Lecture 6 note (9/29/20)

Moment about the mean:

$$\mu_r = E[(X-\mu)^r]$$

Moment about the origin:

$$\mu'_r = E[X^r]$$

$$e^{uX} = 1 + uX + \frac{u^2X^2}{2!} + \cdots\cdots$$

$$M(u) = 1 + uE[X] + \frac{u^2}{2!}E[X^2] + \frac{u^3}{3!}E[X^3] + \cdots$$

$$\frac{d}{du}M(u) = 0 + E(X) + \underset{0}{u}E[X^2] + \underset{0}{\frac{u^2}{2!}}E[X^3] + \cdots$$

$$\left.\frac{d}{du}M(u)\right|_{u=0} = E(X)$$

$$E[X^2] = \left.\frac{d^2}{du^2}M(u)\right|_{u=0}$$

$\vdots$

moment about the origin

x	1	-1
$f(x)$	$1/2$	$1/2$

Maclaurin series

$$f(t) = f(0) + f'(0) \times t + f''(0) \times \frac{t^2}{2!} + \dots$$

a) $$M_X(t) = E\left[e^{tX}\right] = \sum e^{tx} f(x)$$

$$= e^{t.1} \times \frac{1}{2} + e^{t.(-1)} \times \frac{1}{2}$$

$$= \frac{1}{2}\left(e^{t} + e^{-t}\right)$$

b) M_1', M_2', M_3', M_4'

$$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

$+ \qquad +$

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots$$

$$\frac{1}{2}\left(e^{t} + e^{-t}\right) = \left(1 + \overset{1}{E[X^2]}\frac{t^2}{2!} + \overset{1}{E(X^4)}\frac{t^4}{4!} + \dots\right)$$

$$= M_X(u)$$

$M_1' = E(X) = 0 \qquad E[X^2] = 1 = M_2'$

$M_3' = E(X^3) = 0 \qquad E[X^4) = 1 = M_4'$

$$M_X(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \qquad (1)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}\underset{0}{E(X^3)} + \frac{t^4}{4!}\underset{1}{E(X^4)} \dots \qquad (2)$$

$$E(X) = 0$$

$$E(X^2) = 1$$

$$2^{nd} \text{ approach: } M_X(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots$$

$$\mu_1' = E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= 0 + \frac{2t}{2!} + \frac{4t^3}{4!} + \dots \Big|_{t=0}$$

$$= 0$$

$$\mu_2' = E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= \frac{2}{2!} + \frac{12t^2}{4!} + \dots \Big|_{t=0}$$

$$= 1$$

$$\vdots$$

In-class exercise:

x	$1/2$	$-1/2$
$f(x)$	$1/2$	$1/2$

a) $M_X(t) = E[e^{tX}] = \sum e^{tx} f(x)$

$$= e^{t \times 1/2} \times \frac{1}{2} + e^{t \times \frac{-1}{2}} \times \frac{1}{2}$$

$$= \frac{1}{2}\left(e^{t/2} + e^{-t/2}\right)$$

b) $$e^{t/2} = 1 + \frac{1}{2}t + \frac{1}{4} \times \frac{t^2}{2!} + \frac{1}{8} \times \frac{t^3}{3!} + \dots$$

$$e^{-t/2} = 1 - \frac{1}{2}t + \frac{1}{4} \times \frac{t^2}{2!} - \frac{1}{8} \times \frac{t^3}{3!} + \dots$$

$$\frac{1}{2}\left(e^{t/2} + e^{-t/2}\right) = 1 + \frac{1}{4}\frac{t^2}{2!} + \frac{1}{16}\frac{t^4}{4!} + \dots$$

$$= M_X(t)$$

$$\mu_1' = E(X) = \frac{d}{dt} M(t)\Big|_{t=0} = 0 + \frac{1}{4} \times \frac{2t}{2!} + \dots = \boxed{0}$$

$$\mu_2' = E(X^2) = \frac{d^2}{dt^2} M(t)\Big|_{t=0} = \frac{1}{4} \times \frac{2}{2!} + \dots = \boxed{\frac{1}{4}}$$

$\mu_3' = \boxed{0}$ $\mu_4' = \boxed{\frac{1}{16}}$

Moment about the mean:

$M_X(t) = E(e^{t(X-\mu)})$

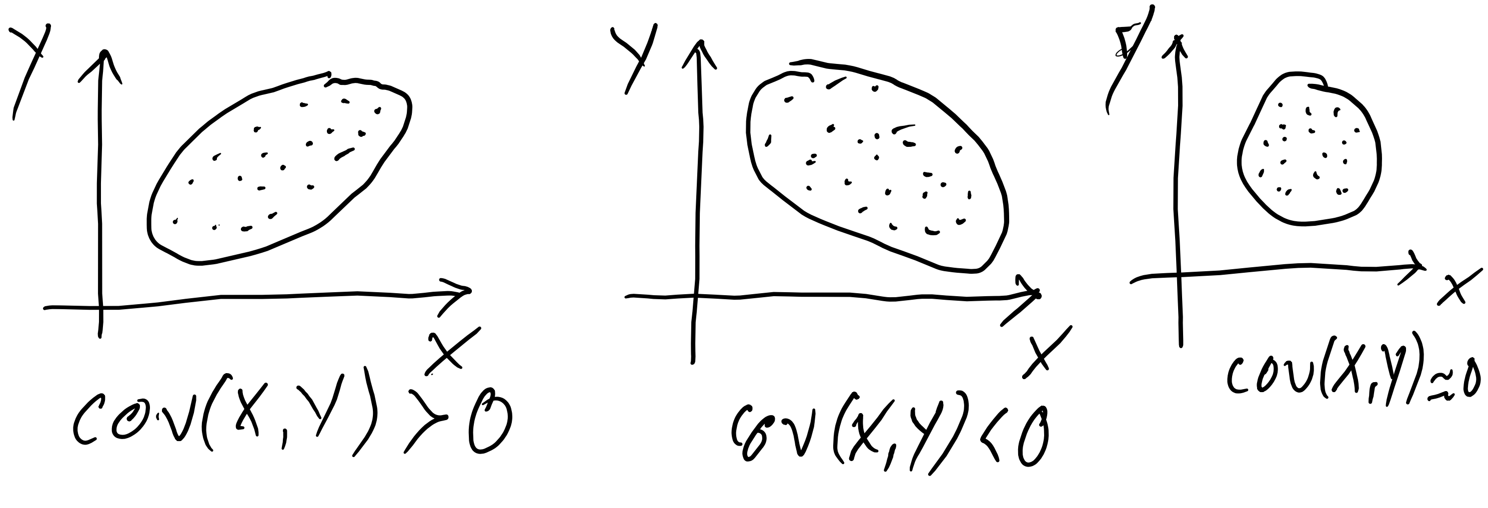
Theorem 3-8)

$$M_{\frac{X+a}{b}}(t) = E\left[e^{t \cdot \left(\frac{X+a}{b}\right)}\right]$$

$$= E\left[e^{\frac{Xt}{b}} e^{\frac{at}{b}}\right]$$

$$= e^{at/b} E\left(e^{Xt/b}\right)$$

$$= e^{at/b} M_X\left(\frac{t}{b}\right)$$



$$\sigma_{XX} = \sigma_X^2 = E[(X-\mu_x)^2]$$

$$\sigma_{YY} = \sigma_Y^2 = E[(Y-\mu_Y)^2]$$

In-class exercise:

$$f(x,y) = \frac{1}{12}$$

$$\sigma_{XY} = E(XY) - E(X) \cdot E(Y)$$

$$E(X) = \sum\sum x f(x,y)$$

$$= 0\left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12}\right) + 1 \cdot \left(\frac{3}{12}\right) + 2 \cdot \left(\frac{3}{12}\right)$$

$$3 \cdot \left(\frac{2}{12}\right) + 4\left(\frac{1}{12}\right)$$

$$= 19/12$$

$$E(Y) = \sum\sum y f(x,y)$$

$$= 0 \cdot \left(\frac{5}{12}\right) + 1\left(\frac{4}{12}\right) + 2\left(\frac{3}{12}\right) = \frac{10}{12}$$

$$E(XY) = \sum\sum xy f(x,y)$$

$$= \overset{x}{0} \times \overset{y}{0} \times \frac{1}{12} + \overset{x}{0} \times \overset{y}{1} \times \frac{1}{12} + \overset{x}{0} \times \overset{y}{2} \times \frac{1}{12}$$

$$+ \ldots\ldots \overset{x}{2} \times \overset{y}{2} \times \frac{1}{12}$$

$$= 1$$

$$\sigma_{xy} = E(XY) - E(X)E(Y)$$

$$= 1 - \frac{19}{12} \cdot \frac{10}{12}$$

$$= \boxed{-\frac{23}{72}}$$

$\rho = 1 \rightarrow$ $X \nearrow 50\%$

$Y \nearrow 50\%$

$\rho = -1 \rightarrow$ $X \nearrow 50\%$ $\approx$ $X \searrow 50\%$

$Y \searrow 50\%$ $\quad$ $Y \nearrow 50\%$

$\rho = 0 \rightarrow$

100\$ on 10/1/20

99\$/101.1\$ on 10/2/20 → −0.01% / 1.1%

10/3/20

Mode:

$X = \{4, 2, 4, 3, 2, 2\}$

Mode = 2

Median

$X = \{0, 1, 2, 4, 5\}$ median = 2

$X = \{10, \underbrace{20, 40}, 50\}$

30 = Median

$P(X \leq 30) = \frac{1}{2}$

$P(X \geq 30) = 1/2$