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List of integrals of irrational functions

The following is a list of integrals (antiderivative functions) of irrational functions. For a complete list of integral functions, see lists of integrals. Throughout this article the constant of integration is omitted for brevity.

Integrals involving $r = \sqrt{a^2 + x^2}$

$$\int r\,dx = \frac{1}{2}\left(xr + a^2\,\ln(x+r)\right)$$

$$\int r^3\,dx = \frac{1}{4}xr^3 + \frac{3}{8}a^2xr + \frac{3}{8}a^4\ln(x+r)$$

$$\int r^5\,dx = \frac{1}{6}xr^5 + \frac{5}{24}a^2xr^3 + \frac{5}{16}a^4xr + \frac{5}{16}a^6\ln(x+r)$$

$$\int xr\,dx = \frac{r^3}{3}$$

$$\int xr^3\,dx = \frac{r^5}{5}$$

$$\int xr^{2n+1}\,dx = \frac{r^{2n+3}}{2n+3}$$

$$\int x^2r\,dx = \frac{xr^3}{4} - \frac{a^2xr}{8} - \frac{a^4}{8}\ln(x+r)$$

$$\int x^2r^3\,dx = \frac{xr^5}{6} - \frac{a^2xr^3}{24} - \frac{a^4xr}{16} - \frac{a^6}{16}\ln(x+r)$$

$$\int x^3r\,dx = \frac{r^5}{5} - \frac{a^2r^3}{3}$$

$$\int x^3r^3\,dx = \frac{r^7}{7} - \frac{a^2r^5}{5}$$

$$\int x^3r^{2n+1}\,dx = \frac{r^{2n+5}}{2n+5} - \frac{a^2r^{2n+3}}{2n+3}$$

$$\int x^4r\,dx = \frac{x^3r^3}{6} - \frac{a^2xr^3}{8} + \frac{a^4xr}{16} + \frac{a^6}{16}\ln(x+r)$$

$$\int x^4r^3\,dx = \frac{x^3r^5}{8} - \frac{a^2xr^5}{16} + \frac{a^4xr^3}{64} + \frac{3a^6xr}{128} + \frac{3a^8}{128}\ln(x+r)$$

$$\int x^5r\,dx = \frac{r^7}{7} - \frac{2a^2r^5}{5} + \frac{a^4r^3}{3}$$

$$\int x^5r^3\,dx = \frac{r^9}{9} - \frac{2a^2r^7}{7} + \frac{a^4r^5}{5}$$

$$\int x^5r^{2n+1}\,dx = \frac{r^{2n+7}}{2n+7} - \frac{2a^2r^{2n+5}}{2n+5} + \frac{a^4r^{2n+3}}{2n+3}$$

$$\int \frac{r\,dx}{x} = r - a\ln\left|\frac{a+r}{x}\right| = r - a \operatorname{arsinh}\frac{a}{x}$$

$$\int \frac{r^3\,dx}{x} = \frac{r^3}{3} + a^2 r - a^3 \ln\left|\frac{a+r}{x}\right|$$

$$\int \frac{r^5\,dx}{x} = \frac{r^5}{5} + \frac{a^2 r^3}{3} + a^4 r - a^5 \ln\left|\frac{a+r}{x}\right|$$

$$\int \frac{r^7\,dx}{x} = \frac{r^7}{7} + \frac{a^2 r^5}{5} + \frac{a^4 r^3}{3} + a^6 r - a^7 \ln\left|\frac{a+r}{x}\right|$$

$$\int \frac{dx}{r} = \operatorname{arsinh}\frac{x}{a} = \ln\left(\frac{x+r}{a}\right)$$

$$\int \frac{dx}{r^3} = \frac{x}{a^2 r}$$

$$\int \frac{x\,dx}{r} = r$$

$$\int \frac{x\,dx}{r^3} = -\frac{1}{r}$$

$$\int \frac{x^2\,dx}{r} = \frac{x}{2}r - \frac{a^2}{2}\operatorname{arsinh}\frac{x}{a} = \frac{x}{2}r - \frac{a^2}{2}\ln\left(\frac{x+r}{a}\right)$$

$$\int \frac{dx}{xr} = -\frac{1}{a}\operatorname{arsinh}\frac{a}{x} = -\frac{1}{a}\ln\left|\frac{a+r}{x}\right|$$

Integrals involving $s = \sqrt{x^2 - a^2}$

Assume $x^2 > a^2$ (for $x^2 < a^2$, see next section):

$$\int s\,dx = \frac{1}{2}\left(xs - a^2 \ln|x+s|\right)$$

$$\int xs\,dx = \frac{1}{3}s^3$$

$$\int \frac{s\,dx}{x} = s - |a| \arccos\left|\frac{a}{x}\right|$$

$\displaystyle\int \frac{dx}{s} = \ln\left|\frac{x+s}{a}\right|$. Here $\displaystyle\ln\left|\frac{x+s}{a}\right| = \operatorname{sgn}(x)\ \operatorname{arcosh}\left|\frac{x}{a}\right| = \frac{1}{2}\ln\left(\frac{x+s}{x-s}\right)$, where the positive value of $\displaystyle\operatorname{arcosh}\left|\frac{x}{a}\right|$ is to be taken.

$$\int \frac{dx}{xs} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right|$$

$$\int \frac{x\,dx}{s} = s$$

$$\int \frac{x\,dx}{s^3} = -\frac{1}{s}$$

$$\int \frac{x\,dx}{s^5} = -\frac{1}{3s^3}$$

$$\int \frac{x\,dx}{s^7} = -\frac{1}{5s^5}$$

$$\int \frac{x\,dx}{s^{2n+1}} = -\frac{1}{(2n-1)s^{2n-1}}$$

$$\int \frac{x^{2m}\,dx}{s^{2n+1}} = -\frac{1}{2n-1}\frac{x^{2m-1}}{s^{2n-1}} + \frac{2m-1}{2n-1}\int \frac{x^{2m-2}\,dx}{s^{2n-1}}$$

$$\int \frac{x^2\,dx}{s} = \frac{xs}{2} + \frac{a^2}{2}\ln\left|\frac{x+s}{a}\right|$$

$$\int \frac{x^2\,dx}{s^3} = -\frac{x}{s} + \ln\left|\frac{x+s}{a}\right|$$

$$\int \frac{x^4\,dx}{s} = \frac{x^3 s}{4} + \frac{3}{8}a^2 xs + \frac{3}{8}a^4\ln\left|\frac{x+s}{a}\right|$$

$$\int \frac{x^4\,dx}{s^3} = \frac{xs}{2} - \frac{a^2 x}{s} + \frac{3}{2}a^2\ln\left|\frac{x+s}{a}\right|$$

$$\int \frac{x^4\,dx}{s^5} = -\frac{x}{s} - \frac{1}{3}\frac{x^3}{s^3} + \ln\left|\frac{x+s}{a}\right|$$

$$\int \frac{x^{2m}\,dx}{s^{2n+1}} = (-1)^{n-m}\frac{1}{a^{2(n-m)}}\sum_{i=0}^{n-m-1}\frac{1}{2(m+i)+1}\binom{n-m-1}{i}\frac{x^{2(m+i)+1}}{s^{2(m+i)+1}} \qquad (n > m \geq 0)$$

$$\int \frac{dx}{s^3} = -\frac{1}{a^2}\frac{x}{s}$$

$$\int \frac{dx}{s^5} = \frac{1}{a^4}\left[\frac{x}{s} - \frac{1}{3}\frac{x^3}{s^3}\right]$$

$$\int \frac{dx}{s^7} = -\frac{1}{a^6}\left[\frac{x}{s} - \frac{2}{3}\frac{x^3}{s^3} + \frac{1}{5}\frac{x^5}{s^5}\right]$$

$$\int \frac{dx}{s^9} = \frac{1}{a^8}\left[\frac{x}{s} - \frac{3}{3}\frac{x^3}{s^3} + \frac{3}{5}\frac{x^5}{s^5} - \frac{1}{7}\frac{x^7}{s^7}\right]$$

$$\int \frac{x^2\,dx}{s^5} = -\frac{1}{a^2}\frac{x^3}{3s^3}$$

$$\int \frac{x^2\,dx}{s^7} = \frac{1}{a^4}\left[\frac{1}{3}\frac{x^3}{s^3} - \frac{1}{5}\frac{x^5}{s^5}\right]$$

$$\int \frac{x^2\,dx}{s^9} = -\frac{1}{a^6}\left[\frac{1}{3}\frac{x^3}{s^3} - \frac{2}{5}\frac{x^5}{s^5} + \frac{1}{7}\frac{x^7}{s^7}\right]$$

Integrals involving $u = \sqrt{a^2 - x^2}$

$$\int u\,dx = \frac{1}{2}\left(xu + a^2\arcsin\frac{x}{a}\right) \qquad (|x| \leq |a|)$$

$$\int xu\,dx = -\frac{1}{3}u^3 \qquad (|x| \leq |a|)$$

$$\int x^2 u\,dx = -\frac{x}{4}u^3 + \frac{a^2}{8}(xu + a^2\arcsin\frac{x}{a}) \qquad (|x| \leq |a|)$$

$$\int \frac{u\,dx}{x} = u - a\ln\left|\frac{a+u}{x}\right| \qquad (|x| \le |a|)$$

$$\int \frac{dx}{u} = \arcsin\frac{x}{a} \qquad (|x| \le |a|)$$

$$\int \frac{x^2\,dx}{u} = \frac{1}{2}\left(-xu + a^2\arcsin\frac{x}{a}\right) \qquad (|x| \le |a|)$$

$$\int u\,dx = \frac{1}{2}\left(xu - \operatorname{sgn} x\ \operatorname{arcosh}\left|\frac{x}{a}\right|\right) \qquad (\text{for } |x| \ge |a|)$$

$$\int \frac{x}{u}\,dx = -u \qquad (|x| \le |a|)$$

Integrals involving $R = \sqrt{ax^2 + bx + c}$

Assume $(ax^2 + bx + c)$ cannot be reduced to the following expression $(px + q)^2$ for some p and q.

$$\int \frac{dx}{R} = \frac{1}{\sqrt{a}}\ln|2\sqrt{a}R + 2ax + b| \qquad (\text{for } a > 0)$$

$$\int \frac{dx}{R} = \frac{1}{\sqrt{a}}\,\operatorname{arsinh}\frac{2ax+b}{\sqrt{4ac-b^2}} \qquad (\text{for } a > 0,\ 4ac - b^2 > 0)$$

$$\int \frac{dx}{R} = \frac{1}{\sqrt{a}}\ln|2ax + b| \quad (\text{for } a > 0,\ 4ac - b^2 = 0)$$

$$\int \frac{dx}{R} = -\frac{1}{\sqrt{-a}}\arcsin\frac{2ax+b}{\sqrt{b^2-4ac}} \qquad (\text{for } a < 0,\ 4ac - b^2 < 0,\ |2ax+b| < \sqrt{b^2-4ac})$$

$$\int \frac{dx}{R^3} = \frac{4ax+2b}{(4ac-b^2)R}$$

$$\int \frac{dx}{R^5} = \frac{4ax+2b}{3(4ac-b^2)R}\left(\frac{1}{R^2} + \frac{8a}{4ac-b^2}\right)$$

$$\int \frac{dx}{R^{2n+1}} = \frac{2}{(2n-1)(4ac-b^2)}\left(\frac{2ax+b}{R^{2n-1}} + 4a(n-1)\int\frac{dx}{R^{2n-1}}\right)$$

$$\int \frac{x}{R}\,dx = \frac{R}{a} - \frac{b}{2a}\int\frac{dx}{R}$$

$$\int \frac{x}{R^3}\,dx = -\frac{2bx+4c}{(4ac-b^2)R}$$

$$\int \frac{x}{R^{2n+1}}\,dx = -\frac{1}{(2n-1)aR^{2n-1}} - \frac{b}{2a}\int\frac{dx}{R^{2n+1}}$$

$$\int \frac{dx}{xR} = -\frac{1}{\sqrt{c}}\ln\left|\frac{2\sqrt{c}R + bx + 2c}{x}\right|,\ c > 0$$

$$\int \frac{dx}{xR} = -\frac{1}{\sqrt{c}}\operatorname{arsinh}\left(\frac{bx+2c}{|x|\sqrt{4ac-b^2}}\right),\ c < 0$$

$$\int \frac{dx}{xR} = \frac{1}{\sqrt{-c}}\arcsin\left(\frac{bx+2c}{|x|\sqrt{b^2-4ac}}\right),\ c < 0, b^2 - 4ac > 0$$

$$\int \frac{dx}{xR} = -\frac{2}{bx}\left(\sqrt{ax^2+bx}\right),\ c=0$$

$$\int \frac{x^2}{R}\,dx = \frac{2ax-3b}{4a^2}R + \frac{3b^2-4ac}{8a^2}\int\frac{dx}{R}$$

$$\int \frac{dx}{x^2R} = -\frac{R}{cx} - \frac{b}{2c}\int\frac{dx}{xR}$$

$$\int R\,dx = \frac{2ax+b}{4a}R + \frac{4ac-b^2}{8a}\int\frac{dx}{R}$$

$$\int xR\,dx = \frac{R^3}{3a} - \frac{b(2ax+b)}{8a^2}R - \frac{b(4ac-b^2)}{16a^2}\int\frac{dx}{R}$$

$$\int x^2R\,dx = \frac{6ax-5b}{24a^2}R^3 + \frac{5b^2-4ac}{16a^2}\int R\,dx$$

$$\int \frac{R}{x}\,dx = R + \frac{b}{2}\int\frac{dx}{R} + c\int\frac{dx}{xR}$$

$$\int \frac{R}{x^2}\,dx = -\frac{R}{x} + a\int\frac{dx}{R} + \frac{b}{2}\int\frac{dx}{xR}$$

$$\int \frac{x^2\,dx}{R^3} = \frac{(2b^2-4ac)x+2bc}{a(4ac-b^2)R} + \frac{1}{a}\int\frac{dx}{R}$$

Integrals involving $S = \sqrt{ax + b}$

$$\int S\,dx = \frac{2S^3}{3a}$$

$$\int \frac{dx}{S} = \frac{2S}{a}$$

$$\int \frac{dx}{xS} = \begin{cases} -\dfrac{2}{\sqrt{b}}\operatorname{arcoth}\left(\dfrac{S}{\sqrt{b}}\right) & (\text{for } b>0, \quad ax>0) \\ -\dfrac{2}{\sqrt{b}}\operatorname{artanh}\left(\dfrac{S}{\sqrt{b}}\right) & (\text{for } b>0, \quad ax<0) \\ \dfrac{2}{\sqrt{-b}}\arctan\left(\dfrac{S}{\sqrt{-b}}\right) & (\text{for } b<0) \end{cases}$$

$$\int \frac{S}{x}\,dx = \begin{cases} 2\left(S-\sqrt{b}\operatorname{arcoth}\left(\dfrac{S}{\sqrt{b}}\right)\right) & (\text{for } b>0, \quad ax>0) \\ 2\left(S-\sqrt{b}\operatorname{artanh}\left(\dfrac{S}{\sqrt{b}}\right)\right) & (\text{for } b>0, \quad ax<0) \\ 2\left(S-\sqrt{-b}\arctan\left(\dfrac{S}{\sqrt{-b}}\right)\right) & (\text{for } b<0) \end{cases}$$

$$\int \frac{x^n}{S}\,dx = \frac{2}{a(2n+1)}\left(x^nS - bn\int\frac{x^{n-1}}{S}\,dx\right)$$

$$\int x^nS\,dx = \frac{2}{a(2n+3)}\left(x^nS^3 - nb\int x^{n-1}S\,dx\right)$$

$$\int \frac{1}{x^n S}\,dx = -\frac{1}{b(n-1)}\left(\frac{S}{x^{n-1}} + \left(n-\frac{3}{2}\right)a\int\frac{dx}{x^{n-1}S}\right)$$

References

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