Divided Difference technique to generate polynomials recursively.

The divided differences of $f(x)$ with respect to $x_0, x_1, x_2, \ldots, x_n$ are used to express $P_n(x)$ in the form:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

for appropriate constants $a_0, a_1, \ldots, a_n$.

To determine 'a_0' we substitute $x = x_0$ in $P_n(x)$.

$$P_n(x_0) = a_0 = f(x_0).$$

Similarly $P_n(x_1) = a_0 + a_1(x_1 - x_0)$

$$\Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

In divided difference notation which is related to Aitken's Δ^2 notation, we write

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots + f[x_0,x_1,\ldots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

where $f[x_0] = f(x_0)$ — 0^{th} Divided diff.

$$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$$ — 1^{st} Div. Dif.

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

2^{nd} Divided difference.

$$f[x_0,x_1,\ldots,x_n] = \frac{f[x_1,x_2,\ldots,x_n]-f[x_0,x_1,\ldots,x_{n-1}]}{x_n-x_0}.$$

Example Complete the divided difference table for the data:

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Construct the Interpolating polynomial that uses all the data.

Solution: 1^{st} Newton divided difference involving x_0 and x_1 is

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$= \frac{0.6200860 - 0.7651977}{1.3 - 1.0}$$

$$= -0.4837057.$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

$$= \frac{0.4554022 - 0.6200860}{1.6 - 1.3}.$$

$$= -0.548926.$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$$

$$= \frac{0.2818186 - 0.4554022}{1.9 - 1.6}$$

$$= -0.578612.$$

$$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$$

$$= \frac{0.1103623 - 0.2818186}{2.2 - 1.9}$$

$$= -0.571521$$

Second Divided differences

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{-0.548926 + 0.4837057}{1.6 - 1.0}.$$

$$= -0.1087005.$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$= \frac{-0.578612 + 0.54892}{1.9 - 1.3}$$

$$= -0.0494767.$$

$$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$$

$$= \frac{-0.571521 + 0.578612}{2.2 - 1.6}.$$

$$= 0.0118183.$$

$$f[x_0, x_1, x_2, x_3]$$

$$= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{-0.0494767 + 0.1087005}{1.9 - 1.0}$$

$$= 0.06580423.$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$$

$$= \frac{0.0118183 + 0.0494767}{2.2 - 1.3}$$

$$= 0.0681056$$

$f[x_0, x_1, x_2, x_3, x_4]$

$$= \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0}$$

$$= \frac{0.0681056 - 0.06580423}{2.2 - 1.0}$$

$$= 0.001917808.$$

The interpolating polynomial that uses all the data in the Newton's forward divided difference form is given as follows:

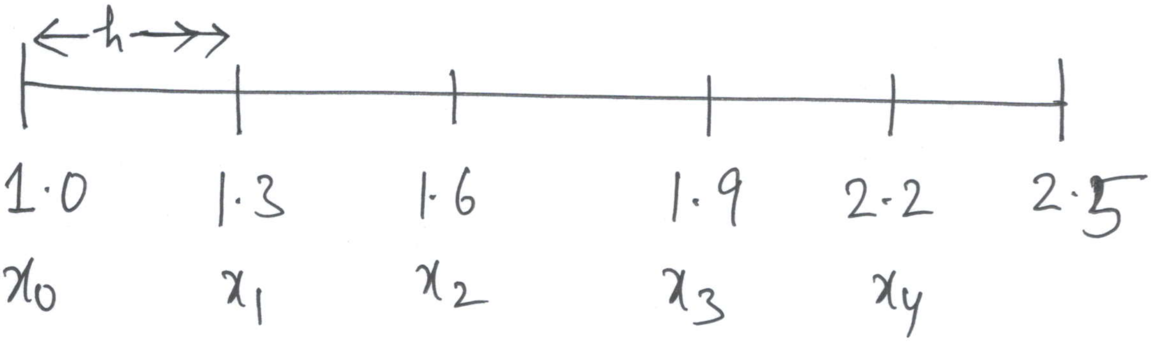
$$P_4(x) = 0.7651977 - 0.4837057(x-1.0)$$
$$- 0.1087005(x-1.0)(x-1.3)$$
$$+ 0.06580423(x-1.0)(x-1.3)(x-1.6)$$
$$+ 0.001917808(x-1.0)(x-1.3)(x-1.6)(x-1.9).$$

$$P_4(1.5) = 0.7651977 - 0.4837057(1.5-1.0)$$
$$- 0.1087005(1.5-1.0)(1.5-1.3)$$
$$+ 0.06580423(1.5-1.0)(1.5-1.3)(1.5-1.6)$$
$$+ 0.001917808(1.5-1.0)(1.5-1.3)(1.5-1.6)(1.5-1.9).$$

$$= 0.7651977 - 0.4837057 \times 0.5$$
$$- 0.1087005 \times 0.5 \times 0.2$$
$$+ 0.06580423 \times 0.5 \times 0.2 \times (-0.1)$$
$$+ 0.001917808 \times 0.5 \times 0.2 \times -0.1 \times -0.4$$

$$= 0.5118244289$$

Newton's divided-difference formula can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing. In this case introduce the notation $h = x_{i+1} - x_i$, for each $i = 0, 1, \cdots n-1$.



Then any $x = x_0 + sh$. and the difference $x - x_i = x_0 + sh - x_0 - ih$
$$= (s-i)h.$$

The interpolating polynomial

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k](x-x_0)\cdots(x-x_{k-1})$$

becomes

$$P_n(x_0+sh) = f[x_0] + sh\, f[x_0, x_1] + s(s-1)h^2 \times f[x_0, x_1, x_2] + \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0, \cdots x_n].$$

$$= f[x_0] + \sum_{k=1}^{n} s(s-1)\cdots(s-k+1)h^k \times f[x_0, x_1, \ldots, x_k].$$

Using binomial-coefficient notation

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$$ we write

$$P_n(x) = P_n(x_0 + sh)$$

$$= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k f[x_0, x_1, \ldots, x_k]$$

Forward Differences

$$f[x_0, x_1] = \frac{1}{h}\left(f(x_1) - f(x_0)\right) = \frac{1}{h}\Delta f(x_0).$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right]$$

$$= \frac{1}{2h^2}\Delta^2 f(x_0).$$

$$\vdots$$

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!\, h^k}\Delta^k f(x_0).$$

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0).$$

Backward Differences

If the interpolating nodes are reordered from last to first as $x_n, x_{n-1}, \dots, x_0$,

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x-x_n)$$
$$+ f[x_n, x_{n-1}, x_{n-2}](x-x_n)(x-x_{n-1})(x-x_{n-2})$$
$$+ \dots + f[x_n, \dots, x_0](x-x_n)(x-x_{n-1}) \dots (x-x_1).$$

If in addition if they are equally spaced

$$x = x_n + sh \quad \& \quad x = x_i + (s+n-i)h.$$
$$x_n = x_i + (n-i)h.$$
$$\Rightarrow x - x_i = (s+n-i)h$$

$$P_n(x) = P_n(x_n + sh)$$

$$= f[x_n] + sh\, f[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, \dots x_{n-2}]$$
$$+ \dots + f[x_n, \dots, x_0](x-x_n)(x-x_{n-1})(x-x_{n-2}) \dots (x-x_1).$$

Because
$$x - x_n = (s+n-n)h = sh$$
$$x - x_{n-1} = (s+n-n+1)h$$
$$= (s+1)h$$
$$\vdots$$
$$x - x_1 = (s+n-1)h$$
$$x - x_0 = (s+n)h.$$

$$P_n(x) = P_n(x_n + sh)$$

$$= f[x_n] + sh\, f[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}]$$

$$+ \cdots + s(s+1)\cdots(s+n-1)h^n f[x_n, x_{n-1}, \ldots x_0]$$

To further simplicify the Newtons Backward divided differences formula we define the Backward differences as:

<u>Definition</u>: Given the sequence $\{p_n\}_{n=0}^{\infty}$ define the backward difference

$$\nabla p_n = p_n - p_{n-1}, \text{ for } n \geq 1.$$

$$\nabla^2 p_n = \nabla(\nabla p_n) = \nabla p_n - \nabla p_{n-1}.$$

or $\nabla^2 p_n$.

$\vdots$

$$\nabla^k p_n = \nabla(\nabla^{k-1} p_n), \quad n \geq 2.$$

The above definition implies that

$$f[x_n, x_{n-1}] = \frac{1}{h} \nabla f(x_n).$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2} \nabla^2 f(x_n).$$

and in general,

$$f[x_n, x_{n-1}, \ldots, x_{n-k}] = \frac{1}{k!\,h^k} \nabla^k f(x_n).$$

Consequently,

$$P_n(x) = f[x_n] + s\nabla f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_n)$$
$$+ \cdots + \frac{s(s+1)\cdots\cdots(s+n-1)}{n!}\nabla^n f(x_n).$$

Using the binomial coefficient notation

$$\binom{-s}{k} = \frac{-s(-s-1)\cdots\cdots(-s-k+1)}{k!}$$
$$= \frac{(-1)^k s(s+1)\cdots\cdots(s+k-1)}{k!}$$

then

$$P_n(x) = f[x_n] + (-1)^1\binom{-s}{1}\nabla f(x_n) + (-1)^2\binom{-s}{2}\cdot \nabla^2 f(x_n)$$
$$+ \cdots + (-1)^n\binom{-s}{n}\nabla^n f(x_n).$$
$$= f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k}\nabla^k f(x_n)$$

The same table produced earlier will be used for the Newton's backward divided difference interpolating poly.

x_i	$f(x_i)$	1st Divided difference	2nd divided differences	3rd divided differences	4th divided difference
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

Only one interpolating polynomial of degree at most 4 uses these five data points, but we will organize the data points to obtain the best approximating of degree 1, 2 and 3. This will give us a sense of accuracy of the 4^{th}-degree approximation for the given value of x.

If an approximation to $f(1.1)$ is required, the reasonable choice for the nodes would be $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$ and $x_4 = 2.2$ since this choice makes the earliest possible use of the data points closest to $x = 1.1$ and also makes use of the 4^{th} divided difference formula.. This implies $h = 0.3$ and

$$x = 1.1 = x_0 + sh$$
$$= 1.0 + s \times 0.3.$$

$$\Rightarrow s = \frac{0.1}{0.3} = \frac{1}{3}.$$

The forward divided difference formula is used with the divided differences that have a solid underline '_______' in the table given in the previous pages.

$$P_4(1.1) = P_4\left(1.0 + \frac{1}{3}(0.3)\right)$$

$$= 0.7651977 + \frac{1}{3} \times (0.3)(-0.4837057)$$

$$+ \frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.1087339)$$

$$+ \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(0.3)^3(0.0658784)$$

$$+ \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(0.3)^4(0.0018251)$$

$$= 0.7196460.$$

To approximate a value when x is close to the end of the tabulated values, say $x = 2.0$, we would again like to make the earliest use of the data points closest to x. This requires using Newtons backward divided-difference formula with $h = 0.3$, $x_n = 2.2$

then $x = 2.0 = 2.2 + s \times 0.3$

$$\Rightarrow s = \frac{-0.2}{0.3} = -\frac{2}{3}.$$

The divided differences in the Table that have double underline '═══' are used. Here too we use the 4^{th} divided difference formula:

$$P_4(2.0) = P_4\left(2.2 - \frac{2}{3}(0.3)\right)$$

$$= 0.1103623 - \frac{2}{3}(0.3)(-0.5715210)$$

$$-\frac{2}{3}\left(\frac{1}{3}\right)(0.3)^2(0.0118183)$$

$$-\frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.0680685)$$

$$-\frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4(0.0018251)$$

$$= 0.2238754.$$