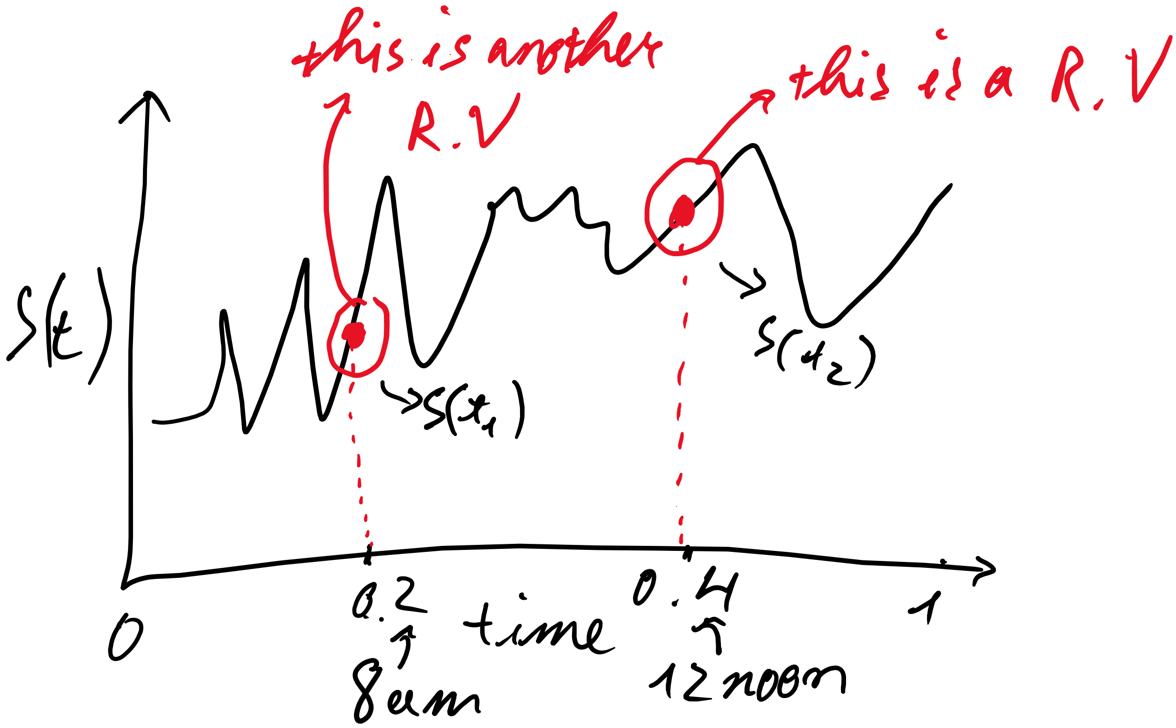
Lecture 13's note (12/8/20)



$(t_1 = 0.2) = a \rightarrow$ PDF for $S(t_1)$

$a_1 \quad a_2 \quad a_3 \ldots\ldots a_n$

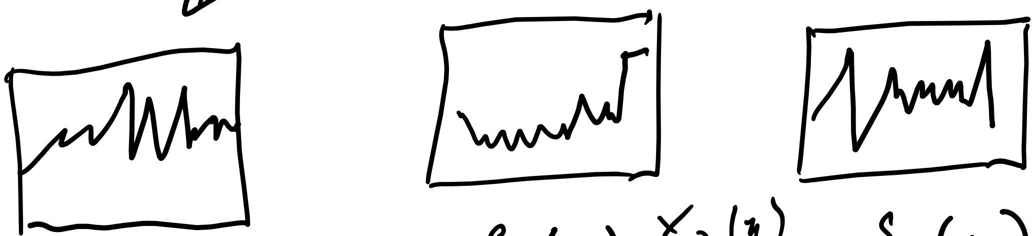
$(t_2 = 0.4) = b \rightarrow$ PDF for $S(t_2)$

$b_1 \quad b_2 \quad b_3 \ldots b_n$

random experiment

Die : [1 , 2 , 3 , 4 , 5 , 6]

n_1 $\quad$ n_2 $\quad$ n_3 $\quad$ ....



$S_1(t)$ $\quad X_1(n)$ $\quad S_2(t)$ $\quad X_2(n)$ $\quad S_3(t)$ $\quad X_3(n)$

$S_1 = a_1 + b_1 t + c_1 t^2$

$S_2(t) = a_2 + b_2 t + c_2 t^2$

equation for showing relationship btw stock price & time → sample functions

$$E(x) = \int x f(x)\,dx$$

$$E(x^3) = \int x^3 f(x)\,dx$$

$$n = 3 \rightarrow X_3 = 1000(1+R)^3$$

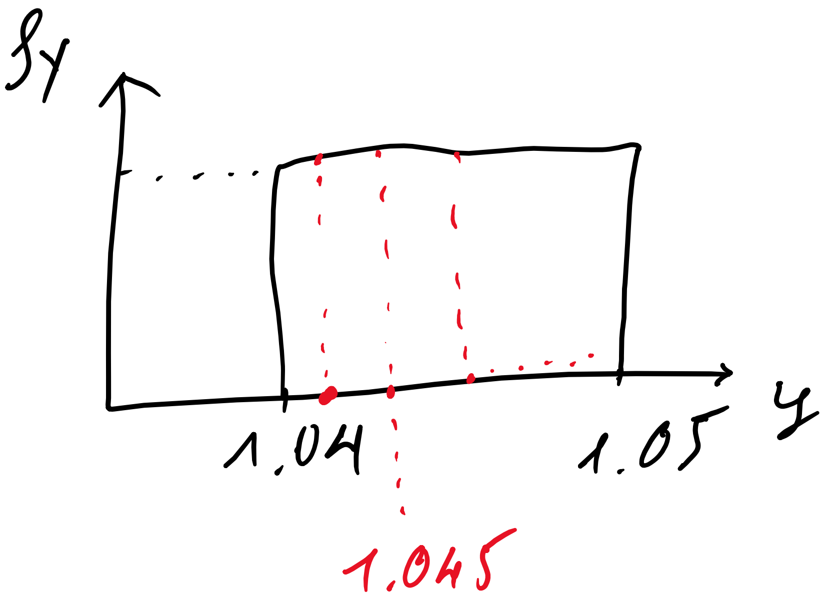
$$Y = 1 + R \rightarrow Y \sim \text{Uniform}(1.04, 1.05)$$

$$\rightarrow f_Y(y) = \begin{cases} \dfrac{1}{1.05 - 1.04} = 100 & 1.04 \leq y \leq 1.05 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_3) = E(1000(1+R)^3) = E(1000 \cdot Y^3)$$

$$= 1000 \cdot E(Y^3)$$

PDF



$X_1 \quad X_2 \quad X_3 \quad \cdots \quad X_m$

They are dependent.

Markov chains: X_{m+1} only depend on X_m, not all previous values.

Example: System contains 3 states: S_0, S_1, S_2

$t_1 = 0 \quad t_2 = 1 \quad t_3 = 2 \quad t_4 = 3$

$S_0 \quad S_1 \quad S_1 \quad S_2 \quad \cdots$

$X_{t_1} = 1 \rightarrow$ at time t_1, the system is in state S_1

States : $S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad \ldots$

$S_1 \to$ 5 people

$S_2 \to$ 3 people

$\ldots$

$X_n = 5 \rightarrow$ the system is in state S_1

$X_n = 3 \rightarrow$ " " S_2

$\vdots$

$X_m = i \rightarrow$ at time step m, the system is in state i

$X_{m+1} = j \rightarrow$ at time step $m+1$, " " j.

Time step

$$P(X_{m+1} = j \mid X_m = i, X_{m-1} = i_{m-1} \ldots)$$

states

$$= P(X_{m+1} = j \mid X_m = i)$$

time step : m | $m+1$

State : i | j

p_{ij} : probability that the system move to state j if it is at state i

$$P(X_2 = j \mid X_1 = i) = p_{ij}$$

$S_1 \quad S_2 \quad S_3 \quad S_4$

time step : $t = 1$ $\quad$ $t = 2$

State : S_2

$$S_2 \xrightarrow{p_{21}} S_1$$
$$S_2 \xrightarrow[p_{23}]{} S_3$$
$$S_2 \xrightarrow[p_{24}]{} S_4$$
$$S_2 \xrightarrow[p_{22}]{} S_2$$

$$P = \begin{array}{cc} & \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \end{array} \begin{array}{c} \to 1 \\ \to 1 \\ \to 1 \end{array}$$

state of system

$$P(X_4 = 3 \mid X_3 = 2) = p_{23} = \frac{2}{3}$$

time step

$P(X_0 = 1) = \frac{1}{3}$: probability of the system to be in state 1 initially is $\frac{1}{3}$.

a) $P(X_0 = 1, X_1 = 2)$

$= P(X_0 = 1) \cdot P(X_1 = 2 \mid X_0 = 1)$

$= \frac{1}{3} \cdot p_{12}$

$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

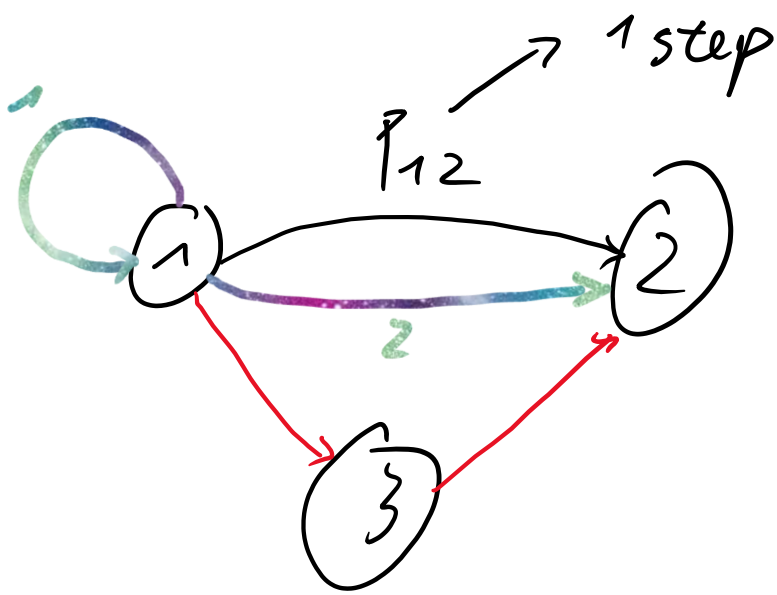
b) $P(X_0 = 1, X_1 = 2, X_2 = 3)$

$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B)$

$P(X_2 = 3 \mid X_1 = 2, X_0 = 1)$

$= P(X_2 = 3 \mid X_1 = 2)$

(because this is Markov chain)



What if you want the system to move from state 1 → state 2 in 2 steps.

$$P = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \end{array} \rightarrow \text{1 step transition}$$

total prob. of moving from 1 → 1 in 2 steps.

$$P^2 = \begin{bmatrix} p'_{11} & p'_{12} \\ p'_{21} & p'_{22} \end{bmatrix} \rightarrow \text{2-step transition}$$

$P^3 =$ 3-step

⋮

P^n : n-step

time step	$t_1 = 0$	$t_2 = 1$	$t_3 = 2$
State	1	2	2

$$\underbrace{1 \quad 2}_{1} \quad \underbrace{2 \quad 2}_{1}$$

1 options

$$P_{12} \times P_{22} \times P_{22} \times P_{22} \times P_{22}$$

$$1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad = P_1$$

$$1 \quad 1 \quad 1 \quad 1 \quad 1$$

5 steps

another option

$$1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad = P_2$$

$$P_{11} \times P_{11} \times P_{11} \times P_{12} \times P_{22}$$

Total prob. for moving from $1 \to 2$ in 5 steps: $P_1 + P_2$

prob. for the system to stay at state 1 at the initial time step

$$\pi^{(0)} = [\, P(X_0 = 1) \quad P(X_0 = 2) \,]$$

Determine the prob. for the system to stay at state 1 at time step $n = 3$

$$\pi^{(3)} = \pi^{(0)} \cdot P^3 = \pi^{(2)} \cdot P^{(1)}$$

Example:

$$P = \begin{array}{c c} & \begin{array}{cc} LR & HR \end{array} \\ \begin{array}{c} LR \\ HR \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

$$\pi^{(0)} = \begin{array}{c} \begin{array}{cc} LR & HR \end{array} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}$$

What is the prob. distribution after 3 years?

$$\pi^{(3)} = [1 \quad 0] \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}^3 = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}$$

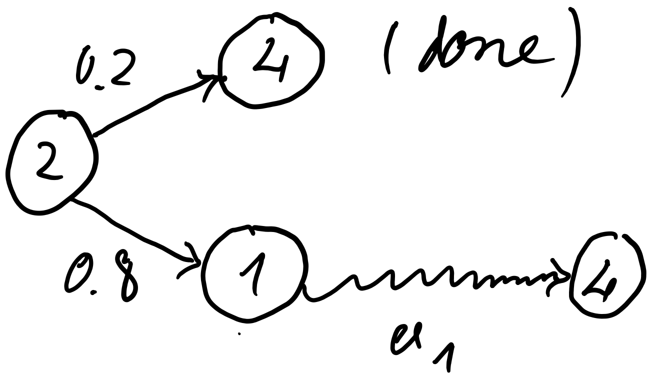
Example :

a_i : the prob. that the system settles in state 4 given it started in state i

$i = 4$: $a_4 = 1$ (start at state 4, stay in 4)

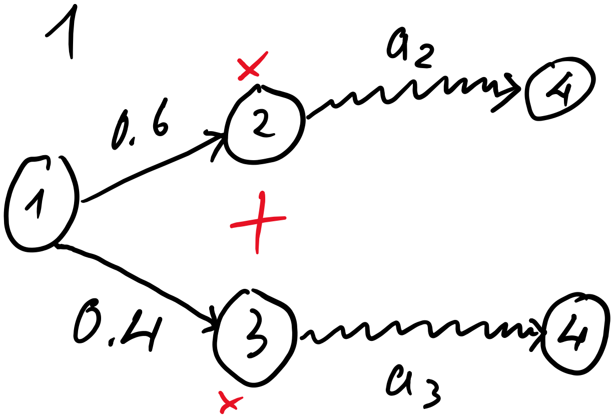
$i = 5$: $a_5 = 0$ (start at 5, never go to 4)

$i = 2$



$$a_2 = 0.2 + 0.8\, a_1 \tag{1}$$

$i = 1$



$$a_1 = 0.6 a_2 + 0.4\, a_3 \tag{2}$$

$i = 3$

$$\begin{aligned} a_3 &= 0.3 a_2 + 0.5 a_1 + \cancel{0.2 a_5} \\ &= 0.3 a_2 + 0.5 a_1 \end{aligned} \tag{3}$$

Solving (1) (2) (3) !

* Mean hitting times :

Example path :

t	0	1	2	3
$x(t)$	2	2	2	1

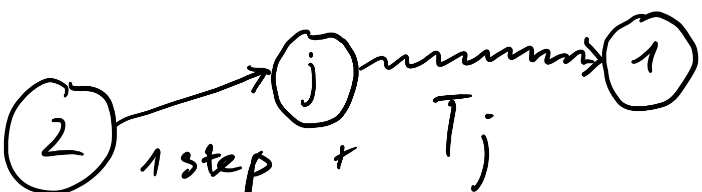
$T = 3$ steps (first hitting time)

T_j : first hitting time to state 1, starting from state j at $t = 0$

We're looking for $t_2 = E[T_2]$

$$t_2 = 1 + \sum_{j=1}^{2} p_{2j} t_j$$

total no. of states in the system



1 step + T_j

$$t_2 = 1 + p_{21} t_1 + p_{22} t_2$$

$t_1 = 0$ (start at state 1, don't need any step to go to state 1)

$$t_2 = 1 + 0.4\, t_2 \;\rightarrow\; t_2 = \frac{5}{3} \,!$$

Mean return times:

Example:

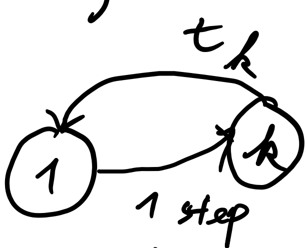
t	0	1	2	3	4
$x(t)$	1	2	2	1	1

first time return to 1

$R_1 = 3$ steps

We're looking for: $r_1 = E(R_1)$

R_ℓ: first passage time to return to state 1 if it started from state ℓ.



total no. of state in the system

$$r_1 = 1 + \sum_{k=1}^{2} p_{1k}\, t_k$$

$$= 1 + p_{11} t_1 + p_{12} t_2$$

$$= 1 + 0 + 0.2 \times \frac{5}{3}$$

$$= \frac{4}{3}$$

$t_1 = 0$ & $t_2 = \frac{5}{3}$ (mean hitting times)

* Steady-state condition:

$$\pi^{(0)} = \begin{bmatrix} x & y \end{bmatrix}$$

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} 0.6x + 0.15y = x \\ 0.4x + 0.85y = y \end{array} \right. \\ * \quad x + y = 1 \end{array} \right. \rightarrow \left\{ \begin{array}{l} x = \dfrac{3}{11} \\ y = \dfrac{8}{11} \end{array} \right.$$

In-class exercise:

a) $i = 0 \to a_0 = 1$

$i = 3 \to a_3 = 0$

$$a_2 = \sum_{j=0}^{3} p_{2j} a_j = p_{20} a_0 + p_{21} a_1 + p_{22} a_2 + p_{23} a_3$$

$$= \frac{1}{2} a_1 \qquad (1)$$

$$a_1 = \sum_{j=0}^{3} p_{1j} a_j = \frac{1}{3} + \frac{2}{3} a_2 \qquad (2)$$

$$a_1 = \frac{1}{2} \qquad a_2 = \frac{1}{4}$$

b) $t_0 = 0 \qquad t_3 = 0$

$$t_1 = 1 + \sum_{j=0}^{3} t_j p_{1j} = 1 + \frac{2}{3} t_2$$

$$t_2 = 1 + \sum_{j=0}^{3} t_j p_{2j} = 1 + \frac{1}{2} t_1$$

$$t_1 = \frac{5}{2} \qquad t_2 = \frac{9}{5} = 2.25$$

How about the steady state condition of this system?

$$P = \begin{array}{c|c} & \begin{matrix} s0 & s1 & s2 & s3 \end{matrix} \\ \begin{matrix} s0 \\ s1 \\ s2 \\ s3 \end{matrix} & \begin{bmatrix} p_{11} & - & \cdots \\ p_{21} & \cdots & \cdots \\ p_{31} & & \cdots \\ p_{41} & & \end{bmatrix} \end{array}$$

$p_{11} = 1$

$p_{21} = \frac{1}{3}$

$p_{31} = 0$

$$\pi^{(0)} = [x \quad y \quad z \quad g]$$

$$\pi^{(0)} \times P = \pi^{(0)}$$

$$\begin{cases} \begin{cases} x p_{11} + y p_{21} + z p_{31} + g p_{41} = x \\ \vdots \end{cases} \\ x + y + z + g = 1 \end{cases}$$

$\rightarrow x! \quad y! \quad z! \quad g!$