Functional Analysis Problems with Solutions

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Contents

Notations:
- $\mathcal{B}(X,Y)$: the space of all bounded (continuous) linear operators from X to Y.
- $\text{Image}\,(T) \equiv \text{Ran}(T)$: the image of a mapping $T : X \to Y$.
- $x_n \xrightarrow{w} x$: x_n converges weakly to x.
- X^*: the space of all bounded (continuous) linear functionals on X.
- $\mathbb{F}$ or $\mathbb{K}$: the scalar field, which is $\mathbb{R}$ or $\mathbb{C}$.
- Re, Im: the real and imaginary parts of a complex number.

Chapter 1

Normed and Inner Product Spaces

Problem 1.
Prove that any ball in a normed space X is convex.

Solution.
Let $B(x_0;r)$ be any ball of radius $r>0$ centered at $x_0 \in X$, and $x, y \in B(x_0;r)$. Then

$$\|x - x_0\| < r \quad \text{and} \quad \|y - x_0\| < r.$$

For every $a \in [0,1]$ we have

$$\begin{aligned} \|ax + (1-a)y - x_0\| &= \|(x - x_0)a + (1-a)(y - x_0)\| \\ &\leq a\|x - x_0\| + (1-a)\|y - x_0\| \\ &< ar + (1-a)r = r. \end{aligned}$$

So $ax + (1-a)y \in B(x_0;r)$. $\blacksquare$.

Problem 2.
Consider the linear space $C[0,1]$ equipped with the norm

$$\|f\|_1 = \int_0^1 |f(x)|dx.$$

Prove that there is no inner product on $C[0,1]$ agreed with this norm.

Solution.
We show that the norm $\|.\|_1$ does not satisfy the parallelogram law. Let

$$f(x) = 1 \text{ and } g(x) = 2x.$$

Then

$$\|f\|_1 = \int_0^1 1.dx = 1, \quad \|g\|_1 = \int_0^1 |2x|dx = 1,$$

while

$$\|f - g\|_1 = \int_0^1 |1 - 2x|dx = \frac{1}{2}, \quad \|f + g\|_1 = \int_0^1 |1 + 2x|dx = 2.$$

Thus,

$$\|f - g\|_1^2 + \|f + g\|_1^2 = \frac{17}{4} \neq 2(\|f\|_1^2 + \|g\|_1^2) = 4. \quad \blacksquare$$

Problem 3.
Consider the linear space $C[0,1]$ equipped with the norm

$$\|f\| = \max_{t\in[0,1]} |f(t)|.$$

Prove that there is no inner product on $C[0,1]$ agreed with this norm.

Solution.
We show that the parallelogram law with respect to the given norm does not hold for two elements in $C[0,1]$.
Let $f(t) = t,\ g(t) = 1 - t,\ t \in [0,1]$. Then $f, g \in C[0,1]$ and

$$\|f\| = \max_{t\in[0,1]} t = 1, \quad \|g\| = \max_{t\in[0,1]} (1 - t) = 1,$$

and

$$\|f + g\| = \max_{t\in[0,1]} 1 = 1, \quad \text{and} \quad \|f - g\| = \max_{t\in[0,1]} |-1 + 2t| = 1.$$

Thus,

$$\|f - g\|_1^2 + \|f + g\|_1^2 = 2 \neq 2(\|f\|_1^2 + \|g\|_1^2) = 4. \quad \blacksquare$$

Problem 4.
Prove that:
If the unit sphere of a normed space X contains a line segment $[x, y]$ where $x, y \in X$ and $x \neq y$, then x and y are linearly independent and $\|x + y\| = \|x\| + \|y\|$.

Solution.
Suppose that the unit sphere contains a line segment $[x, y]$ where $x, y \in X$ and $x \neq y$. Then

$$\|ax + (1-a)y\| = 1 \text{ for any } a \in [0,1].$$

Choose $a = 1/2$ then we get $\|\frac{1}{2}(x+y)\| = 1$, that is $\|x+y\| = 2$. Since x and y belong to the unit sphere, we have $\|x\| = \|y\| = 1$. Hence

$$\|x+y\| = \|x\| + \|y\|.$$

Let us show that x, y are linearly independent. Assume $y = \beta x$ for some $\beta \in \mathbb{C}$. We have

$$1 = \|ax + (1-a)\beta x\| = |a + (1-a)\beta|.$$

For $a = 0$ we get $|\beta| = 1$ and for $a = 1/2$ we get $|1+\beta| = 2$. These imply that $\beta = 1$, and so $x = y$, which is a contradiction. ■

Problem 5.
Prove that two any norms in a finite dimensional space X are equivalent.

Solution.
Since equivalence of norms is an equivalence relation, it suffices to show that an arbitrary norm $\|.\|$ on X is equivalent to the Euclidian norm $\|.\|_2$. Let $\{e_1, ..., e_n\}$ be a basis for X. Every $x \in X$ can be written uniquely as $x = \sum_{k=1}^n c_k e_k$. Therefore,

$$\|x\| \leq \sum_{k=1}^n |e_k| \|e_k\| \leq \left(\sum_{k=1}^n |c_k|^2\right)^{1/2} \left(\sum_{k=1}^n \|e_k\|^2\right)^{1/2} \leq A\|x\|_2,$$

where $A = \left(\sum_{k=1}^n |e_k|^2\right)^{1/2}$ is a non-zero constant. This shows that the map $x \mapsto \|x\|$ is continuous w.r.t. the Euclidian norm. Now consider $S = \{x : \ \|x\|_2 = 1\}$. This is just the unit sphere in $(X, \|.\|_2)$, which is compact. The map

$$S \to \mathbb{R} \text{ defined by } x \mapsto \|x\|$$

is continuous, so it attains a minimum m and a maximum M on S. Note that $m > 0$ because $S \neq \emptyset$. Thus, for all $x \in S$, we have

$$m \leq \|x\| \leq M.$$

Now, for $x \in X, \ x \neq 0, \ \frac{x}{\|x\|_2} \in S$, so

$$m \leq \frac{\|x\|}{\|x\|_2} \leq M.$$

That is

$$m\|x\|_2 \leq \|x\| \leq M\|x\|_2.$$

Hence, the two norms are equivalent. ■

Problem 6.
Let X be a normed space.
(a) Find all subspaces of X which are contained in some ball $B(a;r)$ of X.
(b) Find all subspaces of X which contain some ball $B(x_0;\rho)$ of X.

Solution.
(a) Let Y be a subspace of X which is contained in some ball $B(a;r)$ of X. Note first that the ball $B(a;r)$ must contain the vector zero of X (and so of Y); otherwise, the question is impossible. For any number $A > 0$ and any $x \in Y$, we have $Ax \in Y$ since Y is a linear space. By hypothesis $Y \subset B(a;r)$, so we have $Ax \in B(a;r)$. This implies that $\|Ax\| < r + \|a\|$. Finally

$$\|x\| < \frac{r + \|a\|}{A}.$$

$A > 0$ being arbitrary, it follows that $\|x\| = 0$, so $x = 0$. Thus, there is only one subspace of X, namely, $Y = \{0\}$, which is contained in some ball $B(a;r)$ of X.
(b) Let Z be a subspace of X which contain some ball $B(x_0;\rho)$ of X. Take any $x \in B(0;\rho)$. Then $x + x_0 \in B(x_0;\rho)$ and so $x + x_0 \in Z$ since $Z \supset B(x_0;\rho)$. Now, since $x_0 \in Z$, $x + x_0 \in Z$ and Z is a linear space, we must have $x \in Z$. Hence $B(0;\rho) \subset Z$.
Now for any nonzero $x \in X$, we have $\frac{\rho x}{2\|x\|} \in B(0;\rho) \subset Z$. Hence $x \in Z$. We can conclude that $Z = X$. In other words, the only subspace of X which contains some ball $B(x_0;\rho)$ of X is X itself. ■

Problem 7.
Prove that any finite dimensional normed space :
(a) is complete (a Banach space),
(b) is reflexive.

Solution.
Let X be a finite dimensional normed space. Suppose $\dim X = d$.

(a) By Problem 5, it suffices to consider the Euclidian norm in X. Let $\{e_1, ..., e_d\}$ be a basis for X. For $x \in X$ there exist numbers $c_1, ..., c_d$ such that

$$x = \sum_{k=1}^{d} c_k e_k \quad \text{and} \quad \|x\| = \left(\sum_{k=1}^{d} |c_k|^2 \right)^{1/2}.$$

Let $(x^{(n)})$ be a Cauchy sequence in X. If for each n, $x^{(n)} = \sum_{k=1}^{d} a_k^{(n)} e_k$ then

$$\|x^{(n)} - x^{(m)}\| = \left(\sum_{k=1}^{d} |a_k^{(n)} - a_k^{(m)}|^2 \right)^{1/2} \to 0 \text{ as } n, m \to \infty.$$

Hence, for every $k = 1, ..., d$,

$$|a_k^{(n)} - a_k^{(m)}| \to 0 \text{ as } n, m \to \infty.$$

Therefore, each sequence of numbers $(a_k^{(n)})$ is a Cauchy sequence, so

$$a_k^{(n)} \to a_k^{(0)} \text{ as } n \to \infty \text{ for every } k = 1, 2, ..., d.$$

Let $a = \sum_{k=1}^{d} a_k^{(0)} e_k$ then $x^{(n)} \to a \in X$.

(b) Let $f \in X^\sharp$ where $X^\sharp$ is the space of all linear functionals on X. We have

$$f(x) = f\left(\sum_{k=1}^{d} c_k e_k \right) = \sum_{k=1}^{d} c_k f(e_k) = \sum_{k=1}^{d} c_k \alpha_k,$$

where $\alpha_k = f(e_k)$. Let us define $f_k \in X^\sharp$ by the relation $f_k(x) = c_k, \quad k = 1, ..., d$. For any $x \in X$ and $f \in X^\sharp$, we get

$$f(x) = \sum_{k=1}^{d} f_k(x) \alpha_k, \ \textit{i.e.}, \ f = \sum_{k=1}^{d} \alpha_k f_k.$$

Hence, $\dim X^\sharp \le d$.
Let $\sum_{k=1}^{d} \alpha_k f_k = 0$. Then, for any $x \in X$, $\sum_{k=1}^{d} \alpha_k f_k(x) = 0$, and by taking $x = \sum_{k=1}^{d} \bar{\alpha}_k e_k$, we obtain $f_k(x) = \bar{\alpha}_k$, and

$$\sum_{k=1}^{d} \alpha_k f_k(x) = \sum_{k=1}^{d} |\alpha_k|^2 = 0.$$

Hence, $\alpha_k = 0$ for all $k = 1, ..., d$ and thus, $\dim X^\sharp = d$. For the space X^* we have $X^* \subset X^\sharp$, so $\dim X^* = n \le d$ and $\dim (X^*)^\sharp = n$. From the relation $X \subset (X^*)^* \subset (X^*)^\sharp$ we conclude that $d \le n$. Thus, $n = d$, and so $X = (X^*)^*$. ■

Problem 8. (Reed-Simon II.4)
(a) Prove that the inner product in a normed space X can be recovered from the ***polarization identity****:*

$$\langle x, y\rangle = \frac{1}{4}\Big[(\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)\Big].$$

(b) Prove that a normed space is an inner product space if and only if the norm satisfies the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Solution.
(a) For the real field case, the polarization identity is

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2). \quad (*)$$

We use the symmetry of the inner product and compute the right hand side of $(*)$:

$$\begin{aligned}\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) &= \frac{1}{4}\big[\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\big]\\ &= \frac{1}{2}\big[\langle x,y\rangle + \langle y,x\rangle\big]\\ &= \langle x,y\rangle.\end{aligned}$$

For the complex field case, we again expand the right hand side, using the relation we just established:

$$\begin{aligned}&\frac{1}{4}\Big[(\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)\Big]\\ &= \frac{1}{2}\big[\langle x,y\rangle + \langle y,x\rangle\big] - \frac{i}{2}\big[\langle x,iy\rangle + \langle iy,x\rangle\big]\\ &= \frac{1}{2}\langle x,y\rangle + \frac{1}{2}\langle y,x\rangle - \frac{i^2}{2}\langle x,y\rangle + \frac{i^2}{2}\langle y,x\rangle\\ &= \langle x,y\rangle.\end{aligned}$$

(b) If the norm comes from an inner product, then we have

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle\\ &= 2\langle x,x\rangle + 2\langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle x,y\rangle - \langle y,x\rangle\\ &= 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

Now suppose that the norm satisfies the parallelogram law. Assume the field is $\mathbb{C}$, and define the inner product via the polarization identity from part (a). If $x, y.z \in X$, we write

$$x+y = x + \frac{y+z}{2} + \frac{y-z}{2}, \quad x+z = x + \frac{y+z}{2} - \frac{y-z}{2},$$

and we have

$$\begin{aligned}
\langle x,y\rangle + \langle x,z\rangle &= \frac{1}{4}\big(\|x+y\|^2 + \|x-y\|^2 - \|x-y\|^2 - \|x-z\|^2\big) \\
&+ \frac{i}{4}\big(\|x+iy\|^2 + \|x+iz\|^2 - \|x-iy\|^2 - \|x-iz\|^2\big) \\
&= \frac{1}{2}\left(\left\|x+\frac{y+z}{2}\right\|^2 + \left\|\frac{y+z}{2}\right\|^2 - \left\|x-\frac{y+z}{2}\right\|^2 - \left\|\frac{y-z}{2}\right\|^2\right) \\
&- \frac{i}{2}\left(\left\|x+i\frac{y+z}{2}\right\|^2 + \left\|i\frac{y+z}{2}\right\|^2 - \left\|x-i\frac{y+z}{2}\right\|^2 - \left\|i\frac{y-z}{2}\right\|^2\right) \\
&= \frac{1}{2}\left(\left\|x+\frac{y+z}{2}\right\|^2 + \left\|\frac{y+z}{2}\right\|^2 - \left\|\frac{y-z}{2}\right\|^2 - \left\|x-\frac{y+z}{2}\right\|^2\right) \\
&- \frac{i}{2}\left(\left\|x+i\frac{y+z}{2}\right\|^2 + \left\|i\frac{y+z}{2}\right\|^2 - \left\|i\frac{y-z}{2}\right\|^2 - \left\|x-i\frac{y+z}{2}\right\|^2\right) \\
&= \frac{1}{4}\big(\|x+y+z\|^2 + \|x\|^2 - \|x-(y+z)\|^2 - \|x\|^2\big) \\
&- \frac{i}{4}\big(\|x+i(y+z)\|^2 + \|x\|^2 - \|x-i(y+z)\|^2 - \|x\|^2\big) \\
&= \langle x, y+z\rangle.
\end{aligned}$$

This holds for all $x, y, z \in X$, so, in particular,

$$\langle x, ny\rangle = n\langle x,y\rangle \quad \text{for} \quad n \in \mathbb{N}.$$

And it also satisfies

$$\langle x, ry\rangle = r\langle x,y\rangle \quad \text{for} \quad r \in \mathbb{Q}.$$

Moreover, again by the polarization identity, we have

$$\begin{aligned}
\langle x, iy\rangle &= \frac{1}{4}\big(\|x+iy\|^2 - \|x-iy\|^2\big) - \frac{i}{4}\big(\|x-y\|^2 - \|x+y\|^2\big) \\
&= i\langle x,y\rangle.
\end{aligned}$$

Combining these results we have

$$\langle x, \alpha y\rangle = \alpha\langle x,y\rangle \quad \text{for} \quad \alpha \in \mathbb{Q} + i\mathbb{Q}.$$

Now, if $\alpha \in \mathbb{C}$, by the density of $\mathbb{Q}+i\mathbb{Q}$ in $\mathbb{C}$, there exists a sequence (α_n) in $\mathbb{Q}+i\mathbb{Q}$ converging to α. It follows that

$$\langle x, \alpha y\rangle = \alpha\langle x, y\rangle \quad \text{for} \quad \alpha \in \mathbb{C}.$$

Thus the $\langle ., .\rangle$ is linear.
Since $\|i(x-iy)\| = \|x-iy\|$, we have

$$\overline{\langle y, x\rangle} = \langle x, y\rangle,$$

and

$$\langle x, x\rangle = \frac{1}{4}(\|2x\|^2) - \frac{i}{4}\big(|1+i|\|x\|^2 - |1-i|^2\|x\|^2\big) = \|x\|^2.$$

So this shows that the norm is induced by $\langle ., .\rangle$ and that it is also positive definite, and thus it is an inner product. ∎

Problem 9. (Least square approximation. Reed-Simon II.5)
Let X be an inner product space and let $\{x_1, ..., x_N\}$ be an orthonormal set. Prove that

$$\left\| x - \sum_{n=1}^{N} c_n x_n \right\|$$

is minimized by choosing $c_n = \langle x_n, x\rangle$.

Solution.
For every $x \in X$, we write

$$x = \sum_{n=1}^{N} \langle x_n, x\rangle x_n + z, \quad \text{for some} \quad z \in X. \qquad (*)$$

We observe that for all $n = 1, ..., N$,

$$\begin{aligned}\langle x_n, z\rangle &= \langle x_n, x\rangle - \sum_{k=1}^{N}\langle x_n, x\rangle\langle x_n, x_k\rangle \\ &= \langle x_n, x\rangle - \langle x_n, x\rangle = 0.\end{aligned}$$

Therefore $z \perp x_n$. Then due to $(*)$ we can write

$$x - \sum_{n=1}^{N} c_n x_n = \underbrace{\sum_{n=1}^{N}\big(\langle x_n, x\rangle - c_n\big)x_n}_{z_N} + z.$$

Since $z \perp z_N$, we have

$$\begin{aligned} \left\| x - \sum_{n=1}^{N} c_n x_n \right\|^2 &= \|z_N\|^2 + \|z\|^2 \\ &= \sum_{n=1}^{N} \left| \langle x_n, x \rangle - c_n \right|^2 + \|z\|^2, \end{aligned}$$

which attains its minimum if

$$c_n = \langle x_n, x \rangle \quad \text{for all} \quad n = 1, ..., N. \qquad \blacksquare$$

Review: Quotient normed space.

• Let X be a vector space, and let M be a subspace of X. We define an equivalence relation on X by

$$x \sim y \quad \text{if and only if} \quad x - y \in M.$$

For $x \in X$, let $[x] = x + M$ denote the equivalence class of x and X/M the set of all equivalence classes. On X/M we define operations:

$$\begin{aligned} [x] + [y] &= [x + y] \\ \alpha[x] &= [\alpha x], \;\; \alpha \in \mathbb{C}. \end{aligned}$$

Then X/M is a vector space.
If the subspace M is closed, then we can define a norm on X/M by

$$\|[x]\| = \inf_{y \in [x]} \|y\| = \inf_{m \in M} \|x + m\| = \inf_{m \in M} \|x - z\| = d(x, M).$$

What a ball in X/M looks like?

$$B([x_0]; r) := \{[x] : \; \|[x] - [x_0]\| < r\} = \{x + M : \; \|x - x_0 + M\| < r\}.$$

• Suppose that M is closed in X. The canonical map (the natural projection) is defined by

$$\pi : X \to X/M, \;\; \pi(x) = [x] = x + M.$$

It can be shown that $\|\pi(x)\| \leq \|x\|, \; \forall x \in X$, so π is continuous.

Problem 10.
Let X be a normed space and M a closed subspace of X. Let $\pi : X \to X/M$ be the canonical map. Show that the topology induced by the standard norm on X/M is the usual quotient topology, i.e. that $O \subset X/M$ is open in X/M if and only if $\pi^{-1}(O)$ is open in X.

Solution.

• If O is open in X/M, then $\pi^{-1}(O)$ is open in X since π is continuous.

• Now suppose that $O \subset X/M$ and that $\pi^{-1}(O)$ is open in X. We show that O is open in X/M. Consider an open ball $B(0;r)$, $r > 0$ in X. Let $x \in B(0;r)$. Then $\|x\| < r$, and so

$$\|[x]\| \leq \|x\| < r.$$

On the other hand, if $\|[x]\| < r$, then there is an $y \in M$ such that $\|x+y\| < r$. Hence $x+y \in B(0;r)$, and so

$$[x] = \pi(x+y) \in \pi\big(B(0;r)\big).$$

If $[x_0] \in O$, then $x_0 \in \pi^{-1}(O)$. Since $\pi^{-1}(O)$ is open in X, there is an $r > 0$ such that

$$B(x_0;r) \subset \pi^{-1}(O).$$

This implies that

$$O = \pi\pi^{-1}(O) \supset \pi\big(B(x_0;r)\big) = \pi\big(x_0 + B(0;r)\big) = \{x + M : \ \|x - x_0 + M\| < r\}.$$

The last set is the open ball of radius $r > 0$ centered at $[x_0] \in O$. Thus O is open in X/M. ■

Problem 11.
Let $X = C[0,1]$, $M = \{f \in C[0,1] : \ f(0) = 0\}$. Show that $X/M = \mathbb{C}$.

Solution.
Given $[f] \in X/M$, let $\varphi([f]) = f(0)$. Then the map $\varphi : X/M \to \mathbb{C}$ is well-defined. Indeed, if $[f] = [g]$, then $(f-g)(0) = 0$ so $f(0) = g(0)$. It is clearly linear.
If $f \in X$, then $g = f - f(0) \in M$, and so $f - g = f(0)$ is constant, which tells us that

$$\|[f]\| = |f(0)| = |\varphi([f])|,$$

so φ is an isometry (and thus injective and continuous). Finally, constants are in $X = C[0,1]$, so φ is surjective and thus an isometric isomorphism. ■

Problem 12.
If $0 < p < 1$ then ℓ^p is a vector space but $\|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p}$ is not a norm for ℓ^p.

Solution.
Recall that if $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in \ell^p$ and $\alpha \in \mathbb{C}$ then

$$x + y = (x_1 + y_1, x_2 + y_2, ...) \quad \text{and} \quad \alpha x = (\alpha x_1, \alpha x_2, ...).$$

It is clear that $\alpha x \in \ell^p$. We show that $x + y \in \ell^p$. For $t \geq 0$ it not hard to see that $(1+t)^p \leq 1 + t^p,\ 0 < p < 1$. This implies that

$$(a+b)^p \leq a^p + b^p,\ 0 < p < 1 \quad \text{and} \quad a, b \geq 0.$$

Therefore,

$$\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p.$$

Since both $\|x\|_p^p$ and $\|y\|_p^p$ are bounded, $\|x + y\|_p^p$ is bounded. Hence $x + y \in \ell^p$.
To show $\|.\|_p$ is not a norm for ℓ^p, let us take an example: If

$$x = (1, 0, ...) \quad \text{and} \quad y = (0, 1, 0, ...)$$

then $\|x\|_p = \|y\|_p = 1$ but $\|x + y\|_p = 2^{1/p} > 2$ since $1/p > 1$. Therefore,

$$\|x + y\|_p > \|x\|_p + \|y\|_p. \quad \blacksquare$$

Problem 13.
Suppose that X is a linear space with inner product $\langle .,. \rangle$. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, prove that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle \quad \text{as} \quad n \to \infty.$$

Solution.
Using the Cauchy-Schwarz and triangle inequalities, we have

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\ &\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\ &\leq \|x_n - x\|(\|y_n - y\| + \|y\|) + \|x\|\|y_n - y\| \\ &\leq \|x_n - x\|\|y_n - y\| + \|x_n - x\|\|y\| + \|x\|\|y_n - y\|. \end{aligned}$$

Since $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$, we see that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle \quad \text{as} \quad n \to \infty. \quad \blacksquare$$

Problem 14.
Prove that if M is a closed subspace and N is a finite dimensional subspace of a normed space X, then $M+N := \{m+n: \ m\in M,\ n\in N\}$ is closed.

Solution.
Assume $\dim N = 1$, say $N = \text{Span}\{x\}$. The case $x \in M$ is trivial. Suppose $x \notin M$. Consider the sequence $z_k := \alpha_k x + m_k$, where $m_k \in M$, $\alpha_k \in \mathbb{C}$, and suppose $z_k \in M + N \to y$ as $k \to \infty$. We want to show $y \in M + N$. The sequence (α_k) is bounded; otherwise, there exists a subsequence $(\alpha_{k'})$ such that $0 < |\alpha_{k'}| \to \infty$ as $k' \to \infty$. Then

$$\frac{z_{k'}}{\alpha_{k'}} \ \text{and} \ \frac{m_{k'}}{\alpha_{k'}} \to 0 \ \text{ as } \ k' \to \infty,$$

so x must be 0, which is in M. This is a contradiction. Consequently, (α_k) is bounded and therefore it has a subsequence $(\alpha_{k'})$ which is converging to some $\alpha \in \mathbb{C}$. Thus

$$m_{k'} = z_{k'} - \alpha_{k'} x \to y - \alpha x \ \text{ as } \ k' \to \infty.$$

Hence, $y - \alpha x$ is in M since M is closed. Thus $y \in M + N$.

The solution now follows by induction. ■

Chapter 2

Banach Spaces

Problem 15.
Let X be a normed space. Prove that X is a Banach space if and only if the series $\sum_{n=1}^{\infty} a_n$ converges, where (a_n) is any sequence in X satisfying $\sum_{n=1}^{\infty} \|a_n\| < \infty$.

Solution.
Suppose that X is complete. Let (a_n) be a sequence in X such that $\sum_{n=1}^{\infty} \|a_n\| < \infty$. Let $S_n = \sum_{i=1}^{n} a_i$ be the partial sum. Then for $m > n$,

$$\|S_m - S_n\| = \left\| \sum_{i=n+1}^{m} a_i \right\| \leq \sum_{i=n+1}^{m} \|a_i\|.$$

By hypothesis, the series $\sum_{n=1}^{\infty} \|a_n\|$ converges, so $\sum_{i=n+1}^{m} \|a_i\| \to 0$ as $n \to \infty$. Therefore, (S_n) is a Cauchy sequence in the Banach space X. Thus, (S_n) converges, that is, the series $\sum_{n=1}^{\infty} a_n$ converges.
Conversely, suppose $\sum_{n=1}^{\infty} a_n$ converges in X whenever $\sum_{n=1}^{\infty} \|a_n\| < \infty$. We show that X is complete. Let (y_n) be a Cauchy sequence in X. Then

$$\exists n_1 \in \mathbb{N} : \ \|y_{n_1} - y_m\| < \frac{1}{2} \ \text{ whenever } \ m > n_1,$$
$$\exists n_2 \in \mathbb{N} : \ \|y_{n_2} - y_m\| < \frac{1}{2^2} \ \text{ whenever } \ m > n_2 > n_1.$$

Continuing in this way, we see that there is a sequence (n_k) strictly increasing such that

$$\|y_{n_k} - y_m\| < \frac{1}{2^k} \ \text{ whenever } \ m > n_k.$$

In particular, we have

$$\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

Set $x_k = y_{n_{k+1}} - y_{n_k}$. Then

$$\sum_{k=1}^{n} \|x_k\| = \sum_{k=1}^{n} \|y_{n_{k+1}} - y_{n_k}\| < \sum_{k=1}^{n} \frac{1}{2^k}.$$

It follows that $\sum_{k=1}^{\infty} \|x_k\| < \infty$. By hypothesis, there is an $x \in X$ such that $\sum_{k=1}^{m} x_k \to x$ as $m \to \infty$. But we have

$$\begin{aligned} \sum_{k=1}^{m} x_k &= \sum_{k=1}^{m} (y_{n_{k+1}} - y_{n_k}) \\ &= y_{n_{m+1}} - y_{n_1}. \end{aligned}$$

Hence $y_{n_m} \to x + y_{n_1}$ in X as $m \to \infty$. Thus, the sequence (y_n) has a convergent subsequence and so must itself converges. ■

Problem 16.
Let X be a Banach space. Prove that the closed unit ball $\overline{B(0;1)} \subset X$ is compact if and only if X is finite dimensional.

Solution.
• Suppose $\dim X = n$. Then X is isomorphic to $\mathbb{R}^n$ (with the standard topology). The result then follows from the Heine-Borel theorem.

• Suppose that X is not finite dimensional. We want to show that $\overline{B(0;1)}$ is not compact. To do this, we construct a sequence in $\overline{B(0;1)}$ which have no convergent subsequence.

We will use the following fact usually known as *Riesz's Lemma:* (See the proof below)
Let M be a closed subspace of a Banach space X. Given any $r \in (0,1)$, there exists an $x \in X$ such that

$$\|x\| = 1 \text{ and } d(x, M) \geq r.$$

Pick $x_1 \in X$ such that $\|x_1\| = 1$. Let $S_1 := \text{Span}\{x_1\}$. Then S_1 is closed. According to Riesz's Lemma, there exists $x_2 \in X$ such that

$$\|x_2\| = 1 \text{ and } d(x_2, S_1) \geq \frac{1}{2}.$$

Now consider the subspace S_2 generated by $\{x_1, x_2\}$. Since X is infinite dimensional, S_2 is a proper closed subspace of X, and we can apply the Riesz's Lemma to find an $x_3 \in X$ such that

$$\|x_3\| = 1 \quad \text{and} \quad d(x_3, S_2) \geq \frac{1}{2}.$$

If we continue to proceed this way, we will have a sequence (x_n) and a sequence of closed subspaces (S_n) such that for all $n \in \mathbb{N}$

$$\|x_n\| = 1 \quad \text{and} \quad d(x_{n+1}, S_n) \geq \frac{1}{2}.$$

It is clear that the sequence (x_n) is in $\overline{B(0;1)}$, and for $m > n$ we have

$$\|x_n - x_m\| \geq d(x_m, S_n) \geq \frac{1}{2}.$$

Therefore, no subsequence of (x_n) can form a Cauchy sequence. Thus, $\overline{B(0;1)}$ is not compact. ■

Proof of Riesz's Lemma:
Take $x_1 \notin M$. Put $d = d(x_1, M) = \inf_{m \in M} \|m - x_1\|$. Then $d > 0$ since M is closed. For any $\varepsilon > 0$, by definition of the infimum, there exists $m_1 \in M$ such that

$$0 < \|m_1 - x_1\| < d + \varepsilon.$$

Set $x = \frac{x_1 - m_1}{\|x_1 - m_1\|}$. Then $\|x\| = 1$ and

$$\|x - m\| = \frac{1}{\|x_1 - m_1\|} \big\| x_1 - \underbrace{(m_1 + \|x_1 - m_1\| m)}_{\in M} \big\|$$

This implies that

$$d(x, M) = \inf_{m \in M} \|x - m\| = \frac{\inf_{m \in M} \|x_1 - m\|}{\|x_1 - m_1\|} \geq \frac{d}{d + \varepsilon}.$$

By choosing $\varepsilon > 0$ small, $\frac{d}{d+\varepsilon}$ can be arbitrary close to 1. ■

Problem 17.
Let X be a Banach space and M a closed subspace of X. Prove that the quotient space X/M is also a Banach space under the quotient norm.

Solution.
We use criterion established above (in problem 15). Suppose that $([x_n])$ is any

sequence in X/M such that $\sum_{n=1}^{\infty} \|[x_n]\| < \infty$. We show that

$$\exists [x] \in X/M : \sum_{n=1}^{k} [x_n] \to [x] \text{ as } k \to \infty.$$

For each n, $\|[x_n]\| = \inf_{z \in M} \|x_n + z\|$, and therefore there is $z_n \in M$ such that

$$\|x_n + z_n\| \le \|[x_n]\| + \frac{1}{2^n}$$

by definition of the infimum. Hence

$$\sum_{n=1}^{\infty} \|x_n + z_n\| \le \sum_{n=1}^{\infty} \|[x_n]\| + \frac{1}{2^n} < \infty.$$

But $(x_n + z_n)$ is a sequence in the Banach space X, and so

$$\sum_{n=1}^{\infty} (x_n + z_n) = x \text{ for some } x \in X.$$

Then we have

$$\begin{aligned}
\left\| \sum_{n=1}^{k} [x_n] - [x] \right\| &= \left\| \sum_{n=1}^{k} [x_n - x] \right\| \\
&= \inf_{z \in E} \left\| \sum_{n=1}^{k} (x_n - x + z) \right\| \\
&\le \left\| \sum_{n=1}^{k} \big((x_n - x) + z_n \big) \right\| \\
&= \left\| \sum_{n=1}^{k} (x_n + z_n) - x \right\| \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Hence $\sum_{n=1}^{k} [x_n] \to [x]$ as $k \to \infty$. ■

SPACE ℓ^p

(Only properties concerning to norms and completeness will be considered. Other properties such as duality will be discussed later.)

Problem 18.
Show that $\ell^p,\ 1 \leq p < \infty$ *equipped with the norm* $\|.\|_p$ *is a Banach space.*

Solution.
Let $x^{(i)} = (x_1^{(i)}, ..., x_k^{(i)}, ...)$ for $i = 1, 2, ...$ be a Cauchy sequence in ℓ^p. Then

$$\|x^{(i)} - x^{(j)}\|_p \to 0 \quad \text{as} \quad i, j \to \infty.$$

Since $\|x^{(i)} - x^{(j)}\|_p \geq |x_k^{(i)} - x_k^{(j)}|$ for every k, it follows that

$$|x_k^{(i)} - x_k^{(j)}| \to 0 \quad \text{for every} \quad k \quad \text{as} \quad i, j \to \infty.$$

This tells us that the sequence $\big(x_k^{(i)}\big)$ is a Cauchy sequence in $\mathbb{F}$, which is complete, so that $\big(x_k^{(i)}\big)$ converges to $x_k \in \mathbb{F}$ as $i \to \infty$ for each k. Set $x = (x_1, ..., x_k, ...)$. We will show that

$$(*) \qquad \|x^{(i)} - x\|_p \to 0 \quad \text{as} \quad i, j \to \infty \quad \text{and} \quad x \in \ell^p.$$

Given $\varepsilon > 0$, for any $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$\left(\sum_{k=1}^{M} |x_k^{(i)} - x_k^{(j)}|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k^{(i)} - x_k^{(j)}|^p\right)^{\frac{1}{p}} < \varepsilon \quad \text{if} \quad i, j > N.$$

Letting $j \to \infty$, for $i > N$ we get

$$(**) \qquad \left(\sum_{k=1}^{M} |x_k^{(i)} - x_k|^p\right)^{\frac{1}{p}} < \varepsilon.$$

By Minkowski's inequality,

$$\begin{aligned}
\left(\sum_{k=1}^{M} |x_k|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{k=1}^{M} |x_k^{(N)} - x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{M} |x_k^{(N)}|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{k=1}^{M} |x_k^{(N)} - x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |x_k^{(N)}|^p\right)^{\frac{1}{p}} \\
&< \varepsilon + \left(\sum_{k=1}^{\infty} |x_k^{(N)}|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Letting $M \to \infty$, since the last sum is finite, we see that

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} < \infty.$$

This shows that $x \in \ell^p$. Finally letting $M \to \infty$ in $(**)$, for $i > N$ we get

$$\|x^{(i)} - x\|_p = \left(\sum_{k=1}^{\infty} |x_k^{(i)} - x_k|^p\right)^{\frac{1}{p}} < \varepsilon.$$

This shows that $x^{(i)} \to x$ in ℓ^p as required. ■

Problem 19.
(a) Show that ℓ^∞ equipped with the norm $\|.\|_\infty$ is a Banach space.
(b) Let c_0 be the space of sequences converging to 0. Show that c_0 is a closed subspace of ℓ^∞.

Solution.
(a) We need to show that ℓ^∞ is complete. Assume that the sequence $(x^{(n)})$ is Cauchy in ℓ^∞. That is, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$(1) \quad n, m \geq N \Rightarrow \|x^{(n)} - x^{(m)}\|_\infty < \epsilon.$$

For a fixed n, we write $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, ...)$. Then for $N = N(\epsilon)$ as above,

$$(2) \quad |x_j^{(n)} - x_j^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_\infty < \epsilon \text{ for all } j.$$

So, for a fixed j, the sequence $(x_j^{(n)})$ is Cauchy, and therefore convergent in $\mathbb{C}$. Denote

$$x_j := \lim_{n\to\infty} x_j^{(n)}, \text{ and } x := (x_1, x_2, ...).$$

We need to show $x \in \ell^\infty$ and $x^{(n)} \to x$ as $n \to \infty$. In (2), for a fixed j, letting $n \to \infty$ yields

$$|x_j - x_j^{(m)}| < \epsilon \text{ for all } m \geq N.$$

Therefore

$$\sup_j |x_j - x_j^{(m)}| := \|x - x^{(m)}\|_\infty \leq \epsilon \text{ for all } m \geq N.$$

That is $x^{(m)} \to x$ as $m \to \infty$ in ℓ^∞. Now for all $j \in \mathbb{N}$, $n \geq N$,

$$|x_j| \leq |x_j - x_j^{(n)}| + |x_j^{(n)}| \leq \|x - x^{(n)}\|_\infty + \|x^{(n)}\|_\infty \leq \epsilon + \|x^{(n)}\|_\infty < \infty.$$

This shows that $\|x\|_\infty < \infty$ and so $x \in \ell^\infty$.
(b) Of course $c_0 \subset \ell^\infty$. Assume $(x^{(n)}) \in c_0$ that converges in ℓ^∞ to x. We have to show that $x \in c_0$. Let $\epsilon > 0$ be arbitrary. Since $x^{(n)} \to x$ in ℓ^∞, we can choose $N \in \mathbb{N}$ such that

$$\|x^{(N)} - x\|_\infty < \frac{\epsilon}{2}.$$

Since $x^{(N)} = (x_1^{(N)}, x_2^{(N)}, ...) \in c_0$ we have $x_j^{(N)} \to 0$ as $j \to \infty$. Therefore, choose $J \in \mathbb{N}$ such that

$$j \geq J \Rightarrow |x_j^{(N)}| < \frac{\epsilon}{2}.$$

Then, for $j \geq J$,

$$|x_j| \leq |x_j - x_j^{(N)}| + |x_j^{(N)}| \leq \|x - x^{(N)}\|_\infty + |x_j^{(N)}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $x = (x_1, x_2, ...) \in c_0$ ■.

Problem 20.
(a) Let c_{00} be the space of sequences such that if $x = (x_n)_{n\in\mathbb{N}} \in c_{00}$ then $x_n = 0$ for all $n \geq n_0$, where n_0 is some integer number. Show that c_{00} with the norm $\|.\|_\infty$ is NOT a Banach space.
(b) What is the closure of c_{00} in ℓ^∞?

Solution.
We observe that $c_{00} \subset c_0 \subset \ell^\infty$.
(a) Consider the sequence $x^{(n)}$ defined by

$$x^{(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, 0, ...\right) \in c_{00}.$$

Then, for $n, m \geq N$,

$$\|x^{(n)} - x^{(m)}\|_\infty = \begin{cases} \frac{1}{m+1} & \text{if } \ n \geq m, \\ \frac{1}{n+1} & \text{if } \ n \leq m. \end{cases}$$

In both cases we have for any $N > 0$

$$\|x^{(n)} - x^{(m)}\|_\infty \leq \frac{1}{N+1} \quad \text{for } \ n, m \geq N.$$

So the sequence $x^{(n)}$ is a Cauchy sequence in c_{00}. Evidently, it is also a Cauchy sequence in ℓ^∞. Now consider the sequence

$$x = \left(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, ...\right) \in \ell^\infty.$$

We see that $x^{(n)} \notin c_{00}$, and

$$\lim_{n\to\infty} \|x^{(n)} - x\|_\infty = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

This tells us that there is a Cauchy sequence in c_{00} which does not converge to *something in* c_{00}. Therefore, c_{00} equipped with the $\|.\|_\infty$ norm is not a Banach space.

(b) We claim that $\overline{c_{00}} = c_0$ (closure taken in ℓ^∞).
According to Problem 19, c_0 is closed, so we have $\overline{c_{00}} \subset c_0$. We show the inverse inclusion. Take an arbitrary sequence $x = (x_1, x_2, ...) \in c_0$. We build a sequence $a^{(n)}$ from x as follows:

$$a^{(n)} = (x_1, x_2, .., x_n, 0, 0....).$$

It is clear that $a^{(n)} \in c_{00}$. Now since the sequence x converges to 0, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_i| < \varepsilon \ \text{ for } \ i \geq N.$$

Then

$$\|a^{(n)} - x\|_\infty = \sup_{i\geq N} |x_i| \leq \varepsilon.$$

hence $a^{(n)}$ converges to x (in ℓ^∞). Hence $x \in \overline{c_{00}}$. Thus, $c_0 \subset \overline{c_{00}}$. ■

Problem 21.
Prove that:
(a) If $1 \leq p < q < \infty$, then $\ell^p \subset \ell^q$ and $\|x\|_q \leq \|x\|_p$.
(b) If $x \in \bigcup_{1\leq p<\infty} \ell^p$ then $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$.

Solution.
(a) Let $x = (x_1, x_2, ...) \in \ell^p$. Then, for n large enough, we have $|x_n| < 1$ and hence $|x_n|^q \leq |x_n|^p$ since $1 \leq p < q < \infty$. That implies $x \in \ell^q$. Now we want to show

$$\left(\sum |x_n|^q\right)^{1/q} \leq \left(\sum |x_n|^p\right)^{1/p}.$$

Let $a_n = |x_n|^p$ and $\alpha = \frac{q}{p} > 1$. The above inequality is equivalent to

$$\sum a_n^\alpha \leq \left(\sum a_n\right)^\alpha,$$

which follows by

$$\sum a_n^\alpha \leq (\max a_n)^{\alpha-1} \sum a_n \leq \left(\sum a_n\right)^{\alpha-1} \sum a_n = \left(\sum a_n\right)^{\alpha}.$$

(b) Let $x = (x_1, x_2, ...) \in \ell^{p_0}$ for some p_0. Clearly, $\|x\|_p \geq \max_n |x_n| = \|x\|_\infty$ for any finite p. On the other hand,

$$\|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p} \leq \left(\|x\|_\infty^{p-p_0} \sum_n |x_n|^{p_0}\right)^{1/p} = \|x\|_\infty^{\frac{p-p_0}{p}} \, \|x\|_{p_0}^{\frac{p_0}{p}} \xrightarrow{p\to\infty} \|x\|_\infty.$$

■

Problem 22.
Prove that:
(a) If $1 \leq p < \infty$ then ℓ^p is separable.
(b) ℓ^∞ is not separable.

Solution.
(a) First we show that $E := \{x \in \ell^p : \ x_n = 0, \ n \geq N \ \text{ for some } N\}$ is dense in ℓ^p. Indeed, if $x \in \ell^p$, $x = \sum_{k=1}^\infty x_k e_k$, where e_k is the sequence such that the k-component is 1 and the others are zero, then

$$\left\| x - \sum_{k=1}^n x_k e_k \right\|_p = \left(\sum_{k=n+1}^\infty |x_k|^p\right)^{1/p} \to 0 \ \text{ as } \ n \to \infty.$$

But $\sum_{k=1}^n x_k e_k \in E$, so E is dense in ℓ^p. Now let $A \subset E$ consisting of elements $x = (x_1, x_2, ..., x_n, 0, 0, ...) \in E$ such that $x_k = a_k + ib_k$, $a_k, b_k \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, A is dense in E. Hence A is dense in ℓ^p. Since A is countable, ℓ^p is separable.
(b) We now show that it is not the case for ℓ^∞.
Let $F := \{x \in \ell^\infty : \ \forall k \geq 1, \ x_k = 0 \ \text{ or } \ x_k = 1\}$. Then F is uncountable. Note that for $x \in F$, $\|x\|_\infty = 1$. Moreover, $x, y \in F$, $x \neq y \Rightarrow \|x - y\|_\infty = 1$. Assume that ℓ^∞ is separable. Then there is a set $A = \{a_1, a_2, ...\}$ dense in ℓ^∞. So, for all $x \in F$, there exists $k \in \mathbb{N}$ such that $\|x - a_k\|_\infty \leq \frac{1}{3}$. Let $\mathcal{F}$ be the family of closed balls $B(x; \frac{1}{3})$, $x \in F$. If $B \neq B'$ then $B \cap B' = \emptyset$. This allows us to construct an injection $f : \mathcal{F} \to A$ which maps each $B \in \mathcal{F}$ with an element $a \in B \cap A$. This is impossible since $\mathcal{F}$ is uncountable and A is countable. ■

* * **

Problem 23. (The space $C[0,1]$)
Let $C[0,1]$ be the space of all continuous functions on $[0,1]$.
(a) Prove that if $C[0,1]$ is equipped with the uniform norm

$$\|f\| = \max_{x\in[0,1]} |f(x)|, \ f \in C[0,1]$$

then $C[0,1]$ is a Banach space.
(b) Give an example to show that $C[0,1]$ equipped with the L^1-norm

$$\|f\|_1 = \int_0^1 |f(x)|dx, \ f \in C[0,1]$$

is not a Banach space.

Solution.
(a) Let (f_n) be a Cauchy sequence in $C[0,1]$ with respect to the uniform norm. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_m - f_n\| < \varepsilon \ \text{ for } \ m, n \geq N.$$

Therefore

$$(*) \qquad |f_m(x) - f_n(x)| < \varepsilon \ \text{ for } \ m, n \geq N \ \text{ and } \ x \in [0,1].$$

This shows that for every $x \in [0,1]$, the sequence $\big(f_n(x)\big)$ is a Cauchy sequence of numbers and therefore converges to a number which depends on x, say, $f(x)$. In $(*)$, fix n and let $m \to \infty$, we have

$$(**) \qquad |f(x) - f_n(x)| < \varepsilon \ \text{ for } \ n \geq N \ \text{ and } \ x \in [0,1].$$

Thus the sequence (f_n) converges uniformly to f on $[0,1]$ so that f is continuous on $[0,1]$, that is, $f \in C[0,1]$. From $(**)$ we obtain

$$\max_{x\in[0,1]} |f(x) - f_n(x)| = \|f - f_n\| \leq \varepsilon \ \text{ for } \ n \geq N.$$

This shows that

$$\lim_{n\to\infty} \|f - f_n\| = 0.$$

(b) For each $n \in \mathbb{N}$, consider the function

$$f_n(x) = \begin{cases} nx & \text{if } \ 0 \leq x < \frac{1}{n} \\ 1 & \text{if } \ \frac{1}{n} \leq x \leq 1. \end{cases}$$

One can check that the sequence (f_n) is a Cauchy sequence with respect to the L^1-norm, but it converges to the function

$$f(x) = \begin{cases} 0 & \text{if } \ x = 0 \\ 1 & \text{if } \ 0 < x \leq 1, \end{cases}$$

which is not continuous, that is, $f \notin C[0,1]$. Thus the space $C[0,1]$ is not complete. ■

Chapter 3

Hilbert Spaces

3.1 Hilbert spaces

Problem 24.
Let M be a closed subspace of a Hilbert space H. Prove that:
(a) $(M^\perp)^\perp = M$.
(b) If $\operatorname{codim} M := \dim H/M = 1$ then $\dim M^\perp = 1$.

Solution.
(a) In general, if M is a subset of H then $M \subset (M^\perp)^\perp$. Indeed,

$$M^\perp := \{x \in X : \ x \perp M\}.$$

So we have

$$x \in M \Rightarrow x \perp M^\perp \Rightarrow x \in (M^\perp)^\perp.$$

Now suppose M is a closed subspace of H and $x \in (M^\perp)^\perp$. Since $x \in H = M \oplus M^\perp$, we have

$$x = u + v, \ u \in M, \ v \in M^\perp.$$

Since $M \subset (M^\perp)^\perp$ we have

$$x = u + v, \ u \in (M^\perp)^\perp, \ v \in M^\perp.$$

Since $x - u \in (M^\perp)^\perp$ and $v \in M^\perp$ and $v = x - u$ we obtain

$$v \in M^\perp \cap (M^\perp)^\perp,$$

which implies $v = 0$. Hence, $x = u \in M$.
(b) Assume that there are two linearly independent vectors $x, y \in M^{\perp}$. Recall that M is the zero vector of the linear space X/M. Consider the cosets $[x], [y]$. Assume $\alpha[x] + \beta[y] = M$, for some scalars α, β. Then $\alpha x + \beta y \in M$ and, since $\alpha x + \beta y \in M^{\perp}$ as well, we conclude that $\alpha x + \beta y = 0$. Hence, $\alpha = \beta = 0$ and therefore $[x], [y]$ are linearly independent. This contradicts the hypothesis $\operatorname{codim} M = 1$. ■

Problem 25.
Let $T : H_1 \to H_2$ be an isometry of two Hilbert spaces H_1 and H_2, i.e., $\|Tx\| = \|x\|$ for every $x \in H_1$. Prove that

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \textit{for every} \quad x, y \in H_1.$$

Solution.
By hypothesis, we have

$$\|T(x + y)\|^2 = \|x + y\|^2 \quad \text{for every} \quad x, y \in H_1.$$

By opening up the norm using the inner product, we obtain

$$Re\langle Tx, Ty \rangle = Re\langle x, y \rangle.$$

Similarly, $\|T(x + iy)\|^2 = \|x + iy\|^2$ gives that

$$Im\langle Tx, Ty \rangle = Im\langle x, y \rangle.$$

Hence,

$$\langle Tx, Ty \rangle = \langle x, y \rangle. \quad \blacksquare$$

Problem 26.
Let C be a closed convex set in a Hilbert space H. Show that C contains a unique element of minimal norm.

Solution.
Let $\eta = \inf_{z \in C} \|z\|$.

• If $\eta = 0$, then, by definition of infimum, there is a sequence (z_j) in C such that $\|z_j\| \to 0$ as $j \to \infty$. Therefore, $z_j \to 0$ as $j \to \infty$. Since C is closed, $0 \in C$, and 0 is the unique element of minimal norm.
• Suppose $\eta > 0$. First we show that C contains an element of minimal norm. Take (z_j) in C such that $\|z_j\| \to \eta$ as $j \to \infty$. The convexity of C implies that $\frac{1}{2}(z_j + z_k) \in C$, so that

$$\|z_j + z_k\|^2 = 4.\frac{1}{4}\|z_j + z_k\|^2 \geq 4\eta^2.$$

Recall now the parallelogram law:

$$\|x - y\|^2 + \|x + y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Applying this we have:

$$\begin{aligned}
\|z_j - z_k\|^2 &= 2\left(\|z_j\|^2 + \|z_k\|^2\right) - \|z_j + z_k\|^2 \\
&\leq 2\left(\|z_j\|^2 + \|z_k\|^2\right) - 4\eta^2 \to 4\eta^2 - 4\eta^2 = 0 \ \text{ as } \ j, k \to \infty.
\end{aligned}$$

Thus the sequence (z_j) is Cauchy, so converges to some $z \in H$. Since C is closed, $z \in C$. The norm function is continuous, so $\|z\| = \eta$. This shows that C contains an element of minimal norm.
Assume that there are two elements $a_1, a_2 \in C$ such that $\|a_1\| = \|a_2\| = \eta$. By the above we have

$$\|a_1 + a_2\|^2 \geq 4\eta^2.$$

By the parallelogram law we have

$$\|a_1 - a_2\|^2 = 2\left(\|a_1\|^2 + \|a_2\|^2\right) - \|a_1 + a_2\|^2 \leq 4\eta^2 - 4\eta^2 = 0.$$

Hence, $a_1 = a_2$. ■

Problem 27.
Let $1 \leq p < \infty$. Prove that ℓ^p is a Hilbert space if and only if $p = 2$.

Solution.
• Suppose $p = 2$. In the space ℓ^2 the inner product is defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} \bar{x}_i y_i \ \text{ for } \ x = (x_i),\ y = (y_i) \in \ell^2.$$

This inner product gives rise to the norm

$$\|x\|_2 = \sqrt{\langle x, x\rangle} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}.$$

According to Problem 18, the normed space ℓ^2 is complete. So ℓ^2 is a Hilbert space.
• Consider the general case where $1 \leq p < \infty$. Assume that ℓ^p with the corresponding inner product is a Hilbert space. Consider two elements $e_n, e_m \in \ell^p$ with $m \neq n$ defined as follows:

$$e_n = (\underbrace{0, .., 0}_{n-1}, 1, 0, ...),$$
$$e_m = (\underbrace{0, .., 0}_{m-1}, 1, 0, ...).$$

Since ℓ^p is a Hilbert space, by the parallelogram law we have

$$\|e_n + e_m\|_p^2 + \|e_n - e_m\|_p^2 = 2\left(\|e_n\|_p^2 + \|e_m\|_p^2\right),$$

That is

$$2^{2/p} + 2^{2/p} = 2^2.$$

The unique solution of this equation is $p = 2$. We conclude that ℓ^p is a Hilbert space if and only if $p = 2$. ■

Problem 28.
Consider the Hilbert space $H = L^2[-1, 1]$ equipped with the usual scalar product:

$$\langle x, y\rangle = \int_{-1}^{1} \overline{x(t)} y(t) dt, \quad x, y \in H.$$

Let $M = \{x \in H : \int_{-1}^{1} x(t)dt = 0\}$.
(a) Show that M is closed in H. Find $M^\perp$.
(b) Calculate the distance from y to M for $y(t) = t^2$.

Solution.
(a) Let $\mathbf{1} \in H$ be the function $\mathbf{1}(t) = 1, \ \forall t \in [-1, 1]$. Define the map $T : H \to \mathbb{C}$ by

$$x \mapsto \langle \mathbf{1}, x\rangle.$$

Then T is linear. We show that T is bounded, so continuous.

$$\begin{aligned}|Tx| = \left|\int_{-1}^{1}\mathbf{1}(t)x(t)dt\right| &\leq \int_{-1}^{1}|\mathbf{1}(t)|\ |x(t)|dt \\ &\leq \left(\int_{-1}^{1}1dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|x^2(t)|^2dt\right)^{\frac{1}{2}} \\ &= \sqrt{2}\|x\|_2.\end{aligned}$$

By definition, $M = \operatorname{Ker} T = T^{-1}(0)$. Since T is continuous, M is closed. Furthermore,

$$x \in M \Leftrightarrow \langle \mathbf{1}, x\rangle = 0 \Leftrightarrow M = (\operatorname{Span}\{\mathbf{1}\})^{\perp}.$$

Since M is closed, (see Problem 23 a).

$$M^{\perp} = \operatorname{Span}\{\mathbf{1}\}.$$

(b) The distance from $y \in H$ to M is the length of the projection vector of y on $M^{\perp}$. We have

$$d(y, M) = \frac{|\langle \mathbf{1}, y\rangle|}{\|\mathbf{1}\|_2} = \left(\int_{-1}^{1}t^2dt\right)\left(\int_{-1}^{1}1dt\right)^{-1/2} = \frac{\sqrt{2}}{3}. \quad \blacksquare$$

Problem 29.
Consider the Hilbert space $H = L^2[-1,1]$ equipped with the usual scalar product:

$$\langle f, g\rangle = \int_{-1}^{1}\overline{f(t)}g(t)dt, \quad f, g \in H.$$

Let $E = \{x \in H : \ f(-t) = f(t), \ \ t \in [-1,1]\}$.
(a) Show that E is closed in H. Find $E^{\perp}$.
(b) Calculate the distance from h to E for $h(t) = e^t$.

Solution.
(a) Define $\tilde{f}(t) := f(-t)$. Define the map

$$S : H \to H \ \text{ defined by } \ Sf = \tilde{f}.$$

Clearly, S is linear. S is bounded, so continuous. Indeed,

$$\|Sf\|_2 = \|\tilde{f}\|_2 = \left(\int_{-1}^{1}|\tilde{f}(t)|^2dt\right)^{\frac{1}{2}} = \left(\int_{-1}^{1}|f(-t)|^2dt\right)^{\frac{1}{2}} = \|f\|_2.$$

In fact, S is an isometry. It follows that $I - S$ is continuous. By definition, $E = \mathrm{Ker}(I - S)$, so E is closed.
By definition, E consists of all even functions, so $E^{\perp}$ is the linear subspace of all odd functions. In fact, we have

$$f(t) = \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{\varphi} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{\psi} \equiv \varphi(t) + \psi(t),$$

with φ is even, ψ is odd, and

$$\langle \varphi, \psi \rangle = \int_{-1}^{1} \overline{\varphi(t)}\psi(t)dt = 0, \quad \text{so that } \ \varphi \perp \psi.$$

(b) The distance from $h \in H$ to E is the length of the projection vector of h on $E^{\perp}$. By the above expression,

$$\mathrm{Proj}_{E^{\perp}}(h) = \frac{1}{2}[h(t) - h(-t)] = \frac{1}{2}(e^t - e^{-t}).$$

Therefore,

$$\begin{aligned} \big(\mathrm{dist}(h, E)\big)^2 &= \left\| \frac{1}{2}(e^t - e^{-t}) \right\|^2 \\ &= \frac{1}{4}\int_{-1}^{1} |e^t - e^{-t}|^2 dt \\ &= \frac{1}{4}(e^2 - e^{-2} - 4). \end{aligned}$$

Thus,

$$\mathrm{dist}(h, E) = \frac{1}{2}\sqrt{e^2 - e^{-2} - 4}. \quad \blacksquare$$

Problem 30.
Let H be a Hilbert space and M be a closed subspace of H. Denoting by $P : H \to M$ the orthogonal projection of H onto M, prove that, for any $x, y \in H$,

$$\langle Px, y \rangle = \langle x, Py \rangle.$$

(This is telling us that P is self-adjoint).

Solution.
We know that if M is a closed subspace of H, then

- For all $u \in H$, there exist unique $u_M \in M$ and $u_{M^\perp} \in M^\perp$ such that

$$u = u_M + u_{M^\perp}.$$

- If $P : H \to M$ the orthogonal projection of H onto M, then

$$Pu = u_M.$$

Now for arbitrary $x, y \in H$, we have

$$\begin{aligned} x &= x_M + x_{M^\perp} & \qquad y &= y_M + y_{M^\perp} \\ Px &= x_M & Py &= y_M. \end{aligned}$$

With these, can have

$$\langle Px, y\rangle = \langle x_M, y_M + y_{M^\perp}\rangle = \langle x_M, y_M\rangle,$$

since $\langle x_M, y_{M^\perp}\rangle = 0$, and

$$\langle x, Py\rangle = \langle x_M + x_{M^\perp}, y_M\rangle = \langle x_M, y_M\rangle,$$

since $\langle x_{M^\perp}, y_M\rangle = 0$. Thus,

$$\langle Px, y\rangle = \langle x, Py\rangle. \quad \blacksquare$$

Problem 31.
Let H be a Hilbert space and $A \subset H$ a closed convex non-empty set. Prove that $P_A : H \to H$ is non-expansive, i.e.,

$$\|P_A(x) - P_A(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

(P_A is the orthogonal projection on A).

Solution.
We claim:

$$(*) \qquad Re\Big\langle x - P_A(x), P_A(x) - a\Big\rangle \geq 0, \quad \forall x \in H,\ a \in A.$$

Let $x_A = P_A(x)$. Then x can be decomposed uniquely as

$$x = x_A + x'_A, \quad \forall x_A \in A,\ x'_A \in A^\perp.$$

We have

$$\begin{aligned} 2Re\Big\langle x - P_A(x), P_A(x) - a\Big\rangle &= \langle x'_A, x_A - a\rangle + \langle x_A - a, x'_A\rangle \\ &= \langle x'_A, x'_A\rangle + \langle x_A - a, x_A - a\rangle \\ &= \|x'_A\|^2 + \|x_A - a\|^2 \geq 0. \end{aligned}$$

Hence $(*)$ is proved.
Replacing a with $P_A(y)$ in $(*)$, we obtain

$$Re\Big\langle x - P_A(x), P_A(x) - P_A(y)\Big\rangle \geq 0. \tag{3.1}$$

Analogously,

$$Re\Big\langle y - P_A(y), P_A(y) - P_A(x)\Big\rangle \geq 0.$$

And therefore,

$$Re\Big\langle P_A(y) - y, P_A(x) - P_A(y)\Big\rangle \geq 0. \tag{3.2}$$

Adding 3.1 and 3.2 we get

$$Re\Big\langle x - y - [P_A(x) - P_A(y)], P_A(x) - P_A(y)\Big\rangle \geq 0, \quad \text{i.e.,}$$

$$Re\Big\langle x - y, P_A(x) - P_A(y)\Big\rangle \geq \|P_A(x) - P_A(y)\|^2. \tag{3.3}$$

Form the Cauchy-Schwarz inequality, we have

$$\begin{aligned} Re\Big\langle x - y, P_A(x) - P_A(y)\Big\rangle &\leq |\langle x - y, P_A(x) - P_A(y)\rangle| \\ &\leq \|x - y\| \, \|P_A(x) - P_A(y)\| \end{aligned}$$

and from here, replacing in 3.3, we obtain that

$$\|P_A(x) - P_A(y)\|^2 \leq \|x - y\| \, \|P_A(x) - P_A(y)\|.$$

Thus,

$$\|P_A(x) - P_A(y)\| \leq \|x - y\|. \quad \blacksquare$$

Problem 32.
Let X be a Hibert space, and $G_1 \subset G_2 \subset ... \subset G_n \subset ...$ be a sequence of closed linear subspaces of X. Let

$$G = \overline{\text{Span}\left(\bigcup_{n\in\mathbb{N}} G_n\right)}.$$

(a) Prove that $d(x,G) = \lim_{n\to\infty} d(x,G_n),\ \forall x \in X.$
(b) Prove that $P_G(x) = \lim_{n\to\infty} P_{G_n}(x),\ \forall x \in X.$ Note: P_G is the orthogonal projection of X on G.

Solution.
(a) Let

$$A = \text{Span}\left(\bigcup_{n\in\mathbb{N}} G_n\right).$$

Then we have

$$d(x,G) = d(x,\bar{A}) = d(x,A).$$

For any $\varepsilon > 0$ we have

$$d(x,G) + \varepsilon = d(x,A) + \varepsilon > d(x,A) = \inf_{a\in A} \|x - a\|.$$

From this, we deduce that there is some $a_\varepsilon \in A$ such that

$$d(x,G) + \varepsilon > \|x - a_\varepsilon\|.$$

Since $a_\varepsilon \in \text{Span}\left(\bigcup_{n\in\mathbb{N}} G_n\right)$, we can find $\lambda_1, ..., \lambda_k \in \mathbb{K}$ and $x_1, ..., x_k \in \bigcup_{n\in\mathbb{N}} G_n$ such that

$$a_\varepsilon = \lambda_1 x_1 + ... + \lambda_k x_k.$$

Then there are $n_1, ..., n_k \in \mathbb{N}$ such that $x_1 \in G_{n_1}, ..., x_k \in G_{n_k}$. Let $n = \max\{n_1, ..., n_k\}$. By hypothesis, the sequence (G_n) is increasing , so we have $G_{n_1}, ..., G_{n_k} \subset G_n$, so $x_1, ..., x_k \in G_n$. And since G_n is a linear space,

$$a_\varepsilon = \lambda_1 x_1 + ... + \lambda_k x_k \in G_n.$$

That is,

$$(*) \qquad \forall \varepsilon > 0,\ \exists n \in \mathbb{N}:\ d(x,G) + \varepsilon > \|x - a_\varepsilon\| \geq d(x,G_n).$$

From $G_n \subset G$ it follows that

$$(**) \qquad d(x,G) \leq d(x,G_n),\ \forall n \in \mathbb{N}.$$

$(*)$ and $(**)$ imply that

$$d(x,G) \leq d(x,G_n) < d(x,G) + \varepsilon.$$

Hence,

$$d(x, G) = \inf_{n\in\mathbb{N}} d(x, G_n).$$

From $G_n \subset G_{n+1}$, it follows that $d(x, G_{n+1}) \leq d(x, G_n)$. The sequence of real numbers $\Big(d(x, G_n)\Big)$, which is decreasing and bounded below, must converge to its infimum. Thus

$$\lim_{n\to\infty} d(x, G_n) = d(x, G).$$

(b) Let $a_n = P_{G_n}(x),\ a = P_G(x)$. Then by part (a) we have

$$\lim_{n\to\infty} \|x - a_n\| = \lim_{n\to\infty} d(x, G_n) = d(x, G).$$

Since $G_n \subset G,\ \forall n \in \mathbb{N}$, then $(a_n) \subset G$ and so $a_n \to P_G(x)$ in norm, that is,

$$\lim_{n\to\infty} \|a_n - a\| = 0,$$

which gives that

$$\lim_{n\to\infty} P_{G_n}(x) = P_G(x). \qquad \blacksquare$$

Problem 33.
Let H be a Hilbert space.
(a) Prove that for any two subspaces M, N of H we have

$$(M + N)^\perp = M^\perp \cap N^\perp.$$

(b) Prove that for any two closed subspaces E, F of H we have

$$(E \cap F)^\perp = \overline{E^\perp + F^\perp}.$$

Solution.
(a) If $x \in (M + N)^\perp$, then for every $m \in M$ and $n \in N$ we have

$$\langle m + n, x\rangle = 0$$

since $m + n \in M + N$. For $n = 0$ we have $\langle m, x\rangle = 0$. This holds for all $m \in M$, so $x \in M^\perp$. Similarly $x \in N^\perp$. Thus $x \in M^\perp \cap N^\perp$, and hence $(M + N)^\perp \subset M^\perp \cap N^\perp$.
If $x \in M^\perp \cap N^\perp$, then we have

$$\langle m, x\rangle = 0 \ \text{ and } \ \langle n, x\rangle = 0, \quad \forall m \in M,\ n \in N.$$

Hence

$$\langle m+n, x\rangle = 0.$$

This means that $x \in (M+N)^\perp$. Hence $M^\perp \cap N^\perp \subset (M+N)^\perp$.

(b) From part (a) it follows that

$$\overline{M+N} = (M^\perp \cap N^\perp)^\perp.$$

Setting $E^\perp$ in the place of M and $F^\perp$ in the place of N, we obtain

$$\overline{E^\perp + F^\perp} = (E\cap F)^\perp. \quad \blacksquare$$

Why in question (b) E and F must be closed?

Problem 34.
A system $\{x_i\}_{i\in\mathbb{N}}$ in a normed space X is called a ***complete system*** *if* Span $\{\sum_{i=1}^n \alpha_i x_i : \ \forall n\in\mathbb{N},\ \alpha_i\in\mathbb{F}\}$ *is dense on X.*
If $\{x_i\}_{i\in\mathbb{N}}$ is a complete system in a Hilbert space H and $x\perp x_i$ for every i, show that $x=0$.

Solution.
Given $x\in X$, if $x\perp x_i$ for every i, then $x\perp$ Span $\{\sum_{i=1}^n \alpha_i x_i\}$. Let $D :=$ Span $\{\sum_{i=1}^n \alpha_i x_i\}$. By definition, $\overline{D} = X$. Then there exists a sequence (x_n) in D converging to x and $x\perp x_n$ for every n. Hence

$$0 = \langle x, x_n\rangle \to \langle x, x\rangle = \|x\|^2 \ \text{ as } \ n\to\infty.$$

Thus $x = 0$. $\blacksquare$

Problem 35.
Let H be a Hilbert space and $\{\varphi_i\}_{i=1}^\infty$ an orthonormal system in H. Show that

$$\|\varphi_n - \varphi_m\| = \sqrt{2} \ \textit{ for } \ m\neq n.$$

Solution.
Using orthogonality and the fact that $\|\varphi_k\| = 1,\ \forall k\in\mathbb{N}$, we get for $n\neq m$,

$$\begin{aligned}
\|\varphi_n - \varphi_m\|^2 &= \langle \varphi_n - \varphi_m, \varphi_n - \varphi_m\rangle \\
&= \langle \varphi_n, \varphi_n\rangle - \langle \varphi_n, \varphi_m\rangle - \langle \varphi_m, \varphi_n\rangle + \langle \varphi_m, \varphi_m\rangle \\
&= \|\varphi_n\|^2 + \|\varphi_m\|^2 = 2. \quad \blacksquare
\end{aligned}$$

Problem 36.
Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal set in a Hilbert space H. Prove that

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle\langle y, e_i\rangle| \leq \|x\|\ \|y\|,\ \forall x, y \in H.$$

Solution.
Using the Cauchy-Schwarz inequality for ℓ^2 and the Bessel inequality for H, we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle x, e_i\rangle\langle y, e_i\rangle| &\leq \left(\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2\right)^{\frac{1}{2}} \\ &\leq \left(\|x\|^2\right)^{\frac{1}{2}} \left(\|y\|^2\right)^{\frac{1}{2}} = \|x\|\ \|y\|.\end{aligned}$$

Problem 37.
Let H be a Hilbert space and A and B be any two subsets of H. Show that
(a) $A^\perp$ is a closed subspace of H.
(b) $A \subset (A^\perp)^\perp := A^{\perp\perp}$.
(c) $A \subset B \Rightarrow B^\perp \subset A^\perp$.
(d) $A^\perp = \overline{A}^\perp = \overline{A^\perp}$.

Solution.
(a) Let $x, y \in A^\perp$ and $\alpha, \beta \in \mathbb{F}$. Then for any $a \in A$,

$$\begin{aligned}\langle \alpha x + \beta y, a\rangle &= \langle \alpha x, a\rangle + \langle \beta y, a\rangle \\ &= \alpha\langle x, a\rangle + \beta\langle y, a\rangle = 0.\end{aligned}$$

Hence $\alpha x + \beta y \in A^\perp$, so $A^\perp$ is a subspace of H.
We show now that $A^\perp$ is closed. Suppose that the sequence (x_n) in $A^\perp$ converges to some $x \in H$. Since $\langle x_n, a\rangle = 0$ for all $a \in A$, by continuity of the inner product, we have

$$\langle x, a\rangle = \langle \lim_{n\to\infty} x_n, a\rangle = \lim_{n\to\infty} \langle x_n, a\rangle = 0.$$

So $x \in A^\perp$, and $A^\perp$ is closed.

(b) For any $x \in A$, we have $x \perp A^{\perp}$. This implies that $x \in (A^{\perp})^{\perp}$. Thus

$$A \subset (A^{\perp})^{\perp} := A^{\perp\perp}.$$

(c) Take any $x \in B^{\perp}$. Then $x \perp y$ for all $y \in B$. But $A \subset B$, so $x \perp y$ for all $y \in A$. Hence $x \in A^{\perp}$.

(d) From (a) we have

$$A^{\perp} = \overline{A^{\perp}},$$

which is the second equality in (d). Now pick any $x \perp A$, that is, $x \in A^{\perp}$. If $a \in \overline{A}$, then there exists a sequence (a_n) is A such that $a_n \to a$. Since $x \perp A$, we have $x \perp a_n$ for all n. Hence

$$\langle x, a \rangle = \langle x, \lim_{n\to\infty} a_n \rangle = \lim_{n\to\infty} \langle x, a_n \rangle = 0.$$

Thus we have $x \perp A$, so $x \perp \overline{A}$. Therefore,

$$A^{\perp} \subset \overline{A}^{\perp}. \qquad (i)$$

On the other hand, by (b) we have

$$A \subset \overline{A} \Longrightarrow \overline{A}^{\perp} \subset A^{\perp}. \qquad (ii)$$

(i) and (ii) complete the proof. ■

Problem 38.
(a) Show that $M := \{x = (x_n) \in \ell^2 : \; x_{2n} = 0, \; \forall n \in \mathbb{N}\}$ is a closed subspace of the Hilbert space ℓ^2.
(b) Find $M^{\perp}$.

Solution.
(a) Take any $x = (x_n), y = (y_n) \in M$. It is clear that for any scalars α, β,

$$(\alpha x + \beta y)_{2n} = \alpha x_{2n} + \beta y_{2n} = 0.$$

That gives that $\alpha x + \beta y \in M$. Hence M is a linear subspace of ℓ^2.
Let us prove that it is closed. Take $x \in \overline{M}$. There exists a sequence $x^{(k)} = (x_n^{(k)}) \in M$ converging to x as $k \to \infty$. Since $x_{2n}^{(k)} = 0$, we obtain

$$x_{2n} = \lim_{k\to\infty} x_{2n}^{(k)} = 0,$$

that is, $x \in M$. Hence M is closed.
(b) Now

$$\begin{aligned} z \in M^\perp &\iff \langle z, x\rangle = 0,\ \forall x \in M \\ &\iff \sum_{n=0}^{\infty} z_{2n+1}\overline{x_{2n+1}} = 0 \text{ for all scalars } x_{2n+1} \text{ such that } \sum_{n=0}^{\infty} |x_{2n+1}|^2 < \infty \\ &\iff z_{2n+1} = 0,\ \forall n = 0, 1, 2, ... \end{aligned}$$

Therefore

$$M^\perp = \{z = (z_n) \in \ell^2 :\ z_{2n+1} = 0,\ \forall n = 0, 1, 2, ...\}. \qquad \blacksquare$$

Problem 39.
Let V be a subspace of a Hilbert space.
(a) Show that $\overline{V}^\perp = V^\perp$.
(b) Show that V is dense in H if and only if $V^\perp = \{0\}$.

Solution.
(a) From Problem 37d we get (a).
(b) If V is dense, then $\overline{V} = H$. Hence

$$V^\perp = \overline{V}^\perp = H^\perp = \{0\}.$$

Conversely, suppose $V^\perp = \{0\}$. If V is not dense in H, that is, $\overline{V} \subsetneq H$, pick $x \in H \setminus \overline{V}$. Let $x' = P_{\overline{V}}x$. Then

$$x - x' \in \overline{V}^\perp = V^\perp = \{0\}.$$

Thus $x = x' \in \overline{V}$. This is a contradiction. Thus V is dense in H. $\blacksquare$

3.2 Weak convergence

Problem 40.
Prove that in any finite dimensional vector space, strong convergence and weak convergence are equivalent.

Solution.
Consider first the case that $X = \mathbb{F}^d$ under the Euclidian norm $\|.\|_2$. Suppose that the sequence (x_n) converges weakly to x in $\mathbb{F}^d$. Then for each standard basis vector e_k, $k = 1, 1, .., d$, we have

$$\langle x_n, e_k\rangle \to \langle x, e_k\rangle \quad \text{as} \quad n \to \infty.$$

That is, weak convergence implies componentwise convergence. But since there are only finitely many components, this implies norm convergence, since

$$\|x_n - x\|_2^2 = \sum_{k=1}^{d} |\langle x_n, e_k\rangle - \langle x, e_k\rangle|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

For the general case, choose any basis $\{e_1, e_2, ..., e_d\}$ for X, and then use the fact that all norms on X are equivalent to define an isomorphism between X and $\mathbb{F}^d$. ∎

Problem 41.
Show that if the sequence (x_n) in a normed space X is weakly convergent to $x_0 \in X$, then

$$\liminf_{n\to\infty} \|x_n\| \geq \|x_0\|.$$

Solution.
If $x_0 = 0$ then $\|x_0\| = 0$ and the statement is obviously true. Now assume $\|x_0\| \neq 0$. By a well known theorem [1], there is some $f \in X^*$ such that

$$\|f\| = 1, \qquad f(x_0) = \|x_0\|.$$

Since (x_n) converges weakly to x_0 and f is continuous, we have

$$\lim_{n\to\infty} f(x_n) = f(x_0) = \|x_0\|.$$

But

$$f(x_n) \leq |f(x_n)| \leq \|f\|\ \|x_n\| = \|x_n\|.$$

[1](Kreyszig, p 223) *Let X be a normed space and $x_0 \neq 0$ be an element in X. Then there exists a bounded linear functional $f \in X^*$ such that*

$$\|f\| = 1, \qquad f(x_0) = \|x_0\|.$$

Hence,

$$\liminf_{n\to\infty} \|x_n\| \geq \lim_{n\to\infty} f(x_n) = \|x_0\|. \quad \blacksquare$$

Note.
If $X = H$ is a Hilbert space, using the definition of weak convergence we can have different solution.

$$\|x_0\|^2 = \langle x_0, x_0 \rangle = \lim_{n\to\infty} \langle x_0, x_n \rangle.$$

Since $\langle x, x_n \rangle \leq \|x_0\| \ \|x_n\|$, so we have

$$\|x_0\|^2 = \lim_{n\to\infty} \langle x, x_n \rangle \leq \|x_0| \liminf_{n\to\infty} \|x_n\|.$$

Thus

$$\|x_0\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Problem 42.
Let X and Y be normed spaces, $T \in \mathcal{B}(X, Y)$ and (x_n) a sequence in X. Show that if $x_n \xrightarrow{w} x$, then $Tx_n \xrightarrow{w} Tx$

Solution.

Recall: Definition of *weak convergence* in a normed space:

$$x_n \xrightarrow{w} x \Longleftrightarrow f(x_n) \to f(x), \ \forall f \in X^*.$$

We must show that

$$\varphi(Tx_n) \to \varphi(Tx), \ \forall \varphi \in Y^*.$$

That is,

$$(\varphi \circ T)x_n \to (\varphi \circ T)x, \ \forall \varphi \in Y^*.$$

But $\varphi \circ T \in X^*$, so our hypothesis $x_n \xrightarrow{w} x$ guarantees our desired conclusion. $\blacksquare$

Problem 43.
Let H be a Hilbert space and (x_n) be a sequence in H. Suppose $x_n \xrightarrow{w} x$. Show that

$$\lim_{n\to\infty} \|x_n - x\| = 0 \Longleftrightarrow \|x\| \geq \limsup_{n\to\infty} \|x_n\| \qquad (1).$$

Solution.
By Problem 41 we have

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

So the right hand side of (1) is equivalent to $\|x\| = \lim_{n\to\infty} \|x_n\|$. Now note that

$$\begin{aligned} &(i) \quad \|x_n - x\|^2 = \|x_n\|^2 - 2Re\langle x_n, x\rangle + \|x\|^2 \\ &(ii) \quad \big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|. \end{aligned}$$

If $\lim_{n\to\infty} \|x_n - x\| = 0$ then by (ii) we get

$$\lim_{n\to\infty} \|x_n\| = \|x\|.$$

If $\lim_{n\to\infty} \|x_n\| = \|x\|$ then (i) give that

$$\lim_{n\to\infty} \|x_n - x\| = 0. \qquad \blacksquare.$$

Chapter 4

Linear Operators - Linear Functionals

4.1 Linear bounded operators

Problem 44.
Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ be norm spaces, and $T \in \mathcal{B}(X, Y)$. We define $\|T\|$ as

$$\|T\| := \inf_{x \in X}\{M : \ \|Tx\|_2 \leq M\|x\|_1\}.$$

(a) Show that

$$\|T\| = \sup_{\|x\|_1 = 1} \|Tx\|_2 = \sup_{\|x\|_1 \leq 1} \|Tx\|_2 = \sup_{x \in X, \ x \neq 0} \frac{\|Tx\|_2}{\|x\|_1}.$$

(b) Show that

$$\|T\| = \sup_{\|x\|_1 < 1} \|Tx\|_2.$$

Solution.
(a) We have

$$\frac{\|Tx\|_2}{\|x\|_1} \leq M, \quad \text{for all} \quad x \neq 0, \ x \in X.$$

By definition,

$$\|T\| := \inf_{x\in X}\{M : \|Tx\|_2 \leq M\|x\|_1\} = \sup_{x\in X,\ x\neq 0} \frac{\|Tx\|_2}{\|x\|_1} \qquad (i).$$

Now, let $y = \frac{x}{\|x\|_1}$ for $x \in X,\ x \neq 0$. Then $y \in X$ and $\|y\| = 1$. By (i) we have

$$\|T\| = \sup_{x\in X,\ x\neq 0} \frac{\|Tx\|_2}{\|x\|_1} = \sup_{\|y\|=1} \left\| T\left(\frac{\|x\|_1 y}{\|x\|_1}\right)\right\|_2 = \sup_{\|y\|=1} \|Ty\|_2 = \sup_{\|x\|=1} \|Tx\|_2 \qquad (ii).$$

From (ii) it follows that

$$\|T\| = \sup_{\|x\|_1=1} \|Tx\|_2 \leq \sup_{\|x\|_1\leq 1} \|Tx\|_2 \leq \sup_{x\in X,\ \|x\|_1\leq 1} \frac{\|Tx\|_2}{\|x\|_1} \leq \sup_{x\in X,\ x\neq 0} \frac{\|Tx\|_2}{\|x\|_1} = \|T\|.$$

Thus,

$$\|T\| = \sup_{\|x\|_1\leq 1} \|Tx\|_2.$$

(b) Let

$$B := \{x \in X : \|x\|_1 \leq 1\} \ \text{ and } \ B^o := \{x \in X : \|x\|_1 < 1\}.$$

Since $\|T\| = \sup_{x\in B} \|Tx\|_2$, there exists a sequence (x_n) in B such that

$$\|T\| = \lim_{n\to\infty} \|Tx_n\|_2.$$

Consider the sequence (y_n) defined by $y_n = (1 - \frac{1}{2^n})x_n$. It is clear that $y_n \in B^o$ for all $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \|Ty_n\|_2 &= \lim_{n\to\infty} \left\| T\left(1 - \frac{1}{2^n}\right) x_n \right\| = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) \|Tx_n\|_2 \\
&= \lim_{n\to\infty} (1 - \frac{1}{2^n}) \lim_{n\to\infty} \|Tx_n\|_2 = \|T\|.
\end{aligned}$$

Thus

$$\sup_{x\in B^o} \|Tx\|_2 \geq \|T\|.$$

On the other hand,

$$\sup_{x\in B^o} \|Tx\|_2 \leq \sup_{x\in B} \|Tx\|_2 = \|T\|.$$

The proof is complete. ■

Problem 46.
Let $H = \ell^2$ be the well known Hilbert space. Consider the left shift defined by

$$L : \ell^2 \to \ell^2, \quad x = (x_1, x_2, x_3, ...) \mapsto Lx = (x_2, x_3, ...).$$

(a) Show that L is a linear bounded operator. Find $\|L\|$.
(b) Define the right shift and answer the same question as in part (a).

Solution.
We answer only the first part.
- It is easy to check the linearity of L.
- For any $x = (x_1, x_2, x_3, ...) \in \ell^2$, we have

$$\begin{aligned} \|Lx\|^2 = \|(x_2, x_3, ...)\|^2 &= \sum_{n=2}^{\infty} |x_n|^2 \\ &\leq \sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2. \end{aligned}$$

Hence
$$\|Lx\| \leq \|x\|, \ \forall x \in \ell^2.$$
This implies that T is bounded, and
$$\|L\| \leq 1. \qquad (*)$$
On the other hand, consider the sequence $e = (0, 1, 0, 0, ...) \in \ell^2$. We have
$$\|e\| = 1 \ \text{ and } \ \|Le\| = \|(1, 0, 0, ...)\| = 1.$$
So (see Problem 39)
$$\|L\| = \sup_{\|x\|=1} \|Lx\| \geq 1. \qquad (**)$$
Combining $(*)$ and $(**)$ we obtain $\|L\| = 1$. ■

Problem 47.
Let $X = C[0, 1]$ with the max-norm (the uniform norm). We define the integral operator
$$K : X \to X \ \text{ by } \ Kf(x) = \int_0^x f(y)dy.$$
Show that K is bounded. Find $\|K\|$.

Solution.
The operator K is bounded. Indeed,

$$\|Kf\| \leq \sup_{x\in[0,1]} \int_0^x |f(y)|dy \leq \int_0^1 |f(y)|dy \leq \|f\|.$$

Hence $\|K\| \leq 1$.
In fact, $\|K\| = 1$, since $1 \in X,\ K1 = x$ and so $\|K1\| = \|x\| = 1.$ ■

****Problem 48.**
Let $X = L^2[0,1]$ with the norm $\|.\|_2$. We define the integral operator

$$A : X \to X \quad by \quad Af(x) = \int_0^x f(y)dy.$$

Show that A is bounded. Find $\|A\|$.

Solution.
Warning: It's completely different from the previous problem!!!

First solution: Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|Af\|_2^2 = \int_0^1 \left|\int_0^t f(s)ds\right|^2 dt &= \int_0^1 \left|\int_0^t \sqrt{\cos\frac{\pi s}{2}}.\frac{f(s)}{\sqrt{\cos\frac{\pi s}{2}}}ds\right|^2 dt \\
&\leq \int_0^1 \left(\int_0^t \cos\frac{\pi s}{2}ds \int_0^t \frac{|f(s)|^2}{cos\frac{\pi s}{2}}ds\right) dt \\
&= \frac{2}{\pi}\int_0^1 \left(\int_0^t \sin\frac{\pi t}{2}.\frac{|f(s)|^2}{cos\frac{\pi s}{2}}ds\right) dt \\
&= \frac{2}{\pi}\int_0^t \left(\int_0^1 \sin\frac{\pi t}{2}\frac{|f(s)|^2}{cos\frac{\pi s}{2}}dt\right) ds \\
&= \frac{2}{\pi}\int_0^t \left(\int_0^1 \sin\frac{\pi t}{2}dt\right)\frac{|f(s)|^2}{cos\frac{\pi s}{2}}ds \\
&= \left(\frac{2}{\pi}\right)^2 \int_0^t \frac{|f(s)|^2}{cos\frac{\pi s}{2}}ds.
\end{aligned}$$

Equality holds for $f(s) = \cos\frac{\pi s}{2}$. Thus

$$\|A\| = \frac{2}{\pi}. \quad ■$$

Second solution: We find the norm of $T = A^*A$, then $\|A\| = \sqrt{\|T\|}$.
Since A is compact (we will see this somewhere later) and A^*A is self adjoint, we have that T is a compact, normal operator. Hence, its spectrum is countable with 0 as the only possible cluster point and $\|T\|$ is equal to the spectral radius of T. These two facts together imply that

$$\|T\| = \max\{|\lambda| : \ Tf = \lambda f \ \text{ for some } \ f \in X\},$$

We have that A^* is given by

$$A^*f(x) = \int_x^1 f(y)dy.$$

Hence

$$(*) \qquad Tf(x) = \int_x^1 \int_0^y f(z)dz \ dy.$$

By the Fundamental Theorem of Calculus, Tf is twice differentiable for all $f \in X$, so any eigenvector of T must satisfy the differential equation

$$\lambda \frac{d^2}{dx^2} f(x) = \frac{d^2}{dx^2} \int_x^1 \int_0^y f(z)dz \ dy = \frac{d}{dx}\left(- \int_0^x f(z)dz \right) = -f(x), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

By a theorem from differential equations, we know that there are only two linearly independent solutions to the above differential equation over $\mathbb{C}$, namely, $e^{i\omega x}$ and $e^{-i\omega x}$, where $\omega^2 = \lambda, \ \lambda, \omega \neq 0$. Hence the general solution to the above equation is

$$f(x) = \alpha e^{i\omega x} + \beta e^{-i\omega x}, \ \alpha, \beta \in \mathbb{C}.$$

When we apply T, we get

$$\begin{aligned}
T(\alpha e^{i\omega x}+\beta e^{-i\omega x}) &= \int_x^1\int_0^y(\alpha e^{i\omega z}+\beta e^{-i\omega z})dz\,dy\\
&= \int_x^1\left(\frac{\alpha e^{i\omega z}}{i\omega}-\frac{\beta e^{-i\omega z}}{i\omega}\right)\Big|_0^y dy\\
&= \int_x^1\left[\left(\frac{\alpha e^{i\omega y}}{i\omega}-\frac{\beta e^{-i\omega y}}{i\omega}\right)-\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)\right]dy\\
&= \left(\frac{\alpha e^{i\omega y}}{-\omega^2}-\frac{\beta e^{-i\omega y}}{-\omega^2}\right)-\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)y\Big|_x^1\\
&= \left(\frac{\alpha e^{i\omega}}{-\omega^2}-\frac{\beta e^{-i\omega}}{-\omega^2}\right)-\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)\\
&- \left[\left(\frac{\alpha e^{i\omega x}}{-\omega^2}-\frac{\beta e^{-i\omega x}}{-\omega^2}\right)-\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)x\right]\\
&= \frac{1}{\omega^2}(\alpha e^{i\omega x}+\beta e^{-i\omega x})+\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)x\\
&- \left(\frac{\alpha e^{i\omega}}{\omega^2}-\frac{\beta e^{-i\omega}}{\omega^2}\right)-\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right).
\end{aligned}$$

Now since $\alpha e^{i\omega x}+\beta e^{-i\omega x}$ is an eigenvector, we must have that

$$\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)x-\left(\frac{\alpha e^{i\omega}}{\omega^2}-\frac{\beta e^{-i\omega}}{\omega^2}\right)-\left(\frac{\alpha}{i\omega}-\frac{\beta}{i\omega}\right)=0,\ \forall x\in[0,1],$$

which implies $\alpha=\beta$. Hence

$$\alpha e^{i\omega x}+\beta e^{-i\omega x}=2\alpha\cos\omega x.$$

Moreover, we must have that $Tf(1)=0$ for any eigenvector f by $(*)$, so $2\alpha\cos\omega=0$. Since $\alpha\neq 0$, we must have that

$$\omega=\frac{(2n+1)\pi}{2},\ n\in\mathbb{Z}.$$

Hence the eigenvalues are of the form

$$\lambda_n=\frac{1}{\omega_n^2}=\frac{2^2}{(2n+1)^2\pi^2},\ n\in\mathbb{Z}.$$

Thus

$$\max\{\lambda_n:\ n\in\mathbb{Z}\}=\frac{4}{\pi^2}\ \ (\text{take}\ \ n=0,-1).$$

We conclude that
$$\|T\| = \frac{4}{\pi^2} \text{ and } \|A\| = \frac{2}{\pi}. \quad \blacksquare$$

Problem 49.
Let a, b be real numbers such that $a < b$. Consider the Hilbert space $L^2[a,b]$ over $\mathbb{R}$ and the operator $T : L^2[a,b] \to \mathbb{R}$ defined by
$$Tf = \int_a^b f(x)dx, \quad f \in L^2[a,b].$$
(a) Show that T is bounded. Compute $\|T\|$.
(b) According to the Riesz's Theorem, there exists a function $g \in L^2[a,b]$ such that
$$Tf = \langle f, g\rangle \ \text{ for all } \ f \in L^2[a,b].$$
Find such a function g and verify that $\|g\|_{L^2} = \|T\|$.

Solution.
(a) By Hölder's inequality, we have
$$\begin{aligned} |Tf| &= \left|\int_a^b f(x)dx\right| \\ &\leq \int_a^b |1.f(x)|dx \\ &\leq \left(\int_a^b 1^2 dx\right)^{1/2} \left(\int_a^b |f(x)|^2\right)^{1/2} \\ &= \sqrt{b-a}\ \|f\|_{L^2}. \end{aligned}$$
Hence, T is bounded, i.e., $T \in \mathcal{B}(L^2, \mathbb{R}) = (L^2)^*$, the dual space of L^2.
From the above we get
$$\|T\|_{(L^2)^*} \leq \sqrt{b-a}.$$
Now consider the function $h(x) = \frac{1}{\sqrt{b-a}},\ x \in (a,b)$. It is obviously that $h \in L^2(a,b)$ and
$$\|h\|_{L^2} = \left(\int_a^b (b-a)^{-1}\right)^{1/2} = 1 \text{ and } |Th| = \left|\int_a^b \frac{1}{\sqrt{b-a}} dx\right| = \sqrt{b-a}.$$
Hence,
$$\sqrt{b-a} = |Th| \leq \|T\|_{(L^2)^*} \|h\|_{L^2} = \|T\|_{(L^2)^*}.$$

Therefore,
$$\|T\|_{(L^2)^*} = \sqrt{b-a}.$$
(b) By the Riesz's Theorem, there exists a function $g \in L^2[a,b]$ such that $Tf = \langle f, g\rangle$ for all $f \in L^2[a,b]$. Here, functions are real-valued, so we can write
$$Tf = \langle f, g\rangle \Leftrightarrow \int_a^b f(x)dx = \int_a^b f(x)g(x)dx.$$
It is evident that the above equation is satisfied for all $f \in L^2[a,b]$ if we choose $g(x) = 1$ on $[a,b]$. By the uniqueness of this representation guaranteed by Riesz's Theorem, we can definitely conclude that
$$g(x) = 1, \quad x \in [a,b].$$
Also, we can verify that
$$\|g\|_{L^2} = \left(\int_a^b 1^2 dx\right)^{1/2} = \sqrt{b-a} = \|T\|_{(L^2)^*}. \quad \blacksquare$$

Problem 50.
Let X, Y be normed spaces and $T \in \mathcal{B}(X,Y)$. Consider the following statement:
$$T \text{ is an isometry } \Leftrightarrow \|T\| = 1.$$
Do you agree with it? Why?

Solution.
- The direct way ($\Rightarrow$) is correct. Indeed, suppose that T is an isometry; then
$$\begin{aligned}\|T\|_{\mathcal{B}(X,Y)} &= \sup_{\substack{x\in X\\ \|x\|_X=1}} \|Tx\|_Y \\ &= \sup_{\substack{x\in X\\ \|x\|_X=1}} \|x\|_X \quad (\text{since } \|Tx\|_Y = \|x\|_X) \\ &= 1.\end{aligned}$$
- The other way ($\Leftarrow$) is false. The left shift on ℓ^2 is a counter-example (see Problem 40). $\blacksquare$

Problem 51.
Define $T : C[0,1] \to C[0,1]$ *by*

$$(Tx)(t) = t\int_0^t x(s)ds.$$

(a) Prove that T *is a bounded linear operator. Compute* $\|T\|$.
(b) Prove that the inverse $T^{-1} : \text{Image}(T) \to C[0,1]$ *exists but not bounded.*

Solution.
(a) For all $x_1, x_2 \in C[0,1]$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$ and all $t \in [0,1]$ we have

$$\begin{aligned} T(\alpha_1 x_1 + \alpha_2 x_2)(t) &= t\int_0^t (\alpha_1 x_1 + \alpha_2 x_2)(s)ds \\ &= \alpha_1 t\int_0^t x_1(s)ds + \alpha_2 t\int_0^t x_2(s)ds \\ &= \alpha_1 T(x_1)(t) + \alpha_2 T(x_2)(t) \\ &= [\alpha_1 T(x_1) + \alpha_2 T(x_2)](t). \end{aligned}$$

Hence,

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2).$$

This shows that T is linear.
For each $x \in C[0,1]$ we have

$$\begin{aligned} \|Tx\| &= \max_{t\in[0,1]} \left| t\int_0^t x(s)ds \right| \leq \max_{t\in[0,1]} |t| \int_0^t |x(s)|ds \\ &\leq \max_{t\in[0,1]} t\int_0^t \|x\|ds = \max_{t\in[0,1]} t^2\|x\| = \|x\|. \end{aligned}$$

Hence T is a bounded with $\|T\| \leq 1$. Moreover, if $x(t) = 1,\ t \in [0,1]$ then $\|x\| = 1$ and $(Tx)(t) = t\int_0^t ds = t^2$, therefore, $\|Tx\| = \max_{t\in[0,1]} t^2 = 1$. Thus,

$$1 = \|Tx\| \leq \|T\|\|x\| \quad \text{and so} \quad \|T\| \geq 1.$$

Hence we have proved that $\|T\| \leq 1$ and $\|T\| \geq 1$, so $\|T\| = 1$.
(b) Suppose that $x \in C[0,1]$ satisfies

$$Tx = 0, \quad i.e., \quad t\int_0^t x(s)ds = 0, \quad \forall t \in [0,1].$$

It follows that $\int_0^t x(s)ds = 0,\ \forall t \in (0,1]$. By differentiation w.r.t. t we get $x(t) = 0, \forall t \in (0,1]$. Since $x(t)$ is continuous, we then also have $x(0) = 0$. Hence, $x = 0$. We have thus proved

$$\forall x \in C[0,1],\ T(x) = 0 \Rightarrow x = 0.$$

Hence, T^{-1} exists.
Now we show that T^{-1} is not bounded. Given any $n \in \mathbb{N}$, we let $x_n(t) = t^n$. Then

$$x_n \in C[0,1] \ \text{ and } \ \|x_n\| = \max_{t\in[0,1]} |t^n| = 1.$$

We let

$$y_n(t) = Tx_n(t) = t\int_0^t s^n ds = \frac{1}{n+1}\ t^{n+2}.$$

Then

$$\|y_n\| = \max_{t\in[0,1]} \left| \frac{1}{n+1}\ t^{n+2} \right| = \frac{1}{n+1}.$$

Also by construction, y_n Image(T) and $T^{-1}y_n = x_n$; thus

$$\|T^{-1}y_n\| = \|x_n\| = 1.$$

This shows that T^{-1} cannot be bounded. (For if T^{-1} were bounded, then we would have

$$\|T^{-1}y_n\| \leq \|T^{-1}\|\|y_n\|,\ i.e.,\ 1 \leq \|T^{-1}\|\ \frac{1}{n+1},\ \forall n \in \mathbb{N}.$$

This is impossible. ■

Problem 53.
Let $1 < p < \infty$ and q be its exponent conjugate. For $f \in L^p(0,\infty)$, let

$$(Tf)(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(tx)dt.$$

Show that
(a) Tf is well-defined on $(0,\infty)$, and that Tf is continuous with respect to x.
(b) $Tf \in L^p(0,\infty)$.
(c) T is a bounded linear operator from $L^p(0,\infty)$ to itself. Calculate its norm.
(Hint: Use $f_n = x^{-1/p}\chi_{\{1\leq x\leq n\}}$.)

Solution.
(a) For $f \in L^p(0,\infty)$, we need to show $\psi : x \mapsto \int_0^x f(t)dt$ is well-defined and continuous w.r.t. x. By Holder's inequality we have

$$\int_0^x |f(t)|dt \leq |x|^{1/q}\|f\|_p < \infty,$$

which shows that ψ is well-defined. Also, for $x, y \in (0,\infty)$, we have

$$\psi(x) - \psi(y)| = \left|\int_x^y f(t)dt\right| \leq \int_x^y |f(t)|dt \leq |x-y|^{1/q}\|f\|_p.$$

This shows that ψ is continuous.
Since $(Tf)(x) = \frac{\psi(x)}{x}$, Tf is also continuous on $(0,\infty)$. Moreover we have

$$|(Tf)(x)| \leq x^{-1/p}\|f\|_p \to 0 \;\text{ as }\; x \to \infty.$$

So $(Tf)(x) \to 0$ as $x \to \infty$.
(b) To prove this part (b) we will use the following theorem:

Given a σ-finite measure space $(X, \mathfrak{A}, \mu)$. Let $1 < p < \infty$ and q be its exponent conjugate. If $|\varphi| < \infty$ a.e on X and if $\int_X \varphi\psi d\mu$ exists in $\mathbb{C}$ for every $\psi \in L^q(X)$, then $f \in L^p(X)$.

Now, for any $g \in L^q(0,\infty)$, by Fubini's theorem we have

$$\begin{aligned}
\int_{(0,\infty)} |(Tf)(x)g(x)|dx &\leq \int_{(0,\infty)}\int_0^1 |(Tf)(x)||g(x)|dtdx \\
&= \int_{(0,\infty)}\int_0^1 \frac{1}{t}\,|f(x)||g(x/t)|dtdx \\
&\leq \|f\|_p \int_0^1 \|g(./t)\|_q\,\frac{dt}{t}.
\end{aligned}$$

But

$$\|g(./t)\|_q = \left(\int_{(0,\infty)} |g(x/t)|^q dx\right)^{1/q} = t^{1/q}\|g\|_q.$$

Thus,

$$\int_{(0,\infty)} |(Tf)(x)g(x)|dx \leq \|f\|_p\|g\|_q \int_0^1 t^{1/q-1}dt = q\|f\|_p\|g\|_q < \infty.$$

Hence, $Tf \in L^p(0,\infty)$.
(c) By the last inequality we have

$$\|Tf\|_p \leq q\|f\|_p.$$

This shows that T is an bounded operator on $L^p(0,\infty)$ with $\|T\|_{p,p} \leq q$. We would like to establish the equality. Consider the function $f_n,\ n \in \mathbb{N}$ defined by $f_n = x^{-1/p}\chi_{\{1\leq x\leq n\}}$. We have

$$\|f_n\|_p^p = \int_1^n \left(x^{-1/p}\right)^p dx = \ln n.$$

For $x > 1$ we have

$$(Tf_n)(x) = \frac{1}{x}\int_1^{\min\{x,n\}} t^{-1/p} dt = q\ \frac{\min\{x,n\}^{1/q} - 1}{x}.$$

For $0 < \varepsilon < 1$, since $(1-\varepsilon)^p \geq 1 - p\varepsilon$ we have

$$\begin{aligned}
\int_{(0,\infty)} [(Tf_n)(x)]^p dx &= q^p \left(\int_1^n (1 - x^{-1/q})^p\ \frac{1}{x}\ dx + (n^{1/q} - 1)^p \int_n^\infty x^{-p} dx\right) \\
&\geq q^p \left(\ln n - \frac{p}{q} + \frac{1}{p-1}\right) \\
&\geq q^p \left(\|f_n\|_p^p - \frac{p}{q} + \frac{1}{p-1}\right).
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \frac{\|Tf_n\|_p}{\|f_n\|_p} \geq q.$$

Hence, $\|T\|_{p,p} \geq q$. Finally, we get

$$\|T\|_{p,p} = q. \quad \blacksquare$$

Problem 54.
Let $(c_j)_{j=1}^\infty$ be a sequence of complex numbers. Define an operator D on ℓ^2 by

$$Dx = (c_1x_1, c_2x_2, ...) \ \ \text{for} \ \ x = (x_1, x_2, ...) \in \ell^2.$$

Prove that D is bounded if and only if $(c_j)_{j=1}^\infty$ is bounded, and in this case $\|D\| = \sup_j |c_j|$.

Solution.

• Suppose $(c_j)_{j=1}^{\infty}$ is bounded. Let $M = \sup_j |c_j| < \infty$. Then

$$\begin{aligned}\|Dx\| &= \left(\sum_{j=1}^{\infty} |c_j x_j|^2\right)^{1/2} = \left(\sum_{j=1}^{\infty} |c_j|^2 |x_j|^2\right)^{1/2} \\ &\leq \left(\sum_{j=1}^{\infty} M^2 |x_j|^2\right)^{1/2} = M\left(\sum_{j=1}^{\infty} |x_j|^2\right)^{1/2} \\ &= M\|x\|.\end{aligned}$$

Hence, D is bounded and $\|D\| \leq M$.
• Suppose D is bounded. We want to show that $(c_j)_{j=1}^{\infty}$ is bounded. Consider the vector $e_j = (0, 0, .., 0, 1, 0, ...)$ where the number 1 appears at the j-th coordinate. Clearly $\|e_j\| = 1$ and $\|De_j\| = |c_j|$ for all $j = 1, 2, ..$ Since D is bounded,

$$|c_j| = \|De_j\| \leq \|D\| \text{ for any } j = 1, 2, ...$$

Hence $(c_j)_{j=1}^{\infty}$ is bounded and $M = \sup_j |c_j| \leq \|D\|$. Finally

$$\|D\| = M. \quad \blacksquare$$

Problem 55.
Prove that $\mathcal{B}(\mathbb{F}, Y)$ is not a Banach space if Y is not complete.

Hint: Take a Cauchy sequence (y_n) in Y which does not converge and consider the sequence of operators (B_n) defined by :

$$B_n\lambda := \lambda y_n; \quad \lambda \in \mathbb{F}.$$

Solution.
We follow the suggestion above. It is easy to see that

$$B_n \in \mathcal{B}(\mathbb{F}, Y) \text{ and } \|B_n\| = \|y_n\|, \ \forall n \in \mathbb{N}.$$

Since $(B_n - B_m)\lambda = \lambda(y_n - y_m)$, we have

$$\|B_n - B_m\| = \|y_n - y_m\|, \quad \forall n \in \mathbb{N}.$$

Therefore (B_n) is a Cauchy sequence in $\mathcal{B}(\mathbb{F}, Y)$. Suppose there exists $B \in \mathcal{B}(\mathbb{F}, Y)$ such that $\|B_n - B\| \to 0$ as $n \to \infty$. Let $y := B1 \in Y$, where 1 is the unit element in $\mathbb{F}$. Then

$$\|y_n - y\| = \|B_n 1 - B1\| \leq \|B_n - B\| \to 0 \text{ as } n \to \infty,$$

i.e., the sequence (y_n) converges to y. This contradiction proves that (B_n) cannot be convergent. Hence, $\mathcal{B}(\mathbb{F}, Y)$ is not a Banach space. ■

Problem 56.
Let $T_1, T_2, ...$ be the following bounded linear operators $\ell^1 \to \ell^\infty$:

$$\begin{aligned} T_1(x_1, x_2, x_3, ...) &= (x_1, x_1, x_1, x_1, ...) \\ T_2(x_1, x_2, x_3, ...) &= (x_1, x_2, x_2, x_2, ...) \\ T_3(x_1, x_2, x_3, ...) &= (x_1, x_2, x_3, x_3, ...) ...etc \end{aligned}$$

Prove that the sequence (T_n) is strongly operator convergent. Also prove that (T_n) is not uniformly operator convergent.

Solution.
Let $T : \ell^1 \to \ell^\infty$ be the bounded linear operator given by

$$T(x_1, x_2, x_3, ...) = (x_1, x_2, x_3, ...).$$

T is obviously linear. It is bounded with norm $\|T\| \leq 1$. Indeed, if $x = (x_1, x_2, x_3, ...) \in \ell^1$ then

$$\sum_{k=1}^{\infty} |x_k| = \|x\|_1 < \infty, \quad \text{so} \quad |x_k| \leq \|x\|_1, \quad \forall k = 1, 2, ...$$

Hence,

$$\|x\|_\infty \leq \|x\|_1.$$

We claim that (T_n) strongly operator converges to T.
For any $x = (x_1, x_2, x_3, ...) \in \ell^1$,

$$\begin{aligned} \|T_n x - Tx\|_\infty &= \|(0, ..0, x_n - x_{n+1}, x_n - x_{n+2}, ...)\|_\infty \\ &= \sup_{j \geq 1} |x_n - x_{n+j}|. \end{aligned}$$

Since $x = (x_1, x_2, x_3, ...) \in \ell^1$, $\sum_{k=1}^{\infty} |x_k| < \infty$, so we have $\lim_{k\to\infty} |x_k| = 0$. Hence, given any $\varepsilon > 0$, there is some $K \in \mathbb{N}$ such that

$$|x_k| \leq \varepsilon, \quad \text{for all} \quad k \geq K.$$

Then if $n \geq K$ we have

$$n + j \geq K, \quad \forall j \in \mathbb{N}.$$

Hence,

$$|x_n - x_{n+j}| \leq |x_n| + |x_{n+j}| \leq 2\varepsilon.$$

Thus, for $n \geq K$,

$$\|T_n x - Tx\|_\infty = \sup_{j\geq 1} |x_n - x_{n+j}| \leq 2\varepsilon.$$

In other words,

$$\lim_{n\to\infty} \|T_n x - Tx\|_\infty = 0.$$

This is true for every $x \in \ell^1$. Hence, the sequence (T_n) is strongly operator convergent to T.
It follows from this that if (T_n) would be uniformly operator convergent, then the limit must be equal to T. We show that this is not true. Now we have

$$(T_n - T)(x) = (0, .., 0, x_n - x_{n+1}, x_n - x_{n+2}, ...).$$

In particular, if $x = (0, .., 0, 1, 0, ..)$ with 1 at the n-th position, then

$$(T_n - T)(x) = (0, .., 0, 1, 1, ...).$$

Here $\|x\|_1 = 1$ and $\|(0, .., 0, 1, 1, ...)\|_\infty = 1$. Hence,

$$\|T_n - T\| \geq 1, \quad \text{for all} \quad n.$$

Thus, (T_n) is not uniformly operator convergent. ■

Problem 57.
Let H_1 and H_2 be two Hilbert spaces. Let $\{a_1, ..., a_n\}$ be an orthonormal system of H_1 and $\{b_1, ...,_n\}$ be an orthonormal system of H_2, and $\lambda_1, ..., \lambda_n \in \mathbb{K}$. Consider the operator

$$U : H_1 \to H_2 \quad \textit{defined by} \quad U(x) = \sum_{i=1} \lambda_i b_i \langle x, a_i \rangle.$$

Calculate $\|U\|$.

Solution.

From the Pythagoras theorem and the Bessel inequality we have

$$\begin{aligned}\|U(x)\|^2 &= \sum_{i=1}^n |\lambda_i|^2\|b_i\|^2|\langle x, a_i\rangle|^2 \\ &= \sum_{i=1}^n |\lambda_i|^2|\langle x, a_i\rangle|^2 \\ &\leq M^2\sum_{i=1}^n |\langle x, a_i\rangle|^2 \leq M^2\|x\|^2,\end{aligned}$$

where $M = \max_{1\leq i\leq n}|\lambda_i|$. Hence, we have that

$$\|U(x)\| \leq M\|x\|,\ \forall x \in H.$$

Therefore, $\|U\| \leq \max_{1\leq i\leq n}|\lambda_i|$.
On the other hand, we have

$$\begin{aligned}&\|U(a_i)\| \leq \|U\|\ \|a_i\| = \|U\|,\\ &U(a_i) = \lambda_i b_i,\ \forall i \in \{1, ..., n\}.\end{aligned}$$

It follows that $|\lambda_i| \leq \|U\|,\ \forall i \in \{1, ..., n\}$. This implies $\max_{1\leq i\leq n}|\lambda_i| \leq \|U\|$. Thus,

$$\|U\| = \max_{1\leq i\leq n}|\lambda_i|. \quad \blacksquare$$

Problem 58.
(a) Let X be a Hilbert space. Let $a, b \in H$ be two non-zero orthogonal elements. Consider the operator

$$U : H \to H,\quad U(x) = a\langle x, b\rangle + b\langle x, a\rangle.$$

Calculate $\|U\|$.
(b) Consider the operator $T : L^2[0, \pi] \to L^2[0, \pi]$ defined by

$$(Tf)(x) = \sin x\int_0^\pi f(t)\cos t dt + \cos x\int_0^\pi f(t)\sin t dt.$$

Calculate $\|T\|$.

Solution.
(a) Note that $a\langle x, b\rangle$ and $b\langle x, a\rangle$ are two orthogonal vectors. Using the Pythagoras theorem we have

$$\|U(x)\|^2 = \|a\|^2|\langle x, b\rangle|^2 + \|b\|^2|\langle x, a\rangle|^2.$$

From the Bessel inequality we deduce that

$$\|U(x)\|^2 = \|a\|^2\|b\|^2\left(\left|\left\langle x, \frac{b}{\|b\|}\right\rangle\right|^2 + \left|\left\langle x, \frac{a}{\|a\|}\right\rangle\right|^2\right) \leq \|a\|^2\|b\|^2\|x\|^2.$$

Hence, $\|U\| \leq \|a\| \; \|b\|$.
But

$$U(a) = a\langle a, b\rangle + b\langle a, a\rangle = \|a\|^2 b.$$

Therefore,

$$\|U(a)\| = \|a\|^2\|b\| \leq \|U\| \; \|a\|.$$

Hence, $\|U\| \geq \|a\| \; \|b\|$. Thus,

$$\|U\| = \|a\| \; \|b\|.$$

(b) Let $H = L^2[0, \pi]$. Then H, with the usual inner product, is a Hilbert space. The two vectors $a = \sin x$ and $b = \cos x$ are orthogonal. Indeed,

$$\langle a, b\rangle = \langle \sin x, \cos x\rangle = \int_0^\pi \sin x \cos x dx = 0.$$

By (a) we get

$$\|T\| = \|\sin x\| \; \|\cos x\|.$$

But

$$\|\sin x\|^2 = \int_0^\pi (\sin x)^2 dx = \frac{\pi}{2}; \; \|\cos x\|^2 = \int_0^\pi (\cos x)^2 dx = \frac{\pi}{2}.$$

Thus,

$$\|T\| = \frac{\pi}{2}. \qquad \blacksquare$$

Problem 59.
(a) Let H be a Hilbert space and $\{e_1, e_2\} \subset H$ an orthonormal system. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be scalar square matrix. Consider the operators $U, V : H \to H$ defined by

$$U(x) = a\langle x, e_1\rangle e_1 + b\langle x, e_2\rangle e_2,$$
$$V(x) = c\langle x, e_1\rangle e_1 + d\langle x, e_2\rangle e_2.$$

Prove that
$$\|U+V\|^2+\|U-V\|^2=2(\|U\|^2+\|V\|^2)$$
if and only if
$$(\max\{|a+c|,|b+d|\})^2+(\max\{|a-c|,|b-d|\})^2=2\left(\max\{|a|,|b|\}^2+\max\{|c|,|d|\}^2\right).$$
(b) Prove that if $\dim H\geq 2$ then $\mathcal{B}(H)$ is not a Hilbert space

Solution.
(a) Using Problem 57 we have

$$\|U\|=\max\{|a|,|b|\};\quad \|V\|=\max\{|c|,|d|\}.$$
$$\|U+V\|=\max\{|a+c|,|b+d|\};\quad \|U-V\|=\max\{|a-c|,|b-d|\}.$$

From here we obtain the statement.
(b) Since $\dim H\geq 2$ we can find $x,y\in H$, two linearly independent vectors. Using the Gram-Schmidt procedure on $\{x,y\}$ we can construct an orthonormal system $\{e_1,e_2\}\subset H$. Now construct U,V using the matrix

$$A=\begin{pmatrix}2&1\\0&2\end{pmatrix}.$$

That is $U,V:H\to H$ defined by

$$U(x)=2\langle x,e_1\rangle e_1+\langle x,e_2\rangle e_2,\quad V(x)=2\langle x,e_2\rangle e_2.$$

If $\mathcal{B}(H)$ were a Hilbert space, then by the parallelogram law we must have

$$\|U+V\|^2+\|U-V\|^2=2(\|U\|^2+\|V\|^2),$$

i,e., by (a)

$$(\max\{2,3|\})^2+(\max\{2,1\})^2=2\left(\max\{2,1\}^2+\max\{0,2\}^2\right).$$

This is not true. ■

4.2 Linear Functionals

Review:

Definition 1 *Let X be a linear space. A map $p : X \to \mathbb{R}$ is called* sublinear *if it is subadditive and positive homogeneous, i.e.,*

$$\begin{aligned} &p(x+y) \leq p(x) + p(y), \ \forall x, y \in X. \\ &p(\lambda x) = \lambda p(x), \ \forall x \in X, \ \forall \lambda \geq 0. \end{aligned}$$

Definition 2 *Let X be a linear space. A map $p : X \to \mathbb{K}$ is called* seminorm *if it is subadditive and absolutely homogeneous, i.e.,*

$$\begin{aligned} &p(x+y) \leq p(x) + p(y), \ \forall x, y \in X. \\ &p(\lambda x) = |\lambda| p(x), \ \forall x \in X, \ \forall \lambda \in \mathbb{K}. \end{aligned}$$

The Hahn-Banach theorems

Theorem 1 *(The case $\mathbb{K} = \mathbb{R}$). Let X be a real linear space, $G \subset X$ a linear subspace, $p : X \to \mathbb{R}$ a sublinear functional and $f : G \to \mathbb{R}$ a linear functional such that $f(x) \leq p(x), \ \forall x \in G$. Then there is a linear functional $\bar{f} : X \to \mathbb{R}$ which extends f, i.e.,*

$$\bar{f}(x) = f(x), \ \forall x \in G \ \text{ and } \ \bar{f}(x) \leq p(x), \ \forall x \in X.$$

Theorem 2 *(The case $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let X be a linear space, $G \subset X$ a linear subspace, $p : X \to \mathbb{R}$ a seminorm and $f : G \to \mathbb{K}$ a linear functional such that $|f(x)| \leq p(x), \ \forall x \in G$. Then there is a linear functional $\bar{f} : X \to \mathbb{K}$ which extends f such that*

$$|\bar{f}(x)| \leq p(x), \ \forall x \in X.$$

Theorem 3 *(The normed space case). Let X be a linear space, $G \subset X$ a linear subspace and $f : G \to \mathbb{K}$ a linear and continuous functional. Then there is $\bar{f} : X \to \mathbb{K}$, a linear and continuous functional which extends f such that $\|\bar{f}\| = \|f\|$. Such an $\bar{f}$ is called a* Hahn-Banach extension *for f.*

Theorem 4 *Let X be a normed space, $x_0 \in X$ and $G \subset X$ a linear subspace such that $\delta = d(x_0, G) > 0$. Then there is $f : X \to \mathbb{K}$, a linear and continuous functional, such that*

$$f = 0 \ \text{ on } \ G, \ f(x_0) = 1 \ \text{ and } \ \|f\| = \frac{1}{\delta}.$$

Three classic problems.

Problem 60.
The dual space of ℓ^1 is ℓ^∞, that is, $(\ell^1)^* = \ell^\infty$.

Solution.
Let (e_k) be the standard basis for ℓ^1, where $e_k = \delta_{kj}$. Every $x \in \ell^1$ has unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

Norms on ℓ^1 and on ℓ^∞ are respectively:

$$\|x\|_1 = \sum_{k=1}^{\infty} |\xi_k|, \quad \|x\|_\infty = \sup_{k \in \mathbb{N}} |\xi_k|.$$

Take $f \in (\ell^1)^*$, that is, $f \in \mathcal{B}(\ell^1, \mathbb{F})$. For $x \in \ell^1$,

$$f(x) = f\left(\sum_{k=1}^{\infty} \xi_k e_k\right) = \sum_{k=1}^{\infty} \xi_k \gamma_k \quad \text{where} \quad \gamma_k = f(e_k).$$

Then for all $k \in \mathbb{N}$,

$$|\gamma_k| = |f(e_k)| \leq \|f\| \, \|e_k\| = \|f\|.$$

So,

$$(*) \quad \sup_{k \in \mathbb{N}} |\gamma_k| \leq \|f\|.$$

On the other hand, for every $b = (\beta_k) \in \ell^\infty$ we can obtain a corresponding bounded linear functional g on ℓ^1. In fact, we may define g on ℓ^1 by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \quad \text{where} \quad x = (\xi_k) \in \ell^1.$$

Then $g : \ell^1 \to \mathbb{F}$ is linear, and the boundedness follows from

$$|g(x)| \leq \sum_{k=1}^{\infty} |\xi_k \beta_k| \leq \sup_{k \in \mathbb{N}} |\beta_k| \sum_{k=1}^{\infty} |\xi_k| = \sup_{k \in \mathbb{N}} |\beta_k| \, \|x\|_1.$$

Hence $g \in (\ell^1)^*$.
We finally show that the norm of f is the norm on ℓ^∞. We have

$$|f(x)| = \left|\sum_{k=1}^{\infty} \xi_k \gamma_k\right| \leq \sup_{k\in\mathbb{N}} |\gamma_k| \sum_{k=1}^{\infty} |\xi_k| = \|x\| \sup_{k\in\mathbb{N}} |\gamma_k|.$$

Taking the supremum over x such that $\|x\| = 1$, we see that

$$\|f\| \leq \sup_{k\in\mathbb{N}} |\gamma_k| = \|(\gamma_k)\|_\infty.$$

From($*$) and this, we obtain

$$\|f\| = \|(\gamma_k)\|_\infty.$$

Thus there is an isometric isomorphism between $(\ell^1)^*$ and ℓ^∞, so that we can write $(\ell^1)^* = \ell^\infty$. ■

Problem 61.
The dual space of ℓ^p is ℓ^q, that is, $(\ell^p)^* = \ell^q$, here $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Solution.
The basis for ℓ^p is (e_k), where $e_k = \delta_{kj}$ as in the previous problem. Every $x \in \ell^p$ has unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

Take $f \in (\ell^p)^*$, that is, $f \in \mathcal{B}(\ell^p, \mathbb{F})$. For $x \in \ell^p$,

$$(1) \qquad f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k \quad \text{where} \quad \gamma_k = f(e_k).$$

Let q be the conjugate of p. Consider $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \begin{cases} \frac{|\gamma_k|^q}{\gamma_k} & \text{if } 1 \leq k \leq n \text{ and } \gamma_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0. \end{cases}$$

By substituting this into (1) we obtain

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q.$$

We also have

$$\begin{aligned} f(x_n) \leq \|f\| \, \|x_n\| &= \|f\| \left(\sum_{k=1}^{n} |\xi_k^{(n)}|^p \right)^{\frac{1}{p}} \\ &= \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^{(q-1)p} \right)^{\frac{1}{p}} \\ &= \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{p}}. \end{aligned}$$

Together,

$$f(x_n) = \sum_{k=1}^{n} |\gamma_k|^q \leq \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{p}}.$$

Dividing the last factor and using $1 - \frac{1}{p} = \frac{1}{q}$, we get

$$\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1-\frac{1}{p}} = \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{q}} \leq \|f\|.$$

Since n is arbitrary, letting $n \to \infty$, we obtain

$$(2) \qquad \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}} \leq \|f\|.$$

Conversely, for any $b = (\beta_k) \in \ell^q$ we can get a corresponding bounded linear functional g on ℓ^p. In fact, we may define g on ℓ^p by setting

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \quad \text{where} \quad x = (\xi_k) \in \ell^p.$$

Then g is linear. The boundedness follows from the Hölder inequality. Hence $g \in (\ell^p)^*$. We finally prove that the norm of f is the norm of (γ_k) in ℓ^q. From (1) and the Hölder inequality we have

$$\begin{aligned} |f(x)| \leq \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| &\leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}} \\ &= \|x\| \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Hence

$$\|f\| \leq \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{\frac{1}{q}}.$$

From (2) we obtain

$$\|f\| = \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{\frac{1}{q}}.$$

This can be written

$$\|f\| = \|(\gamma_k)\|_q \quad \text{where} \quad \gamma_k = f(e_k).$$

Thus there is an isometric isomorphism between $(\ell^p)^*$ and ℓ^q. ■

Problem 62.
The dual space of c_0 is ℓ^1, that is, $(c_0)^* = \ell^1$.

Solution.
Recall that $c_0 \subset \ell^\infty$, and if $x = (\lambda_n) \in c_0$ the norm of x in c_0 is $\|x\| = \sup_{n \in \mathbb{N}} |\lambda_n|$. Let $x = (\lambda_n) \in c_0$ and (e_k), where $e_k = \delta_{kj}$, be the basis for ℓ^∞ as in preceding examples. Then x has unique representation

$$x = \sum_{k=1}^{\infty} \lambda_k e_k, \quad \text{and} \quad \lim_{k \to \infty} \lambda_k = 0.$$

Consider any $f \in (c_0)^*$. Since f is linear,

$$f(x) = \sum_{k=1}^{\infty} \lambda_k \gamma_k \quad \text{where} \quad \gamma_k = f(e_k).$$

For a given $N \in \mathbb{N}$, take a special sequence $x_0^N = (\lambda_k^{(0)}) \in c_0$ where

$$\lambda_k^{(0)} = \begin{cases} \frac{\gamma_k}{|\gamma_k|} & \text{if } 1 \leq k \leq N \text{ and } \gamma_k \neq 0 \\ 0 & \text{if } k > N \text{ or } \gamma_k = 0. \end{cases}$$

Note that $\|x_0^N\| = 1$, and we have

$$|f(x_0^N)| = \sum_{k=1}^{N} |f(e_k)| \leq \|f\| \, \|x_0^N\| = \|f\| < \infty.$$

Since N is arbitrary,

$$\sum_{k=1}^{\infty} |f(e_k)| = \sum_{k=1}^{\infty} |\gamma_k| = \|f\| < \infty.$$

This shows that $(\gamma_k) \in \ell^1$ and

$$(i) \quad \|(\gamma_k)\| \leq \|f\|.$$

On the other hand,

$$|f(x)| = \left|\sum_{k=1}^{\infty} \lambda_k \gamma_k\right| \leq \sup_{k\in\mathbb{N}} |\lambda_k| \sum_{k=1}^{\infty} |\gamma_k| = \|x\| \; \|(\gamma_k)\| < \infty.$$

It follows that

$$(ii) \qquad \sup_{\|x\|=1} |f(x)| = \|f\| \leq \|(\gamma_k)\|.$$

From (i) and (ii) we obtain $\|f\| = \|(\gamma_k)\|$.
Now given $(\xi_k) \in \ell^1$, we want to construct a linear bounded functional g on c_0. Let $x = (\lambda_k) \in c_0$. Consider the function $g : c_0 \to \mathbb{F}$ defined by

$$g(x) = \sum_{k=1}^{\infty} \lambda_k \xi_k.$$

It is clear that g is linear. Its boundedness follows from

$$|g(x)| \leq \sup_{k\in\mathbb{N}} |\lambda_k| \sum_{k=1}^{\infty} |\xi_k| = \|x\| \sum_{k=1}^{\infty} |\xi_k| < \infty.$$

Hence $g \in (c_0)^*$. Thus there is an isometric isomorphism between $(c_0)^*$ and ℓ^1, so we can write $(c_0)^* = \ell^1$. ■

* * **

Problem 63.
Let X be a normed space and f, g are nonzero linear functionals on X. Show that

$$\ker(f) = \ker(g) \iff f = cg \text{ for some nonzero scalar } c.$$

Solution.
The reverse way ($\Leftarrow$) is trivial.
We show the direct way. Suppose $\ker(f) = \ker(g)$. Since $f \neq 0$, there exists some $x_0 \in X$ such that $f(x_0) \neq 0$, and by rescaling, we can assume that $f(x_0) = 1$. Since $x_0 \notin \ker(f) = \ker(g)$, we have $g(x_0) \neq 0$. Given any $y \in X$, we have

$$f\big(y - f(y)x_0\big) = f(y) - f(y)f(x_0) = 0.$$

Therefore, $y - f(y)x_0 \in \ker(f) = \ker(g)$. Hence,

$$g(y) - f(y)g(x_0) = g\big(y - f(y)x_0\big) = 0.$$

This implies that

$$g(y) = g(x_0)f(y), \quad \forall y \in X.$$

Hence $g = cf$ with $c = g(x_0) \neq 0$. ■

Problem 64.
Let X be a normed space and f are nonzero linear functional on X. Show that f is continuous if and only if $\ker(f)$ is closed.

Solution.
Suppose that f is continuous. Since $\ker(f) = f^{-1}(\{0\})$, so $\ker(f)$ is closed.
Conversely, suppose that $\ker(f)$ is closed. Pick an $x_0 \in X$ such that $f(x_0) = 1$. Assume that f is not continuous, that is, f is not bounded. Then there exists a sequence (x_n) in X such that

$$\|x_n\| = 1 \text{ and } f(x_n) \geq n, \ \forall n \in \mathbb{N}.$$

Define $y_n = x_0 - \frac{x_n}{f(x_n)}$. Then

$$y_n \in \ker(f) \text{ for all } n \text{ and } \lim_{n\to\infty} y_n = x_0.$$

By hypothesis, $\ker(f)$ is closed, so $x_0 \in \ker(f)$, that is, $f(x_0) = 0$. This contradicts our assumption that $f(x_0) = 1$. Thus f is continuous. ■

Problem 65
Let Z be a subspace of a normed space X, and $y \in X$. Let $d = d(y, Z)$. Prove that there exists $\Lambda \in X^*$ such that $\|\Lambda\| \leq 1$, $\Lambda(y) = d$ and $\Lambda(z) = 0$ for all $z \in Z$.

Solution.
If $y \in Z$, then $d = 0$ and so the zero functional works, so we may assume that $y \notin Z$. Consider the subspace $Y = \mathbb{C}y + Z \subset X$. Since $y \notin Z$, so for every $x \in Y$, there is a unique $\alpha \in \mathbb{C}$ and $z \in Z$ such that $x = \alpha y + z$. Define

$$\lambda(x) = \alpha d.$$

We observe that $\lambda(z) = 0$ since

$$\begin{aligned} x = z \quad &\Rightarrow \quad \alpha y = 0 \\ &\Rightarrow \quad \alpha = 0 \quad (\text{otherwise} \quad y = 0 \quad \text{so} \quad y \in Z). \end{aligned}$$

Also we have $\lambda(y) = d$ since $y = 1.y + 0$. It is clearly that λ is linear in x and

$$\|x\| = |\alpha| \|y + \frac{1}{\alpha}\, z\| \geq \alpha d = |\lambda(x)|.$$

Thus, λ is continuous, that is, $\lambda \in Y^*$ and $\|\lambda\| \leq 1$. By the Hahn-Banach theorem we may extend λ to an element $\Lambda \in X^*$ with the same norm. ■

Problem 66
Let X be a normed space and (x_n) be a sequence in X. Set $V := \text{Span}\,\{x_1, x_2, ...\}$. Let W be the set of all continuous $f \in X^*$ such that $f(x_n) = 0, \quad \forall n \in \mathbb{N}$. Prove that

$$\overline{V} = \{x \in X : \ f(x) = 0, \ \forall f \in W\}.$$

Solution.
Let $Y := \{x \in X : \ f(x) = 0, \ \forall f \in W\}$.
• We show that $\overline{V} \subset Y$. Take any $x_0 \in \overline{V}$. There exists a sequence (u_k) in V such that $u_k \to x_0$ as $k \to \infty$. Since $V := \text{Span}\,\{x_1, x_2, ...\}$, for each $k \in \mathbb{N}$, there are scalars $c_1^{(k)}, ..., c_{n_k}^{(k)}$ such that $u_k = \sum_{n=1}^{n_k} c_n^{(k)} x_n$. By linearity of f and by definition of W, we have

$$f(u_k) = \sum_{n=1}^{n_k} c_n^{(k)} f(x_n) = 0, \ \forall f \in W.$$

By continuity of f, we have

$$f(x_0) = 0, \ \forall f \in W.$$

Hence $x_0 \in Y$, and so $\overline{V} \subset Y$.

• Now given any $x_0 \in Y$, we want to show $x_0 \in \overline{V}$. Assume that $x_0 \notin \overline{V}$. Then $d(x_0, \overline{V}) = d > 0$. Using the result in Problem 65, there is an $F \in X^*$ such that

$$F(x_0) = d \text{ and } F(x) = 0, \ \forall x \in \overline{V}.$$

It follows that

$$F(x_0) = d \text{ and } F(x_n) = 0, \ \forall n \in \mathbb{N}.$$

Hence $F \in W$ but $F(x_0) \neq 0$. It means that $x_0 \notin Y$: a contradiction. Thus $x_0 \in \overline{V}$, and so $Y \subset \overline{V}$.

The proof is complete. ■

Problem 67.

Let X be a normed space and $\{x_1, ..., x_n\} \subset X$ a linearly independent system. Prove that for any $\alpha_1, ..., \alpha_n \in \mathbb{K}$ there exists $x^* \in X^*$ such that

$$x^*(x_i) = \alpha_i, \ \forall i \in \{1, ..., n\}.$$

Solution.

Let $Y = \text{Span}\{x_1, ..., x_n\}$. Since the system $\{x_1, ..., x_n\} \subset X$ a linearly independent, it follows that $\{x_1, ..., x_n\}$ is an algebraic basis for Y. Define $f : Y \to \mathbb{K}$ linear with $f(x_i) = \alpha_i, \ \forall i \in \{1, ..., n\}$. Since Y is of finite dimension, f is continuous. By the Hahn-Banach theorem there is an extension $x^* \in X^*$ for f. Then

$$x^*(x_i) = f(x_i) = \alpha_i, \ \forall i \in \{1, ..., n\}. \qquad ■$$

Problem 68.

Let X be a normed space and $Y \subset X$ a linear subspace. For $x_0 \in X \setminus Y$ we define

$$f : \text{Span}\{Y, x_0\} \to \mathbb{K}, \ f(y + \lambda x_0) = \lambda, \forall y \in Y, \ \forall \lambda \in \mathbb{K}.$$

(a) Prove that f is well defined and linear.
(b) Prove that f is continuous if and only if $x_0 \notin \overline{Y}$.
(c) Prove that $\overline{Y} = \bigcap\{\ker x^* : \ x^* \in X^*, \ Y \subset \ker x^*\}$.

Solution.
(a) Let us first observe that

$$\mathrm{Span}\{Y, x_0\} = \{y + \lambda x_0 : \; y \in Y, \; \lambda \in \mathbb{K}\}.$$

To show that f is well defined, we show that

$$(y + \lambda x_0 = y' + \lambda' x_0) \; \Rightarrow \; (y = y', \; \lambda = \lambda').$$

Assume that $\lambda \neq \lambda'$. Then

$$x_0 = \frac{y - y'}{\lambda - \lambda'} \in Y$$

since $y, y' \in Y$ and Y is a linear subspace. This is a contradiction. Thus, $\lambda = \lambda'$ implies that $y = y'$. The linearity of f is obvious.
(b) Suppose that f is continuous. We will prove that $x_0 \notin \overline{Y}$. Assume, for a contradiction, that $x_0 \in \overline{Y}$. Then there is a sequence $(y_n) \subset Y$ such that $y_n \to x_0$. Since $y_n, x_0 \in Y$, $y_n \to x_0$ and f is continuous, $f(y_n) \to f(x_0)$. By definition of f, $f(y_n) = 0, \; \forall n \in \mathbb{N}$. It follows that $f(x_0) = 0$, which is false, since by definition of f, we have $f(x_0) = 1$.
Suppose now that that $x_0 \notin \overline{Y}$, i.e., $x_0 \in (\overline{Y})^c$, which is open. Then

$$\exists \varepsilon > 0 : B(x_0; \varepsilon) \subset (\overline{Y})^c.$$

The inclusion is equivalent to $\overline{Y} \cap B(x_0; \varepsilon) = \emptyset$. Let $x = y + \lambda x_0 \in \mathrm{Span}\{Y, x_0\}$. If $\lambda = 0$, then

$$(*) \; |f(x)| = 0 \leq \frac{\|x\|}{\varepsilon}.$$

If $\lambda \neq 0$, then $\frac{y}{-\lambda} \in Y$ so $\frac{y}{-\lambda} \notin B(x_0; \varepsilon)$, i.e., $\left\| -\frac{y}{\lambda} - x_0 \right\| \geq \varepsilon$. Therefore,

$$(**) \; \|x\| = |\lambda| \; \| - \frac{y}{\lambda} - x_0 \| \geq |\lambda| \varepsilon = \varepsilon |f(x)|.$$

Both cases (*) and (**) show that f is continuous.
(c) If $x^* \in X^*$ and $Y \subset \ker x^*$ then $\overline{Y} \subset \ker x^*$ (since $\ker x^*$ is closed) and therefore

$$\overline{Y} \subset \bigcap \{\ker x^* : \; x^* \in X^*, \; Y \subset \ker x^*\}.$$

Conversely, let $x_0 \in \bigcap \{\ker x^* : \; x^* \in X^*, \; Y \subset \ker x^*\}$, and assume that $x_0 \notin \overline{Y}$. By (b), there is an $f : \mathrm{Span}\{Y, x_0\} \to \mathbb{K}$ linear and continuous such that $f|_Y = 0, \; f(x_0) = 1$. Let $x^* \in X^*$ be a Hahn-Banach extension for f given by the Hahn-Banach theorem. Then $x^* = f$ on $\mathrm{Span}\{Y, x_0\}$. In particular, $x^* = 0$ on Y. Hence $Y \subset \ker x^*$ and $x^*(x) = f(x_0) = 1$, that is $x \notin \ker x^*$ and so

$$x_0 \notin \bigcap \{\ker x^* : \; x^* \in X^*, \; Y \subset \ker x^*\},$$

which is a contradiction. ■

Chapter 5

Fundamental Theorems

Review some main points:

1. The Baire category theorem:

Definition 3 *Let X be a metric space and E be a subset of X.*
(a) E is called a set of the first category if E is a countable union of nowhere dense (non-dense) sets in X.
(b) If E is not a set of the first category, then E is called a set of the second category.

Theorem 5 *(Baire category theorem - form I)*
If X is a complete metric space, then X is a set of the second category.

Let X be a complete metric space. The Baire category theorem tells us that if $X = \bigcup_{n=1}^{\infty} A_n$, then some of the set $\overline{A}_n$ must have non-empty interior.

Theorem 6 *(Baire category theorem - form II)*
Let X be a complete metric space. If $(U_n)_{n\in\mathbb{N}}$ is a sequence of open dense subsets of X, then $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

It can be shown that these two forms of Baire category theorem are equivalent (try it!).

2. The principle of uniform boundedness (P.U.B.)

Theorem 7 *Let X be a Banach space and Y be a normed space. Let $\mathcal{F}$ be a family of bounded linear operators from X to Y. Suppose that for each $x \in X$, $\{\|Tx\| : T \in \mathcal{F}\}$ is bounded, then $\{\|T\| : T \in \mathcal{F}\}$ is bounded.*

One of the P.U.B. consequences is:

Theorem 8 *(Banach-Steinhaus theorem)*
Let X be a Banach space and Y be a normed space and (T_n) be a sequence in $\mathcal{B}(X,Y)$. Suppose for every $x \in X$, $T_n x \to Tx$ as $n \to \infty$. Then the family $\{\|T_n\|,\ n \in \mathbb{N}\}$ is bounded, i.e., $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$, and T is linear and bounded, i.e., $T \in \mathcal{B}(X,Y)$.

3. The open mapping theorem

Theorem 9 *Let X and Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is onto (surjective), then T is an open mapping, (that is, if U is open in X, then $T(U)$ is open in Y).*

Consequence:

Theorem 10 *(The inverse mapping theorem)*
Let X and Y be Banach spaces. If $T \in \mathcal{B}(X, Y)$ is bijective, then T^{-1} exists and $T^{-1} \in \mathcal{B}(Y, X)$.

5. The closed graph theorem

Theorem 11 *Let X and Y be Banach spaces and T be a linear map from X to Y. If we define the graph of T by*
$$\Gamma(T) := \{(x, y) \in X \times Y : \ y = Tx\},$$
then T is bounded (continuous) if and only if $\Gamma(T)$ is closed in $X \times Y$.

Problem 69.
(a) If X is a normed space, prove that any proper closed linear subspace of X is a nowhere dense set.
(b) Prove that c_0 is a nowhere dense set of c.

Solution.
(a) Let A be a proper closed linear subspace of X. If A is not nowhere dense, then $\mathring{\bar{A}} = \mathring{A} \neq \emptyset$ since A is closed. Therefore, A contain an open ball $B(a; \varepsilon)$. Take $x \in B(0; \varepsilon)$ then we have
$$\|x\| < \varepsilon \Rightarrow a + x \in B(a; \varepsilon) \subset A.$$
Since $a, a + x \in A$ and A is a linear space, x must be in A. Therefore $B(0; \varepsilon) \subset A$. Now take any $y \in X$ with $y \neq 0$, then we have
$$\frac{\varepsilon}{2\|y\|}\ y \in B(0; \varepsilon).$$
This implies that $\frac{\varepsilon}{2\|y\|}\ y \in A$, and so $y \in A$. Thus $X = A$. This is a contradiction.
(b) We know that c_0 is a proper closed linear subspace of c. So c_0 is nowhere dense in c. $\blacksquare$

Problem 70.
Show that $L^2[0,1]$ is a subset of the first category in $L^1[0,1]$.

Solution.
For short, we write $L^1 := L^1[0,1]$ and $L^2 := L^2[0,1]$. We know that $L^2 \subset L^1$. For each $n \in \mathbb{N}$, let

$$A_n = \{f : \|f\|_2 \leq n\}.$$

Every function in L^2 will be in some A_n, Thus

$$L^2 = \bigcup_{n\in\mathbb{N}} A_n.$$

We first prove that A_n is closed. Fix n. Let (f_k) be a sequence in A_n such that $\|f_k - f\|_1 \to 0$. We show that $f \in A_n$. Since $\|f_k - f\|_1 \to 0$, (f_k) converges in measure to f. Hence there exists a subsequence (f_{k_j}) converging to f almost everywhere. By definition of A_n we have

$$\int_0^1 f_{k_j}^2(t)dt \leq n^2 \text{ for } j = 1, 2, ...$$

Applying Fatou's lemma, we obtain

$$\int_0^1 f(t)dt \leq n^2.$$

Thus $f \in A_n$, and A_n is closed.
Now we show that A_n is nowhere dense, i.e. $\left(\overline{A}_n\right)^0 = (A_n)^0 = \emptyset$. To do this, it suffices to show that for any open ball $B_\varepsilon(f)$ in L^1, there exists a point $g \in B_\varepsilon(f)$ but $g \notin A_n$. Take

$$h(t) = \frac{\varepsilon}{4}\frac{1}{\sqrt{t}}.$$

Then $h \notin L^2$ and

$$\int_0^1 h(t)dt = \frac{\varepsilon}{2},$$

so that

$$h \in L^1 \text{ and } \|h\|_1 = \frac{\varepsilon}{2}.$$

Let $g = f + h$. Then $g \notin L^2$ so $g \notin A_n$, and

$$\|f - g\|_1 = \|h\|_1 = \frac{\varepsilon}{2}.$$

Hence $g \in B_\varepsilon(f)$. Thus A_n is nowhere dense, and L^2 is of the first category in L^1. ■

Problem 71.
Let $X \neq \{0\}$ be a normed space, and $A \subset X$ which is not nowhere dense. We denote by A' the derivative set of A, i.e., the set of all accumulation points of A. Prove that

$$\exists x \in X \text{ and } \varepsilon > 0 \text{ such that } B(x;\varepsilon) \subset A'.$$

(Note: $\bar{A} = A \cup A'$).

Solution.
Since $\mathring{\bar{A}} \neq \emptyset$,

$$\exists x \in X \text{ and } \varepsilon > 0 \text{ such that } B(x;\varepsilon) \subset \bar{A}.$$

We will prove $B(x;\varepsilon) \subset A'$. Let $y \in B(x;\varepsilon)$. If $y \notin A'$, then there is $r > 0$ such that

$$B(y;r) \cap (A \setminus \{y\}) = \emptyset.$$

Let $\delta = \min\{r, \frac{1}{2}\,(\varepsilon - \|y-x\|)\} > 0$. We claim $B(y;\delta) \subset B(x;\varepsilon)$. Indeed, if $z \in B(y;\delta)$ then

$$\begin{aligned}
\|z-x\| &\leq \|z-y\| + \|y-x\| \\
&< \delta + \|y-x\| \\
&\leq \frac{1}{2}\,(\varepsilon - \|y-x\|) + \|y-x\| \\
&= \frac{1}{2}\,(\varepsilon + \|y-x\|) < \varepsilon.
\end{aligned}$$

Hence $z \in B(x;\varepsilon)$.
Obviously, $B(y;\delta) \subset B(y;r)$. So

$$B(y;\delta) \cap (A \setminus \{y\}) \subset B(y;r) \cap (A \setminus \{y\}) = \emptyset.$$

That is $B(y;\delta) \cap A \subset \{y\}$. Since $y \in B(x;\varepsilon) \subset \bar{A}$, so $y \in \bar{A}$, from whence $B(y;\delta) \cap A \neq \emptyset$, and therefore

$$B(y;\delta) \cap A = \{y\}.$$

Since $B(y;\delta)\cap\bar{A}\subset\overline{B(y;\delta)\cap A}$ and $B(y;\delta)\subset B(x;\varepsilon)\subset\bar{A}$, i.e., $B(y;\delta)\cap\bar{A}=B(y;\delta)$. Hence, $B(y;\delta)=\{y\}$, which is false, since $X\neq\{0\}$. (Indeed, $\exists a\in X,\ a\neq 0$. Take $b=\frac{a}{\|a\|}\in X$. Then $\|b\|=1$ and

$$y+\frac{\delta}{2}\,b\in B(y;\delta)=\{y\},$$

which implies that $b=0$: a contradiction). $\blacksquare$

Problem 72.
Let X is a Banach space and $A\subset X$ a dense set. Can we find a function $f:X\to\mathbb{R}$ such that, for every $x\in A$, we have $\lim_{t\to x}|f(t)|=\infty$?

Solution.
Suppose that such a function exists. Since f takes finite values, for every $x\in X$, there is $k\in\mathbb{N}$ such that $|f(x)|\leq k$, i.e.,

$$X=\bigcup_{k\in\mathbb{N}}A_k\ \text{ where }\ A_k=\{x\in X:\ |f(x)|\leq k\}.$$

Since X is a Banach space, X is of the second Baire category. So there is $k\in\mathbb{N}$ such that $\mathring{\bar{A}}_k\neq\emptyset$. Using Problem 74, it follows that there is $x\in X$ and $\varepsilon>0$ such that $B(x;\varepsilon)\subset A_k'$. Since A is dense, we have

$$\emptyset\neq B(x;\varepsilon)\cap A\subset A_k'\cap A.$$

It follows that there is an element $a\in A$ which is in A_k'. By definition of accumulation points, there is a sequence $(x_n)\subset A_k$ with $x_n\neq a,\ \forall n\in\mathbb{N}$ such that $x_n\to a$. Since $a\in A$, by hypothesis we have $\lim_{t\to a}|f(t)|=\infty$. Hence, $\lim_{n\to\infty}|f(x_n)|=\infty$. This is not possible since $(x_n)\subset A_k$ implies that $|f(x_n)|\leq k,\ \forall k\in\mathbb{N}$. $\blacksquare$

Problem 73.
Show that $\mathbb{Q}$ is a subset of the first category of $\mathbb{R}$.

Solution.
Since $\mathbb{Q}$ is countable, we have

$$\mathbb{Q}=\bigcup_{n\in\mathbb{N}}\{x_n\}\ \text{ and }\ (\overline{\{x_n\}})^0=(\{x_n\})^0=\emptyset.\qquad\blacksquare$$

Problem 74.
Show that the set of piecewise linear functions on $\mathbb{R}$ is of the first category.

Solution.
Let P denote the set of piecewise linear functions on $\mathbb{R}$. Let P_n be the set of piecewise linear functions having n intervals of linearity. We have

$$P = \bigcup_{n=1}^{\infty} P_n.$$

We need to prove that P_n is nowhere dense for every n. Fix an arbitrary n. Let $f \in P_n$. Consider the ball $B_r(f),\ r > 0$. Let

$$g(t) = \frac{r}{2}\sin(2\pi 4nt) + f(t).$$

Then

$$|f(t) - g(t)| = \frac{r}{2}|\sin(2\pi 4nt)|.$$

Hence $d(f,g) = \frac{r}{2}$, and so $g \in B_r(f)$. We claim that the ball $B_{\frac{r}{2}}(g)$ contains no element from P_n. Pick $h \in B_r(f) \cap P_n$ and suppose $d(g,h) < \frac{r}{2}$. Then

$$(*) \quad d(g,h) = \sup|g(t) - h(t)| = \sup\left|\frac{r}{2}\sin(2\pi 4nt) + f(t) - h(t)\right| < \frac{r}{2}.$$

Observing that the term $\frac{r}{2}\sin(2\pi 4nt)$ oscillates between $-\frac{r}{2}$ and $\frac{r}{2}$ $4n$ times on $[0,1]$. Thus the term $f(t) - h(t)$ must also oscillate between negative and positive values $4n$ times for (*) to hold. But this is impossible since the term $f(t) - h(t)$ is a piecewise linear function with at most $2n$ intervals of linearity. So, the open ball $B_{\frac{r}{2}}(g)$ contains no element from P_n. Since n is arbitrary, we see that P_n is nowhere dense, and hence P is of the first category. ■

Problem 75. (Inverse mapping theorem)
Let X and Y be Banach spaces and $T \in \mathcal{B}(X,Y)$. Suppose T is bijective. Show that there exist real numbers $a, b > 0$ such that

$$a\|x\| \leq \|Tx\| \leq b\|x\|,\ \forall x \in X.$$

Solution.
Since T is linear, bijective and bounded, T^{-1} exists, is linear and bounded by the inverse mapping theorem. Let $\|T^{-1}\| = \frac{1}{a}$ and $\|T\| = b$. Note that $T \neq 0,\ a, b > 0$. Now since T is bounded,

$$(i) \qquad \|Tx\| \leq \|T\|\ \|x\| = b\|x\|,\ \forall x \in X.$$

Also, since T^{-1} is bounded,

$$(ii) \qquad \|x\| = \|T^{-1}(Tx)\| \leq \|T^{-1}\|\ \|Tx\| = \frac{1}{a}\|Tx\|,\ \forall x \in X.$$

(i) and (ii) imply that

$$a\|x\| \leq \|Tx\| \leq b\|x\|,\ \forall x \in X. \qquad \blacksquare$$

Problem 76. *Let $X = C^1[0,1]$ be the space of continuously differentiable functions on $[0,1]$ and $Y = C[0,1]$. The norm on $C[0,1]$ and $C^1[0,1]$ is the sup-norm. Consider the map*

$$T : C^1[0,1] \to C[0,1] \ \text{ defined by } \ Tx(t) = \frac{dx(t)}{dt}, \quad t \in [0,1].$$

Show that the the graph of T is closed but T is not bounded. Does this contradict the closed graph theorem ?

Solution.
- It is clear that T is linear.
- We show that $\Gamma(T)$ is closed. Suppose $x_n \to x$ in $X = C^1[0,1]$ and $Tx_n \to y$ in $Y = C[0,1]$. We must show that $y = Tx$. For any $t \in [0,1]$, we have

$$\begin{aligned}
\int_0^t y(s)ds &= \int_0^t \lim_{n\to\infty} \frac{dx_n}{ds} ds, \quad y \in C[0,1] \\
&= \lim_{n\to\infty} \int_0^t \frac{dx_n}{ds} ds \quad \text{(uniform convergence)} \\
&= \lim_{n\to\infty} \big(x_n(t) - x_n(0)\big) = x(t) - x(0).
\end{aligned}$$

Thus, with $y \in C[0,1]$, we have

$$x(t) = x(0) + \int_0^t y(s)ds, \qquad t \in [0,1].$$

Hence

$$x \in C^1[0,1] \text{ and } \frac{dx}{ds} = y \text{ on } [0,1].$$

That is $Tx = y$, and so $(x, y) \in \Gamma(T)$, and $\Gamma(T)$ is closed.
• We show that T is not bounded. Take $f_n(t) = t^n$, $n \in \mathbb{N}$, $t \in [0,1]$. Then $f_n \in C^1[0,1]$ and $Tf_n = nf_{n-1}$ for $n > 1$. But we have

$$\|f_n\| = 1 \text{ and } \|Tf_n\| = n,$$

which shows that T is unbounded. The reason? Is $C^1[0,1]$ with the sup-norm a Banach space? ■

Problem 77. (P.U.B.)
Let H be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $\{x_n\}$ be a sequence in H. Prove that the following two statements are equivalent:
(1) $\lim_{n\to\infty}\langle x, x_n\rangle = 0, \quad \forall x \in H.$
(2) $\lim_{n\to\infty}\langle e_m, x_n\rangle = 0, \quad \forall m \in \mathbb{N}$ *and* $\{\|x_n\|\}$ *is bounded.*

Solution.
• (1) ⇒ (2)
Assume that (1) is true . Then the first part of (2) is automatically true. We have only to show that $\{\|x_n\|\}$ is bounded. Consider, for each n, the functional $f_n(x) = \langle x, x_n\rangle$. This is a bounded functional by Schwarz inequality, and, because $f_n(x) \to 0$ for each x, we have that the set $\{\|f_n(x)\| : n \in \mathbb{N}\}$ is bounded. *The principle of uniform boundedness* then gives us that the set $\{\|f_n\|\}$ is bounded. But $\|f_n\| = \|x_n\|$, so $\{\|x_n\|\}$ is bounded.
• (2) ⇒ (1)
Assume that (2) holds. Let B be the bound of $\{\|x_n\|\}$ and let $x \in X$. We write $x = \sum_n \langle e_n, x\rangle e_n$. For every $\varepsilon > 0$, let K be such that

$$\sum_{m>K} |\langle e_m, x\rangle|^2 < \frac{\varepsilon}{B}.$$

We know that, for m fixed, $\langle e_m, x_n\rangle \to 0$ as $n \to \infty$. So we may find N such that if $n > N$ then $|\langle e_m, x_n\rangle| < \frac{\varepsilon}{K \sup|\langle e_r, x\rangle|}$ (the denominator is finite because the sequence

$\langle e_r, x\rangle$ is an ℓ^2-sequence). Then, if $n > N$, we have

$$\begin{aligned}
|\langle x, y\rangle| &= \left|\sum_{m=1}^{\infty} \overline{\langle e_m, x\rangle}\langle e_m, x_n\rangle\right| \\
&\leq \sum_{m=1}^{K} \frac{\varepsilon}{K\sum|\langle e_r, x\rangle|}\,|\langle e_m, x\rangle| + \sum_{m>K} \overline{\langle e_m, x\rangle}\langle e_m, x_n\rangle \\
&\leq \varepsilon + \left(\sum_{m>K}|\langle e_m, x\rangle|^2\right)^{1/2} + \left(\sum_{m>K}|\langle e_m, x_n\rangle|^2\right)^{1/2} \\
&\leq \varepsilon + \frac{\varepsilon}{B}\,\|x_n\| \leq 2\varepsilon. \quad \blacksquare
\end{aligned}$$

Problem 78. (Closed graph theorem)
Let H be a Hilbert space and $T : H \to H$ a linear operator which is symmetric, i.e.,

$$\langle Tx, y\rangle = \langle x, Ty\rangle,\ \forall x, y \in H.$$

Prove that T is continuous.

Solution.
Since H is a complete space and T is a linear operator, it is sufficient to prove that T is a closed graph operator. Let (x_n) be a sequence in H such that $x_n \to x \in H$ and $Tx_n \to y \in H$. We will show that $y = Tx$.
By hypothesis, we have

$$\langle T(x_n - x), y\rangle = \langle x_n - x, Ty\rangle,\ \forall y \in H.$$

Hence

$$\langle T(x_n - x), y\rangle \to 0,\ \forall y \in H.$$

That is

$$T(x_n - x) \xrightarrow{w} 0, \quad \text{or, equivalently} \quad Tx_n \xrightarrow{w} Tx.$$

But $Tx_n \to y$, so $Tx_n \xrightarrow{w} y$. Since the limit is unique, we have $Tx = y$. Thus, T is a closed graph operator. $\blacksquare$

Problem 79. (Banach-Steinhaus theorem)
Let $a = (a_1, a_2, ...) = (a_n)_{n\in\mathbb{N}}$ be a sequence of scalars such that the sequence $(a_n x_n)_{n\in\mathbb{N}} \in c_0$ for all sequences $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Prove that $a \in \ell^\infty$.

Solution.
For every $n \in \mathbb{N}$, consider the operator

$$U_n : c_0 \to c_0 \text{ defined by } U_n(x) = U_n(x_1, x_2, ...) = (a_1x_1, a_2x_2, ..., a_nx_n, 0, ...).$$

Then U_n is linear, and

$$\begin{aligned} \|U_n(x_1, x_2, ...)\| &= \|(a_1x_1, a_2x_2, ..., a_nx_n, 0, ...)\| \\ &= \max(|a_1x_1|, ..., |a_nx_n|) \\ &\leq \|x\| \max(|a_1|, ..., |a_n|), \ \forall x \in c_0. \end{aligned}$$

So U_n is continuous, and $\|U_n\| \leq \max(|a_1|, ..., |a_n|)$.
For $1 \leq k \leq n$, we have

$$\|U_n\| \geq \|U_n(e_k)\| = \|(0, ..., 0, a_k, 0, ...)\| = |a_k|,$$

where $e_k = (\underbrace{0, .., 0}_{k-1}, 1, 0, ...) \in c_0$. Therefore $\max(|a_1|, ..., |a_n|) \leq \|U_n\|$. Hence,

$$\|U_n\| = \max(|a_1|, ..., |a_n|), \forall n \in \mathbb{N}.$$

Now consider the operator

$$U : c_0 \to c_0 \text{ defined by } U(x) = (a_1x_1, a_1x_2, ...) = (a_nx_n)_{n\in\mathbb{N}}.$$

We have that $U_n(x) \to U(x), \ \forall x \in c_0$ because

$$\|U_n(x) - U(x)\| = \sup_{k\geq n+1} |a_kx_k| \to 0.$$

Now from the Banach-Steinhaus theorem we get that $\sup_{n\in\mathbb{N}} \|U_n\| < \infty$, i.e., $\sup_{n\in\mathbb{N}} |a_n| < \infty$. In other words, $a \in \ell^\infty$. ■

Problem 80. (Similar problem)
Let $a = (a_1, a_2, ...) = (a_n)_{n\in\mathbb{N}}$ be a sequence of scalars such that the sequence $(a_nx_n)_{n\in\mathbb{N}} \in c_0$ for all sequences $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$. Prove that $a \in c_0$.

Problem 81. (P.U.B.)
Let $x = (x_1, x_2, ...) = (x_i)_{i=1}^\infty$ be a sequence of scalars such that the series $\sum_{i=1}^\infty x_iy_i$ is convergent for all $y = (y_1, y_2, ...) \in c_0$. Prove that $x \in \ell^1$.

Solution.
For every $n \in \mathbb{N}$, we define the linear operator

$$T_n : c_0 \to \mathbb{C}, \quad T_n(y) = \sum_{i=1}^{n} x_i y_i.$$

Then we have

$$|T_n(y)| \leq \sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i| \right) \|y\|_\infty.$$

This shows that T_n is bounded with

$$\|T_n\| \leq \sum_{i=1}^{n} |x_i|.$$

By hypothesis, $\sum_{i=1}^{\infty} x_i y_i < \infty$, the sequence $\Big(T_n(y)\Big)_{n\in\mathbb{N}}$ converges for every $y \in c_o$, and c_0 is a Banach space, the principle of uniform boundedness implies that

$$\exists M > 0 : \quad \|T_n\| \leq M, \ \forall n \in \mathbb{N}.$$

Now let $x^{(n)} = \big(x_1^{(n)}, x_2^{(n)}, ..., x_n^{(n)}, 0, 0, ...\big)$ be a truncated version of x (so that $x^{(n)} \in \ell^1$). Let

$$T_n(y) = \sum_{i=1}^{n} x_i^{(n)} y_i, \quad y = (y_1, y_2, ...) \in c_0.$$

Define $y^{(n)} = \big(y_1^{(n)}, y_2^{(n)}, ...,\big)$ by

$$y_k^{(n)} = \begin{cases} \frac{\overline{x_k^{(n)}}}{|x_k^{(n)}|} & \text{if } \ x_k^{(n)} \neq 0, \\ 0 & \text{if } \ x_k^{(n)} = 0. \end{cases}$$

Then

$$|T_n(y^{(n)})| = \sum_{k=1}^{n} |x_k^{(n)}| = \|x^{(n)}\|_1 = \|x^{(n)}\|_1 \|y^{(n)}\|_\infty.$$

Hence

$$\|T_n\| \geq \|x^{(n)}\|_1,$$

which in turn implies that

$$\|x^{(n)}\|_1 \leq M, \ \forall n \in \mathbb{N}.$$

But it is clear from the definition of $x^{(n)}$ that $\big(\|x^{(n)}\|_1\big)$ is an increasing sequence of real numbers. Being bounded above by M, it must converge. Hence

$$\sum_{i=1}^{\infty} |x_i| < \infty,$$

and so $x \in \ell^1$. ■

Problem 82. (Very similar problem)
Let $c = (c_1, c_2, ...) = (c_i)_{i=1}^{\infty}$ be a sequence of scalars such that the series $\sum_{i=1}^{\infty} c_i a_i$ is convergent for all $a = (a_1, a_2, ...) \in \ell^1$. Prove that $x \in \ell^{\infty}$.

Problem 83. (Closed graph theorem)
Let X, Y and Z be Banach spaces. Suppose that $T : X \to Y$ is linear, that $J : Y \to Z$ is linear, bounded and injective, and that $JT \equiv J \circ T : X \to Z$ is bounded.
Show that T is also bounded.

Solution.
We will show that the graph $\Gamma(T)$ is closed. Then by the closed graph theorem, this implies that T is bounded (continuous).
Let $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ be a convergent sequence in $X \times Y$, that is,

$$x_n \to x \text{ in } X, \quad y_n \to y \text{ in } Y \text{ and } Tx_n = y_n.$$

Since J and JT are continuous,

$$\big[y_n \to y \Rightarrow Jy_n \to Jy\big] \text{ and } \big[x_n \to x \Rightarrow JTx_n \to JTx\big].$$

Since $Tx_n = y_n$, we have $Jy_n \to JTx$. Since the limit is unique, this gives that $Jy = JTx$. But by hypothesis J is injective, so we have

$$Jy = JTx \Rightarrow y = Tx.$$

This shows that $(x, y) \in \Gamma(T)$, and $\Gamma(T)$ is closed. ■

Problem 84. (Closed graph theorem)
Let X be a Banach space and E, F two closed subspaces of X such that $X = E \oplus F$. Consider the projections on E and on F defined by

$$P_E : X \to E, \quad P_E(u) = x,$$
$$P_F : X \to F, \quad P_F(u) = y, \quad \textit{where} \quad u = x + y, \ x \in E, \ y \in F.$$

Use the closed graph theorem to show that $P_E \in \mathcal{B}(X, E)$ and $P_F \in \mathcal{B}(X, F)$.

Solution.
The linearity of these two maps are easy to check. Let us prove that they are bounded by using the closed graph theorem. Denote the graph of P_E by Γ_E. We can write

$$\Gamma_E = \{(x, y) \in X \times E : \ x - y \in F\}.$$

Let $(x_n, y_n) \in \Gamma_E$ for every $n \in \mathbb{N}$. Suppose $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since $x_n - y_n \in F$ for every $n \in \mathbb{N}$, and F is a closed subspace of X, $\lim_{n\to\infty}(x_n - y_n) = x - y \in F$. It follows that $(x, y) \in \Gamma_E$. Thus, Γ_E is closed, and so P_E is bounded. The proof for P_F is the same. ■

Problem 85. (Inverse mapping theorem)
Let $(X, \|.\|_1)$ and $(X, \|.\|_2)$ be Banach spaces. Suppose that

$$\exists C \geq 0 : \ \|x\|_2 \leq C\|x\|_1, \ \forall x \in X.$$

Show that the two norms $\|.\|_1$ and $\|.\|_2$ are equivalent.

Solution.
Consider the identity map

$$id : (X, \|.\|_1) \to (X, \|.\|_2), \quad id(x) = x.$$

It is clear that the identity map is linear and bijective. It is continuous since by hypothesis we have

$$(*) \qquad \|id(x)\|_2 = \|x\|_2 \leq C\|x\|_1, \ \forall x \in X.$$

By the inverse mapping theorem, the inverse map id^{-1} exists and continuous. That is

$$(**) \qquad \exists C' \geq 0 : \ \|x\|_1 \leq C'\|x\|_2, \ \forall x \in X.$$

$(*)$ and $(**)$ together imply that $\|.\|_1$ and $\|.\|_2$ are equivalent. ■

Chapter 6

Linear Operators on Hilbert Spaces

Review.

1. Definition and elementary properties

Let $T : H \to K$ be a linear operator between Hilbert spaces H and K.
• The following statements are equivalent:
1. T is continuous at 0,
2. T is continuous,
3. T is bounded on H.
• An isomorphism between H and K is a linear surjection $U : H \to K$ such that

$$\langle Ux, Uy\rangle = \langle x, y\rangle, \quad \forall x, y \in H.$$

An isomorphism is an isometry and so preserves completeness, but an isometry need not to be an isomorphism.

Proposition 1 *Two Hilbert spaces are isomorphic if and only if they have the same dimension.*

2. Adjoint of an Operator
• Let $A \in \mathcal{B}(H)$. Then A^* is called the adjoint operator of A if

$$\langle Ax, x\rangle = \langle x, A^*x\rangle, \quad \forall x \in H.$$

Proposition 2 *If $A, B \in \mathcal{B}(H)$ and $\alpha \in \mathbb{F}$, then*

(a) $(\alpha A + B)^* = \bar{\alpha}A^* + B^*$.

(b) $(AB)^* = B^*A^*$.

(c) $A^{**} := (A^*)^* = A$.

(d) If A is invertible in $\mathcal{B}(H)$ and A^{-1} is its inverse, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proposition 3 *It $A \in \mathcal{B}(H)$ then*

$$\|A\| = \|A^*\| = \|A^*A\|^{1/2}.$$

3. Self-adjoint, normal, unitary operators

Definition 4 *If $A \in \mathcal{B}(H)$, then*

(a) *A is Hermitian or self-adjoint if $A^* = A$.*

(b) *A is normal if $AA^* = A^*A$.*

(c) *A is unitary if it is a surjective isometry.*

Proposition 4 *Let $A \in \mathcal{B}(H)$. The following statements are equivalent.*

(a) *$A^*A = AA^* = I$.*

(b) *A is unitary.*

(c) *A is a normal isometry.*

4. Positive operators

Definition 5 *Let H be a Hilbert space. An operator $A \in \mathcal{B}(X)$ is called positive if*

$$\langle Ax, x\rangle \geq 0, \ \forall x \in X.$$

We write $A \geq 0$. If $A, B \in \mathcal{B}(X)$ and $A - B \geq 0$ then we write $B \leq A$.

Proposition 5 *If $A \in \mathcal{B}(X)$ then $A^*A \geq 0$. In addition, if $A \geq 0$, then*
1. A is self adjoint,
2. there exists a unique $B \in \mathcal{B}(X)$ such that $B \geq 0$ and $B^2 = A$. Furthermore, B is also self adjoint and commutes with every bounded operator which commutes with A. We write $B = \sqrt{A}$.

We define $|A| = \sqrt{A^*A}$.

5. Projection, Orthogonal projection

Definition 6

If $P \in \mathcal{B}(H)$ and $P^2 = P$, then P is called a projection.
If $P \in \mathcal{B}(H)$, $P = P^2$ and $P^* = P$, then P is called an orthogonal projection.

Proposition 6

If $P : H \to H$ is a projection then $H = \text{Image}\, P \oplus \ker P$.
If $H = M \oplus N$, where M, N are subspaces of H, then there is a projection $P : H \to H$ with $\text{Image}\, P = M$ and $\ker P = N$.

* * * * *

Problem 86. *Let P be an orthogonal projection defined on a Hilbert space H. Show that $\|P\| = 1$.*

Solution.
If $x \in H$ and $Px \neq 0$, then the use of the Cauchy-Schwarz inequality implies that

$$\begin{aligned} \|Px\| &= \frac{\langle Px, Px\rangle}{\|Px\|} \\ &= \frac{\langle x, P^2x\rangle}{\|Px\|} \quad (\text{since} \quad P^* = P) \\ &= \frac{\langle x, Px\rangle}{\|Px\|} \leq \frac{\|x\|\ \|Px\|}{\|Px\|} \\ &\leq \|x\|. \end{aligned}$$

Therefore $\|P\| \leq 1$.
Now, if $P \neq 0$, then there is an $x_0 \in H$ such that

$$Px_0 \neq 0 \ \text{ and } \ \|P(Px_0)\| = \|Px_0\|.$$

This implies that $\|P\| \geq 1$. Thus $\|P\| = 1$. ■

Problem 87. *Given a function $\phi : [0,1] \to \mathbb{C}$, consider the operator*

$$P : L^2[0,1] \to L^2[0,1] \ \text{ defined by } \ Pf(x) = \phi(x)f(x).$$

Find necessary and sufficient conditions on the function ϕ for P to be an orthogonal projection.

Solution.
First, in order for P to be a well-defined operator acting on $L^2[0,1]$, the function ϕf needs to be in $L^2[0,1]$ for all $f \in L^2[0,1]$. In particular ϕf is measurable, and taking $f \equiv 1$, it follows that ϕ is a measurable function on $[0,1]$.
Secondly, P is an orthogonal projection if and only if $P^* = P$ and $P^2 = P$. The last equality is equivalent to $\phi^2(x)f(x) = \phi(x)f(x)$, $\forall f \in L^2[0,1]$. Again by taking $f \equiv 1$, we have $a^2(x) = a(x)$ for almost every $x \in [0,1]$. Thus

$$a(x) = 0 \quad \text{or} \quad a(x) = 1 \ \text{ for almost all } \ x \in [0,1].$$

In particular ϕ takes real values. Then

$$\begin{aligned}
\langle Pf, g\rangle &= \int_0^1 Pf(x)\overline{g(x)}dx \\
&= \int_0^1 \phi(x)f(x)\overline{g(x)}dx \\
&= \int_0^1 f(x)\overline{\phi(x)g(x)}dx \\
&= \int_0^1 f(x)\overline{Pg(x)}dx \\
&= \langle f, Pg\rangle,
\end{aligned}$$

which proves that P is self-adjoint. Since $0 \leq \phi(x) \leq 1$ for a.e. on $[0,1]$, we have that

$$\|Pf\|_{L^2} = \left(\int_0^1 \phi(x)^2|f(x)|^2dx\right)^{\frac{1}{2}} \leq \left(\int_0^1 |f(x)|^2dx\right)^{\frac{1}{2}} = \|f\|_{L^2}.$$

Thus, P is bounded.
In conclusion, the necessary and sufficient conditions for P to be an orthogonal projection is ϕ a measurable satisfying $\phi(x) = 0$ or $\phi(x) = 1$ for almost all $x \in [0,1]$. ■

Problem 88. *Consider the right-shift on the Hilbert space ℓ^2:*

$$S : \ell^2 \to \ell^2, \quad S(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...).$$

Define its adjoint operator.

Solution.
For $(\alpha_n) = (\alpha_1, \alpha_2, ...)$ and $(\beta_n) = (\beta_1, \beta_2, ...)$ in ℓ^2,

$$\begin{aligned}
\big\langle S^*(\alpha_n), (\beta_n)\big\rangle &= \big\langle (\alpha_n), S(\beta_n)\big\rangle \\
&= \langle (\alpha_1, \alpha_2, ...), (0, \beta_1, \beta_2, ...)\rangle \\
&= \alpha_2\bar{\beta}_1 + \alpha_3\bar{\beta}_2 + ... \\
&= \langle (\alpha_2, \alpha_3, ...), (\beta_1, \beta_2, ...)\rangle.
\end{aligned}$$

Thus

$$S^*(\alpha_1, \alpha_2, ...) = (\alpha_2, \alpha_3, ...).$$

Hence, the adjoint of the right-shift is the left-shift. ■

Problem 89. *Let $A : \ell^2 \to \ell^2$ be defined by*

$$Ax = A(x_1, x_2, x_3, ...) = (0, 0, x_3, x_4, ...).$$

Prove A is linear, continuous, self-adjoint and positive. Find $\sqrt{A}$.

Solution.
With similar argument as the previous problem, we can show that A is linear. We have

$$\|Ax\|^2 = \sum_{k=3}^{\infty} |x_k|^2 \leq \sum_{k=1}^{\infty} |x_k|^2 = \|x\|^2, \ \forall x = (x_1, x_2, x_3, ...) \in \ell^2.$$

Therefore,

$$\|Ax\| \leq \|x\|, \ \forall x \in \ell^2.$$

This shows that A is continuous. Also, for all $x, y \in \ell^2$,

$$\langle Ax, y\rangle = \sum_{k=3}^{\infty} x_k \bar{y_k} = \langle x, Ay\rangle.$$

Hence, A is self-adjoint. And,

$$\langle Ax, x\rangle = \sum_{k=3}^{\infty} |x_k|^2 \geq 0, \ \forall x \in \ell^2.$$

So A is positive. Then there exists the square root $\sqrt{A} : \ell^2 \to \ell^2$. We have

$$A^2 x = A(Ax) = A(0, 0, x_3, x_4, ...) = (0, 0, x_3, x_4, ...), \ \forall x \in \ell^2.$$

It follows that $A^2 = A$. Hence, $\sqrt{A} = A$. ■

Problem 90.(Multiplication operator)
Let (X, Ω, μ) be a σ-finite measure space. Consider the Hilbert space $H = L^2(X, \Omega, \mu) =: L^2(\mu)$. If $\phi \in L^\infty(\mu)$, define

$$M_\phi : L^2(\mu) \to L^2(\mu) \ \text{ by } \ M_\phi f = \phi f.$$

(a) Show that

$$M_\phi \in \mathcal{B}(H) \ \text{ and } \ \|M_\phi\| = \|\phi\|_\infty.$$

Here $\|\phi\|_\infty$ is the μ-essential supremum norm.
(b) Show that $M_\phi^* = M_{\bar{\phi}}$.
(c) Show that M_ϕ is normal. When M_ϕ is self adjoint? unitary?

Solution.
(a) The linearity of the operator M_ϕ is evident. We show that M_ϕ is bounded and calculate its norm.
By definition, $\|\phi\|_\infty$ is the infimum of all $c > 0$ such that $|\phi(x)| \leq c$ a.e. $[\mu]$, and so $|\phi(x)| \leq \|\phi\|_\infty$ a.e. $[\mu]$. Thus we can assume that ϕ is a bounded measurable and $|\phi(x)| \leq \|\phi\|_\infty$ for all x. If $f \in L^2(\mu)$, then

$$\int |\phi f|^2 d\mu \leq \|\phi\|_\infty^2 \int |f|^2 d\mu.$$

That is,

$$M_\phi \in \mathcal{B}\big(L^2(\mu)\big) \text{ and } \|M_\phi\| \leq \|\phi\|_\infty \quad (*).$$

If $\varepsilon > 0$, the σ-finiteness of the measure space implies that

$$\exists \Delta \in \Omega \text{ such that } 0 < \mu(\Delta) < \infty \text{ and } |\phi(x)| \geq \|\phi\|_\infty - \varepsilon, \quad \forall x \in \Delta.$$

If we take $f = \frac{1}{\sqrt{\mu(\Delta)}}\chi_\Delta$, then we have

$$f \in L^2(\mu) \text{ and } \|f\|_2 = 1.$$

So

$$\|M_\phi\|^2 \geq \|\phi f\|^2 = \frac{1}{\mu(\Delta)} \int_\Delta |\phi|^2 d\mu \geq (\|\phi\|_\infty - \varepsilon)^2.$$

Letting $\varepsilon \longrightarrow 0$, we get that

$$\|M_\phi\| \geq \|\phi\|_\infty \quad (**).$$

(*) and (**) give that $\|M_\phi\| = \|\phi\|_\infty$.
(b) For $f, g \in L^2(\mu)$, we have

$$\begin{aligned}
\langle f, M_\phi g\rangle &= \int f\left(\overline{M_\phi g}\right) d\mu \\
&= \int f(\overline{\phi g}) d\mu \\
&= \int (\bar{\phi} f)\bar{g} d\mu \\
&= \langle M_{\bar{\phi}} f, g\rangle.
\end{aligned}$$

This shows that

$$M_\phi^* = M_{\bar{\phi}}.$$

(c) Every multiplication operator M_ϕ is normal. Indeed,

$$M_\phi M_\phi^* = M_\phi M_{\bar{\phi}} = M_{\bar{\phi}} M_\phi = M_\phi^* M_\phi.$$

M_ϕ is self-adjoint if and only if $\phi = \bar{\phi}$, that is, ϕ is real-valued.
M_ϕ is unitary if and only if $|\phi| = 1$ a.e. $[\mu]$. ■

Problem 91. *Let H be a Hilbert space and $A \in \mathcal{B}(H)$. Show that*
(a) $\overline{\text{Image}\, A} = (\ker A^*)^\perp$.
(b) $\ker A = (\text{Image}\, A^*)^\perp$.

Solution.
(a) Take any $x \in \text{Image}\, A$. Then there is a $y \in H$ such that $x = Ay$. For any $z \in \ker A^*$, we have

$$\langle x, z\rangle = \langle Ay, z\rangle = \langle y, A^*z\rangle = \langle y, 0\rangle = 0.$$

hence $x \in (\ker A^*)^\perp$. This proves that $\text{Image}\, A \subset (\ker A^*)^\perp$. Since $(\ker A^*)^\perp$ is closed, it follows that

$$(i) \qquad \overline{\text{Image}\, A} \subset (\ker A^*)^\perp.$$

On the other hand, if $x \in (\text{Image}\, A)^\perp$, then for all $y \in H$, we have

$$0 = \langle Ay, x\rangle = \langle y, A^*x\rangle.$$

Therefore $A^*x = 0$, that is, $x \in \ker A^*$. This prove that $(\text{Image}\, A)^\perp \subset \ker A^*$. Taking orthogonal complements both sides, we obtain

$$(ii) \qquad (\ker A^*)^\perp \subset \text{Image}\, A \subset \overline{\text{Image}\, A}.$$

From (i) and (ii) it follows that

$$\overline{\text{Image}\, A} = (\ker A^*)^\perp.$$

(b) Replacing A by A^* in (a), we get

$$(\ker A)^\perp = \overline{\text{Image}\, A^*}.$$

Taking orthogonal complements both sides and using a result in Problem 39, we obtain

$$\ker A = (\overline{\text{Image}\, A^*})^\perp = (\text{Image}\, A^*)^\perp. \qquad ■$$

Problem 92.(Integral operator)
Let (X, Ω, μ) be a measure space. Let $k : X \times X \to \mathbb{F}$ be an $\Omega \times \Omega$-measurable function for which there are constants c_1 and c_2 such that

$$\int_X |k(x,y)| d\mu(y) \leq c_1 \quad a.e.[\mu],$$
$$\int_X |k(x,y)| d\mu(x) \leq c_2 \quad a.e.[\mu].$$

Consider the operator $K : L^2(\mu) \to L^2(\mu)$ defined by

$$(Kf)(x) = \int k(x,y) f(y) d\mu(y).$$

The function k is called the kernel of the operator K.
(a) Show that K is a bounded linear operator and $\|K\| \leq \sqrt{c_1 c_2}$.
(b) Show that K^* is the integral operator with kernel $k^*(x,y) = \overline{k(x,y)}$.

Solution.
(a) Linearity of K comes from linearity of the integral $\int$. It suffices to show that K is bounded. Actually it must be shown first that $Kf \in L^2(\mu)$, but this will follow from the argument that demonstrates the boundedness of K. If $f \in L^2(\mu)$,

$$\begin{aligned}
|Kf(x)| &\leq \int |k(x,y)|\, |f(y)| d\mu(y) \\
&= \int |k(x,y)|^{1/2} |k(x,y)|^{1/2} |f(y)| d\mu(y) \\
&\leq \left[\int |k(x,y)| d\mu(y)\right]^{1/2} \left[\int |k(x,y)|\, |f(y)|^2 d\mu(y)\right]^{1/2} \\
&\leq \sqrt{c_1} \left[\int |k(x,y)|\, |f(y)|^2 d\mu(y)\right]^{1/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int |Kf(x)|^2 d\mu(y) &\leq c_1 \int\int |k(x,y)|\, |f(y)|^2 d\mu(y) d\mu(x) \\
&= c_1 \int |f(y)|^2 \int |k(x,y)| d\mu(x) d\mu(y) \\
&\leq c_1 c_2 \|f\|^2.
\end{aligned}$$

Now this shows that the formula used to define Kf is finite a.e. $[\mu]$, and so

$$Kf \in L^2(\mu) \text{ and } \|Kf\|^2 \le c_1c_2\|f\|^2.$$

(b) By definition,

$$\begin{aligned}\langle Kf, g\rangle &= \int k(x,y)f(y)\overline{g(y)}d\mu(y)\\ &= \int f(y)\overline{\overline{k(x,y)}g(y)}d\mu(y)\\ &= \langle f, K^*g\rangle, \quad \text{where } K^*g(y) = \int \overline{k(x,y)}g(y)d\mu(y).\end{aligned}$$

Hence, the kernel of K^* is $k^*(x,y) = \overline{k(x,y)}$. ■

Problem 93. *Let $\mathcal{H} = H \oplus H$ where H be a Hilbert space. Let $A \in \mathcal{B}(H)$ and B be the operator defined on $\mathcal{H}$ by*

$$B = \begin{pmatrix} 0 & iA \\ -iA^* & 0 \end{pmatrix}$$

Prove that $\|A\| = \|B\|$ and that B is self-adjoint.

Solution.
For any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $x_1, x_2, y_1, y_2 \in H$ we have

$$\begin{aligned}\langle Bx, y\rangle &= \left\langle \begin{pmatrix} 0 & iA \\ -iA^* & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle\\ &= \langle iAx_2, y_1\rangle + \langle -iA^*x_1, y_2\rangle\\ &= \langle x_2, -iA^*y_1\rangle + \langle x_1, iAy_2\rangle\\ &= \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 & iA \\ -iA^* & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle\\ &= \langle x, By\rangle.\end{aligned}$$

Moreover,

$$\begin{aligned}\|Bx\|^2 &= \|iAx_2\|^2 + \|-iA^*x_1\|^2\\ &\le (\max\{\|A\|, \|A^*\|\})^2\|x\|^2\\ &= \|A\|^2\|x\|^2.\end{aligned}$$

Hence, $\|B\| \leq \|A\|$. Conversely, one can take $\tilde{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ and obtain

$$\|B\tilde{x}\| = \|Ax_2\| \leq \|B\|\|\tilde{x}\| = \|B\|\|x_2\|.$$

Therefore, $\|A\| \leq \|B\|$. Finally, we obtain $\|A\| \leq \|B\|$. ■

Remark:
Note that norm on $H \oplus H$ is

$$\|(a,b)\| = \|a\| + \|b\|, \;\; a, b \in H.$$

Problem 94. *Let T be a self-adjoint operator on a Hilbert space H. Show that its norm is given by*

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Solution.
For $\|x\| = 1$ we have

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| = \|Tx\| \leq \|T\|.$$

Therefore

$$(i) \qquad \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|.$$

In order to establish the inverse inequality, we consider the case:

$$z \in H, \;\; \|z\| = 1, \;\; Tz \neq 0 \quad \text{and} \quad u = \frac{1}{\lambda}Tz \quad \text{where} \quad \lambda = \sqrt{\|Tz\|}.$$

If we denote by $\alpha := \sup_{\|x\|=1} |\langle Tx, x\rangle|$, then we have

$$\begin{aligned}
\|Tz\|^2 &= \big\langle T(\lambda z), u\big\rangle \\
&= \frac{1}{4}\big[\big\langle T(\lambda z + u), \lambda z + u\big\rangle - \big\langle T(\lambda z - u), \lambda z - u\big\rangle\big] \\
&\leq \frac{\alpha}{4}\big[\|\lambda z + u\|^2 + \|\lambda z - u\|^2\big] \\
&= \frac{\alpha}{2}\big[\|\lambda z\|^2 + \|u\|^2\big] \\
&= \frac{\alpha}{2}\big[\|\lambda\|^2 + \|Tz\|^2\big] = \alpha\|Tz\|.
\end{aligned}$$

This implies that, for any $z \in H$ with $\|z\| = 1$, we have $\|Tz\| \leq \alpha$, and hence

$$(ii) \qquad \|T\| \leq \alpha = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

(i) and (ii) completes the proof. ■

Problem 95
Let H be a Hilbert space and A a positive self-adjoint operator on H. Prove that the following assertions are equivalent:
(i) $A(H)$ is dense in H.
(ii) $\mathrm{Ker}\, A = \{0\}$.
(iii) A is positive definite, i.e., $\langle Ax, x\rangle > 0,\ \forall x \in H \setminus \{0\}$.

Solution.
• (i) $\Rightarrow$ (ii)
Suppose $Ax = 0$. Then, for any $y \in H$,

$$\begin{aligned}
\langle Ax, y\rangle &= \langle x, Ay\rangle = 0 \text{ (since } A \text{ is self-adjoint)} \\
&\Rightarrow x \perp A(H) \\
&\Rightarrow x \perp \overline{A(H)} = H \text{ (since } A(H) \text{ is dense)} \\
&\Rightarrow x = 0.
\end{aligned}$$

• (ii) $\Rightarrow$ (iii)
Since A is positive, $\sqrt{A} = B$ exists. It is also a self-adjoint operator on H. To show A is positive definite, we show $\langle Ax, x\rangle = 0 \Rightarrow x = 0$. Now

$$0 = \langle Ax, x\rangle = \langle B^2x, x\rangle = \langle B(Bx), x\rangle = \langle Bx, Bx\rangle = \|Bx\|^2.$$

This implies that $Bx = 0$. Therefore,

$$Ax = B(Bx) = 0.$$

Since $\mathrm{Ker}\, A = \{0\}$, we have $x = 0$.
• (iii) $\Rightarrow$ (i)
Assume that $A(H)$ is not dense in H. Then there is $x \in H \setminus \{0\}$ such that $x \perp A(H)$. In particular, $x \perp Ax$, i.e., $\langle Ax, x\rangle = 0$. But A is positive definite, so $x = 0$, a contradiction. ■

Problem 96
Let H be a Hilbert space. If $A, B : H \to H$ are self-adjoint operators with $0 \leq A \leq B$ and B is compact, prove that A is compact.

Solution.
Let (x_n) be a sequence in the closed unit ball B_H. Since B is compact, there is a subsequence (x_{n_k}) such that (Bx_{n_k}) converges. From the Cauchy-Schwarz inequality we have

$$\langle Bx, x\rangle \leq \|Bx\|\ \|x\|,\ \forall x \in H.$$

It follows that

$$\begin{aligned}
\langle Bx_{n_k} - Bx_{m_k}, x_{m_k} - x_{n_k}\rangle &\leq \|Bx_{m_k} - Bx_{n_k}\|\ \|x_{m_k} - x_{n_k}\| \\
&\leq \|Bx_{m_k} - Bx_{n_k}\|\ (\|x_{m_k}\| + \|x_{n_k}\|) \\
&\leq 2\|Bx_{m_k} - Bx_{n_k}\|.
\end{aligned}$$

From $0 \leq A \leq B$ we get

$$\begin{aligned}
\langle A(x_{n_k} - x_{m_k}), x_{m_k} - x_{n_k}\rangle &\leq \langle B(x_{n_k} - x_{m_k}), x_{m_k} - x_{n_k}\rangle \\
&\leq 2\|Bx_{m_k} - Bx_{n_k}\|.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\|\sqrt{A}(x)\|^2 &= \langle \sqrt{A}(x), \sqrt{A}(x)\rangle \\
&= \langle \sqrt{A}^2(x), x\rangle \\
&= \langle Ax, x\rangle \\
&\leq \|Ax\|\ \|x\|,\ \forall x \in H.
\end{aligned}$$

Then

$$\begin{aligned}
\|\sqrt{A}x_{m_k} - \sqrt{A}x_{n_k}\|^2 &\leq \|A(x_{m_k} - x_{n_k})\|\ \|x_{m_k} - x_{n_k}\| \\
&\leq \|A(x_{m_k} - x_{n_k})\|(\|x_{m_k}\| + \|x_{n_k}\|) \\
&\leq 2\|A(x_{m_k} - x_{n_k})\|.
\end{aligned}$$

Therefore,

$$\|\sqrt{A}x_{m_k} - \sqrt{A}x_{n_k}\|^2 \leq \langle A(x_{m_k} - x_{n_k}), x_{m_k} - x_{n_k}\rangle.$$

Hence

$$\|\sqrt{A}x_{m_k} - \sqrt{A}x_{n_k}\|^2 \leq 2\|Bx_{m_k} - Bx_{n_k}\|.$$

From this we see that the sequence $(\sqrt{A}x_{n_k})$ is Cauchy, hence converges. The operator $\sqrt{A}$ is compact. And so is the operator $A = \sqrt{A}^2$. ∎

Chapter 7

Compact Operators

In this chapter we study general properties of compact operators on Banach and Hilbert spaces. Spectral properties of these operators will be discussed later.

Definition 7 *Let X and Y be Banach spaces. An operator $T \in \mathcal{B}(X,Y)$ is called compact operator if the image of every bounded set in X has compact closure in Y (relatively compact set). Equivalently, $T \in \mathcal{B}(X,Y)$ is compact if and only if for every bounded sequence (x_n) in X, (Tx_n) has a convergent subsequence in Y.*
The set of all compact operators is denoted by $\mathcal{B}_0(X,Y)$.

Proposition 7 *Let X and Y be Banach spaces. Then $\mathcal{B}_0(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$. That is, if (T_n) is a sequence of compact operators and $T \in \mathcal{B}(X,Y)$ such that $\|T_n - T\| \to 0$, then $T \in \mathcal{B}_0(X,Y)$.*

Definition 8 *(Finite rank operators)*
An operator $T : X \to Y$ has finite rank if $\mathrm{Image}\, T := T(X)$ is finite dimensional.

Proposition 8 *Let X and Y be Banach spaces. Every finite rank operator from X to Y is compact.*

Proposition 9 *Let X and Y be Banach spaces, and $T : X \to Y$ be a compact operator. If $x_n \xrightarrow{w} x$ then $Tx_n \to Tx$.*

Proposition 10 *Let H and K be Hilbert spaces. Then T is compact if and only if T^* is compact.*

Proposition 11 *Let H and K be Hilbert spaces and $T \in \mathcal{B}(H,K)$. Then T is compact if and only if for any sequence $(x_n) \subset H$ converging weakly to x, the sequence (Tx_n) converges (strongly) to Tx in K.*

Problem 97. *Let X be a Banach space. Prove that if $T \in \mathcal{B}(X)$ is arbitrary and $A \in \mathcal{B}_0(X)$, then AT and TA are compact operators. (This is called the two sides ideal property for compact operators).*

Solution.
Suppose (x_n) is a sequence in H such that $\|x_n\| \leq 1$ for every $n \in \mathbb{N}$. Since T is continuous,

$$\|Tx_n\| \leq \|T\| \; \|x_n\| \leq \|T\|, \quad \forall n \in \mathbb{N}.$$

If we set $y_n = \frac{Tx_n}{\|T\|}$, and then we have $\|y_n\| \leq 1$ for every $n \in \mathbb{N}$. Since A is compact, the sequence (Ay_n) has a convergent subsequence. Now we have

$$\|T\| Ay_n = \frac{\|T\| ATx_n}{\|T\|} = ATx_n, \quad \forall n \in \mathbb{N}.$$

It follows that the sequence (ATx_n) also has a convergent subsequence. Thus AT is compact. The similar argument for TA ■

Problem 98.
(a) Let X be a Banach space. Show that the identity $I : X \to X$ is compact if and only if X has finite dimensional.
(b) Let X, Y be Banach spaces and $A \in \mathcal{B}(X, Y)$. Suppose that A has the property:

$$\exists c > 0 : \; \|Ax\| \geq c\|x\|, \; \forall x \in X,$$

Find condition(s) for X so that A can be a compact operator.

Solution.
(a) See Problem 16.

(b) First we note that A is injective. Indeed,

$$Ax = 0 \Rightarrow cx = 0 \Rightarrow x = 0.$$

Let $Z = A(X)$, then $U : X \to Z$ defined by $U(x) = A(x)$ is bijective. Let us consider $U^{-1} : Z \to X$. Clearly U^{-1} is linear. we claim:

$$(*) \; \|U^{-1}(y)\| \leq \frac{1}{c}\|y\|, \; \forall y \in Z$$

Proof: Since $y \in A(X)$, there is an $x \in X$ such that $A(x) = U(x) = y$. This implies that $x = U^{-1}(y)$. By our hypothesis,

$$\|Ax\| = \|U(x)\| = \|y\| \geq c\|x\|.$$

Thus,

$$\|y\| \geq c\|U^{-1}(y)\|.$$

Hence, (*) is proved. It follows that U^{-1} is linear and continuous. If U is a compact operator, then by the ideal property for the compact operators (Problem 97), it follows that $I = U^{-1}U : X \to X$ is compact, which means that X is finite dimensional. Conversely, if X is finite dimensional, then every $A \in \mathcal{B}(X,Y)$ is compact (in a finite dimensional normed space, a set is compact if and only if it is closed and bounded). Hence A is compact if and only if X is finite dimensional. ■

Problem 99.
Let H and K be Hilbert spaces and $A \in \mathcal{B}(H,K)$. Show that A is compact if and only if A^*A is compact.

Solution.
• Suppose $A \in \mathcal{B}(H,K)$ is compact. Let (x_n) be a sequence in X converging weakly to 0. We have

$$\|A^*Ax_n\| \leq \|A^*\| \; \|Ax_n\|.$$

Since A is compact, $Ax_n \to 0$ (strongly) in Y. Thus $A^*Ax_n \to 0$, and so A^*A is compact.

• Reciprocally, suppose A^*A is compact. For any sequence (x_n) such that $x_n \xrightarrow{w} 0$, we have

$$\|Ax_n\|^2 = \langle Ax_n, Ax_n\rangle = \langle x_n, A^*Ax_n\rangle \leq \|A^*Ax_n\| \; \|x_n\|.$$

Since $\|x\|$ is uniformly bounded and A^*A is compact, $A^*Ax_n \to 0$. Therefore, $Ax_n \to 0$, and hence A is compact. ■

Problem 100.
Let X be c_0 or ℓ^p, $1 \leq p \leq \infty$. Consider the operator

$$U : X \to X, \; U(x) = U(x_1, x_2, ...) = (0, x_1, 0, x_3, 0, x_5, ...).$$

Prove that U is not compact but U^2 is compact.

Solution.
We first note that c_0 and ℓ^p, $1 \leq p \leq \infty$ (with appropriate norms) are Banach spaces (see Problems 18,19). We have

$$U^2(x) = U(U(x)) = U(0, x_1, 0, x_3, 0, x_5, ...) = (0, 0, ...).$$

Thus, $U^2 = 0$, therefore U^2 is compact.
On the other hand, if

$$e_n = (\underbrace{0, ..., 0}_{n}, 1, 0, ...) \in X$$

then

$$U(e_{2n-1}) = e_{2n}, \ \forall n \in \mathbb{N}.$$

Now, we have explicitly

$$e_{2n} = (\underbrace{0, ..., 0}_{2n}, 1, 0, ...), \quad e_{2(n+k)} = (\underbrace{0, ..., 0}_{2n}, \underbrace{0, ..., 0}_{2k}, 1, 0, ...),$$

so that

$$e_{2(n+k)} - e_{2n} = (\underbrace{0, ..., 0}_{2n}, \underbrace{-1, 0, ..., 0}_{2k}, 1, 0, ...).$$

For $X = c_0$ or $X = \ell^\infty$ we have

$$\|e_{2(n+k)} - e_{2n}\|_\infty = 1.$$

For $X = \ell^p$, $1 \leq p < \infty$ we have

$$\|e_{2(n+k)} - e_{2n}\|_p = 2^{1/p}.$$

It follows that, in both cases, the sequence $\Big(U(e_{2n-1})\Big)$ cannot have any convergent subsequence. Thus, U is not a compact operator. ■

Problem 101
Let $1 \leq p < \infty$, and $\lambda = (\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ with $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$. We define the multiplication operator

$$M_\lambda : \ell^p \to \ell^p, \ M_\lambda(x) = (\lambda_1 x_1, \lambda_2 x_2, ...), \ x = (x_1, x_2, ...) \in \ell^p.$$

Prove that:
(a) M_λ is continuous and $\|M_\lambda\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.
(b) M_λ is a compact operator if and only if $\lambda \in c_0$.

Solution.
(a) We have

$$|\lambda_n x_n|^p \leq \|\lambda\|_\infty^p |x_n|^p, \ \forall n \in \mathbb{N}.$$

Since the series $\sum_{n=1}^\infty |x_n|^p$ converges and $\|\lambda\|_\infty < \infty$ by hypothesis, the series $\sum_{n=1}^\infty |\lambda_n x_n|^p$ converges. Moreover,

$$\begin{aligned}
\|M_\lambda(x)\| &= \left(\sum_{n=1}^\infty |\lambda_n x_n|^p\right)^{1/p} \\
&\leq \|\lambda\|_\infty \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} \\
&= \|\lambda\|_\infty \|x\|, \ \forall x \in \ell^p.
\end{aligned}$$

This shows that M_λ is continuous and $\|M_\lambda\| \leq \|\lambda\|_\infty$. Also, for any $n \in \mathbb{N}$,

$$|\lambda_n| = |M_\lambda(e_n)| \leq \|M_\lambda\| \ \|e_n\| = \|M_\lambda\|.$$

Here $e_n = (\underbrace{0, ..., 0}_{n-1}, 1, 0, ...) \in \ell^p$. Therefore, $\|\lambda\|_\infty = \sup_{n\in\mathbb{N}} |\lambda_n| \leq \|M_\lambda\|$. Thus, $\|M_\lambda\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.

(b) Suppose M_λ is a compact operator. i.e., $M_\lambda(B_{\ell^p})$ is relatively compact (B_{ℓ^p} the closed unit ball in ℓ^p). Then

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N} : \sum_{k=n_\varepsilon}^\infty |M_\lambda(x)|^p \leq \varepsilon^p, \ \forall x \in B_{\ell^p}.$$

Let $n \geq n_\varepsilon$. Then for $e_n \in B_{\ell^p}$ we have

$$\sum_{k=n_\varepsilon}^\infty |M_\lambda(e_n)|^p \leq \varepsilon^p,$$

that is

$$n \geq n_\varepsilon \Rightarrow |\lambda_n| < \varepsilon.$$

So $\lambda_n \to 0$, that is, $\lambda \in c_0$.
Conversely, if $\lambda_n \to 0$, then

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow |\lambda_n| < \varepsilon.$$

Let $x = (x_1, x_2, ...) \in B_{\ell^p}$, then

$$\sum_{k=n_\varepsilon}^\infty |M_\lambda(x)|^p = \sum_{k=n_\varepsilon}^\infty |\lambda_k x_k|^p \leq \varepsilon^p \sum_{k=n_\varepsilon}^\infty |x_k|^p \leq \varepsilon^p \sum_{k=1}^\infty |x_k|^p \leq \varepsilon^p.$$

Therefore $M_\lambda(B_{\ell^p})$ is relatively compact. ■

Problem 102
Consider the linear operator defined by

$$T : \ell^2 \to \ell^2, \quad x = (\xi_1, \xi_2, \xi_3, ...) \mapsto Tx = \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, ...\right).$$

Show that T is compact.

Solution.
It is clear that T is linear. To show that it is compact, we will show that it is the norm limit of a sequence of compact operators. Let

$$T_n : \ell^2 \to \ell^2, \quad T_n x = \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, ..., \frac{\xi_n}{n}, 0, 0, ...\right).$$

Then T_n is linear, bounded, and of finite rank so compact. Furthermore,

$$\begin{aligned} \|(T - T_n)x\|^2 &= \sum_{i=n+1}^{\infty} \frac{1}{i^2} |\xi_i|^2 \\ &\leq \frac{1}{(n+1)^2} \sum_{i=1}^{\infty} |\xi_i|^2 \\ &= \frac{\|x\|^2}{(n+1)^2}. \end{aligned}$$

Taking the supremum over all x of norm 1, we see that

$$\|T - T_n\| \leq \frac{1}{n+1}.$$

Hence, $T_n \to T$ in norm. Thus, T is compact. ■

Problem 103
Let $(c_j)_{j=1}^{\infty}$ be a sequence of complex numbers. Define an operator D on ℓ^2 by

$$Dx = (c_1 x_1, c_2 x_2, ...), \ x = (x_1, x_2, ...) \in \ell^2.$$

Prove that D is compact if and only if $\lim_{j \to \infty} c_j = 0$.

Solution.

• We note that D is linear. To show that it is compact, we will show that it is the norm limit of a sequence of compact operators. Suppose $\lim_{j\to\infty} c_j = 0$. Define D_n by

$$D_n = (c_1x_1, ..., c_nx_n, 0, 0, ...).$$

We obtain that

$$(D - D_n) = (0, ..., 0.c_{n+1}x_{n+1}, c_{n+2}x_{n+2}, ...)$$

and moreover,

$$\|D - D_n\| = \sup_{j\geq n+1} |c_j| \to 0 \text{ as } n \to \infty.$$

Since each D_n has finite rank and hence is compact, the operator D is compact.

• Assume that (c_j) does not converge to zero as $j \to \infty$. Then, for a given $\varepsilon > 0$, there exists a subsequence (c_{j_k}) such that $|c_{j_k}| \geq \varepsilon$. Consider the sequence of vectors (e_j) of the standard basis. We have $\|e_{j_k}\| = 1$ and for any indices m, k we have

$$\|De_{j_m} - De_{j_k}\|^2 = \|c_{j_m}e_{j_m} - c_{j_k}e_{j_k}\|^2 = |c_{j_m}|^2 + |c_{j_k}|^2 \geq 2\varepsilon^2 > 0.$$

We conclude that the sequence (De_{j_k}) does not contain a convergent subsequence and thus the operator D is not compact. ■

Trick used in problems 102 and 103 is called "cut off " method: From the sequence $x = (x_1, x_2, ...)$ we get the sequence $(x_1, ..., x_n, 0, 0, ...)$ by cutting off the tail of x.

Problem 104
Let $g \in C[0,1]$ be a fixed function. Consider the operator $A \in \mathcal{B}(C[0,1])$ defined by

$$(Au)(s) := g(s)u(s),$$

i.e., the operator of multiplication by g. Is this operator compact?

Solution.

Note first that $C[0,1]$, equipped with the sup-norm, is a Banach space.
It is clear that if $g \equiv 0$ then A is compact. Let us prove that if g is not identically zero then A is not compact. Indeed, since g is not identically zero, there exists a subinterval $[a,b] \subset [0,1]$ such that

$$m := \min_{s\in[a,b]} |g(s)| > 0.$$

Consider the sequence (u_n):

$$u_n \in C[0,1];\ u_n(s) := \sin\left(2^n \frac{s-a}{b-a}\,\pi\right);\ s \in [0,1],\ n \in \mathbb{N}.$$

It is clear that (u_n) is a bounded sequence. On the other hand, (Au_n) does not have Cauchy subsequences. Indeed, take arbitrary $k, n \in \mathbb{N}$ with $k > n$. Let

$$s_n := a + \frac{1}{2^{n+1}}\,(b-a)$$

Then $s_n \in [a,b]$ and

$$\begin{aligned}
\|Au_k - Au_n\| &= \max_{s\in[a,b]} |g(s)(u_k(s) - u_n(s))| \\
&\geq m \max_{s\in[a,b]} |u_k(s) - u_n(s)| \\
&\geq m|u_k(s_n) - u_n(s_n)| \\
&= m|\sin(2^{k-n-1}\pi) - \sin(\pi/2)| \\
&= m|0-1| = m > 0.
\end{aligned}$$

Hence (Au_n) cannot have any convergent subsequence, A is not compact. ■

Problem 105
Given $k \in L^2([0,1]\times[0,1])$, define the operator $A : L^2([0,1]) \to L^2([0,1])$ by

$$(Af)(x) = \int_0^1 k(x,y)f(y)dy.$$

(a) Show that A is bounded.
(b) Under what condition on k, the operator A is self-adjoint.
(c) Show that A is compact.

Solution.
(Look at Problem 92! They are different!)
(a) We estimate $\|A\|$ to see A is bounded.

$$\begin{aligned}
\|Af\|^2 &= \int_0^1 \left|\int_0^1 k(x,y)f(y)dy\right|^2 dx \\
&\leq \int_0^1 \left(\int_0^1 |k(x,y)|^2 dy\right) dx . \int_0^1 |f(y)|^2 dy \quad \text{(Cauchy-Schwarz)} \\
&\leq \|f\|^2 . \int_0^1 \int_0^1 |k(x,y)|^2 dydx.
\end{aligned}$$

Since $k \in L^2([0,1] \times [0,1])$, $\int_0^1 \int_0^1 |k(x,y)|^2 dydx < \infty$; hence,

$$\|A\| \leq \left(\int_0^1 \int_0^1 |k(x,y)|^2 dydx \right)^{1/2} < \infty.$$

Thus A is bounded.

(b) We have

$$(Af)(x) = \int_0^1 k(x,y) f(y) dy.$$
$$(A^* g)(x) = \int_0^1 \overline{k(x,y)} g(y) dy.$$

Therefore,

$$\begin{aligned} \langle Af, g \rangle &= \int_0^1 \int_0^1 k(x,y) f(y) dy \, \overline{g(x)} \, dx \\ &= \int_0^1 f(y) \overline{\int_0^1 \overline{k(x,y)} \, g(x) \, dxdy} \\ &= \langle f, A^* g \rangle. \end{aligned}$$

Hence, A is self-adjoint if

$$k(x,y) = \overline{k(y,x)}.$$

(c) Let $(u_j)_{j=1}^{\infty}$ be an orthonormal basis in $L^2[0,1]$. Then

$$k(x,y) = \sum_{j=1}^{\infty} k_j(y) u_j(x), \quad \text{where} \quad k_j(y) = \int_0^1 k(x,y) \overline{u_j(x)} \, dx,$$

for almost all y. Due to the Parseval identity, we have, for almost all y

$$\int_0^1 |k(x,y)|^2 dx = \sum_{j=1}^{\infty} |k_j(y)|^2,$$

and

$$(1) \qquad \int_0^1 \int_0^1 |k_j(y)|^2 dxdy = \sum_{j=1}^{\infty} \int_0^1 |k_j(y)|^2 dy.$$

We now define the following operator of rank N

$$k_N f(x) = \int_0^1 k_N(x,y) f(y) dy,$$

where $k_N(x,y)=\sum_{j=1}^N k_j(y)u_j(x)$. By Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\|(A-k_N)f\|^2 &= \int_0^1 \left|\int_0^1 (k(x,y)-k_N(x,y))f(y)dy\right|^2 dx\\
&\le \left(\int_0^1\int_0^1 |k(x,y)-k_N(x,y)|^2 dxdy\right)\left(\int_0^1 |f(y)|^2 dy\right).\\
&\le \|f\|^2\left(\int_0^1\int_0^1 |k(x,y)-k_N(x,y)|^2 dxdy\right).
\end{aligned}$$

Thus by using that the right hand side in (1) is absolutely convergent, we find

$$\begin{aligned}
\|(A-k_N)\|^2 &\le \int_0^1\int_0^1 |k(x,y)-k_N(x,y)|^2 dxdy\\
&= \int_0^1\int_0^1 |k(x,y)|^2 dxdy - \int_0^1\int_0^1 k(x,y)\sum_{j=1}^N \overline{k_j(y)u_j(x)}\, dxdy\\
&- \int_0^1\int_0^1 \overline{k(x,y)}\sum_{j=1}^N k_j(y)u_j(x)dxdy + \sum_{j=1}^N\int_0^1 |k_j(y)|^2 dy\\
&= \int_0^1\int_0^1 |k(x,y)|^2 dxdy - \sum_{j=1}^N |k_j(y)|^2 dy \to 0 \quad \text{as} \quad N\to\infty. \quad \blacksquare
\end{aligned}$$

Problem 106
Part I
Consider the operator

$$U: C[0,1]\to C[0,1] \quad \textit{defined by} \quad (Uf)(x)=\int_0^x e^t f(t)dt,\ x\in[0,1],$$

and the sequence of operators

$$U_n: C[0,1]\to C[0,1] \quad \textit{defined by} \quad (U_nf)(x)=\int_0^x \left(\sum_{k=0}^n \frac{t^k}{k!}\right) f(t)dt,\ x\in[0,1].$$

Prove that $\lim_{n\to\infty}\|U_n-U\|=0$.

Part II
1. Let M be a set of C^1-functions f on $[0,1]$. Prove that M is relatively compact in $C[0,1]$ if f satisfies following conditions

$$|f(0)|\le k_1 \quad \textit{and} \quad \int_0^1 |f'(x)|^2 dx \le k_2$$

where k_1, k_2 are positive constants. (Hint: Use Arzela-Ascoli theorem).
2. Show that the operator U in Part I is compact.

Solution.
Part I
From Calculus we know that if $\phi \in C[0,1]$ then $x \mapsto \int_0^x \phi(t)dt$ is continuous. Hence, U, U_n take their values in $C[0,1]$. Using the Taylor expansion $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ we obtain

$$(Uf - U_n f)(x) = \int_0^x \left(\sum_{k=n+1}^{\infty} \frac{t^k}{k!} \right) f(t)dt, \ \forall x \in [0,1].$$

Then

$$\begin{aligned} |(Uf - U_n f)(x)| &\leq \int_0^x \left(\sum_{k=n+1}^{\infty} \frac{t^k}{k!} \right) |f(t)|dt \\ &\leq \|f\|_\infty \int_0^1 \left(\sum_{k=n+1}^{\infty} \frac{t^k}{k!} \right) dt, \ \forall x \in [0,1]. \end{aligned}$$

Thus,

$$\begin{aligned} \|U_n - U\| &\leq \|f\|_\infty \int_0^1 \left(\sum_{k=n+1}^{\infty} \frac{t^k}{k!} \right) dt, \ \forall f \in C[0,1], \\ &\leq \int_0^1 \left(\sum_{k=n+1}^{\infty} \frac{t^k}{k!} \right) dt. \end{aligned}$$

But if $u_n(t) = \sum_{k=n+1}^{\infty} \frac{t^k}{k!}$ then $u_n : [0,1] \to \mathbb{R}$ and

$$|u_n(t)| \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} \to 0,$$

that is, $u_n \to 0$ uniformly on $[0,1]$. Hence, $\int_0^1 u_n(t)dt \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \|U_n - U\| = 0.$$

Part II

1. For all $x, y \in [0,1]$ with $x < y$, and all $f \in M$, we have

$$\begin{aligned}
|f(x) - f(y)| &= \left| \int_x^y |f'(x)| \right| dt \\
&\leq \left(\int_x^y |f'(x)|^2 dt \right)^{1/2} \left(\int_x^y 1 dt \right)^{1/2} \\
&\leq \sqrt{y-x} \left(\int_x^y |f'(x)|^2 dt \right)^{1/2} \\
&\leq \sqrt{k_2}\, \sqrt{y-x}.
\end{aligned}$$

This shows that M is qui-continuous.
Now for all $x \in M$, and all $f \in M$, we have

$$\begin{aligned}
|f(x)| &\leq |f(x) - f(0)| + |f(0)| \\
&\leq k_1 + \sqrt{k_2}.
\end{aligned}$$

So M is uniformly bounded. Thus, by Arzela-Ascoli Theorem[1], M is relatively compact in $C[0,1]$.

2. For $\|f\| \leq 1$, let $g(x) = \int_0^1 e^{tx} f(t) dt, \ \forall x \in [0,1]$. Using the differentiation theorem for the Riemann integral with parameter, we have

$$g'(x) = \int_0^1 \frac{\partial}{\partial x}(e^{tx} f(t)) dt = \int_0^1 t e^{tx} f(t) dt,$$
$$|g'(x)| \leq \int_0^1 t e^{tx} |f(t)| dt \leq \int_0^1 t e^{tx} dt \leq e, \ \forall x \in [0,1].$$

It follows that the set of all functions g is uniformly Lipschitz. We also have

$$|g(0)| = \left| \int_0^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt \leq \|f\| \leq 1, \ \forall g.$$

From previous question, it follows that $A = \{Uf : \ \|f\| \leq 1\}$ is relatively compact. Thus the operator U is compact. ■

[1]Arzela-Ascoli Theorem: Let (X, d) be a compact metric space and $A \subset C(X)$. Then the following assertions are equivalent:

1. A is relatively compact.
2. A is uniformly bounded and equi-continuous,
3. Any sequence $(f_n) \subset A$ contains a uniformly convergent subsequence.

Problem 107
(a) Let X be an infinite dimensional Banach space, and A be a compact operator on X. Prove that there is $y \in X$ such that the equation $A(x) = y$ has no solution, i.e., A is not surjective.
(b) Let $1 \leq p < \infty$, and the operator

$$U : \ell^p \to \ell^p, \; U(x) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...\right), \; x = (x_1, x_2, ...) \in \ell^p.$$

Find an element $y \in \ell^p$ for which the equation $U(x) = y$ has no solution.
(c) Consider the operator

$$A : c_{00} \to c_{00} \quad \text{defined by} \quad A(x) = A(x_1, x_2, ...) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...\right).$$

Prove that A is compact and bijective. On c_{00} we have the ℓ^∞- norm.

Solution.
(a) Let us suppose, for a contradiction, that A is surjective. From the open mapping theorem it follows that A is an open operator, in particular $A(B(0;1)) \subset X$ is an open set, i.e.,

$$\exists \varepsilon > 0 : \; \overline{B}(0; \varepsilon) = \varepsilon \overline{B}(0; 1) \subset A(B(0; 1)),$$

where $B(0;1) = \{x \in X : \; \|x\| < 1\}$. Since A is compact it follows that $A(B(0;1))$ is relatively compact. Hence, $\overline{B}(0;\varepsilon)$ is relatively compact, from whence $\overline{B}(0;1)$ is relatively compact, therefore compact. But if $\overline{B}(0;1)$ is compact, then, by Problem 16, X must be finite dimensional, which is a contradiction. Thus, A is not surjective.
(b) Choose $y = \left(1, \frac{1}{2^\alpha}, \frac{1}{3^\alpha}, ...\right) \in \ell^p$. If $x = (x_1, x_2, ...) \in \ell^p$ has the property that $U(x) = y$, then

$$\frac{1}{n} x_n = \frac{1}{n^\alpha}, \; \forall n \in \mathbb{N},$$

that is,

$$x_n = \frac{1}{n^{\alpha - 1}}, \; \forall n \in \mathbb{N}.$$

Since $x = (x_1, x_2, ...) \in \ell^p$, the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^{(\alpha-1)p}}$ converges[2], therefore, $(\alpha - 1)p > 1 \Leftrightarrow \alpha > 1 + \frac{1}{p}$. From here, it follows that for

$$y = \left(1, \frac{1}{2^{\frac{1}{p+1}}}, \frac{1}{3^{\frac{1}{p+1}}}, ...\right),$$

the equation $U(x) = y$ has no solution.

(c) For $n \in \mathbb{N}$, consider

$$A_n : c_{00} \to c_{00},\ A_n(x) = A_n(x_1, x_2, ...) = \left(x_1, \frac{x_2}{2}, .., \frac{x_n}{n}, 0, ...\right).$$

Then

$$\|A - A_n\| = \frac{1}{n+1},\ \forall n \in \mathbb{N}.$$

Therefore $A_n \to A$ in norm. Since every A_n is a finite rank operator, so A_n is compact. The sequence of compact operators (A_n) converges to A in norm, so A must be compact. The fact that A is bijective is obvious from the expression which defines A. $\blacksquare$

[2] The generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha p}}$ converges if and only if

$$\alpha p > 1 \Leftrightarrow \alpha > \frac{1}{p}.$$

Chapter 8

Bounded Operators on Banach Spaces and Their Spectra

Review:

1. Definitions

Let X be a Banach space and $T \in \mathcal{B}(X),\ \lambda \in \mathbb{C}$.

• *Resolvent and spectrum of T:*
Set $T_\lambda = T - \lambda I$. The set $\rho(T)$ of all λ such that T_λ has an inverse $R_\lambda(T) = (T - \lambda I)^{-1}$ is called the resolvent of T. The set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the spectrum of T.

• *Eigenvalues and eigenvectors of T:*
An $x \neq 0$ which satisfies $Tx = \lambda x$ for some λ is called an eigenvector of T. The corresponding λ is an eigenvalue of T. It is evident that $\lambda \in \sigma(T)$.

2. Basic properties

Theorem 12 *(Spectrum)*
If T is a bounded linear operator on a Banach space X, then its spectrum $\sigma(T)$ is compact and lies in the disk given by

$$|\lambda| \leq \|T\|.$$

Theorem 13 *(Resolvent equation)*
Let $T \in B(X, X)$ where X is a Banach space. Then

1. *The resolvent $R_\lambda(T)$ satisfies the following equation called the resolvent equation*

$$R_\mu - R_\lambda = (\mu - \lambda)R_\mu R_\lambda \quad \textit{for} \quad \mu, \lambda \in \rho(T).$$

2. *R_λ commutes with any $S \in B(X, X)$ which commutes with T.*

3. *We have*

$$R_\mu R_\lambda = R_\lambda R_\mu \quad \textit{for} \quad \mu, \lambda \in \rho(T).$$

3. Classification of spectrum

- $\lambda \in \mathbb{C}$ is called a regular point of $A \in \mathcal{B}(X)$ iff $(A - \lambda I)^{-1}$ exists and is bounded.
- If λ is not a regular point, it is called a spectrum point. All such points form the spectrum $\sigma(A)$.
- Every $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$ is a regular point.
- Classification of spectrum:
  1. The point spectrum: $\sigma_p(A)$ is the set of eigenvalues of A.
  2. The continuous spectrum: $\lambda \in \sigma_c(A)$ iff $\lambda \in \sigma(A) \setminus \sigma_p(A)$ and Image$(A - \lambda I)$ is dense in X.
  3. The residual spectrum: $\sigma_r(A) = \sigma(A) \setminus (\sigma_p(A) \cup \sigma_c(A))$. If $\lambda \in \sigma_r(A)$ then
  $$\overline{\text{Image}(A - \lambda I)} \neq X \quad \text{and} \quad \ker(A - \lambda I) = 0$$
  .

4. Spectral radius

Definition 9 *Let $T \in B(X, X)$ where X is a Banach space. The spectral radius $r_\sigma(T)$ of T is the radius of the smallest closed disk centered at the orgin and containing $\sigma(T)$.*

$$r_\sigma(T) := \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Formula for spectral radius:
It can be shown that

$$r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}.$$

5. Spectral mapping theorem

Theorem 14 *(Spectral theorem for polynomials)*
Let $T \in B(X, X)$ where X is a Banach space, and

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0, \ \alpha_n \neq 0.$$

Then

$$\sigma(p(T)) = p(\sigma(T)),$$

that is, the spectrum of the operator

$$p(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \ldots + \alpha_0 I$$

consists precisely of all those values which the polynomial p assumes on the spectrum $\sigma(T)$ of T.

* * * * *

Problem 108
Let X be a Banach space. Suppose that $A \in \mathcal{B}(X)$ is an invertible operator. Show that

$$\sigma(A^{-1}) = \{\lambda^{-1} : \ \lambda \neq 0, \ \lambda \in \sigma(A)\}.$$

Solution.
For $\lambda \neq 0$, we can write

$$A^{-1} - \lambda^{-1} I = (\lambda I - A)\lambda^{-1} A^{-1}.$$

From this equality we conclude that $A^{-1} - \lambda^{-1} I$ is invertible if and only if $A - \lambda I$ is invertible. Hence, we have

$$\sigma(A^{-1}) = \{\lambda^{-1} : \ \lambda \in \sigma(A)\}. \quad \blacksquare$$

Problem 109
Let X be a Banach space, let $A \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$. Assume that there exists a sequence (x_n) in X such that

$$\|x_n\| = 1, \ \forall n \in \mathbb{N} \ \text{ and } \ Ax_n - \lambda x_n \to 0 \ \text{ as } \ n \to \infty.$$

Prove that $\lambda \in \sigma(A)$.

Solution.
Assume that $A - \lambda I$ is invertible. Then, there exists a number $c > 0$ such that (see problem 75)

$$\|(A - \lambda I)x\| \geq c\|x\|, \ \forall x \in X.$$

Replace x by x_n with $\|x_n\| = 1$ for all n, we have

$$\|Ax_n - \lambda x_n\| = \|(A - \lambda I)x_n\| \geq c\|x_n\| = c.$$

This contradicts the condition in the statement of the problem. $\blacksquare$

Problem 110
Let (a_n) and (b_n) be complex sequences such that

$$|a_{n-1}| > |a_n| \xrightarrow{(n\to\infty)} 0 \quad \text{and} \quad |b_{n-1}| > |b_n| \xrightarrow{(n\to\infty)} 0.$$

Consider the operator $T : \ell^2 \to \ell^2$ defined by

$$Tx = (a_1x_1, a_2x_2 + b_1x_1, a_3x_3 + b_2x_2, ...), \quad x = (x_j) \in \ell^2.$$

(a) Show that T is compact.
(b) Find all eigenvalues and eigenvectors of T.

Solution.
(a) Let the sequence of operators $T_n : \ell^2 \to \ell^2, \quad n = 1, 2, ...$ be defined by for any $x \in \ell^2$:

$$(T_nx)_j = \begin{cases} (Tx)_j & \text{if } 1 \le j \le n \\ 0 & \text{if } j > n. \end{cases}$$

All the T_n's are operators of finite rank and hence compact. Moreover, we have

$$\|T_nx - Tx\| \le (|a_{n+1}| + |b_n|)\|x\|,$$

which implies that

$$\|T_n - T\| \to 0 \text{ as } n \to \infty.$$

But a uniform limit of a sequence of compact operators is compact. Hence, T is compact.
(b) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T and that $x \neq 0$ is the corresponding eigenvector. Then

$$0 = Tx - \lambda x = \big((a_1 - \lambda)x_1, (a_2 - \lambda)x_2 + b_1x_1, ..., (a_n - \lambda)x_n + b_{n-1}x_{n-1}, ...\big).$$

If λ coincides with none of the a_n's, then $x = 0$: impossible. So it is necessary that $\lambda = a_n$ for some n. In this case, x_n can be chosen arbitrary, and $x_1 = ... = x_{n-1} = 0$, and for $k = 1, 2, ...$ we have $0 = (a_{n+k} - \lambda)x_{n+k} + b_{n+k-1}x_{n+k-1}$. If we choose $x_n = 1$ then we get

$$x_{n+k} = \frac{b_{n+k-1}}{\lambda - a_{n+k}} \frac{b_{n+k-2}}{\lambda - a_{n+k-1}} ... \frac{b_n}{\lambda - a_{n+1}}, \quad k = 1, 2, ...$$

Thus, for any n, $\lambda = a_n$ is a simple eigenvalue. The corresponding eigenvector is $x = (0, ..0, x_n, x_{n+1}, ...)$ defined as above. ■

Problem 111
Let X be a Banach space and let $A \in \mathcal{B}(X)$ such that $A^n = 0$ for some $n \in \mathbb{N}$ (A is nilpotent). Find $\sigma(A)$.

Solution.
The spectral mapping theorem implies

$$\{\lambda^n : \ \lambda \in \sigma(A)\} = \sigma(A^n) = \sigma(0) = \{0\}.$$

Therefore,

$$\lambda \in \sigma(A) \Leftrightarrow \lambda^n = 0 \Leftrightarrow \lambda = 0.$$

Thus, $\sigma(A) = \{0\}$. ■

Problem 112
Let $P \in \mathcal{B}(X)$ be a projection, i.e., a linear operator on X such that $P^2 = P$. Construct the resolvent $R(P;\lambda)$ of P

Solution.
If $P = 0$, then obviously

$$P - \lambda I = -\lambda I; \ \ \sigma(P) = \{0\}; \ \ R(P;\lambda) := (P - \lambda I)^{-1} = -\lambda^{-1} I.$$

If $P = I$, then

$$P - \lambda I = (1-\lambda) I; \ \ \sigma(P) = \{1\}; \ \ R(P;\lambda) := (P - \lambda I)^{-1} = (1-\lambda)^{-1} I.$$

Suppose P is non- trivial, i.e., $P \neq 0, I$. Take any $\lambda \neq 0, 1$. Then using the equalities $P^2 = P$; $Q^2 = Q$ and $QP = PQ$ where $Q = I - P$, we obtain

$$\begin{aligned} \Big((1-\lambda)^{-1}P - \lambda^{-1}Q\Big)(P - \lambda I) &= \Big((1-\lambda)^{-1}P - \lambda^{-1}Q\Big)((1-\lambda)P - \lambda Q) \\ &= P + Q = I. \end{aligned}$$

Similarly, we have

$$(P - \lambda I)\Big((1-\lambda)^{-1}P - \lambda^{-1}Q\Big) = I.$$

Thus,

$$R(P;\lambda) = (1-\lambda)^{-1}P - \lambda^{-1}Q = \lambda^{-1}\Big((1-\lambda)^{-1}P - I\Big). \quad ■$$

Problem 113
Let $C_{\mathbb{R}}$ be the space of all continuous and bounded functions $x(t)$ on $\mathbb{R}$ with norm $\|x\| = \sup_{\mathbb{R}} |x(t)|$. On the space $C_{\mathbb{R}}$ we define the operator A by

$$(Ax)(t) = x(t+c),$$

where $c \in \mathbb{R}$ is a constant. Prove that

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}.$$

Solution.
Notice that

$$\|Ax\| = \sup_{t\in\mathbb{R}} |x(t+c)| = \sup_{\tau\in\mathbb{R}} |x(\tau)| = \|x\|.$$

It follows that $\|A\| = 1$ and therefore all of the point of $\{\lambda \in \mathbb{C} : \ |\lambda| > 1\}$ are regular points of A. The operator A is invertible since the operator defined by

$$(A^{-1}x)(t) = x(t-c)$$

is bounded and is the inverse of A. Next $\|A^{-1}\| = 1$ and hence all of the point of $\{\lambda \in \mathbb{C} : \ |\lambda| > 1\}$ are regular points of A^{-1}. From Problem 65 we deduce that all of the points $\{\lambda \in \mathbb{C} : \ |\lambda| < 1\}$ are regular points of A.
Consider $|\lambda| = 1$. This means that

$$\lambda = e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi.$$

Set $a = \frac{i\varphi}{c}$ and $x_a(t) = e^{at}$. We obtain $x_a \in C_{\mathbb{R}}$ and

$$(Ax_a)(t) = e^{a(t+c)} = e^{at}e^{ac} = \lambda x_a.$$

This means that λ is an eigenvalue of A. Thus,

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}. \quad \blacksquare$$

Problem 114
(a) Let H be a Hilbert space and $a, b \in H$. Consider the rank-one operator

$$U : H \to H, \quad U(x) = \langle x, a\rangle b, \ x \in H.$$

Calculate $\|U\|$ and the spectral radius $r(U)$. Show that

$$r(U) = \|U\| \;\Leftrightarrow\; a, b \;\; \text{are linearly independent.}$$

(b) Let X be a Banach space, $x^* \in X^*$ and $y \in X$. Consider the rank-one operator

$$V : X \to X, \;\; V(x) = x^*(x)y, \; x \in X.$$

Calculate $\|V\|$ and the spectral radius $r(V)$. Show that

$$r(V) = \|V\| \;\Leftrightarrow\; |x^*(y)| = \|x^*\| \; \|y\|.$$

Solution.
(a) We have

$$\|U(x)\| = |\langle x, a\rangle| \; \|b\| \le \|a\| \; \|b\| \; \|x\|, \; \forall x \in H.$$

Hence, $\|U\| \le \|a\| \; \|b\|$. We also have

$$\|U\| \; \|a\| \ge \|U(a)\| = |\langle a, a\rangle| \; \|b\| = \|a\|^2 \; \|b\|.$$

Therefore, $\|U\| \ge \|a\| \; \|b\|$. Thus,

$$\|U\| = \|a\| \; \|b\|.$$

For $x \in H$, let $y = U(x)$. Then

$$U^2(x) = U(y) = \langle y, a\rangle b, \quad \text{where} \;\; \langle y, a\rangle = \langle Ux, a\rangle = \langle x, a\rangle \overline{\langle a, b\rangle}.$$

Denoting $\lambda = \overline{\langle a, b\rangle}$, we have

$$U^2(x) = \lambda \langle x, a\rangle b = \lambda U(x), \; \forall x \in H.$$

Therefore

$$U^2 = \lambda U.$$

From here, by induction, we get

$$U^n = \lambda^{n-1} U, \; \forall n \in \mathbb{N}.$$

Now

$$r(U) = \lim_{n\to\infty} (\|U\|^n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(|\lambda|^{\frac{n-1}{n}} \|U\|^{\frac{1}{n}} \right) = |\lambda| = |\langle a, b\rangle|.$$

The last assertion is a consequence of the fact proved above and the fact that in the Cauchy-Schwarz inequality we have equality if and only if $\{a, b\}$ are linearly independent.

$$r(U) = \|U\| \;\Leftrightarrow \|a\| \, \|b\| = |\langle a, b\rangle| \Leftrightarrow \; a, b \; \text{ are linearly independent.}$$

(b) We have

$$\|V(x)\| = |x^*(x)| \, \|y\| \leq \|x^*\| \, \|y\| \, \|x\|, \; \forall x \in X.$$

Therefore, $\|V\| \leq \|x^*\| \, \|y\|$.
If $y = 0$ then $\|V\| = 0$. Suppose $y \neq 0$. For $x \in X$, we have

$$|x^*(x)| \, \|y\| = \|V(x)\| \leq \|V\| \, \|x\|.$$

Then

$$\begin{aligned} |x^*(x)| \leq \frac{\|V\|}{\|y\|}\|x\| \quad &\Rightarrow \quad \|x^*\| \leq \frac{\|V\|}{\|y\|} \\ &\Rightarrow \quad \|V\| \geq \|x^*\| \, \|y\|. \end{aligned}$$

Thus,

$$\|V\| = \|x^*\| \, \|y\|.$$

To calculate $r(V)$ we use the same procedure as in (a). For $x \in H$, let $z = V(x)$. Then

$$\begin{aligned} &V^2(x) = V(z) = x^*(z)y \\ &z = V(x) = x^*(x)y, \\ &x^*(z) = x^*(x)x^*(y). \end{aligned}$$

Therefore,

$$V^2(x) = x^*(y)x^*(x)y = \lambda x^*(x)y, \quad \lambda = x^*(y).$$

Thus, $V^2 = \lambda V$. And from here, by induction, we get

$$V^n = \lambda^{n-1}V, \; \forall n \in \mathbb{N}.$$

And as above we obtain

$$r(V) = |\lambda| = |x^*(y)|. \qquad \blacksquare$$

Problem 115
Let $k \in C([0,1] \times [0,1])$ be a given function. Consider the operator

$$B \in \mathcal{B}(C[0,1]) \quad \textit{defined by} \;\; (Bu)(s) = \int_0^s k(s,t)u(t)dt.$$

Find $\sigma(B)$ and $r(B)$.

Solution.
Let us prove by induction that

$$(*) \qquad |(B^n u)(s)| \leq \frac{M^n}{n!}\, s^n \|u\|_\infty,\ \forall s \in [0,1],\ \forall n \in \{0,1,2,...\}$$

where

$$M := \max_{(s,t)\in[0,1]^2} |k(s,t)|.$$

For $n = 0$, then $B^0 = I$, and (*) is trivial. Suppose (*) holds for $n = k$. Then for $n = k+1$ we have

$$\begin{aligned}
\left|(B^{k+1}u)(s)\right| &= \left|\int_0^s k(s,t)(B^k u)(t)dt\right| \\
&\leq \int_0^s |k(s,t)|\ \left|(B^k u)(t)\right| dt \\
&\leq M \int_0^s \left|(B^k u)(t)\right| dt \\
&\leq M \int_0^s \frac{M^k}{k!}\, t^k \|u\|_\infty dt \\
&= \frac{M^{k+1}}{k!}\, \|u\|_\infty \int_0^s t^k dt \\
&= \frac{M^{k+1}}{(k+1)!}\, s^{k+1}\, \|u\|_\infty,\ \forall s \in [0,1].
\end{aligned}$$

Hence, (*) is proved by induction.
It follows from (*) that

$$\|B^n u\|_\infty \leq \frac{M^n}{n!}\, \|u\|_\infty,\ \forall u \in C[0,1],$$

i.e.,

$$\|B^n\| \leq \frac{M^n}{n!},\ \forall n \in \{0,1,2,...\}.$$

Therefore

$$r(B) = \lim_{n\to\infty} \|B^n\|^{1/n} \leq \lim_{n\to\infty} \frac{M}{(n!)^{1/n}} = 0.$$

Since $r(B) = 0$, $\sigma(B)$ cannot contain nonzero elements. Taking into account that $\sigma(B)$ is nonempty, we conclude that $\sigma(B) = \{0\}$. ■

Problem 116
Let X be a Banach space and $A, B \in \mathcal{B}(X)$. Suppose $AB = BA$. Prove that

$$r(A+B) \leq r(A) + r(B),$$

where $r(T)$ is the spectral radius of an operator $T \in \mathcal{B}(X)$.

Solution.
Recall

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} \text{ and } r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Take an arbitrary $\varepsilon > 0$. The spectral radius formula implies that

$$\|A^n\| \leq \big(r(A) + \varepsilon\big)^n, \quad \|B^n\| \leq \big(r(B) + \varepsilon\big)^n$$

for sufficiently large $n \in \mathbb{N}$. Therefore there exists a constant $M \geq 1$ such that

$$\|A^n\| \leq M\big(r(A) + \varepsilon\big)^n, \quad \|B^n\| \leq M\big(r(B) + \varepsilon\big)^n, \ \forall n \in \mathbb{N}.$$

Since $AB = BA$, we have

$$(A+B)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \ A^{n-k}B^k.$$

Hence

$$\begin{aligned} \|(A+B)^n\| &\leq \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \ \|A^{n-k}\| \ \|B^k\| \\ &\leq M^2 \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \ \big(r(A) + \varepsilon)^{n-k}(r(B) + \varepsilon\big)^k \\ &= M^2\big(r(A) + r(B) + 2\varepsilon\big)^n, \ \forall n \in \mathbb{N}. \end{aligned}$$

Consequently,

$$r(A+B) = \lim_{n\to\infty} \|(A+B)^n\|^{1/n} \leq r(A) + r(B) + 2\varepsilon, \ \forall \varepsilon > 0.$$

This implies that

$$r(A+B) \leq r(A) + r(B). \quad \blacksquare$$

Problem 117
Let $K \subset \mathbb{C}$ be an arbitrary non-empty compact set. Construct an operator $B \in \mathcal{B}(\ell^p)$, $1 \leq p \leq \infty$, such that $\sigma(B) = K$.

Solution.
Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a dense subset of K. (Recall that every metric compact space is separable, that is, it contains a countable dense subset.) Consider the operator $B : \ell^p \to \ell^p$ defined by

$$Bx = (\lambda_1 x_1, \lambda_2 x_2, ...), \ \forall x = (x_1, x_2, ...) \in \ell^p.$$

Then B is a bounded operator and λ_k's are its eigenvalues. Consequently,

$$\{\lambda_k\}_{k\in\mathbb{N}} \subset \sigma(B).$$

Since $\sigma(B)$ is closed and $\{\lambda_k\}_{k\in\mathbb{N}}$ is dense in K,

$$K \subset \sigma(B).$$

On the other hand, let $\lambda \in \mathbb{C} \setminus K$. Then $d := \inf_{k\in\mathbb{N}} |\lambda_k - \lambda| > 0$ and $B - \lambda I$ has a bounded inverse $(B - \lambda I)^{-1} : \ell^p \to \ell^p$ defined by

$$(B - \lambda I)^{-1} x = \left(\frac{1}{\lambda_1 - \lambda}\, x_1, ..., \frac{1}{\lambda_k - \lambda}\, x_k, ... \right), \ \forall x = (x_1, x_2, ...) \in \ell^p.$$

Hence, $\lambda \notin \sigma(B)$. Therefore, $\sigma(B) \subset K$. Finally,

$$\sigma(B) = K. \quad \blacksquare$$

Problem 118
Let $g \in C[0,1]$ be a fixed function and $A \in \mathcal{B}(C[0,1])$ be defined by

$$(Af)(t) = g(t)f(t), \ t \in [0,1].$$

Find $\sigma(A)$ and construct effectively the resolvent $R(A;\lambda)$. Find the eigenvalues and eigenvectors of A.

Solution.
Let $\lambda \in \mathbb{C},\ \lambda \notin g([0,1]) := \{g(t);\ t \in [0,1]\}$. Then, since $g \in C[0,1]$,

$$\frac{1}{g-\lambda} \in C[0,1]$$

and $A - \lambda I$ has an inverse

$$R(A;\lambda) = (A-\lambda I)^{-1} \in \mathcal{B}(C[0,1])$$

defined by

$$R(A;\lambda)f(t) = (g(t)-\lambda)^{-1}f(t),\ t \in [0,1].$$

Hence, $\sigma(A) \subset g[0,1]$.
Suppose now $\lambda \in g([0,1])$, i.e., $\lambda = g(t_0)$ for some $t_0 \in [0,1]$. Then

$$(A-\lambda I)f(t_0) = (g(t_0)-\lambda)f(t_0) = 0,$$

i.e., Image$(A-\lambda I)$ consists of functions vanishing at t_0.
Consequently, Image$(A-\lambda I) \neq C[0,1]$ and $A-\lambda I$ is not invertible. Therefore, $g([0,1]) \subset \sigma(A)$. Finally,

$$\sigma(A) = g([0,1]).$$

Take an arbitrary $\lambda \in g([0,1])$. Let $g^{-1}(\lambda) := \{\tau \in [0,1]:\ g(\tau) = \lambda\}$. The equation $Af = \lambda f$, i.e., $(g(t)-\lambda)f(t) = 0$ is equivalent to $f(t) = 0,\ \forall t \in [0,1] \setminus g^{-1}(\lambda)$. If $g^{-1}(\lambda)$ contains an interval of positive length, then it is easy to see that the set

$$\{f \in C[0,1] \setminus \{0\}:\ f(t) = 0,\ \forall t \in [0,1] \setminus g^{-1}(\lambda)\}$$

is non-empty and coincides with the set of all eigenvectors corresponding to the eigenvalues λ. If $g^{-1}(\lambda)$ does not contain an interval of positive length, then $[0,1] \setminus g^{-1}(\lambda)$ is dense in $[0,1]$ and $f(t) = 0,\ \forall t \in [0,1] \setminus g^{-1}(\lambda)$ implies by continuity that $f \equiv 0$. In this case λ is not an eigenvalue. ■

Problem 119
Let X be a Banach space and $A, B \in \mathcal{B}(X)$. Show that for any $\lambda \in \rho(A) \cap \rho(B)$,

$$R(B;\lambda) - R(A;\lambda) = R(B;\lambda)(A-B)R(A;\lambda).$$

Solution.

$$\begin{aligned}
R(B;\lambda)(A-B)R(A;\lambda) &= R(B;\lambda)\big[(A-\lambda I)-(B-\lambda I)\big]R(A;\lambda)\\
&= \big[R(B;\lambda)(A-\lambda I)-R(B;\lambda)(B-\lambda I)\big]R(A;\lambda)\\
&= R(B;\lambda)(A-\lambda I)R(A;\lambda)-R(B;\lambda)(B-\lambda I)R(A;\lambda)\\
&= R(B;\lambda)-R(A;\lambda). \quad \blacksquare
\end{aligned}$$

Problem 120
Let $k \in C([0,1]\times[0,1])$ be given. Consider the operator $B \in \mathcal{B}(C[0,1])$ defined by

$$(Bu)(s) = \int_0^s k(s,t)u(t)dt.$$

*Find the spectral radius of B. What is the spectrum of B. (*Hint: *Prove by induction that*

$$|(B^n u)(s)| \le \frac{M^n}{n!}\, s^n \|u\|_\infty, \ \forall n \in \mathbb{N},$$

for some constant $M > 0$).

Solution.
Let us first prove by induction that

$$|(B^n u)(s)| \le \frac{M^n}{n!}\, s^n \|u\|_\infty, \quad \forall s \in [0,1],\ \forall n \in \mathbb{N}, \qquad (1)$$

where

$$M := \max_{(s,t)\in[0,1]^2} |k(s,t)|.$$

(1) is true for $n=1$. Indeed,

$$|(Bu)(s)| \le M\int_0^s |u(t)|dt \le M\|u\|_\infty \int_0^s dt = \frac{M^1}{1!}s^1\|u\|_\infty.$$

Suppose (1) holds for $n = k$. Then for $n = k+1$ we have

$$\begin{aligned}
|(B^{k+1}u)(s)| &= \left| \int_0^s k(s,t)(B^k u)(t)dt \right| \\
&\leq \int_0^s |k(s,t)||(B^k u)(t)|dt \\
&\leq M \int_0^s |(B^k u)(t)|dt \\
&\leq M \int_0^s \frac{M^k}{k!} t^k \|u\|_\infty dt \\
&= \frac{M^{k+1}}{k!} \|u\|_\infty \int_0^s t^k dt \\
&= \frac{M^{k+1}}{(k+1)!} s^{k+1} \|u\|_\infty, \quad \forall s \in [0,1].
\end{aligned}$$

Hence, (1) is true for $n = k+1$. Thus (1) is proved by induction.
It follows from (1) that

$$\|B^n u\|_\infty \leq \frac{M^n}{n!} \|u\|_\infty, \quad \forall u \in C[0,1].$$

It follows that

$$\|B^n\| \leq \frac{M^n}{n!}, \quad \forall n \in \mathbb{N}.$$

Therefore,

$$r(B) = \lim_{n\to\infty} \|B^n\|^{1/n} \leq \lim_{n\to\infty} \frac{M}{(n!)^{1/n}} = 0.$$

Since $r(B) = 0$, $\sigma(B)$ cannot contain non-zero elements. Taking into account that $\sigma(B)$ is not empty, we conclude that $\sigma(B) = \{0\}$. ■

Problem 121
Determine the spectra of the left and the right shift operators on ℓ^2:

$$\begin{aligned}
R(x_1, x_2, x_3, ...) &= (0, x_1, x_2, ...), \\
L(x_1, x_2, x_3, ...) &= (x_2, x_3, x_4, ...).
\end{aligned}$$

Classify them into point, continuous, and residual spectrum.

Solution.
We have shown that $\|R\| = \|L\| = 1$ (problem 46). It follows that

$$\{\lambda \in \mathbb{C} : \ |\lambda| > 1\} \subset \rho(R) \ \text{ and } \ \{\lambda \in \mathbb{C} : \ |\lambda| > 1\} \subset \rho(L).$$

Now I prove the following four claims, and then I will state the conclusion.

Claim(1): $R - \lambda I$ is injective (one-to-one) for all $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$.

Proof.
If $\lambda = 0$, then

$$Rx = 0 \Rightarrow x_i = 0, \ \forall i \in \mathbb{N}.$$

Hence $x = 0$, and so $R - \lambda I$ is injective.
Suppose $0 < |\lambda| \leq 1$. Then

$$(R - \lambda I)x = 0 \Rightarrow (0, x_1, x_2, ...) = (\lambda x_1, \lambda x_2, \lambda x_3, ...).$$

Since $\lambda \neq 0$, this implies that $x_1 = x_2 = ... = 0$. Hence $x = 0$, and $R - \lambda I$ is injective. $\square$

Claim(2): $R - \lambda I$ is not surjective (onto) for all $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$.

Proof.
Note that if $\lambda = 0$, then $e_1 = (1, 0, ...) \notin \text{Image}(R - \lambda I) = \text{Image}\, R$.
Suppose $0 < |\lambda| \leq 1$. Then

$$\begin{aligned}(R - \lambda I)x = e_1 \ &\Rightarrow \ x_n = -\frac{1}{\lambda^n}, \ \forall n \in \mathbb{N} \\ &\Rightarrow \ \|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{|\lambda|^2}\right)^n .\end{aligned}$$

The above series cannot be convergent because $0 < |\lambda| \leq 1$ implies that $\frac{1}{|\lambda|^2} \geq 1$. And hence $e_1 \notin \text{Image}(R - \lambda I)$. Therefore, $R - \lambda I$ is not surjective. $\square$

Claim(3): $L - \lambda I$ injective (one-to-one) for all $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

Proof.
Suppose it was not injective. There would be some nonzero $x \in \ell^2$ such that $(L - \lambda I)x = 0$. Then

$$(x_2, x_3, x_4, ...) = (\lambda x_1, \lambda x_2, \lambda x_3, ...).$$

Hence

$$x_n = \lambda^{n-1} x_1, \ \forall n \in \mathbb{N}.$$

Since $x \neq 0$, $x_1 \neq 0$. Since $|\lambda| = 1$, we have

$$\|x\| = \sum_{n=0}^{\infty} |x_1|^2 |\lambda|^{2n} = \sum_{n=0}^{\infty} |x_1|^2.$$

This sum cannot be finite since $x_1 \neq 0$, but this is impossible. So x must be zero, and hence $L - \lambda I$ injective. $\square$

Claim(4): $L - \lambda I$ is not injective (one-to-one) for all $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$.

Proof.
By a similar argument as above, we see that any nonzero x that satisfies the equation $(L - \lambda I)x = 0$ is of the form

$$x = (x_1, \lambda x_1, \lambda^2 x_1, \lambda_3 x_1, ...).$$

Choose $x_1 = 1$, then

$$\|x\| = \sum_{n=0}^{\infty} (|\lambda|^2)^n.$$

The series is convergent since $|\lambda| < 1$, so $x \in \ell^2$ is nonzero and satisfies the equation $(L - \lambda I)x = 0$. Thus $L - \lambda I$ is not injective. $\square$

Conclusion:

- Claims (1) and (2) show that

$$\sigma(R) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}.$$

Recall that Image$(R - \lambda I)$ is dense if and only if $\ker(R - \lambda I)^* = \ker(L - \bar{\lambda} I) = \{0\}$. Since $|\bar{\lambda}| = |\lambda|$, claims (3) and (4) show that Image$(R - \lambda I)$ is dense if and only if $|\lambda| = 1$. Therefore

$$\sigma_c(R) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\} \ \text{ and } \ \sigma_r(R) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}.$$

Also from the above results we get

$$\sigma_p(R) = \varnothing.$$

- Note that claim (4) shows us that

$$\sigma_p(L) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}.$$

Since $\sigma(L)$ is a closed set, we get

$$\sigma(L) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}.$$

(As from the above we know that $|\lambda| > 1$ implies that $\lambda \in \rho(L)$.)
For $|\lambda| = 1$ we know that $\ker(L - \lambda I)^* = \ker(R - \bar{\lambda} I) = \{0\}$ by claim (1). Hence Image$(L - \lambda I)$ is dense. Therefore

$$\sigma_c(L) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\} \ \text{ and } \ \sigma_r(L) = \varnothing. \qquad \blacksquare$$

* * **

Alternate solution.

Consider $R, L : \ell^2 \to \ell^2$ defined by

$$Rx = (0, x_1, x_2, ...); \quad Lx = (x_2, x_3, ...); \quad x = (x_1, x_2, x_3, ...) \in \ell^2.$$

It is clear that $\|R\| = \|L\| = 1$. So, every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is a regular point for both of the operators R and L. Concerning the eigenvalues of these operators, we obtain the following:

$$\begin{aligned} Lx = \lambda x \ (x \neq 0) \ &\Rightarrow \ x_2 = \lambda x_1; \quad x_3 = \lambda x_2; ... \\ &\Rightarrow \ x = (1, \lambda, \lambda^2, \lambda^3, ...)x_1. \end{aligned}$$

Such a vector belongs to ℓ^2 iff $|\lambda| < 1$. Hence,

$$\sigma_p(L) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}.$$

From the above result we also have $\dim \ker(L - \lambda I) = 1$.
For R we have

$$\begin{aligned} Rx = \lambda x \ (x \neq 0) \ &\Rightarrow \ 0 = \lambda x_1; \quad x_1 = \lambda x_2; \quad x_2 = \lambda x_3; ... \\ &\Rightarrow \ x_1 = x_2 = x_3 = ... = 0... \\ &\Rightarrow \ x = 0 : \quad \text{a contradiction.} \end{aligned}$$

Hence, $\sigma_p(R) = \emptyset$.
Next, since $L^* = R$ and $R^* = L$, we obtain

$$\begin{aligned} &(1) \quad \text{Image}(R - \lambda I)^{\perp} = \ker(L - \bar{\lambda} I), \\ &(2) \quad \text{Image}(L - \lambda I)^{\perp} = \ker(R - \bar{\lambda} I). \end{aligned}$$

For $|\lambda| < 1$ the relation (1) yields

$$\text{codim}\,\text{Image}(R - \lambda I) = \dim \ker(L - \bar{\lambda} I) = 1.$$

Hence, $\{\lambda \in \mathbb{C} : \ |\lambda| < 1\} \subset \sigma_r(R)$. Since the spectrum of an operator is closed, we conclude that

$$\{\lambda \in \mathbb{C} : \ |\lambda| = 1\} \subset \sigma(L) \ \text{ and } \ \{\lambda \in \mathbb{C} : \ |\lambda| = 1\} \subset \sigma(R).$$

Moreover, for $|\lambda| = 1$, form (1) and (2) we have

$$\overline{\text{Image}(R - \lambda I)} = \overline{\text{Image}(L - \lambda I)} = \ell^2.$$

Hence, $\sigma_c(R) = \sigma_c(L) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}$.
Conclusion:

$$\begin{aligned} \sigma(R) = \sigma(L) &= \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\} \\ \sigma_p(R) = \sigma_p(L) &= \{\lambda \in \mathbb{C} : \ |\lambda| < 1\} \\ \sigma_r(L) = \emptyset, \quad \sigma_r(R) &= \{\lambda \in \mathbb{C} : \ |\lambda| < 1\} \\ \sigma_c(L) = \sigma_c(R) &= \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}. \qquad \blacksquare \end{aligned}$$

Chapter 9

Compact Operators and Their Spectra

As bounded linear operators, compact operators share spectral properties of bounded linear operators. Besides, compact operators have some more particular spectral properties.
Let $T \in \mathcal{B}(X)$ be a compact operator on a Banach space X. Suppose $\dim X = \infty$.

1. $0 \in \sigma(T)$. Every spectral value $\lambda \neq 0$ is an eigenvalue.

2. For $\lambda \neq 0$, $\dim \ker(T_\lambda) \equiv \dim \ker(T - \lambda I) < \infty$.

3. For $\lambda \neq 0$, the range of $T_\lambda \equiv T - \lambda I$ is closed.

4. The set of eigenvalues of T, namely $\sigma_p(T)$, is at most countable. The value $\lambda = 0$ is the only possible point of accumulation of that set.

Problem 122
Let $T \in \mathcal{B}(X)$ be a compact operator on a Banach space X. Suppose $\dim X = \infty$. Show that

$$\begin{aligned} &(a) \qquad \dim \ker(T_\lambda^n) < \infty \quad \forall n \in \mathbb{N}, \\ &(b) \qquad \{0\} = \ker(T_\lambda^0) \subset \ker(T_\lambda^1) \subset \ker(T_\lambda^2) \subset ... \end{aligned}$$

Solution.
Since T_λ is linear, $T_\lambda 0 = 0$. By induction we get

$$T_\lambda^n x = 0 \Rightarrow T_\lambda^{n+1} x = 0, \quad \forall n \in \mathbb{N},$$

and so (b) follows.
We now prove (a). By the binomial formula,

$$\begin{aligned} T_\lambda^n = (T-\lambda I)^n &= \sum_{k=0}^{n} \binom{n}{k} T^k (-\lambda)^{n-k} \\ &= (-\lambda)^n I + T \underbrace{\sum_{k=1}^{n} \binom{n}{k} T^{k-1} (-\lambda)^{n-k}}_{S}. \end{aligned}$$

This can be written

$$T_\lambda^n = W - \mu I, \quad \text{where} \quad \mu = -(-\lambda)^n.$$

Note that T is compact and S is bounded, so $W = TS = ST$ is compact. The property 2 above gives that $\dim \ker(T_\lambda^n) < \infty$. ∎

Problem 123
Let $T \in \mathcal{B}(X)$ be a compact operator on a Banach space X. Suppose $\dim X = \infty$. Show that

$$0 \in \sigma(T).$$

Solution.
If $0 \notin \sigma(T)$ then T is invertible, and we have $TT^{-1} = I$. But T and T^{-1} are compact, so I is compact. This requires that the dimension of X is finite (problems 16, 98): a contradiction. Thus $0 \in \sigma(T)$. ∎

Problem 124
Let $T : X \to X$ be a compact operator on a normed space X and let $\lambda \neq 0$. Then there exists a smallest integer r (depending on λ) such that from $n = r$ on, the kernels $\ker(T_\lambda^n)$ are equal, and if $r > 0$, the inclusions

$$ker(T_\lambda^0) \subset \ker(T_\lambda^1) \subset \ldots \subset \ker(T_\lambda^r)$$

are all proper (strict).

Solution.
For simplicity, we let $N_n := \ker(T_\lambda^n)$.
• We know that $N_m \subset N_{m+1}$ (Problem 122). Suppose that $N_m = N_{m+1}$ for no m. Then N_n is a proper subspace of N_{n+1} for every n. Since these kernels are closed, by Riesz lemma, there is a sequence (y_n) in N_n such that

$$\|y_n\| = 1 \text{ and } \|y_n - x\| \geq \frac{1}{2} \;\; \forall x \in N_{n-1}.$$

We show that

$$(*) \qquad \|Ty_n - Ty_m\| \geq \frac{1}{2}|\lambda| \text{ for } m < n,$$

so that the sequence (Ty_n) has no convergence subsequences. This contradicts the compactness of T.
From $T_\lambda = T - \lambda I$ we have $T = T_\lambda + \lambda I$ and

$$Ty_n - Ty_m = \lambda y_n - \tilde{x} \text{ where } \tilde{x} = T_\lambda y_m + \lambda y_m - T_\lambda y_n.$$

Let $m < n$. We show that $\tilde{x} \in N_{n-1}$. Since $m \leq n-1$, we clearly have $\lambda y_m \in N_m \subset N_{n-1}$. Also $y_m \in N_m$ implies

$$0 = T_\lambda^m y_m = T_\lambda^{m-1}(T_\lambda y_m),$$

that is, $T_\lambda y_m \in N_{m-1} \subset N_{n-1}$. Similarly, $y_n \in N_n$ implies $T_\lambda y_n \in N_{n-1}$. Together, $\tilde{x} \in N_{n-1}$. Also $x = \frac{1}{\lambda}\tilde{x} \in N_{n-1}$. Hence

$$\|\lambda y_n - \tilde{x}\| = |\lambda| \; \|y_n - x\| \geq \frac{1}{2}|\lambda|.$$

Thus we have $(*)$. Therefore, we must have $N_m = N_{m+1}$ for some m.
• We now prove that

$$(**) \qquad N_m = N_{m+1} \Longrightarrow N_n = N_{n+1} \text{ for all } n > m.$$

Assume that this does not hold. Then N_n is a proper subspace of N_{n+1} for some $n > m$. We consider an $x \in N_{n+1} \setminus N_n$. By definition,

$$T_\lambda^{n+1} x = 0 \text{ but } T_\lambda^n x \neq 0.$$

Set $z = T_\lambda^{n-m} x$. Then

$$T_\lambda^{m+1} z = T_\lambda^{n+1} x = 0 \text{ but } T_\lambda^m z = T_\lambda^n x \neq 0.$$

Hence

$$z \in N_{m+1} \text{ but } z \notin N_m.$$

So N_m is a proper subspace of N_{m+1}. This contradicts $(**)$. The first statement is proved, where r is the smallest n such that $N_n = N_{n+1}$. Consequently, if $r > 0$, the inclusions in the theorem are strict. ■

Problem 125
Let $T : X \to X$ be a compact operator on a Banach space X and let $\lambda \neq 0$. Then there exists a smallest integer q (depending on λ) such that from $n = q$ on, the ranges $T_\lambda^n(X)$ are equal, and if $q > 0$, the inclusions

$$T_\lambda^0(X) \supset T_\lambda^1(X) \supset \dots \supset T_\lambda^q(X)$$

are all proper (strict).

Solution.
For simplicity, we let $R_n := T_\lambda^n(X)$. Suppose that $R_s = R_{s+1}$ for no s. Then R_{n+1} is a proper subspace of R_n for every n. Since these ranges are closed, by Riesz lemma, there is a sequence (x_n) in R_n such that

$$\|x_n\| = 1 \text{ and } \|x_n - x\| \geq \frac{1}{2} \ \ \forall x \in R_{n+1}.$$

Let $m < n$. Since $T = T_\lambda + \lambda I$, we can write

$$(i) \qquad Tx_m - Tx_n = \lambda x_m - (-T_\lambda x_m + T_\lambda x_n + \lambda x_n).$$

On the right hand side, $\lambda x_m \in R_m,\ x_m \in R_m$, so that $T_\lambda x_m \in R_{m+1}$. Since $n > m$, also $T_\lambda x_n + \lambda x_n \in R_n \subset R_{n+1}$. Hence (i) is of the form

$$Tx_m - Tx_n = \lambda(x_m - x) \text{ with } x \in R_{m+1}.$$

Consequently,

$$\|Tx_m - Tx_n\| = |\lambda|\ \|x_m - x\| \geq \frac{1}{2}|\lambda|.$$

This contradicts the fact that (Tx_n) has a convergent subsequence since (x_n) is bounded and T is compact. Thus, $R_s = R_{s+1}$ for some s. Let q be the smallest s such that $R_s = R_{s+1}$. Then, if $q > 0$, the inclusions in the theorem are proper. Furthermore, $R_{q+1} = R_q$ means that T_λ maps R_q onto itself. Hence repeated application of T_λ gives $R_{n+1} = R_n$ for every $n > q$. ∎

Problem 126
Let A be an invertible operator, and let K be a compact operator in a Banach space. Prove that
(a) $\dim(\ker(A + K)) < \infty$.
(b) $\operatorname{codim}(\operatorname{Image}(A + K)) < \infty$.

Solution.
(a) Since A is invertible, we can write

$$A+K=A(I+A^{-1}K).$$

The operator $A^{-1}K$ is compact (see note above), so $\dim(\ker(I+A^{-1}K))<\infty$. This implies that

$$A \text{ invertible } \Rightarrow \ker(AB)=\ker(B), \quad \forall B\in\mathcal{B}(X).$$

Indeed,

$$\begin{aligned} x\in\ker(B) &\Rightarrow ABx=A0=0 \\ &\Rightarrow x\in\ker(AB). \end{aligned}$$

And

$$\begin{aligned} x\in\ker(AB) &\Rightarrow ABx=0 \\ &\Rightarrow Bx=0 \\ &\Rightarrow x\in\ker(B). \end{aligned}$$

It follows that

$$\begin{aligned} \dim(\ker(A+K)) &= \dim(A(I+A^{-1}K)) \\ &= \dim(I+A^{-1}K)<\infty. \end{aligned}$$

(b) One can write

$$A+K=(I+KA^{-1})A.$$

The operator KA^{-1} is compact, so $\operatorname{codim}(I+KA^{-1})<\infty$. This implies that if A is invertible, and then $\operatorname{Image}(BA)=\operatorname{Image}(B)$. Indeed,

$$\begin{aligned} x\in\operatorname{Image}(B) &\Rightarrow x=By \\ &\Rightarrow x=(BA)A^{-1}y\in\operatorname{Image}(BA). \end{aligned}$$

And

$$\begin{aligned} x\in\operatorname{Image}(BA) &\Rightarrow x=BAy \\ &\Rightarrow x=B(Ay)\in\operatorname{Image}(B). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \operatorname{codim}(\operatorname{Image}(A+K)) &= \operatorname{codim}(\operatorname{Image}((I+KA^{-1})A)) \\ &= \operatorname{codim}(\operatorname{Image}(I+KA^{-1}))<\infty. \quad \blacksquare \end{aligned}$$

Problem 127
Consider the operator $K : L^2([0,1]) \to L^2([0,1])$ *defined by*

$$(Kf)(t) = \int_0^1 k(t,s)f(s)ds$$

where $k(s,t) = \min\{t, s : \ t, s \in [0,1]\}$.
(a) Prove that K *is a compact self-adjoint operator.*
(b) Find the spectrum $\sigma(K)$ *and the norm* $\|K\|$.

Solution.
(a) Since $k(t,s) = \overline{k(s,t)}$ and $k(t,s)$ is a continuous function, the operator is self-adjoint and compact (see Problem 105).
(b) Since K is compact and self-adjoint, the spectrum of K consists of zero and real eigenvalues[1]. Assume that $\lambda y = Ky$. This means that

$$\begin{aligned}(1) \quad \lambda y(t) &= \int_0^t \min\{t,s\}y(s)ds + \int_t^1 \min\{t,s\}y(s)ds \\ &= \int_0^t sy(s)ds + \int_t^1 ty(s)ds \\ &= \int_0^t sy(s)ds + t\int_t^1 y(s)ds.\end{aligned}$$

Taking the derivative twice, we obtain

$$\begin{aligned}(2) \quad & \lambda y'(t) = ty(t) + \int_t^1 y(s)ds - ty(t) = \int_t^1 y(s)ds \\ & \lambda y''(t) = -y(t).\end{aligned}$$

Clearly, $\lambda \neq 0$; otherwise, $y = 0$ so $\ker(K) = 0$. We have the differential equation

$$(3) \quad \lambda y'' + y = 0 \quad \text{with b.v.c.} \ \ y'(0) = y(0) = 0$$

because of (1). Let us prove that $\lambda > 0$, which means that the operator K is positive. Multiplying (3) by $\bar{y}$ and integrating we obtain

$$\lambda \int_0^1 y''(t)\bar{y}(t)dt + \|y\|^2 = 0.$$

[1] If $T \in \mathcal{B}(H)$ is self adjoint, all its eigenvalues are real. (We will see this in the next chapter.)

Integrating by parts we obtain

$$\lambda\left(y'\bar{y}\big|_0^1 - \int_0^1 |y'|^2 dt\right) + \|y\|^2 = 0.$$

The b.v.c. yield

$$-\lambda\int_0^1 |y'|^2 dt + \|y\|^2 = 0.$$

Hence $\lambda > 0$.
The solution of the differential equation is

$$y = C_1 \cos\frac{t}{\sqrt{\lambda}} + C_2 \sin\frac{t}{\sqrt{\lambda}}.$$

From the b.v.c. it follows that $C_1 = 0$ and $\frac{C_2}{\sqrt{\lambda}} \cos\frac{1}{\sqrt{\lambda}} = 0$. Therefore, the eigenvalues of K are

$$\lambda_k = \frac{4}{\pi^2(2k-1)^2}, \quad k = 1, 2, \ldots$$

Since K is seft-adjoint, we obtain

$$\|K\| = \max_{k\in\mathbb{N}} |\lambda_k| = |\lambda_1| = \frac{4}{\pi^2}. \quad \blacksquare$$

Problem 128(Similar problem)
Consider the operator $K : L^2([0,1]) \to L^2([0,1])$ defined by

$$(Kf)(t) = \int_0^1 k(t,s)f(s)ds$$

where $k(s,t) = \max\{t, s : \ t, s \in [0,1]\}$.
(a) Prove that K is a compact self-adjoint operator.
(b) Find the spectrum $\sigma(K)$ and the norm $\|K\|$.
(c) Is K a positive operator?

Problem 129
Let S be the operator defined on $C[0,1]$ by

$$(Sf)(x) = \int_0^x f(y)dy.$$

(a) Compute the spectrum of S.
(b) Show that S is compact.

Solution.
(a) First we show that S is continuous. For $f, g \in C[0,1]$ we have

$$\begin{aligned}\|Sf - Sg\| = \|S(f-g)\| &= \left|\int_0^x (f(y) - g(y))dy\right| \\ &\leq \int_0^x |f(y) - g(y)|dy \\ &\leq \sup_{x\in[0,1]} \int_0^x |f(y) - g(y)|dy \\ &\leq |[0,1]|.\|f-g\| = \|f-g\|.\end{aligned}$$

Hence, S is Lipschitz continuous with constant 1. Thus, $\|S\| \leq 1$. Next, we show that $\|S^n\|^{1/n} \to 0$ as $n \to \infty$. Observe that

$$|(Sf)(x)| \leq \int_0^x |f(t)|dt \leq \|f\|x = \|f\| \, \frac{x^1}{1!}, \quad x \in [0,1].$$

By induction

$$\begin{aligned}|(S^{n+1}f)(x)| &\leq \int_0^x |S^n f(t)|dt \\ &\leq \|f\| \int_0^x \frac{t^n}{n!} \, dt \\ &= \|f\| \, \frac{x^{n+1}}{(n+1)!}, \quad x \in [0,1], \quad n = 1, 2, \ldots\end{aligned}$$

Thus,

$$\|S^n f\| \leq \|f\| \frac{1}{n!}, \quad n = 1, 2, \ldots$$

So

$$\|S^n\|^{1/n} \leq \left(\frac{1}{n!}\right)^{1/n} \to 0 \ \text{ as } \ n \to \infty.$$

Recall that the spectral radius of S is given by

$$r(S) = \lim_{n\to\infty} \|S^n\|^{1/n}.$$

This implies that $r(S) = 0$. Thus $\sigma(S) = \{0\}$.

(b) Suppose $F \subset C[0,1]$ is a bounded subset. Put

$$\|F\| := \sup_{f\in F} \|f\|.$$

Then SF is equi-continuous by the Fundamental Theorem of the Calculus:

$$\forall f \in F, \ |(Sf)'(x)| \leq \|f\| \leq \|F\|.$$

SF is bounded since:

$$\forall f \in F, \ |(Sf)(x)| \leq \|f\| \leq \|F\|.$$

By Ascoli-Arzela theorem SF is relatively compact. Thus S is compact. ■

Problem 130
Set

$$(Tf)(x) = \int_0^{1-x} f(y)dy, \quad f \in C[0,1], \ x \in [0,1].$$

(a) Prove that T is a linear bounded and compact operator on $C[0,1]$.
(b) Calculate $\sigma(T)$ and the eigenvalues of T.

Solution.
(a)

- Linearity of T: trivial.
- Boundedness of T:

$$|(Tf)(x)| = \left| \int_0^{1-x} f(y)dy \right| \leq \|f\|,$$

where $\|f\| = \max_{x\in[0,1]} |f(x)|$. Hence,

$$\|T\| \leq 1.$$

- Compactness of T: let $(f_n)_{n=1}^{\infty}$ be a bounded sequence in $C[0,1]$. Hence,

$$\|f_n\| \leq M \quad \forall n \in \mathbb{N}$$

for some $M \geq 0$. By Arzela-Ascoli theorem, it suffices to show that $A := \{Tf_n; \ n \in \mathbb{N}\}$ is bounded and equicontinuous. We have

$$\|Tf_n\| \leq \|T\|\|f_n\| \leq \|f_n\| \leq M \quad \forall n \in \mathbb{N}.$$

Given any $\varepsilon > 0$, without lost of generality, we can assume $x < y$, then

$$|(Tf_n)(x) - (Tf_n)(y)| = \left| \int_{1-y}^{1-x} f(y)dy \right| \leq M|x - y|.$$

Thus,

$$|(Tf_n)(x) - (Tf_n)(y)| < \varepsilon \quad \forall n \in \mathbb{N} \ \text{ provided } \ |x - y| < \frac{\varepsilon}{M}.$$

(b) First we see that $\lambda = 0$ is an eigenvalue. Assume that $\lambda \neq 0$ is an eigenvalue, i.e.,

$$\lambda g(x) = (Tg)(x) = \int_0^{1-x} g(y)dy, \quad x \in [0, 1]$$

for some $0 \neq g \in C[0, 1]$. This implies that

$$Tg \in C^1[0, 1] \ \text{ and } \ \lambda g'(x) = -g(1 - x), \quad x \in [0, 1].$$

Moreover, we have $g(1) = 0$. But $g \in C^1[0, 1]$ implies that $g \in C^2[0, 1]$. By differentiating once more we get

$$\lambda g''(x) = \frac{g(x)}{\lambda}, \quad x \in [0, 1] \ \text{ and } \ g(1) = g'(0) = 0.$$

Hence

$$g(x) = A\cos(x/\lambda) \ \text{ with } \ g(1) = 0.$$

This gives that

$$\lambda_k = \frac{1}{\frac{\pi}{2} + k\pi}, \quad k \in \mathbb{Z}.$$

Check if all these λ's are eigenvalues. We calculate

$$\begin{aligned}
(Tg_k)(x) &= \int_0^{1-x} \cos(t/\lambda_k)dt \\
&= \lambda_k[\sin(t/\lambda_k)]_0^{1-x} \\
&= \lambda_k \sin\left[\left(\frac{\pi}{2} + k\pi\right)(1 - x)\right] \\
&= \lambda_k(-1)^k g_k(x).
\end{aligned}$$

Hence, $\lambda = \lambda_{2l}, \ \ l \in \mathbb{Z}$ are the eigenvalues of T, i.e.,

$$\sigma_p(T) = \{\lambda_{2l}, \ \ l \in \mathbb{Z}\}.$$

We know that $\sigma(T)$ is closed and $\sigma(T) \setminus \{0\} \subset \sigma_p(T)$. This yields

$$\sigma(T) = \{0\} \cup \sigma_p(T). \qquad \blacksquare$$

Problem 131
Let T be a compact operator on a Hilbert space H and (λ_n) be a sequence of complex numbers. Suppose there exists a nested sequence of distinct subspaces (M_n) such that for all $n \in \mathbb{N}$

$$\begin{aligned} &(i) \qquad M_n \subsetneq M_{n+1} \\ &(ii) \qquad (T - \lambda_n I)M_{n+1} \subset M_n. \end{aligned}$$

Prove that $\lim_{n\to\infty} \lambda_n = 0$.

Solution.
Since M_n is a subspace of M_{n+1}, we can write $M_{n+1} = M_n \oplus (M_n^{\perp})_M$, where $(M_n^{\perp})_M$ is the orthogonal complement of M_n in M_{n+1}. For short we write $(M_n^{\perp})_M = M_{n+1} \ominus M_n$. Let $\{e_n\}$ be a sequence of unit vectors defined by

$$e_1 \in M_1, \quad e_{n+1} \in M_{n+1} \ominus M_n, \quad \forall n \in \mathbb{N}.$$

Clearly, that is an orthonormal system. Moreover,

$$\big\langle (T - \lambda_n I)e_n, e_n \big\rangle = 0, \quad \text{for all} \quad n \geq 2,$$

which implies that

$$\begin{aligned} \|Te_n\| \geq |\langle Te_n, e_n\rangle| &= \big|\big\langle (T - \lambda_n I)e_n, e_n \big\rangle\big| + |\langle \lambda_n e_n, e_n\rangle| \\ &= 0 + |\langle \lambda_n e_n, e_n\rangle| = |\lambda_n|. \end{aligned}$$

Since T is compact and $e_n \xrightarrow{w} 0$, it follows that $\lim_{n\to\infty} Te_n = 0$. Thus

$$\lim_{n\to\infty} \lambda_n = 0. \qquad \blacksquare$$

Problem 132
Let T be a compact operator on a Hilbert space H and any $C > 0$. Prove that there is a finite number of linearly independent eigenvectors $x_1, ..., x_n$ of T corresponding to eigenvalues $\lambda_1, ..., \lambda_n$ such that $\lambda_i > C$ for all $i = 1, ..., n$.

Solution.
We can rescale to get $\|x_i\| = 1$ for all $i = 1, ..., n$. Suppose to the contrary that

there is an infinite sequence $\{x_n\}$ of unit vectors, and a sequence of eigenvalues $\{\lambda_n\}$ satisfying

$$\lambda_n > C \text{ and } Tx_n = \lambda_n x_n, \ \forall n \in \mathbb{N}.$$

Let $M_n = \text{Span}\{x_1, ..., x_n\}$, then $\{M_n\}$ is a nested sequence of subspaces of H and the inclusions $M_n \subset M_{n+1}$ are strict. Let $x \in M_n$, then there are $c_1, ..., c_n \in \mathbb{C}$ such that $x = \sum_{i=1}^n c_i x_i$. So we have

$$\begin{aligned}
(T - \lambda_n I)x &= (T - \lambda_n I) \sum_{i=1}^n c_i x_i \\
&= \sum_{i=1}^n c_i (T - \lambda_n I) x_i \\
&= \sum_{i=1}^n c_i (T x_i - \lambda_n x_i) \\
&= \sum_{i=1}^n c_i (\lambda_i - \lambda_n) x_i \in M_{n-1}.
\end{aligned}$$

This implies that

$$(T - \lambda_n I) M_n \subset M_{n-1}, \quad n \geq 2.$$

From Problem 130, we obtain

$$\lim_{n\to\infty} \lambda_n = 0.$$

This contradicts the assumption $\lambda_n > C$ for all $n \in \mathbb{N}$. ■

Note: Argument in problems 130, 131 is the proof of Proposition 4 in the review at the beginning of this chapter.

Chapter 10

Bounded Self Adjoint Operators and Their Spectra

Review some main points.

Bounded self adjoint operators on Hilbert spaces were defined and considered before. This chapter is devoted to their spectral properties.

Definition:

Let $T \in \mathcal{B}(H)$ where H is a complex Hilbert space. The adjoint operator of T is the operator $T^* : H \to H$ defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,\ \forall x, y \in H.$$

T is said to be self adjoint if $T = T^*$. We can say that T is self adjoint if and only if

$$\langle Tx, y\rangle = \langle x, Ty\rangle,\ \forall x, y \in H.$$

Another equivalent condition is:
T is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$.

Let $T \in \mathcal{B}(H)$ be a bounded self adjoint operator on the complex Hilbert space H.

Proposition 12 *(Eigenvalues and eigenvectors)*
1. All eigenvalues of T (if they exist) are real.
2. Eigenvectors corresponding to different eigenvalues of T are orthogonal.

Proposition 13 *(Spectrum)*
1. $\sigma(T) \subset \mathbb{R}$.
2. $\sigma(T) \subset [m, M]$ *where* $m = \inf_{\|x\|=1}\langle Tx, x\rangle$ *and* $M = \sup_{\|x\|=1}\langle Tx, x\rangle$.
3. $m, M \in \sigma(T)$.

Proposition 14 *(Norm)*
$\|T\| = \max\{|m|, |M|\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

Problem 133
Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Prove that the residual spectrum of T is empty, that is,

$$\sigma_r(T) = \varnothing.$$

Solution.
Assume that $\sigma_r(T) \neq \emptyset$. Take $\lambda \in \sigma_r(T)$. By the definition of $\sigma_r(T)$, T_λ^{-1} exists but its domain is not dense in H. Hence, by the projection theorem, there is a $y \neq 0$ in H such that y is perpendicular to the domain $D(T_\lambda^{-1})$ of T_λ^{-1}. But $D(T_\lambda^{-1})$ is the range of T_λ, hence

$$\langle T_\lambda x, y\rangle = 0, \quad \forall x \in H.$$

Since λ is real and T is self-adjoint, we obtain

$$\langle x, T_\lambda y\rangle = 0, \quad \forall x \in H.$$

Taking $x = T_\lambda y$, we get $\|T_\lambda y\|^2 = 0$, so that

$$T_\lambda y = Ty - \lambda y = 0.$$

Since $y \neq 0$, this shows that λ is an eigenvalue of T. But this contradicts $\lambda \in \sigma_r(T)$. ■

Second solution:
By Problem 91, noting that T is self adjoint, for any $\lambda \in \mathbb{C}$ we have

$$\text{Image}(T - \lambda I)^\perp = \ker(T^* - \bar{\lambda} I) = \ker(T - \bar{\lambda} I).$$

And hence, if $\overline{\text{Image}(T - \lambda I)} \neq H$, then $\bar{\lambda}$ is an eigenvalue of T. Since T is self adjoint, λ is real. Thus $\lambda = \bar{\lambda}$ is an eigenvalue of T. Therefore λ does not belong to the residual spectrum of T. ■

Problem 134
Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Prove that

$$(*) \qquad \lambda \in \rho(T) \Longleftrightarrow \exists c > 0 : \quad \forall x \in H, \ \|T_\lambda x\| \geq c\|x\|.$$

Solution.

• If $\lambda \in \rho(T)$ then $R_\lambda := T_\lambda^{-1} : H \to H$ exists and is bounded, say $\|R_\lambda\| = k > 0$. Now since $I = R_\lambda T_\lambda$, we have for every $x \in H$

$$\|x\| = \|R_\lambda T_\lambda x\| \leq \|R_\lambda\| \, \|T_\lambda x\| = k\|T_\lambda x\|.$$

This gives

$$\|T_\lambda x\| \geq c\|x\|, \quad \text{where} \quad c = \frac{1}{k}.$$

• Conversely, suppose $(*)$ holds. We shall show:

(a) $T_\lambda : H \to T_\lambda(H)$ is bijective;

(b) $T_\lambda(H)$ is dense in H;

(c) $T_\lambda(H)$ is closed in H;

so that $T_\lambda(H) = H$ and $R_\lambda := T_\lambda^{-1}$ is bounded by the bounded inverse theorem.[1]

(a) By (7.1), we have for $x_1, x_2 \in H$

$$\|T_\lambda x_1 - T_\lambda x_2\| = \|T_\lambda(x_1 - x_2)\| \geq c\|x_1 - x_2\|.$$

Therefore,

$$T_\lambda x_1 = T_\lambda x_2 \Longrightarrow x_1 = x_2.$$

Thus $T_\lambda : H \to T_\lambda(H)$ is bijective.

(b) We show that $x_0 \perp \overline{T_\lambda(H)}$ implies $x_0 = 0$, so that $\overline{T_\lambda(H)} = H$ by the projection theorem.[2] Let $x_0 \perp \overline{T_\lambda(H)}$ Then for all $x \in H$ we have

$$0 = \langle T_\lambda x, x_0 \rangle = \langle Tx, x_0 \rangle - \lambda \langle x, x_0 \rangle.$$

Since T is self-adjoint,

$$\langle Tx, x_0 \rangle = \langle x, Tx_0 \rangle.$$

It follows that

$$\langle x, Tx_0 \rangle = \lambda \langle x, x_0 \rangle = \langle x, \bar{\lambda} x_0 \rangle.$$

Thus $Tx_0 = \bar{\lambda} x_0$. So $x_0 = 0$ since otherwise, $\bar{\lambda} = \lambda$ would be an eigenvalue of T, and $T_\lambda x_0 = 0$, which would imply

$$0 = \|T_\lambda x_0\| \geq c\|x_0\| > 0 : \quad \text{a contradiction.}$$

[1] A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, then T is continuous and thus bounded.

[2] If Y is a closed subspace of H, then $H = Y \oplus Y^\perp$.

(c) To show $T_\lambda(H)$ is closed we show

$$y \in \overline{T_\lambda(H)} \Longrightarrow y \in T_\lambda(H).$$

Let $y \in \overline{T_\lambda(H)}$. There is a sequence (y_n) in $T_\lambda(H)$ which converges to y. For every n we have $y_n = T_\lambda x_n$ for some $x_n \in H$. By (7.1),

$$\|x_n - x_m\| \leq \frac{1}{c}\|T_\lambda(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\|.$$

Hence (x_n) is Cauchy. Since H is complete, (x_n) converges, say, $x_n \to x$. Since T_λ is continuous, $y_n = T_\lambda x_n \to T_\lambda x$. Since the limit is unique, $T_\lambda x = y$. Hence $y \in T_\lambda(H)$. Thus $T_\lambda(H)$ is closed in H. ■

Problem 135
(a) Let $A \in \mathcal{B}(X)$ where X is a Banach space. Suppose there exists $m > 0$ such that

$$\|Ax\| \geq m\|x\|, \ \forall x \in X.$$

Show that Image A *is closed in X.*
(b) Let $A \in \mathcal{B}(H)$ be self adjoint, where H is a Hilbert space. Let $\lambda \in \mathbb{C}$ such that $Im\lambda \neq 0$. Prove that

$$\|Ax - \lambda x\| \geq |Im\lambda| \ \|x\|, \ \forall x \in H.$$

Prove that λ is a regular point of A.

Solution.
(a) Let (x_n) be a sequence in X. Suppose $Ax_n \to y$ as $n \to \infty$. From the hypothesis we get

$$\|Ax_n - Ax_m\| \geq m\|x_n - x_m\| \ \text{ for } \ n \neq m.$$

Since (Ax_n) is a Cauchy sequence in X, it follows that (x_n) is also a Cauchy sequence in X. Hence $x \to x$ as $n \to \infty$. But A is continuous, so $Ax_n \to Ax$ as $n \to \infty$. By uniqueness of the limit, we obtain $y = Ax$. This shows that Image A is closed.

(b) Let $\lambda = a + ib$ with $a, b \in \mathbb{R}$ and $b = Im\lambda \neq 0$. We have for all $x \in H$

$$\begin{aligned} \|Ax - \lambda x\|^2 &= \langle Ax - (a+ib)x, Ax - (a+ib)x\rangle \\ &= \langle (A - aI)x - ibx, (A - aI)x - ibx\rangle \\ &= \|(A - aI)x\|^2 + b^2\|x\|^2, \end{aligned}$$

which implies that

$$\|Ax - \lambda x\| \geq |b|\|x\|, \ \forall x \in H.$$

By part (a), Image$(A - \lambda I)$ is closed and therefore $\lambda \notin \sigma_c(A)$. By Problem 91 $\sigma_r(A) = \varnothing$, thus λ is a regular point of A. ■

Problem 136
Let $A \in \mathcal{B}(H)$ be self adjoint, where H is a Hilbert space.
(a) Prove that

$$\|A\| = \sup_{x \neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2}.$$

(b) Prove that at least one of $\|A\|$ or $-\|A\|$ is an element of $\sigma(A)$.

Solution.
(a) Using the Cauchy-Schwarz inequality, we have

$$|\langle Ax, x\rangle| \leq \|Ax\| \ \|x\| \leq \|A\| \ \|x\|^2, \ \forall x \in H.$$

Hence

$$\sup_{x \neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2} \leq \|A\|. \qquad (i)$$

Now we establish the reverse. Notice first that for all $x, y \in H$ we have

$$\big\langle A(x+y), x+y \big\rangle - \big\langle A(x-y), x-y \big\rangle = 2\big[\langle Ax, y\rangle + \langle Ay, x\rangle\big].$$

Using the triangle inequality, we get

$$2\big|\langle Ax, y\rangle + \langle Ay, x\rangle\big| \leq \big|\langle A(x+y), x+y\rangle\big| + \big|\langle A(x-y), x-y\rangle\big|.$$

If we let $C = \sup_{x \neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2}$, then the Parallelogram Law gives that

$$\begin{aligned} \big|\langle Ax, y\rangle + \langle Ay, x\rangle\big| &\leq \frac{1}{2} C\big(\|x+y\|^2 + \|x-y\|^2\big) \\ &= C\big(\|x\|^2 + \|y\|^2\big). \qquad (*) \end{aligned}$$

Now let x be any vector with $\|x\| = 1$ and let $y = \frac{Ax}{\|Ax\|}$ (the case $Ax = 0$ does not give the supremum, hence we may assume that $Ax \neq 0$). Then $\|y\| = 1$. From $(*)$ we get

$$\left| \frac{\langle Ax, Ax\rangle}{\|Ax\|} + \frac{\langle Ax, Ax\rangle}{\|Ax\|} \right| \leq 2C.$$

Hence $\|Ax\| \leq C$. This holds for all $x \in H$ with $\|x\| = 1$. Thus $\|A\| \leq C$. (ii)
Combine (i) and (ii) we obtain

$$\|A\| = \sup_{x \neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2}.$$

(b) If we take x arbitrary with $\|x\| = 1$, then we get

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|. \qquad (**)$$

Let (x_n) be a sequence in H with $\|x_n\| = 1$ such that $|\langle Ax_n, x_n\rangle|$ converges to $\|A\|$ (this is possible by $(**)$). Let $\langle Ax_n, x_n\rangle \to \lambda$ (it may be necessary to pass to subsequence). Clearly, $\lambda = \pm\|A\|$. Now,

$$\begin{aligned} 0 \leq \|Ax_n - \lambda x_n\|^2 &= \|Ax_n\|^2 - 2\lambda\langle Ax_n, x_n\rangle + \lambda^2\|x_n\|^2 \\ &\leq 2\lambda^2 - 2\lambda\langle Ax_n, x_n\rangle, \end{aligned}$$

which converges to 0 as $n \to \infty$. Thus $\lambda \in \sigma(A)$ (see Problem 109). ■

* * **

To close this chapter, we introduce a well known theorem relative to compact self adjoint operators on Hilbert spaces:

The spectral theorem for compact self adjoint operators on Hilbert spaces.

Problem 137
Let $T \in \mathcal{B}(H)$ be a compact self adjoint operator on a Hilbert space H.
(a) There exists a system (finite or infinite) of orthonormal eigenvectors $\{e_1, e_2, ...\}$ of T and corresponding eigenvalues $\{\lambda_1, \lambda_2, ...\}$ of T such that $|\lambda_1| \geq |\lambda_2| \geq ....$ If the system is infinite then $\lambda_n \to 0$ as $n \to \infty$.
(b) Eigenvectors and eigenvalues mentioned above satisfy the following equation:

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle e_k, \ \forall x \in H.$$

Solution.
(a) We use Proposition 13-3 (that we proved in Problem 136b) repeatedly for constructing eigenvalues and eigenvectors.

Let $H_1 = H$ and $T_1 = T$. Then by Proposition 13-3, there exists an eigenvalue λ_1 of T_1 and a corresponding eigenvector e_1 such that

$$\|e_1\| = 1 \text{ and } |\lambda_1| = \|T_1\|.$$

Now Span$\{e_1\}$ is a closed subspace of H_1 hence, by the projection theorem,

$$H_1 = \text{Span}\{e_1\} \oplus \text{Span}\{e_1\}^\perp.$$

Let $H_2 = \text{Span}\{e_1\}^\perp$. Clearly H_2 is a closed subspace of H_1 and $T(H_2) \subset H_2$. Indeed, if $x \in H_2$ then $x \perp e_1$, hence $Tx = \lambda x \Rightarrow Tx \perp e_1$. Let T_2 be the restriction of T_1 on H_2, that is, $T_2 = T_1|_{H_2} = T|_{H_2}$. Then T_2 is a compact and self adjoint operator in $\mathcal{B}(H_2)$. If $T_2 = 0$, then there is nothing to prove. Assume that $T_2 \neq 0$. Then by Proposition 13-3, there exists an eigenvalue λ_2 of T_2 and a corresponding eigenvector e_2 such that

$$\|e_2\| = 1 \text{ and } |\lambda_2| = \|T_2\|.$$

Since T_2 is a restriction of T_1,

$$|\lambda_2| = \|T_2\| \leq \|T_1\| = \lambda_1.$$

By construction e_1 and e_2 are orthonormal.
Now let $H_3 = \text{Span}\{e_2, e_2\}^\perp$. Clearly $H_3 \subset H_2$ and $T(H_3) \subset H_3$. The operator $T_3 = T|_{H_3}$ is compact and self adjoint....If we continue to proceed in this way, either after some stage, say n, we get $T_n = 0$ or there exists an infinite sequence (λ_n) of eigenvalues of T and corresponding eigenvectors (e_n) satisfying

$$\|e_n\| = 1, \ \ \forall n \in \mathbb{N} \text{ and } |\lambda_1| \geq |\lambda_2| \geq ...$$

If the sequence (λ_n) is infinite, we show that $\lambda_n \to 0$ as $n \to \infty$. Suppose that $\lambda_n \nrightarrow 0$ as $n \to \infty$. Then there exists $\varepsilon > 0$ such that $|\lambda_n| > \varepsilon$ for infinitely many n. For $n \neq m$, we have

$$\|Te_n - Te_m\|^2 = \|\lambda_n e_n - \lambda_m e_m\|^2 = \lambda_n^2 + \lambda_m^2 > \varepsilon^2.$$

This shows that the sequence (Te_n) has no convergent subsequence, a contradiction to the compactness of T. Hence $\lambda_n \to 0$ as $n \to \infty$.

(b) There are two cases to consider:
Case 1. $T_n = 0$ for some n.
Let $x_n = x - \sum_{k=1}^n \langle x, e_k \rangle e_k$ for all $x \in H$. Then $x_n \perp e_k$ for $1 \leq k \leq n$, since $\{e_1, e_2, ...\}$ is an orthonormal system and

$$\langle x_n, e_k \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0.$$

Hence

$$0 = T_n x_n = Tx - \sum_{k=1}^{n} \langle x, e_k \rangle T_k e_k = \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k.$$

That is,

$$Tx = \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k, \ \forall x \in H.$$

Case 2. $T_n \neq 0$ for infinitely many n.
For $x \in H$, by Case 1, we have

$$\begin{aligned} \left\| Tx - \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k \right\| &= \|T_n x_x\| \leq \|T_n\| \ \|x_n\| \\ &= |\lambda_n| \ \|x_n\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

Hence

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k, \ \forall x \in H. \qquad \blacksquare$$

THANK YOU and GOOD LUCK!

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