EE 381 Probability & Statistic with Applications to Computing (Fall 2020)

Lecture 4
Discrete Random Variables (p1)

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Discrete Random Variable’s topics

- **Discrete probability distributions**
- **Joint distributions**
- Mathematical expectation
- Variance, Standardized random variables
- Covariance & correlation
- Special probability distributions

Random Variables

- Basically, a random variable, usually denoted by X or Y, is used to represent the value of the outcome of a certain experiment.
- Recall: the sample space of the experiment is the set of all possible outcomes.
- In general, a random variable has some specified physical, geometrical, or other significance.
- 2 types of random variables:
  - ✓Discrete random variable: If its sample space is finite or countably infinite.
  - ✓Non-discrete (continuous) random variable: If its sample space is non-countably infinite number.

Random Variables

- **Example**: Suppose that a coin is tossed twice so that the sample space is $S = \{HH, HT, TH, TT\}$. Let X represents the number of heads that can come up.

Sample Point	HH	HT	TH	TT
X	2	1	1	0

- X is a random variable. $X=\{0, 1, 2\}$.
- Many other random variables could also be defined on this sample space. For example, the square of the number of heads, or number of heads minus the number of tails, etc.

Discrete Probability Distributions

Let X be a discrete random variable, and suppose that the possible values that it can assume are given by $x_1, x_2, x_3, \ldots$, arranged in some order. The probability of these assumed values are given by:

$$P(X = x_k) = f(x_k) \qquad k = 1, 2, \ldots \qquad (*)$$

The probability function, also referred to as *probability distribution*, is given by:

$$P(X = x) = f(x) \qquad (**)$$

For $x = x_k$, (**) reduces to (*), while for other values of *x*, *f(x)=0*.

In general, *f(x)* is a probability function if:

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$

Where the sum in (**) is taken over all possible values of *x*.

Discrete Probability Distributions

Example: Find the probability function (probability distribution) corresponding to the random variable X.

Sample Point	HH	HT	TH	TT
X	2	1	1	0

Assuming that the coin is fair, we have:

$$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$$

Then,

$$\mathrm{P}(\mathrm{X} = 0) = P(TT) = \frac{1}{4}$$

$$\mathrm{P}(\mathrm{X} = 1) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4}$$

$$\mathrm{P}(\mathrm{X} = 2) = P(HH) = \frac{1}{4}$$

x	0	1	2
$f(x)$	1/4	1/2	1/4

Probability distribution table

Distribution Functions for Random Variables

The *cumulative distribution function* (CDF), or briefly the *distribution function*, for a random variable X is defined by:

$$F(x) = P(X \leq x)$$

Where *x* is any real number, i.e., $-\infty \leq x \leq \infty$

The distribution function *F(x)* has the following properties:

1. *F(x)* is non-decreasing as *x* increasing [i.e., $\mathrm{F}(x) \leq F(y)$ if $\mathrm{x} \leq y$].
2. *F(−∞)=0* and *F(∞)=1*
3. *F(x)* is continuous from the right [i.e., $\lim_{h \to o^+} F(x+h) = F(x)$ for all *x*].

Distribution Functions for Discrete Random Variables

The distribution function for a discrete random variable X can be obtained from its probability function by noting that, for all x in $(-\infty, \infty)$,

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

Where the sum is taken over all values u taken on by X for which $u \leq x$.

If X takes on only a finite number of values $x_1, x_2, \dots, x_n$, then the distribution function is given by:

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \leq x < x_2 \\ f(x_1) + f(x_2) & x_2 \leq x < x_3 \\ \vdots & \vdots \\ f(x_1) + \cdots + f(x_n) & x_n \leq x < \infty \end{cases}$$

Distribution Functions for Discrete Random Variables

Example 1: Given the following information:

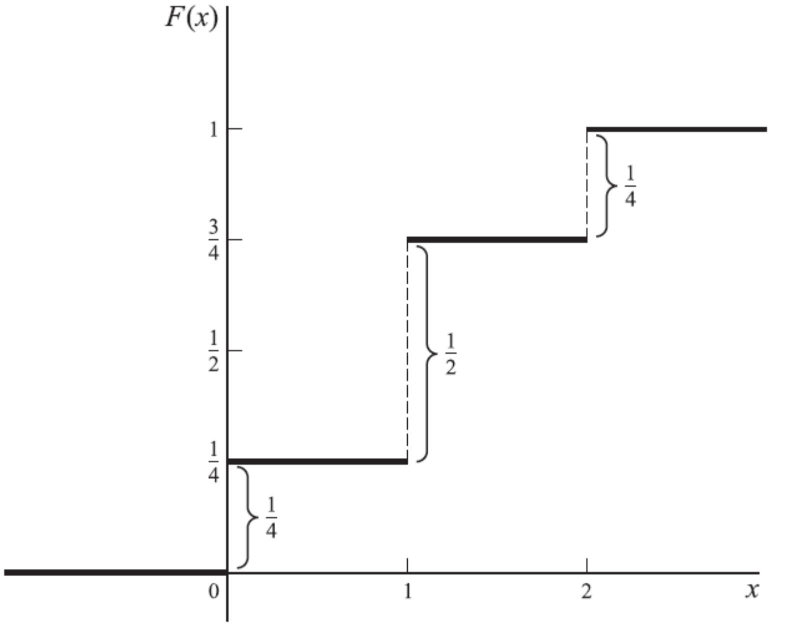
Sample Point	HH	HT	TH	TT
X	2	1	1	0

x	0	1	2
$f(x)$	1/4	1/2	1/4

a) Find the distribution function for the random variable X?

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & 2 \leq x < \infty \end{cases}$$

Distribution Functions for Discrete Random Variables



Example 1 (ctn.):

b) Obtain the distribution function's graph.

Note:

- The magnitudes of the jumps at 0, 1, 2 are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, which are precisely the probability function of X → we can obtain the probability function from the distribution function.

- It is also called *staircase function* or *step function*. The distribution function is continuous from the right at 0 , 1, 2. For example, the value of the function at an integer is obtained from the higher step (value at 1 is $\frac{3}{4}$).
- As proceed from left to right, the distribution function either remains the same or increase, taking on values from 0 to 1 → called *monotonically increasing function*.

Distribution Functions for Discrete Random Variables

Example 2: Suppose a pair of fair dice are to be tossed, and let the random variable X denote the sum of the points.

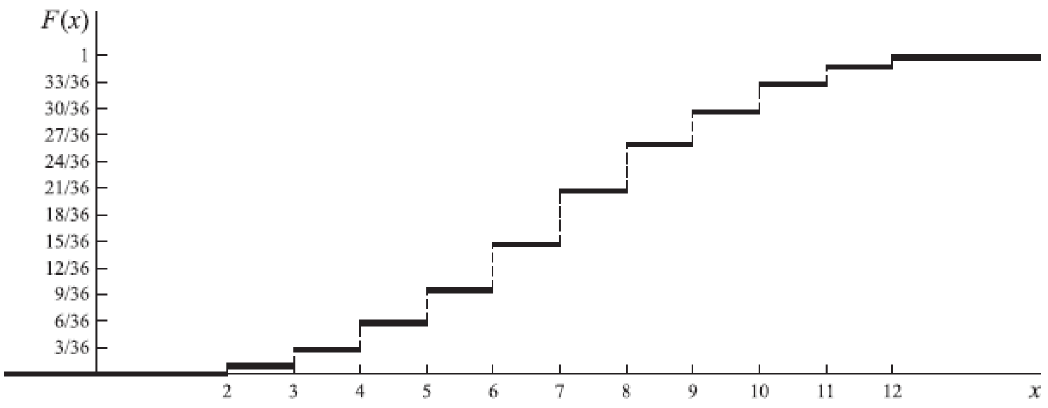
a) Obtain the probability distribution for X

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

b) Find the distribution function for X?

We have $F(x) = P(X \le x) = \sum_{u \le x} f(u)$.

$$F(x) = \begin{cases} 0 & -\infty < x < 2 \\ 1/36 & 2 \le x < 3 \\ 3/36 & 3 \le x < 4 \\ 6/36 & 4 \le x < 5 \\ \vdots & \vdots \\ 35/36 & 11 \le x < 12 \\ 1 & 12 \le x < \infty \end{cases}$$



Joint Distributions

- The discrete probability distribution can be generalized to 2 ore more random variables.
- If X and Y are two discrete random variables, we define the *joint probability function* of X and Y by:

$$P(X = x, Y = y) = f(x, y)$$

where 1. $f(x, y) \geq 0$

2. $\sum_x \sum_y f(x, y) = 1$

i.e., the sum over all values of *x* and *y* is 1.

- Suppose that X can assume any one of m values $x_1, x_2, \ldots, x_m$ and Y can assume any one of n values $y_1, y_2, \ldots, y_n$. Then the probability of the event that $X = x_j$ and $Y = y_k$ is given by

$$P\left(X = x_j, Y = y_k\right) = f\left(x_j, y_k\right)$$

Joint Distributions

- The probability that $X = x_j$ is obtained by adding all entries in the row corresponding to x_j and is given by:

$$P(X = x_j\ ) = f_1(x_j) = \sum_{k=1}^{n} f\,(x_j, y_k)$$

- Similarly, $P(Y = y_k\ ) = f_2(y_k) = \sum_{j=1}^{m} f\,(x_j, y_k)$

$X \backslash Y$	y_1	y_2	...	y_n	Totals ↓
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	...	$f(x_1, y_n)$	$f_1(x_1)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	...	$f(x_2, y_n)$	$f_1(x_2)$
⋮	⋮	⋮		⋮	⋮
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	...	$f(x_m, y_n)$	$f_1(x_m)$
Totals →	$f_2(y_1)$	$f_2(y_2)$	...	$f_2(y_n)$	1 ← Grand Total

The *joint probability table* for X and Y.

Joint Distributions

- It should be noted that:

$$\sum_{j=1}^{m} f_1(x_j) = 1 \quad \sum_{k=1}^{n} f_2(y_k) = 1$$

- Which can be written

$$\sum_{j=1}^{m} \sum_{k=1}^{n} f(x_j, y_k) = 1$$

- This equation just shows that the total probability of all entries is 1.
- The joint cumulative distribution function of *X* and *Y* is defined by:

$$F(x, y) = P(X \leq x, Y \leq y) = \sum_{u \leq x} \sum_{v \leq y} f(u, v)$$

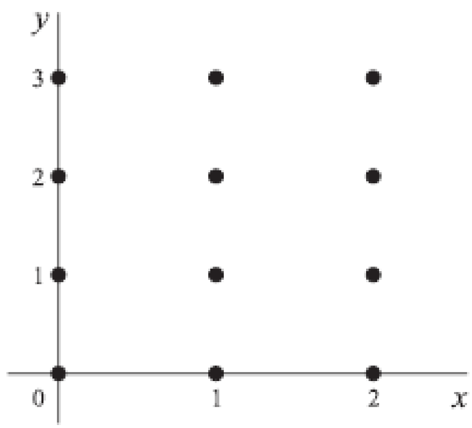
- *F(x, y)* is the sum of all entries for which $x_j \leq x$ and $y_k \leq y$

Joint Distributions

Example: The joint probability function of 2 discrete random variable X and Y is: $f(x, y) = c(2x+y)$, where x and y are integers such that $0 \leq x \leq 2$, $0 \leq y \leq 3$, and $f(x, y) = 0$ otherwise.

a) Find the value of the constant c.

The sample points (x, y) for probabilities are different from zero and the probabilities associated with these points are indicated in below figure and table.



X \ Y	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$5c$	$6c$	$7c$	$22c$
Totals →	$6c$	$9c$	$12c$	$15c$	$42c$

Since the grand total, 42c, must equal 1, then **c = 1/42.**

Joint Distributions

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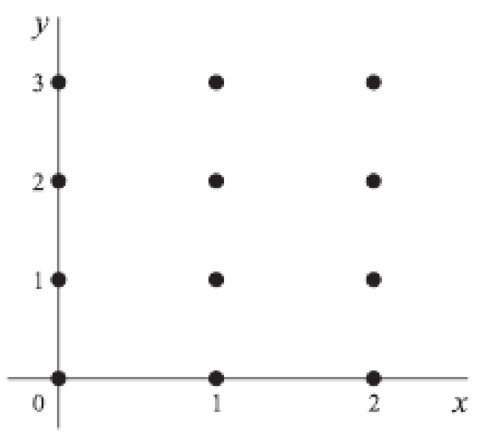

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0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
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Totals →	$6c$	$9c$	$12c$	$15c$	$42c$

Since the grand total, 42c, must equal 1, then **c = 1/42.**

Joint Distributions

Example (ctn.):

b) Find *P(X = 2, Y = 1)?*

Based on the table,

$$P(X = 2, Y = 1) = 5c = \frac{5}{42}$$

X \ Y	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$\boxed{5c}$	$6c$	$7c$	$22c$
Totals →	$6c$	$9c$	$12c$	$15c$	$42c$

Joint Distributions

Example (ctn.):

c) Find $P(X \geq 1, Y \leq 2)$?

Based on the table,

$$P(X \geq 1, Y \leq 2) = \sum_{x \geq 1} \sum_{y \leq 2} f(x, y) = (2c + 3c + 4c + 4c + 5c + 6c)$$

$$= 24c = \frac{24}{42} = \frac{4}{7}$$

X \ Y	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$5c$	$6c$	$7c$	$22c$
Totals →	$6c$	$9c$	$12c$	$15c$	$42c$

Independent Random Variables

- Suppose that X and Y are discrete random variables. If the events X=x and Y=y are independent events for all x and y → X and Y are independent random variables.

$$P(X = x, Y = y) = P(X = x) * P(Y = y) \qquad (*)$$

- Or equivalently: $$f(x, y) = f_1(x) f_2(y) \qquad (**)$$
- Conversely, if equation (**) is satisfied, then X and Y are independent. If not, they are dependent.

Independent Random Variables

- Example: Show that random variable X and Y whose the joint probability table is given below are dependent.
- Solution: If X and Y are independent, then we must have:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

- Based on the table, $P(X = 2, Y = 1) = \frac{5}{42}$ $P(X = 2) = \frac{11}{21}$ $P(Y = 1) = \frac{3}{14}$
- So that $P(X = 2, Y = 1) \neq P(X = 2)P(Y = 1)$

Therefore, X and Y are dependent.

$X \backslash Y$	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$\boxed{5c}$	$6c$	$7c$	$\boxed{22c}$
Totals →	$6c$	$\boxed{9c}$	$12c$	$15c$	$42c$

Change of Variable

- Given the probability distributions of one or more random variables, we are often interested in finding distributions of other random variables that depend on them.

Theorem 2-1: Let X be a discrete random variable whose probability function is f(x). Suppose that a discrete random variable U is defined in terms of X by $U = \phi(X)$, where to each value of X there corresponds one and only value of U and conversely, so that X $= \psi(U)$. Then the probability function for U is given by

$$g(u) = f[\psi(u)]$$

Change of Variable

Example: The probability function of a random variable *X* is:

$$f(x) = \begin{cases} 2^{-x} & x = 1,2,3, \ldots \\ 0 & otherwise \end{cases}$$

Find the probability function for the random variable $U = X^4 + 1$.

Solution: The relationship between the values *u* and *x* of the random variables *U* and *X* is given by u $= x^4 + 1$ or $x = \sqrt[4]{u-1}$ where *u*= *2*, *17*, *82*,…Then the required probability function for *U* is given by:

$$g(u) = \begin{cases} 2^{-\sqrt[4]{u-1}} & u = 2, 17, 82, \ldots \\ 0 & otherwise \end{cases}$$

Change of Variable

Theorem 2-2: Let X and Y be a discrete random variables having joint probability function *f(x, y)*. Suppose that two discrete random variables U and V are defined in terms of X and Y by $U = \phi_1(X, Y), V = \phi_2(X, Y)$, where to each pair of X and Y there corresponds one and only one pair of values of U and V conversely, so that $X = \psi_1(U, V), Y = \psi_2(U, V)$. Then the joint probability function for U and V is given by

$$g(u, v) = f[\psi_1(u, v), \psi_2(u, v)]$$

Conditional Distributions

Recall: If P(A)>0, then: $P(B|A) = \frac{P(A \cap B)}{P(A)}$

If X and Y are discrete random variables and we have the events *(A: X=x)*, *(B: Y=y)*, then

$$f(y|x) = P(Y = y|X = x) = \frac{f(x, y)}{f_1(x)}$$

where

$f(x, y) = P(X = x, Y = y)$: the joint probability function

$f_1(x)$: the marginal probability function for X.

$f(y|x)$: the *conditional probability function* of Y given X.

Similarly,

$$f(x|y) = \frac{f(x, y)}{f_2(y)}$$

Reference

Notes, equations, and figures in the lecture are based on or taken from materials in the course textbook:

“Probability and Statistics”, by Spiegel, Schiller and Srinivasan, ISBN 987-007-179557-9 (McGraw-Hill/Schaun’s)