



EE 381 Probability & Statistic with Applications to Computing (Fall 2020)



Lecture 8 Continuous Random Variables (p1)

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Discrete Random Variable's topics

- **Probability distributions, Expectation**
- **Variance, Covariance & Correlation**
- Uniform Distribution, Normal Distribution
- Other Distributions, Central Limit Theorem

Continuous Random Variable

- **Example:** when measuring height, weight, and temperature, you have continuous data, because they can be divided into smaller increments, including fractional and decimal values.

Continuous Random Variable

- A non-discrete random variable X is said to be *absolute continuous*, or simply *continuous*, if its distribution function may be represented as:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u)du \quad (-\infty < x < \infty)$$

Where the function $f(x)$ has the properties

$$\diamond f(x) \geq 0$$

$$\diamond \int_{-\infty}^{\infty} f(x)dx = 1$$

- The probability that X takes on any one particular value is zero.
- The *interval probability* that X lies between two different values, a and b , is given by:

$$P(a < X < b) = \int_a^b f(x)dx$$

Continuous Random Variable

- **Example:** Assuming we have function $f(x)$ as follow:

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

- Find the constant c such that $f(x)$ is a density function.
- Compute $P(1 < x < 2)$.

Solution:

- $f(x)$ must satisfy two properties: $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$

Satisfying property 1 $\rightarrow c \geq 0$

Satisfying property 2 $\rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_0^3 cx^2 dx = \frac{cx^3}{3} \Big|_0^3 = 9c = 1 \rightarrow c=1/9$.

$$\text{b) } P(1 < X < 2) = \int_1^2 \frac{1}{9} x^2 dx = \frac{x^3}{27} \Big|_1^2 = \frac{7}{27}$$

Continuous Random Variable

- **Example (ctn.):** Assuming we have function $f(x)$ as follow:

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

c) Find the distribution function $F(x)$.

d) Compute $P(1 < x \leq 2)$.

Solution:

c) $F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$

If $x < 0$, then $F(x) = 0$. If $0 \leq x < 3$, then:

$$F(x) = \int_0^x f(u) du = \int_0^x \frac{1}{9} u^2 du = \frac{x^3}{27}$$

If $x \geq 3$, then

$$F(x) = \int_0^3 f(u) du + \int_3^x f(u) du = \int_0^3 \frac{1}{9} u^2 du + \int_3^x 0 du = 1$$

The required distribution function $F(x)$ is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^3}{27} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

Continuous Random Variable

d) Compute $P(1 < x \leq 2)$.

Solution:

d) We have:

$$\begin{aligned} P(1 < X \leq 2) &= P(X \leq 2) - P(X \leq 1) \\ &= F(2) - F(1) \\ &= \frac{2^3}{27} - \frac{1^3}{27} = \frac{7}{27} \end{aligned}$$

Using this result, we can easily prove that:

$$P(1 \leq X \leq 2) = P(1 \leq X < 2) = P(1 < X \leq 2) = P(1 < X < 2) = \frac{7}{27}$$

Continuous Random Variable

The probability that X is between x and $x + \Delta x$ is given by:

$$P(x \leq X \leq x + \Delta x) = \int_x^{x+\Delta x} f(u) du$$

So that if Δx is small, we have: $P(x \leq X \leq x + \Delta x) = f(x)\Delta x$

It is also observed that when $f(x)$ is continuous, the derivative of the distribution function is the density function:

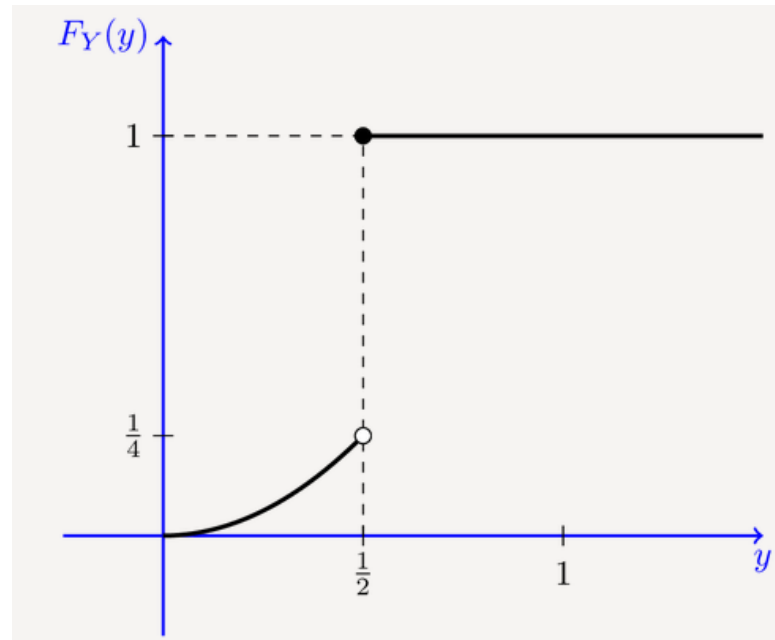
$$\frac{dF(x)}{dx} = f(x)$$

Continuous Random Variable

Note that there are also random variables exist that are neither discrete nor continuous.

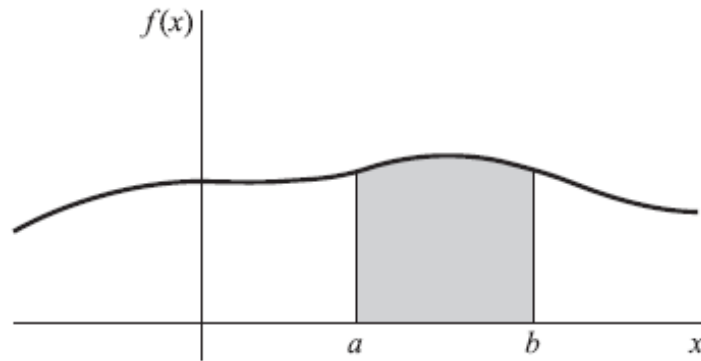
We called them mixed random variables, which contain a discrete part and a continuous part.

$$F_Y(y) = \begin{cases} 1 & y \geq \frac{1}{2} \\ y^2 & 0 \leq y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



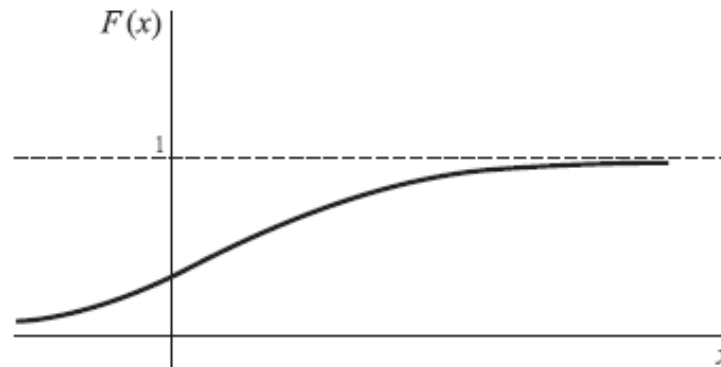
Graphical Interpretations

- If $f(x)$ is the density function for a random variable X , then we can represent $y=f(x)$ graphically by a curve as follow:



$f(x) \geq 0$ so the curve cannot fall below the x axis. $P(a < X < b)$ is the shaded area.

- The distribution function is a monotonically increasing function which increases from 0 to 1 and is represented by a curve as follow:



Joint Distribution for Continuous Random Variables

- The joint probability function for random variables X and Y is defined by:

1. $f(x, y) \geq 0$

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$

- The probability that X lies btw a & b while Y lies btw c & d is given by:

$$P(a < X < b, c < Y < d) = \int_{x=a}^b \int_{y=c}^d f(x, y) dx dy$$

- The joint distribution function of X and Y is defined by:

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{u=-\infty}^x \int_{v=-\infty}^y f(u, v) du dv$$

- Then the density function is obtained by differentiating the distribution function with respect to x and y :

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

Joint Distribution for Continuous Random Variables

- The distributions functions of X and Y are defined by:

$$P(X \leq x) = F_1(x) = \int_{u=-\infty}^x \int_{v=-\infty}^{\infty} f(u, v) du dv$$

$$P(Y \leq y) = F_2(y) = \int_{u=-\infty}^{\infty} \int_{v=-\infty}^y f(u, v) du dv$$

The density functions of X and Y are defined by:

$$f_1(x) = \int_{v=-\infty}^{\infty} f(x, v) dv \qquad f_2(y) = \int_{u=-\infty}^{\infty} f(u, y) du$$

Changes of Variables

- **Theorem 2-3:** Let X be a continuous random variable with probability density $f(x)$. Let us define $U = \phi(X)$, where to each value of X there corresponds one and only one value of U and conversely, so that $X = \psi(U)$. Then the probability density of U is given by $g(u)$ where

$$g(u)|du| = f(x)|dx|$$

or

$$g(u) = f(x) \left| \frac{dx}{du} \right| = f[\psi(U)]|\psi'(u)|$$

Changes of Variables

Example: The probability function of random variable X is given by:

$$f(x) = \begin{cases} \frac{x^2}{81} & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density for the random variable $U = \frac{1}{3}(12 - X)$.

Solution: $x = 12 - 3u$. So $u = 5$ when $x = -3$, and $u = 2$ when $x = 6$.

Since $\psi'(u) = \frac{dx}{du} = -3$, follows Theorem 2-3, the density function for U is:

$$g(u) = \begin{cases} \frac{(12 - 3u)^2}{27} & 2 < u < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Recheck: } \int_2^5 \frac{(12-3u)^2}{27} du = -\frac{(12-3u)^3}{243} \Big|_2^5 = 1$$

Changes of Variable

Theorem 2-4: Let X and Y be a continuous random variables having joint probability function $f(x, y)$. Let's define $U = \phi_1(X, Y), V = \phi_2(X, Y)$, where to each pair of X and Y there corresponds one and only one pair of values of U and V conversely, so that $X = \psi_1(U, V), Y = \psi_2(U, V)$. Then the joint probability function for U and V is given by $g(u, v)$ where

$$g(u, v)|du dv| = f(x, y)|dx dy|$$

Or
$$g(u, v) = f(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = f[\psi_1(u, v), \psi_2(u, v)]|J|$$

The Jacobian determinant, or briefly Jacobian, is given by:

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Changes of Variables

Example: If the random variables X and Y have joint density function:

$$f(x, y) = \begin{cases} \frac{xy}{96} & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density for the random variable $U = (X + 2Y)$.

Solution: Let $u = x + 2y$, $v = x$ (chosen arbitrarily). This yields $x = v$, $y = \frac{1}{2}(u-v)$

$0 < x < 4, 1 < y < 5$ corresponds to $0 < v < 4, 2 < u - v < 10$.

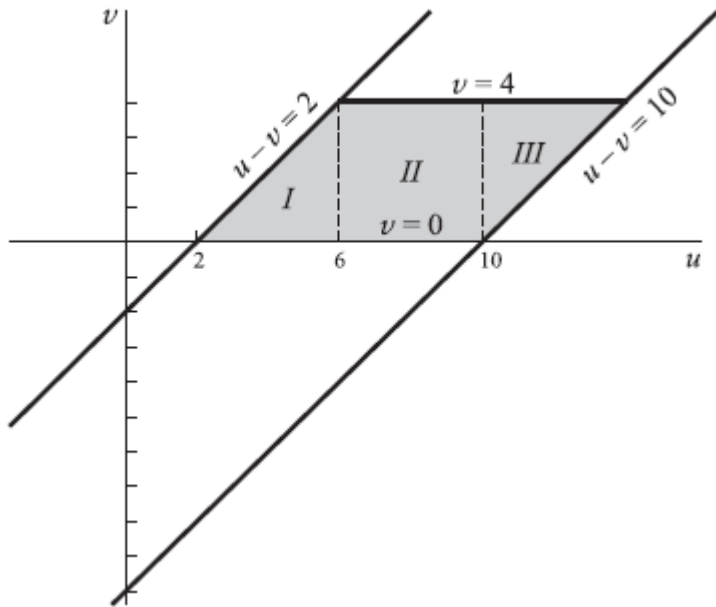
The Jacobian is given by: $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

By Theorem 2-4, the joint density function of U and V is:

$$g(u, v) = \begin{cases} \frac{v(u-v)}{384} & 2 < u - v < 10, 0 < v < 4 \\ 0 & \text{otherwise} \end{cases}$$

Changes of Variables

Example (ctn.): The marginal density function of U is given by:



$$g_1(u) = \begin{cases} \int_{v=0}^{u-2} \frac{v(u-v)}{384} dv & 2 < u < 6 \\ \int_{v=0}^4 \frac{v(u-v)}{384} dv & 6 < u < 10 \\ \int_{v=u-10}^4 \frac{v(u-v)}{384} dv & 10 < u < 14 \\ 0 & \text{otherwise} \end{cases}$$

Carrying out the integrations:

$$g_1(u) = \begin{cases} (u-2)^2(u+4)/2304 & 2 < u < 6 \\ (3u-8)/144 & 6 < u < 10 \\ (348u - u^3 - 2128)/2304 & 10 < u < 14 \\ 0 & \text{otherwise} \end{cases}$$

Conditional Distributions

- Let X and Y are continuous random variables. The conditional density function of Y given X is:

$$f(y|x) = \frac{f(x, y)}{f_1(x)}$$

Where $f(x, y)$ is the joint density function of X and Y ; $f_1(x)$ is the marginal density function of X .

- Example: Find the probability of Y being between c and d given that $x < X < x + dx$.

$$P(c < Y < d | x < X < x + dx) = \int_c^d f(y|x) dy$$

Conditional Distributions

Example: if X and Y have the joint density function

$$f(x, y) = \begin{cases} \frac{3}{4} + xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find (a) $f(y|x)$, (b) $P(Y > \frac{1}{2} | \frac{1}{2} < X < \frac{1}{2} + dx)$.

Solution:

a) For $0 < x < 1$, we have $f_1(x) = \int_0^1 \left(\frac{3}{4} + xy \right) dy = \frac{3}{4} + \frac{x}{2}$

$$f(y|x) = \frac{f(x, y)}{f_1(x)} = \begin{cases} \frac{3 + 4xy}{3 + 2x} & 0 < y < 1 \\ 0 & \text{other } y \end{cases}$$

b) $P(Y > \frac{1}{2} | \frac{1}{2} < X < \frac{1}{2} + dx) = \int_{1/2}^{\infty} f(y | \frac{1}{2}) dy = \int_{1/2}^1 \frac{3 + 2y}{4} dy = \frac{9}{16}$

Expectation, Variance, Covariance & Correlation

- Expectation of discrete random variables: $E(X) = \mu_X = \sum xf(x)$
- Expectation of continuous random variables: $E(X) = \mu_X = \int_{-\infty}^{\infty} xf(x)dx$

Example: The density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of X is:

$$E(X) = \int_{-\infty}^{\infty} xf(x) dx = \int_0^2 x \left(\frac{1}{2}x \right) dx = \int_0^2 \frac{x^2}{2} dx = \frac{x^3}{6} \Big|_0^2 = \frac{4}{3}$$

Expectation, Variance, Covariance & Correlation

- *Variance* of continuous random variables:

$$\sigma_X^2 = E[(x - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Standard deviation: $\sigma = \sqrt{\sigma^2}$

Example: The density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

The variance of X is ($\mu = 4/3$):

$$\sigma^2 = E\left[\left(X - \frac{4}{3}\right)^2\right] = \int_{-\infty}^{\infty} \left(x - \frac{4}{3}\right)^2 f(x) dx = \int_0^2 \left(x - \frac{4}{3}\right)^2 \left(\frac{1}{2}x\right) dx = \frac{2}{9}$$

Expectation, Variance, Covariance & Correlation

Moment generating function of random variable X :

$$M_X(t) = \sum e^{tx} f(x) \quad (\text{discrete variable})$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \quad (\text{continuous variable})$$

Expectation, Variance, Covariance & Correlation

Covariance (variance for joint distributions):

If X and Y are 2 continuous random variables having joint density function $f(x,y)$.
The means of X and Y are:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y) dx dy, \quad \mu_Y = E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y) dx dy$$

And the variance are:

$$\sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x, y) dx dy$$

$$\sigma_Y^2 = E[(Y - \mu_Y)^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y - \mu_Y)^2 f(x, y) dx dy$$

The covariance is defined as:

$$\sigma_{XY} = \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f(x, y) dx dy$$

Expectation, Variance, Covariance & Correlation

Correlation for continuous random variable has the same formula as correlation for discrete random variables.

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Conditional Expectation, Conditional Variance

If X and Y have joint density function $f(x,y)$, the conditional density function of Y given X is $f(y|x) = \frac{f(x,y)}{f_1(x)}$.

- The *conditional expectation* of Y given X is defined as:

$$E(Y|X = x) = \int_{-\infty}^{\infty} yf(y|x)dy$$

Where “ $X=x$ ” is to be interpreted as $x < X \leq x + dx$ in the continuous case.

We note the following properties:

1. $E(Y|X = x) = E(Y)$ when X and Y are independent.
2. $E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_1(x)dx$

Briefly, $E(Y|X = x)$ is a number depending on y . if Y has an influence on the value of X , then Y will have an influence on the average value of X .

- The *conditional variance* of Y given X is:

$$E[(Y - \mu_2)^2 | X = x] = \int_{-\infty}^{\infty} (y - \mu_2)^2 f(y|x) dy$$

Reference

Notes, equations, and figures in the lecture are based on or taken from materials in the course textbook:

“Probability and Statistics”, by Spiegel, Schiller and Srinivasan, ISBN 987-007-179557-9 (McGraw-Hill/Schaun’s)