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List of integrals of Gaussian functions

In the expressions in this article,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

is the standard normal probability density function,

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$

is the corresponding cumulative distribution function (where **erf** is the error function), and

$$T(h,a) = \varphi(h) \int_0^a \frac{\varphi(hx)}{1+x^2}\,dx$$

is Owen's T function.

Owen[1] has an extensive list of Gaussian-type integrals; only a subset is given below.

Indefinite integrals

$$\int \varphi(x)\,dx = \Phi(x) + C$$

$$\int x\varphi(x)\,dx = -\varphi(x) + C$$

$$\int x^2\varphi(x)\,dx = \Phi(x) - x\varphi(x) + C$$

$$\int x^{2k+1}\varphi(x)\,dx = -\varphi(x) \sum_{j=0}^{k} \frac{(2k)!!}{(2j)!!} x^{2j} + C$$ [2]

$$\int x^{2k+2}\varphi(x)\,dx = -\varphi(x) \sum_{j=0}^{k} \frac{(2k+1)!!}{(2j+1)!!} x^{2j+1} + (2k+1)!!\,\Phi(x) + C$$

In the previous two integrals, $n!!$ is the double factorial: for even n it is equal to the product of all even numbers from 2 to n, and for odd n it is the product of all odd numbers from 1 to n; additionally it is assumed that $0!! = (-1)!! = 1$.

$$\int \varphi(x)^2\,dx = \frac{1}{2\sqrt{\pi}} \Phi\left(x\sqrt{2}\right) + C$$

$$\int \varphi(x)\varphi(a+bx)\,dx = \frac{1}{t}\varphi\left(\frac{a}{t}\right)\Phi\left(tx+\frac{ab}{t}\right)+C, \qquad t=\sqrt{1+b^2}$$ [3]

$$\int x\varphi(a+bx)\,dx = -\frac{1}{b^2}\left(\varphi(a+bx)+a\Phi(a+bx)\right)+C$$

$$\int x^2\varphi(a+bx)\,dx = \frac{1}{b^3}\left((a^2+1)\Phi(a+bx)+(a-bx)\varphi(a+bx)\right)+C$$

$$\int \varphi(a+bx)^n\,dx = \frac{1}{b\sqrt{n(2\pi)^{n-1}}}\Phi\left(\sqrt{n}(a+bx)\right)+C$$

$$\int \Phi(a+bx)\,dx = \frac{1}{b}\left((a+bx)\Phi(a+bx)+\varphi(a+bx)\right)+C$$

$$\int x\Phi(a+bx)\,dx = \frac{1}{2b^2}\left((b^2x^2-a^2-1)\Phi(a+bx)+(bx-a)\varphi(a+bx)\right)+C$$

$$\int x^2\Phi(a+bx)\,dx = \frac{1}{3b^3}\left((b^3x^3+a^3+3a)\Phi(a+bx)+(b^2x^2-abx+a^2+2)\varphi(a+bx)\right)+C$$

$$\int x^n\Phi(x)\,dx = \frac{1}{n+1}\left(\left(x^{n+1}-nx^{n-1}\right)\Phi(x)+x^n\varphi(x)+n(n-1)\int x^{n-2}\Phi(x)\,dx\right)+C$$

$$\int x\varphi(x)\Phi(a+bx)\,dx = \frac{b}{t}\varphi\left(\frac{a}{t}\right)\Phi\left(xt+\frac{ab}{t}\right)-\varphi(x)\Phi(a+bx)+C, \qquad t=\sqrt{1+b^2}$$

$$\int \Phi(x)^2\,dx = x\Phi(x)^2+2\Phi(x)\varphi(x)-\frac{1}{\sqrt{\pi}}\Phi\left(x\sqrt{2}\right)+C$$

$$\int e^{cx}\varphi(bx)^n\,dx = \frac{e^{\frac{c^2}{2nb^2}}}{b\sqrt{n(2\pi)^{n-1}}}\Phi\left(\frac{b^2xn-c}{b\sqrt{n}}\right)+C, \qquad b\neq 0, n>0$$

Definite integrals

$$\int_{-\infty}^{\infty} x^2\varphi(x)^n\,dx = \frac{1}{\sqrt{n^3(2\pi)^{n-1}}}$$

$$\int_{-\infty}^{\infty} \varphi(x)\varphi(a+bx)\,dx = \frac{1}{\sqrt{1+b^2}}\varphi\left(\frac{a}{\sqrt{1+b^2}}\right)$$

$$\int_{-\infty}^{0} \varphi(ax)\Phi(bx)\,dx = \frac{1}{2\pi|a|}\left(\frac{\pi}{2}-\arctan\left(\frac{b}{|a|}\right)\right)$$

$$\int_{0}^{\infty} \varphi(ax)\Phi(bx)\,dx = \frac{1}{2\pi|a|}\left(\frac{\pi}{2}+\arctan\left(\frac{b}{|a|}\right)\right)$$

$$\int_{0}^{\infty} x\varphi(x)\Phi(bx)\,dx = \frac{1}{2\sqrt{2\pi}}\left(1+\frac{b}{\sqrt{1+b^2}}\right)$$

$$\int_{0}^{\infty} x^2\varphi(x)\Phi(bx)\,dx = \frac{1}{4}+\frac{1}{2\pi}\left(\frac{b}{1+b^2}+\arctan(b)\right)$$

$$\int_{-\infty}^{\infty} x\varphi(x)^2\Phi(x)\,dx = \frac{1}{4\pi\sqrt{3}}$$

$$\int_{0}^{\infty} \Phi(bx)^2\varphi(x)\,dx = \frac{1}{2\pi}\left(\arctan(b)+\arctan\sqrt{1+2b^2}\right)$$

$$\int_{-\infty}^{\infty} \Phi(a+bx)^2 \varphi(x)\,dx = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right) - 2T\left(\frac{a}{\sqrt{1+b^2}}, \frac{1}{\sqrt{1+2b^2}}\right)$$

$$\int_{-\infty}^{\infty} x\Phi(a+bx)^2 \varphi(x)\,dx = \frac{2b}{\sqrt{1+b^2}}\varphi\left(\frac{a}{t}\right)\Phi\left(\frac{a}{\sqrt{1+b^2}\sqrt{1+2b^2}}\right)$$ [4]

$$\int_{-\infty}^{\infty} \Phi(bx)^2 \varphi(x)\,dx = \frac{1}{\pi}\arctan\sqrt{1+2b^2}$$

$$\int_{-\infty}^{\infty} x\varphi(x)\Phi(bx)\,dx = \int_{-\infty}^{\infty} x\varphi(x)\Phi(bx)^2\,dx = \frac{b}{\sqrt{2\pi(1+b^2)}}$$

$$\int_{-\infty}^{\infty} \Phi(a+bx)\varphi(x)\,dx = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$$

$$\int_{-\infty}^{\infty} x\Phi(a+bx)\varphi(x)\,dx = \frac{b}{t}\varphi\left(\frac{a}{t}\right), \qquad t=\sqrt{1+b^2}$$

$$\int_{0}^{\infty} x\Phi(a+bx)\varphi(x)\,dx = \frac{b}{t}\varphi\left(\frac{a}{t}\right)\Phi\left(-\frac{ab}{t}\right) + \frac{1}{\sqrt{2\pi}}\Phi(a), \qquad t=\sqrt{1+b^2}$$

$$\int_{-\infty}^{\infty} \ln(x^2)\frac{1}{\sigma}\varphi\left(\frac{x}{\sigma}\right)\,dx = \ln(\sigma^2) - \gamma - \ln 2 \approx \ln(\sigma^2) - 1.27036$$

References

1. Owen 1980.
2. Patel & Read (1996) lists this integral above without the minus sign, which is an error. See calculation by WolframAlpha (http://www.wolframalpha.com/input/?fp=1&i=D(-e^(-x^2/2)/sqrt(2pi)*Sum((2k)!!/(2j)!!*x^(2j),{j,0,k}),x)&s=40&incTime=true).
3. Patel & Read (1996) report this integral with error, see WolframAlpha (http://www.wolframalpha.com/input/?i=Integrate(1/sqrt(2pi)*e^(-x^2/2)*1/sqrt(2pi)*e^(-(a%2Bb*x)^2/2),x)).
4. Patel & Read (1996) report this integral incorrectly by omitting *x* from the integrand.

- Owen, D. (1980). "A table of normal integrals". *Communications in Statistics: Simulation and Computation*. **B9** (4): 389–419. doi:10.1080/03610918008812164 (https://doi.org/10.1080%2F03610918008812164).
- Patel, Jagdish K.; Read, Campbell B. (1996). *Handbook of the normal distribution* (2nd ed.). CRC Press. ISBN 0-8247-9342-0.

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