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# Random Polynomials

KAMBIZ FARAHMAND

Department of Mathematical Statistics, University of Cape Town

**Abstract.** The behaviour of an algebraic and a trigonometric polynomial with real random coefficients is reviewed, and the number of times that the curves representing these polynomials cross any real line in the $xy$-plane is presented.

Let $a_0, a_1, a_2, \ldots, a_{n-1}$ be a sequence of independent normally distributed random variables with mathematical expectation $\mu$ and variance one. The set of equations $y = f(x)$ where

$$f(x) = \sum_{i=0}^{n-1} a_i x^i$$

represent a family of curves in the $xy$-plane. Kac [7] shows that for $\mu = 0$ the number of times that this family crosses the line $x$-axis, on average, is $(2/\pi)\log_n$. Ibragimov and Maslova [5] obtained the same number of crossings for a case which includes the results due to Kac [7, 8], Littlewood and Offord [9], and others. They consider the case when the coefficients $a_j (j = 0, 1, \ldots, n-1)$ belong to the domain of attraction of normal law. Their method also allowed them [6] to show that the number of crossings reduces by half when $\mu \neq 0$. Farahmand [2] studied the number of times that this family of curves crosses the level $K = o(\sqrt{n})$ (crossing with line $y = K$) and showed that this number decreases as $K$ increases. He [3] also showed that even in this case the number of crossings for $\mu \neq 0$ is half of those for $\mu = 0$.

Denote by $N_K(a, b)$ and $N_{KX}(a, b)$ the number of times that this family crosses the line $y = K$ and $y = Kx$, respectively, where $K$ is any constant independent of $x$. The following theorem shows the behaviour of the family of curves $f(x)$ in the $xy$-plane.

THEOREM 1. *If the coefficients of $f(x)$ are independent normally distributed random variables with mean $\mu$ and variance one, then for $\mu = 0$ and $K$ as any constant such that $(K^2/n) \to 0$ as $n \to \infty$, the mathematical expectation of $N_K$ and $N_{KX}$ satisfies*

$$EN_{KX}(-1, 1) \sim EN_K(-1, 1) \sim (1/\pi)\log(n/K^2) \quad \text{if } K \to \infty \text{ as } n \to \infty$$

$$EN_{KX}(-1, 1) \sim EN_K(-1, 1) \sim (1/\pi)\log n \quad \text{if } K \text{ is unbounded}$$

$$EN_{KX}(-\infty, -1) + EN_{KX}(1, \infty) \sim EN_K(-\infty, -1) + EN_K(1, \infty) \sim (1/\pi)\log n.$$

*For $\mu \neq 0$ the right-hand side of the above formulae reduce by half.*

From the above theorem it is interesting to note that for sufficiently large $n$ we still obtain a sizeable number of crossings even when the line tends to be perpendicular to the $x$-axis (i.e., when $K \to \infty$). The following theorem reveals the behaviour of $f(x)$ for the extreme values of $K$.

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THEOREM 2. *If the coefficients of $f(x)$ follow the assumption of Theorem 1 then for $K$ being any constant such that $\exp(nf) < K < \infty$ where $f(n)$ is any function of $n$ such that $f(n) \to \infty$ as $n \to \infty$, then for all sufficiently large $n$*

$$EN_{KX}(-\infty,\infty) \sim EN_K(-\infty,\infty) \sim 1.$$

The following theorem considers the random trigonometric polynomial of the form

$$f(x) = \sum_{i=1}^{n} a_i \cos ix.$$

THEOREM 3. *If the coefficients of the random trigonometric polynomial $f(x)$ follow the assumption of Theorem 1 then for any $\mu$ (zero or nonzero) and any $K$ such that $K^2/n \to 0$ as $n \to \infty$ we have*

$$EN_K(0,2\pi) \sim EN_{KX}(0,2\pi) \sim 2n/\sqrt{3}.$$

Comparing the results of Theorem 1 and Theorem 3 shows several interesting differences of behaviour between the algebraic and the trigonometric curves. First, the trigonometric curves on average oscillate more frequently than the algebraic one. Second, the number of $K$ level crossings of the trigonometric curves is the same as $x$-axis crossings which for the algebraic case decreases as $K$ increases. Finally, changing $\mu = 0$ to nonzero $\mu$ will not affect the number of crossings of trigonometric curves, whereas in the case of algebraic curves they are reduced by half.

The proofs of the above theorems are based on a formula known as the Kac–Rice [10] formula.

## A FORMULA FOR THE EXPECTED NUMBER OF CROSSINGS

The number of real roots of the equation $f(x) - K = 0$ has the expectation given by the Kac–Rice formula

$$EN(a,b) = \int_a^b dx \int_{-\infty}^{\infty} |y| p(0,y)dy,$$

where $p(t,y)$ is the joint probability density for $f(x) - K$ and its derivative. For normally distributed coefficients Cramer and Leadbetter [1, p. 285] have calculated $EN(a,b)$ as

$$EN_K(a,b) = \int_a^b (B/A)(1-\lambda^2)^{1/2}\phi(\xi/A)[2\phi(\eta) + \eta\{2\Phi(\eta) - 1\}]dx$$

where

$$A^2 = \mathrm{Var}\{f(x) - K\}, \qquad B^2 = \mathrm{Var}\{f'(x)\}$$
$$\lambda = (AB)^{-1}\mathrm{cov}[\{f(x) - K\}, f'(x)], \quad \xi = E[f(x) - K]$$
$$\gamma = E[f'(x)] \text{ and } \eta = B^{-1}(1-\lambda^2)^{-1/2}(\gamma - B\lambda\xi/A).$$

Since the coefficients of $f(x)$ are independent normal random variables we can easily calculate $A$, $B$, $\lambda$, $\xi$, $\lambda$ and $\eta$ (for example see [2] and [4]). Using $\Phi(t) = 1/2 + (\pi)^{-1/2}\mathrm{erf}(t/\sqrt{2})$ simplifies the above Kac–Rice formula as the sum of one single and one double integral

$$\begin{aligned} EN_K(a,b) &= \int_a^b (\Delta/\pi A^2)\exp(-B^2K^2/2\pi) \\ &\quad + \sqrt{2}|KC|/(\pi A^3)\exp(-K^2/2A^2)\mathrm{erf}\{|KC|/A\Delta\sqrt{2}\}dx \\ &= I_1(a,b) + I_2(a,b) \text{ (say)}, \end{aligned}$$

where $C = \text{cov}[\{f(x) - K\},\ f'(x)]$ and $\Delta^2 = A^2B^2 - C^2$. In the same way the Kac–Rice formula for the equation $f(x) - Kx = 0$ can be found (see [4])

$$\begin{aligned} EN_{KX}(a,b) = \int_a^b & (\Delta/\pi A^2)\exp\{K^2(-A^2 + 2Cx - B^2x^2)/2\Delta^2\} \\ & + (\sqrt{2}/\pi)|K(Cx - A^2)|A^{-3}\exp(-K^2x^2/2A^2)\text{erf}\{|K(Cx - A^2)|/(\sqrt{2}A^2\Delta)\}dx \\ = {} & I_1'(a,b) + I_2'(a,b)\ (\text{say}). \end{aligned}$$

It can be shown that $I_2(a,b)$ and $I_2'(a,b)$ are small compared to $I_1(a,b)$ and $I_1'(a,b)$ of which the latter are the main contributors to $EN_K(a,b)$ and $EN_{KX}(a,b)$. This leads to an asymptotic formula for $EN_K(a,b)$ and $EN_{KX}(a,b)$ for Theorems 1, 2, and 3.

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Department of Mathematical Statistics, University of Cape Town, Rondebosch, 7700, SOUTH AFRICA