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1. INTRODUCTION

Kac's theorem [1] on the average number of real roots belonging to a random polynomial whose coefficients are independent Gaussian random variables with zero mean and unit variance is a well-known classical result. In 1995, Eldman and Kostlan [2] found an elementary geometric derivation of Kac's formula; specifically, they showed that the average number of roots is the length of a certain curve on a multidimensional sphere. In this author's opinion, this derivation is very important, because it reveals the relation between various problems related to random polynomials and the methods of integral geometry.

This paper begins with an example, borrowed from [3], that is concerned with the density of a set of straight lines in a plane. All further considerations are based on the approach used in this example; however, we apply it to sets of polynomials rather than to sets of lines. As a result, we obtain a formula for the average number of real roots belonging to a random polynomial whose coefficients have an arbitrary joint density. This formula is generalized to polynomials in several variables and to random point and vector fields.

2. DENSITY FOR A SET OF STRAIGHT LINES IN A PLANE

A straight line G in a plane is determined by two coordinates: the angle φ between the abscissa and the perpendicular from the origin to the line ($0 \le \varphi \le 2\varphi$) and the length ρ of this perpendicular ($\rho \ge 0$). The measure of a set of straight lines is defined as the integral over this set expressed in the differential form

$$dG = d\rho \wedge d\varphi,$$

which is called the density of this set of lines. It is easy to show (see, e.g., [3, p. 33 of the Russian translation]) that this measure is unique (up to multiplication by a constant) and invariant with respect to the motions of a plane.

Consider a smooth curve C of finite length L. If the line G intersects C, then its position is determined by two parameters, the length s of the fragment of the curve from its origin to the intersection point with G and the angle θ between C and G.

The relation between the old and new coordinates of G can be simply written as:

$$dG = |\sin\theta| ds \wedge d\theta. \quad (1)$$

Let us integrate (1) along all the straight lines intersecting C. On the right-hand side, we obtain $2L$. On the left-hand side, each line is counted as many times as there are intersection points of this line with C. We denote this number by $n(C)$. As a result, we obtain

$$\int n(C) dG = 2L, \quad (2)$$

where the integration occurs over all the straight lines in a plane.

3. DENSITY FOR A SET OF POLYNOMIALS

The real polynomial

$$G(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 \quad (3)$$

is determined by $n + 1$ coordinates (i.e., its coefficients). The measure of a set of polynomials can be defined as the integral over this set expressed in the differential form

$$dG = p(a_0, a_1, \ldots, a_n) da_0 \wedge \ldots \wedge da_n,$$

where p is a nonnegative measurable function, which is called the coordinate density.

To establish an analogy with the density of a set of lines, we consider each polynomial as a plane curve (i.e., its graph).

Similarly, consider the smooth curve C as the graph of some smooth function $h(t)$. Suppose that the graph of the polynomial G intersects C at some point with the abscissa s. Let us replace the first coordinate (the free

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term a_0) of G by s. In the new coordinates, dG has the form

$$dG = q(s, a_1, \dots, a_n)ds \wedge da_1 \wedge \dots \wedge da_n, \qquad (4)$$

where the coordinate density is defined by

$$q(s, a_1, \dots, a_n) = p(h(s) - a_n s^n - \dots - a_1 s, a_1, \dots, a_n) \times \left|n a_n s^{n-1} + \dots + a_1 - h'(s)\right|.$$

Integrating (4) over all the polynomials whose graphs intersect C inside the strip $H = \{(x, y) | \alpha \le x \le \beta\}$, where $-\infty \le \alpha < \beta \le \infty$, we obtain the following analog of (2):

$$\int n(h; \alpha, \beta)dG = \int_\alpha^\beta Q(s)dt.$$

The integrand $n(h; \alpha, \beta)$, on the left-hand side, is the number of roots of the equation $G(s) = h(s)$ on the interval $[\alpha, \beta]$. The function $Q(s)$, which is called the density of the number of roots, is defined by

$$Q(s) = \int_{\mathbf{R}^n} q(s, a_1, \dots, a_n)da_1 \wedge \dots \wedge da_n.$$

If the density p is considered as a probability, then $Q(s)$ has the following probabilistic interpretation: if $[s, s + ds]$ is an infinitesimal interval, then $Q(s)ds$ is the probability that the equation $G(s) = h(s)$ has a root in $[s, s + ds]$. The following assertion is valid.

Theorem 1. *Suppose that a polynomial of form* (3) *has real random coefficients with the joint density* $p(a_0, a_1, \dots, a_n)$.

Then, the average number of roots of the equation $G(t) = h(t)$ *in the interval* $[\alpha, \beta]$ *is given by the formula*

$$\mathbf{E}n(h; \alpha, \beta) = \int_\alpha^\beta dt \int_{\mathbf{R}^n} p(h(t) - a_n t^n - \dots - a_1 t, a_1, \dots, a_n)\left|n a_n t^{n-1} + \dots + a_1 - h'(t)\right| da_1 \dots da_n.$$

Remark 1. In the case where the coefficients of the polynomial are independent Gaussian random variables with zero mean and unit variance and the function h identically vanishes, Theorem 1 implies Kac's formula [1], which was mentioned above.

4. ALGEBRAIC SURFACES

Let us generalize the results of the preceding section to a case with several variables. Suppose that the following real polynomial in several variables is given:

$$G(\bar{x}) = \sum_\alpha a_\alpha \bar{x}^\alpha, \qquad (5)$$

where $\bar{x} = (x_1, x_2, \dots, x_d)$ is a point in $\mathbf{R}^d$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is a multi-index. The summation is over all α such that $0 \le \alpha_j \le n$ for $j = 1, 2, \dots, d$.

Consider the algebraic surface M determined by the equation

$$M = \{\bar{x} | \ G(\bar{x}) = 0\}.$$

Let P denote the pair (G, S), where S is a point of M. If a set of polynomials is endowed with the measure

$$dG = p(a_{(0\dots0)}, \dots, a_{(n\dots n)})da_{(0\dots0)} \wedge \dots \wedge da_{(n\dots n)},$$

where $p(a_{(0\dots0)}, \dots, a_{(n\dots n)})$ is some density, then it is natural to endow the set of pairs P with the measure

$$dP = dG \wedge dS,$$

where dS is a $(d-1)$-dimensional Lebesgue measure on M. As is known from mathematical analysis (see, e.g., [4, p. 313]), dS can be represented in the form

$$dS = \frac{dx_1 dx_2 \dots dx_{d-1}}{|\cos(v)|},$$

where v is the angle between the OX_d axis and the normal to M. Given an equation of a surface, it is easy to find its normal; thus, we obtain

$$dP = \frac{|\nabla G|}{\left|\frac{\partial G}{\partial x_d}\right|} p(a_{(0\dots0)}, \dots, a_{(n\dots n)}) \times da_{(0\dots0)} \wedge \dots \wedge da_{(n\dots n)} \wedge dx_1 \wedge \dots \wedge dx_{d-1}. \qquad (6)$$

Since the point S belongs to M, its coordinates satisfy the equality

$$a_{(0\dots0)} = -\sum_{\alpha \ne (0\dots0)} a_\alpha \bar{x}^\alpha. \qquad (7)$$

Taking the total differentials of the both sides of this equality, we obtain

$$da_{(0\dots0)} = I - \frac{\partial G}{\partial x_d} dx_d, \qquad (8)$$

where I does not contain dx_d and $da_{(0\dots0)}$. The substitution of (7) and (8) into (6) yields

$$dP = |\nabla G| p\left(-\sum_{\alpha \ne (0\dots0)} a_\alpha \bar{x}^\alpha, \dots, a_{(n\dots n)}\right) \times \bigwedge_{\alpha \ne (0\dots0)} da_\alpha \wedge dx_1 \wedge \dots \wedge dx_d. \qquad (9)$$

Let Ω be a compact set in $\mathbf{R}^d$. Integrating (9) over all the pairs (G, S) with S belonging to Ω, we obtain the relation

$$\int \mathrm{mes}(M \cap \Omega)dG = \int_\Omega Q(x_1, x_2, \dots, x_d)dx_1 \wedge dx_2 \wedge \dots \wedge dx_d, \qquad (10)$$

where the density is

$$Q(x_1, x_2, \ldots, x_d) = \int |\nabla G| p\Big(-\sum_{\alpha \neq (0\ldots0)} a_\alpha \bar{x}^\alpha, \ldots, a_{(n\ldots n)}\Big) \bigwedge_{\alpha \neq (0\ldots0)} da_\alpha. \qquad (11)$$

For the probability density p, we obtain the following result.

Theorem 2. *Suppose that a polynomial of form* (7) *has real random coefficients with the joint density* $p(a_{(0\ldots0)}, \ldots, a_{(n\ldots n)})$.

Then, the average area of the intersection of the algebraic surface M, determined by the equation $G(t) = 0$ with the domain Ω, equals

$$\mathbf{E}(\mathrm{mes}(M \cap \Omega)) = \int_\Omega (dx_1) dx_2 \ldots dx_d \int_{\mathbf{R}^{(n+1)^d - 1}} |\nabla G| \times p\Big(-\sum_{\alpha \neq (0\ldots0)} a_\alpha \bar{x}^\alpha, \ldots, a_{(n\ldots n)}\Big) \prod_{\alpha \neq (0\ldots0)} da_\alpha.$$

Remark 2. In a case where the coefficients of the polynomial are independent Gaussian random variables with zero mean and unit variance, Theorem 2 yields, after simple calculations, the same result as Ibragimov and Podkorytov [5].

5. THE CORRELATION FUNCTION OF A POINT FIELD

In this section, we use the concepts of a random point field and its correlation function. The necessary information can be found in, for example, [6].

Suppose that the graph of the polynomial G (in one variable) intersects C at k points with the abscissas s_1, s_2, …, s_k, where $k \leq n$. Let us replace the first k coordinates (the coefficients $a_0, a_1, \ldots, a_{k-1}$) of G with $s_1, s_2, \ldots, s_k$ and write dG in the new coordinates.

The functional dependence between the polynomial G and the curve C is expressed by the k equations

$$a_n s_j^n + \ldots + a_k s_j^k = h(s_j) - a_{k-1} s_j^{k-1} - \ldots - a_1 s_j - a_0, \quad j = 1, 2, \ldots, k. \qquad (12)$$

Taking the total differentials of both sides of each equation and taking all the terms containing ds_j to the left-hand side, we obtain

$$I_j + (n a_n s_j^{n-1} + \ldots + a_1 - h'(s_j)) ds_j = -s_j^{k-1} da_{k-1} - \ldots - s_j da_1 - da_0, \quad j = 1, 2, \ldots, k,$$

where I_j is a differential form only containing terms with $da_n, \ldots, da_k$. Let us multiply both sides of the product of all the k equations by $da_k \wedge \ldots \wedge da_n$. In the resulting equation, all the factors on the left-hand side that contain I_j vanish, and the right-hand side includes the Vandermonde determinant as a coefficient:

$$\prod_{j=1}^k (n a_n s_j^{n-1} + \ldots + a_1 - h'(s_j)) \times ds_1 \wedge \ldots \wedge ds_k \wedge da_k \wedge \ldots \wedge da_n = (-1)^k \prod_{1 \leq i < j \leq k} (s_i - s_j) da_0 \wedge \ldots \wedge da_n.$$

This equality implies that, in the new coordinates, the measure dG on the set of polynomials has the form

$$dG = \left| \prod_{1 \leq i < j \leq k} \frac{1}{s_i - s_j} \prod_{j=1}^k (n a_n s_j^{n-1} + \ldots + a_1 - h'(s_j)) \right| \times p(\tilde{a}_0, \ldots, \tilde{a}_{k-1}, a_k, \ldots, a_n) ds_1 \wedge \ldots \wedge ds_k \wedge da_k \wedge \ldots \wedge da_n.$$

Here, $\tilde{a}_0, \ldots, \tilde{a}_{k-1}$ denote the functions of $s_1, \ldots, s_k$ and $a_k, \ldots, a_n$ obtained by solving (12).

If the measure dG is considered as probability, then the abscissas of the intersection points of the polynomial with the curve can be understood as a random point field.

It is easy to see that the k-point correlation function of this field is given by

$$\rho(s_1, s_2, \ldots, s_k) = \int_{\mathbf{R}^{n-k+1}} \left| \prod_{1 \leq i < j \leq k} \frac{1}{s_i - s_j} \prod_{j=1}^k (n a_n s_j^{n-1} + \ldots + a_1 - h'(s_j)) \right| \times p(\tilde{a}_0, \ldots, \tilde{a}_{k-1}, a_k, \ldots, a_n) da_k \wedge \ldots \wedge da_n.$$

This formula implies, in particular, the following theorem.

Theorem 3. *Suppose that $B_1, B_2, \ldots, B_m$ are disjoint Borel subsets of the line and $k_1, k_2, \ldots, k_m$ are positive integers such that $k_1 + k_2 + \ldots + k_m = k$. Let $n(B_i)$ denote the number of roots of the equation $G(s) = h(s)$ belonging to B_i. Then, the following formula is valid*:

$$\mathbf{E}\prod_{i=1}^{m}\frac{n(B_i)!}{(n(B_i)-k_i)!} = \int\limits_{B_1^{k_1}\times\ldots\times B_m^{k_m}}\int\limits_{\mathbf{R}^{n-k+1}}\left|\prod_{1\le i<j\le k}\frac{1}{s_i-s_j}\prod_{j=1}^{k}(na_ns_j^{n-1}+\ldots+a_1-h'(s_j))\right|$$
$$\times p(\tilde{a}_0, \ldots, \tilde{a}_{k-1}, a_k, \ldots, a_n)ds_1\ldots ds_k da_k\ldots da_n.$$

6. VECTOR FIELDS

We now return to polynomials in d variables. In this section, we consider a system of d polynomials determining a vector field V:

$$G_j(\bar{x}) = \sum_{\alpha}a_\alpha^j\bar{x}^\alpha, \quad j = 1, 2, \ldots, d. \tag{13}$$

Suppose that the set of vector fields V is endowed with a measure dV expressed in the form

$$dV = p(a^1_{(0\ldots0)}, \ldots, a^d_{(n\ldots n)})da^1_{(0\ldots0)}\wedge\ldots\wedge da^d_{(n\ldots n)},$$

where p represents a certain density.

Let $(x_1, x_2, \ldots, x_d)$ be a point at which the field V vanishes. Then, we have the system of equalities

$$a^j_{\varepsilon_j} = -\sum_{\alpha\ne\varepsilon_j}a_\alpha^j\bar{x}^{\alpha-\varepsilon_j}, \quad j = 1, 2, \ldots, d, \tag{14}$$

where ε_j denotes the multi-index in which the jth position is occupied by one and all the other positions are occupied by zeros. Then, take the total differentials of both sides:

$$da^j_{\varepsilon_j} = \sum_{i=1}^{d}\left(-\sum_{\alpha\ne\varepsilon_j}(\alpha_i-\delta_{ij})a_\alpha^j\bar{x}^{\alpha-\varepsilon_j-\varepsilon_i}\right)dx_i + I_j,$$
$$j = 1, 2, \ldots, d,$$

where I_j does not contain $d\alpha^j_{\varepsilon_j}$ and dx_i. Let us multiply the left- and right-hand sides of the product of these d equalities by $\bigwedge\limits_{\alpha\ne\varepsilon_1,\ldots,\varepsilon_d}da^j_\alpha$. As a result, we obtain a new expression for dV, namely,

$$dV = \left|\det\left(\sum_{\alpha\ne\varepsilon_j}(\alpha_i-\delta_{ij})a_\alpha^j\bar{x}^{\alpha-\varepsilon_j-\varepsilon_i}\right)_{i,\,j=1,2,\ldots,d}\right|$$
$$\times p(a^1_{(0\ldots0)}, \ldots, a^d_{(n\ldots n)})\bigwedge_{\alpha\ne\varepsilon_1,\ldots,\varepsilon_d}da^j_\alpha\wedge dx_1\wedge\ldots\wedge dx_d.$$

It should be noted that the argument of the density p in this expression contains the expressions obtained from (14) and not $a^1_{\varepsilon_1}, \ldots, a^d_{\varepsilon_d}$.

Let us integrate the obtained equality over all the fields V that have at least one zero in Ω, where Ω is a compact set in $\mathbf{R}^d$. On the left-hand side, each V is counted as many times as there are its zeros in Ω. We denote this number by $N(\Omega)$. As a result, we obtain

$$\int N(\Omega)dV = \int\limits_{\Omega}Q(x_1, x_2, \ldots, x_d)dx_1\wedge dx_2\wedge\ldots\wedge dx_d,$$

where the density $Q(s)$ of the number of zeros is given by

$$Q(x_1, x_2, \ldots, x_d)$$
$$= \int\left|\det\left(\sum_{\alpha\ne\varepsilon_j}(\alpha_i-\delta_{ij})a_\alpha^j\bar{x}^{\alpha-\varepsilon_j-\varepsilon_i}\right)_{i,\,j=1,2,\ldots,d}\right|$$
$$\times p(a^1_{(0\ldots0)}, \ldots, a^d_{(n\ldots n)})\bigwedge_{\alpha\ne\varepsilon_1,\ldots,\varepsilon_d}da^j_\alpha.$$

For the probability density p, the following theorem is valid.

Theorem 4. *Suppose that the d polynomials constituting a system of form* (13) *have real random coefficients with the joint density $p(a^1_{(0\ldots0)}, \ldots, a^d_{(n\ldots n)})$.*

Then, the average number of zeros of the random field in Ω determined by this system is given by the formula

$$\mathbf{E}N(\Omega) = \int\limits_{\Omega}dx_1dx_2\ldots dx_d$$
$$\times\int\limits_{\mathbf{R}^{(n+1)^d-d}}\left|\det\left(\sum_{\alpha\ne\varepsilon_j}(\alpha_i-\delta_{ij})a_\alpha^j\bar{x}^{\alpha-\varepsilon_j-\varepsilon_i}\right)_{i,\,j=1\ldots d}\right|$$
$$\times p(a^1_{(0\ldots0)}, \ldots, a^d_{(n\ldots n)})\prod_{\alpha\ne\varepsilon_1,\ldots,\varepsilon_d}da^j_\alpha.$$

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