EE 381 HW #7

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1. use Binomial distribution, find probability of ~~tossing~~ fair coin 3 times there will appear:

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

p = prob. of Success
$q = 1 - p$ = prob. of failure
n = num. of tosses
x = num. of heads

Coin: H or T is 1/2 bc there are 2 sides of a coin and only lands on 1.

$2^3 = 8$ possiblities

a) 3 heads

$S = \{HHH, HHT, HTH, HTT, TTH, THH, THT, TTT\}$

$n = 3$ = total tosses
p = success = 1/2 success $P(H) = 1/2$
$q = 1 - p = 1 - 1/2 = 1/2$ failure $P(T) = 1/2$
$x = 3$ = num heads

$$f(3) = P(X=3) = \binom{3}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^{3-3} = \frac{3!}{3!(3-3)!} \cdot \frac{1}{8} \cdot \left(\tfrac{1}{2}\right)^0$$

$$= \frac{\cancel{3!}}{\cancel{3!}\,0!} \cdot \frac{1}{8} \cdot 1 = \boxed{\frac{1}{8} = P(3 \text{ heads})}$$

$0! = 1$

b) 2 tails and 1 head

$n = 3$ = total tosses
p = success = 1/2 success
$q = 1 - p = 1 - 1/2 = 1/2$ failure
$X = 2$ = num tails

$$f(X=2) = P(X=2) = \binom{3}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{3-2}$$

$$= \frac{3!}{2!(3-2)!} \cdot \frac{1}{4} \cdot \left(\tfrac{1}{2}\right)^1$$

$$= \frac{3!}{2!\,1!} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3 \cdot \cancel{2!}}{\cancel{2!} \cdot 1!} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

$$\boxed{\frac{3}{8} = P(2 \text{ heads}, 1 \text{ tails})}$$

c) At Least 1 head

In this case, we will consider all the cases where we have at least 1 head.

$P(\text{At least 1 head}) = \cancel{P(0 \text{ heads})} + P(1 \text{ heads}) + P(2 \text{ heads}) + P(3 \text{ heads})$

1 head:

$n = 3 = \text{num tosses}$

$p = 1/2 = \text{success}$

$q = 1 - p = 1 - 1/2 = 1/2$ failure

$x = \text{num heads} = 1$

$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$

$= \frac{n!}{x!(n-x)!} p^x q^{n-x}$

$$f(1) = P(X=1) = \frac{\overset{n}{3!}}{\underset{x}{1}!(\underset{n}{3}-\underset{x}{1})!} \cdot \overset{p}{\left(\frac{1}{2}\right)}^{\overset{x}{1}} \cdot \overset{q}{\left(\frac{1}{2}\right)}^{\overset{n\ \ x}{3-1}} = \frac{3!}{1!\,2!} \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$$

$$= \frac{3 \cdot \cancel{2!}}{1!\,\cancel{2!}} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8} = P(1 \text{ heads})$$

2 heads:

$n = 3 = \text{num tosses}$

$p = 1/2 = \text{success}$

$q = 1 - p = 1 - 1/2 = 1/2$ failure

$x = \text{num heads} = 2$

$$f(2) = P(X=2) = \frac{\overset{n}{3!}}{\underset{x}{2}!(\underset{n}{3}-\underset{x}{2})!} \cdot \overset{p}{\left(\frac{1}{2}\right)}^{\overset{x}{2}} \cdot \overset{q}{\left(\frac{1}{2}\right)}^{\overset{n\ \ x}{3-2}} = \frac{3!}{2!\,1!} \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\right)^1$$

$$= \frac{3 \cdot \cancel{2!}}{\cancel{2!}\,1!} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(2 \text{ heads})$$

What I learned in the past 2 yrs at CB

Aggregation: has-a relationship

Inheritance: is-a relationship

Composition: one can't exist w/o the other

<<interface>>: driveable

...le, Car, Boat

(continued!)

c) continued!

3 heads:

$n = 3 =$ num tosses

$p = 1/2 =$ success

$q = 1 - p = 1 - 1/2 = 1/2$ failure

$X = 3 =$ num heads

$$f(3) = P(X=3) = \frac{3!}{3!(3-3)!} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{3-3}$$

$$= \frac{\cancel{3!}}{\cancel{3!}\,0!} \cdot \frac{1}{8} \cdot \left(\frac{1}{2}\right)^0 = \frac{1}{8} \cdot 1 = \boxed{\frac{1}{8} = P(3 \text{ heads})}$$

$$P(\text{at least 1 head}) = P(1 \text{ head}) + P(2 \text{ heads}) + P(3 \text{ heads})$$

$$= \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \boxed{\frac{7}{8} = P(\text{at least 1 head})}$$

d) P(not more than 1 tail)

$$P(\text{not more than 1 tail}) = P(\text{no tails}) + P(1 \text{ tail}) + \cancel{P(2 \text{ tails}) + P(3 \text{ tails})}$$

0 tails:

$n = 3 =$ num tosses

$p = 1/2$ success

$q = 1 - p = 1 - 1/2 = 1/2$

$X = 0$ tails

$$f(0) = P(X=0) = \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{3-0}$$

$$= \frac{3!}{0!(3-0)!} \cdot 1 \cdot \left(\frac{1}{2}\right)^3 = \frac{\cancel{3!}}{0!\,\cancel{3!}} \cdot 1 \cdot \frac{1}{8} = \frac{1}{8}$$

$$\frac{1}{8} = P(0 \text{ tails})$$

1. (continued!)

d) (continued!)

<u>1 tail:</u>

$n = 3$ tosses

$p = 1/2$ success

$q = 1 - p = 1 - 1/2 = 1/2$ fail

$X = 1$ = num tails

$$f(1) = P(X=1) = \binom{\overset{n}{3}}{\underset{x}{1}} \cdot \left(\frac{1}{2}\right)^{\overset{x}{1}} \cdot \left(\frac{1}{2}\right)^{\overset{n}{3} - \overset{x}{1}}$$

$$= \frac{\overset{n}{3!}}{\underset{x}{1}!\,(\underset{n}{3}-\underset{x}{1})!} \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$$

$$= \frac{3!}{1!\,2!} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3 \cdot \cancel{2!}}{1!\,\cancel{2!}} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8} = P(1 \text{ tail})$$

$$P(\text{not more than 1 tail}) = P(0 \text{ tail}) + P(1 \text{ tail})$$

$$= \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \boxed{\frac{1}{2} = P(\text{not more than 1 tail})}$$

If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random 1, 0, and less than 2 bolts are defective.

(Distribution)
Binomial Thm: $f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$

$= \dfrac{n!}{x!(n-x)!}$

Given:

P(defective) = 20% = 0.2

P(undefective) = 1 − P(defective) = 1 − 0.2 = 0.8

total num. bolts = 4

a) 1 bolt defective

n = total num bolts = 4

$p = 0.2$ defective = success

$q = 1 - p = 1 - 0.2 = 0.8$ = undefective unsuccess

X = num defective = 1

$f(1) = P(X = 1) = \binom{4}{1} 0.2^1 \cdot 0.8^{4-1}$

$= \dfrac{4!}{1!(4-1)!} \cdot 0.2 \cdot (0.8)^3 = \dfrac{4!}{1!3!} \cdot 0.2 \cdot 0.512 = \dfrac{4 \cdot \cancel{3!}}{1!\cancel{3!}} \cdot 0.2 \cdot 0.512$

$\boxed{= 0.4096 = P(\text{1 bolt defective})}$

2. (continued!)

b) 0 bolts defective

Given:

P(defective) = 20% = 0.2 = success = p

P(undefective) = 1 − P(defective) = 1 − 0.2 = 0.8 = q = failure

n = total num bolts = 4

X = num defective = 0

Binomial Distribution: $f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$

$f(0) = P(X=0) = \binom{4}{0}(0.2)^0 \cdot (0.8)^{4-0}$

$= \frac{4!}{0!(4-0)!} \cdot 1 \cdot 0.8^4 = \frac{\cancel{4!}}{0!\cancel{4!}} \cdot 1 \cdot 0.4096 = \boxed{0.4096 = P(0 \text{ bolts defective})}$

c) less than 2 bolts defective

P(less than 2 bolts defective) = P(0 bolt defective) + P(1 bolt defective) + ~~P(2 bolts defective)~~ + ~~P(3 bolts defective)~~ + ~~P(4 bolts defective)~~

// we eliminate 2, 3, and 4 b/c we only want the probability of 0 bolts defective and only 1 bolt defective.

P(less than 2 bolts defective) = P(0 bolts defective) + P(1 bolt defective)

$= 0.4096 + 0.4096 = \boxed{0.8192 = P(\text{less than 2 bolts defective})}$

. A box contains 6 blue marbles and 4 red marbles. An experiment is performed in which a marble is chosen at random and its color observed, but the marble is not replaced. Find probability that after 5 trials of the experiment, 3 blue marbles will have been chosen.

Given:

$n = \text{num trials} = 5$

$x = \text{num blue marbles} = 3$

$p = \text{success of getting a blue marble} = \frac{\text{num blue}}{|S|} = \frac{6}{10} = \frac{3}{5}$

$|S| = \{6 \text{ blue}, 4 \text{ red}\} = 10$

$q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$ failure of getting a blue

Binomial Distribution:

$$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \frac{5!}{3!(5-3)!} \cdot \left(\frac{3}{5}\right)^5 \cdot \left(\frac{2}{5}\right)^{5-3}$$

$$= \frac{5!}{3!2!} \cdot \left(\frac{3}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2$$

$$= \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}\,2!} \cdot \frac{27}{125} \cdot \frac{4}{25} = \frac{216\cancel{0}}{625\cancel{0}} = \boxed{\frac{216}{625} = 0.3456}$$

4. Find the probability of (a) 2 or more heads and (b) fewer than 4 heads, in a single toss of 6 fair coins.

Given: each coin has 2 outcomes.

B/c we have 6 coins, we have $2^{\text{num coins}} = 2^6 = 64$ outcomes. $P(T) = 1/2$, $P(H) = 1/2$

Binomial Distribution: $f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$

a) 2 or more heads

num heads ≥ 2: ~~0~~ ~~1~~ 2 3 4 5 6

2 heads:

n = num tosses = 6

x = num heads = 2

p = success = 1/2

$q = 1-p = 1-1/2 = 1/2$ failure

$f(2) = P(X=2) = \binom{6}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{6-2} = \frac{6!}{2!(6-2)!} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^4$

$= \frac{6!}{2!4!} \cdot \frac{1}{4} \cdot \frac{1}{16} = \frac{6 \cdot 5 \cdot \cancel{4!}}{2!\cancel{4!}} \cdot \frac{1}{64} = \frac{30}{128} = \frac{15}{64}$

3 heads:

n = num tosses = 6

x = num heads = 3

p = success = 1/2

q = failure = $1 - 1/2 = 1/2$

$f(3) = P(X=3) = \binom{6}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{6-3}$

$= \frac{6!}{3!(6-3)!} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 = \frac{6!}{3!3!} \cdot \frac{1}{8} \cdot \frac{1}{8}$

$= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{3!\cancel{3!}} \cdot \frac{1}{64} = \frac{120}{384} = \frac{20}{64}$

..(continued!)

4 heads:

n = num tosses = 6
x = num heads = 4
p = success = 1/2 of getting heads
$q = 1 - p = 1 - 1/2 = 1/2$ of not getting heads

$f(4) = P(X=4) = \binom{6}{4} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{6-4} = \frac{6!}{4!(6-4)!} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2$

$= \frac{6!}{4!2!} \cdot \frac{1}{16} \cdot \frac{1}{4} = \frac{6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}2!} \cdot \frac{1}{64} = \frac{30}{128} = \frac{15}{64}$

5 heads:

n = num tosses = 6
x = num heads = 5
p = success = 1/2 of getting heads
$q = 1 - p = 1 - 1/2 = 1/2$ not getting heads = failure

$f(5) = P(X=5) = \binom{6}{5} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{6-5} = \frac{6!}{5!(6-5)!} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^1$

$= \frac{6!}{5!1!} \cdot \frac{1}{32} \cdot \frac{1}{2} = \frac{6 \cdot \cancel{5!}}{\cancel{5!}1!} \cdot \frac{1}{64} = \frac{6}{64}$

6 heads:

n = num tosses = 6
x = num heads = 6
p = success = 1/2 of getting heads
$q = 1 - p = 1 - 1/2 = 1/2$ not getting heads = failure

$f(6) = P(X=6) = \binom{6}{6} \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^{6-6} = \frac{6!}{6!(6-6)!} \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^0$

$= \frac{\cancel{6!}}{\cancel{6!}0!} \cdot \frac{1}{64} \cdot 1 = \frac{1}{64}$

4. (continued!)

$P(2 \text{ or more heads}) = P(X=2) + P(X=3) + P(X=4) + P(X = 5) + P(X=6)$

$= \frac{15}{64} + \frac{20}{64} + \frac{15}{64} + \frac{6}{64} + \frac{1}{64} = \boxed{\frac{57}{64} = P(2 \text{ or more heads})}$

b) Fewer than 4 heads

num heads < 4: 0 1 2 3 $\cancel{4}$ $\cancel{5}$ $\cancel{6}$

// < 4 does NOT include 4!

0 heads:

$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$

n = num tosses = 6

x = num heads = 0

p = success = 1/2 = success of heads

$q = 1-p = 1-1/2 = 1/2$ not heads = failure

$f(0) = P(X=0) = \binom{6}{0} \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^{6-0} = \frac{6!}{0!(6-0)!} \cdot 1 \cdot \left(\frac{1}{2}\right)^6$

$= \frac{\cancel{6!}}{0!\cancel{6!}} \cdot 1 \cdot \frac{1}{64} = \frac{1}{64}$

1 head:

n = num tosses = 6

x = num heads = 1

p = success = 1/2 success of heads

$q = 1-p = 1-1/2 = 1/2$ not getting heads = failure

$f(1) = P(X=1) = \binom{6}{1} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^{6-1} = \frac{6!}{1!(6-1)!} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^5 = \frac{6!}{1!5!} \cdot \frac{1}{2} \cdot \frac{1}{32}$

$= \frac{6 \cdot \cancel{5!}}{1!\cancel{5!}} \cdot \frac{1}{64} = \frac{6}{64}$

6) (continued!)

2 heads:

n = num tosses = 6

x = num heads = 2

p = success = 1/2 getting heads

$q = 1 - p = 1 - 1/2$ = not getting heads = failure

$$f(2) = P(x=2) = \binom{6}{2} \cdot \left(\tfrac{1}{2}\right)^2 \cdot \left(\tfrac{1}{2}\right)^{6-2} = \frac{6!}{2!(6-2)!} \cdot \left(\tfrac{1}{2}\right)^2 \cdot \left(\tfrac{1}{2}\right)^4$$

$$= \frac{6!}{2!\,4!} \cdot \frac{1}{4} \cdot \frac{1}{16} = \frac{6 \cdot 5 \cdot \cancel{4!}}{2!\,\cancel{4!}} \cdot \frac{1}{64} = \frac{30}{128} = \frac{15}{64}$$

3 heads:

n = num tosses = 6

x = num heads = 3

p = success = 1/2 = getting heads

$q = 1 - p = 1 - 1/2 = 1/2$ = failure = not heads

$$f(3) = P(x=3) = \binom{6}{3} \cdot \left(\tfrac{1}{2}\right)^3 \cdot \left(\tfrac{1}{2}\right)^{6-3} = \frac{6!}{3!(6-3)!} \cdot \left(\tfrac{1}{2}\right)^3 \cdot \left(\tfrac{1}{2}\right)^3$$

$$= \frac{6!}{3!\,3!} \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{3!\,\cancel{3!}} \cdot \frac{1}{64} = \frac{120}{384} = \frac{20}{64}$$

P(less than 4 heads) = P(0 heads) + P(1 head) + P(2 heads) + P(3 heads) =

$$\frac{1}{64} + \frac{6}{64} + \frac{15}{64} + \frac{20}{64} = \boxed{\frac{42}{64} = \frac{21}{32} = P(<4 \text{ heads})}$$

5. Out of 800 families w/5 children each, how many would you expect to have... Assume equal prob. for boys and girls!

a) 3 boys

Binomial Distribution: $f(x) = P(X) = \binom{n}{x} p^x q^{n-x}$

$= \frac{n!}{x!(n-x)!} p^x q^{n-x}$

Given:

$P(B) = 1/2$

$P(G) = 1/2$

$n = \text{num children} = 5$

$x = \text{num boys} = 3$

$p = \text{success} = 1/2 = \text{a boy}$

$q = \text{failure} = 1-p = 1-1/2 = 1/2 = \text{failure/girl}$

$f(3) = P(X=3) = \binom{5}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{5-3}$

$= \frac{5!}{3!(5-3)!} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 = \frac{5!}{3!2!} \cdot \frac{1}{8} \cdot \frac{1}{4}$

$= \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}\, 2!} \cdot \frac{1}{32} = \frac{20}{64} = \frac{5}{16} = P(3 \text{ boys})$

// multiply number of families by P(3 boys) to get how many families have 3 boys.

$\overset{50}{\cancel{800}} \cdot \frac{5}{\cancel{16}} = \boxed{250 \text{ Families w/3 boys}}$

(continued:)

b) 5 girls

n = num children = 5

x = num girls = 5

$p = 1/2$ = success of getting girls

$q = 1/2$ of failure = boys

$$f(5) = P(X=5) = \binom{5}{5}\cdot\left(\tfrac{1}{2}\right)^{5}\cdot\left(\tfrac{1}{2}\right)^{5-5}$$

$$= \frac{5!}{5!\,(5-5)!}\cdot\frac{1}{32}\cdot\left(\tfrac{1}{2}\right)^{0} = \frac{\cancel{5!}}{\cancel{5!}\,0!}\cdot\frac{1}{32}\cdot 1 = \boxed{\frac{1}{32} = P(5G)}$$

Actual families w/5 girls = total num families $\cdot$ P(5 girls)

$$= \cancel{800}^{\,25}\cdot\frac{1}{\cancel{32}} = \boxed{25 \text{ families w/ 5 girls}}$$

c) Either 2 or 3 boys

$\boxed{P(2 \text{ or } 3 \text{ boys}) = P(2B) \cup P(3B) = P(2B) + P(3B)}$

2 boys:

n = num children = 5

x = num boys = 2

$p = 1/2$ = success of getting boys

$q = 1 - p = 1 - 1/2 = 1/2$ of getting girls = failure

$$f(2) = P(X=2) = \binom{5}{2}\cdot\left(\tfrac{1}{2}\right)^{2}\cdot\left(\tfrac{1}{2}\right)^{5-2} = \frac{5!}{2!\,(5-2)!}\cdot\left(\tfrac{1}{2}\right)^{2}\cdot\left(\tfrac{1}{2}\right)^{3}$$

$$= \frac{5!}{2!\,3!}\cdot\frac{1}{4}\cdot\frac{1}{8} = \frac{5\cdot 4\cdot \cancel{3!}}{2!\,\cancel{3!}}\cdot\frac{1}{32} = \boxed{\frac{20}{64} = P(2B)}$$

5. (c) continued!

3 boys:

n = num children = 5
x = num boys = 3
p = 1/2 = success of getting boys
$q = 1 - p = 1 - 1/2 = 1/2$ of failure = not getting a boy

$$f(3) = P(X=3) = \binom{5}{3}\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{5-3} = \frac{5!}{3!(5-3)!} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2$$

$$= \frac{5!}{3!\,2!} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}\,2!} \cdot \frac{1}{32} = \frac{20}{64} = P(3B)$$

$$P(2B \cup 3B) = P(2B) + P(3B)$$
$$= \frac{20}{64} + \frac{20}{64} = \frac{40}{64} = \boxed{\frac{5}{8} = P(2B \cup 3B)}$$

Total Families w/ either 2 or 3 boys = total num families · $P(2B \cup 3B)$

$$= \cancel{800}^{100} \cdot \frac{5}{\cancel{8}} = \boxed{\text{500 families w/ 2 or 3 BOYS}}$$

6. Find the probability of guessing correctly at least 6 of the 10 awnsers on true/false exam. ($P(T) = 1/2$, $P(F) = 1/2$)

Binomial Distribution; ~~xxx xxx~~ 6 7 8 9 10

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

6 corr:
n = num quest = 10
X = num correct = 6
p = 1/2 = correct = success
$q = 1 - p = 1 - 1/2 = 1/2$ = failure

$$f(6) = P(X=6) = \binom{10}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{10-6}$$
$$= \frac{10!}{6!(10-6)!} \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^4 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}\,4!} \cdot \frac{1}{64} \cdot \frac{1}{16}$$

continued!)

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} \cdot \frac{1}{64} \cdot \frac{1}{16} = \frac{5040}{24{,}576} = P(X=6)$$

7 correct:

n = total questions = 10
x = num correct = 7
$p = 1/2$ = success of correct
$q = 1 - p = 1/2$ failure = incorrect

$$f(7) = P(X=7) = \binom{10}{7} \cdot \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^{10-7}$$

$$= \frac{10!}{7!(10-7)!} \cdot \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^3 = \frac{10!}{7!3!} \cdot \frac{1}{128} \cdot \frac{1}{8}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}3!} \cdot \frac{1}{128} \cdot \frac{1}{8} = \frac{720}{6144} = P(X=7)$$

8 correct:

n = total num quest = 10
x = num correct = 8
$p = 1/2$ = success of correct
$q = 1 - p = 1 - 1/2 = 1/2$ failure = incorrect

$$f(8) = P(X=8) = \binom{10}{8} \cdot \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^{10-8} = \frac{10!}{8!(10-8)!} \cdot \left(\frac{1}{2}\right)^8 \cdot \left(\frac{1}{2}\right)^2$$

$$= \frac{10!}{8!2!} \cdot \frac{1}{256} \cdot \frac{1}{4} = \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!}2!} \cdot \frac{1}{256} \cdot \frac{1}{4} = \frac{90}{2048} = P(X=8)$$

=

6. (continued!)

9 correct:

n = num questions = 10
x = num correct = 9
p = 1/2 = success = correct
$q = 1 - p = 1 - 1/2 = 1/2$ = incorrect = failure

$$f(9) = P(X=9) = \binom{10}{9} \cdot \left(\frac{1}{2}\right)^{9} \cdot \left(\frac{1}{2}\right)^{10-9}$$

$$= \frac{10!}{9!(10-9)!} \cdot \left(\frac{1}{2}\right)^{9} \cdot \left(\frac{1}{2}\right)^{1} = \frac{10!}{9!1!} \cdot \frac{1}{512} \cdot \frac{1}{2}$$

$$= \frac{10 \cdot \cancel{9!}}{\cancel{9!}\,1!} \cdot \frac{1}{512} \cdot \frac{1}{2} = \frac{10}{1024} = P(X=9)$$

10 correct:

n = num quests. = 10
x = num correct = 10
p = 1/2 = success = correct
$q = 1 - p = 1 - 1/2 = 1/2$ = incorrect = failure

$$f(10) = P(X=10) = \binom{10}{10} \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{10-10}$$

$$= \frac{10!}{10!(10-10)!} \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{0} = \frac{\cancel{10!}}{\cancel{10!}\,0!} \cdot \frac{1}{1024} \cdot 1 = \frac{1}{1024} = P(X=10)$$

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10) = \frac{5040}{24576} + \frac{720}{6144} + \frac{90}{2048} + \frac{10}{1024} + \frac{1}{1024} = \boxed{0.377 = P(X \geq 6)}$$

A box contains a very large number of red, white, blue, and yellow marbles in the ratio 4:3:2:1. Find the probability that in 10 drawings, 8 red and 2 yellow marbles will be drawn.

[num red marbles [num yellow marbles

Multinomial distribution: $P(X_1=n_1, X_2=n_2, \ldots X_k=n_k)$

$$= \frac{n!}{n_1!\, n_2! \ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$$

Given:

we have a total of 10 marbles $= 4+3+2+1 = 10$ marbles

num red marbles $= 4$

num white marbles $= 3$

num blue marbles $= 2$

num yellow marbles $= 1$

$$P(\text{red}) = \frac{\text{num red}}{|\text{total marbles}|} = \frac{4}{10} = \frac{2}{5}$$

$$P(\text{white}) = \frac{\text{num white}}{|\text{total marbles}|} = \frac{3}{10}$$

$$P(\text{blue}) = \frac{\text{num blue}}{|\text{total marbles}|} = \frac{2}{10} = \frac{1}{5}$$

$$P(\text{yellow}) = \frac{\text{num yellow}}{|\text{total marbles}|} = \frac{1}{10}$$

$P(X_1=n_1, X_2=n_2, X_3=n_3, X_4=n_4) =$

$$\frac{\text{total num marbles!}}{n_1!\, n_2!\, n_3!\, n_4!} \cdot P(\text{red})^{n_1} \cdot P(\text{white})^{n_2} \cdot P(\text{blue})^{n_3} \cdot P(\text{yellow})^{n_4}$$

$$= \frac{10!}{8!\,0!\,0!\,2!} \left(\frac{2}{5}\right)^8 \cdot \left(\frac{3}{10}\right)^0 \cdot \left(\frac{1}{5}\right)^0 \cdot \left(\frac{1}{10}\right)^2$$

$$= \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!}\,0!\,0!\,2!} \cdot \left(\frac{2}{5}\right)^8 \cdot 1 \cdot 1 \cdot \left(\frac{1}{10}\right)^2$$

$$= \boxed{0.0002949}$$

n_1 = number of red marbles $= 8$

n_2 = number of whites $= 0$

n_3 = number of blues $= 0$

n_4 = number of yellows $= 2$

8. out of 60 applicants to a university, 40 are from the East. If 20 applicants are to be selected at random, find the probability that

a) 10 will be from the East

Given:

total applicants = 60

East = 40

remaining = total applicants − east = 60 − 40 = 20

1) Hypergeometric w/ replacement: $P(X=x) = \binom{n}{x} \frac{b^x r^{n-x}}{(b+r)^n}$, $x = 0, 1, \ldots, n$

since $p = \frac{b}{b+r}$ and $q = 1 - p = \frac{r}{b+r}$

2) Hypergeometric w/o replacement: $\frac{\binom{b}{x}\binom{r}{n-x}}{\binom{b+r}{n}}$, $x = \max(0, n-r), \ldots, \min(n, b)$

binomial coefficient

P(selecting 20 applicants at random) $= \binom{60}{20} = {}_{60}C_{20} = \frac{n!}{r!(n-r)!}$

$= \frac{60!}{20!(60-20)!}$ $= \binom{\text{Total applicants}}{\text{num applicants}}$

P(selecting 10 from the east) $= \binom{\text{total east}}{\text{num east choosing}} = \binom{40}{10} = {}_{40}C_{10} = \frac{n!}{r!(n-r)!} = \frac{40!}{10!(40-10)!}$

P(selecting 10 remaining) $= \binom{\text{Total ppl left}}{\text{num remaining}} = \binom{20}{10} = {}_{20}C_{10} = \frac{n!}{r!(n-r)!} = \frac{20!}{10!(20-10)!} = \frac{20!}{10!\,10!}$

$= \frac{\binom{40}{10}\binom{20}{10}}{\binom{60}{20}} = \boxed{0.0374}$

What I learned in the past 3 yrs at CB

(continued!)

not more than 2 will be from the east

requirements:

East: 0, 1, 2

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

P(X=0): no one from the east:

✓ P(selecting 20 applicants at random) = $\binom{\text{total ppl}}{\text{20 applicants}} = \binom{60}{20}$

$= {}_{60}C_{20} = \frac{60!}{20!(60-20)!}$

✓ P(selecting 0 from the East) = $\binom{\text{Total num from East}}{\text{number of ppl East}} = \binom{40}{0}$

$= {}_{40}C_{0} = \frac{40!}{0!(40-0)!}$

✓ P(selecting 20 from the remaining = $\binom{\text{Total ppl remaining}}{\text{num remaining}} = \binom{20}{20}$

$= {}_{20}C_{20} = \frac{20!}{20!(20-20)!}$

* $P(X=0) = \frac{{}_{40}C_{0} \cdot {}_{20}C_{20}}{{}_{60}C_{20}}$

P(X=1): 1 from the east:

✓ P(selecting 20 applicants at random) = $\binom{\text{total ppl}}{\text{20 applicants}} = \binom{60}{20} = {}_{60}C_{20} = \frac{60!}{20!(60-20)!}$

✓ P(selecting 1 from the East) = $\binom{\text{Total East}}{\text{num select East}} = \binom{40}{1} = {}_{40}C_{1} = \frac{40!}{1!(40-1)!}$

✓ P(selecting 19 from remaining) = $\binom{\text{Total ppl remaining}}{\text{num remaining}} = \binom{20}{19} = {}_{20}C_{19} = \frac{20!}{1!(20-19)!}$

* $P(X=1) = \frac{{}_{40}C_{1} \cdot {}_{20}C_{19}}{{}_{60}C_{20}}$

8. (continued!)

$P(X=2)$: 2 people from the East:

✓ P(selecting 20 applicants at random) $= \binom{\text{Total ppl}}{\text{20 applicants}} = \binom{60}{20}$

$= {}_{60}C_{20} = \dfrac{60!}{20!(60-20)!}$

✓ P(choosing 2 ppl from the east) $= \binom{\text{Total East}}{\text{num select East}} = \binom{40}{2}$

$= {}_{40}C_{2} = \dfrac{40!}{2!(40-2)!}$

✓ P(choosing 18 ppl from remaining) $= \binom{\text{Total ppl remaining}}{\text{num remaining}}$

$= \binom{20}{18} = {}_{20}C_{18} = \dfrac{20!}{18!(20-18)!}$

* $P(X=2) = \dfrac{{}_{40}C_{2} \cdot {}_{20}C_{18}}{{}_{60}C_{20}}$

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \dfrac{{}_{40}C_{0} \cdot {}_{20}C_{20}}{{}_{60}C_{20}} + \dfrac{{}_{40}C_{1} \cdot {}_{20}C_{19}}{{}_{60}C_{20}} + \dfrac{{}_{40}C_{2} \cdot {}_{20}C_{18}}{{}_{60}C_{20}}$

$= 3.554 \times 10^{-11} = P(X \le 2)$ = no more than 2 from the East

…ove that when N is larger, the hypergeometric distribution is identical w/ binomial distribution.

hypergeometric distribution:

$$P(X=x) = \frac{\binom{b}{x}\binom{r}{n-x}}{\binom{b+r}{n}}, \quad x = \max(0, n-r), \ldots, \min(n, b)$$

$$= \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \frac{r!}{x!(r-x)!} \cdot \frac{(N-r)!}{(n-x)!(N-n(r-x))!} \cdot \frac{n!\cdot(N-n)!}{N!}$$

$$= \binom{n}{x} \cdot \frac{r!/(r-x)!}{N!/(N-x)!} \cdot \frac{(N-r)!\cdot(N-n)!}{(N-x)!\cdot(N-r-(n-x))!}$$

$$= \binom{n}{x} \cdot \frac{r!/(r-x)!}{N!/(N-x)!} \cdot \frac{(N-r)!/(N-r-(n-x))!}{(N-n+(n-x))!/(N-n)!}$$

$$= \binom{n}{x} \cdot \prod_{k=1}^{x} \frac{(r-x+k)}{(N-x+k)} \cdot \prod_{m=1}^{n-x} \frac{(N-r-(n-x)+m)}{(N-n+m)}$$

Now, we take the large N limit for fixed r/N, n and x we get the binomial pmf since,

$$\lim_{N\to\infty} \frac{(r-x+k)}{(N-x+k)} = \lim_{N\to\infty} \frac{r}{N} = p$$

and

$$\lim_{N\to\infty} \frac{(N-r(n-x)+m)}{(N-n+m)} = \lim_{N\to\infty} \frac{N-r}{N} = 1-p$$