Homework 5 Solution

1) $f(x) = \left(\frac{1}{2}\right)^x \quad (x = 1, 2, 3 \dots)$

$$E(X) = \sum x f(x) = \sum_{x=1}^{\infty} x\left(\frac{1}{2}\right)^x$$

$$= \frac{1}{2} + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{32}\right) + \dots$$

We have:

$$\frac{E(X)}{2} = \frac{\frac{1}{2} + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{32}\right) + \dots}{2}$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \dots$$

$$= \frac{1}{4} + 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{16}\right) + 4\left(\frac{1}{32}\right) + 5\left(\frac{1}{64}\right) \dots$$

$$E(X) - \frac{1}{2}E(X) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1$$

$$\Rightarrow \quad \frac{1}{2} E(X) = 1$$

$$\Rightarrow \quad E(X) = \boxed{2}$$

2)

x	1	0
$f(x)$	$1/3$	$2/3$

y	2	-3
	$3/4$	$1/4$

a) $E(X) = 1 \times \frac{1}{3} + 0 \times \frac{2}{3} = \frac{1}{3}$

$E(Y) = 2 \times \frac{3}{4} + (-3) \times \frac{1}{4} = \frac{3}{4}$

$$E(3X+2Y) = 3E(X) + 2E(Y)$$

$$= 3 \times \frac{1}{3} + 2 \times \frac{3}{4} = \boxed{\frac{5}{2}}$$

b) $E(X^2) = \frac{1}{3}$ $\qquad E(Y^2) = \frac{21}{4}$

$$E(2X^2 - Y^2) = 2E(X^2) - E(Y^2)$$

$$= 2 \times \frac{1}{3} - \frac{21}{4} = \boxed{-\frac{55}{12}}$$

c) $E(XY) = E(X).E(Y) = \frac{1}{3} \times \frac{3}{4} = \boxed{\frac{1}{4}}$

since X & Y are independent

d) $E(X^2Y) = E(X^2).E(Y) = \frac{1}{3} \times \frac{3}{4} = \boxed{\frac{1}{4}}$

3)

x	1	2	-1
$f(x)$	$1/2$	$1/3$	$1/6$

$$E(X_1) = E(X_2) = \dots = E(X_n) = 1 \times \frac{1}{2} + 2 \times \frac{1}{3} + (-1)\frac{1}{6}$$

$$= 1$$

a) $E(X_1 + X_2 + \dots + X \quad)$

$$= E(X_1) + E(X_2) + \dots E(X_n)$$

$$= 1 + 1 + \dots + 1$$

$$= \boxed{n}$$

b) $E(X_1^2) = \dots = E(X_n^2) = 1^2 \times \frac{1}{2} + 4 \times \frac{1}{3} + 1 \times \frac{1}{6}$

$$= 2$$

$E(X_1^2 + X_2^2 + \dots + X_n^2)$

$$= E(X_1^2) + E(X_2^2) + \dots + E(X_n^2)$$

$$= 2 + 2 + \dots + 2$$

$$= \boxed{2n}$$

4)

x	-2	3	1
$f(x)$	$1/3$	$1/2$	$1/6$

$$E(X) = (-2)\times\frac{1}{3} + 3\times\frac{1}{2} + 1\times\frac{1}{6}$$
$$= 1$$

$$Var(X) = E[(X-\mu)^2] = 9\times\frac{1}{3} + 4\times\frac{1}{2} + 0\times\frac{1}{6}$$
$$= \boxed{5}$$

$$\sigma = \sqrt{Var(X)} = \boxed{\sqrt{5}}$$

5) $E[(X-1)^2] = 10$

$\Rightarrow E(X^2) - 2E(X) + 1 = 10$

$\Rightarrow E(X^2) - 2E(X) = 9 \quad (1)$

$E[(X-2)^2] = 6$

$\Rightarrow E(X^2) - 4E(X) = 2 \quad (2)$

From (1) & 2, we have $\begin{cases} E(X^2) = 16 \\ E(X) = 3.5 \end{cases}$

a) $E(X) = \boxed{3.5}$

b) $Var(X) = E(X^2) - (E(X))^2$

$= 16 - (3.5)^2$

$= \boxed{3.75}$

c) $\sigma_X = \boxed{\sqrt{3.75}}$

6)

x	1	2	3	4	5	6
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$E(X_1) = E(X_2) = E(X_3) = \frac{21}{6}$

a) $E(X_1 + X_2 + X_3)$

$= E(X_1) + E(X_2) + E(X_3)$

$= \frac{21}{6} \times 3 = \boxed{\frac{21}{2}}$

b) $Var(X_1 + X_2 + X_3)$

$= Var(X_1) + Var(X_2) + Var(X_3)$

$Var(X_1) = Var(X_2) = Var(X_3)$

$Var(X_1) = 1.71^2$

$\Rightarrow \quad Var(X_1 + X_2 + X_3) = 3 \times 1.71^2$

$= \boxed{8.7723}$

7) Applying Chebyshev's inequality:

a) $P(|X-3| \geq 2) \leq \frac{1}{2}$

b) $P(|X-3| \geq 1) \leq 2$ (useless)