Daniel Pvong
10/3/20

EE 381 HW #6

1. Random variable X defined:

$$X = \begin{cases} -2 & \text{prob. } 1/3 \\ 3 & \text{prob. } 1/2 \\ 1 & \text{prob. } 1/6 \end{cases}$$

X	-2	3	1
f(X)	1/3	1/2	1/6

$$E(x) = \sum_{j=1}^{n} x_j f(x_j) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3)$$

$$= -2\left(\frac{1}{3}\right) + 3\left(\frac{1}{2}\right) + 1\left(\frac{1}{6}\right) = \frac{-2}{3} + \frac{3}{2} + \frac{1}{6} =$$

$$-\frac{4}{6} + \frac{9}{6} + \frac{1}{6} = \frac{6}{6} = 1 = E(X) = \mu_X$$

MCLAVRIN

Moment Generating Function: $\mu_X(u) = E[e^{uX}] = \sum_k e^{uk} p(k)$

$$\mu_X(u) = E[e^{tX}] = \sum_k e^{tk} p(k)$$

$$= e^{t(-2)}(1/3) + e^{t(3)}(1/2) + e^{t(1)}(1/6)$$

$$= \frac{1}{3}e^{-2t} + \frac{1}{2}e^{3t} + \frac{1}{6}e^{t}$$ = Moment Generating Function For MCLAVRIN

1. (continued)

Mcclaurin Series: $f(t) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$

$e^{-2t} = 1 - 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \dots$

$e^{3t} = 1 + 3t + \frac{9t^2}{2!} + \frac{27t^3}{3!} + \frac{81t^4}{4!} + \dots$

$e^{3t} = 1 + 3t + \frac{9t^2}{2!} + \frac{27t^3}{3!} + \frac{81t^4}{4!} + \dots$

$e^{t} = 1 + t + \frac{t^2}{!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots$

$= \frac{1}{3}e^{-2t} = \frac{1}{3} - \frac{1}{3} \times 2t + \frac{4t^2}{2!} \times \frac{1}{3} - \frac{8t^3}{3!} \times \frac{1}{3} + \frac{16t^4}{4!} \times \frac{1}{3} + \dots$

Moment Generating Function about the Mean ($\mu = 1$)

$M_X(t) = E(e^{t(X-\mu)}) = \sum e^{t(X-\mu)} f(X)$

$= e^{t((-2)-1)}\left(\frac{1}{3}\right) + e^{t((3)-1)}\left(\frac{1}{2}\right) + e^{t((1)-1)}\left(\frac{1}{6}\right)$

$= e^{t(-3)}\left(\frac{1}{3}\right) + e^{t(2)}\left(\frac{1}{2}\right) + e^{t(0)}\frac{1}{6}$

$= \frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t} + \frac{1}{6}\cancel{e^{0t}}^{\,1} = \boxed{\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t} + \frac{1}{6} = M_X(t) = \text{MGF about the mean}}$

// Beware: chain Rule!

1st Moment about mean: $\frac{d}{dt} M_X(t)\Big|_{t=0} = \frac{d}{dt}\left[\frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t} + \frac{1}{6}\Big|_{t=0}\right]$

$= -3 \cdot \frac{1}{3}e^{-3t} + 2 \cdot \frac{1}{2}e^{2t} + 0 \,\Big|_{t=0}$

plug in $t=0$: $-\frac{3}{3}e^{-3(0)} + \frac{2}{2}e^{2(0)} + 0 = 1(1) + 1(1) + 0 = -1 + 1 = \boxed{0}$

$\boxed{E(X) = 0}$

1. (continued!)

1st Moment Mean $E(X) = 0$

2nd Moment about mean:

$$\frac{d^2}{dt^2} \mu_X(t) \Big|_{t=0} = \frac{d}{dt}\left[\frac{-3}{3}e^{-3t} + \frac{2}{2}e^{2t} + 0 \Big|_{t=0} \right]$$

$$= \frac{d}{dt}\left[-e^{-3t} + e^{2t} + 0 \Big|_{t=0} \right] = (-3) \cdot -e^{-3t} + 2e^{2t} + 0 \Big|_{t=0}$$

$$= 3e^{-3t} + 2e^{2t} + 0 \Big|_{t=0}$$

plug in $t=0$: $3e^{-3(0)} + 2e^{2(0)} + 0 \Big|_{t=0}$

$$= 3e^0 + 2e^0 + 0 \Big|_{t=0} = 3(1) + 2(1) + 0 \Big|_{t=0} = 3 + 2 + 0 \Big|_{t=0} = \boxed{5}$$

$E(x^2) = 5$

2nd moment about mean $= 5$

3rd Moment about mean:

$$\frac{d^3}{dt^3} \mu_X(t) \Big|_{t=0} = \frac{d}{dt}\left[3e^{-3t} + 2e^{2t} + 0 \Big|_{t=0} \right]$$

$$= (-3)3e^{-3t} + (2)2e^{2t} + 0 \Big|_{t=0} = -9e^{-3t} + 4e^{2t} + 0 \Big|_{t=0}$$

plug in $t=0$: $-9e^{-3(0)} + 4e^{2(0)} + 0 \Big|_{t=0} = -9e^0 + 4e^0 + 0 \Big|_{t=0}$

1. (continued!)

$= -9(1) + 4(1) + 0\Big|_{t=0} = -9+4+0 = \boxed{-5 = E(X^3)}$

$\boxed{\text{3rd Moment about mean} = -5}$

4th moment about the mean:

$\frac{d^4}{dt^4} = \frac{d}{dt}\left[-9e^{-3t} + 4e^{2t} + 0\Big|_{t=0}\right]$

$= (-3)(-9)e^{-3t} + (2)4e^{2t} + 0\Big|_{t=0} = 27e^{-3t} + 8e^{2t} + 0\Big|_{t=0}$

Plug in $t=0$: $27e^{-3(0)} + 8e^{2(0)} + 0\Big|_{t=0}$

$= 27e^{0} + 8e^{0} + 0\Big|_{t=0} = 27(1) + 8(1) + 0\Big|_{t=0} = \boxed{35 = E(X^4)}$

$\boxed{\text{4th moment about mean} = 35}$

2. The joint probability function for the random variables X and Y are given in the following table:

← X →

X \ Y	0	1	2	Total ↓
0	1/18	1/9	1/6	1/3
1	1/9	1/18	1/9	5/18
2	1/6	1/6	1/18	7/18
Total →	1/3	1/3	1/3	1

↑ Y ↓

$P(X=0) = \sum_{j=0}^{n} f(x_j)$
$= 1/18 + 1/9 + 1/6 = 1/3$

$P(Y=0) = \sum_{j=0}^{n} f(y_j) = 1/3$

← Grand total = 1

2. (continued!)

a) var(X) and var(Y)

$var(X) = E(X^2) - [E(X)^2]$

// we need to find expectation first

$E(X^2) = \sum_X \sum_Y x^2 f(x,y) = \sum_x x^2 \left[\sum_y f(x,y) \right]$

$= x_0^2 f_0(x,y) + x_1^2 f_1(x,y) + x_2^2 f_2(x,y)$

$= (0)^2 (1/3) + (1)^2 (5/18) + (2)^2 (7/18)$

$= (0)(1/3) + (1)(5/18) + (4)(7/18)$

$= 0 + \frac{5}{18} + \frac{28}{18} = \boxed{\frac{33}{18} = E(x^2)}$

$E(X) = \sum_X \sum_Y X f(x,y) = \sum_x x \left[\sum_y f(x,y) \right]$

$= x_0 f_0(x,y) \; x_1 f_1(x,y) + x_2 f_2(x,y)$

$= (0)(1/3) + (1)(5/18) + (2)(7/18)$

$= 0 + \frac{5}{18} + \frac{14}{18} = \boxed{\frac{19}{18} = E(X)}$

$var(X) = \theta^2 x = E(X^2) - [E(X)^2] = \frac{33}{18} - \left(\frac{19}{18}\right)^2$

$= \frac{33}{18} - \frac{361}{324} = \frac{33}{18}\left(\frac{18}{18}\right) - \frac{361}{324} = \frac{594}{324} - \frac{361}{324} = \boxed{\frac{233}{324} = var(X)}$

2. (continued!)

$$E(Y^2) = \sum_X \sum_Y Y^2 f(x,y) = \sum_Y Y^2 \left[\sum_X f(x,y)\right]$$

$$= Y_0^2 f_0(x,y) + Y_1^2 f_1(x,y) + Y_2^2 f_2(x,y)$$
$$= (0)^2 (1/3) + (1)^2 (1/3) + (2)^2 (1/3)$$
$$= 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) = 0 + \frac{1}{3} + \frac{4}{3} = \boxed{\frac{5}{3} = E(Y^2)}$$

$$E(Y) = \sum_X \sum_Y Y f(x,y) = \sum_Y Y \left[\sum_X f(x,y)\right]$$

$$= Y_0 f_0(x,y) + Y_1 f_1(x,y) + Y_2 f_2(x,y)$$
$$= (0)(1/3) + (1)(1/3) + (2)(1/3)$$
$$= 0 + \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = \boxed{1 = E(Y)}$$

$$Var(Y) = \sigma^2 Y = E(Y^2) - [E(Y)^2]$$

$$= \frac{5}{3} - (1)^2 = \frac{5}{3} - \frac{1}{1} = \frac{5}{3} - \frac{1}{1}\left(\frac{3}{3}\right) = \frac{5}{3} - \frac{3}{3} = \boxed{\frac{2}{3} = var(Y)}$$

b) σX and σY

σX = Standard deviation of $X = \sqrt{var(X)} = \sqrt{\frac{233}{324}} = \boxed{\frac{\sqrt{233}}{18}}$

σY = Standard deviation of $Y = \sqrt{var(Y)} = \boxed{\sqrt{2/3} = \sigma Y}$

covariance

c) $\sigma_{XY} = E(XY) - E(X)E(Y)$ //Thm 3.15

we need to determine E(XY) first.

$$E(XY) = \sum_X \sum_Y XY f(x,y)$$

2. (continued!)

$= x_0 Y_0 f_0(x,y) + x_0 Y_1 f_1(x,y) + x_0 Y_2 f_2(x,y) +$
$x_1 Y_0 f_0(x,y) + x_1 Y_1 f_1(x,y) + x_1 Y_2 f_2(x,y) +$
$x_2 Y_0 f_0(x,y) + x_2 Y_1 f_1(x,y) + x_2 Y_2 f_2(x,y) =$

$(0)(0)(1/18)+(0)(1)(1/9)+(0)(2)(1/6)+$
$(1)(0)(1/9)+(1)(1)(1/18)+(1)(2)(1/9)+$
$(2)(0)(1/6)+(2)(1)(1/6)+(2)(2)(1/18) =$

$0+0+0+$
$0+1/18+2/9+$
$0+2/6+4/18$

$$= \frac{1}{18}+\frac{2}{9}+\frac{2}{6}+\frac{4}{18} = \frac{1}{18}+\frac{2}{9}\left(\frac{2}{2}\right)+\frac{2}{6}\left(\frac{3}{3}\right)+\frac{4}{18}$$

$$= \frac{1}{18}+\frac{4}{18}+\frac{6}{18}+\frac{4}{18} = \frac{15}{18} = \boxed{\frac{5}{6} = E(XY)}$$

σ_{XY} = covariance

$E(X) = 19/18$
$E(Y) = 1$ from part A

$\sigma_{XY} = E(XY) - E(X)E(Y)$

$$\sigma_{XY} = \frac{5}{6} - \left(\frac{19}{18}\right)(1) = \frac{5}{6} - \frac{19}{18} = \frac{5}{6}\left(\frac{3}{3}\right) - \frac{19}{18}$$

$$= \frac{15}{18} - \frac{19}{18} = \frac{-4}{18} = \boxed{\frac{-2}{9} = \sigma_{XY} = COV(X,Y)}$$

d) ρ

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = \frac{-2/9}{(\sqrt{233}/18)(\sqrt{2/3})} = \frac{-2/9}{0.6924}$$

$\boxed{\sigma_{XY} = -2/9 \text{ from part C}}$ $\boxed{\sigma_X = \sqrt{233}/18,\ \sigma_Y = \sqrt{2/3} \text{ from part B}}$ $= \boxed{-0.32094}$

3. Find (a) covariance, (b) the correlation coefficient of two random variables X and Y if

$E(X)=2$, $E(Y)=3$, $E(XY)=10$, $E(X^2)=9$, $E(Y^2)=16$

Given: $E(X)=2$
$E(Y)=3$
$E(XY)=10$
$E(X^2)=9$
$E(Y^2)=16$

a) covariance

Thm 3.14

$$\text{covariance} = \sigma_{XY} = \overline{E(XY)} - E(X)\,E(Y)$$
$$= 10-(2)(3) = 10-6 = \boxed{4 = \text{covariance} = E(XY)}$$

b) coefficient of two random variables X and Y

correlation coefficient: $\rho = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y} = \dfrac{\text{cov}(XY)}{\text{std dev } X \cdot \text{std dev } Y}$

So, we backtrace to find the variance of X and Y. and then determine std dev of X and Y.

$$\sigma_X^2 = E(X^2) - [E(X)^2] = 9-[(2)^2] = 9-4 = 5 = \sigma_X^2 = \text{var}(X)$$

$$\sigma_Y^2 = E(Y^2) - [E(Y)^2] = 16-[(3)^2] = 16-9 = 7 = \sigma_Y^2 = \text{var}(Y)$$

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\text{var}(X)} = \sqrt{5} = \sigma_X = \text{std dev } X$$
$$\sigma_Y = \sqrt{\sigma_Y^2} = \sqrt{\text{var}(Y)} = \sqrt{7} = \sigma_Y = \text{std dev } Y$$

$4 = \text{covariance} = E(XY) = \sigma_{XY}$ from part a

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{4}{\sqrt{5}\sqrt{7}} = \boxed{\frac{4}{\sqrt{35}} = \rho}$$

4. The correlation coefficient of two random X and Y is $-1/4$ while their variances are 3 and 5. Find covariance.

Given:

$\rho = -1/4$: correlation coefficient
$var(X) = \sigma_X^2 = 3$
$var(Y) = \sigma_Y^2 = 5$

unknown: covariance: σ_{XY}

Formula: standard deviation $= \sqrt{var} = \sqrt{\sigma^2}$

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

solve:

standard deviation of $X = \sqrt{var(X)} = \sqrt{\sigma_X^2} = \sqrt{3}$ = std dev. X

standard deviation of $Y = \sqrt{var(Y)} = \sqrt{\sigma_Y^2} = \sqrt{5}$ = std dev. Y

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \Rightarrow \sigma_{XY} = \sigma_X \sigma_Y \rho$$

$$(\sqrt{3}\cdot\sqrt{5}) \cdot \frac{-1}{4} = \frac{\sigma_{XY}}{\cancel{\sqrt{3}\cdot\sqrt{5}}} \cdot \cancel{(\sqrt{3}\cdot\sqrt{5})}$$

$$\sigma_{XY} = \frac{\sqrt{3}\sqrt{5}}{-4} = \boxed{\frac{-\sqrt{15}}{4} = \sigma_{XY} = cov(XY)}$$