



EE 381 Probability & Statistic with Applications to Computing (Fall 2020)



Lecture 1 Basic Concepts of Probability (p1)

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Chapter 1's topics

- **Discrete Sample Spaces**
- **Important Theorems on Probability**
- Conditional Probability
- Independence
- Bayes' Rule
- Combinatorial Analysis

Introduction about Probability & Statistics

- Probability: the study of the likeliness of result, action or event occurring. Often based on prior knowledge or the statistics of similar or past events.
- Statistics: the study of and the dealing with data.
- An understanding of Probability and Statistics is necessary in most if not all work related to science and engineering.

Introduction about Probability

- Engineering Applications include:
 - ✓ Realistic signals - with noise or characteristic “unknown” parts
 - ✓ Signal-to-noise ratios, noise-power measurements
 - ✓ Reliability, quality, failure rates, etc.
- Probability theory is necessary for engineering system modeling and simulations:
 - ✓ Unknown initial conditions (random)
 - ✓ Noisy measurements, expected inaccuracies, etc. during operation.

Random Experiments

- In science and engineering, experiments are important because we based on those to control the value of the variables that affect the outcome of the experiment.
- However, in some experiments, even though they are conducted in the same conditions, the outcomes are different each time. They are call *random experiments*.

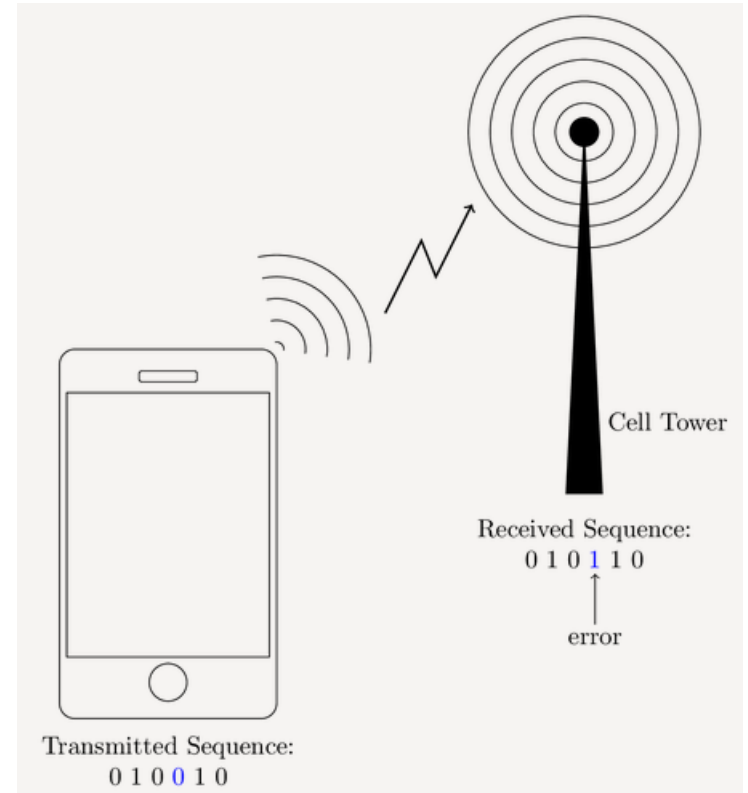
Examples:

1. Tossing a coin. The result of the experiment will either come up “Tail” or “Head”. That will be one of the elements of the set $\{H, T\}$.
2. Tossing a dice. The result of the experiment will produce one of the numbers in the set $\{1, 2, 3, 4, 5, 6\}$.
3. Tossing a coin twice. There will be 4 possible results, as indicated by $\{HH, HT, TH, TT\}$.

Random Experiments

Example: Communication Systems

- Conversation on phone (converted to information bits 0 and 1) when transmitted to cell tower is affected by noise.
- Noise is a random phenomenon.
- Probability theory is used to understand the behavior of noise in this system, and take measures to correct the errors.



[1]

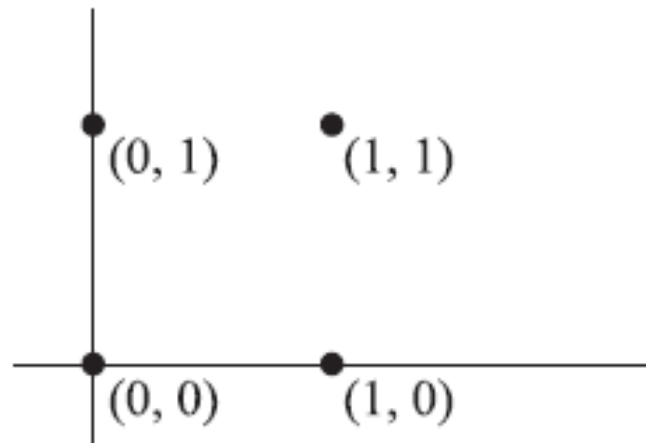
Sample spaces

- A set S that consists of all possible outcomes of a random experiment is called a sample space, and each outcome is called a sample point.

Example: Tossing a dice. One sample space is given by $\{1, 2, 3, 4, 5, 6\}$. Or another sample space is $\{\text{odd}, \text{even}\}$. The first sample space is preferred, since it is more comprehensive.

Different types of sample spaces

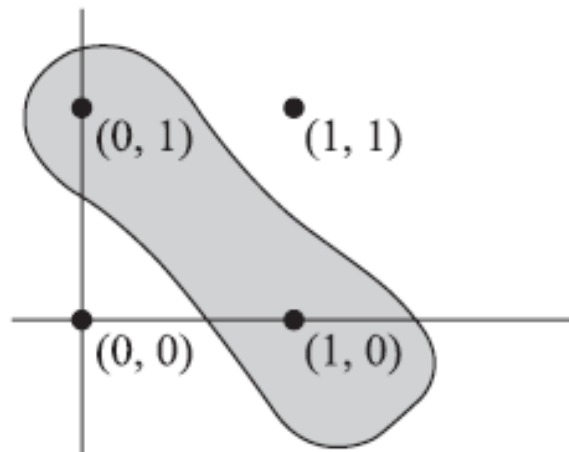
- *Discrete sample space:*
 - ✓ *Finite sample space:* has a finite number of points.
 - ✓ *Countably infinite sample space:* has as many points as there are natural numbers 1, 2, 3,...
- *Nondiscrete sample space:*
 - ✓ *Noncountably infinite sample space:* has as many points as there are in some interval on the x axis, such as $0 \leq x \leq 1$.



Example of finite sample space: tossing a coin twice.
0 for Tail, and 1 for Head.

Events

- An *event* is a subset A of a sample space S (a set of possible outcomes).
- If the outcome of an experiment is an element of A , meaning that the event A has occurred.
- An event consisting of a single point of S is called a *simple* or *elementary event*.
- The empty set \emptyset is called the *impossible event*.



Example of an event: tossing a coin twice. Bolded area shows the event that only one head comes up.

Events

- By using set operations on events in S , we can obtain other events in S .
- For example, if A and B are events, then:
 1. $A \cup B$ is the event “either A or B or both”. $A \cup B$ is called the union of A and B .
 2. $A \cap B$ is the event “both A and B ”. $A \cap B$ is called the intersection of A and B .
 3. A' is the event “not A ”. A' is called the complement of A .
 4. $A - B = A \cap B'$ is the event “ A but not B ”. In particular, $A' = S - A$
- If A and B are disjoint, i.e., $A \cap B = \emptyset$, we say that the events are mutually exclusive, meaning that they cannot both occur.

Example: Tossing the coin twice. Let $A = \{HT, TH, HH\}$ and $B = \{HT, TT\}$

$$A \cup B = \{HT, TH, HH, TT\}$$

$$A \cap B = \{HT\}$$

$$A' = \{TT\}$$

$$A - B = \{TH, HH\}$$

The Concept of Probability

- In random experiment, there is always uncertainty as to whether a particular event will or will not occur.
- We define probability as a measure of the chance with which we can expect the event will occur.
- Example: for a certain event
 - ✓ If we say the probability is 100%, we are sure that the event will occur.
 - ✓ If we say the probability is $\frac{1}{4}$ or 25%, then we would say there is a 25% chance that the event will occur, and 75% chance it will not occur.

The Concept of Probability

Two important procedures for estimating the probability of an event:

- **Classical approach:** If an event can occur in h different ways out of a total number of n possible ways, all of which are equally likely, then the probability of the event is h/n .
 - ✓ Example: probability that a head will turn up in a single toss of a coin is $\frac{1}{2}$ because there are 2 equally like ways, and one of them (a head) can arise in only one way.
- **Frequency approach:** if after n repetitions of an experiment, where n is very large, an event is observed to occur in h of these, then the probability of the event is h/n . This is also called the *empirical probability* of the event.
 - ✓ Example: Tossing a coin 1000 times and find that it comes up heads 532 times, we estimate the probability of a head coming up to be $532/1000=0.532$.

Both approaches have serious drawbacks: “equally likely” are vague, and “large number” is vague.

The Axioms of Probability

Suppose we have sample space S , and event A belong to sample space S .

- Axiom 1: $P(A) \geq 0$
- Axiom 2: $P(S)=1$
- Axiom 3: for any mutually exclusive events A_1, A_2, \dots
$$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$$

Important Theorems on Probability

- **Theorem 1-1:** If $A_1 \subset A_2$, then $P(A_1) \leq P(A_2)$
and $P(A_2 - A_1) = P(A_2) - P(A_1)$
- **Theorem 1-2:** For every event A , $0 \leq P(A) \leq 1$
- **Theorem 1-3:** $P(\emptyset) = 0$
- **Theorem 1-4:** If A' is the complement of A , then $P(A') = 1 - P(A)$
- **Theorem 1-5:**
If $A = A_1 \cup A_2 \cup \dots \cup A_n$, where A_1, A_2, \dots, A_n are mutually exclusive, then
$$P(A) = P(A_1) + P(A_2) + \dots P(A_n)$$

If $A = S$, the sample space, then $P(A_1) + P(A_2) + \dots P(A_n) = 1$

Important Theorems on Probability

- **Theorem 1-6:** If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally, if A_1, A_2, A_3 are any three events, then

$$\begin{aligned} &P(A_1 \cup A_2 \cup A_3) \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \end{aligned}$$

Generalizations to n events can also be made.

- **Theorem 1-7:** For any event A and B, $P(A) = P(A \cap B) + P(A \cap B')$
- **Theorem 1-8:** If an event A must result in the occurrence of one of the mutually exclusive events A_1, A_2, \dots, A_n , then

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \dots + P(A \cap A_n)$$

Assignment of Probabilities

- If a sample space S consists of a finite number of outcomes, then

$$P(A_1) + P(A_2) + \cdots P(A_n) = 1 \quad (*)$$

Where A_1, A_2, \dots, A_n are elementary events given by $A_i = \{a_i\}$

- Arbitrary probabilities can be assigned for these simple events as long as (*) is satisfied. In particular, if we assume equal probabilities for all simple events, then

$$P(A_k) = \frac{1}{n}, \quad k = 1, 2, 3, \dots, n$$

And if A is any event made up of h such simple events, we have

$$P(A) = \frac{h}{n}$$

Assignment of Probabilities

Example: A single dice is tossed once. Find the probability of a 2 or 5 turning up.

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. If we assign equal probabilities to the sample points (the dice is fair), then

$$P(1) = P(2) = \dots = P(6) = \frac{1}{6}$$

The event that either 2 or 5 turns up is indicated by $2 \cup 5$. Therefore,

$$P(2 \cup 5) = P(2) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Summary

Below are definitions used in probability that should be understood/remembered:

Experiment

- ✓ An experiment is some action that results in an outcome.
- ✓ A random experiment is one in which the outcome is uncertain before the experiment is performed.

Possible outcomes/sample space

- ✓ A description of all possible experimental outcomes.
- ✓ The set of possible outcomes may be discrete or form a continuum.

Event

- ✓ An elementary event is one for which there is only one outcome.
- ✓ A composite event is one for which the desired result can be achieved in multiple ways. Multiple outcomes result in the event described.

Equally Likely Events/Outcomes

- ✓ When the set of events or each of the possible outcomes is equally likely to occur.
- ✓ A term that is used synonymously to equally likely outcomes is a uniform random variable.

Reference

Notes, equations, and figures in the lecture are based on or taken from materials in the course textbook:

“Probability and Statistics”, by Spiegel, Schiller and Srinivasan, ISBN 987-007-179557-9 (McGraw-Hill/Schaun’s)