Homework 7 Solution

1) a) $P(3 \text{ heads}) = \binom{3}{3} \cdot \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = \boxed{\frac{1}{8}}$

b) $P(2 \text{ tails } \& \text{ } 1 \text{ head}) = \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \boxed{\frac{3}{8}}$

c) $P(\text{at least } 1 \text{ head}) = 1 - P(\text{no head})$

$$= 1 - \binom{3}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3$$

$$= \boxed{7/8}$$

d) $P(\text{not more than } 1 \text{ tail}) = P(0 \text{ tail}) + P(1 \text{ tail})$

$$= \binom{3}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 + \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

2) Prob. of defective bolt : $p = 0.2$

Prob. of non-defective bolt : $q = 1 - 0.2 = 0.8$

X : no. of defective bolts

$n = 4$

a) $P(X=1) = \binom{4}{1}(0.2)^1(0.8)^3 = \boxed{0.4096}$

b) $P(X=0) = \binom{4}{0}(0.2)^0(0.8)^4 = \boxed{0.4096}$

c) $P(X<2) = P(X=0) + P(X=1)$

$= 0.4096 + 0.4096$

$= \boxed{0.8192}$

3) $b=6$, $r=4$, $n=5$, $x=3$

$$P(X=3) = \frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = \boxed{\frac{10}{21}}$$

4) a) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - (P(X=0) + P(X=1))$

$= 1 - \left[\binom{6}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5\right]$

$= \boxed{0.890625}$

b) $P(X < 4) = P(X=0) + P(X=1) + P(X=2)$
$+ P(X=3)$

$$= \binom{6}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$$

$$= 0.015625 + 0.09375 + 0.234375 + 0.3125$$

$$= \boxed{0.65625}$$

5) $p = q = \frac{1}{2}$ (Equal probs. for boys & girls)

a) $P(X=3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 0.3125$

There are $0.3125 \times 800 = \boxed{250}$ families with 3 boys.

b) $P(X=0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 0.03125$

There are $0.03125 \times 800 = \boxed{25}$ families with 5 girls.

c) $P(X=2 \text{ or } X=3) = P(X=2) + P(X=3)$

$= \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

$= 0.3125 + 0.3125$

$= 0.625$

There are $\boxed{500}$ families with either 2 or 3 boys.

6) $P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= \binom{10}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4 + \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3$

$+ \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1$

$+ \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0$

$$= \left(\frac{1}{2}\right)^{10} \times \big(C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10) \big)$$

$$= \boxed{0.377}$$

7)

$$P(X_1 = 8, X_2 = 2) = \frac{10!}{8!\,2!}(0.4)^8(0.1)^2$$

$$= \boxed{0.00029}$$

8) a)

$$\frac{C(40,10) \times C(20,10)}{C(60,20)} = \boxed{0.0374}$$

b)

$$\frac{C(40,0) \cdot C(20,20) + C(40,1)\,C(20,19) + C(40,2)\,C(20,18)}{C(60,20)}$$

$$= \boxed{3.5 \times 10^{-11}}$$

9) Hypergeometric distribution

$$\begin{cases} P(X=x) = \dfrac{\binom{b}{x}\binom{r}{n-x}}{\binom{b+r}{n}} \\ \mu = \dfrac{nb}{b+r} \qquad \sigma^2 = \dfrac{nbr(b+r-n)}{(b+r)^2(b+r-1)} \end{cases} \qquad (1)$$

$$p = \frac{b}{b+r} = \frac{b}{N} \Rightarrow b = Np$$

$$q = \frac{r}{b+r} = \frac{r}{N} \Rightarrow r = Nq$$

where $N = b + r$

$$(1) \Rightarrow \begin{cases} P(X=x) = \dfrac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}} \\ \mu = np \qquad \sigma^2 = \dfrac{npq(N-n)}{N-1} \end{cases}$$

If N is a very large number:

$$\binom{Np}{x} = \frac{(Np)!}{(Np-x)!\,x!} = \frac{Np.(Np-1)\ldots.(Np-x+1)}{x!}$$

$$\simeq \frac{(Np)^x}{x!}$$

$$\binom{Nq}{n-x} = \frac{(Nq)^{n-x}}{n-x!}$$

$$\binom{N}{n} = \frac{N^{n}}{n!}$$

$$P(X=x) = \frac{\dfrac{(Np)^{x}}{x!} \cdot \dfrac{(Nq)^{n-x}}{(n-x)!}}{\dfrac{N^{n}}{n!}}$$

$$= \frac{n!}{(n-x)!\,x!} \cdot \frac{N^{x} \cdot p^{x} \cdot N^{n-x} \cdot q^{n-x}}{N^{n}}$$

$$= \boxed{\frac{n!}{(n-x)!\,x!} \cdot p^{x} \cdot q^{n-x}}$$

$$\mu = \boxed{np} \qquad \sigma^{2} = \frac{npq(N-n)}{N-1} = \boxed{npq}$$

$\Rightarrow$ identical with binomial distribution!!