EE 381 Probability & Statistic with Applications to Computing (Fall 2020)

Lecture 3
Basic Concepts of Probability (p3)

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Chapter 1’s topics

- Discrete Sample Spaces
- Important Theorems on Probability
- Conditional Probability
- Independence
- Bayes’ Rule
- **Combinatorial Analysis**

Combinatorial Analysis

- Permutations with and without repetition
- Combinations with and without repetition
- Binomial coefficient
- Stirling's Approximation

Permutations

Permutations: A *permutation* of a set of distinct objects is an **ordered** arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

- The number of r-permutations of a set with n elements is denoted by $P(n ,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

Permutations

Permutations: If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Rewrite $P(n, r)$ in terms of factorials as:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

In-class Exercise

One hundred tickets, numbered 1, 2, 3,…, 100 are sold to 100 different people for drawing. Four different prizes are awarded, including a grand prize. How many ways are there to award the 4 different prizes if

a) There are no restrictions?

b) The person holding ticket 47 wins the grand prize?

c) The person holding ticket 47 wins one of the prizes?

Combinations

Combinations: An *r-combination* of elements of a set is an **unordered** selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of *r*-combinations of a set with *n* distinct elements is denoted by *C*(*n*, *r*), where:

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Example: Let *S* be the set {*a*, *b*, *c*, *d*}. Then {*a*, *c*, *d*} is a 3-combination from S. It is the same as {*d*, *c*, *a*} since the order listed does not matter.

C(4,2) = 6 because the 2-combinations of {*a*, *b*, *c*, *d*} are the six subsets {*a*, *b*}, {*a*, *c*}, {*a*, *d*}, {*b*, *c*}, {*b*, *d*}, and {*c*, *d*}.

Combinations

Combination's example 1: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: The number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: The number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 593,775\ .$$

In-class Exercise

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if

a) The committee must have the same number of men and women?

b) The committee must have more women than men?

Permutations with Repetition

Permutations with Repetition: The number of r-permutations of a set of n objects with repetition allowed is n^r

This is just like product rule.

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

Combinations with Repetition:

The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1)$$

Example 1: How many ways are there to select 4 pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: Let's list them down.

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

There are 15 ways, which is the number of 4-combinations with repetition allowed from a 3-element set {apple, orange, pear}.

Combinations with Repetition

Example 2: Suppose that a cookie shop has four different kinds of cookies. How many ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements.

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

In-class Exercise

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1, x_2,$ and x_3 are nonnegative integers?

Combinatorial Analysis

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Summarizing the formulas for counting permutations and combination with and without repetition

Binomial Coefficient

Binomial Coefficient:

The combination formula $\binom{n}{r}$ are often called *binomial coefficients* because they arise in the *binomial expansion*.

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

Example:

$$(x+y)^4 = x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Stirling’s Approximation

Stirling’s Approximation to n!

When n is large, a direct evaluation of $n!$ may be impractical. In such cases, use the Stirling’s approximation formula:

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

Where $e = 2.71828\ldots$, which is the base of natural logarithms.

The symbol ~ means that the ratio of the left side to the right side approaches 1 as $n \to \infty$

Reference

Notes, equations, and figures in the lecture are based on or taken from materials in the course textbook:

"Probability and Statistics", by Spiegel, Schiller and Srinivasan, ISBN 987-007-179557-9 (McGraw-Hill/Schaun's)