Finite Difference Method

Anh Ha LE

University of Sciences

April 18, 2025

Contents

Math Modeling and Simulation of Physical Processes

- Describe the physical phenomenon
- Model the physical phenomenon to become mathematical equations(PDE)
- Simulate the mathematic equations (discrete solution)
- Compare the discrete solution and experiment result

Some kind of Partial Differential Equation (PDE)

- Elliptic equation
  - Diffusion equation
  - Poisson's equation
- Parabolic equation
  - Heat equations

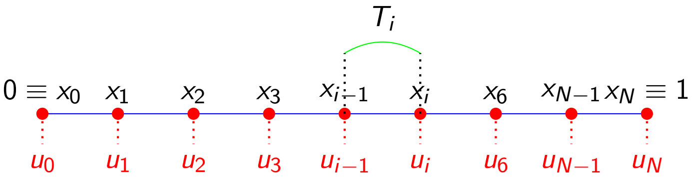
Laplace equation

We consider the partial differential equation on $(0, 1)[$

$$\begin{cases} -u_{xx}(x) & = f(x) \quad \text{for all } x \in (0,1) \\ u(0) & = 0 \\ u(1) & = 0 \end{cases} \tag{1}$$

To find the dicrete solution of this equation, there are many methods, we will choose a method which is the simplest methed, it is the finite difference scheme.

Mesh



Let us consider a uniform partion with $N+1$ points x_i for all $i = 0, 1, 2, \cdots, N$ (see figure). We have space step is $\Delta x = \frac{1}{N}$, then

$$x_i = i\Delta x$$

Our purpose is the value of the function at points x_i

$$u_i \simeq u(x_i) \text{ for all } i = 0, 1, 2, \cdots, N$$

Approximation of derivatives

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_{i+1}) - u(x_i)}{\Delta x} \text{ forward difference}$$

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_i) - u(x_{i-1})}{\Delta x} \text{ backward difference}$$

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x} \text{ central difference}$$

Approximation of derivatives (Cont.)

Use the Taylor series expansion at x_i

$$u(x_{i+1}) = u(x_i) + \frac{\partial u}{\partial x}(x_i)(x_{i+1} - x_i) + \frac{\frac{\partial^2 u}{\partial x^2}(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{\frac{\partial^3 u}{\partial x^3}(x_i)}{3!}(x_{i+1} - x_i)^3 + 0((x_{i+1} - x_i)^4)$$

Or

$$u(x_{i+1}) = u(x_i) + \frac{\partial u}{\partial x}(x_i)\Delta x + \frac{\frac{\partial^2 u}{\partial x^2}(x_i)}{2!}\Delta^2 x + \frac{\frac{\partial^3 u}{\partial x^3}(x_i)}{3!}\Delta^3 x + 0(\Delta^4 x) \tag{2}$$

We can approximate the derivative $\frac{\partial u}{\partial x}(x_i)$ that

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_{i+1}) - u(x_i)}{\Delta x} + 0(\Delta x)$$

Approximation of derivatives

It is similar, we obtain

$$u(x_{i-1}) = u(x_i) - \frac{\partial u}{\partial x}(x_i)\Delta x + \frac{\frac{\partial^2 u}{\partial x^2}(x_i)}{2!}\Delta^2 x - \frac{\frac{\partial^3 u}{\partial x^3}(x_i)}{3!}\Delta^3 x + 0(\Delta^4 x) \tag{3}$$

We can approximate the derivative $\frac{\partial u}{\partial x}(x_i)$ that

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_i) - u(x_{i-1})}{\Delta x} + 0(\Delta x)$$

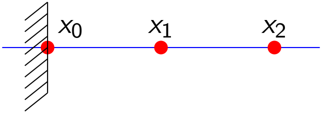
Let (2)-(3), we have

$$u(x_{i-1}) - u(x_{i-1}) = 2\frac{\partial u}{\partial x}(x_i)\Delta x + 2\frac{\frac{\partial^3 u}{\partial x^3}(x_i)}{3!}\Delta^3 x + 0(\Delta^4 x)$$

We can also approximate the derivative $\frac{\partial u}{\partial x}(x_i)$ that

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x} + 0(\Delta^2 x)$$

Approximation of derivative at boundary



We use the Taylor series expansion at x_0

$$u(x_1) = u(x_0) + \frac{\partial u}{\partial x}(x_0)(x_1 - x_0) + \frac{\frac{\partial^2 u}{\partial x^2}}{2!}(x_1 - x_0)^2 + 0((x_1 - x_0)^3)$$

Or

$$u(x_1) = u(x_0) + \frac{\partial u}{\partial x}(x_0)\Delta x + \frac{\frac{\partial^2 u}{\partial x^2}}{2!}\Delta^2 x + 0(\Delta^3 x) \qquad (4)$$

And

$$u(x_2) = u(x_0) + 2\frac{\partial u}{\partial x}(x_0)\Delta x + 2\frac{\partial^2 u}{\partial x^2}\Delta^2 x + 0(\Delta^3 x) \qquad (5)$$

Approximation of the derivatives at boundary (Cont.)

From (4), we have

$$\frac{\partial u}{\partial x}(x_0) = \frac{u(x_1) - u(x_0)}{\Delta x} + 0(\Delta x) \tag{6}$$

Combining (4) and (5), there holds

$$u(x_2) - 4u(x_1) = -3u(x_0) - 2\frac{\partial u}{\partial x}(x_0) + 0(\Delta^3 x)$$

or

$$\frac{\partial u}{\partial x}(x_0) = \frac{-3u(x_0) + 4u(x_1) - u(x_2)}{2\Delta x} + 0(\Delta^2 x) \tag{7}$$

Approximation of the second order derivatives

Using again the Taylor series expansion, there holds

$$u(x_{i+1}) = u(x_i) + \frac{\partial u}{\partial x}(x_i)\Delta x + \frac{\frac{\partial^2 u}{\partial x^2}(x_i)}{2!}\Delta^2 x + \frac{\frac{\partial^3 u}{\partial x^3}(x_i)}{3!}\Delta^3 x + 0(\Delta^4 x)$$

and

$$u(x_{i-1}) = u(x_i) - \frac{\partial u}{\partial x}(x_i)\Delta x + \frac{\frac{\partial^2 u}{\partial x^2}(x_i)}{2!}\Delta^2 x - \frac{\frac{\partial^3 u}{\partial x^3}(x_i)}{3!}\Delta^3 x + 0(\Delta^4 x)$$

Adding two previous approximate equations side by side, we have

$$u(x_{i+1}) + u(x_{i-1}) = 2u(x_i) + \frac{\partial^2 u}{\partial x^2}(x_i)\Delta^2 x + 0(\Delta^4 x) \tag{8}$$

or

$$\frac{\partial^2 u}{\partial x^2}(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta^2 x} + 0(\Delta^2 x) \tag{9}$$

Discretizing Laplace equation

From the first equation of (1), we have

$$-\frac{\partial^2 u}{\partial x^2}(x_i) = f(x_i) \qquad \text{for all } i = \overline{1, N-1}$$

Using the approximation in (9), there holds for all $i = \overline{1, N-1}$

$$-\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta^2 x} + 0(\Delta^2 x) = f_i$$

where $f_i = f(x_i)$ for $i = 1, ..., N-1$. Or we can write

$$-\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta^2 x} \approx f_i$$

Then we give the following scheme

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta^2 x} = f_i, \tag{10}$$

Dicrete equations

Using the Dirichlet boundary condition, we obtain

$$u_0 = u(x_0) = 0 \qquad \text{and} \qquad u(x_N) = u_N = 0$$

Linear system for the scheme

$$\begin{cases} i = 1, \frac{2u_1 - u_2}{\Delta^2 x} & = f_1 \\ i = 2, \frac{-u_1 + 2u_2 - u_3}{\Delta^2 x} & = f_2 \\ i = 3, \qquad \frac{-u_2 + 2u_3 - u_4}{\Delta^2 x} & = f_3 \\ \qquad \qquad \cdots & \\ i = N - 2 \qquad \frac{-u_{N-2} + 2u_{N-2} - u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i = N - 1, \qquad \frac{-u_{N-2} + 2u_{N-1}}{\Delta^2 x} & = f_{N-1} \end{cases}$$

Matrix form $AU = F$, $\quad A \in \mathbb{R}^N \times \mathbb{R}^N$, $\quad U, F \in \mathbb{R}^N$,

$$A = \frac{1}{\Delta^2 x}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{bmatrix}$$

The matrix A remains tridiagonal and symmetric positive definite

Other types of boundary condition

- Dirichlet Neumann Boundary Condition: $u(0) = \frac{\partial u}{\partial x}(1) = 0$.
  - Using the backward diffence at 1, it means that

$$\frac{\partial u}{\partial x}(1) = \frac{u_N - u_{N-1}}{\Delta x} = 0 \quad \Rightarrow u_{N-1} = u_N$$

Only changing the last equation in the linear system:

$$\frac{-u_{N-2} + u_{N-1}}{\Delta^2 x} = f_{N-1}$$

Other types of boundary condition

Then the linear system for the scheme

$$\begin{cases} i=1, \frac{2u_1-u_2}{\Delta^2 x} & = f_1 \\ i=2, \frac{-u_1+2u_2-u_3}{\Delta^2 x} & = f_2 \\ i=3, \qquad \frac{-u_2+2u_3-u_4}{\Delta^2 x} & = f_3 \\ \qquad \cdots & \\ i=N-2 \qquad \frac{-u_{N-3}+2u_{N-2}-u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i=N-1, \qquad \frac{-u_{N-2}+u_{N-1}}{\Delta^2 x} & = f_{N-1} \end{cases}$$

Other types of boundary condition (Cont.)

- Using the second order approximation of the derivative at 1, it means that

$$\frac{\partial u}{\partial x}(1) = \frac{-3u_N + 4u_{N-1} - u_{N-2}}{2\Delta x} = 0$$

Implying

$$u_N = \frac{4u_{N-1} - u_{N-2}}{3}$$

Changing only the last equation in the linear system, the last equation becomes

$$\frac{-u_{N-2} + u_{N-1}}{\Delta^2 x} = \frac{3}{2} f_{N-1}$$

Other types of boundary condition (Cont.)

Then the linear system for the scheme

$$\begin{cases} i = 1, \frac{2u_1 - u_2}{\Delta^2 x} & = f_1 \\ i = 2, \frac{-u_1 + 2u_2 - u_3}{\Delta^2 x} & = f_2 \\ i = 3, \quad \frac{-u_2 + 2u_3 - u_4}{\Delta^2 x} & = f_3 \\ \qquad \cdots & \\ i = N-2 \quad \frac{-u_{N-3} + 2u_{N-2} - u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i = N-1, \quad \frac{-u_{N-2} + u_{N-1}}{\Delta^2 x} & = \frac{3}{2} f_{N-1} \end{cases}$$

Other types of boundary condition (Cont.)

- Using the central diffrence at 1, it means that

$$\frac{\partial u}{\partial x}(1) = \frac{u_{N+1} - u_{N-1}}{2\Delta x}$$

Implying

$$u_{N+1} = u_{N-1}$$

We discretize additionally at point $x_N = 1$, there holds

$$\frac{-u_{N-1} + 2u_N - u_{N+1}}{\Delta^2 x} = f_N$$

where $f_N = f(x_N)$. Combining with discrete boundary condition, we have

$$\frac{-u_{N-1} + u_N}{\Delta^2 x} = \frac{f_N}{2}$$

Other types of boundary condition (Cont.)

Then the linear system for the scheme

$$\begin{cases} i=1, \frac{2u_1-u_2}{\Delta^2 x} & = f_1 \\ i=2, \frac{-u_1+2u_2-u_3}{\Delta^2 x} & = f_2 \\ i=3, \quad \frac{-u_2+2u_3-u_4}{\Delta^2 x} & = f_3 \\ \qquad \cdots & \\ i=N-1 \quad \frac{-u_{N-2}+2u_{N-1}-u_N}{\Delta^2 x} & = f_{N-1} \\ i=N, \quad \frac{-u_{N-1}+u_N}{\Delta^2 x} & = \frac{1}{2} f_N \end{cases}$$

Other types of boundary condition (Cont.)

- Non-homogeneous Dirichlet Boundary Condition:

$$u(0) = \alpha, \quad u(1) = \beta.$$

The first and last equations will be changed in the linear system, it means that

$$u_0 = \alpha \Rightarrow \frac{2u_1 - u_2}{\Delta^2 x} = f_1 + \frac{\alpha}{\Delta^2 x},$$

$$u_N = \beta \Rightarrow \frac{-u_{N-2} + 2u_{N-1}}{\Delta^2 x} = f_{N-1} + \frac{\beta}{\Delta^2 x}$$

Other types of boundary condition (Cont.)

Then the linear system for the scheme

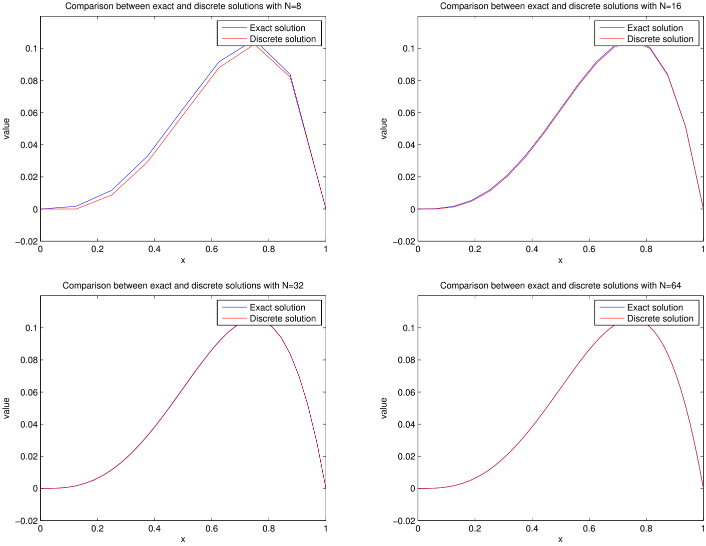
$$\begin{cases} i=1, \frac{2u_1-u_2}{\Delta^2 x} & = f_1 + \frac{\alpha}{\Delta^2 x} \\ i=2, \frac{-u_1+2u_2-u_3}{\Delta^2 x} & = f_2 \\ i=3, \quad \frac{-u_2+2u_3-u_4}{\Delta^2 x} & = f_3 \\ \cdots & \\ i=N-2 \quad \frac{-u_{N-3}+2u_{N-2}-u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i=N-1, \quad \frac{-u_{N-2}+2u_{N-1}}{\Delta^2 x} & = f_{N-1} + \frac{\beta}{\Delta^2 x} \end{cases}$$

Experiment test

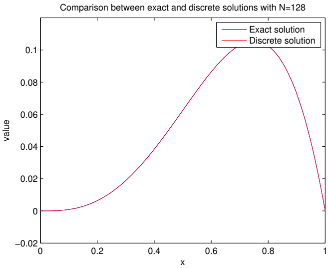
We set up with the following exact solution $u(x)$ and function $f(x)$

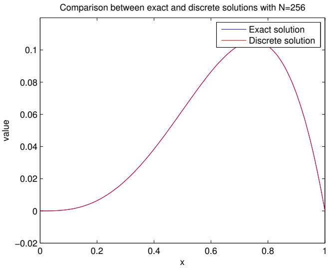
$$f(x) = 12x^2 - 6x$$
$$u(x) = x^3(1 - x)$$

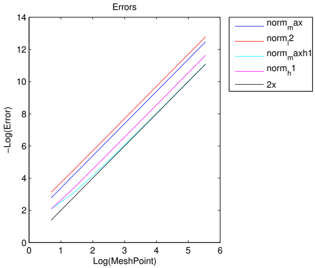
Experiment test



Experiment test







Norms

We definite

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \text{ and } \widehat{U} = \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \\ u(x_N) \end{bmatrix}$$

and Error $E = U - \widehat{U}$ containt the errors at each grid point.
To estimate the amplitude of error vector, we define some norms on it.

Norms

Given $V \in \mathbb{R}^{N+1}$, $V = (V_0, V_1, \cdots, V_N)^T$

$$\|V\|_{\infty,h} = \max_{0\le i\le N} |V_i| \quad (\text{ discrete } L_h^\infty)$$

$$\|V\|_{1,h} = \sum_{i=0}^{N-1} |V_i| h \quad (\text{ discrete } L_h^1)$$

$$\|V\|_{2,h}^2 = \sum_{i=0}^{N-1} |V_i|^2 h \quad (\text{ discrete } L_h^2)$$

Local Truncation Error

We can replace discrete solution u_i by exact solution $u(x_i)$ in (10). In general, the exact solution won't satisfy this equation, which define τ_i

$$\tau_i = -\frac{1}{h^2}(u(x_{i-1}) - 2u(x_i) + u(x_{i+1})) - f(x_i) \text{ for all } i = 1, \cdots, N-1 \tag{11}$$

Using Taylor series, we get

$$\tau_i = -\left[u''(x_i) + \frac{1}{12}h^2 u''''(x_i) + O(h^4)\right] - f(x_i) \tag{12}$$

Using our original differential equation (1) this becomes

$$\tau_i = -\frac{1}{12}h^2 u''''(x_i) - O(h^4) = O(h^2)$$

Global Error

We define τ to be the vector with component τ_i then

$$\tau = A\widehat{U} - F \tag{13}$$

also

$$A\widehat{U} = \tau + F \tag{14}$$

To obtain a relation between the local error τ and the global error $E = U - \widehat{U}$, we get

$$AE = -\tau \tag{15}$$

This is simply the matrix form of the system of equations

$$\frac{1}{h^2}(E_{i-1} - 2E_i + E_{i+1}) = -\tau_i \text{ for all } i \in [1, N-1] \tag{16}$$

with the boundary conditions

$$E_0 = E_N = 0 \tag{17}$$

Let A^{-1} be the inverse of the matrix A. Then solving the system (15) gives

$$E = -A^{-1}\tau$$

and taking norms gives

$$\|E\| = \|A^{-1}\tau\| \leq \|A^{-1}\|\|\tau\| \tag{18}$$

We know that $\|\tau\| = O(h^2)$ and we are hoping the same will be true of $\|E\| = O(h^2)$. It is clear what we need for this to be true: we need $\|A^{-1}\|$ to be bounded by some constant independent of h as $h \to 0$:

$$\|A^{-1}\| \leq C \text{ for } h \text{ sufficiently small}$$

Stability

Then we will have

$$\|E\| \leq C\|\tau\| \tag{19}$$

so $\|E\|$ goes to zero at least as fast as $\|\tau\|$.

Definition

Suppose a finite difference method for Laplace equation gives a sequence of matrix equations of the form $AU = F$. We say that the method is stable if A^{-1} exists for all h sufficiently small (for $h < h_0$, say) and if there is a constant C , independent of h, such that

$$\|A^{-1}\| \leq C \text{ for all } h < h_0 \tag{20}$$

Consistency

We say that a method is consistent with the differential equation and boundary conditions if

$$\|\tau\| \to 0 \text{ as } h \to 0 \tag{21}$$

Convergence

A method is said to be convergent if $\|E\| \to 0$ as $h \to 0$.
Combining the ideas introduced above we arrive at the conclusion that

$$\textit{consistency} + \textit{stability} \Longrightarrow \textit{convergence} \tag{22}$$

This is easily proved by using (20) and (21) to obtain the bound

$$\|E\| \leq \|A^{-1}\|\|\tau\| \leq C\|\tau\| \to 0 \text{ as } h \to 0 \tag{23}$$

Stability in L^2 norm

Since the matrix A is symmetric, the L_h^2-norm of A is equal to its spectral radius

$$\|A\|_{2,h} = \rho(A) = \max_{1 \le p \le N-1} \lambda_p \tag{24}$$

where λ_p refers to the p^{th} eigenvalue of the matrix A.
The matrix A^{-1} is also symmetric, and the eigenvalues of A^{-1} are simply the inverses of the eigenvalues of A, so

$$\|A^{-1}\|_{2,h} = \max_{1 \le p \le N-1} \lambda_p^{-1} = (\min_{1 \le p \le N-1} \lambda_p)^{-1} \tag{25}$$

So all we need to do is compute the eigenvalues of A and show that they are bounded away from zero as $h \to 0$

Stability in L^2 norm

We will now focus on one particular value of $h = \frac{1}{N}$. Then the $N-1$ eigenvalues of A are given by

$$\lambda_p = \frac{2}{h^2}(1 - \cos(\pi p h)) \text{ for all } p = 1, \cdots, N-1 \tag{26}$$

The eigenvector u^p corresponding to p has components u^p for $j = 1, \cdots, N-1$ given by

$$u_j^p = \sin(\pi p j h) \tag{27}$$

This can be verified by checking that $Au^p = \lambda_p u^p$. The j th component of the vector Au^p is

Stability in L^2 norm

$$
\begin{aligned}
(Au^p)_j &= -\frac{1}{h^2}(u^p_{j-1} - 2u^p_j + u^p_{j+1}) \\
&= -\frac{1}{h^2}(\sin(\pi p(j-1)h) - 2\sin(\pi pjh) + \sin(\pi p(j+1)h)) \\
&= -\frac{1}{h^2}(2\sin(\pi pjh)\cos(\pi ph) - 2\sin(\pi pjh)) \\
&= \lambda_p u^p_j
\end{aligned}
$$

From (26), we see that the smallest eigenvalue of A is

$$
\begin{aligned}
\lambda_1 &= \frac{2}{h^2}(1-\cos(\pi h)) \\
&= \frac{2}{h^2}(\frac{1}{2}\pi^2 h^2 - \frac{1}{24}\pi^4 h^4 + O(h^6)) \\
&= \pi^2 + O(h^2)
\end{aligned}
$$

Stability in L^2 norm

This is clearly bounded away from zero as $h \to 0$, so we see that the method is stable in the L_h^2-norm. Moreover we get an error bound from this:

$$\|E\|_{2,h} \leq \|A^{-1}\|_{2,h}\|\tau\|_{2,h} \approx \frac{1}{\pi^2}\|\tau\|_{2,h} \tag{28}$$

Since $\tau_j \approx \frac{h^2}{12}u''''(x_j)$, we expect $\|\tau\|_{2,h} \approx \frac{h^2}{12}\|u''''\|_{2,h} = \frac{h^2}{12}\|f''\|_{2,h}$

Stability

we define discrete L_h^2-norm

$$\|u\|_{2,h}^2 = \sum_{i=0}^{N-1} u_i^2 h$$

Multiplying (10) by u_i then sum over $i = \cdots, N-1$, we get

$$\sum_{i=1}^{N-1} \frac{(u_i - u_{i-1})u_i}{h^2} + \frac{(u_i - u_{i+1})u_i}{h^2} = \sum_{i=1}^{N-1} f_i u_i$$

$$\sum_{i=1}^{N-1} \frac{(u_i - u_{i-1})u_i}{h^2} + \sum_{i=1}^{N-1} \frac{(u_i - u_{i+1})u_i}{h^2} = \sum_{i=1}^{N-1} f_i u_i$$

We can change the index in the sum, we have

$$\sum_{i=0}^{N-2} \frac{(u_{i+1} - u_i)u_{i+1}}{h^2} + \sum_{i=1}^{N-1} \frac{(u_i - u_{i+1})u_i}{h^2} = \sum_{i=1}^{N-1} f_i u_i$$

Stability

$$\sum_{i=0}^{N-1} \frac{(u_{i+1} - u_i)u_{i+1}}{h^2} - \frac{(u_N - u_{N-1})u_N}{h^2}$$
$$+ \sum_{i=0}^{N-1} \frac{(u_i - u_{i+1})u_i}{h^2} - \frac{(u_0 - u_1)u_0}{h^2} = \sum_{i=1}^{N-1} f_i u_i$$

Sine $u_0 = u_N = 0$, then

$$\sum_{i=0}^{N-1} \frac{(u_{i+1} - u_i)u_{i+1}}{h^2} + \sum_{i=0}^{N-1} \frac{(u_i - u_{i+1})u_i}{h^2} = \sum_{i=1}^{N-1} f_i u_i$$
$$\sum_{i=0}^{N-1} \frac{(u_{i+1} - u_i)^2}{h^2} = \sum_{i=0}^{N-1} f_i u_i$$

Stability

We can write again

$$\sum_{i=0}^{N-1} (\delta_x^+ u)_i^2 h = \sum_{i=1}^{N-1} f_i u_i h, \tag{29}$$

where

$$(\delta_x^+ u)_i = \frac{u_{i+1} - u_i}{h}$$

Let's define the discrete H_h^1-norm

$$\|\delta_x^+\|_{2,h}^2 = \sum_{i=0}^{N-1} (\delta_x^+ u)_i^2 h$$

Stability

Applying Holder inequality, there holds

$$\sum_{i=1}^{N-1} f_i u_i h \leq \left(\sum_{i=0}^{N-1} h f_i^2\right)^{1/2} \left(\sum_{i=0}^{N-1} h u_i^2\right)^{1/2} = \|f\|_{2,h}\|u\|_{2,h}$$

From (29), we get

$$\|\delta_x^+\|_{2,h}^2 \leq \|f\|_{2,h}\|u\|_{2,h} \tag{30}$$

Stability

Lemma

There exists a constant positive C_Ω such that

$$\|u\|_{2,h} \leq C_\Omega \|\delta_x^+ u\|\|_{2,h}$$

Proof: Since $u_0 = 0$ then

$$u_i = u_i - u_0 = (u_i - u_{i-1}) + \cdots + (u_1 - u_0) = \sum_{j=0}^{i-1} (u_{j+1} - u_j)$$

$$= \sum_{j=0}^{i-1} \frac{u_{j+1} - u_j}{h} h = \sum_{j=0}^{i-1} (\delta_x^+ u)_i . h$$

Stability

Applying Holder inequality, there holds

$$u_i^2 \leq \sum_{j=0}^{i-1} h \sum_{j=0}^{i-1} (\delta_x^+ u)_j^2 h \leq Nh \sum_{j=0}^{N-1} (\delta_x^+ u)_j^2 h = \|\delta_x^+ u\|_{2,h}^2$$

So

$$\|u\|_{2,h}^2 = \sum_{i=0}^{N-1} h u_i^2 \leq \sum_{i=1}^{N-1} h\|\delta_x^+ u\|_{2,h}^2 = h(N-1)\|\delta_x^+ u\|_{2,h}^2 \leq \|\delta_x^+ u\|_{2,h}^2$$

We have completed the proof of the lemma. Using the lemma and (30), we get

$$\|\delta_x^+ u\|_{2,h} \leq \|f\|_{2,h}$$

Consistency

Let L be the differential operator, $\widehat{u}$ be a exact solution of the following equation:

$$L\widehat{u}(x) = f(x), \text{ for all } x \in \Omega$$

Let L_h be the discrete differential operator of L, and $\{u_i\}_{i=0}^{N}$ be the discrete solution, we have

$$L_h u_i = f_i \text{ for all } i \in [1, N-1]$$

Consistency (Cont.)

Definition

A finite differential scheme is said to be consistent with the partial differential equation it present, if for any smooth solution u, the truncation error of the scheme:

$$\tau_i = L_h\widehat{u}(x_i) - f(x_i) \text{ for all } i \in [1, N-1]$$

tends uniformly forward to zero when h tends to zero, that mean that

$$\lim_{h\to 0} \|\tau\|_{\infty,h} = 0$$

Consistency (Cont.)

Lemma

Suppose $\widehat{u} \in C^4(\Omega)$. Then, the numerical scheme in (10) is cosistent and second-order accuracy for the norm $\|\cdot\|_\infty$

Proof: We write again the definition L, L_h operators of our case:

$$L(\widehat{u})(x_i) = -\frac{\partial^2 \widehat{u}}{\partial x^2}(x_i)$$

$$L_h(\widehat{u})(x_i) = -\frac{\widehat{u}(x_{i-1}) - 2\widehat{u}(x_i) + \widehat{u}(x_{i+1})}{h^2}$$

By using the fact that

$$L(\widehat{u})(x_i) = -\frac{\partial^2 \widehat{u}}{\partial x^2}(x_i) = f(x_i)$$

Consistency (Cont.)

We have

$$\tau_i = L_h(\widehat{u})(x_i) - f(x_i) = L_h(\widehat{u})(x_i) - L(\widehat{u})(x_i)$$

Using the defintion of L and L_h, there holds

$$\tau_i = -\frac{\widehat{u}(x_{i-1}) - 2\widehat{u}(x_i) + \widehat{u}(x_{i+1})}{h^2} + \frac{\partial^2 \widehat{u}}{\partial x^2}(x_i)$$

Using the Taylor series expansion respect x, there exists $\eta_i \in [x_{i-1}, x_{i+1}]$ such that

$$-\frac{\widehat{u}(x_{i-1}) - 2\widehat{u}(x_i) + \widehat{u}(x_{i+1})}{h^2} + \frac{\partial^2 \widehat{u}}{\partial x^2}(x_i) = \frac{-h^2}{12}\frac{\partial^4 \widehat{u}}{\partial x^4}(\eta_i)$$

Consistency (Cont.)

we get

$$\tau_i = -\frac{h^2}{12}\frac{\partial^4 \widehat{u}}{\partial x^4}(\eta_i) = -\frac{h^2}{12}\frac{\partial^2 f}{\partial x^2}(\eta_i)$$

Thus,

$$\|\tau\|_{\infty,h} \leq \frac{h^2}{12}\|\frac{\partial^2 f}{\partial x^2}\|_{\infty}$$

and

$$\|\tau\|_{2,h} \leq \frac{h^2}{12}\|\frac{\partial^2 f}{\partial x^2}\|_{2,h}$$

Convergence

Lemma

Let u be the exact solution and u_h be the discrete solution, there holds

$$\lim_{h\to 0} \|\delta_x^+(\widehat{u}-u)\|_{2,h} = 0.$$

Proof: We have

$$\tau_i = L_h(\widehat{u})(x_i) - f(x_i) = L_h(\widehat{u})(x_i) - L_h(u)(x_i) = L_h(\widehat{u}-u)(x_i)$$

Using the proof of stability, we have

$$\|\delta_x^+(\widehat{u}-u)\|_{2,h} \le \|\tau\|_{2,h} \le \frac{h^2}{12}\|\frac{\partial f^2}{\partial x^2}\|_{2,h}$$