

Homework 6 solution

$$1) \quad \begin{array}{c|ccc} x & -2 & 3 & 1 \\ \hline f(x) & 1/3 & 1/2 & 1/6 \end{array}$$

$$\mu = -2 \times \frac{1}{3} + 3 \times \frac{1}{2} + 1 \times \frac{1}{6} = 1$$

$$\begin{aligned} M_x(t) &= E[e^{t(x-\mu)}] = \sum e^{t(x-\mu)} f(x) \\ &= e^{-3t} \times \frac{1}{3} + e^{2t} \times \frac{1}{2} + e^{0t} \times \frac{1}{6} \\ &= \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} + \frac{1}{6} \end{aligned}$$

$$\begin{aligned} E[(X-\mu)] = M_1 &= \left. \frac{d}{dt} M_x(t) \right|_{t=0} = \left. -\frac{3}{3} e^{-3t} + \frac{2}{2} e^{2t} + 0 \right|_{t=0} \\ &= \boxed{0} \end{aligned}$$

$$\begin{aligned} E[(X-\mu)^2] = M_2 &= \left. \frac{d^2}{dt^2} M_x(t) \right|_{t=0} = \left. 3e^{-3t} + 2e^{2t} + 0 \right|_{t=0} \\ &= \boxed{5} \end{aligned}$$

$$\begin{aligned} E[(X-\mu)^3] = M_3 &= \left. \frac{d^3}{dt^3} M_x(t) \right|_{t=0} = \left. -9e^{-3t} + 4e^{2t} + 0 \right|_{t=0} \\ &= \boxed{-5} \end{aligned}$$

$$\begin{aligned} E[(X-\mu)^4] = M_4 &= \left. \frac{d^4}{dt^4} M_x(t) \right|_{t=0} = \left. 27e^{-3t} + 8e^{2t} + 0 \right|_{t=0} \\ &= \boxed{35} \end{aligned}$$

2) a)

$$\begin{aligned}
 E(X) &= \sum \sum x f(x, y) \\
 &= 0\left(\frac{1}{18} + \frac{1}{9} + \frac{1}{6}\right) + 1\left(\frac{1}{9} + \frac{1}{18} + \frac{1}{9}\right) + 2\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{18}\right) \\
 &= 0 + \frac{5}{18} + \frac{14}{18} \\
 &= \frac{19}{18}
 \end{aligned}$$

$$\begin{aligned}
 E(Y) &= \sum \sum y f(x, y) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum \sum x^2 f(x, y) \\
 &= \frac{33}{18}
 \end{aligned}$$

$$E(Y^2) = \frac{5}{3}$$

$$\begin{aligned}
 \text{Var}(X) &= \sigma_x^2 = E(X^2) - (E(X))^2 \\
 &= \frac{33}{18} - \left(\frac{19}{18}\right)^2 = \frac{233}{324}
 \end{aligned}$$

$$\text{Var}(Y) = \frac{5}{3} - 1^2 = \frac{2}{3}$$

$$b) \quad \sigma_x = \sqrt{\frac{233}{324}} = \frac{\sqrt{233}}{18}$$

$$\sigma_y = \sqrt{\frac{2}{3}}$$

$$c) E(XY) = \sum \sum xy f(x, y)$$

$$= 0 \times 0 \times \frac{1}{18} + 0 \times 1 \times \frac{1}{9} + 0 \times 2 \times \frac{1}{6} + 1 \times 0 \times \frac{1}{9} + 1 \times 1 \times \frac{1}{18} \\ + 1 \times 2 \times \frac{1}{9} + 2 \times 0 \times \frac{1}{6} + 2 \times 1 \times \frac{1}{6} + 2 \times 2 \times \frac{1}{18}$$

$$= \frac{5}{6}$$

$$\sigma_{xy} = E(XY) - E(X)E(Y)$$

$$= \frac{5}{6} - \frac{19}{18} \times 1$$

$$= -\frac{2}{9}$$

$$d) \rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{-2/9}{\sqrt{\frac{233}{324} \cdot \frac{2}{3}}} = -0.32$$

$$3) E(X) = 2 \quad E(Y) = 3 \quad E(XY) = 10$$

$$E(X^2) = 9 \quad E(Y^2) = 16$$

$$a) \sigma_{xy} = E(XY) - E(X)E(Y) = 10 - 2 \times 3$$

$$= 4$$

$$b) \quad \sigma_x^2 = E(X^2) - (E(X))^2 \\ = 9 - 2^2 = 5$$

$$\sigma_x = \sqrt{5}$$

$$\sigma_y^2 = E(Y^2) - (E(Y))^2 \\ = 16 - 9 = 7$$

$$\sigma_y = \sqrt{7}$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{4}{\sqrt{5 \cdot 7}} = \boxed{\frac{4}{\sqrt{35}}}$$

$$4) \quad \rho = -\frac{1}{4} \quad \text{Var}(X) = 3 \quad \text{Var}(Y) = 5$$

$$\sigma_x = \sqrt{3} \quad \sigma_y = \sqrt{5}$$

$$\Rightarrow \sigma_{xy} = \rho \times \sigma_x \times \sigma_y = -\frac{1}{4} \times \sqrt{3} \times \sqrt{5} \\ = \boxed{-\frac{\sqrt{15}}{4}}$$