Homework 8 Solution

1) $$f(x) = \begin{cases} c e^{-3x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

a) $c \geq 0$

$$\int_0^\infty c e^{-3x} dx = -\frac{c}{3} e^{-3x} \Big|_0^\infty$$

$$= \left(-\frac{c}{3} e^{-\infty}\right) - \left(-\frac{c}{3} e^{0}\right)$$

$$= \frac{c}{3} = 1$$

$$\Rightarrow \boxed{c = 3}$$

b) $$P(1 < X < 2) = \int_1^2 3 e^{-3x} dx = \boxed{e^{-3} - e^{-6}}$$

c) $$P(X \geq 3) = \int_3^\infty 3 e^{-3x} dx = \boxed{e^{-9}}$$

d) $$P(X < 1) = \int_0^1 3 e^{-3x} dx = \boxed{1 - e^{-3}}$$

2) $$f(x) = \begin{cases} cx^2 & 1 \leqslant x \leqslant 2 \\ cx & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

a) $c \geqslant 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \iff \int_1^2 cx^2\,dx + \int_2^3 cx\,dx = 1$$

So, $c = \boxed{6/29}$

b) $$P(X > 2) = \int_2^{\infty} f(x)\,dx$$

$$= \int_2^3 \frac{6x}{29}\,dx = \frac{3x^2}{29}\Big|_2^3$$

$$= \boxed{\frac{15}{29}}$$

c) $$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_1^{3/2} \frac{6x^2}{29}\,dx = \frac{2x^3}{29}\Big|_1^{3/2}$$

$$= \boxed{\frac{19}{116}}$$

3) $F(x) = \begin{cases} c(1-x^2) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$$f(x) = \frac{dF(x)}{dx} = -2cx$$

$$\int_0^1 f(x)\,dx = 1 \iff \int_0^1 -2cx\,dx = 1$$

$$\iff -cx^2\Big|_0^1 = 1$$

$$\iff -c = 1$$

$$\iff c = -1$$

$$\boxed{\text{Yes, } F(x) \text{ can be a distribution function if } c = -1}$$

4) $$f(x,y) = \begin{cases} c(x^2+y^2) & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $c \geq 0$

$$\int_0^1 \int_0^1 c(x^2+y^2)\,dx\,dy = 1$$

$$\Leftrightarrow \int_0^1 \left(cx^2 y + c\frac{y^3}{3} \Big|_0^1 \right) dx = 1$$

$$\Leftrightarrow \int_0^1 \left(cx^2 + \frac{c}{3} \right) dx = 1$$

$$\Leftrightarrow \frac{cx^3}{3} + \frac{cx}{3} \Big|_0^1 = 1$$

$$\Leftrightarrow c = \boxed{\frac{3}{2}}$$

b) $$P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right) = \int_{x=0}^{x=1/2} \int_{y=\frac{1}{2}}^{y=1} \frac{3}{2}(x^2+y^2)\,dx\,dy$$

$$= \boxed{\frac{1}{4}}$$

c) $$f_1(x) = \int_{y=0}^{y=1} \frac{3}{2}(x^2+y^2)\,dy$$

$$= \frac{3}{2}x^2 y + \frac{y^3}{2} \Big|_0^1$$

$$= \frac{3}{2}x^2 + \frac{1}{2}$$

$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{1/4}^{3/4} f_1(x)\,dx = \int_{1/4}^{3/4} \left(\frac{3}{2}x^2 + \frac{1}{2}\right)dx$$

$$= \frac{x^3}{2} + \frac{x}{2} \Big|_{1/4}^{3/4} = \boxed{\frac{29}{64}}$$

d) $$f_1(y) = \int_{x=0}^{x=1} \frac{3}{2}(x^2+y^2)\,dx$$

$$= \frac{3}{2}y^2 + \frac{1}{2}$$

$$P\left(Y < \frac{1}{2}\right) = \int_0^{1/2} \left(\frac{3}{2}y^2 + \frac{1}{2}\right)dy$$

$$= \boxed{\frac{5}{16}}$$

e) $$P\left(X < \frac{1}{2}\right) = \int_0^{1/2} \left(\frac{3}{2}x^2 + \frac{1}{2}\right)dx = \frac{5}{16}$$

$$P\left(Y > \frac{1}{2}\right) = 1 - P\left(Y < \frac{1}{2}\right)$$

$$= 1 - \frac{5}{16} = \frac{11}{16}$$

$$P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right) = \frac{1}{4}$$

Because $P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right) = \frac{1}{4} \neq P\left(X < \frac{1}{2}\right) \cdot P\left(Y > \frac{1}{2}\right)$

$$= \frac{55}{256}$$

$\Rightarrow X$ & Y are dependent.

5) $$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$Y = X^2 \Rightarrow \psi(Y) = X = \sqrt{Y} = Y^{1/2}$$

$$x > 0 \Rightarrow y > 0$$

$$\psi'(Y) = (Y^{1/2})' = \frac{1}{2} Y^{-1/2}$$

$$f(y) = f[\psi(Y)] \, |\psi'(Y)|$$

$$= e^{-\sqrt{y}} \cdot \frac{1}{2} y^{-1/2} = \boxed{\frac{1}{2} \frac{e^{-\sqrt{y}}}{\sqrt{y}}} \quad y > 0$$

6) $f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$u = \frac{X}{Y} \qquad V = X + Y$$

$$u = \frac{X}{Y} = \frac{X+Y}{Y} - \frac{Y}{Y} = \frac{X+Y}{Y} - 1 = \frac{V}{Y} - 1$$

$$\Rightarrow Y = \Psi_2(u,v) = \frac{V}{u+1}$$

$$X = \Psi_1(u,v) = \frac{uV}{u+1}$$

$$\left.\begin{array}{l} x \geq 0 \Rightarrow \dfrac{uv}{u+1} \geq 0 \\ y \geq 0 \Rightarrow \dfrac{v}{u+1} \geq 0 \end{array}\right\} \Rightarrow u \geq 0, \ v \geq 0$$

$$\frac{\partial x}{\partial u} = \left(\frac{uv}{u+1}\right)' = v\left(\frac{(u+1) - u}{(u+1)^2}\right) = \frac{v}{(u+1)^2}$$

$$\frac{\partial x}{\partial v} = \left(\frac{uv}{u+1}\right)' = \frac{u}{u+1}$$

$$\frac{\partial y}{\partial u} = \left(\frac{v}{u+1}\right)' = v(-1)(u+1)^{-2} = \frac{-v}{(u+1)^2}$$

$$\frac{\partial y}{\partial v} = \left(\frac{v}{u+1}\right)' = \frac{1}{u+1}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{(u+1)^2} & \frac{u}{u+1} \\ -\frac{v}{(u+1)^2} & \frac{1}{u+1} \end{vmatrix}$$

$$= \frac{v}{(u+1)^3} + \frac{uv}{(u+1)^3} = \frac{v}{(u+1)^2}$$

$$g(u,v) = f\left[\Psi_1(u,v), \Psi_2(u,v)\right] \cdot |J|$$

$$= e^{-\left(\frac{uv}{u+1} + \frac{v}{u+1}\right)} \cdot \left|\frac{v}{(u+1)^2}\right|$$

$$= \boxed{\frac{v \cdot e^{-v}}{(u+1)^2}} \qquad u \geq 0, \quad v \geq 0$$

7) $f(x,y) = \begin{cases} x+y & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

a) $f(X \mid Y) = \dfrac{f(x,y)}{f_2(y)}$

$$f_1(x) = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$$

$$f_2(y) = y + \frac{1}{2}$$

$$f(x \mid y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & 0 \leq x \leq 1 \\ 0 & \text{other } x \end{cases}$$

b) $f(y \mid x) = \dfrac{f(x,y)}{f_1(x)}$

$$= \begin{cases} \dfrac{x+y}{x+1/2} & 0 \leq y \leq 1 \\ 0 & \text{other } y \end{cases}$$

8) $f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

a) $E(X) = \int_0^\infty x e^{-x}\, dx$ $\quad \begin{cases} u = x \\ v = -e^{-x} \end{cases}$

$$= -xe^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})\,dx$$

$$= \left[-xe^{-x} - (e^{-x})\right]\Big|_0^\infty$$

$$= \lim_{x\to\infty}\left[-xe^{-x} - e^{-x}\right] - \left[-0e^{-0} - e^{-0}\right]$$

$$= [0 - 0] - [-0 - 1]$$

$$= \boxed{1}$$

* Infinity isn't a number, it is a concept so treat it as a limit.

$$E(X^2) = \int_0^\infty x^2 e^{-x}\, dx = \boxed{2}$$

$$E(X-1)^2 = \int_0^\infty (X-1)^2 e^{-x}\, dx = \boxed{1}$$

b) $Var(X) = E(X^2) - \mu^2$

$= 2 - 1^2 = \boxed{1}$

$\sigma_X = \sqrt{1} = \boxed{1}$

9) $f(x) = \begin{cases} x/2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

a) $M_X(t) = \int_0^2 e^{tx} \cdot \frac{x}{2} \, dx$

$= \frac{1}{2} \int_0^2 x \cdot e^{tx} \, dx$

$= \frac{1}{2} \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^2$

$= \boxed{\frac{2te^{2t} - e^{2t} + 1}{2t^2}}$

b) $\mu_1 = \left. \frac{d M_X(t)}{dt} \right|_{t=0} = \boxed{\frac{4}{3}}$

$$\mu_2' = \frac{d^2 M_X(t)}{dt^2}\Big|_{t=0} = \boxed{2}$$

$$\mu_3' = \frac{d^3 M_X(t)}{dt^3}\Big|_{t=0} = \boxed{\frac{16}{5}}$$

$$\mu_4' = \frac{d^4 M_X(t)}{dt^4}\Big|_{t=0} = \boxed{\frac{16}{3}}$$

10) $$f(x,y) = \begin{cases} x+y & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $$f_1(x) = x + \frac{1}{2}$$

$$f_2(y) = y + \frac{1}{2}$$

$$E(X) = \int_0^1 x\left(x + \frac{1}{2}\right) dx = \frac{x^3}{3} + \frac{x^2}{4}\Big|_0^1 = \frac{7}{12}$$

$$E(X^2) = \int_0^1 x^2\left(x + \frac{1}{2}\right) dx = \frac{x^4}{4} + \frac{x^3}{6}\Big|_0^1 = \frac{5}{12}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$Var(Y) = Var(X) = \boxed{\frac{11}{144}}$$

b) $\sigma_x = \sigma_y = \sqrt{\frac{11}{144}} = \boxed{\frac{\sqrt{11}}{12}}$

c) $E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy$

$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3} \Big|_0^1 \right) dx$

$= \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3} \right) dx$

$= \frac{x^3}{6} + \frac{x^2}{6} \Big|_0^1 = \frac{1}{3}$

$\sigma_{XY} = E(XY) - E(X) \cdot E(Y)$
$= 1/3 - \frac{7}{12} \cdot \frac{7}{12} = \boxed{-\frac{1}{144}}$

d) $\rho = \frac{\sigma_{XY}}{\sigma_x \sigma_Y} = \frac{-1/144}{11/144} = \boxed{-\frac{1}{11}}$