EE 381 Probability & Statistic with Applications to Computing (Fall 2020)

Lecture 1
Basic Concepts of Probability (p1)

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Chapter 1’s topics

- **Discrete Sample Spaces**
- **Important Theorems on Probability**
- Conditional Probability
- Independence
- Bayes’ Rule
- Combinatorial Analysis

Introduction about Probability & Statistics

- Probability: the study of the likeliness of result, action or event occurring. Often based on prior knowledge or the statistics of similar or past events.
- Statistics: the study of and the dealing with data.
- An understanding of Probability and Statistics is necessary in most if not all work related to science and engineering.

Introduction about Probability

- Engineering Applications include:
  - ✓Realistic signals - with noise or characteristic "unknown" parts
  - ✓Signal-to-noise ratios, noise-power measurements
  - ✓Reliability, quality, failure rates, etc.
- Probability theory is necessary for engineering system modeling and simulations:
  - ✓Unknown initial conditions (random)
  - ✓Noisy measurements, expected inaccuracies, etc. during operation.

Random Experiments

- In science and engineering, experiments are important because we based on those to control the value of the variables that affect the outcome of the experiment.
- However, in some experiments, even though they are conducted in the same conditions, the outcomes are different each time. They are call *random experiments*.

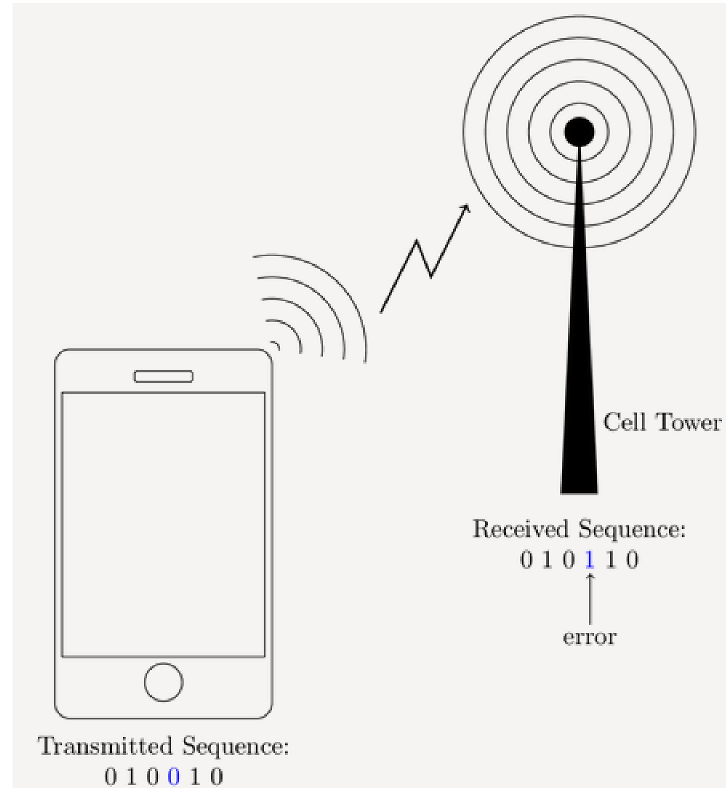
Examples:

1. Tossing a coin. The result of the experiment will either come up "Tail" or "Head". That will be one of the elements of the set {H,T}.
2. Tossing a dice. The result of the experiment will produce one of the numbers in the set {1, 2, 3, 4, 5, 6}.
3. Tossing a coin twice. There will be 4 possible results, as indicated by {HH, HT, TH, TT}.

Random Experiments

Example: Communication Systems

- Conversation on phone (converted to information bits 0 and 1) when transmitted to cell tower is affected by noise.
- Noise is a random phenomenon.
- Probability theory is used to understand the behavior of noise in this system, and take measures to correct the errors.



[1]

[1]: "Introduction to Probability, Statistics and Random Processes" by H. Pishro-Nik, ISBN-9780990637202

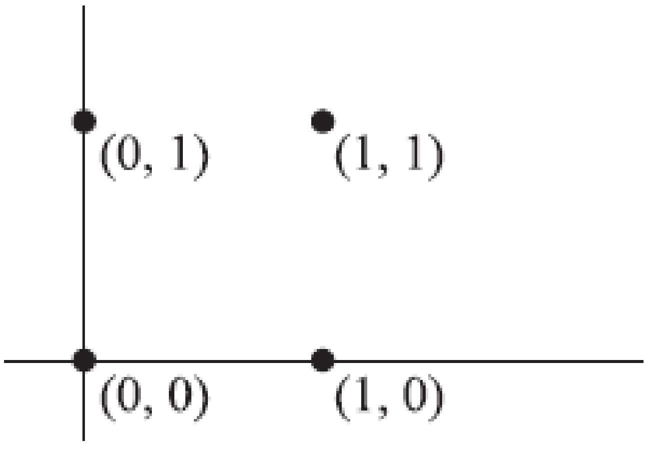
Sample spaces

- A set S that consists of all possible outcomes of a random experiment is called a sample space, and each outcome is called a sample point.

Example: Tossing a dice. One sample space is given by {1, 2, 3, 4, 5, 6}. Or another sample space is {odd, even}. The first sample space is preferred, since it is more comprehensive.

Different types of sample spaces

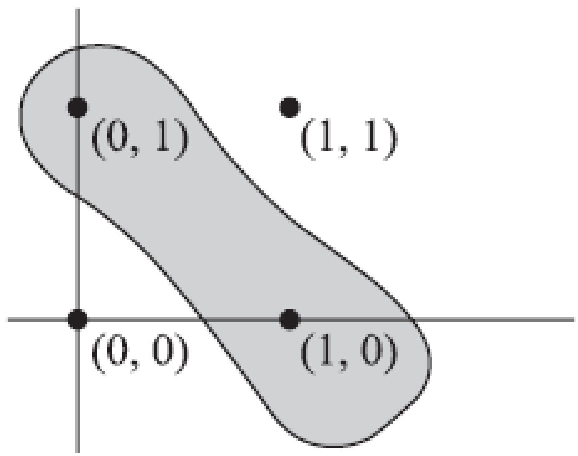
- *Discrete sample space:*
  ✓*Finite sample space*: has a finite number of points.
  ✓*Countably infinite sample space*: has as many points as there are natural numbers 1, 2, 3,…
- *Nondiscrete sample space*:
  ✓*Noncountably infinite sample space*: has as many points as there are in some interval on the x axis, such as $0 \leq x \leq 1$.



Example of finite sample space: tossing a coin twice.
0 for Tail, and 1 for Head.

Events

- An *event* is a subset A of a sample space S (a set of possible outcomes).
- If the outcome of an experiment is an element of A, meaning that the event A has occurred.
- An event consisting of a single point of S is called a *simple* or *elementary event*.
- The empty set ∅ is called the *impossible event*.



Example of an event: tossing a coin twice. Bolded area shows the event that only one head comes up.

Events

- By using set operations on events in S, we can obtain other events in S.
- For example, if A and B are events, then:
  1. $A \cup B$ is the event "either A or B or both". $A \cup B$ is called the union of A and B.
  2. $A \cap B$ is the event "both A and B". $A \cap B$ is called the intersection of A and B.
  3. A' is the event "not A". A' is called the complement of A.
  4. $\mathrm{A} - \mathrm{B} = A \cap B'$ is the event "A but not B". In particular, $A' = S - A$
- If A and B are disjoint, i.e., $A \cap B = \emptyset$, we say that the events are mutually exclusive, meaning that they cannot both occur.

Example: Tossing the coin twice. Let A={HT, TH, HH} and B={HT, TT}

$A \cup B$={HT, TH, HH, TT} $\quad$ $A \cap B = \{HT\}$

$A' = \{TT\}$ $\quad$ $\mathrm{A} - \mathrm{B} = \{\mathrm{TH}, \mathrm{HH}\}$

The Concept of Probability

- In random experiment, there is always uncertainty as to whether a particular event will or will not occur.
- We define probability as a measure of the chance with which we can expect the event will occur.
- Example: for a certain event
  - ✓If we say the probability is 100%, we are sure that the event will occur.
  - ✓If we say the probability is ¼ or 25%, then we would say there is a 25% chance that the event will occur, and 75% chance it will not occur.

The Concept of Probability

Two important procedures for estimating the probability of an event:

- **Classical approach**: If an event can occur in h different ways out of a total number of n possible ways, all of which are equally likely, then the probability of the event is h/n.
  - ✓Example: probability that a head will turn up in a single toss of a coin is ½ because there are 2 equally like ways, and one of them (a head) can arise in only one way.
- **Frequency approach**: if after n repetitions of an experiment, where n is very large, an event is observed to occur in h of these, then the probability of the event is h/n. This is also called the *empirical probability* of the event.
  - ✓Example: Tossing a coin 1000 times and find that it comes up heads 532 times, we estimate the probability of a head coming up to be 532/1000=0.532.

Both approaches have serious drawbacks: "equally likely" are vague, and "large number" is vague.

The Axioms of Probability

Suppose we have sample space S, and event A belong to sample space S.

- Axiom 1: $P(A) \geq 0$
- Axiom 2: $P(S)=1$
- Axiom 3: for any mutually exclusive events A_1, A_2,…

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

Important Theorems on Probability

- **Theorem 1-1**: If $A_1 \subset A_2$, then $P(A_1) \leq P(A_2)$ and $P(A_2 - A_1) = P(A_2) - P(A_1)$

- **Theorem 1-2**: For every event A, $0 \leq P(A) \leq 1$

- **Theorem 1-3**: $P(\emptyset) = 0$

- **Theorem 1-4**: If A' is the complement of A, then $P(A') = 1 - P(A)$

- **Theorem 1-5:**

If $A = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A) = P(A_1) + P(A_2) + \cdots P(A_n)$$

If $A = S$, the sample space, then $P(A_1) + P(A_2) + \cdots P(A_n) = 1$

Important Theorems on Probability

- **Theorem 1-6**: If A and B are any two events, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally, if A_1, A_2, A_3 are any three events, then
$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - \mathrm{P}(A_1 \cap A_2) - \mathrm{P}(A_1 \cap A_3) - \mathrm{P}(A_2 \cap A_3) + \mathrm{P}(A_1 \cap A_2 \cap A_3)$$

Generalizations to n events can also be made.

- **Theorem 1-7**: For any event A and B, $P(A) = P(A \cap B)$+ $P(A \cap B')$

- **Theorem 1-8**: If an event A must result in the occurrence of one of the mutually exclusive events $A_1, A_2, \dots, A_n$, then
$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n)$$

Assignment of Probabilities

- If a sample space S consists of a finite number of outcomes, then

$$P(A_1) + P(A_2) + \cdots P(A_n) = 1 \qquad (*)$$

Where $A_1, A_2, \ldots, A_n$ are elementary events given by $A_i = \{a_i\}$

- Arbitrary probabilities can be assigned for these simple events as long as (*) is satisfied. In particular, if we assume equal probabilities for all simple events, then

$$P(A_k) = \frac{1}{n}, \qquad k = 1,2,3,\ldots,n$$

And if A is any event made up of h such simple events, we have

$$P(A) = \frac{h}{n}$$

Assignment of Probabilities

Example: A single dice is tossed once. Find the probability of a 2 or 5 turning up.

The sample space is S={1, 2, 3, 4, 5, 6}. If we assign equal probabilities to the sample points (the dice is fair), then

$$P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$$

The event that either 2 or 5 turns up is indicated by $2 \cup 5$. Therefore,

$$P(2 \cup 5) = P(2) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Summary

Below are definitions used in probability that should be understood/remembered:

Experiment

- ✓An experiment is some action that results in an outcome.
- ✓A random experiment is one in which the outcome is uncertain before the experiment is performed.

Possible outcomes/sample space

- ✓A description of all possible experimental outcomes.
- ✓The set of possible outcomes may be discrete or form a continuum.

Event

- ✓An elementary event is one for which there is only one outcome.
- ✓A composite event is one for which the desired result can be achieved in multiple ways. Multiple outcomes result in the event described.

Equally Likely Events/Outcomes

- ✓When the set of events or each of the possible outcomes is equally likely to occur.
- ✓A term that is used synonymously to equally likely outcomes is a uniform random variable.

In-class Exercise

A card is drawn at random from an ordinary deck of 52 playing cards. Using sample space, events and theorems on probability to find the probability that it is:

a) An ace.

b) A jack of hearts

Reference

Notes, equations, and figures in the lecture are based on or taken from materials in the course textbook:

“Probability and Statistics”, by Spiegel, Schiller and Srinivasan, ISBN 987-007-179557-9 (McGraw-Hill/Schaun’s)