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EE 381 HW 2

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1. A pair of dice loaded. The probability that a 4 appears on the first die is 2/7, and the probability that a 3 appears on the 2nd die is 2/7. Other outcomes for each die appear with probability 1/7th. What is the probability of 7 appearing as the sum of the numbers when two dice are rolled.

Given:

two dices are biased.

$S = \{1, 2, 3, 4, 5, 6\}$

//Independent Events:
If $P(B|A) = B$, then probability of B occuring is not affected by the occurence of A (or non-occurence), then we say A and B are independent events.
$P(A \wedge B) = P(A) * P(B)$, $P(A_j \wedge A_k) = P(A_j) * P(A_k)$,
and $P(A_1 \wedge A_2 \wedge A_3) = P(A_1) * P(A_2) * P(A_3)$

//Die 1

$1 = \{P(d1\ 1) + P(d1\ 2) + P(d1\ 3) + P(d1\ 4) + P(d1\ 5) + P(d1\ 6)\}$

$P(d1\ 4) = 2/7$ //Assign equal probabilities to everything besides 4 for die 1

$1 = \{P(d1\ 1) + P(d1\ 2) + P(d1\ 3) + \cancel{2/7} + P(d1\ 5) + P(d1\ 6)\}$

$-2/7 \qquad\qquad -\cancel{2/7}$

$1 - 2/7 = P(d1\ 1) + P(d1\ 2) + P(d1\ 3) + P(d1\ 5) + P(d1\ 6)$

$\frac{7}{7} - \frac{2}{7} = P(d1\ 1) + P(d1\ 2) + P(d1\ 3) + P(d1\ 5) + P(d1\ 6)$

$\frac{5}{7} = P(d1\ 1) = P(d1\ 2) = P(d1\ 3) + P(d1\ 5) + P(d1\ 6)$

$\frac{\frac{5}{7}}{5} = \frac{\cancel{5}\, P(d1\ 1)}{\cancel{5}}$; $P(d1\ 1) = \frac{\cancel{5}}{7} \cdot \frac{1}{\cancel{5}} = \frac{1}{7}$

$P(d1\ 1) = P(d1\ 2) = P(d1\ 3) = P(d1\ 5) = P(d1\ 6) = 1/7$

1. (continued!) $P(d2\ 3) = 2/7$ //Assign equal probabili... for die 2

$1 = \{P(d2\ 1)+P(d2\ 2)+P(d2\ 3)+P(d2\ 4)+P(d2\ 5)+P(d2\ 6)\}$

$1 = \{P(d2\ 1)+P(d2\ 2)+2/7+P(d2\ 4)+P(d2\ 5)+P(d2\ 6)\}$

$\frac{5}{7} = \{P(d2\ 1)+P(d2\ 2)+P(d2\ 4)+P(d2\ 5)+P(d2\ 6)\}$

$\frac{\frac{5}{7}}{5} = \frac{\cancel{5}P(d2\ 1)}{\cancel{5}}$; $P(d2\ 1) = \frac{8}{7} \cdot \frac{1}{8} = \frac{1}{7}$

$P(d2\ 1) = P(d2\ 2) = P(d2\ 4) = P(d2\ 5) = P(d2\ 6) = 1/7$

$\boxed{\text{1st parameter = 1st dice}}$, $\boxed{\text{2nd parameter = 2nd dice}}$

possible sevens $= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$P(\text{sum of } 7) = \{P(1,6)+P(2,5)+P(3,4)+P(4,3)+P(5,2)+P(6,1)\}$

//NOTE: Probability of a number on the 1st die dosen't affect the probability of a number on the 2nd die; thus, is an independent event.

$P(1,6) = P(1) \cdot P(6) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$

$P(2,5) = P(2) \cdot P(5) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$

$P(3,4) = P(3) \cdot P(4) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$

$P(4,3) = P(4) \cdot P(3) = \frac{2}{7} \cdot \frac{2}{7} = \frac{4}{49}$

$P(5,2) = P(5) \cdot P(2) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$

$P(6,1) = P(6, P(1) = \frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$

$$\boxed{P(\text{sum of } 7) = \frac{1}{49} + \frac{1}{49} + \frac{1}{49} + \frac{4}{49} + \frac{1}{49} + \frac{1}{49} = \frac{9}{49}}$$

...it string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive zeros, given that its first bit is a 0? (Assume that 0 bits and 1 are equally likely!)

// conditional probability:

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A \cap B) = \frac{|A \cap B|}{S}$

$P(A|B) = \frac{P(B \cap A)}{P(B)}$

Entire subset S
= {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}

$|S| = 16$

$A \cap B$ = there are two consecutive zeros AND the first bit is a 0.

{0000, 0001, 0010, 0011, 0100}

$|A \cap B| = 5$

solution:

$P(A|B) = \frac{P(B \cap A)}{P(B)} = \frac{|A \cap B| / |S|}{1/2}$ $\frac{5}{16} / \frac{1}{2} = \frac{5}{8\cancel{16}} \cdot \frac{\cancel{2}}{1} = \boxed{5/8}$ $\boxed{P(A|B) = 5/8}$

let A be the event where there are two consecutive 0s.

A = {0000, 0001, 0010, 0011, 0100, 1000, 1001, 1100}

$|A| = 8$; $P(A) = \frac{|A|}{|S|} = \frac{8}{16} = 1/2$

let B be the event where it's first bit is a 0

B = {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111}

$|B| = 8$; $P(B) = \frac{|B|}{|S|} = 8/16 = 1/2$

3. A fair die is tossed twice. Find the probability of getting a 4, 5, or 6 on the first toss and a 1, 2, 3, or 4 on the second toss.

sample space $S = \{1, 2, 3, 4, 5, 6\}$
let A be the 1st toss.
we will Assign equal probability; thus $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$

Using Thm 1.5: $A_1 \cup A_2 \cup \dots \cup A_n$ are mutually exclusive, then
$P(A) = P(A_1) + P(A_2) + \dots + P(A_n)$

$P(A) = P(4 \cup 5 \cup 6) = P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$P(A) = \frac{1}{2}$

let B be the 2nd toss.
we will assign equal probability; thus $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$

$P(B) = P(1 \cup 2 \cup 3 \cup 4) = P(1) + P(2) + P(3) + P(4)$
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$P(B) = 2/3$

// Independent Events: Rolling the 2nd dice does not effect the probability of dice 1.

$$\boxed{P(A \cap B) = P(A) * P(B) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}}$$

4. Let E be the event that a randomly generated bit string length four begins at 1 and let F be the event that this bit string contains an even number of 1's. Are E and F independent, if the bit strings of length four are equally likely.

//Independent Events:
If $P(B|A) = P(B)$; the probability is not affected

Bit Format: _ _ _ _

$S = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$

$|S| = 16$

$E = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$

$|E| = 8$ //NOTE: 0 is an even number

$P(E) = \frac{|E|}{|S|} = \frac{8}{16} = \frac{1}{2}$

$F = \{0000, 1011, 0101, 0110, 1001, 1010, 1100, 1111\}$

$|F| = 8$ $P(F) = \frac{|F|}{|S|} = \frac{8}{16} = \frac{1}{2}$

$E \wedge F = \{1001, 1010, 1100, 1111\}$

$|E \wedge F| = 4$

$P(E \wedge F) = \frac{|E \wedge F|}{|S|} = \frac{4}{16} = \frac{1}{4}$

$E \wedge F$ means that a random generated bit starts 1 and the bit strings has an even number of 1.

4. (continued!)

To check if Events E and F are independent, we will multiply the probability of events E and F to see if it is equal to the probability of the intersection of E and F [P(E ∧ F)].

$P(E) \cdot P(F) \stackrel{?}{=} P(E \wedge F)$

$\frac{1}{2} \cdot \frac{1}{2} \stackrel{?}{=} \frac{1}{4}$

$\frac{1}{4} \stackrel{\checkmark}{=} \frac{1}{4}$

$P(E \wedge F) = P(E)P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Yes, Events E and F are independent

1. Two cards are drawn from a well-shuffled ordinary deck of 52 cards.
Find probability that they are both aces if the first card is ...

Given: There are 13 ranks of cards
ranks = {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}
There are 4 suits; 4 aces in the deck
suits = {Spades, clubs, Diamonds, Hearts}
By assigning equally likelihood, each card has a $\frac{1}{52}$ chance

a) replaced

$P(\text{card 1 Ace}) = \frac{4}{52} = \frac{1}{13}$; However, by replacing the deck, the deck resets, thus we will have 52 cards again

$P(\text{card 2 Ace}) = \frac{4}{52} = \frac{1}{13}$

Solution:
$\boxed{P(\text{card 1 Ace})\,P(\text{card 2 Ace}) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}}$

5. (continued!)

b. not replaced

$P(\text{card 1 Ace}) = \frac{4}{52} = \frac{1}{13}$

After pulling an ace, w/o replacing the deck, the deck size becomes 51, so the probability of pulling an ace after the first pull is 3/51 b/c there are 3 aces left in the deck.

$P(\text{card 2 Ace}) = \frac{3}{51}$

solution:

$\boxed{P(\text{card 1 Ace})\,P(\text{card 2 Ace}) = \frac{1}{13} \cdot \frac{3}{51} = \frac{3}{663} = \frac{1}{221}}$

6. a) Box 1 contains 3 red and 2 blue marbles; yet Box 2 contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up heads, a marble is chosen from box 1. If tails, a marble is chosen from box 2.
Find the probability that the red marble is chosen.

Bayes' Thm: $P(A_k \mid A) = \dfrac{P(A_k)\,P(A \mid A_k)}{P(A)} = \dfrac{P(A_k)\,P(A \mid A_k)}{\sum_{j=1}^{n=1} P(A_j)\,P(A \mid A_j)}$

Given:

$P(\text{heads}) = 1/2$ heads = Box 1

$P(\text{tails}) = 1/2$ tails = Box 2

$\sum_{j=1}^{n=1} P(A_j)\,P(A \mid A_j) = P(\text{heads})\,P(\text{reds} \mid \text{Box1}) + P(\text{tails})\,P(\text{reds} \mid \text{Box2}) = P(\text{red})$

$\text{Box1} = \{3 \text{ red}, 2 \text{ blue}\} = 5$ total marbles $= |\text{Box1}|$

$\text{Box2} = \{2 \text{ red}, 8 \text{ blue}\} = 10$ total marbles $= |\text{Box2}|$

$P(\text{Box1 red}) = \dfrac{\text{red in box1}}{|\text{Box1}|} = \dfrac{3}{5} = P(\text{red} \mid \text{Box1})$ // red given that it's in box 1

$P(\text{Box2 red}) = \dfrac{\text{red in box2}}{|\text{Box2}|} = \dfrac{2}{10} = \dfrac{1}{5} = P(\text{red} \mid \text{Box 2})$ // red given

Applying bayes thm: // List all cases of pulling a red marble

solution:

$P(\text{red}) = P(\text{heads})\,P(\text{red} \mid \text{Box1}) + P(\text{tails})\,P(\text{red} \mid \text{Box2})$

$= \dfrac{1}{2} \cdot \dfrac{3}{5} + \dfrac{1}{2} \cdot \dfrac{1}{5} = \dfrac{3}{10} + \dfrac{1}{10} = \dfrac{4}{10} = \dfrac{2}{5} = 0.4$

6. b) Assume that one who tosses the coin does not reveal whether it has turned up heads or tails (so, the box from which a marble was chosen is not revealed), but does reveal that a red marble was chosen. What is the probability that Box 1 was chosen.

Given: heads = Box 1
tails = Box 2

$P(\text{heads}) = 1/2$

$P(\text{tails}) = 1/2$

Bayes' Thm:

$$P(A_k|A) = \frac{P(A_k)P(A|A_k)}{P(A)} = \frac{P(A_k)P(A|A_k)}{\sum_{j=1}^{m=1} P(A_j)P(A|A_k)}$$

Box 1 = {3 red, 2 blue} = |Box 1| = 5

Box 2 = {2 red, 8 blue} = |Box 2| = 10

$P(\text{red}|\text{Box1}) = \frac{\text{red in box1}}{|\text{Box 1}|} = \frac{3}{5}$

$P(\text{red}|\text{Box2}) = \frac{\text{red in box2}}{|\text{Box 2}|} = \frac{2}{10} = \frac{1}{5}$

// we obtained the probability of a red marble was chosen from part A

$P(\text{red}) = P(\text{red}|\text{Box})P(\text{heads}) + P(\text{red}|\text{Box2})P(\text{tails})$

$= \frac{3}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2} = \frac{3}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

// we are finding the probability of Box1 was chosen given that a red marble was chosen

$$P(\text{Box1}|\text{red}) = \frac{P(\text{red}|\text{Box1}) \cdot P(\text{Box 1})}{P(\text{red})} = \frac{3/5 \cdot 1/2}{2/5}$$

$$= \frac{\frac{3}{10}}{\frac{2}{5}} = \frac{3}{\cancel{10}_{2}} \cdot \frac{\cancel{5}}{2} = \frac{3}{4}$$

$$\boxed{P(\text{Box1}|\text{red}) = \frac{3}{4}}$$

7. How many different 3-letter initials are there that begin with an A?

a_1 a_2 a_3

__ __ __

Given: The 1st initial is an A; thus, having 2 spaces left for any initial.

a_1 a_2 a_3

A __ __

letters $= \{A-Z\} = 26$

By using the product rule: there are $n_1 \cdot n_2$ ways to do it

So, $26 \cdot 26 = 26^2$ = 576 different ways to initialize 3-letters initials given that the 1st initial begins w/ an A.

8. How many strings of three decimal digits

a) Do not contain same digit 3 times?

Digits = {0 - 9} = 10

Format: $\underline{a_1}$ $\underline{a_2}$ $\underline{a_3}$

Total num 3 decimal digits = 10 · 10 · 10 = 1000

same Digit 3 times = {000, 111, 222, 333, 444, 555, 666, 777, 888, 999}

|same Digit 3 times| = 10

// subtraction Rule: $n_1 + n_2$ - the common number of ways to the task that are common to the different ways.

= 1000 - |sameDigits3Times| = 1000 - 10

= $\boxed{\text{990 ways not containing same digits 3 times}}$

b) Begin with an odd digit?

~~oddDigits = {100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120~~

Format: $\underline{a_1}$ $\underline{a_2}$ $\underline{a_3}$

Given: 1st digit is odd

Format: $\underline{\text{odd}}$ $\underline{a_2}$ $\underline{a_3}$

$\boxed{\text{possible numbers} = 5 \cdot 10 \cdot 10 = 500 \text{ numbers that start with an odd digit.}}$

Digits = {0 - 9} = 10

odds = {1, 3, 5, 7, 9} = |odds| = 5

Product Rule: $n_1 \cdot n_2$ ways

8. c) Have exactly two digits that are 4's?

$\underline{9_1}$ $\underline{9_2}$ $\underline{9_3}$

digits $= \{0-9\} = 10$

possible cases $= \{ \underline{4}\,\underline{4}\,\underline{d}, \underline{4}\,\underline{d}\,\underline{4}, \underline{d}\,\underline{4}\,\underline{4} \} = |\text{possible cases}| = 3$

$(\text{digits} \cap 4) = 1$

// we subtract 1 from the set b/c we don't want 3 copies of #4. using subtraction Rule: $10 - 1 = 9$

we multiply the number of possible cases by all possible digits besides #4

$\boxed{= 3 \times 9 = 27 \text{ possible ways of having exactly two digits that are 4's.}}$

7) How many license plates can be made using either two uppercase english letters, followed by, four digits OR two digits, followed by, four uppercase letters

let A = 2 uppercase followed by 4 digits

let B = 2 digits followed by 4 uppercase letters

digits = {0-9} = 10

letters = {A-Z} = 26

$A = \underbrace{\_\ \_}_{\text{2 uppercase letters}}^{a1\ a2}\ \underbrace{\_\ \_\ \_\ \_}_{\text{4 digits}}^{d1\ d2\ d3\ d4}$

//Product rule

$B = \underbrace{\_\ \_}_{\text{2 digits}}^{d1\ d2}\ \underbrace{\_\ \_\ \_\ \_}_{\text{4 uppercase letters}}^{a1\ a2\ a3\ a4}$

//Product rule

$A = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$
$= 6,760,000$

$B = 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 45,697,600$

//Sum Rule:

$|A| + |B| = 6,760,000 + 45,697,600 =$

$\boxed{52,457,600}$ ways to create license plates by either having 2 uppercase letters followed by 4 digits or having 2 digits followed by 4 uppercase letters.