{336}$

b) $P(\overset{p}{15},\overset{r}{1}) = \dfrac{15!}{(15-1)!} = \dfrac{15!}{14!} = \dfrac{15\cdot \cancel{14!}}{\cancel{14!}} = \boxed{15}$

c) $P(\overset{p}{3},\overset{r}{3}) = \dfrac{3!}{(3-3)!} = \dfrac{3!}{0!} = \dfrac{3\cdot 2\cdot 1}{1} = \boxed{6}$

d) $C(\overset{p}{7},\overset{r}{4}) = \dfrac{7!}{(7-4)!\,4!} = \dfrac{7!}{3!\,4!} = \dfrac{7\cdot 6\cdot 5\cdot \cancel{4!}}{3!\,\cancel{4!}} = \dfrac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} =$

$\dfrac{210}{6} = \boxed{35}$

e) $C(\overset{r}{4},\overset{r}{4}) = \dfrac{4!}{(4-4)!\,4!} = \dfrac{4!}{0!\,4!} = \dfrac{\cancel{4!}\,1}{\cancel{4!}} = \boxed{1}$

Daniel Duong

3) Four different math books, 6 different physics books, and 2 different chemistry books need to be arranged on a shelf. How many different arrangements possible if

↳ permutation

a) The books in each particular must stand all together?

math books = 4

physics books = 6

chemistry books = 2

// permutation!

// permutation ways to sort the categories

categories arrangement:

cat1	cat2	cat3
↓	↓	↓
3	2	1

$= 3 \cdot 2 \cdot 1 =$ 6 to sort by subjects

Math book arrangements:

book1	book2	book3	book4
↓	↓	↓	↓
4	3	2	1

$= 4!\ 4 \cdot 3 \cdot 2 \cdot 1$

Physics book arrangements:

book1	book2	book3	book4	book5	book6
↓	↓	↓	↓	↓	↓
6	5	4	3	2	1

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

= 720 ways for physics books

chemistry books:

book1	book2
↓	↓
2	1

$P(\overset{n}{2}, \overset{r}{2}) = \frac{2!}{(2-2)!} = \frac{2!}{0!} =$ 2 ways for chemistry books

3 a) (continued) // HARD!

total ways to arrange all books =

$6 \cdot 24 \cdot 720 \cdot 2 =$ $\boxed{207,360 \text{ ways to arrange together.}}$

b) only the mathematic books must stand together?

math books

Given: //permutation

The math books must be grouped together

book 1 book 2 book 3 book 4

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$4 \quad 3 \quad 2 \quad 1 = 4! = 4 \cdot 3 \cdot 2 \cdot 1 =$ 24 ways for math books

So, there are no constraints on the physic books and chemistry books, so they can be grouped freely to such where they can be grouped in between other subjects. Additionally, the constraint whereas only math books must stand together, so they can be arranged before or after any subject, thus we count it as 1 book. there are 6 different physics books and 2 different chemistry books, thus we add it together and put it into a factorial.

$(6+2+1)! = 9!$ to arrange books (math books as 1 set)

$9!\,4! = 362,880 \cdot 24 =$ $\boxed{8,709,120 \text{ ways possible if only the math books must stand together.}}$

4) How many different salads can be made from lettuce, escarole, endive, watercress, and chicory?

Greens = {lettuce, escarole, endive, watercress, chicory}

= 5 options = n   //combination: $C(n,r) = \frac{n!}{(n-r)!\,r!}$

//NOTE: A salad should have at least 1 green. You should be able to mix up to at most the number of different greens provided.

$\boxed{\text{Total different salads} = \text{case 1} + \text{case 2} + \text{case 3} + \text{case 4} + \text{case 5} = 5 + 10 + 10 + 5 + 1 = 31 \text{ ways}}$

case 1: Lettuce only

$$5C1 = C(\overset{n}{5}, \overset{r}{1}) = \frac{5!}{(5-1)!\,1!} = \frac{5!}{4!\,1!} = \frac{5 \cdot \cancel{4!}}{\cancel{4!}\,1!} = 5 \text{ ways}$$

case 2: Lettuce and escarole

$$5C2 = C(\overset{n}{5}, \overset{r}{2}) = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}\,2!} = \frac{20}{2!} = 10 \text{ ways}$$

case 3: Lettuce, escarole, and endive

$$5C3 = C(\overset{n}{5}, \overset{r}{3}) = \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!\,3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2!\,\cancel{3!}} = \frac{20}{2!} = 10 \text{ ways}$$

case 4: Lettuce, escarole, endive, and watercress

$$5C4 = C(\overset{n}{5}, \overset{r}{4}) = \frac{5!}{(5-4)!\,4!} = \frac{5!}{1!\,4!} = \frac{5 \cdot \cancel{4!}}{1!\,\cancel{4!}} = 5 \text{ ways}$$

case 5: ALL GREENS

$$5C5 = C(\overset{n}{5}, \overset{r}{5}) = \frac{5!}{(5-5!)\,5!} = \frac{\cancel{5!}}{0!\,\cancel{5!}} = \frac{1}{1} = 1 \text{ way}$$

5) How many permutations of {a, b, c, d, e, f, g} end with a?

$S = 7$

Given:

letter formatting: a_1 a_2 a_3 a_4 a_5 a_6 a_7

RULES: a_1 a_2 a_3 a_4 a_5 a_6 a

w/a being the last letter, you cannot use the a again; thus you have 6 options left.

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 6 & 5 & 4 & 3 & 2 & 1 & \end{array}$$

$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$ 720 ways of permutations w/a being the last character

6) How many bit strings of length 10 contain:

Bit Strings = $\{0, 1\}$

Bit string format: _ _ _ _ _ _ _ _ _ _

a) Exactly four ones.

a1 a2 a3 a4 a5 a6 a7 a8 a9 a10

_ 1 _ _ 1 _ 1 1 _ _

NOTE: The order where the four one's does not order, so it can be placed however you like.

// combination: $C(n, r) = \frac{n!}{(n-r)!r!}$

n = length 10 bit string = 10

r = four number ones = 4

$$C(\overset{n}{10}, \overset{r}{4}) = \frac{10!}{(10-4)!4!} = \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} =$$

$\boxed{\text{210 ways to arrange length 10 bit string w/ exactly four ones.}}$

b) At most four ones? $n = 10$, $r \leq 4$

NOTE: someone can decide to have no ones in their strings, one one, two one's, three one's or four one's; so we must add the combinations of possibilities. order doesn't matter b/c

$C(10, r)$ [0 ones] + $C(10, 1)$ [1 ones] + $C(10, 2)$ [2 ones] + $C(10, 3)$ [3 ones] + $C(10, 4)$ [4 ones]

let n be the number of length bit string = 10

let r be the number of ones selected

6. continued(!) $C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(1\ldots$

b. 0 ones

$C(\overset{n}{10},\overset{r}{0}) = \dfrac{10!}{(10-0)!\,0!} = \dfrac{\cancel{10!}}{\cancel{10!}} = 1$

1 one

$C(\overset{n}{10},\overset{r}{1}) = \dfrac{10!}{(10-1)!\,1!} = \dfrac{10!}{9!} = \dfrac{10\cdot\cancel{9!}}{\cancel{9!}} = 10$

2 ones

$C(10,2) = \dfrac{10!}{(10-2)!\,2!} = \dfrac{10!}{8!\,2!} = \dfrac{10\cdot 9\cdot\cancel{8!}}{\cancel{8!}\,2!} = \dfrac{90}{2} = 45$

3 ones

$C(10,3) = \dfrac{10!}{(10-3)!\,3!} = \dfrac{10!}{7!\,3!} = \dfrac{10\cdot 9\cdot 8\cdot\cancel{7!}}{\cancel{7!}\,3!} = \dfrac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}$

$= \dfrac{720}{6} = 120$

4 ones

$C(10,4) = \dfrac{10!}{(10-4)!\,4!} = \dfrac{10!}{6!\,4!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot\cancel{6!}}{\cancel{6!}\,4!} = \dfrac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$

$1+10+45+120+121 =$ $\boxed{\text{386 ways to have up to four ones}}$

c. At least four ones?

NOTE: we filter out the options that have anything four ones.

so, we count the four ones and up..

$C(\overset{n}{10},\overset{r}{4}) = \dfrac{10!}{(10-4)!\,4!} = \dfrac{10!}{6!\,4!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot\cancel{6!}}{\cancel{6!}\,4!} = \dfrac{10\cdot 9\cdot 8\cdot 7}{4!} = 210$

$C(\overset{n}{10},\overset{r}{5}) = \dfrac{10!}{6!\,4!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot\cancel{6!}}{\cancel{6!}\,4!} = 252$

b. (continued!)

c. (continued!)

$n = 10$
$r \geq 4$

$C(10, 6) = \frac{10!}{(10-6)!6!} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{4! \cancel{6!}} = 210$

$C(10, 7) = \frac{10!}{(10-7)!7!} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{3! \cancel{7!}} = 120$

$C(10, 8) = \frac{10!}{(10-8)!8!} = \frac{10!}{2!8!} = \frac{10 \cdot 9 \cdot \cancel{8!}}{2! \cancel{8!}} = 45$

$C(10, 9) = \frac{10!}{(10-9!) 9!} = \frac{10!}{1!9!} = \frac{10 \cdot \cancel{9!}}{1! \cancel{9!}} = 10$

$C(10, 10) = \frac{10!}{(10-10)!10!} = \frac{\cancel{10!}}{0! \cancel{10!}} = \frac{1}{1} = 1$

$C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) +$
$C(10,9) + C(10,10) = 210 + 252 + 210 + 120 + 45 + 10 + 1 =$

848 ways to have at least four ones.

d. An equal number of 0's and 1's?

_ _ _ _ _ _ _ _ _ _

// we know that $10/2 = 5$, so we have to have 5 heads and 5 tails, regardless of the order. (split heads and tails evenly)

$n = 10$ $r =$ num heads or num tails)

$C(10, 5) = \frac{10!}{(10-5)!5!} = \frac{10!}{5!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}\, 5!} =$ 252 ways of having even amount of heads and tails.

7. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes:

a) are there in total?

Formatting ex:

c1	c2	c3	c4	c5	c6	c7	c8	c9	c10
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
H	T	H	T	T	T	T	H	T	T

result = heads or tails = 2

// Product rule: $n_1 \cdot n_2$ ways

total possible ways $= 2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2 = 2^{10} =$ $\boxed{\text{1024 possible results}}$

b) contain exactly two heads?

Given: we have to have exactly 2 heads, but the order does not matter, so it's a combination.

combination: $C(n,r) = \dfrac{n!}{(n-r)!\,r!}$

n = 10 b/c 10 flips

r = 2 heads

$C(10,2) = \dfrac{10!}{(10-2)!\,2!} = \dfrac{10!}{8!\,2!} = \dfrac{10\cdot9\cdot\cancel{8!}}{\cancel{8!}\,2!} = \dfrac{90}{2} =$ $\boxed{\text{45 ways w/ exactly two heads}}$

c) contain at most three tails?

n = 10
$r \leq 3$

// NOTE: we have at most 3 tails, so we to consider 3 cases where there's no tails, 1 tail, 2 tails, and 3 tails. order does not matter, so it's a combination.

$C(10,0) = \dfrac{10!}{(10-0)!\,0!} = \dfrac{\cancel{10!}}{\cancel{10!}} =$ 1 way for no tails

$C(10,1) = \dfrac{10!}{(10-1)!\,1!} = \dfrac{10!}{9!\,1!} = \dfrac{10\cdot\cancel{9!}}{\cancel{9!}\,1!} =$ 10 ways for 1 tail

$C(10,2) = \dfrac{10!}{(10-2)!\,2!} = \dfrac{10!}{8!\,2!} = \dfrac{10\cdot9\cdot\cancel{8!}}{\cancel{8!}\,2!} = \dfrac{90}{2} =$ $\boxed{\text{45 ways for 2 tails}}$

$C(10,3) =$ $\boxed{\text{120 ways for 3 tails}}$

$= 1+10+45+120 =$ $\boxed{176}$ ways w/ at most 3 tails continued!)

d) contain the same number of heads and tails?

//NOTE: There are a total number of 10 coin tosses and 10 results. B/C there a coin either has heads or tails, it has two possible results. so we divide number of coin tosses by the # of possible results : $\frac{10}{2} = 5$ Meaning that there are 5 heads and 5 tails; Yet order is not indicated, thus order does not matter, so its a combination!

n=10
r=5

$$C(10, 5) = \frac{10!}{(10-5)!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{5!\cancel{5!}} =$$

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \boxed{252 \text{ ways to have the number of heads and tails.}}$$

*8. How many ways are there for 8 men and 5 women in a line, *** so that no two women are standing next to each other.

Hint! position the women first and then consider possible positions for the men.

//NOTE: we have a constraint where no women stand next to each other.

women arrangements = 5! = 120
men arrangements = 8! = 40,320

Total = 13 ppl

1	2	3	4	5	6	7	8	9	10	11	12	13
W	M	W	M	W	M	W	M	W	M	M	M	M

NOTE that the easiest pattern was to alternate between women and men. with that there are 4 men behind the last women. leaves us w/ 4 men to be positioned among those women.

For each man, there are 6 possibilities w/repitition allowed: 4 gaps between the women and 2 sides of the women. Do not count 6^4 b/c it doesn't count order of men seated in the same group.

$$C(9,4) = \frac{9!}{(9-4)!4!} = \frac{9\cdot8\cdot7\cdot6\cdot\cancel{5!}}{\cancel{5!}4!} = 126$$

$= C(9,4)\cdot$ women arrangements $\cdot$ men arrangements

$$= 126\cdot5!\cdot8! = 126\cdot120\cdot40{,}320 = \boxed{609{,}638{,}400}$$

9. Find the constant term for the expansion of $(x^2+\frac{1}{x})^{12}$ $\quad \binom{n}{r}=C(n,r) \quad \frac{1}{x}=x^{-1}$

//use Bionomial coefficient: $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \dots \binom{n}{n}y^n$

let $n=12$

$$= (x^2+\tfrac{1}{x})^{12} = (x^2)^{12} + \binom{12}{1}(x^2)^{12-1}\tfrac{1}{x} + \binom{12}{2}(x^2)^{12-2}\tfrac{1}{x^2} + \binom{12}{3}(x^2)^{12-3}\tfrac{1}{x^3} +$$

$$+\binom{12}{4}(x^2)^{12-4}\tfrac{1}{x^4} + \binom{12}{5}(x^2)^{12-5}\tfrac{1}{x^5} + \binom{12}{6}(x^2)^{12-6}\tfrac{1}{x^6} + \binom{12}{7}(x^2)^{12-7}\tfrac{1}{x^7} +$$

$$\binom{12}{8}(x^2)^{12-8}\tfrac{1}{x^8} + \binom{12}{9}(x^2)^{12-9}\tfrac{1}{x^9} + \binom{12}{10}(x^2)^{12-10}\tfrac{1}{x^{10}} + \binom{12}{11}(x^2)^{12-11}\tfrac{1}{x^{11}} +$$

$$\binom{12}{12}(x^2)^{12-12}\tfrac{1}{x^{12}}$$

$$= x^{24} + 12(x^2)^{11}\tfrac{1}{x} + 66(x^2)^{10}\tfrac{1}{x^2} +$$

$$220(x^2)^9\tfrac{1}{x^3} + 495(x^2)^8\tfrac{1}{x^4} + 792(x^2)^7\tfrac{1}{x^5}$$

$$+ 924(x^2)^6\tfrac{1}{x^6} + 792(x^2)^5\tfrac{1}{x^7} + 495(x^2)^4\tfrac{1}{x^8}$$

$$+ 220(x^2)^3\tfrac{1}{x^9} + 66(x^2)^2\tfrac{1}{x^{10}} + 12(x^2)^1\tfrac{1}{x^{11}}$$

$$+ 1(x^2)^0\tfrac{1}{x^{12}}$$

$C(12,1) = 12$
$C(12,2) = 66$
$C(12,3) = 220$
$C(12,4) = 495$
$C(12,5) = 792$
$C(12,6) = 924$
$C(12,7) = 792$
$C(12,8) = 495$
$C(12,9) = 220$
$C(12,10) = 66$
$C(12,11) = 12$
$C(12,12) = 1$

7. (continued!)

$$= x^{24} + 12x^{22}\frac{1}{x} + 66x^{20}\frac{1}{x^2} + 220x^{18}\frac{1}{x^3} + 495x^{16}\frac{1}{x^4} + 792x^{14}\frac{1}{x^5} + 924x^{12}\frac{1}{x^6} + 792x^{10}\frac{1}{x^7} + 495x^{8}\frac{1}{x^8} + 220x^{6}\frac{1}{x^9} + 66x^{4}\frac{1}{x^{10}} + 12x^{2}\frac{1}{x^{11}} + \frac{1}{x^{12}}$$

$$\boxed{= x^{24} + 12x^{21} + 66x^{18} + 220x^{15} + 495x^{12} + 792x^{9} + 924x^{6} + 792x^{3} + 495 + \frac{220}{x^3} + \frac{66}{x^6} + \frac{12}{x^9} + \frac{1}{x^{12}}}$$ ✓

Given:

$n = 12$

$\frac{1}{x} = x^{-1}$

$a = x^2$

$b = x^{-1}$

$$(\underbrace{x^2}_{a} + \underbrace{\tfrac{1}{x}}_{b})^{12 \leftarrow n} \quad (a+b)^n$$

$$\boxed{*\binom{n}{r} a^{n-r} b^{r}}$$ // multiply by constant!

$$= \binom{12}{r}(x^{2})^{12-r}(x^{-1})^{r}$$ // pull the constants out (ignored)

$2(12-r) =$

$x^{24-2r}\, x^{-r} = x^{0}$ // set eval to 1

// combine like terms

//FIND r

$x^{24-2r-r} = x^{0}$ // same bases

$x^{24-3r} = x^{0}$

$24 - 3r = 0$

$+3r \quad +3r$

$24 = 3r$

$\frac{24}{3} = \frac{3r}{3}$; $\boxed{r = 8}$

9. (continued!)

Since we found $r=8$ we plug back into the original equation!

$n=12$ $a=x^2$ $b=x^{-1}$

$\binom{n}{r} a^{n-r} b^{r}$

original Equation: $(x^2+\frac{1}{x})^{12}$

$\binom{12}{8}(x^2)^{12-8}(\frac{1}{x})^{8}$ $\quad C(12,8)=495$

$= C(12,8)\cdot(x^2)^4(x^{-1})^8$

$= 495 \cdot \cancel{x^8} \cdot \cancel{x^{-8}} = \boxed{495}$