EE 381 Probability & Statistic with Applications to Computing (Fall 2020)

Lecture 2
Basic Concepts of Probability (p2)

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Chapter 1’s topics

- Discrete Sample Spaces
- Important Theorems on Probability
- **Conditional Probability**
- **Independence**
- **Bayes’ Rule**
- Combinatorial Analysis

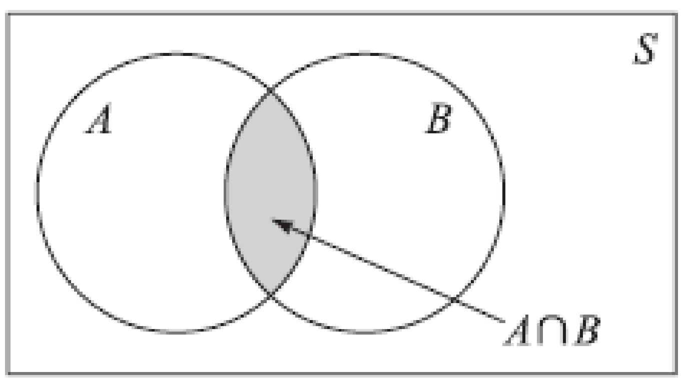
Conditional Probability

- Let A and B be 2 events such that $P(A) > 0$. Probability of B given that A has occurred, denoted by $P(B|A)$
- A is known to have occurred → becomes the new sample space replacing the original S. Then:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{or } P(A \cap B) = P(A) * P(B|A)$$

where $P(A \cap B)$: the probability that both A and B occur.

- $P(B|A)$ is called the *conditional probability* of B given A. In other words, it means the probability that B will occur given that A has occurred.



Venn diagram of events A and B.

Conditional Probability

Example: Find the probability that a single toss of a dice will result in a number less that 4 if

a) No other information is given

b) It is given that the toss resulted in an odd number.

Solution:

Assuming equal probabilities for the sample points.

a) Let B denote the event {less than 4}. B is the union of the events 1,2,3 turning up. Applying theorem 1-5:

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

b) Let A be the event {odd number}, we see that $P(A) = \frac{3}{6} = \frac{1}{2}$. Also $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$. Then:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Hence, the added knowledge that the toss results in an odd number raises the probability from $\frac{1}{6}$ to $\frac{2}{3}$.

Theorem on Conditional Probability

- **Theorem 1-9**: For any three events A_1, A_2, A_3,we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1\,)P(A_2|\,A_1)P(A_3|A_1 \cap A_2)$$

Probability that all A_1, A_2, A_3 occur

Probability that A_1 occurs

Probability that A_2 occurs given that A_1 has occurred

Probability that A_3 occurs given that both A_1&A_2has occurred

This theorem can be generalized to *n* events.

- **Theorem 1-10**: If an event A must result in one of the mutually exclusive $A_1, A_2,\ldots, A_n$, then

$$P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + \cdots + P(A_n)P(A|A_n)$$

Independent Events

- If $P(B|A)=P(B)$, i.e., the probability of B occurring is not affected by the occurrence or non-occurrence of A, then we say that A and B are independent events.

This is equivalent to

$$P(A \cap B) = P(A)P(B) \qquad (*)$$

- Conversely, if (*) holds, then A and B are independent events.

Independent Events

- We say that three events A_1, A_2, A_3 are independent if they are pairwise independent:

$$P(A_j \cap A_k) = P(A_j)P(A_k) \qquad j \neq k \qquad \text{where } j, k = 1, 2, 3 \qquad (*)$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \qquad (**)$$

- Note that neither (*) or (**) is by itself sufficient.

Bayes’ Theorem or Rule

- Suppose that $A_1, A_2,..., A_n$ are mutually exclusive events whose union is the sample space S, i.e., one of the events must occur. Then if A is any event, we have the following important theorem:

$$P(A_k|A) = \frac{P(A_k)P(A|A_k)}{\sum_{j=1}^{n} P\left(A_j\right)P(A|A_j)}$$

- This is called **Bayes’ theorem or rule**, or *theorem on the probability of causes*.
- This enable us to find the probabilities of the various events $A_1, A_2,..., A_n$ that can *cause A to occur*.

Bayes' Theorem or Rule

Bayes' theorem example 1: Given the below information:

	Bin 1	Bin 2	Bin 3	Bin 4	Bin 5	Bin 6	Subtotal
10 ohm	500	0	200	800	1200	1000	3700
100 ohm	300	400	600	200	800	0	2300
1000 ohm	200	600	200	600	0	1000	2600
Subtotal	1000	1000	1000	1600	2000	2000	8600

Assuming bin marginal probability is equally likely, meaning P(bin#)=1/6

a) What is the probability **P(B)** of selecting a 10 Ohm resistor from a random bin?

$$\mathrm{P}\left(10\Omega \mid Bin1\right)=\frac{500}{1000} \qquad \mathrm{P}\left(10\Omega \mid Bin2\right)=\frac{0}{1000} \qquad \mathrm{P}\left(10\Omega \mid Bin3\right)=\frac{200}{1000}$$

$$\mathrm{P}\left(10\Omega \mid Bin4\right)=\frac{800}{1600} \qquad \mathrm{P}\left(10\Omega \mid Bin5\right)=\frac{1200}{2000} \qquad \mathrm{P}\left(10\Omega \mid Bin6\right)=\frac{1000}{2000}$$

$$\mathrm{P}\left(B\right)=\mathrm{P}\left(B \mid A_1\right)\cdot \mathrm{P}\left(A_1\right)+\mathrm{P}\left(B \mid A_2\right)\cdot \mathrm{P}\left(A_2\right)+\cdots+\mathrm{P}\left(B \mid A_n\right)\cdot \mathrm{P}\left(A_n\right)$$

Theorem 1-10

$$\mathrm{P}\left(B\right)=\frac{500}{1000}\cdot\frac{1}{6}+\frac{0}{1000}\cdot\frac{1}{6}+\frac{200}{1000}\cdot\frac{1}{6}+\frac{800}{1600}\cdot\frac{1}{6}+\frac{1200}{2000}\cdot\frac{1}{6}+\frac{1000}{2000}\cdot\frac{1}{6}$$

$$\mathrm{P}\left(B\right)=\frac{5}{10}\cdot\frac{1}{6}+\frac{0}{10}\cdot\frac{1}{6}+\frac{2}{10}\cdot\frac{1}{6}+\frac{5}{10}\cdot\frac{1}{6}+\frac{6}{10}\cdot\frac{1}{6}+\frac{5}{10}\cdot\frac{1}{6}=\frac{23}{10}\cdot\frac{1}{6}=0.3833$$

Bayes' Theorem or Rule

Bayes' theorem example 1 (ctn.):

	Bin 1	Bin 2	Bin 3	Bin 4	Bin 5	Bin 6	Subtotal
10 ohm	500	0	200	800	1200	1000	3700
100 ohm	300	400	600	200	800	0	2300
1000 ohm	200	600	200	600	0	1000	2600
Subtotal	1000	1000	1000	1600	2000	2000	8600

b) Assuming a 10 Ohm resistor is selected, what is the probability it came from bin 3?

Recall Bayes' theorem

$$\mathrm{P}\left(A_i \mid B\right) = \frac{\mathrm{P}\left(B \mid A_i\right) \cdot \mathrm{P}\left(A_i\right)}{\mathrm{P}\left(B \mid A_1\right) \cdot \mathrm{P}\left(A_1\right) + \mathrm{P}\left(B \mid A_2\right) \cdot \mathrm{P}\left(A_2\right) + \cdots + \mathrm{P}\left(B \mid A_n\right) \cdot \mathrm{P}\left(A_n\right)}$$

$$\mathrm{P}\left(Bin3 \mid 10\Omega\right) = \frac{\mathrm{P}\left(10\Omega \mid Bin3\right) \cdot \mathrm{P}\left(Bin3\right)}{\mathrm{P}\left(10\Omega \mid Bin1\right) \cdot \mathrm{P}\left(Bin1\right) + \cdots + \mathrm{P}\left(10\Omega \mid Bin6\right) \cdot \mathrm{P}\left(Bin6\right)}$$

$$\mathrm{P}\left(Bin3 \mid 10\Omega\right) = \frac{{}^{2}\!/_{10} \cdot {}^{1}\!/_{6}}{0.3833} = 0.08696$$

Bayes’ Theorem or Rule

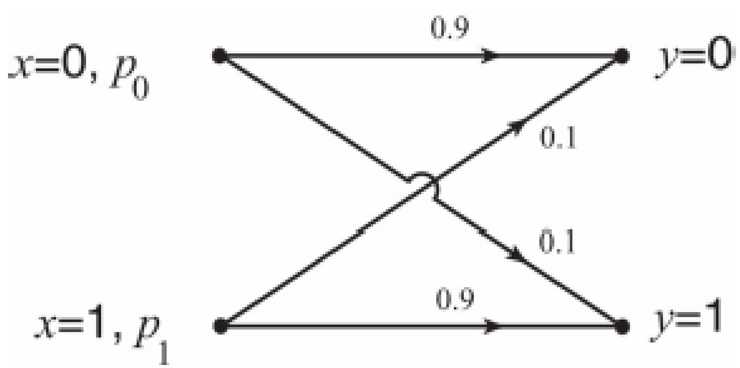
Bayes’ theorem example 2:

A digital communication system sends a sequence of 0 and 1, each of which are received at the other end of a link. Assume that the probability that 0 is received correctly is 0.9, and that a 1 is received correctly is 0.9. Alternately, the probability that a 0 or 1 is not received correctly is 0.1 (the cross-over probability, β). Within the sequence, the probability that a 0 is sent is 60% and that a 1 is sent is 40%. (S means Send, and R means Receive)

$$\Pr(S_0) = 0.60 \qquad \Pr(S_1) = 0.40$$

$$\Pr(R_0 \mid S_0) = 0.90 = 1 - \beta \qquad \Pr(R_1 \mid S_1) = 0.90 = 1 - \beta$$

$$\Pr(R_1 \mid S_0) = 0.10 = \beta \qquad \Pr(R_0 \mid S_1) = 0.10 = \beta$$



Bayes' Theorem or Rule

Bayes' theorem example 2 (ctn.):

a) What is the probability that a "0" is received?

Recall theorem 1-10:

$$\Pr(B) = \Pr(B \mid A_1) \cdot \Pr(A_1) + \Pr(B \mid A_2) \cdot \Pr(A_2) + \cdots + \Pr(B \mid A_n) \cdot \Pr(A_n)$$

$$\Pr(R_0) = \Pr(R_0 \mid S_0) \cdot \Pr(S_0) + \Pr(R_0 \mid S_1) \cdot \Pr(S_1)$$

$$\Pr(R_0) = 0.90 \cdot 0.60 + 0.10 \cdot 0.40$$

$$\Pr(R_0) = 0.54 + 0.04 = 0.58$$

b) What is the probability that a "1" is received?

$$\Pr(R_1) = \Pr(R_1 \mid S_0) \cdot \Pr(S_0) + \Pr(R_1 \mid S_1) \cdot \Pr(S_1)$$

$$\Pr(R_1) = 0.10 \cdot 0.60 + 0.90 \cdot 0.40$$

$$\Pr(R_1) = 0.06 + 0.36 = 0.42$$

Bayes' Theorem or Rule

Bayes' theorem example 2 (ctn.):

c) What is the probability that a received "0" was transmitted as a "0"?

Recall Bayes' theorem:

$$\mathrm{P}\left(A_i \mid B\right)=\frac{\mathrm{P}\left(B \mid A_i\right)\cdot \mathrm{P}\left(A_i\right)}{\mathrm{P}\left(B \mid A_1\right)\cdot \mathrm{P}\left(A_1\right)+\mathrm{P}\left(B \mid A_2\right)\cdot \mathrm{P}\left(A_2\right)+\cdots+\mathrm{P}\left(B \mid A_n\right)\cdot \mathrm{P}\left(A_n\right)}$$

$$\Pr\left(S_0 \mid R_0\right)=\frac{\Pr\left(R_0 \mid S_0\right)\cdot \Pr\left(S_0\right)}{\Pr\left(R_0\right)}=\frac{\Pr\left(R_0 \mid S_0\right)\cdot \Pr\left(S_0\right)}{\Pr\left(R_0 \mid S_0\right)\cdot \Pr\left(S_0\right)+\Pr\left(R_0 \mid S_1\right)\cdot \Pr\left(S_1\right)}$$

$\Pr\left(S_0\right)=0.60$ $\Pr\left(S_1\right)=0.40$ $\Pr\left(R_0 \mid S_0\right)=0.90=1-\beta$

$\Pr\left(R_1 \mid S_1\right)=0.90=1-\beta$ $\Pr\left(R_1 \mid S_0\right)=0.10=\beta$ $\Pr\left(R_0 \mid S_1\right)=0.10=\beta$

$$\Pr\left(S_0 \mid R_0\right)=\frac{\Pr\left(R_0 \mid S_0\right)\cdot \Pr\left(S_0\right)}{0.58}=\frac{0.90\cdot 0.60}{0.58}=\frac{0.54}{0.58}=0.931$$

d) What is the probability that a "1" is received?

$$\Pr\left(S_1 \mid R_1\right)=\frac{\Pr\left(R_1 \mid S_1\right)\cdot \Pr\left(S_1\right)}{\Pr\left(R_1\right)}=\frac{\Pr\left(R_1 \mid S_1\right)\cdot \Pr\left(S_1\right)}{\Pr\left(R_1 \mid S_0\right)\cdot \Pr\left(S_0\right)+\Pr\left(R_1 \mid S_1\right)\cdot \Pr\left(S_1\right)}$$

$$\Pr\left(S_1 \mid R_1\right)=\frac{\Pr\left(R_1 \mid S_1\right)\cdot \Pr\left(S_1\right)}{0.42}=\frac{0.90\cdot 0.40}{0.42}=\frac{0.36}{0.42}=0.857$$

Combinatorial Analysis

- In some cases, number of sample points in the sample space is small, and easy to count.
- However, there are cases that counting becomes a practical impossibility.
- In such cases, we need to use *combinatorial analysis*, or also called a *sophisticated way of counting*.
- Counting methods:
  - ✓Tree diagrams
  - ✓Permutations
  - ✓Combinations
  - ✓Binomial Coefficient
  - ✓Stirling's Approximation

Combinatorial Analysis

Basic counting methods:

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule

Basic Counting Methods

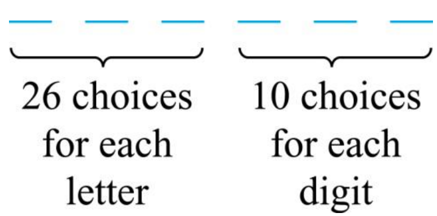
The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task **and** n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example 1: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

Example 2: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule, there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 =$ 17,576,000 different possible license plates.



Basic Counting Methods

The Sum Rule: If a task can be done either in one of n_1 ways **or** in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 1: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick a representative.

Example 2: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the sum and product rule:

$$26 + 26 \cdot 10 = 286$$

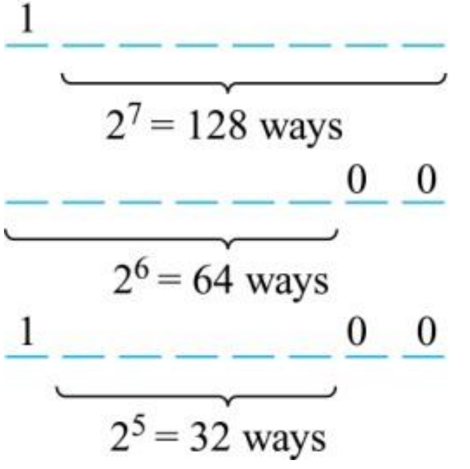
Basic Counting Methods

The Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

- Number of bit strings of length eight and start with 1: $2^7 = 128$
- Number of bit strings of length eight and end start with bits 00: $2^6 = 64$
- Number of bit strings of length eight, start with start with a 1 bit and end with bits 00 : $2^5 = 32$



Hence, the number is $128 + 64 - 32 = 160$.

Basic Counting Methods

The Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Example: How many ways are there to arrange 6 blocks (4 reds and 2 whites)?

Total possible ways: 6!

But, switching between red blocks or white block is meaningless.

Total ways of arranging 4 red blocks: 4!

Total ways of arranging 2 white blocks: 2!

Hence, total ways of arranging 6 blocks (4 reds, 2 whites) is:

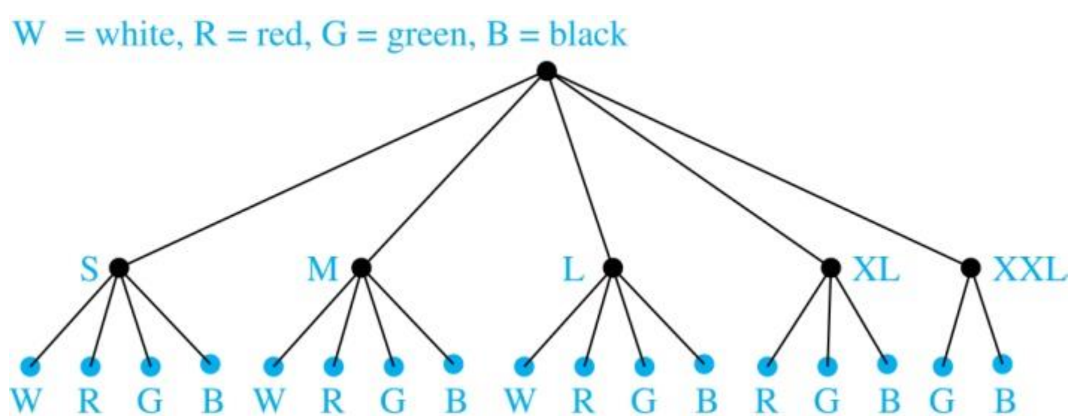
$$\frac{6!}{4! \times 2!} = 15$$

Basic Counting Methods

Tree Diagrams: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "CECS CSULB" T-shirts come in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus bookstore needs to stock to have one of each size and color available?

Solution:



The store must stock 17 T-shirts.

Reference

Notes, equations, and figures in the lecture are based on or taken from materials in the course textbook:

“Probability and Statistics”, by Spiegel, Schiller and Srinivasan, ISBN 987-007-179557-9 (McGraw-Hill/Schaun’s)