Absolute continuity

In calculus and real analysis, **absolute continuity** is a smoothness property of functions that is stronger than continuity and uniform continuity. The notion of absolute continuity allows one to obtain generalizations of the relationship between the two central operations of calculus—differentiation and integration. This relationship is commonly characterized (by the fundamental theorem of calculus) in the framework of Riemann integration, but with absolute continuity it may be formulated in terms of Lebesgue integration. For real-valued functions on the real line, two interrelated notions appear: **absolute continuity of functions** and **absolute continuity of measures**. These two notions are generalized in different directions. The usual derivative of a function is related to the *Radon–Nikodym derivative*, or *density*, of a measure. We have the following chains of inclusions for functions **over a compact subset** of the real line:

absolutely continuous ⊆ *uniformly continuous* = *continuous*

and, for a compact interval,

continuously differentiable ⊆ *Lipschitz continuous* ⊆ *absolutely continuous* ⊆ *bounded variation* ⊆ *differentiable almost everywhere*.

Absolute continuity of functions

A continuous function fails to be absolutely continuous if it fails to be uniformly continuous, which can happen if the domain of the function is not compact – examples are tan(x) over $[0, \pi/2)$, x^2 over the entire real line, and sin($1/x$) over (0, 1]. But a continuous function f can fail to be absolutely continuous even on a compact interval. It may not be "differentiable almost everywhere" (like the Weierstrass function, which is not differentiable anywhere). Or it may be differentiable almost everywhere and its derivative f' may be Lebesgue integrable, but the integral of f' differs from the increment of f (how much f changes over an interval). This happens for example with the Cantor function.

Definition

Let I be an interval in the real line $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is **absolutely continuous** on I if for every positive number ε, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I with $x_k < y_k \in I$ satisfies[1]

$$\sum_{k=1}^{N} (y_k - x_k) < \delta$$

then

$$\sum_{k=1}^{N} |f(y_k) - f(x_k)| < \varepsilon.$$

The collection of all absolutely continuous functions on I is denoted $\mathbf{AC}(I)$.

Equivalent definitions

The following conditions on a real-valued function *f* on a compact interval [*a*,*b*] are equivalent:[2]

1. *f* is absolutely continuous;
2. *f* has a derivative *f* ' almost everywhere, the derivative is Lebesgue integrable, and
$$f(x) = f(a) + \int_a^x f'(t)\,dt$$
for all *x* on [*a*,*b*];
3. there exists a Lebesgue integrable function *g* on [*a*,*b*] such that
$$f(x) = f(a) + \int_a^x g(t)\,dt$$
for all *x* in [*a*,*b*].

If these equivalent conditions are satisfied, then necessarily any function *g* as in condition 3. satisfies *g* = *f* ' almost everywhere.

Equivalence between (1) and (3) is known as the **fundamental theorem of Lebesgue integral calculus**, due to Lebesgue.[3]

For an equivalent definition in terms of measures see the section Relation between the two notions of absolute continuity.

Properties

- The sum and difference of two absolutely continuous functions are also absolutely continuous. If the two functions are defined on a bounded closed interval, then their product is also absolutely continuous.[4]
- If an absolutely continuous function is defined on a bounded closed interval and is nowhere zero then its reciprocal is absolutely continuous.[5]
- Every absolutely continuous function (over a compact interval) is uniformly continuous and, therefore, continuous. Every (globally) Lipschitz-continuous function is absolutely continuous.[6]
- If *f*: [*a*,*b*] → **R** is absolutely continuous, then it is of bounded variation on [*a*,*b*].[7]

- If $f: [a,b] \to \mathbf{R}$ is absolutely continuous, then it can be written as the difference of two monotonic nondecreasing absolutely continuous functions on $[a,b]$.
- If $f: [a,b] \to \mathbf{R}$ is absolutely continuous, then it has the Luzin *N* property (that is, for any $N \subseteq [a,b]$ such that $\lambda(N) = 0$, it holds that $\lambda(f(N)) = 0$, where λ stands for the Lebesgue measure on **R**).
- $f: I \to \mathbf{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the Luzin *N* property. This statement is also known as the Banach-Zareckiĭ theorem.[8]
- If $f: I \to \mathbf{R}$ is absolutely continuous and $g: \mathbf{R} \to \mathbf{R}$ is globally Lipschitz-continuous, then the composition $g \circ f$ is absolutely continuous. Conversely, for every function g that is not globally Lipschitz continuous there exists an absolutely continuous function f such that $g \circ f$ is not absolutely continuous.[9]

Examples

The following functions are uniformly continuous but **not** absolutely continuous:

- The Cantor function on [0, 1] (it is of bounded variation but not absolutely continuous);
- The function:
$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \sin(1/x), & \text{if } x \neq 0 \end{cases}$$
on a finite interval containing the origin.

The following functions are absolutely continuous but not α-Hölder continuous:

- The function $f(x) = x^\beta$ on $[0, c]$, for any $0 < \beta < \alpha < 1$

The following functions are absolutely continuous and α-Hölder continuous but not Lipschitz continuous:

- The function $f(x) = \sqrt{x}$ on $[0, c]$, for $\alpha \le 1/2$.

Generalizations

Let (X, d) be a metric space and let I be an interval in the real line **R**. A function $f: I \to X$ is **absolutely continuous** on I if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals $[x_k, y_k]$ of I satisfies:

$$\sum_k |y_k - x_k| < \delta$$

then:

$$\sum_k d\left(f(y_k), f(x_k)\right) < \epsilon.$$

The collection of all absolutely continuous functions from I into X is denoted $AC(I; X)$.

A further generalization is the space AC^p(I; X) of curves f: $I \to X$ such that:[10]

$$d\left(f(s), f(t)\right) \leq \int_s^t m(\tau)\, d\tau \text{ for all } [s,t] \subseteq I$$

for some m in the L^p space L^p(I).

Properties of these generalizations

- Every absolutely continuous function (over a compact interval) is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
- If f: [a,b] → X is absolutely continuous, then it is of bounded variation on [a,b].
- For $f \in$ AC^p(I; X), the metric derivative of f exists for λ-almost all times in I, and the metric derivative is the smallest $m \in L^p$(I; **R**) such that:[11]

$$d\left(f(s), f(t)\right) \leq \int_s^t m(\tau)\, d\tau \text{ for all } [s,t] \subseteq I.$$

Absolute continuity of measures

Definition

A measure μ on Borel subsets of the real line is absolutely continuous with respect to the Lebesgue measure λ if for every λ-measurable set A, $\lambda(A) = 0$ implies $\mu(A) = 0$. Equivalently, $\mu(A) > 0$ implies $\lambda(A) > 0$. This condition is written as $\mu \ll \lambda$. We say μ is *dominated* by λ.

In most applications, if a measure on the real line is simply said to be absolutely continuous — without specifying with respect to which other measure it is absolutely continuous — then absolute continuity with respect to the Lebesgue measure is meant.

The same principle holds for measures on Borel subsets of $\mathbb{R}^n, n \geq 2$.

Equivalent definitions

The following conditions on a finite measure μ on Borel subsets of the real line are equivalent:[12]

1. μ is absolutely continuous;
2. For every positive number ε there is a positive number $\delta > 0$ such that $\mu(A) < \varepsilon$ for all Borel sets A of Lebesgue measure less than δ;

3. There exists a Lebesgue integrable function g on the real line such that:

$$\mu(A) = \int_A g\,d\lambda$$

for all Borel subsets A of the real line.

For an equivalent definition in terms of functions see the section Relation between the two notions of absolute continuity.

Any other function satisfying (3) is equal to g almost everywhere. Such a function is called Radon–Nikodym derivative, or density, of the absolutely continuous measure μ.

Equivalence between (1), (2) and (3) holds also in $\mathbb{R}^n$ for all $n = 1, 2, 3, \ldots$.

Thus, the absolutely continuous measures on $\mathbb{R}^n$ are precisely those that have densities; as a special case, the absolutely continuous probability measures are precisely the ones that have probability density functions.

Generalizations

If μ and ν are two measures on the same measurable space $(X, \mathcal{A})$, μ is said to be **absolutely continuous with respect to** ν if $\mu(A) = 0$ for every set A for which $\nu(A) = 0$.[13] This is written as "$\mu \ll \nu$". That is:

$$\mu \ll \nu \qquad \text{if and only if} \qquad \text{for all } A \in \mathcal{A}, \quad (\nu(A) = 0 \ \text{ implies } \ \mu(A) = 0).$$

When $\mu \ll \nu$, then ν is said to be **dominating** μ.

Absolute continuity of measures is reflexive and transitive, but is not antisymmetric, so it is a preorder rather than a partial order. Instead, if $\mu \ll \nu$ and $\nu \ll \mu$, the measures μ and ν are said to be equivalent. Thus absolute continuity induces a partial ordering of such equivalence classes.

If μ is a signed or complex measure, it is said that μ is absolutely continuous with respect to ν if its variation $|\mu|$ satisfies $|\mu| \ll \nu$; equivalently, if every set A for which $\nu(A) = 0$ is μ-null.

The Radon–Nikodym theorem[14] states that if μ is absolutely continuous with respect to ν, and both measures are σ-finite, then μ has a density, or "Radon-Nikodym derivative", with respect to ν, which means that there exists a ν-measurable function f taking values in $[0, +\infty)$, denoted by $f = d\mu/d\nu$, such that for any ν-measurable set A we have:

$$\mu(A) = \int_A f\,d\nu.$$

Singular measures

Via Lebesgue's decomposition theorem,[15] every σ-finite measure can be decomposed into the sum of an absolutely continuous measure and a singular measure with respect to another σ-finite measure. See singular measure for examples of measures that are not absolutely continuous.

Relation between the two notions of absolute continuity

A finite measure μ on Borel subsets of the real line is absolutely continuous with respect to Lebesgue measure if and only if the point function:

$$F(x) = \mu((-\infty, x])$$

is an absolutely continuous real function. More generally, a function is locally (meaning on every bounded interval) absolutely continuous if and only if its distributional derivative is a measure that is absolutely continuous with respect to the Lebesgue measure.

If absolute continuity holds then the Radon–Nikodym derivative of μ is equal almost everywhere to the derivative of F.[16]

More generally, the measure μ is assumed to be locally finite (rather than finite) and $F(x)$ is defined as $\mu((0,x])$ for $x > 0$, 0 for $x = 0$, and $-\mu((x,0])$ for $x < 0$. In this case μ is the Lebesgue–Stieltjes measure generated by F.[17] The relation between the two notions of absolute continuity still holds.[18]

Notes

1. Royden 1988, Sect. 5.4, page 108; Nielsen 1997, Definition 15.6 on page 251; Athreya & Lahiri 2006, Definitions 4.4.1, 4.4.2 on pages 128,129. The interval I is assumed to be bounded and closed in the former two books but not the latter book.
2. Nielsen 1997, Theorem 20.8 on page 354; also Royden 1988, Sect. 5.4, page 110 and Athreya & Lahiri 2006, Theorems 4.4.1, 4.4.2 on pages 129,130.
3. Athreya & Lahiri 2006, before Theorem 4.4.1 on page 129.
4. Royden 1988, Problem 5.14(a,b) on page 111.
5. Royden 1988, Problem 5.14(c) on page 111.
6. Royden 1988, Problem 5.20(a) on page 112.
7. Royden 1988, Lemma 5.11 on page 108.
8. Bruckner, Bruckner & Thomson 1997, Theorem 7.11.

9. Fichtenholz 1923.
10. Ambrosio, Gigli & Savaré 2005, Definition 1.1.1 on page 23
11. Ambrosio, Gigli & Savaré 2005, Theorem 1.1.2 on page 24
12. Equivalence between (1) and (2) is a special case of Nielsen 1997, Proposition 15.5 on page 251 (fails for σ-finite measures); equivalence between (1) and (3) is a special case of the Radon–Nikodym theorem, see Nielsen 1997, Theorem 15.4 on page 251 or Athreya & Lahiri 2006, Item (ii) of Theorem 4.1.1 on page 115 (still holds for σ-finite measures).
13. Nielsen 1997, Definition 15.3 on page 250; Royden 1988, Sect. 11.6, page 276; Athreya & Lahiri 2006, Definition 4.1.1 on page 113.
14. Royden 1988, Theorem 11.23 on page 276; Nielsen 1997, Theorem 15.4 on page 251; Athreya & Lahiri 2006, Item (ii) of Theorem 4.1.1 on page 115.
15. Royden 1988, Proposition 11.24 on page 278; Nielsen 1997, Theorem 15.14 on page 262; Athreya & Lahiri 2006, Item (i) of Theorem 4.1.1 on page 115.
16. Royden 1988, Problem 12.17(b) on page 303.
17. Athreya & Lahiri 2006, Sect. 1.3.2, page 26.
18. Nielsen 1997, Proposition 15.7 on page 252; Athreya & Lahiri 2006, Theorem 4.4.3 on page 131; Royden 1988, Problem 12.17(a) on page 303.

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External links

- Absolute continuity (https://www.encyclopediaofmath.org/index.php/Absolute_continuity) at Encyclopedia of Mathematics (http://www.encyclopediaofmath.org/)
- Topics in Real and Functional Analysis (https://www.mat.univie.ac.at/~gerald/ftp/book-fa/index.html) by Gerald Teschl