Lecture 8 note (10/20/20)

Discrete R.Vs:

$$F(x) = P(X \leq x) = \underline{\underline{\sum}} f(x)$$

Continuous R.Vs:

$$F(x) = P(X \leq x) = \underline{\underline{\int}} f(x)\,dx$$

$$P(X = \underset{a}{\overset{\shortparallel}{x}}) = 0$$

$$P(a \leq x \leq a) = \int_a^a f(x)\,dx = 0$$

$$\int_0^3 cx^2\,dx = c\int_0^3 x^2\,dx = c\,\frac{x^3}{3}\Big|_0^3$$

$$= c\left(\frac{3^3}{3} - \frac{0^3}{3}\right) = 9c$$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$$

$0 \quad | \ 0 \quad \frac{1}{9}x^2 \quad | \ 3 \quad 0$

$F(x) = 0 \qquad\qquad F(x) = 1$

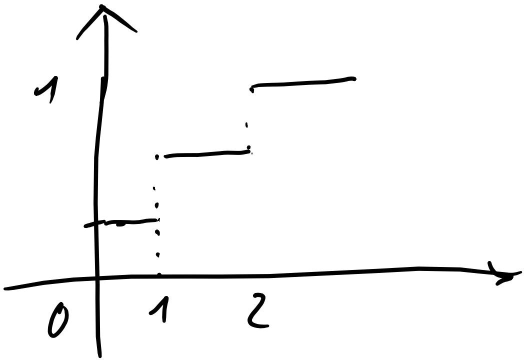
$$\frac{1}{9}\,\frac{u^3}{3}\Bigg|_{u=0}^{u=x} = \frac{x^3}{27} = F(x) \qquad 0 < x < 3$$

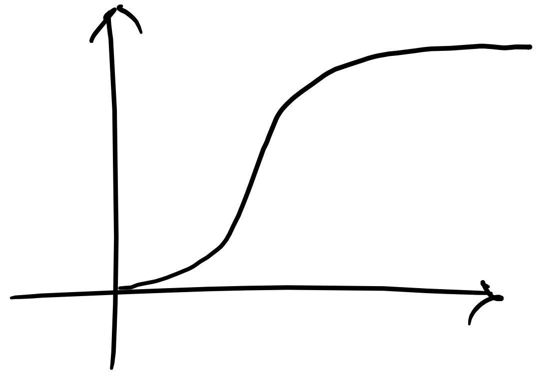
x	1	2	3	4
$f(x)$	$\frac{1}{2}$ +	$\frac{1}{4}$ +	$\frac{1}{4}$	
$F(x)$	$\frac{1}{2}$	$\frac{3}{4}$	1	1

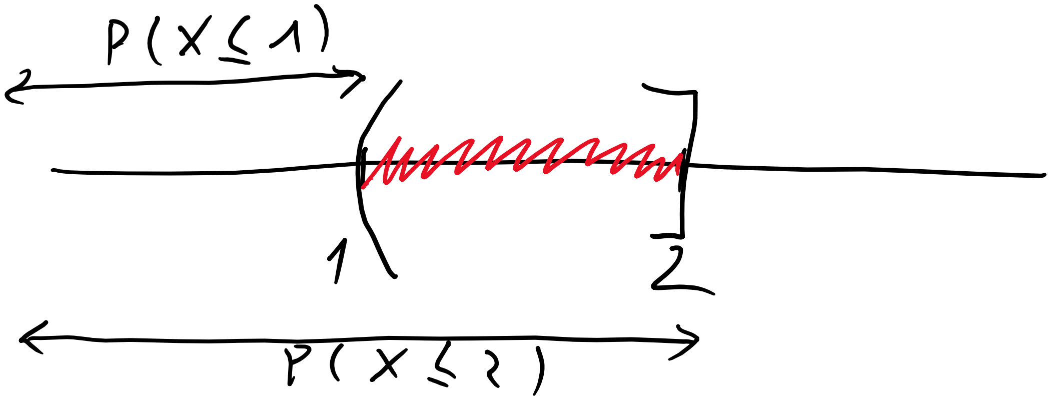
x

$$F(0 \le x < 3) = \int_{-\infty}^{0} \cancel{f(x)}^{\,0}\,dx + \int_{0}^{3 \to x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} f(u)\,du$$







$$P(X \le 2) - P(X \le 1) \qquad \frac{x^3}{27}$$

$$F(2) - F(1)$$

$$\frac{2^3}{27} - \frac{1^3}{27} = \frac{7}{27}$$

$$P(X = x) = 0$$

$$P(1 \le X \le 2) = \underbrace{P(X = 1)}_{0} + \underbrace{P(1 < X \le 2)}_{7/27}$$

In-class exercise:

a) $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{1/2}^{3/2} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_{1/2}^{3/2}$

$= \boxed{\frac{1}{2}}$

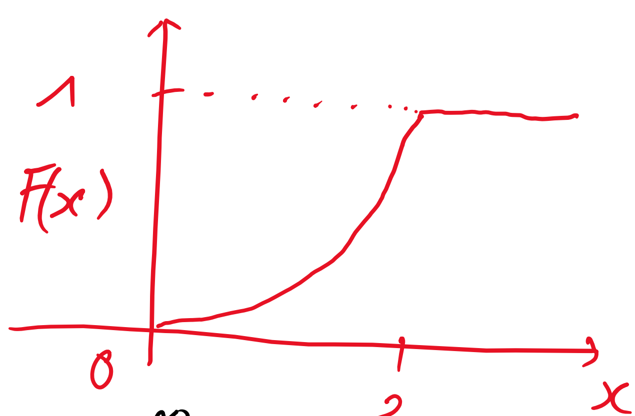
b) If $0 \leq x \leq 2$, then:

$P(0 \leq X \leq 2) = \int_0^x \frac{u}{2}\,du = \frac{x^2}{4}$

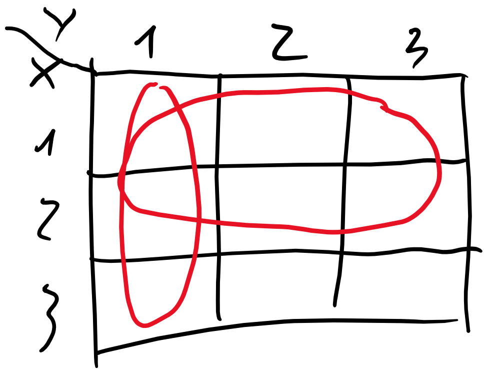
If $x \geq 2$, then:

$F(x) = \int_0^2 f(u)\,du + \int_2^x \underset{0}{\cancel{f(u)}}\,du = 1$

$+ \int_{-\infty}^0 \underset{0}{\cancel{f(u)}}\,du$



$P(X > 1) = \int_1^2 f(x)\,dx + \int_2^{\infty} \underset{0}{\cancel{f(x)}}\,dx$



$P(X \leq 2) =$

$P(Y \leq 1) =$

In-class exercise:

a) $P(3 < X < 4, Y > 2)$

$$= \frac{1}{210} \int_3^4 \int_2^5 (2x + y)\, dx\, dy$$

$$= \frac{1}{210} \int_3^4 \left(2xy + \frac{y^2}{2} \Big|_{y=2}^{y=5} \right) dx$$

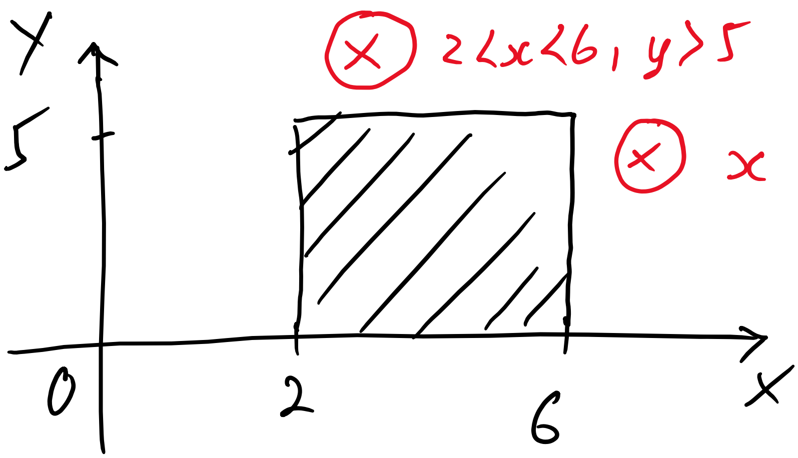
$$= \frac{1}{210} \int_3^4 \left(\frac{21}{2} + 6x \right) dx = \frac{3}{20}$$

b) $F(x,y) = P(X \leq x, Y \leq y)$

$$= \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u,v)\, du\, dv$$

$$F(x,y) = \int_{x=2}^{x=6} \int_{v=0}^{y} \frac{2u+v}{210}\, du\, dv$$

$$= \frac{16y + y^2}{105}$$



$\otimes$ $2<x<6,\ y>5$

$\otimes$ $x>6,\ 0<y<5$

$$X = \begin{cases} 0 & f(0) = \frac{1}{2} \\ 1 & f(1) = 1/4 \\ 2 & f(2) = 1/4 \end{cases}$$

$$Y = 3X^2 + 2$$

Discrete R.Vs :

$$g(u) = f(\underline{\psi(u)})$$

Continuous R.Vs :

$$g(u) = f(\underline{\psi(u)}) \cdot |\underline{\psi'(u)}|$$

$$u = \frac{1}{3}(12 - x) \Rightarrow x = \underline{\psi(u)} = 12 - 3u$$

When $x = -3 \Rightarrow u = \frac{1}{3}(12-(-3)) = 5$

$x = 6 \Rightarrow u = \frac{1}{3}(12-6) = 2$

$$\psi'(u) = (12-3u)' = -3$$

$$g(u) = f(\psi(u)) \cdot |\psi'(u)|$$

$$g(u) = \frac{(12-3u)^2}{81} \cdot |-3|$$

$$= \frac{(12-3u)^2}{27}$$

Discrete R.Vs

$x \backslash y$	0	1	2	3	4	5
2						
3						
4						
5						
6						

$$P(3 \leq x \leq 4, \; y > 2)$$

$$= \sum_{x=3}^{4} \sum_{y=3}^{5} f(x,y)$$

Continuous RVs:

$$P(3 < x < 4, \; y > 2)$$

$$= \int_{x=3}^{4} \int_{y=2}^{5} f(x,y) \, dx \, dy$$

$$= \int_{x=3}^{4} \int_{y=2}^{5} \frac{2x+y}{210} \, dx \, dy$$

$\int c \, dy = cy$

$$\int_{2}^{5} (\overbrace{2x}^{c}+y) \, dy$$

$$= 2xy + \frac{y^2}{2} \Big|_{2}^{5}$$

$$= \frac{21}{2} + 6x$$

$$= \frac{1}{210} \int_{x=3}^{4} \left(\frac{21}{2} + 6x \right) dx$$

$$= 3/20$$

$X \qquad f(x)$

$X = \psi(u)$

$U = \phi(X) \rightarrow g(u)\,?$

$$g(u) = f[\psi(u)] \cdot |\psi'(u)|$$

$$f(x) = \begin{cases} \frac{x^2}{81} & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

$u = \frac{1}{3}(12 - X) \qquad g(u)$

$\psi(u) = X = 12 - 3u$

when $x = -3 \Rightarrow u = \frac{1}{3}(12-(-3)) = 5$

$x = 6 \Rightarrow u = \frac{1}{3}(12-6) = 2$

$$\psi'(u) = (12-3u)' = -3$$

$$g(u) = \frac{(12-3u)^2}{81} \cdot |-3|$$

$$= \frac{(12-3u)^2}{27} \qquad 2 < u < 5$$

$$g(u) = 0 \qquad \text{otherwise}$$

2 or more R.Vs :

$$g(u,v) = f[\psi_1(u,v), \psi_2(u,v)] \cdot |J|$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$f(x,y) = \begin{cases} \frac{xy}{96} & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$U = X + 2Y \rightarrow \psi_1(u,v) = Y = \frac{u-v}{2}$$

$$V = X \rightarrow \psi_2(u,v) = X = V$$

(chosen arbitrarily)

$$0 < x < 4 \Rightarrow 0 < v < 4$$

$$1 < y < 5 \Rightarrow 2 < u - v < 10$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

function

$$\frac{\partial x}{\partial u} = (v)' = 0 \qquad \frac{\partial x}{\partial v} = (v)' = 1$$

variable

$$\frac{\partial y}{\partial u} = \left(\frac{u-v}{2}\right)' = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = \left(\frac{u-v}{2}\right)' = -\frac{1}{2}$$

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = 0 \cdot \left(-\frac{1}{2}\right) - 1 \cdot \frac{1}{2}$$

$$= -\frac{1}{2}$$

$$g(u,v) = \frac{v \cdot \left(\frac{u-v}{2}\right)}{96} \cdot \left|-\frac{1}{2}\right|$$

$$= \frac{v(u-v)}{384} \qquad 2 < u-v < 10$$

$$0 < v < 4$$

$$g(u,v) = 0 \qquad \text{otherwise}$$

$$\int_{v=0}^{v=u-2} \frac{v(u-v)}{384}\, dv \qquad 2 < u < 6$$

$$u - v = 2 \Rightarrow v = u - 2$$

$$\int_{v=0}^{v=4} \frac{v(u-v)}{384}\, dv \qquad 6 < u < 10$$

$$g_2(v) = \int g(u,v)\, du$$

$$= \int_{u=2+v}^{u=v+10} \frac{v(u-v)}{384}\, du \qquad 0 < v < 4$$

$$g_2(v) = 0 \quad \text{otherwise}$$

In-class exercise:

$$U = X + Y \rightarrow \psi_1(u,v) = Y = U - V$$

$$V = X \rightarrow \psi_2(u,v) = X = V$$

$$\rightarrow 2 < v < 6 \qquad 0 < u - v < 5$$

$$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \qquad |J| = 1$$

$$g(u,v) = \begin{cases} \frac{u+v}{210} & \begin{array}{l} 0 < u - v < 5 \\ 2 < v < 6 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

$$g(u,v,t) = f(\psi_1(u,v,t), \psi_2(u,v,t)$$
$$\psi_3(u,v,t)\,|J|$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,t)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{vmatrix}$$

$$f(x,y) = \begin{cases} \frac{xy}{96} & 0 < x < 4,\ 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(x) = \int_{y=1}^{y=5} f(x,y)\,dy$$

$$f_2(y) = \int_{x=0}^{x=4} f(x,y)\,dx$$

$$P(1 < Y < 3 \mid 1 < X < 5)$$

$$\downarrow$$

$$P(\underset{c}{1} < Y < \underset{d}{3} \mid 1 < X < 1 + \underset{dx}{4})$$

$$= \int_1^3 f(y \mid \overset{1}{x})\, dy$$

$$= \int_1^3 \frac{f(x, y)}{f_1(1)}\, dy$$

Example: a) $f_1(x) = \frac{3}{4} + \frac{x}{2}$

b) $P\left(Y > \frac{1}{2} \mid \frac{1}{2} < X < \frac{1}{2} + dx\right)$

$$= \int_{1/2}^{1} f\left(y \mid \frac{1}{2}\right) = \int_{1/2}^{1} \frac{f\left(\frac{1}{2}, y\right)}{f_1\left(\frac{1}{2}\right)}\, dy$$

$$= \int_{1/2}^{1} \frac{\frac{3}{4} + \frac{1}{2} \cdot y}{\underbrace{\frac{3}{4} + \frac{1/2}{2}}_{1}} \, dy$$

$$= \int_{1/2}^{1} \frac{3}{4} + \frac{y}{2} \, dy = \frac{9}{16}$$