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List of integrals of exponential functions

The following is a list of integrals of exponential functions. For a complete list of integral functions, please see the list of integrals.

Indefinite integral

Indefinite integrals are antiderivative functions. A constant (the constant of integration) may be added to the right hand side of any of these formulas, but has been suppressed here in the interest of brevity.

Integrals of polynomials

$$\int xe^{cx}\,dx = e^{cx}\left(\frac{cx-1}{c^2}\right) \qquad \text{for } c \neq 0;$$

$$\int x^2 e^{cx}\,dx = e^{cx}\left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right)$$

$$\begin{aligned}\int x^n e^{cx}\,dx &= \frac{1}{c}x^n e^{cx} - \frac{n}{c}\int x^{n-1}e^{cx}\,dx \\ &= \left(\frac{\partial}{\partial c}\right)^n \frac{e^{cx}}{c} \\ &= e^{cx}\sum_{i=0}^{n}(-1)^i \frac{n!}{(n-i)!c^{i+1}}x^{n-i} \\ &= e^{cx}\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!c^{n-i+1}}x^i\end{aligned}$$

$$\int \frac{e^{cx}}{x}\,dx = \ln|x| + \sum_{n=1}^{\infty}\frac{(cx)^n}{n\cdot n!}$$

$$\int \frac{e^{cx}}{x^n}\,dx = \frac{1}{n-1}\left(-\frac{e^{cx}}{x^{n-1}} + c\int \frac{e^{cx}}{x^{n-1}}\,dx\right) \qquad (\text{for } n \neq 1)$$

Integrals involving only exponential functions

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)}$$

$$\int e^{cx}\,dx = \frac{1}{c}e^{cx}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \qquad \text{for } a > 0,\ a \neq 1$$

Integrals involving the error function

In the following formulas, erf is the error function and Ei is the exponential integral.

$$\int e^{cx}\ln x\,dx = \frac{1}{c}\left(e^{cx}\ln|x| - \operatorname{Ei}(cx)\right)$$

$$\int xe^{cx^2}\,dx = \frac{1}{2c}e^{cx^2}$$

$$\int e^{-cx^2}\,dx = \sqrt{\frac{\pi}{4c}}\,\operatorname{erf}(\sqrt{c}x)$$

$$\int xe^{-cx^2}\,dx = -\frac{1}{2c}e^{-cx^2}$$

$$\int \frac{e^{-x^2}}{x^2}\,dx = -\frac{e^{-x^2}}{x} - \sqrt{\pi}\,\operatorname{erf}(x)$$

$$\int \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

Other integrals

$$\int e^{x^2}\,dx = e^{x^2}\left(\sum_{j=0}^{n-1} c_{2j}\frac{1}{x^{2j+1}}\right) + (2n-1)c_{2n-2}\int \frac{e^{x^2}}{x^{2n}}\,dx \quad \text{valid for any } n>0,$$

where $c_{2j} = \dfrac{1\cdot 3\cdot 5\cdots(2j-1)}{2^{j+1}} = \dfrac{(2j)!}{j!2^{2j+1}}$.

(Note that the value of the expression is *independent* of the value of n, which is why it does not appear in the integral.)

$$\int \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{m}\,dx = \sum_{n=0}^{m} \frac{(-1)^n(n+1)^{n-1}}{n!}\Gamma(n+1,-\ln x) + \sum_{n=m+1}^{\infty} (-1)^n a_{mn}\Gamma(n+1,-\ln x) \qquad (\text{for } x>0)$$

where $a_{mn} = \begin{cases} 1 & \text{if } n=0, \\ \dfrac{1}{n!} & \text{if } m=1, \\ \dfrac{1}{n}\sum_{j=1}^{n} ja_{m,n-j}a_{m-1,j-1} & \text{otherwise} \end{cases}$

and $\Gamma(x,y)$ is the upper incomplete gamma function.

$$\int \frac{1}{ae^{\lambda x}+b}\,dx = \frac{x}{b} - \frac{1}{b\lambda}\ln\left(ae^{\lambda x}+b\right) \text{ when } b\neq 0,\ \lambda\neq 0, \text{ and } ae^{\lambda x}+b>0.$$

$$\int \frac{e^{2\lambda x}}{ae^{\lambda x}+b}\,dx = \frac{1}{a^2\lambda}\left[ae^{\lambda x}+b-b\ln\left(ae^{\lambda x}+b\right)\right] \text{ when } a\neq 0,\ \lambda\neq 0, \text{ and } ae^{\lambda x}+b>0.$$

$$\int \frac{ae^{cx}-1}{be^{cx}-1}\,dx = \frac{(a-b)\log(1-be^{cx})}{bc} + x.$$

$$\int e^x\left(f(x)+f'(x)\right)\mathrm{dx} = e^x f(x) + C$$

$$\int e^x\left(f(x) - (-1)^n\frac{d^n f(x)}{dx^n}\right)\,dx = e^x\sum_{k=1}^{n}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}} + C$$

$$\int e^{-x}\left(f(x) - \frac{d^n f(x)}{dx^n}\right)\,dx = -e^{-x}\sum_{k=1}^{n}\frac{d^{k-1}f(x)}{dx^{k-1}} + C$$

$$\int e^{ax}\left((a)^n f(x) - (-1)^n \frac{d^n f(x)}{dx^n}\right) dx = e^{ax}\sum_{k=1}^{n}(a)^{n-k}(-1)^{k-1}\frac{d^{k-1} f(x)}{dx^{k-1}} + C$$

Definite integrals

$$\begin{aligned}\int_0^1 e^{x\cdot\ln a + (1-x)\cdot\ln b}\,dx &= \int_0^1 \left(\frac{a}{b}\right)^x \cdot b\,dx \\ &= \int_0^1 a^x \cdot b^{1-x}\,dx \\ &= \frac{a-b}{\ln a - \ln b} \qquad \text{for } a>0,\ b>0,\ a\neq b\end{aligned}$$

The last expression is the logarithmic mean.

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \quad (\operatorname{Re}(a)>0)$$

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a>0)$$ (the Gaussian integral)

$$\int_{-\infty}^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad (a>0)$$

$$\int_{-\infty}^\infty e^{-ax^2} e^{-\frac{b}{x^2}}\,dx = \sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}} \quad (a,b>0)$$

$$\int_{-\infty}^\infty e^{-(ax^2+bx)}\,dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \quad (a>0)$$

$$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}-c} \quad (a>0)$$

$$\int_{-\infty}^\infty e^{-ax^2} e^{-2bx}\,dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{a}} \quad (a>0)$$ (see Integral of a Gaussian function)

$$\int_{-\infty}^\infty x e^{-a(x-b)^2}\,dx = b\sqrt{\frac{\pi}{a}} \quad (\operatorname{Re}(a)>0)$$

$$\int_{-\infty}^\infty x e^{-ax^2+bx}\,dx = \frac{\sqrt{\pi}b}{2a^{3/2}}e^{\frac{b^2}{4a}} \quad (\operatorname{Re}(a)>0)$$

$$\int_{-\infty}^\infty x^2 e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \quad (a>0)$$

$$\int_{-\infty}^\infty x^2 e^{-(ax^2+bx)}\,dx = \frac{\sqrt{\pi}(2a+b^2)}{4a^{5/2}}e^{\frac{b^2}{4a}} \quad (\operatorname{Re}(a)>0)$$

$$\int_{-\infty}^\infty x^3 e^{-(ax^2+bx)}\,dx = \frac{\sqrt{\pi}(6a+b^2)b}{8a^{7/2}}e^{\frac{b^2}{4a}} \quad (\operatorname{Re}(a)>0)$$

$$\int_0^\infty x^n e^{-ax^2}\,dx = \begin{cases} \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{2\left(a^{\frac{n+1}{2}}\right)} & (n>-1,\ a>0) \\ \dfrac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\dfrac{\pi}{a}} & (n=2k,\ k \text{ integer},\ a>0) \\ \dfrac{k!}{2(a^{k+1})} & (n=2k+1,\ k \text{ integer},\ a>0)\end{cases}$$

(the operator $!!$ is the Double factorial)

$$\int_0^\infty x^n e^{-ax}\,dx = \begin{cases} \dfrac{\Gamma(n+1)}{a^{n+1}} & (n>-1,\ \operatorname{Re}(a)>0) \\ \\ \dfrac{n!}{a^{n+1}} & (n=0,1,2,\ldots,\ \operatorname{Re}(a)>0) \end{cases}$$

$$\int_0^1 x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}\left[1-e^{-a}\sum_{i=0}^{n}\frac{a^i}{i!}\right]$$

$$\int_0^b x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}\left[1-e^{-ab}\sum_{i=0}^{n}\frac{(ab)^i}{i!}\right]$$

$$\int_0^\infty e^{-ax^b}\,dx = \frac{1}{b}\,a^{-\frac{1}{b}}\,\Gamma\left(\frac{1}{b}\right)$$

$$\int_0^\infty x^n e^{-ax^b}\,dx = \frac{1}{b}\,a^{-\frac{n+1}{b}}\,\Gamma\left(\frac{n+1}{b}\right)$$

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \quad (a>0)$$

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \quad (a>0)$$

$$\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2} \quad (a>0)$$

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} \quad (a>0)$$

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \arctan\frac{b}{a}$$

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$$

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin px\,dx = \arctan\frac{b}{p}-\arctan\frac{a}{p}$$

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos px\,dx = \frac{1}{2}\ln\frac{b^2+p^2}{a^2+p^2}$$

$$\int_0^\infty \frac{e^{-ax}(1-\cos x)}{x^2}\,dx = \operatorname{arccot} a-\frac{a}{2}\ln\left(\frac{1}{a^2}+1\right)$$

$$\int_{-\infty}^\infty e^{ax^4+bx^3+cx^2+dx+f}\,dx = e^f\sum_{n,m,p=0}^{\infty}\frac{b^{4n}}{(4n)!}\frac{c^{2m}}{(2m)!}\frac{d^{4p}}{(4p)!}\frac{\Gamma(3n+m+p+\frac{1}{4})}{a^{3n+m+p+\frac{1}{4}}}$$ (appears in several models of extended superstring theory in higher dimensions)

$$\int_0^{2\pi} e^{x\cos\theta}d\theta = 2\pi I_0(x)$$ (I_0 is the modified Bessel function of the first kind)

$$\int_0^{2\pi} e^{x\cos\theta+y\sin\theta}d\theta = 2\pi I_0\left(\sqrt{x^2+y^2}\right)$$

$$\int_0^\infty \frac{x^{s-1}}{e^x/z-1}\,dx = \operatorname{Li}_s(z)\Gamma(s),$$

where $\operatorname{Li}_s(z)$ is the Polylogarithm.

$$\int_0^\infty \frac{\sin mx}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{m}{2} - \frac{1}{2m}$$

$$\int_0^\infty e^{-x}\ln x\,dx = -\gamma,$$

where γ is the Euler–Mascheroni constant which equals the value of a number of definite integrals.

Finally, a well known result,

$$\int_0^{2\pi} e^{i(m-n)\phi}\,d\phi = 2\pi\delta_{m,n} \qquad \text{for } m,n\in\mathbb{Z}$$

where $\delta_{m,n}$ is the Kronecker delta.

See also

- Gradshteyn and Ryzhik

References

Toyesh Prakash Sharma, Etisha Sharma, "Putting Forward Another Generalization Of The Class Of Exponential Integrals And Their Applications.," International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol.10, Issue.2, pp.1-8, 2023.[1] (https://www.isroset.org/pdf_paper_view.php?paper_id=3100&1-ISROSET-IJSRMSS-08692.pdf)

Further reading

- Moll, Victor Hugo (2014-11-12). *Special Integrals of Gradshteyn and Ryzhik: the Proofs – Volume I* (http://www.crcpress.com/Special-Integrals-of-Gradshteyn-and-Ryzhik-the-Proofs---Volume-I/Moll/9781482256512). Vol. I (1 ed.). Chapman and Hall/CRC Press. ISBN 978-1-48225-651-2. Retrieved 2016-02-12. {{cite book}}: |work= ignored (help)
- Moll, Victor Hugo (2015-10-27). *Special Integrals of Gradshteyn and Ryzhik: the Proofs – Volume II* (http://www.crcpress.com/Special-Integrals-of-Gradshteyn-and-Ryzhik-the-Proofs---Volume-II/Moll/9781482256536). Vol. II (1 ed.). Chapman and Hall/CRC Press. ISBN 978-1-48225-653-6. Retrieved 2016-02-12. {{cite book}}: |work= ignored (help)
- Toyesh Prakash Sharma, https://www.isroset.org/pdf_paper_view.php?paper_id=2214&7-ISROSET-IJSRMSS-05130.pdf

External links

- Wolfram Mathematica Online Integrator (http://www.wolframalpha.com/calculators/integral-calculator/)
- Moll, Victor Hugo. "List with the formulas and proofs in GR" (http://www.math.tulane.edu/~vhm/Table.html). Retrieved 2016-02-12.

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