Daniel Puong
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EE 381 HW #8

1. A random variable X has density function:

$$f(x) = \begin{cases} ce^{-3x} & x>0 \\ 0 & x \le 0 \end{cases}$$ (this impies the range of 0 to ∞)

$\int e^x \, dx = e^x + c$: General Antiderative

a) constant c

f(x) must satisfy $f(x) \ge 0$ and $\int_{\infty}^{\infty} f(x)\,dx = 1$

property 1: $c \ge 0$? ✓

property 2: $\int_{-\infty}^{\infty} f(x)\,dx = 1 = \int_{-\infty}^{\infty} ce^{-3x}\,dx =$

$$\int_0^{\infty} ce^{-3x}\,dx$$

we changed it to 0 b/c its a lower limit,
USUALLY, WHEN NO UPPER LIMIT IS GIVEN, treat it as ∞

$$\int_0^{\infty} ce^{-3x}\,dx = c\int_0^{\infty} e^{-3x}\,dx = c\int_0^{\infty} e^{u}\,du$$

we will let $u = -3x$ (substitution rule)
$du = -3\,dx$

$$= c \cdot -\frac{1}{3}\int_0^{\infty} e^u\,du = -\frac{1}{3}c\int_0^{\infty} e^u\,du = -\frac{1}{3}ce^{u}\Big|_0^{\infty}$$

plug back in for u

$$= -\frac{1}{3}ce^{-3x}\Big|_0^{\infty} = -\frac{1}{3}ce^{-3(\infty)} - \left(-\frac{1}{3}ce^{-3(0)}\right) = -\frac{1}{3}ce^{-\infty} - \left(-\frac{1}{3}ce^{0}\right) = 1$$

$$= -\frac{1}{3}ce^{-\infty} + \frac{1}{3}c = 1 \rightarrow \frac{-c}{3e^{\infty}} + \frac{1}{3}c = 1 = 0 + \frac{1}{3}c = 1 \rightarrow \cancel{3} \cdot \frac{1}{3}c = 1 \cdot 3; \boxed{c=3}$$

1. b) $P(1<X<2)$

To find $P(1<X<2)$, we take the antiderivative of the function w/ the found constant from part A.

$1<X<2$ means the lower limit is 1 and the upper limit 2.

$$P(X) = ce^{-3X} = 3e^{-3X}$$
$$c=3$$

$$= \int_1^2 3e^{-3X}\,dX = 3\int_1^2 e^{-3X}\,dX$$

$$= \cancel{3}\cdot -\frac{1}{\cancel{3}}e^{-3X}\Big|_1^2 = -e^{-3X}\Big|_1^2 =$$

$$-e^{-3(2)} - [-e^{-3(1)}] = -e^{-6} + e^{-3} = \boxed{0.0473 = P(1<X<2)}$$

c) $P(X \geq 3)$

$X \geq 3$ means the lower limit is 3 and the upper limit is $+\infty$. we found $c=3$ from part a.

$$P(X) = ce^{-3X} = 3e^{-3X}$$
$$c=3$$

$$= \int_3^\infty 3e^{-3X} = 3\int_3^\infty e^{-3X}\,dX$$

$$= 3\cdot\left[-\frac{1}{3}e^{-3X}\right]_3^\infty = 3\cdot\left[-\frac{1}{3}e^{-3(\infty)} - \left(-\frac{1}{3}e^{-3(3)}\right)\right]$$

$$= 3\cdot\left[-\frac{1}{3}e^{-\infty} + \frac{1}{3}e^{-9}\right] = 3\cdot\left[\frac{-1}{3e^{\infty}} + \frac{1}{3}e^{-9}\right] = 3\left[0 + \frac{1}{3}e^{-9}\right]$$

$$= \cancel{3}\cdot\frac{1}{\cancel{3}}e^{-9} = \boxed{e^{-9} = 0.0001234}$$

1. d) $(y<1)$

$$f(x) = \begin{cases} ce^{-3x} & x>0 \\ 0 & x \le 0 \end{cases}$$

$c=3 \rightarrow f(x) = 3e^{-3x}$

$x<1$ means the lower limit is -0 and the upper limit is 1. we found $c=3$ from part a.

$p(x) = ce^{-3x} = 3e^{-3x} =$

$$\int_0^1 3e^{-3x} = 3 \cdot \int_0^1 e^{-3x}\,dx$$

$$= 3 \cdot \left[-\frac{1}{3} e^{-3x} \Big|_{-0}^{1} \right] = 3 \cdot \left[-\frac{1}{3} e^{-3(1)} - \left(-\frac{1}{3} e^{-3(0)} \right) \right]$$

$$= 3 \cdot \left[-\frac{1}{3} e^{-3} + \frac{1}{3} e^{0} \right] = 3 \cdot \left[-\frac{1}{3} e^{-3} + \frac{1}{3}(1) \right]$$

$$= 3 \cdot \left[-\frac{1}{3} e^{-3} + \frac{1}{3} \right] = -\frac{3}{3} e^{-3} + \frac{3}{3} = -e^{-3} + 1 =$$

$$\boxed{1 - e^{-3} = 0.9502}$$

2. A random variable X has the density function

$$f(x) = \begin{cases} cx^2 & 1 \le x \le 2 \\ cx & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

a) The constant c

$$= \int_1^2 cx^2\,dx + \int_2^3 cx\,dx$$

$$= c\int_1^2 x^2\,dx + c\int_2^3 x\,dx$$

$$= c \cdot \frac{x^{2+1}}{2+1}\Big|_1^2 + c \cdot \frac{x^{1+1}}{1+1}\Big|_2^3 = \frac{cx^3}{3}\Big|_1^2 + \frac{cx^2}{2}\Big|_2^3$$

$$= \left[\frac{c(2)^3}{3} - \frac{c(1)^3}{3}\right] + \left[\frac{c(3)^2}{2} - \frac{c(2)^2}{2}\right]$$

$$= \left[\frac{8c}{3} - \frac{c}{3}\right] + \left[\frac{9c}{2} - \frac{4c}{2}\right] = \left[\frac{7c}{3}\right] + \left[\frac{5c}{2}\right]$$

$$= \left(\frac{2}{2}\right) \cdot \frac{7c}{3} + \frac{5c}{2} \cdot \left(\frac{3}{3}\right) = \frac{14c}{6} + \frac{15c}{6} = \frac{29c}{6} = 1$$

$$\left(\frac{6}{29}\right) \cdot \frac{\cancel{29}c}{\cancel{6}} = 1 \cdot \left(\frac{6}{29}\right);\ \boxed{c = \frac{6}{29}}$$

2. (continued!)

b. $P(X > 2)$

$$f(x) = \begin{cases} cx^2 & 1 \le x \le 2 \to [1,2] \\ cx & 2 < x < 3 \to (2,3) \\ 0 & 0 \text{ otherwise} \end{cases} \quad (c = 6/29)$$

For b, if $x > 2$, then we choose the equation that fits within the range of $2 < x < 3$, thus, its eq. 2.

$$P(X > 2) = \int_2^3 cx \, dx = c \int_2^3 x^1 \, dx =$$

$$\frac{6}{29} \cdot \frac{x^{1+1}}{1+1}\Big|_2^3 = \frac{6}{29} \cdot \frac{x^2}{2}\Big|_2^3 = \frac{6}{29} \cdot \left[\frac{(3)^2}{2} - \frac{(2)^2}{2}\right]$$

$$= \frac{6}{29} \cdot \left[\frac{9}{2} - \frac{4}{2}\right] = \frac{\cancel{6}^{3}}{29} \left[\frac{5}{\cancel{2}}\right] = \boxed{\frac{15}{29} = P(X > 2)}$$

c. $P(1/2 < X < 3/2)$

* NOTE: integral from $1 - 3/2$ b/c $1/2$ falls under "out of bounds". we use eq. 1 b/c it falls under $1 \le x \le 2$

$$P(1/2 < X < 3/2) = \int_1^{3/2} cx^2 \, dx = c \int_1^{3/2} x^2 \, dx$$

$$= \frac{6}{29} \int_1^{3/2} x^2 \, dx = \frac{6}{29} \cdot \frac{x^{2+1}}{2+1}\Big|_1^{3/2} = \frac{6}{29} \cdot \frac{x^3}{3}\Big|_1^{3/2}$$

$$= \frac{6}{29} \cdot \left[\frac{(\frac{3}{2})^3}{3} - \frac{(1)^3}{3}\right] = \frac{6}{29} \cdot \left[\frac{\frac{27}{8}}{3} - \frac{1}{3}\right] = \frac{6}{29}\left[\frac{27}{24} - \frac{1}{3}\right]$$

$$= \frac{6}{29}\left[\frac{27}{24} - \frac{8}{24}\right] = \frac{\cancel{6}}{29}\left[\frac{19}{\cancel{24}\,4}\right] = \boxed{\frac{19}{116} = P(1/2 < X < 3/2)}$$

3. can the function:

$$F(x) = \begin{cases} c(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

be a distribution function?

1. $F(-\infty) = 0$
2. F is continuous when $0 \le x \le 1$

$f(x) = \frac{d}{dx}[c(1-x^2)] = c\frac{d}{dx}(1-x^2)$

$= c \cdot -2x = -2xc = -2cx$

$= \int_0^1 -2cx\, dx = -2c\int_0^1 x^1 dx = -2c \cdot \frac{x^{1+1}}{1+1}\Big|_0^1$

$= -\frac{2cx^2}{2}\Big|_0^1 = -2c \cdot \frac{x^2}{2}\Big|_0^1 = -2c \cdot \left[\left(\frac{(1)^2}{2}\right) - \left(\frac{(0)^2}{2}\right)\right]$

$= -2c \cdot \left[\frac{1}{2} - \frac{0}{2}\right] = -2c \cdot \frac{1}{2} = -c = 1$

$\frac{\cancel{-}c}{\cancel{-x}} = \frac{1}{-1} \to \boxed{c = -1}$

$F(+\infty) \ne 1$, $\int_{-\infty}^{\infty} f(x)\,dx \ne 1$, so $F(x) = \begin{cases} c(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ can't be a distribution function

must satisfy $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$-2c \cdot \frac{x^2}{2}\Big|_{-\infty}^{\infty} = \left(-2(-1) \cdot \frac{(\infty)^2}{2}\right) - \left(-2(-1) \cdot \frac{(-\infty)^2}{2}\right)$

$= \left(\cancel{2} \cdot \frac{\infty}{\cancel{2}}\right) - \left(\cancel{2} \cdot \frac{\infty}{\cancel{2}}\right) = \infty - \infty = \boxed{0 \ne 1}$

4. Let X and Y to be continuous random variables having joint density function

$$f(x,y) = \begin{cases} c(x^2+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

// Double Intergrae: $\int_{x\,LL}^{x\,UL} \int_{y\,LL}^{y\,UL} f(x,y)\,dx\,dy = 1$

Outer, than inner

a) the constant c

$$\int_0^1 \int_0^1 c(x^2+y^2)\,dx\,dy$$

$$= c \int_0^1 \int_0^1 (x^2+y^2)\,dx\,dy$$

$\int x^2 + y^2\,dy$
$= \int \underbrace{x^2}_{\text{constant}} dy + \int y^2\,dy$
$= x^2 y$, $\frac{y^{2+1}}{2+1} = \frac{y^3}{3}$

$$= c \int_0^1 x^2 y + \frac{y^3}{3} \Big|_{0=y}^{1=y} dx$$

$$= c \int_0^1 \left[\left(x^2(1) + \frac{(1)^3}{3}\right) - \left(x^2(0) + \frac{(0)^3}{3}\right) \right] dx$$

$$= c \int_0^1 x^2 + \frac{1}{3} - 0 + 0\ dx = c \int_0^1 x^2 + \frac{1}{3}\ dx$$

$\int x^2\,dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$

$$= c \cdot \left[\frac{x^3}{3} + \frac{1}{3}x \right]_{0=x}^{1=x}$$

$\int \frac{1}{3}\,dx = \frac{1}{3}x$

$$= c \cdot \left[\frac{(1)^3}{3} + \frac{1}{3}(1) - \cancel{\frac{(0)^3}{3}} + \cancel{\frac{1}{3}(0)} \right] = c \cdot \left[\frac{1}{3} + \frac{1}{3} \right] = \frac{2}{3}c = 1 \rightarrow$$

$c = 3/2$

UL = upper limit
LL = lower limit

4. (continued!) $C = 3/2$

$P(x \overset{UL}{<} \tfrac{1}{2}, y \overset{LL}{>} \tfrac{1}{2})$ // Double integrals

$$f(x,y) = \begin{cases} C(x^2+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3}{2}(x^2+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_0^{1/2} \int_{1/2}^{1} \frac{3}{2}(x^2+y^2)\, dx\, dy$$

$$= \frac{3}{2} \int_0^{1/2} \int_{1/2}^{1} x^2+y^2 \, dx\, dy$$ // outer, than inner

$$= \frac{3}{2} \int_0^{1/2} x^2 y + \frac{y^3}{3} \Big|_{\frac{1}{2}=y}^{1=y} dx$$

$\int x^2 + y^2\, dy = \int x^2\, dy + \int y^2\, dy$
constant
$= x^2 y$, $\frac{y^{2+1}}{2+1} = \frac{y^3}{3}$

$$= \left[\left(x^2(1) + \frac{(1)^3}{3}\right) - \left(x^2(\tfrac{1}{2}) + \frac{(\frac{1}{2})^3}{3}\right)\right]$$

$$= x^2 + \frac{1}{3} - \frac{x^2}{2} - \frac{\frac{1}{8}}{3} = \frac{2x^2}{2} + \frac{1}{3} - \frac{x^2}{2} - \frac{1}{24} = \frac{2x^2}{2} + \frac{8}{24} - \frac{x^2}{2} - \frac{1}{24}$$

$$= \frac{x^2}{2} + \frac{7}{24}$$

$$= \frac{3}{2} \int_0^{1/2} \frac{x^2}{2} + \frac{7}{24}\, dx$$

$\int \frac{x^2}{2} dx = \frac{1}{2}\int x^2 dx = \frac{1}{2} \cdot \frac{x^{2+1}}{2+1} = \frac{x^3}{2 \cdot 3} = \frac{x^3}{6}$
$\int \frac{7}{24} dx = \frac{7}{24}x$

$$= \frac{3}{2} \cdot \left[\frac{x^3}{6} + \frac{7}{24}x\right]_{x=0}^{x=1/2} = \left[\left(\frac{(\frac{1}{2})^3}{6} + \frac{7}{24}\left(\frac{1}{2}\right)\right) - \left(\frac{(0)^3}{6} + \frac{7}{24}(0)\right)\right]$$

$$= \frac{3}{2} \cdot \left[\frac{\frac{1}{8}}{6} + \frac{7}{48} - 0\right] = \frac{3}{2} \cdot \left[\frac{1}{48} + \frac{7}{48}\right] = \frac{3}{2}\left[\frac{8}{48}\right] = \frac{3}{2}\left[\frac{1}{6}\right]_2 = \frac{1}{4} = P(x < \tfrac{1}{2}, Y > \ldots$$

4. (continued!) $\boxed{c = 3/2}$

$$f(x,y) = \begin{cases} c(x^2+y^2) & x \in [0,1],\ y \in [0,1] \quad 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3}{2}(x^2+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

c) $P(\frac{1}{4} < X < \frac{3}{4})$ //single intergal

lower limit upper limit

*** Density Function for X:** $f_1(x) = \int_{v=-\infty}^{\infty} f(x,v)\, dv$

y's interval

$$f(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\, dy$$

$$= \frac{3}{2}\int_0^1 x^2+y^2\, dy \qquad \boxed{c = 3/2}$$

$$\int x^2+y^2\, dy = \int x^2\, dy + \int y^2\, dy$$

constant $\| \quad \|$

$$x^2 y \qquad \frac{y^{2+1}}{2+1} = \frac{y^3}{3}$$

$$= \frac{3}{2} \cdot \left[x^2 y + \frac{y^3}{3} \right]_{0=y}^{1=y}$$

$$= \frac{3}{2} \cdot \left[\left(x^2(1) + \frac{(1)^3}{3}\right) - \left(x^2(0) + \frac{(0)^3}{3}\right) \right]$$

$$= \frac{3}{2} \cdot \left[x^2 + \frac{1}{3} - 0 - 0 \right] = \boxed{\frac{3}{2}\left(x^2+\frac{1}{3}\right) = f(x);\ 0 \le x \le 1}$$

$P(\frac{1}{4} < X < \frac{3}{4})$

$$= \int_{1/4}^{3/4} f(x)\, dx = \int_{1/4}^{3/4} \frac{3}{2}\left(x^2+\frac{1}{3}\right) dx = \frac{3}{2}\int_{1/4}^{3/4} x^2 + \frac{1}{3}\, dx$$

$$\int x^2 + \frac{1}{3}\, dx = \int x^2\, dx + \int \frac{1}{3}\, dx$$

$$\frac{x^{2+1}}{2+1} = \frac{x^3}{3} \qquad \frac{1}{3}x$$

$$\frac{3}{2}\left[\frac{x^3}{3} + \frac{1}{3}x \right]_{1/4=x}^{3/4=x} = \left[\left(\frac{(\frac{3}{4})^3}{3} + \frac{1}{3}\left(\frac{3}{4}\right) \right) - \left(\frac{(\frac{1}{4})^3}{3} + \frac{1}{3}\left(\frac{1}{4}\right) \right) \right] = \frac{3}{2}\left[\frac{\frac{27}{64}}{3} + \frac{3}{12} - \frac{\frac{1}{64}}{3} - \frac{1}{12} \right] =$$

4. (continued!)

$= \frac{\cancel{3}}{2} \cdot \left[\frac{29}{\cancel{96}_{32}}\right] = \boxed{\frac{29}{64} = P\left(\frac{1}{4} < X < \frac{3}{4}\right)}$

d) $P\left(Y < \frac{1}{2}\right)$

$c = 3/2$

$f_2(y) = \int_{x=-\infty}^{\infty} cf(x, y)\, dx$

$f(y) = \int_{x=0}^{1} \frac{3}{2} (x^2 + y^2)\, dX$ → x's interval

$\int x^2 + y^2\, dx = \int x^2\, dx + \int y^2\, dx$

$\frac{x^{2+1}}{2+1} = \frac{x^3}{3}$, constant $y^2 X$

$f(y) = \frac{3}{2} \int_{x=0}^{1} x^2 + y^2\, dX$

$= \frac{3}{2} \cdot \left[\frac{x^3}{3} + y^2 X\right]_{0=x}^{1=x} = \frac{3}{2} \cdot \left[\left(\frac{(1)^3}{3} + y^2(1)\right) - \left(\frac{(0)^3}{3} + y^2(0)\right)\right]$

$= \frac{3}{2} \cdot \left[\frac{1}{3} + y^2 - 0 - 0\right] = \frac{3}{2}\left(\frac{1}{3} + y^2\right) = f(y);\ 0 \le y \le 1$

$= \int_{y=0}^{1/2} \frac{3}{2}\left(\frac{1}{3} + y^2\right) dy = \frac{3}{2} \int_0^{1/2} \frac{1}{3} + y^2\, dy =$

$\frac{3}{2} \cdot \left[\frac{1}{3}y + \frac{y^3}{3}\right]_{0=y}^{1/2=y}$

$\frac{1}{3} + y^2\, dy = \int \frac{1}{3}\, dy + \int y^2\, dy$

$\frac{1}{3}y$, $\frac{y^{2+1}}{2+1} = \frac{y^3}{3}$

$= \frac{3}{2} \cdot \left[\left(\frac{1}{3}\left(\frac{1}{2}\right) + \frac{\left(\frac{1}{2}\right)^3}{3}\right) - \left(\frac{1}{3}(0) + \frac{(0)^3}{3}\right)\right]$

$= \frac{3}{2} \cdot \left[\frac{1}{6} + \frac{1}{24} - 0 - 0\right] = \frac{3}{2} \cdot \left[\frac{4}{24} + \frac{1}{24}\right] = \frac{\cancel{3}}{2} \cdot \left[\frac{5}{\cancel{24}_{8}}\right] = \boxed{\frac{5}{16}}$

$\boxed{5/16 = P\left(Y < \frac{1}{2}\right)}$

4. (continued!)

e) Are X and Y independent?

$f(x) \cdot f(y) = g(x)\, g(y)$

$f(x) \cdot f(y) \neq \frac{3}{2}(x^2+\frac{1}{3})(y^2+\frac{1}{3})$

$\boxed{\text{X and Y are not independent.}}$

5. Let X have the density function

$f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x \leq 0 \end{cases}$ Thm. 2.3

change of variable

Find the density function of $Y = X^2$.

Thm 2.3: $g(u)|du| = f(x)\,|dx|$ OR
$g(u) = f(x)\left|\frac{dx}{du}\right| = f[\psi(u)] \cdot |\psi'(u)|$

$Y = X^2 \rightarrow \psi(u) = \sqrt{Y} = \sqrt{x^2} = X = \psi(u) = \sqrt{y} = y^{1/2}$

when $X=0$: $x(0) = \sqrt{0} = (0)^{1/2} = 0 = y$; $x=0 \rightarrow y=0$

$\psi'(u)' = (\sqrt{y})' = (y^{1/2})' = \frac{1}{2}y^{1/2-1} = \frac{1}{2}y^{-1/2} = \frac{1}{2y^{1/2}}$

$= \frac{1}{2\sqrt{y}} = \psi'(u)$ $\quad g(y) = f(x) \cdot \left|\frac{dx}{dy}\right| = f[\psi(u)] \cdot |\psi'(u)|$

$X = \psi(u) = \sqrt{y} = y^{1/2}$

$$\boxed{g(y) = \begin{cases} e^{-(\sqrt{y})} \cdot \frac{1}{2\sqrt{y}} & y<0 = \frac{e^{-\sqrt{y}}}{2\sqrt{y}} \quad y>0 \\ 0 & y \leq 0 = 0 \quad y<0 \end{cases}}$$

6. Let X and Y have joint density function

$$f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

// there's x and Y; thus, we use thm. 2.4:

$$\begin{cases} u = \dfrac{x}{y} \\ v = x+y \end{cases} \Longleftrightarrow \begin{cases} \cdot\, u = \dfrac{x}{y} \cdot y \rightarrow x = uy = \psi_1 = x = uy \\ v = x+y \rightarrow y = v - x = \psi_2 = y = v-(uy) \end{cases}$$

$-x \; -x$ $\Longleftrightarrow (1+u)y = v \Longleftrightarrow y = \dfrac{v}{1+u}$

$$\Longleftrightarrow \begin{cases} x = \dfrac{uv}{1+u} \\ y = \dfrac{v}{1+u} \end{cases}$$

Thm 2.4: $g(u,v) = f(x,y)\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = f[\psi_1(u,v), \psi_2(u,v)] \cdot |J|$

Since $x \geq 0$ and $y \geq 0 \rightarrow \dfrac{x}{y} = u \geq 0$

Since $y \geq 0 \rightarrow \dfrac{v}{1+u} \geq 0$ and $u \geq 0 \rightarrow v \geq 0$

$|ad - bc|$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{array}{c|cc|} & a & b \\ & \dfrac{v}{(1+u)^2} & \dfrac{u}{1+u} \\ & \dfrac{-v}{(1+u)^2} & \dfrac{1}{1+u} \\ & c & d \end{array} = \left| \overset{-(-)}{\dfrac{v}{(1+u)^3} + \dfrac{uv}{(1+u)^3}} \right|$$

$= \boxed{\left|\dfrac{v(1+u)}{(1+u)^3}\right|}$; Thus, joint density function of U and V is:

$$\begin{cases} e^{-\left(\frac{uv+v}{1+u}\right)} \times \dfrac{v}{(1+u)^2} & u \geq 0 \text{ and } v \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{OR} \quad \begin{cases} e^{-v} \times \dfrac{v}{(1+u)^2} & u \geq 0 \text{ and } v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

7. let

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional density functions

* conditional density function of Y given X is:

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

where $f(x,y)$ is the joint density function of X and Y; $f_1(x)$ is the marginal density function of X.

a) X given y

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \quad 0 \le y \le 1;\ f_2(y) = \int_0^1 (x+y)\, dx$$

$$= \int x^1\, dx + \int y\, dx$$

$\int y\, dx \to$ constant

$$\frac{x^{1+1}}{1+1} = \frac{x^2}{2} \qquad yx$$

$$= \frac{x^2}{2} + yx \Big|_{0=x}^{1=x} = \left[\left(\frac{(1)^2}{2} + y(1)\right) - \left(\frac{(0)^2}{2} + y(0)\right)\right]$$

$$= \boxed{\frac{1}{2} + y = f(y|x)}$$

$$\boxed{f(x|y) = \begin{cases} \dfrac{x+y}{\frac{1}{2}+y} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

7. (continued!)

Y given $X = \frac{f(x,y)}{f_1(x)} = f(y|x)$

$0 \le x \le 1$; $f_1(x) = \int_0^1 (x+y)\,dy$

$\int x\,dy + \int y\,dy$

constant $\to$ ‖ ‖

$xy \qquad \frac{y^{1+1}}{1+1} = \frac{y^2}{2}$

$= xy + \frac{y^2}{2} \Big|_{0=y}^{1=y} = \left[\left(x(1) + \frac{(1)^2}{2}\right) - \left(x(0) + \frac{(0)^2}{2}\right)\right]$

$= \left[x + \frac{1}{2} - 0 - 0\right] = x + \frac{1}{2}$

$$\boxed{f(y|x) = \begin{cases} \frac{x+y}{x+1/2} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}}$$