Answers to Exercises

Linear Algebra

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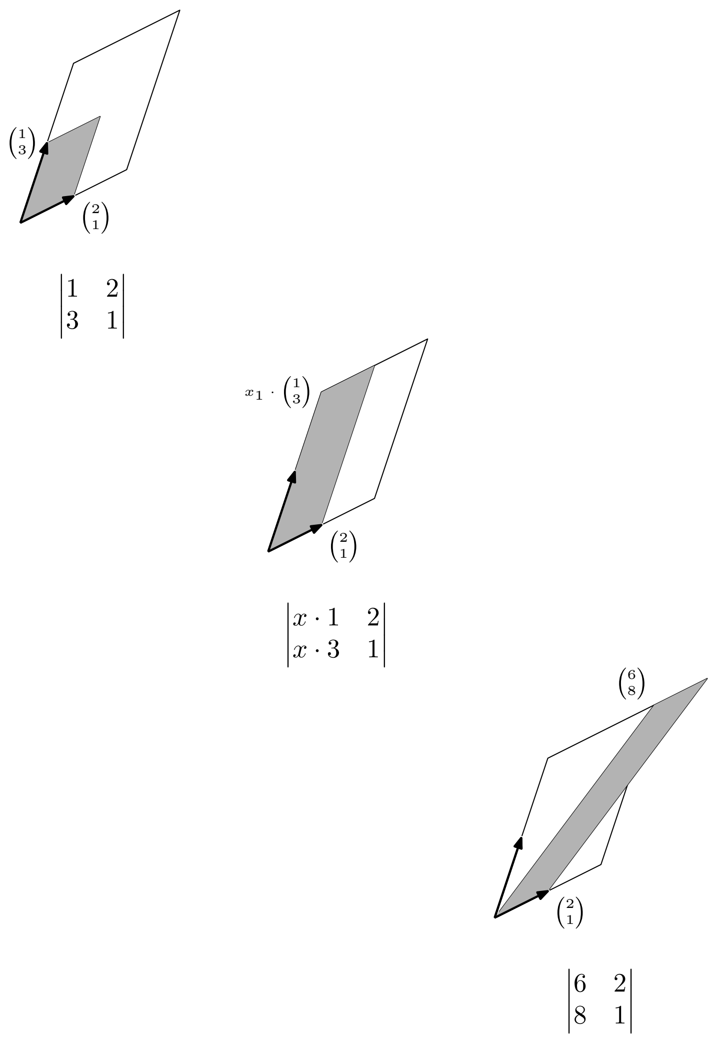


Notation

$\mathbb{R}$	real numbers
$\mathbb{N}$	natural numbers: $\{0, 1, 2, \ldots\}$
$\mathbb{C}$	complex numbers
$\{\ldots \mid \ldots\}$	set of ... such that ...
$\langle\ldots\rangle$	sequence; like a set but order matters
V, W, U	vector spaces
$\vec{v}, \vec{w}$	vectors
$\vec{0}$, $\vec{0}_V$	zero vector, zero vector of V
B, D	bases
$\mathcal{E}_n = \langle\vec{e}_1, \ldots, \vec{e}_n\rangle$	standard basis for $\mathbb{R}^n$
$\vec{\beta}, \vec{\delta}$	basis vectors
$\mathrm{Rep}_B(\vec{v})$	matrix representing the vector
$\mathcal{P}_n$	set of n-th degree polynomials
$\mathcal{M}_{n\times m}$	set of $n\times m$ matrices
$[S]$	span of the set S
$M \oplus N$	direct sum of subspaces
$V \cong W$	isomorphic spaces
h, g	homomorphisms, linear maps
H, G	matrices
t, s	transformations; maps from a space to itself
T, S	square matrices
$\mathrm{Rep}_{B,D}(h)$	matrix representing the map h
$h_{i,j}$	matrix entry from row i, column j
$\lvert T\rvert$	determinant of the matrix T
$\mathscr{R}(h), \mathscr{N}(h)$	rangespace and nullspace of the map h
$\mathscr{R}_\infty(h), \mathscr{N}_\infty(h)$	generalized rangespace and nullspace

Lower case Greek alphabet

name	character	name	character	name	character
alpha	α	iota	ι	rho	ρ
beta	β	kappa	κ	sigma	σ
gamma	γ	lambda	λ	tau	τ
delta	δ	mu	μ	upsilon	υ
epsilon	ϵ	nu	ν	phi	ϕ
zeta	ζ	xi	ξ	chi	χ
eta	η	omicron	o	psi	ψ
theta	θ	pi	π	omega	ω

Cover. This is Cramer's Rule for the system $x_1 + 2x_2 = 6$, $3x_1 + x_2 = 8$. The size of the first box is the determinant shown (the absolute value of the size is the area). The size of the second box is x_1 times that, and equals the size of the final box. Hence, x_1 is the final determinant divided by the first determinant.

These are answers to the exercises in Linear Algebra *by J. Hefferon. Corrections or comments are very welcome, email to jimjoshua.smcvt.edu*

An answer labeled here as, for instance, One.II.3.4, matches the question numbered 4 from the first chapter, second section, and third subsection. The Topics are numbered separately.

Contents

Chapter One: Linear Systems

Subsection One.I.1: Gauss' Method

One.I.1.16 Gauss' method can be performed in different ways, so these simply exhibit one possible way to get the answer.

(a) Gauss' method

$$\xrightarrow{-(1/2)\rho_1+\rho_2}\ \begin{array}{rcrcl} 2x & + & 3y & = & 7 \\ & - & (5/2)y & = & -15/2 \end{array}$$

gives that the solution is $y = 3$ and $x = 2$.

(b) Gauss' method here

$$\xrightarrow[\rho_1+\rho_3]{-3\rho_1+\rho_2}\ \begin{array}{rcrcl} x & & - z & = & 0 \\ & y & + 3z & = & 1 \\ & y & & = & 4 \end{array} \ \xrightarrow{-\rho_2+\rho_3}\ \begin{array}{rcrcl} x & & - z & = & 0 \\ & y & + 3z & = & 1 \\ & & -3z & = & 3 \end{array}$$

gives $x = -1$, $y = 4$, and $z = -1$.

One.I.1.17 **(a)** Gaussian reduction

$$\xrightarrow{-(1/2)\rho_1+\rho_2}\ \begin{array}{rcrcl} 2x & + & 2y & = & 5 \\ & & -5y & = & -5/2 \end{array}$$

shows that $y = 1/2$ and $x = 2$ is the unique solution.

(b) Gauss' method

$$\xrightarrow{\rho_1+\rho_2}\ \begin{array}{rcrcl} -x & + & y & = & 1 \\ & & 2y & = & 3 \end{array}$$

gives $y = 3/2$ and $x = 1/2$ as the only solution.

(c) Row reduction

$$\xrightarrow{-\rho_1+\rho_2}\ \begin{array}{rcl} x - 3y + z & = & 1 \\ 4y + z & = & 13 \end{array}$$

shows, because the variable z is not a leading variable in any row, that there are many solutions.

(d) Row reduction

$$\xrightarrow{-3\rho_1+\rho_2}\ \begin{array}{rcl} -x - y & = & 1 \\ 0 & = & -1 \end{array}$$

shows that there is no solution.

(e) Gauss' method

$$\xrightarrow{\rho_1\leftrightarrow\rho_4}\ \begin{array}{rcl} x + y - z & = & 10 \\ 2x - 2y + z & = & 0 \\ x \qquad + z & = & 5 \\ 4y + z & = & 20 \end{array} \ \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\ \begin{array}{rcl} x + y - z & = & 10 \\ -4y + 3z & = & -20 \\ -y + 2z & = & -5 \\ 4y + z & = & 20 \end{array} \ \xrightarrow[\rho_2+\rho_4]{-(1/4)\rho_2+\rho_3}\ \begin{array}{rcl} x + y - z & = & 10 \\ -4y + 3z & = & -20 \\ (5/4)z & = & 0 \\ 4z & = & 0 \end{array}$$

gives the unique solution $(x, y, z) = (5, 5, 0)$.

(f) Here Gauss' method gives

$$\xrightarrow[-2\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3}\ \begin{array}{rcl} 2x + z + w & = & 5 \\ y - w & = & -1 \\ -(5/2)z - (5/2)w & = & -15/2 \\ y - w & = & -1 \end{array} \ \xrightarrow{-\rho_2+\rho_4}\ \begin{array}{rcl} 2x + z + w & = & 5 \\ y - w & = & -1 \\ -(5/2)z - (5/2)w & = & -15/2 \\ 0 & = & 0 \end{array}$$

which shows that there are many solutions.

One.I.1.18 **(a)** From $x = 1 - 3y$ we get that $2(1 - 3y) + y = -3$, giving $y = 1$.

(b) From $x = 1 - 3y$ we get that $2(1 - 3y) + 2y = 0$, leading to the conclusion that $y = 1/2$.

Users of this method must check any potential solutions by substituting back into all the equations.

One.I.1.19 Do the reduction

$$\xrightarrow{-3\rho_1+\rho_2} \begin{aligned} x-y&=1\\ 0&=-3+k \end{aligned}$$

to conclude this system has no solutions if $k\neq 3$ and if $k=3$ then it has infinitely many solutions. It never has a unique solution.

One.I.1.20 Let $x=\sin\alpha$, $y=\cos\beta$, and $z=\tan\gamma$:

$$\begin{aligned} 2x-y+3z&=3\\ 4x+2y-2z&=10\\ 6x-3y+z&=9 \end{aligned} \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} 2x-y+3z&=3\\ 4y-8z&=4\\ -8z&=0 \end{aligned}$$

gives $z=0$, $y=1$, and $x=2$. Note that no α satisfies that requirement.

One.I.1.21 **(a)** Gauss' method

$$\xrightarrow[\substack{-\rho_1+\rho_3\\-2\rho_1+\rho_4}]{-3\rho_1+\rho_2} \begin{aligned} x-3y&=b_1\\ 10y&=-3b_1+b_2\\ 10y&=-b_1+b_3\\ 10y&=-2b_1+b_4 \end{aligned} \xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{aligned} x-3y&=b_1\\ 10y&=-3b_1+b_2\\ 0&=2b_1-b_2+b_3\\ 0&=b_1-b_2+b_4 \end{aligned}$$

shows that this system is consistent if and only if both $b_3=-2b_1+b_2$ and $b_4=-b_1+b_2$.

(b) Reduction

$$\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} x_1+2x_2+3x_3&=b_1\\ x_2-3x_3&=-2b_1+b_2\\ -2x_2+5x_3&=-b_1+b_3 \end{aligned} \xrightarrow{2\rho_2+\rho_3} \begin{aligned} x_1+2x_2+3x_3&=b_1\\ x_2-3x_3&=-2b_1+b_2\\ -x_3&=-5b_1++2b_2+b_3 \end{aligned}$$

shows that each of b_1, b_2, and b_3 can be any real number — this system always has a unique solution.

One.I.1.24 Because $f(1)=2$, $f(-1)=6$, and $f(2)=3$ we get a linear system.

$$\begin{aligned} 1a+1b+c&=2\\ 1a-1b+c&=6\\ 4a+2b+c&=3 \end{aligned}$$

Gauss' method

$$\xrightarrow[-4\rho_1+\rho_2]{-\rho_1+\rho_2} \begin{aligned} a+b+c&=2\\ -2b&=4\\ -2b-3c&=-5 \end{aligned} \xrightarrow{-\rho_2+\rho_3} \begin{aligned} a+b+c&=2\\ -2b&=4\\ -3c&=-9 \end{aligned}$$

shows that the solution is $f(x)=1x^2-2x+3$.

One.I.1.27 We take three cases, first that $a =\neq 0$, second that $a=0$ and $c\neq 0$, and third that both $a=0$ and $c=0$.

For the first, we assume that $a\neq 0$. Then the reduction

$$\xrightarrow{-(c/a)\rho_1+\rho_2} \begin{aligned} ax+\phantom{(-\tfrac{cb}{a}+d)}by&=j\\ (-\tfrac{cb}{a}+d)y&=-\tfrac{cj}{a}+k \end{aligned}$$

shows that this system has a unique solution if and only if $-(cb/a)+d\neq 0$; remember that $a\neq 0$ so that back substitution yields a unique x (observe, by the way, that j and k play no role in the conclusion that there is a unique solution, although if there is a unique solution then they contribute to its value). But $-(cb/a)+d=(ad-bc)/a$ and a fraction is not equal to 0 if and only if its numerator is not equal to 0. This, in this first case, there is a unique solution if and only if $ad-bc\neq 0$.

In the second case, if $a=0$ but $c\neq 0$, then we swap

$$\begin{aligned} cx+dy&=k\\ by&=j \end{aligned}$$

to conclude that the system has a unique solution if and only if $b\neq 0$ (we use the case assumption that $c\neq 0$ to get a unique x in back substitution). But — where $a=0$ and $c\neq 0$ — the condition "$b\neq 0$" is equivalent to the condition "$ad-bc\neq 0$". That finishes the second case.

Finally, for the third case, if both a and c are 0 then the system

$$\begin{aligned} 0x+by&=j\\ 0x+dy&=k \end{aligned}$$

might have no solutions (if the second equation is not a multiple of the first) or it might have infinitely many solutions (if the second equation is a multiple of the first then for each y satisfying both equations, any pair (x,y) will do), but it never has a unique solution. Note that $a=0$ and $c=0$ gives that $ad-bc=0$.

One.I.1.28 Recall that if a pair of lines share two distinct points then they are the same line. That's because two points determine a line, so these two points determine each of the two lines, and so they are the same line.

Thus the lines can share one point (giving a unique solution), share no points (giving no solutions), or share at least two points (which makes them the same line).

One.I.1.31 Yes. This sequence of operations swaps rows i and j

$$\xrightarrow{\rho_i+\rho_j}\ \xrightarrow{-\rho_j+\rho_i}\ \xrightarrow{\rho_i+\rho_j}\ \xrightarrow{-1\rho_i}$$

so the row-swap operation is redundant in the presence of the other two.

One.I.1.35 *This is how the answer was given in the cited source.* A comparison of the units and hundreds columns of this addition shows that there must be a carry from the tens column. The tens column then tells us that $A < H$, so there can be no carry from the units or hundreds columns. The five columns then give the following five equations.

$$\begin{aligned} A+E&=W\\ 2H&=A+10\\ H&=W+1\\ H+T&=E+10\\ A+1&=T \end{aligned}$$

The five linear equations in five unknowns, if solved simultaneously, produce the unique solution: $A = 4$, $T = 5$, $H = 7$, $W = 6$ and $E = 2$, so that the original example in addition was $47474+5272 = 52746$.

Subsection One.I.2: Describing the Solution Set

One.I.2.15 **(a)** 2 **(b)** 3 **(c)** -1 **(d)** Not defined.

One.I.2.16 **(a)** 2×3 **(b)** 3×2 **(c)** 2×2

One.I.2.17 **(a)** $\begin{pmatrix}5\\1\\5\end{pmatrix}$ **(b)** $\begin{pmatrix}20\\-5\end{pmatrix}$ **(c)** $\begin{pmatrix}-2\\4\\0\end{pmatrix}$ **(d)** $\begin{pmatrix}41\\52\end{pmatrix}$ **(e)** Not defined.
(f) $\begin{pmatrix}12\\8\\4\end{pmatrix}$

One.I.2.18 **(a)** This reduction

$$\left(\begin{array}{cc|c}3&6&18\\1&2&6\end{array}\right)\xrightarrow{(-1/3)\rho_1+\rho_2}\left(\begin{array}{cc|c}3&6&18\\0&0&0\end{array}\right)$$

leaves x leading and y free. Making y the parameter, we have $x = 6-2y$ so the solution set is

$$\{\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}y \bigm| y\in\mathbb{R}\}.$$

(b) This reduction

$$\left(\begin{array}{cc|c}1&1&1\\1&-1&-1\end{array}\right)\xrightarrow{-\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&1\\0&-2&-2\end{array}\right)$$

gives the unique solution $y = 1$, $x = 0$. The solution set is

$$\{\begin{pmatrix}0\\1\end{pmatrix}\}.$$

(c) This use of Gauss' method

$$\left(\begin{array}{ccc|c}1&0&1&4\\1&-1&2&5\\4&-1&5&17\end{array}\right)\xrightarrow[-4\rho_1+\rho_3]{-\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&0&1&4\\0&-1&1&1\\0&-1&1&1\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&0&1&4\\0&-1&1&1\\0&0&0&0\end{array}\right)$$

leaves x_1 and x_2 leading with x_3 free. The solution set is

$$\{\begin{pmatrix}4\\-1\\0\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3 \bigm| x_3\in\mathbb{R}\}.$$

(d) This reduction

$$\left(\begin{array}{ccc|c}2&1&-1&2\\2&0&1&3\\1&-1&0&0\end{array}\right)\xrightarrow[-(1/2)\rho_1+\rho_3]{-\rho_1+\rho_2}\left(\begin{array}{ccc|c}2&1&-1&2\\0&-1&2&1\\0&-3/2&1/2&-1\end{array}\right)\xrightarrow{(-3/2)\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&1&-1&2\\0&-1&2&1\\0&0&-5/2&-5/2\end{array}\right)$$

shows that the solution set is a singleton set.

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}$$

(e) This reduction is easy

$$\left(\begin{array}{cccc|c}1&2&-1&0&3\\2&1&0&1&4\\1&-1&1&1&1\end{array}\right)\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&2&-1&0&3\\0&-3&2&1&-2\\0&-3&2&1&-2\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&2&-1&0&3\\0&-3&2&1&-2\\0&0&0&0&0\end{array}\right)$$

and ends with x and y leading, while z and w are free. Solving for y gives $y=(2+2z+w)/3$ and substitution shows that $x+2(2+2z+w)/3-z=3$ so $x=(5/3)-(1/3)z-(2/3)w$, making the solution set

$$\{\begin{pmatrix}5/3\\2/3\\0\\0\end{pmatrix}+\begin{pmatrix}-1/3\\2/3\\1\\0\end{pmatrix}z+\begin{pmatrix}-2/3\\1/3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

(f) The reduction

$$\left(\begin{array}{cccc|c}1&0&1&1&4\\2&1&0&-1&2\\3&1&1&0&7\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&0&1&1&4\\0&1&-2&-3&-6\\0&1&-2&-3&-5\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&0&1&1&4\\0&1&-2&-3&-6\\0&0&0&0&1\end{array}\right)$$

shows that there is no solution — the solution set is empty.

One.I.2.19 **(a)** This reduction

$$\left(\begin{array}{ccc|c}2&1&-1&1\\4&-1&0&3\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{ccc|c}2&1&-1&1\\0&-3&2&1\end{array}\right)$$

ends with x and y leading while z is free. Solving for y gives $y=(1-2z)/(-3)$, and then substitution $2x+(1-2z)/(-3)-z=1$ shows that $x=((4/3)+(1/3)z)/2$. Hence the solution set is

$$\{\begin{pmatrix}2/3\\-1/3\\0\end{pmatrix}+\begin{pmatrix}1/6\\2/3\\1\end{pmatrix}z \mid z\in\mathbb{R}\}.$$

(b) This application of Gauss' method

$$\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&-1&3\\1&2&3&-1&7\end{array}\right)\xrightarrow{-\rho_1+\rho_3}\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&-1&3\\0&2&4&-1&6\end{array}\right)\xrightarrow{-2\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&-1&3\\0&0&0&1&0\end{array}\right)$$

leaves x, y, and w leading. The solution set is

$$\{\begin{pmatrix}1\\3\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}z \mid z\in\mathbb{R}\}.$$

(c) This row reduction

$$\left(\begin{array}{cccc|c}1&-1&1&0&0\\0&1&0&1&0\\3&-2&3&1&0\\0&-1&0&-1&0\end{array}\right)\xrightarrow{-3\rho_1+\rho_3}\left(\begin{array}{cccc|c}1&-1&1&0&0\\0&1&0&1&0\\0&1&0&1&0\\0&-1&0&-1&0\end{array}\right)\xrightarrow[\rho_2+\rho_4]{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&-1&1&0&0\\0&1&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)$$

ends with z and w free. The solution set is

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

(d) Gauss' method done in this way

$$\left(\begin{array}{ccccc|c}1&2&3&1&-1&1\\3&-1&1&1&1&3\end{array}\right)\xrightarrow{-3\rho_1+\rho_2}\left(\begin{array}{ccccc|c}1&2&3&1&-1&1\\0&-7&-8&-2&4&0\end{array}\right)$$

ends with c, d, and e free. Solving for b shows that $b=(8c+2d-4e)/(-7)$ and then substitution $a+2(8c+2d-4e)/(-7)+3c+1d-1e=1$ shows that $a=1-(5/7)c-(3/7)d-(1/7)e$ and so the solution set is

$$\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-5/7\\-8/7\\1\\0\\0\end{pmatrix}c+\begin{pmatrix}-3/7\\-2/7\\0\\1\\0\end{pmatrix}d+\begin{pmatrix}-1/7\\4/7\\0\\0\\1\end{pmatrix}e \mid c,d,e\in\mathbb{R}\}.$$

One.I.2.20 For each problem we get a system of linear equations by looking at the equations of components.

(a) $k=5$

(b) The second components show that $i=2$, the third components show that $j=1$.

(c) $m=-4$, $n=2$

One.I.2.23 **(a)** Gauss' method here gives

$$\left(\begin{array}{cccc|c}1&2&0&-1&a\\2&0&1&0&b\\1&1&0&2&c\end{array}\right)\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&2&0&-1&a\\0&-4&1&2&-2a+b\\0&-1&0&3&-a+c\end{array}\right)$$

$$\xrightarrow{-(1/4)\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&2&0&-1&a\\0&-4&1&2&-2a+b\\0&0&-1/4&5/2&-(1/2)a-(1/4)b+c\end{array}\right),$$

leaving w free. Solve: $z=2a+b-4c+10w$, and $-4y=-2a+b-(2a+b-4c+10w)-2w$ so $y=a-c+3w$, and $x=a-2(a-c+3w)+w=-a+2c-5w$. Therefore the solution set is this.

$$\{\begin{pmatrix}-a+2c\\a-c\\2a+b-4c\\0\end{pmatrix}+\begin{pmatrix}-5\\3\\10\\1\end{pmatrix}w \mid w\in\mathbb{R}\}$$

(b) Plug in with $a=3$, $b=1$, and $c=-2$.

$$\{\begin{pmatrix}-7\\5\\15\\0\end{pmatrix}+\begin{pmatrix}-5\\3\\10\\1\end{pmatrix}w \mid w\in\mathbb{R}\}$$

One.I.2.24 Leaving the comma out, say by writing a_{123}, is ambiguous because it could mean $a_{1,23}$ or $a_{12,3}$.

One.I.2.25 **(a)** $\begin{pmatrix}2&3&4&5\\3&4&5&6\\4&5&6&7\\5&6&7&8\end{pmatrix}$ **(b)** $\begin{pmatrix}1&-1&1&-1\\-1&1&-1&1\\1&-1&1&-1\\-1&1&-1&1\end{pmatrix}$

One.I.2.27 **(a)** Plugging in $x=1$ and $x=-1$ gives

$$\begin{array}{l}a+b+c=2\\a-b+c=6\end{array}\xrightarrow{-\rho_1+\rho_2}\begin{array}{l}a+b+c=2\\-2b=4\end{array}$$

so the set of functions is $\{f(x)=(4-c)x^2-2x+c \mid c\in\mathbb{R}\}$.

(b) Putting in $x=1$ gives

$$a+b+c=2$$

so the set of functions is $\{f(x)=(2-b-c)x^2+bx+c \mid b,c\in\mathbb{R}\}$.

Subsection One.I.3: General = Particular + Homogeneous

One.I.3.15 For the arithmetic to these, see the answers from the prior subsection.

(a) The solution set is

$$\{\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}y \mid y\in\mathbb{R}\}.$$

Here the particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}6\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-2\\1\end{pmatrix}y \mid y\in\mathbb{R}\}.$$

(b) The solution set is

$$\{\begin{pmatrix}0\\1\end{pmatrix}\}.$$

The particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}0\\1\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}0\\0\end{pmatrix}\}$$

(c) The solution set is

$$\{\begin{pmatrix}4\\-1\\0\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3 \mid x_3\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}4\\-1\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-1\\1\\1\end{pmatrix}x_3 \mid x_3\in\mathbb{R}\}.$$

(d) The solution set is a singleton

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}0\\0\\0\end{pmatrix}t \mid t\in\mathbb{R}\}.$$

(e) The solution set is

$$\{\begin{pmatrix}5/3\\2/3\\0\\0\end{pmatrix}+\begin{pmatrix}-1/3\\2/3\\1\\0\end{pmatrix}z+\begin{pmatrix}-2/3\\1/3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

A particular solution and the solution set for the associated homogeneous system are

$$\begin{pmatrix}5/2\\2/3\\0\\0\end{pmatrix}\quad\text{and}\quad\{\begin{pmatrix}-1/3\\2/3\\1\\0\end{pmatrix}z+\begin{pmatrix}-2/3\\1/3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

(f) This system's solution set is empty. Thus, there is no particular solution. The solution set of the associated homogeneous system is

$$\{\begin{pmatrix}-1\\2\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\3\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}.$$

One.I.3.17 Just plug them in and see if they satisfy all three equations.

(a) No.

(b) Yes.

(c) Yes.

One.I.3.18 Gauss' method on the associated homogeneous system gives

$$\left(\begin{array}{cccc|c}1&-1&0&1&0\\2&3&-1&0&0\\0&1&1&1&0\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&-1&0&1&0\\0&5&-1&-2&0\\0&1&1&1&0\end{array}\right)\xrightarrow{-(1/5)\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&-1&0&1&0\\0&5&-1&-2&0\\0&0&6/5&7/5&0\end{array}\right)$$

so this is the solution to the homogeneous problem:

$$\{\begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix}w \bigm| w\in\mathbb{R}\}.$$

(a) That vector is indeed a particular solution so the required general solution is

$$\{\begin{pmatrix}0\\0\\0\\4\end{pmatrix}+\begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix}w \bigm| w\in\mathbb{R}\}.$$

(b) That vector is a particular solution so the required general solution is

$$\{\begin{pmatrix}-5\\1\\-7\\10\end{pmatrix}+\begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix}w \bigm| w\in\mathbb{R}\}.$$

(c) That vector is not a solution of the system since it does not satisfy the third equation. No such general solution exists.

One.I.3.20 **(a)** Nonsingular:

$$\xrightarrow{-\rho_1+\rho_2}\begin{pmatrix}1&2\\0&1\end{pmatrix}$$

ends with each row containing a leading entry.

(b) Singular:

$$\xrightarrow{3\rho_1+\rho_2}\begin{pmatrix}1&2\\0&0\end{pmatrix}$$

ends with row 2 without a leading entry.

(c) Neither. A matrix must be square for either word to apply.

(d) Singular.

(e) Nonsingular.

One.I.3.21 In each case we must decide if the vector is a linear combination of the vectors in the set.

(a) Yes. Solve

$$c_1\begin{pmatrix}1\\4\end{pmatrix}+c_2\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$$

with

$$\left(\begin{array}{cc|c}1&1&2\\4&5&3\end{array}\right)\xrightarrow{-4\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&1&-5\end{array}\right)$$

to conclude that there are c_1 and c_2 giving the combination.

(b) No. The reduction

$$\left(\begin{array}{cc|c}2&1&-1\\1&0&0\\0&1&1\end{array}\right)\xrightarrow{-(1/2)\rho_1+\rho_2}\left(\begin{array}{cc|c}2&1&-1\\0&-1/2&1/2\\0&1&1\end{array}\right)\xrightarrow{2\rho_2+\rho_3}\left(\begin{array}{cc|c}2&1&-1\\0&-1/2&1/2\\0&0&2\end{array}\right)$$

shows that

$$c_1\begin{pmatrix}2\\1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

has no solution.

(c) Yes. The reduction

$$\left(\begin{array}{cccc|c}1&2&3&4&1\\0&1&3&2&3\\4&5&0&1&0\end{array}\right)\xrightarrow{-4\rho_1+\rho_3}\left(\begin{array}{cccc|c}1&2&3&4&1\\0&1&3&2&3\\0&-3&-12&-15&-4\end{array}\right)\xrightarrow{3\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&2&3&4&1\\0&1&3&2&3\\0&0&-3&-9&5\end{array}\right)$$

shows that there are infinitely many ways

$$\{\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}=\begin{pmatrix}-10\\8\\-5/3\\0\end{pmatrix}+\begin{pmatrix}-9\\7\\-3\\1\end{pmatrix}c_4\;\big|\;c_4\in\mathbb{R}\}$$

to write

$$\begin{pmatrix}1\\3\\0\end{pmatrix}=c_1\begin{pmatrix}1\\0\\4\end{pmatrix}+c_2\begin{pmatrix}2\\1\\5\end{pmatrix}+c_3\begin{pmatrix}3\\3\\0\end{pmatrix}+c_4\begin{pmatrix}4\\2\\1\end{pmatrix}.$$

(d) No. Look at the third components.

One.I.3.24 Assume $\vec{s},\vec{t}\in\mathbb{R}^n$ and write

$$\vec{s}=\begin{pmatrix}s_1\\\vdots\\s_n\end{pmatrix}\quad\text{and}\quad\vec{t}=\begin{pmatrix}t_1\\\vdots\\t_n\end{pmatrix}.$$

Also let $a_{i,1}x_1+\cdots+a_{i,n}x_n=0$ be the i-th equation in the homogeneous system.

(a) The check is easy:

$$\begin{aligned}a_{i,1}(s_1+t_1)+\cdots+a_{i,n}(s_n+t_n)&=(a_{i,1}s_1+\cdots+a_{i,n}s_n)+(a_{i,1}t_1+\cdots+a_{i,n}t_n)\\&=0+0.\end{aligned}$$

(b) This one is similar:

$$a_{i,1}(3s_1)+\cdots+a_{i,n}(3s_n)=3(a_{i,1}s_1+\cdots+a_{i,n}s_n)=3\cdot 0=0.$$

(c) This one is not much harder:

$$\begin{aligned}a_{i,1}(ks_1+mt_1)+\cdots+a_{i,n}(ks_n+mt_n)&=k(a_{i,1}s_1+\cdots+a_{i,n}s_n)+m(a_{i,1}t_1+\cdots+a_{i,n}t_n)\\&=k\cdot 0+m\cdot 0.\end{aligned}$$

What is wrong with that argument is that any linear combination of the zero vector yields the zero vector again.

Subsection One.II.1: Vectors in Space

One.II.1.1 **(a)** $\begin{pmatrix}2\\1\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\2\end{pmatrix}$ **(c)** $\begin{pmatrix}4\\0\\-3\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$

One.II.1.2 **(a)** No, their canonical positions are different.

$$\begin{pmatrix}1\\-1\end{pmatrix}\qquad\begin{pmatrix}0\\3\end{pmatrix}$$

(b) Yes, their canonical positions are the same.

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

One.II.1.3 That line is this set.

$$\{\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}+\begin{pmatrix}7\\9\\-2\\4\end{pmatrix}t\;\big|\;t\in\mathbb{R}\}$$

Note that this system

$$\begin{aligned}-2+7t&=1\\1+9t&=0\\1-2t&=2\\0+4t&=1\end{aligned}$$

has no solution. Thus the given point is not in the line.

One.II.1.4 **(a)** Note that

$$\begin{pmatrix}2\\2\\2\\0\end{pmatrix}-\begin{pmatrix}1\\1\\5\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\-3\\1\end{pmatrix}\qquad\begin{pmatrix}3\\1\\0\\4\end{pmatrix}-\begin{pmatrix}1\\1\\5\\-1\end{pmatrix}=\begin{pmatrix}2\\0\\-5\\5\end{pmatrix}$$

and so the plane is this set.

$$\{\begin{pmatrix}1\\1\\5\\-1\end{pmatrix}+\begin{pmatrix}1\\1\\-3\\1\end{pmatrix}t+\begin{pmatrix}2\\0\\-5\\5\end{pmatrix}s \bigm| t,s\in\mathbb{R}\}$$

(b) No; this system

$$\begin{aligned}1+1t+2s&=0\\1+1t&=0\\5-3t-5s&=0\\-1+1t+5s&=0\end{aligned}$$

has no solution.

One.II.1.6 The points of coincidence are solutions of this system.

$$\begin{aligned}t&=1+2m\\t+s&=1+3k\\t+3s&=4m\end{aligned}$$

Gauss' method

$$\left(\begin{array}{cccc|c}1&0&0&-2&1\\1&1&-3&0&1\\1&3&0&-4&0\end{array}\right)\xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2}\left(\begin{array}{cccc|c}1&0&0&-2&1\\0&1&-3&2&0\\0&3&0&-2&-1\end{array}\right)\xrightarrow{-3\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&0&0&-2&1\\0&1&-3&2&0\\0&0&9&-8&-1\end{array}\right)$$

gives $k=-(1/9)+(8/9)m$, so $s=-(1/3)+(2/3)m$ and $t=1+2m$. The intersection is this.

$$\{\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\3\\0\end{pmatrix}(-\tfrac{1}{9}+\tfrac{8}{9}m)+\begin{pmatrix}2\\0\\4\end{pmatrix}m \bigm| m\in\mathbb{R}\}=\{\begin{pmatrix}1\\2/3\\0\end{pmatrix}+\begin{pmatrix}2\\8/3\\4\end{pmatrix}m \bigm| m\in\mathbb{R}\}$$

One.II.1.7 **(a)** The system

$$\begin{aligned}1&=1\\1+t&=3+s\\2+t&=-2+2s\end{aligned}$$

gives $s=6$ and $t=8$, so this is the solution set.

$$\{\begin{pmatrix}1\\9\\10\end{pmatrix}\}$$

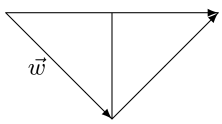
(b) This system

$$\begin{aligned}2+t&=0\\t&=s+4w\\1-t&=2s+w\end{aligned}$$

gives $t=-2$, $w=-1$, and $s=2$ so their intersection is this point.

$$\begin{pmatrix}0\\-2\\3\end{pmatrix}$$

One.II.1.11 *This is how the answer was given in the cited source.* The vector triangle is as follows, so $\vec{w}=3\sqrt{2}$ from the north west.



Subsection One.II.2: Length and Angle Measures

One.II.2.10 **(a)** $\sqrt{3^2+1^2}=\sqrt{10}$ **(b)** $\sqrt{5}$ **(c)** $\sqrt{18}$ **(d)** 0 **(e)** $\sqrt{3}$

One.II.2.11 **(a)** $\arccos(9/\sqrt{85})\approx 0.22$ radians **(b)** $\arccos(8/\sqrt{85})\approx 0.52$ radians **(c)** Not defined.

One.II.2.12 We express each displacement as a vector (rounded to one decimal place because that's the accuracy of the problem's statement) and add to find the total displacement (ignoring the curvature of the earth).

$$\begin{pmatrix}0.0\\1.2\end{pmatrix}+\begin{pmatrix}3.8\\-4.8\end{pmatrix}+\begin{pmatrix}4.0\\0.1\end{pmatrix}+\begin{pmatrix}3.3\\5.6\end{pmatrix}=\begin{pmatrix}11.1\\2.1\end{pmatrix}$$

The distance is $\sqrt{11.1^2+2.1^2}\approx 11.3$.

One.II.2.15 **(a)** We can use the x-axis.

$$\arccos(\frac{(1)(1)+(0)(1)}{\sqrt{1}\sqrt{2}})\approx 0.79\text{ radians}$$

(b) Again, use the x-axis.

$$\arccos(\frac{(1)(1)+(0)(1)+(0)(1)}{\sqrt{1}\sqrt{3}})\approx 0.96\text{ radians}$$

(c) The x-axis worked before and it will work again.

$$\arccos(\frac{(1)(1)+\cdots+(0)(1)}{\sqrt{1}\sqrt{n}})=\arccos(\frac{1}{\sqrt{n}})$$

(d) Using the formula from the prior item, $\lim_{n\to\infty}\arccos(1/\sqrt{n})=\pi/2$ radians.

One.II.2.17 Assume that $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^n$ have components $u_1,\ldots,u_n,v_1,\ldots,w_n$.

(a) Dot product is right-distributive.

$$\begin{aligned}(\vec{u}+\vec{v})\cdot\vec{w}&=[\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix}+\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}]\cdot\begin{pmatrix}w_1\\\vdots\\w_n\end{pmatrix}\\&=\begin{pmatrix}u_1+v_1\\\vdots\\u_n+v_n\end{pmatrix}\cdot\begin{pmatrix}w_1\\\vdots\\w_n\end{pmatrix}\\&=(u_1+v_1)w_1+\cdots+(u_n+v_n)w_n\\&=(u_1w_1+\cdots+u_nw_n)+(v_1w_1+\cdots+v_nw_n)\\&=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}\end{aligned}$$

(b) Dot product is also left distributive: $\vec{w}\cdot(\vec{u}+\vec{v})=\vec{w}\cdot\vec{u}+\vec{w}\cdot\vec{v}$. The proof is just like the prior one.

(c) Dot product commutes.

$$\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix}\cdot\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}=u_1v_1+\cdots+u_nv_n=v_1u_1+\cdots+v_nu_n=\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}\cdot\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix}$$

(d) Because $\vec{u}\cdot\vec{v}$ is a scalar, not a vector, the expression $(\vec{u}\cdot\vec{v})\cdot\vec{w}$ makes no sense; the dot product of a scalar and a vector is not defined.

(e) This is a vague question so it has many answers. Some are (1) $k(\vec{u}\cdot\vec{v})=(k\vec{u})\cdot\vec{v}$ and $k(\vec{u}\cdot\vec{v})=\vec{u}\cdot(k\vec{v})$, (2) $k(\vec{u}\cdot\vec{v})\neq(k\vec{u})\cdot(k\vec{v})$ (in general; an example is easy to produce), and (3) $\|k\vec{v}\,\|=k^2\|\vec{v}\,\|$ (the connection between norm and dot product is that the square of the norm is the dot product of a vector with itself).

One.II.2.20 We prove that a vector has length zero if and only if all its components are zero.

Let $\vec{u}\in\mathbb{R}^n$ have components $u_1,\ldots,u_n$. Recall that the square of any real number is greater than or equal to zero, with equality only when that real is zero. Thus $\|\vec{u}\,\|^2={u_1}^2+\cdots+{u_n}^2$ is a sum of numbers greater than or equal to zero, and so is itself greater than or equal to zero, with equality if and only if each u_i is zero. Hence $\|\vec{u}\,\|=0$ if and only if all the components of $\vec{u}$ are zero.

One.II.2.21 We can easily check that

$$(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})$$

is on the line connecting the two, and is equidistant from both. The generalization is obvious.

One.II.2.24 Assume that $\vec{u},\vec{v}\in\mathbb{R}^n$ both have length 1. Apply Cauchy-Schwartz: $|\vec{u}\cdot\vec{v}|\leq\|\vec{u}\,\|\,\|\vec{v}\,\|=1$.

To see that 'less than' can happen, in $\mathbb{R}^2$ take

$$\vec{u}=\begin{pmatrix}1\\0\end{pmatrix}\qquad\vec{v}=\begin{pmatrix}0\\1\end{pmatrix}$$

and note that $\vec{u}\cdot\vec{v}=0$. For 'equal to', note that $\vec{u}\cdot\vec{u}=1$.

One.II.2.32 Suppose that $\vec{u},\vec{v}\in\mathbb{R}^n$. If $\vec{u}$ and $\vec{v}$ are perpendicular then

$$\|\vec{u}+\vec{v}\,\|^2=(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})=\vec{u}\cdot\vec{u}+2\,\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}=\vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}=\|\vec{u}\,\|^2+\|\vec{v}\,\|^2$$

(the third equality holds because $\vec{u}\cdot\vec{v}=0$).

One.II.2.37 Let

$$\vec{u}=\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix},\quad\vec{v}=\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}\quad\vec{w}=\begin{pmatrix}w_1\\\vdots\\w_n\end{pmatrix}$$

and then

$$\begin{aligned}
\vec{u}\cdot\big(k\vec{v}+m\vec{w}\big)&=\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix}\cdot\big(\begin{pmatrix}kv_1\\\vdots\\kv_n\end{pmatrix}+\begin{pmatrix}mw_1\\\vdots\\mw_n\end{pmatrix}\big)\\
&=\begin{pmatrix}u_1\\\vdots\\u_n\end{pmatrix}\cdot\begin{pmatrix}kv_1+mw_1\\\vdots\\kv_n+mw_n\end{pmatrix}\\
&=u_1(kv_1+mw_1)+\cdots+u_n(kv_n+mw_n)\\
&=ku_1v_1+mu_1w_1+\cdots+ku_nv_n+mu_nw_n\\
&=(ku_1v_1+\cdots+ku_nv_n)+(mu_1w_1+\cdots+mu_nw_n)\\
&=k(\vec{u}\cdot\vec{v})+m(\vec{u}\cdot\vec{w})
\end{aligned}$$

as required.

One.II.2.38 For $x,y\in\mathbb{R}^+$, set

$$\vec{u}=\begin{pmatrix}\sqrt{x}\\\sqrt{y}\end{pmatrix}\qquad\vec{v}=\begin{pmatrix}\sqrt{y}\\\sqrt{x}\end{pmatrix}$$

so that the Cauchy-Schwartz inequality asserts that (after squaring)

$$\begin{aligned}
(\sqrt{x}\sqrt{y}+\sqrt{y}\sqrt{x})^2&\leq(\sqrt{x}\sqrt{x}+\sqrt{y}\sqrt{y})(\sqrt{y}\sqrt{y}+\sqrt{x}\sqrt{x})\\
(2\sqrt{x}\sqrt{y})^2&\leq(x+y)^2\\
\sqrt{xy}&\leq\frac{x+y}{2}
\end{aligned}$$

as desired.

Subsection One.III.1: Gauss-Jordan Reduction

One.III.1.7 These answers show only the Gauss-Jordan reduction. With it, describing the solution set is easy.

(a) $\left(\begin{array}{cc|c}1&1&2\\1&-1&0\end{array}\right)\xrightarrow{-\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&-2&-2\end{array}\right)\xrightarrow{-(1/2)\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&1&1\end{array}\right)\xrightarrow{-\rho_2+\rho_1}\left(\begin{array}{cc|c}1&0&1\\0&1&1\end{array}\right)$

(b) $\left(\begin{array}{ccc|c}1&0&-1&4\\2&2&0&1\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{ccc|c}1&0&-1&4\\0&2&2&-7\end{array}\right)\xrightarrow{(1/2)\rho_2}\left(\begin{array}{ccc|c}1&0&-1&4\\0&1&1&-7/2\end{array}\right)$

(c)

$$\left(\begin{array}{cc|c}3&-2&1\\6&1&1/2\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{cc|c}3&-2&1\\0&5&-3/2\end{array}\right)\xrightarrow[(1/5)\rho_2]{(1/3)\rho_1}\left(\begin{array}{cc|c}1&-2/3&1/3\\0&1&-3/10\end{array}\right)\xrightarrow{(2/3)\rho_2+\rho_1}\left(\begin{array}{cc|c}1&0&2/15\\0&1&-3/10\end{array}\right)$$

(d) A row swap here makes the arithmetic easier.

$$\left(\begin{array}{ccc|c}2&-1&0&-1\\1&3&-1&5\\0&1&2&5\end{array}\right)\xrightarrow{-(1/2)\rho_1+\rho_2}\left(\begin{array}{ccc|c}2&-1&0&-1\\0&7/2&-1&11/2\\0&1&2&5\end{array}\right)\xrightarrow{\rho_2\leftrightarrow\rho_3}\left(\begin{array}{ccc|c}2&-1&0&-1\\0&1&2&5\\0&7/2&-1&11/2\end{array}\right)$$

$$\xrightarrow{-(7/2)\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&-1&0&-1\\0&1&2&5\\0&0&-8&-12\end{array}\right)\xrightarrow[-(1/8)\rho_2]{(1/2)\rho_1}\left(\begin{array}{ccc|c}1&-1/2&0&-1/2\\0&1&2&5\\0&0&1&3/2\end{array}\right)$$

$$\xrightarrow{-2\rho_3+\rho_2}\left(\begin{array}{ccc|c}1&-1/2&0&-1/2\\0&1&0&2\\0&0&1&3/2\end{array}\right)\xrightarrow{(1/2)\rho_2+\rho_1}\left(\begin{array}{ccc|c}1&0&0&1/2\\0&1&0&2\\0&0&1&3/2\end{array}\right)$$

One.III.1.8 Use Gauss-Jordan reduction.

(a) $\xrightarrow{-(1/2)\rho_1+\rho_2}\begin{pmatrix}2&1\\0&5/2\end{pmatrix}\xrightarrow[(2/5)\rho_2]{(1/2)\rho_1}\begin{pmatrix}1&1/2\\0&1\end{pmatrix}\xrightarrow{-(1/2)\rho_2+\rho_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(b) $\xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2}\begin{pmatrix}1&3&1\\0&-6&2\\0&0&-2\end{pmatrix}\xrightarrow[-(1/2)\rho_3]{-(1/6)\rho_2}\begin{pmatrix}1&3&1\\0&1&-1/3\\0&0&1\end{pmatrix}\xrightarrow[-\rho_3+\rho_1]{(1/3)\rho_3+\rho_2}\begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{-3\rho_2+\rho_1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

(c)

$$\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}\begin{pmatrix}1&0&3&1&2\\0&4&-1&0&3\\0&4&-1&-2&-4\end{pmatrix}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&0&3&1&2\\0&4&-1&0&3\\0&0&0&-2&-7\end{pmatrix}$$

$$\xrightarrow[-(1/2)\rho_3]{(1/4)\rho_2}\begin{pmatrix}1&0&3&1&2\\0&1&-1/4&0&3/4\\0&0&0&1&7/2\end{pmatrix}\xrightarrow{-\rho_3+\rho_1}\begin{pmatrix}1&0&3&0&-3/2\\0&1&-1/4&0&3/4\\0&0&0&1&7/2\end{pmatrix}$$

(d)

$$\xrightarrow{\rho_1\leftrightarrow\rho_3}\begin{pmatrix}1&5&1&5\\0&0&5&6\\0&1&3&2\end{pmatrix}\xrightarrow{\rho_2\leftrightarrow\rho_3}\begin{pmatrix}1&5&1&5\\0&1&3&2\\0&0&5&6\end{pmatrix}\xrightarrow{(1/5)\rho_3}\begin{pmatrix}1&5&1&19/5\\0&1&3&2\\0&0&1&6/5\end{pmatrix}$$

$$\xrightarrow[-\rho_3+\rho_1]{-3\rho_3+\rho_2}\begin{pmatrix}1&3&0&-6/5\\0&1&0&-8/5\\0&0&1&6/5\end{pmatrix}\xrightarrow{-5\rho_2+\rho_1}\begin{pmatrix}1&0&0&59/5\\0&1&0&-8/5\\0&0&1&6/5\end{pmatrix}$$

One.III.1.9 For the "Gauss" halves, see the answers to Exercise 19.

(a) The "Jordan" half goes this way.

$$\xrightarrow[-(1/3)\rho_2]{(1/2)\rho_1}\left(\begin{array}{ccc|c}1&1/2&-1/2&1/2\\0&1&-2/3&-1/3\end{array}\right)\xrightarrow{-(1/2)\rho_2+\rho_1}\left(\begin{array}{ccc|c}1&0&-1/6&2/3\\0&1&-2/3&-1/3\end{array}\right)$$

The solution set is this

$$\{\begin{pmatrix}2/3\\-1/3\\0\end{pmatrix}+\begin{pmatrix}1/6\\2/3\\1\end{pmatrix}z \bigm| z\in\mathbb{R}\}$$

(b) The second half is

$$\xrightarrow{\rho_3+\rho_2}\left(\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&0&3\\0&0&0&1&0\end{array}\right)$$

so the solution is this.

$$\{\begin{pmatrix}1\\3\\0\\0\end{pmatrix}+\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}z \bigm| z\in\mathbb{R}\}$$

(c) This Jordan half

$$\xrightarrow{\rho_2+\rho_1}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)$$

gives

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}$$

(of course, the zero vector could be omitted from the description).

(d) The "Jordan" half

$$\xrightarrow{-(1/7)\rho_2}\left(\begin{array}{ccccc|c}1&2&3&1&-1&1\\0&1&8/7&2/7&-4/7&0\end{array}\right)\xrightarrow{-2\rho_2+\rho_1}\left(\begin{array}{ccccc|c}1&0&5/7&3/7&1/7&1\\0&1&8/7&2/7&-4/7&0\end{array}\right)$$

ends with this solution set.

$$\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-5/7\\-8/7\\1\\0\\0\end{pmatrix}c+\begin{pmatrix}-3/7\\-2/7\\0\\1\\0\end{pmatrix}d+\begin{pmatrix}-1/7\\4/7\\0\\0\\1\end{pmatrix}e \mid c,d,e\in\mathbb{R}\}$$

One.III.1.11 In the cases listed below, we take $a,b\in\mathbb{R}$. Thus, some canonical forms listed below actually include infinitely many cases. In particular, they includes the cases $a=0$ and $b=0$.

(a) $\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}1&a\\0&0\end{pmatrix}$, $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(b) $\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&a&b\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&1&a\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&0&a\\0&1&b\end{pmatrix}$, $\begin{pmatrix}1&a&0\\0&0&1\end{pmatrix}$, $\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}$

(c) $\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}1&a\\0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$

(d) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&a&b\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&1&a\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&0&a\\0&1&b\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&a&0\\0&0&1\\0&0&0\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

One.III.1.12 A nonsingular homogeneous linear system has a unique solution. So a nonsingular matrix must reduce to a (square) matrix that is all 0's except for 1's down the upper-left to lower-right diagonal, e.g.,

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad\text{or}\quad\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad\text{etc.}$$

Subsection One.III.2: Row Equivalence

One.III.2.11 Bring each to reduced echelon form and compare.

(a) The first gives

$$\xrightarrow{-4\rho_1+\rho_2}\begin{pmatrix}1&2\\0&0\end{pmatrix}$$

while the second gives

$$\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}1&2\\0&1\end{pmatrix}\xrightarrow{-2\rho_2+\rho_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

The two reduced echelon form matrices are not identical, and so the original matrices are not row equivalent.

(b) The first is this.

$$\xrightarrow[-5\rho_1+\rho_3]{-3\rho_1+\rho_2}\begin{pmatrix}1&0&2\\0&-1&-5\\0&-1&-5\end{pmatrix}\xrightarrow{-\rho_2+\rho_3}\begin{pmatrix}1&0&2\\0&-1&-5\\0&0&0\end{pmatrix}\xrightarrow{-\rho_2}\begin{pmatrix}1&0&2\\0&1&5\\0&0&0\end{pmatrix}$$

The second is this.

$$\xrightarrow{-2\rho_1+\rho_3}\begin{pmatrix}1&0&2\\0&2&10\\0&0&0\end{pmatrix}\xrightarrow{(1/2)\rho_2}\begin{pmatrix}1&0&2\\0&1&5\\0&0&0\end{pmatrix}$$

These two are row equivalent.

(c) These two are not row equivalent because they have different sizes.

(d) The first,

$$\xrightarrow{\rho_1+\rho_2}\begin{pmatrix}1&1&1\\0&3&3\end{pmatrix}\xrightarrow{(1/3)\rho_2}\begin{pmatrix}1&1&1\\0&1&1\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}$$

and the second.

$$\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}2&2&5\\0&3&-1\end{pmatrix}\underset{(1/3)\rho_2}{\xrightarrow{(1/2)\rho_1}}\begin{pmatrix}1&1&5/2\\0&1&-1/3\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&0&17/6\\0&1&-1/3\end{pmatrix}$$

These are not row equivalent.

(e) Here the first is

$$\xrightarrow{(1/3)\rho_2}\begin{pmatrix}1&1&1\\0&0&1\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}$$

while this is the second.

$$\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}1&-1&1\\0&1&2\end{pmatrix}\xrightarrow{\rho_2+\rho_1}\begin{pmatrix}1&0&3\\0&1&2\end{pmatrix}$$

These are not row equivalent.

One.III.2.17 Here are two.

$$\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}$$

One.III.2.18 Any two $n\times n$ nonsingular matrices have the same reduced echelon form, namely the matrix with all 0's except for 1's down the diagonal.

$$\begin{pmatrix}1&0&&0\\0&1&&0\\&&\ddots&\\0&0&&1\end{pmatrix}$$

Two 2×2 singular matrices need not row equivalent.

$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

One.III.2.19 Since there is one and only one reduced echelon form matrix in each class, we can just list the possible reduced echelon form matrices.

For that list, see the answer for Exercise 11.

One.III.2.21 **(a)** As in the base case we will argue that ℓ_2 isn't less than k_2 and that it also isn't greater. To obtain a contradiction, assume that $\ell_2\leq k_2$ (the $k_2\leq\ell_2$ case, and the possibility that either or both is a zero row, are left to the reader). Consider the $i=2$ version of the equation that gives each row of B as a linear combination of the rows of D. Focus on the ℓ_1-th and ℓ_2-th component equations.

$$b_{2,\ell_1}=c_{2,1}d_{1,\ell_1}+c_{2,2}d_{2,\ell_1}+\cdots+c_{2,m}d_{m,\ell_1}b_{2,\ell_2}\qquad=c_{2,1}d_{1,\ell_2}+c_{2,2}d_{2,\ell_2}+\cdots+c_{2,m}d_{m,\ell_2}$$

The first of these equations shows that $c_{2,1}$ is zero because δ_{1,ℓ_1} is not zero, but since both matrices are in echelon form, each of the entries d_{2,ℓ_1}, ..., d_{m,ℓ_1}, and b_{2,ℓ_1} is zero. Now, with the second equation, b_{2,ℓ_2} is nonzero as it leads its row, $c_{2,1}$ is zero by the prior sentence, and each of d_{3,ℓ_2}, ..., d_{m,ℓ_2} is zero because D is in echelon form and we've assumed that $\ell_2\leq k_2$. Thus, this second equation shows that d_{2,ℓ_2} is nonzero and so $k_2\leq\ell_2$. Therefore $k_2=\ell_2$.

(b) For the inductive step assume that $\ell_1=k_1$, ..., $\ell_j=k_j$ (where $1\leq j<m$); we will show that implies $\ell_{j+1}=k_{j+1}$.

We do the $\ell_{j+1}\leq k_{j+1}<\infty$ case here—the other cases are then easy. Consider the ρ_{j+1} version of the vector equation:

$$\begin{aligned}
&\begin{pmatrix}0&\ldots&0&\beta_{j+1,\ell_{j_1}}&\ldots&\beta_{j+1,n}\end{pmatrix}\\
&\quad=\; c_{j+1,1}\begin{pmatrix}0&\ldots&\delta_{1,k_1}&\ldots&\delta_{1,k_j}&\ldots&\delta_{1,k_{j+1}}&\ldots&\delta_{1,k_m}&\ldots\end{pmatrix}\\
&\qquad\vdots\\
&\qquad+c_{j+1,j}\begin{pmatrix}0&\ldots&0&\ldots&\delta_{j,k_j}&\ldots&\delta_{j,k_{j+1}}&\ldots&\delta_{j,k_m}&\ldots\end{pmatrix}\\
&\qquad+c_{j+1,j+1}\begin{pmatrix}0&\ldots&0&\ldots&0&\ldots&\delta_{j+1,k_{j+1}}&\ldots&\delta_{j+1,k_m}&\ldots\end{pmatrix}\\
&\qquad\vdots\\
&\qquad+c_{j+1,m}\begin{pmatrix}0&\ldots&0&\ldots&0&\ldots&0&\ldots&\delta_{m,k_m}&\ldots\end{pmatrix}
\end{aligned}$$

Knowing that $\ell_1 = k_1, \ldots, \ell_j = k_j$, consider the ℓ_1-th, ..., ℓ_j-th component equations.

$$\begin{aligned}
0 &= c_{j+1,1}\delta_{1,k_1} + c_{j+1,2}\cdot 0 + \cdots + c_{j+1,j}\cdot 0 + c_{j+1,j+1}\cdot 0 + \cdots + c_{j+1,m}\cdot 0 \\
0 &= c_{j+1,1}\delta_{1,k_2} + c_{j+1,2}\delta_{2,k_j}\cdots + c_{j+1,j}\cdot 0 + c_{j+1,j+1}\cdot 0 + \cdots + c_{j+1,m}\cdot 0 \\
&\vdots \\
0 &= c_{j+1,1}\delta_{1,k_j} + c_{j+1,2}\delta_{2,k_2}\cdots + c_{j+1,j}\delta_{j,k_j} + c_{j+1,j+1}\cdot 0 + \cdots + c_{j+1,m}\cdot 0
\end{aligned}$$

We can conclude that $c_{j+1,1}, \ldots, c_{j+1,j}$ are all zero.

Now look at the ℓ_{j+1}-th component equation:

$$\beta_{j+1,\ell_{j+1}} = c_{j+1,j+1}\delta_{j+1,\ell_{j+1}} + c_{j+1,j+2}\delta_{j+2,\ell_{j+1}} + \cdots + c_{j+1,m}\delta_{m,\ell_{j+1}}.$$

Because D is in echelon form and because $\ell_{j+1} \leq k_{j+1}$, each of $\delta_{j+2,\ell_{j+1}}, \ldots, \delta_{m,\ell_{j+1}}$ is zero. But $\beta_{j+1,\ell_{j+1}}$ is nonzero since it leads its row, and so $\delta_{j+1,\ell_{j+1}}$ is nonzero.

Conclusion: $k_{j+1} \leq \ell_{j+1}$ and so $k_{j+1} = \ell_{j+1}$.

(c) From the prior answer, we know that for any echelon form matrix, if this relationship holds among the non-zero rows:

$$\rho_i = c_1\rho_1 + \cdots + c_{i-1}\rho_{i-1} + c_{i+1}\rho_{i+1} + \cdots + c_n\rho_n$$

(where $c_1, \ldots, c_n \in \mathbb{R}$) then $c_1, \ldots, c_{i-1}$ must all be zero (in the $i = 1$ case we don't know any of the scalars are zero).

To derive a contradiction suppose the above relationship exists and let ℓ_i be the column index of the leading entry of ρ_i. Consider the equation of ℓ_i-th components:

$$\rho_{i,\ell_i} = c_{i+1}\rho_{i+1,\ell_i} + \cdots + c_n\rho_{n,\ell_i}$$

and observe that because the matrix is in echelon form each of $\rho_{i+1,\ell_i}, \ldots, \rho_{n,\ell_i}$ is zero. But that's a contradiction as ρ_{i,ℓ_i} is nonzero since it leads the i-th row.

Hence the linear relationship supposed to exist among the rows is not possible.

One.III.2.24 We know that $4s + c + 10d = 8.45$ and that $3s + c + 7d = 6.30$, and we'd like to know what $s + c + d$ is. Fortunately, $s + c + d$ is a linear combination of $4s + c + 10d$ and $3s + c + 7d$. Calling the unknown price p, we have this reduction.

$$\left(\begin{array}{ccc|c} 4 & 1 & 10 & 8.45 \\ 3 & 1 & 7 & 6.30 \\ 1 & 1 & 1 & p \end{array}\right) \xrightarrow[-(1/4)\rho_1+\rho_3]{-(3/4)\rho_1+\rho_2} \left(\begin{array}{ccc|c} 4 & 1 & 10 & 8.45 \\ 0 & 1/4 & -1/2 & -0.0375 \\ 0 & 3/4 & -3/2 & p - 2.1125 \end{array}\right) \xrightarrow{-3\rho_2+\rho_3} \left(\begin{array}{ccc|c} 4 & 1 & 10 & 8.45 \\ 0 & 1/4 & -1/2 & -0.0375 \\ 0 & 0 & 0 & p - 2.00 \end{array}\right)$$

The price paid is \$2.00.

One.III.2.26 (1) An easy answer is this:

$$0 = 3.$$

For a less wise-guy-ish answer, solve the system:

$$\left(\begin{array}{cc|c} 3 & -1 & 8 \\ 2 & 1 & 3 \end{array}\right) \xrightarrow{-(2/3)\rho_1+\rho_2} \left(\begin{array}{cc|c} 3 & -1 & 8 \\ 0 & 5/3 & -7/3 \end{array}\right)$$

gives $y = -7/5$ and $x = 11/5$. Now any equation not satisfied by $(-7/5, 11/5)$ will do, e.g., $5x + 5y = 3$.

(2) Every equation can be derived from an inconsistent system. For instance, here is how to derive "$3x + 2y = 4$" from "$0 = 5$". First,

$$0 = 5 \xrightarrow{(3/5)\rho_1} 0 = 3 \xrightarrow{x\rho_1} 0 = 3x$$

(validity of the $x = 0$ case is separate but clear). Similarly, $0 = 2y$. Ditto for $0 = 4$. But now, $0 + 0 = 0$ gives $3x + 2y = 4$.

One.III.2.28 **(a)** The three possible row swaps are easy, as are the three possible rescalings. One of the six possible pivots is $k\rho_1 + \rho_2$:

$$\begin{pmatrix} 1 & 2 & 3 \\ k\cdot 1 + 3 & k\cdot 2 + 0 & k\cdot 3 + 3 \\ 1 & 4 & 5 \end{pmatrix}$$

and again the first and second columns add to the third. The other five pivots are similar.

(b) The obvious conjecture is that row operations do not change linear relationships among columns.

(c) A case-by-case proof follows the sketch given in the first item.

Topic: Computer Algebra Systems

Topic: Input-Output Analysis

Topic: Accuracy of Computations

Topic: Analyzing Networks

Chapter Two: Vector Spaces

Subsection Two.I.1: Definition and Examples

Two.I.1.18 **(a)** $3 + 2x - x^2$ **(b)** $\begin{pmatrix} -1 & +1 \\ 0 & -3 \end{pmatrix}$ **(c)** $-3e^x + 2e^{-x}$

Two.I.1.19 Most of the conditions are easy to check; use Example 1.3 as a guide. Here are some comments.

(a) This is just like Example 1.3; the zero element is $0 + 0x$.
(b) The zero element of this space is the 2×2 matrix of zeroes.
(c) The zero element is the vector of zeroes.
(d) Closure of addition involves noting that the sum

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}$$

is in L because $(x_1{+}x_2)+(y_1{+}y_2)-(z_1{+}z_2)+(w_1{+}w_2) = (x_1{+}y_1{-}z_1{+}w_1)+(x_2{+}y_2{-}z_2{+}w_2) = 0{+}0$. Closure of scalar multiplication is similar. Note that the zero element, the vector of zeroes, is in L.

Two.I.1.20 In each item the set is called Q. For some items, there are other correct ways to show that Q is not a vector space.

(a) It is not closed under addition.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

(b) It is not closed under addition.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Q$$

(c) It is not closed under addition.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in Q \qquad \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \notin Q$$

(d) It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

(e) It is empty.

Two.I.1.22 No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

Two.I.1.26 For each "yes" answer, a check of all the conditions given in the definition of a vector space should be given. For each "no" answer, a specific example of the failure of one of the conditions must be given.

(a) Yes.
(b) Yes.
(c) No, it is not closed under addition. The vector of all $1/4$'s, when added to itself, makes a nonmember.
(d) Yes.
(e) No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2 \cdot f$ is not.

Two.I.1.27 It is a vector space. Most conditions of the definition of vector space are routine; we here check only closure. For addition, $(f_1 + f_2)\,(7) = f_1(7) + f_2(7) = 0 + 0 = 0$. For scalar multiplication, $(r \cdot f)\,(7) = rf(7) = r0 = 0$.

Two.I.1.28 We check Definition 1.1.

For (1) there are five conditions. First, closure holds because the product of two positive reals is a positive real. The second condition is satisfied because real multiplication commutes. Similarly, as real multiplication associates, the third checks. For the fourth condition, observe that multiplying a number by $1 \in \mathbb{R}^+$ won't change the number. Fifth, any positive real has a reciprocal that is a positive real.

In (2) there are five conditions. The first, closure, holds because any power of a positive real is a positive real. The second condition is just the rule that v^{r+s} equals the product of v^r and v^s. The third condition says that $(vw)^r = v^r w^r$. The fourth condition asserts that $(v^r)^s = v^{rs}$. The final condition says that $v^1 = v$.

Two.I.1.32 A *vector space* over $\mathbb{R}$ consists of a set V along with two operations '$\vec{+}$' and '$\vec{\cdot}$' such that

(1) if $\vec{v}, \vec{w} \in V$ then their *vector sum* $\vec{v} \mathbin{\vec{+}} \vec{w}$ is in V and
- $\vec{v} \mathbin{\vec{+}} \vec{w} = \vec{w} \mathbin{\vec{+}} \vec{v}$
- $(\vec{v} \mathbin{\vec{+}} \vec{w}) \mathbin{\vec{+}} \vec{u} = \vec{v} \mathbin{\vec{+}} (\vec{w} \mathbin{\vec{+}} \vec{u})$ (where $\vec{u} \in V$)
- there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} \mathbin{\vec{+}} \vec{0} = \vec{v}$ for all $\vec{v} \in V$
- each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} \mathbin{\vec{+}} \vec{v} = \vec{0}$

(2) if r, s are *scalars* (i.e., members of $\mathbb{R}$) and $\vec{v}, \vec{w} \in V$ then the *scalar product* $r \mathbin{\vec{\cdot}} \vec{v}$ is in V and
- $(r + s) \mathbin{\vec{\cdot}} \vec{v} = r \mathbin{\vec{\cdot}} \vec{v} \mathbin{\vec{+}} s \mathbin{\vec{\cdot}} \vec{v}$
- $r \mathbin{\vec{\cdot}} (\vec{v} \mathbin{\vec{+}} \vec{w}) = r \mathbin{\vec{\cdot}} \vec{v} \mathbin{\vec{+}} r \mathbin{\vec{\cdot}} \vec{w}$
- $(r \cdot s) \mathbin{\vec{\cdot}} \vec{v} = r \mathbin{\vec{\cdot}} (s \mathbin{\vec{\cdot}} \vec{v})$
- $1 \mathbin{\vec{\cdot}} \vec{v} = \vec{v}$.

Two.I.1.33 **(a)** Let V be a vector space, assume that $\vec{v} \in V$, and assume that $\vec{w} \in V$ is the additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} = \vec{0}$. Because addition is commutative, $\vec{0} = \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$.

(b) Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which says that $\vec{0} + \vec{s} = \vec{0} + \vec{t}$ and so $\vec{s} = \vec{t}$.

Two.I.1.35 It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

Two.I.1.38 Assume that $\vec{v} \in V$ is not $\vec{0}$.

(a) One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} = \vec{0}$. For the other way, let r be a nonzero scalar. If $r\vec{v} = \vec{0}$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot \vec{0}$ shows that $\vec{v} = \vec{0}$, contrary to the assumption.

(b) Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} = \vec{0}$. By the prior item, then $r_1 - r_2 = 0$.

(c) A nontrivial space has a vector $\vec{v} \neq \vec{0}$. Consider the set $\{k \cdot \vec{v} \mid k \in \mathbb{R}\}$. By the prior item this set is infinite.

(d) The solution set is either trivial, or nontrivial. In the second case, it is infinite.

Two.I.1.42 **(a)** A small rearrangement does the trick.

$$\begin{aligned}
(\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) + \vec{v}_4 &= ((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \vec{v}_4 \\
&= (\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) \\
&= \vec{v}_1 + (\vec{v}_2 + (\vec{v}_3 + \vec{v}_4)) \\
&= \vec{v}_1 + ((\vec{v}_2 + \vec{v}_3) + \vec{v}_4)
\end{aligned}$$

Each equality above follows from the associativity of three vectors that is given as a condition in the definition of a vector space. For instance, the second '=' applies the rule $(\vec{w}_1 + \vec{w}_2) + \vec{w}_3 = \vec{w}_1 + (\vec{w}_2 + \vec{w}_3)$ by taking $\vec{w}_1$ to be $\vec{v}_1 + \vec{v}_2$, taking $\vec{w}_2$ to be $\vec{v}_3$, and taking $\vec{w}_3$ to be $\vec{v}_4$.

(b) The base case for induction is the three vector case. This case $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$ is required of any triple of vectors by the definition of a vector space.

For the inductive step, assume that any two sums of three vectors, any two sums of four vectors, ..., any two sums of k vectors are equal no matter how the sums are parenthesized. We will show that any sum of $k + 1$ vectors equals this one $((\cdots((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \cdots) + \vec{v}_k) + \vec{v}_{k+1}$.

Any parenthesized sum has an outermost '+'. Assume that it lies between $\vec{v}_m$ and $\vec{v}_{m+1}$ so the sum looks like this.

$$(\cdots\ \vec{v}_1 \cdots \vec{v}_m\ \cdots) + (\cdots\ \vec{v}_{m+1} \cdots \vec{v}_{k+1}\ \cdots)$$

The second half involves fewer than $k+1$ additions, so by the inductive hypothesis we can re-parenthesize it so that it reads left to right from the inside out, and in particular, so that its outermost '+' occurs right before $\vec{v}_{k+1}$.

$$= (\cdots\vec{v}_1\cdots\vec{v}_m\cdots) + ((\cdots(\vec{v}_{m+1}+\vec{v}_{m+2})+\cdots+\vec{v}_k)+\vec{v}_{k+1})$$

Apply the associativity of the sum of three things

$$= ((\cdots\vec{v}_1\cdots\vec{v}_m\cdots) + (\cdots(\vec{v}_{m+1}+\vec{v}_{m+2})+\cdots\vec{v}_k)) + \vec{v}_{k+1}$$

and finish by applying the inductive hypothesis inside these outermost parenthesis.

Subsection Two.I.2: Subspaces and Spanning Sets

Two.I.2.20 By Lemma 2.9, to see if each subset of $\mathcal{M}_{2\times 2}$ is a subspace, we need only check if it is nonempty and closed.

(a) Yes, it is easily checked to be nonempty and closed. This is a paramatrization.

$$\{a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}0&0\\0&1\end{pmatrix} \mid a,b\in\mathbb{R}\}$$

By the way, the paramatrization also shows that it is a subspace, it is given as the span of the two-matrix set, and any span is a subspace.

(b) Yes; it is easily checked to be nonempty and closed. Alternatively, as mentioned in the prior answer, the existence of a paramatrization shows that it is a subspace. For the paramatrization, the condition $a+b=0$ can be rewritten as $a=-b$. Then we have this.

$$\{\begin{pmatrix}-b&0\\0&b\end{pmatrix} \mid b\in\mathbb{R}\}=\{b\begin{pmatrix}-1&0\\0&1\end{pmatrix} \mid b\in\mathbb{R}\}$$

(c) No. It is not closed under addition. For instance,

$$\begin{pmatrix}5&0\\0&0\end{pmatrix}+\begin{pmatrix}5&0\\0&0\end{pmatrix}=\begin{pmatrix}10&0\\0&0\end{pmatrix}$$

is not in the set. (This set is also not closed under scalar multiplication, for instance, it does not contain the zero matrix.)

(d) Yes.

$$\{b\begin{pmatrix}-1&0\\0&1\end{pmatrix}+c\begin{pmatrix}0&1\\0&0\end{pmatrix} \mid b,c\in\mathbb{R}\}$$

Two.I.2.21 No, it is not closed. In particular, it is not closed under scalar multiplication because it does not contain the zero polynomial.

Two.I.2.22 **(a)** Yes, solving the linear system arising from

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix}+r_2\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}$$

gives $r_1=2$ and $r_2=1$.

(b) Yes; the linear system arising from $r_1(x^2)+r_2(2x+x^2)+r_3(x+x^3)=x-x^3$

$$\begin{aligned}2r_2+r_3&=1\\ r_1+r_2&=0\\ r_3&=-1\end{aligned}$$

gives that $-1(x^2)+1(2x+x^2)-1(x+x^3)=x-x^3$.

(c) No; any combination of the two given matrices has a zero in the upper right.

Two.I.2.24 **(a)** Yes, for any $x,y,z\in\mathbb{R}$ this equation

$$r_1\begin{pmatrix}1\\0\\0\end{pmatrix}+r_2\begin{pmatrix}0\\2\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\3\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

has the solution $r_1=x$, $r_2=y/2$, and $r_3=z/3$.

(b) Yes, the equation

$$r_1\begin{pmatrix}2\\0\\1\end{pmatrix}+r_2\begin{pmatrix}1\\1\\0\end{pmatrix}+r_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives rise to this

$$\begin{array}{rl}2r_1+r_2&=x\\ r_2&=y\\ r_1+r_3&=z\end{array}\xrightarrow{-(1/2)\rho_1+\rho_3}\xrightarrow{(1/2)\rho_2+\rho_3}\begin{array}{rl}2r_1+r_2&=x\\ r_2&=y\\ r_3&=-(1/2)x+(1/2)y+z\end{array}$$

so that, given any x, y, and z, we can compute that $r_3=(-1/2)x+(1/2)y+z$, $r_2=y$, and $r_1=(1/2)x-(1/2)y$.

(c) No. In particular, the vector

$$\begin{pmatrix}0\\0\\1\end{pmatrix}$$

cannot be gotten as a linear combination since the two given vectors both have a third component of zero.

(d) Yes. The equation

$$r_1\begin{pmatrix}1\\0\\1\end{pmatrix}+r_2\begin{pmatrix}3\\1\\0\end{pmatrix}+r_3\begin{pmatrix}-1\\0\\0\end{pmatrix}+r_4\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left(\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\1&0&0&5&z\end{array}\right)\xrightarrow{-\rho_1+\rho_3}\xrightarrow{3\rho_2+\rho_3}\left(\begin{array}{cccc|c}1&3&-1&2&x\\0&1&0&1&y\\0&0&1&6&-x+3y+z\end{array}\right)$$

We have infinitely many solutions. We can, for example, set r_4 to be zero and solve for r_3, r_2, and r_1 in terms of x, y, and z by the usual methods of back-substitution.

(e) No. The equation

$$r_1\begin{pmatrix}2\\1\\1\end{pmatrix}+r_2\begin{pmatrix}3\\0\\1\end{pmatrix}+r_3\begin{pmatrix}5\\1\\2\end{pmatrix}+r_4\begin{pmatrix}6\\0\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

leads to this reduction.

$$\left(\begin{array}{cccc|c}2&3&5&6&x\\1&0&1&0&y\\1&1&2&2&z\end{array}\right)\xrightarrow[-(1/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-(1/3)\rho_2+\rho_3}\left(\begin{array}{cccc|c}2&3&5&6&x\\0&-3/2&-3/2&-3&-(1/2)x+y\\0&0&0&0&-(1/3)x-(1/3)y+z\end{array}\right)$$

This shows that not every three-tall vector can be so expressed. Only the vectors satisfying the restriction that $-(1/3)x-(1/3)y+z=0$ are in the span. (To see that any such vector is indeed expressible, take r_3 and r_4 to be zero and solve for r_1 and r_2 in terms of x, y, and z by back-substitution.)

Two.I.2.25 **(a)** $\{\begin{pmatrix}c&b&c\end{pmatrix}\mid b,c\in\mathbb{R}\}=\{b\begin{pmatrix}0&1&0\end{pmatrix}+c\begin{pmatrix}1&0&1\end{pmatrix}\mid b,c\in\mathbb{R}\}$ The obvious choice for the set that spans is $\{\begin{pmatrix}0&1&0\end{pmatrix},\begin{pmatrix}1&0&1\end{pmatrix}\}$.

(b) $\{\begin{pmatrix}-d&b\\c&d\end{pmatrix}\mid b,c,d\in\mathbb{R}\}=\{b\begin{pmatrix}0&1\\0&0\end{pmatrix}+c\begin{pmatrix}0&0\\1&0\end{pmatrix}+d\begin{pmatrix}-1&0\\0&1\end{pmatrix}\mid b,c,d\in\mathbb{R}\}$ One set that spans this space consists of those three matrices.

(c) The system

$$\begin{array}{rl}a+3b&=0\\ 2a-c-d&=0\end{array}$$

gives $b=-(c+d)/6$ and $a=(c+d)/2$. So one description is this.

$$\{c\begin{pmatrix}1/2&-1/6\\1&0\end{pmatrix}+d\begin{pmatrix}1/2&-1/6\\0&1\end{pmatrix}\mid c,d\in\mathbb{R}\}$$

That shows that a set spanning this subspace consists of those two matrices.

(d) The $a=2b-c$ gives $\{(2b-c)+bx+cx^3\mid b,c\in\mathbb{R}\}=\{b(2+x)+c(-1+x^3)\mid b,c\in\mathbb{R}\}$. So the subspace is the span of the set $\{2+x,-1+x^3\}$.

(e) The set $\{a+bx+cx^2\mid a+7b+49c=0\}$ paramatrized as $\{b(-7+x)+c(-49+x^2)\mid b,c\in\mathbb{R}\}$ has the spanning set $\{-7+x,-49+x^2\}$.

Two.I.2.26 Each answer given is only one out of many possible.

(a) We can paramatrize in this way

$$\{\begin{pmatrix}x\\0\\z\end{pmatrix} \bigm| x,z\in\mathbb{R}\} = \{x\begin{pmatrix}1\\0\\0\end{pmatrix}+z\begin{pmatrix}0\\0\\1\end{pmatrix} \bigm| x,z\in\mathbb{R}\}$$

giving this for a spanning set.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

(b) Paramatrize it with $\{y\begin{pmatrix}-2/3\\1\\0\end{pmatrix}+z\begin{pmatrix}-1/3\\0\\1\end{pmatrix} \bigm| y,z\in\mathbb{R}\}$ to get $\{\begin{pmatrix}-2/3\\1\\0\end{pmatrix},\begin{pmatrix}-1/3\\0\\1\end{pmatrix}\}$.

(c) $\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}-1/2\\0\\0\\1\end{pmatrix}\}$

(d) Paramatrize the description as $\{-a_1+a_1x+a_3x^2+a_3x^3 \bigm| a_1,a_3\in\mathbb{R}\}$ to get $\{-1+x,x^2+x^3\}$.

(e) $\{1,x,x^2,x^3,x^4\}$

(f) $\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\}$

Two.I.2.28 Of course, the addition and scalar multiplication operations are the ones inherited from the enclosing space.

(a) This is a subspace. It is not empty as it contains at least the two example functions given. It is closed because if f_2, f_2 are even and c_1, c_2 are scalars then we have this.

$$(c_1f_1+c_2f_2)\,(-x) = c_1\,f_1(-x)+c_2\,f_2(-x) = c_1\,f_1(x)+c_2\,f_2(x) = (c_1f_1+c_2f_2)\,(x)$$

(b) This is also a subspace; the check is similar to the prior one.

Two.I.2.34 As the hint suggests, the basic reason is the Linear Combination Lemma from the first chapter. For the full proof, we will show mutual containment between the two sets.

The first containment $[[S]]\supseteq[S]$ is an instance of the more general, and obvious, fact that for any subset T of a vector space, $[T]\supseteq T$.

For the other containment, that $[[S]]\subseteq[S]$, take m vectors from $[S]$, namely $c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1}$, $\ldots$, $c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m}$, and note that any linear combination of those

$$r_1(c_{1,1}\vec{s}_{1,1}+\cdots+c_{1,n_1}\vec{s}_{1,n_1})+\cdots+r_m(c_{1,m}\vec{s}_{1,m}+\cdots+c_{1,n_m}\vec{s}_{1,n_m})$$

is a linear combination of elements of S

$$=(r_1c_{1,1})\vec{s}_{1,1}+\cdots+(r_1c_{1,n_1})\vec{s}_{1,n_1}+\cdots+(r_mc_{1,m})\vec{s}_{1,m}+\cdots+(r_mc_{1,n_m})\vec{s}_{1,n_m}$$

and so is in $[S]$. That is, simply recall that a linear combination of linear combinations (of members of S) is a linear combination (again of members of S).

Two.I.2.42 **(a)** Always.

Assume that A, B are subspaces of V. Note that their intersection is not empty as both contain the zero vector. If $\vec{w},\vec{s}\in A\cap B$ and r, s are scalars then $r\vec{v}+s\vec{w}\in A$ because each vector is in A and so a linear combination is in A, and $r\vec{v}+s\vec{w}\in B$ for the same reason. Thus the intersection is closed. Now Lemma 2.9 applies.

(b) Sometimes (more precisely, only if $A\subseteq B$ or $B\subseteq A$).

To see the answer is not 'always', take V to be $\mathbb{R}^3$, take A to be the x-axis, and B to be the y-axis. Note that

$$\begin{pmatrix}1\\0\end{pmatrix}\in A \text{ and } \begin{pmatrix}0\\1\end{pmatrix}\in B \quad\text{but}\quad \begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\notin A\cup B$$

as the sum is in neither A nor B.

The answer is not 'never' because if $A\subseteq B$ or $B\subseteq A$ then clearly $A\cup B$ is a subspace.

To show that $A\cup B$ is a subspace only if one subspace contains the other, we assume that $A\not\subseteq B$ and $B\not\subseteq A$ and prove that the union is not a subspace. The assumption that A is not a subset of B means that there is an $\vec{a}\in A$ with $\vec{a}\notin B$. The other assumption gives a $\vec{b}\in B$ with $\vec{b}\notin A$. Consider $\vec{a}+\vec{b}$. Note that sum is not an element of A or else $(\vec{a}+\vec{b})-\vec{a}$ would be in A, which it is not. Similarly the sum is not an element of B. Hence the sum is not an element of $A\cup B$, and so the union is not a subspace.

(c) Never. As A is a subspace, it contains the zero vector, and therefore the set that is A's complement does not. Without the zero vector, the complement cannot be a vector space.

Two.I.2.43 The span of a set does not depend on the enclosing space. A linear combination of vectors from S gives the same sum whether we regard the operations as those of W or as those of V, because the operations of W are inherited from V.

Two.I.2.45 **(a)** Always; if $S \subseteq T$ then a linear combination of elements of S is also a linear combination of elements of T.

(b) Sometimes (more precisely, if and only if $S \subseteq T$ or $T \subseteq S$).

The answer is not 'always' as is shown by this example from $\mathbb{R}^3$

$$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}, \quad T = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

because of this.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \in [S \cup T] \qquad \begin{pmatrix}1\\1\\1\end{pmatrix} \notin [S] \cup [T]$$

The answer is not 'never' because if either set contains the other then equality is clear. We can characterize equality as happening only when either set contains the other by assuming $S \not\subseteq T$ (implying the existence of a vector $\vec{s} \in S$ with $\vec{s} \notin T$) and $T \not\subseteq S$ (giving a $\vec{t} \in T$ with $\vec{t} \notin S$), noting $\vec{s} + \vec{t} \in [S \cup T]$, and showing that $\vec{s} + \vec{t} \notin [S] \cup [T]$.

(c) Sometimes.

Clearly $[S \cap T] \subseteq [S] \cap [T]$ because any linear combination of vectors from $S \cap T$ is a combination of vectors from S and also a combination of vectors from T.

Containment the other way does not always hold. For instance, in $\mathbb{R}^2$, take

$$S = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}, \quad T = \{\begin{pmatrix}2\\0\end{pmatrix}\}$$

so that $[S] \cap [T]$ is the x-axis but $[S \cap T]$ is the trivial subspace.

Characterizing exactly when equality holds is tough. Clearly equality holds if either set contains the other, but that is not 'only if' by this example in $\mathbb{R}^3$.

$$S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}, \quad T = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

(d) Never, as the span of the complement is a subspace, while the complement of the span is not (it does not contain the zero vector).

Subsection Two.II.1: Definition and Examples

Two.II.1.18 For each of these, when the subset is independent it must be proved, and when the subset is dependent an example of a dependence must be given.

(a) It is dependent. Considering

$$c_1 \begin{pmatrix}1\\-3\\5\end{pmatrix} + c_2 \begin{pmatrix}2\\2\\4\end{pmatrix} + c_3 \begin{pmatrix}4\\-4\\14\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

gives rise to this linear system.

$$\begin{aligned} c_1 + 2c_2 + 4c_3 &= 0 \\ -3c_1 + 2c_2 - 4c_3 &= 0 \\ 5c_1 + 4c_2 + 14c_3 &= 0 \end{aligned}$$

Gauss' method

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -3 & 2 & -4 & 0 \\ 5 & 4 & 14 & 0 \end{array}\right) \xrightarrow[-5\rho_1+\rho_3]{3\rho_1+\rho_2} \xrightarrow{(3/4)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

yields a free variable, so there are infinitely many solutions. For an example of a particular dependence we can set c_3 to be, say, 1. Then we get $c_2 = -1$ and $c_1 = -2$.

(b) It is dependent. The linear system that arises here

$$\left(\begin{array}{ccc|c}1&2&3&0\\7&7&7&0\\7&7&7&0\end{array}\right)\xrightarrow[-7\rho_1+\rho_3]{-7\rho_1+\rho_2}\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&2&3&0\\0&-7&-14&0\\0&0&0&0\end{array}\right)$$

has infinitely many solutions. We can get a particular solution by taking c_3 to be, say, 1, and back-substituting to get the resulting c_2 and c_1.

(c) It is linearly independent. The system

$$\left(\begin{array}{cc|c}0&1&0\\0&0&0\\-1&4&0\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_2}\xrightarrow{\rho_3\leftrightarrow\rho_1}\left(\begin{array}{cc|c}-1&4&0\\0&1&0\\0&0&0\end{array}\right)$$

has only the solution $c_1=0$ and $c_2=0$. (We could also have gotten the answer by inspection — the second vector is obviously not a multiple of the first, and vice versa.)

(d) It is linearly dependent. The linear system

$$\left(\begin{array}{cccc|c}9&2&3&12&0\\9&0&5&12&0\\0&1&-4&-1&0\end{array}\right)$$

has more unknowns than equations, and so Gauss' method must end with at least one variable free (there can't be a contradictory equation because the system is homogeneous, and so has at least the solution of all zeroes). To exhibit a combination, we can do the reduction

$$\xrightarrow{-\rho_1+\rho_2}\xrightarrow{(1/2)\rho_2+\rho_3}\left(\begin{array}{cccc|c}9&2&3&12&0\\0&-2&2&0&0\\0&0&-3&-1&0\end{array}\right)$$

and take, say, $c_4=1$. Then we have that $c_3=-1/3$, $c_2=-1/3$, and $c_1=-31/27$.

Two.II.1.19 In the cases of independence, that must be proved. Otherwise, a specific dependence must be produced. (Of course, dependences other than the ones exhibited here are possible.)

(a) This set is independent. Setting up the relation $c_1(3-x+9x^2)+c_2(5-6x+3x^2)+c_3(1+1x-5x^2)=0+0x+0x^2$ gives a linear system

$$\left(\begin{array}{ccc|c}3&5&1&0\\-1&-6&1&0\\9&3&-5&0\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{(1/3)\rho_1+\rho_2}\xrightarrow{3\rho_2}\xrightarrow{-(12/13)\rho_2+\rho_3}\left(\begin{array}{ccc|c}3&5&1&0\\0&-13&2&0\\0&0&-128/13&0\end{array}\right)$$

with only one solution: $c_1=0$, $c_2=0$, and $c_3=0$.

(b) This set is independent. We can see this by inspection, straight from the definition of linear independence. Obviously neither is a multiple of the other.

(c) This set is linearly independent. The linear system reduces in this way

$$\left(\begin{array}{ccc|c}2&3&4&0\\1&-1&0&0\\7&2&-3&0\end{array}\right)\xrightarrow[-(7/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2}\xrightarrow{-5\rho_2+\rho_3}\left(\begin{array}{ccc|c}2&3&4&0\\0&-5/2&-2&0\\0&0&-7&0\end{array}\right)$$

to show that there is only the solution $c_1=0$, $c_2=0$, and $c_3=0$.

(d) This set is linearly dependent. The linear system

$$\left(\begin{array}{cccc|c}8&0&2&8&0\\3&1&2&-2&0\\3&2&2&5&0\end{array}\right)$$

must, after reduction, end with at least one variable free (there are more variables than equations, and there is no possibility of a contradictory equation because the system is homogeneous). We can take the free variables as parameters to describe the solution set. We can then set the parameter to a nonzero value to get a nontrivial linear relation.

Two.II.1.20 Let Z be the zero function $Z(x)=0$, which is the additive identity in the vector space under discussion.

(a) This set is linearly independent. Consider $c_1\cdot f(x)+c_2\cdot g(x)=Z(x)$. Plugging in $x=1$ and $x=2$ gives a linear system

$$\begin{aligned}c_1\cdot 1+{}\quad c_2\cdot 1&=0\\c_1\cdot 2+c_2\cdot(1/2)&=0\end{aligned}$$

with the unique solution $c_1=0$, $c_2=0$.

(b) This set is linearly independent. Consider $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plug in $x = 0$ and $x = \pi/2$ to get

$$\begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= 0 \\ c_1 \cdot 0 + c_2 \cdot 1 &= 0 \end{aligned}$$

which obviously gives that $c_1 = 0$, $c_2 = 0$.

(c) This set is also linearly independent. Considering $c_1 \cdot f(x) + c_2 \cdot g(x) = Z(x)$ and plugging in $x = 1$ and $x = e$

$$\begin{aligned} c_1 \cdot e + c_2 \cdot 0 &= 0 \\ c_1 \cdot e^e + c_2 \cdot 1 &= 0 \end{aligned}$$

gives that $c_1 = 0$ and $c_2 = 0$.

Two.II.1.21 In each case, that the set is independent must be proved, and that it is dependent must be shown by exihibiting a specific dependence.

(a) This set is dependent. The familiar relation $\sin^2(x)+\cos^2(x) = 1$ shows that $2 = c_1 \cdot (4\sin^2(x)) + c_2 \cdot (\cos^2(x))$ is satisfied by $c_1 = 1/2$ and $c_2 = 2$.

(b) This set is independent. Consider the relationship $c_1 \cdot 1 + c_2 \cdot \sin(x) + c_3 \cdot \sin(2x) = 0$ (that '0' is the zero function). Taking $x = 0$, $x = \pi/2$ and $x = \pi/4$ gives this system.

$$\begin{array}{rcl} c_1 & & = 0 \\ c_1 + \quad\quad c_2 & & = 0 \\ c_1 + (\sqrt{2}/2)c_2 + c_3 & & = 0 \end{array}$$

whose only solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

(c) By inspection, this set is independent. Any dependence $\cos(x) = c \cdot x$ is not possible since the cosine function is not a multiple of the identity function (we are applying Corollary 1.17).

(d) By inspection, we spot that there is a dependence. Because $(1 + x)^2 = x^2 + 2x + 1$, we get that $c_1 \cdot (1 + x)^2 + c_2 \cdot (x^2 + 2x) = 3$ is satisfied by $c_1 = 3$ and $c_2 = -3$.

(e) This set is dependent. The easiest way to see that is to recall the triginometric relationship $\cos^2(x) - \sin^2(x) = \cos(2x)$. (*Remark.* A person who doesn't recall this, and tries some x's, simply never gets a system leading to a unique solution, and never gets to conclude that the set is independent. Of course, this person might wonder if they simply never tried the right set of x's, but a few tries will lead most people to look instead for a dependence.)

(f) This set is dependent, because it contains the zero object in the vector space, the zero polynomial.

Two.II.1.24 We have already showed this: the Linear Combination Lemma and its corollary state that in an echelon form matrix, no nonzero row is a linear combination of the others.

Two.II.1.25 **(a)** Assume that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so that any relationship $d_0\vec{u} + d_1\vec{v} + d_2\vec{w} = \vec{0}$ leads to the conclusion that $d_0 = 0$, $d_1 = 0$, and $d_2 = 0$.

Consider the relationship $c_1(\vec{u}) + c_2(\vec{u} + \vec{v}) + c_3(\vec{u} + \vec{v} + \vec{w}) = \vec{0}$. Rewrite it to get $(c_1 + c_2 + c_3)\vec{u} + (c_2 + c_3)\vec{v} + (c_3)\vec{w} = \vec{0}$. Taking d_0 to be $c_1 + c_2 + c_3$, taking d_1 to be $c_2 + c_3$, and taking d_2 to be c_3 we have this system.

$$\begin{array}{rcl} c_1 + c_2 + c_3 &=& 0 \\ c_2 + c_3 &=& 0 \\ c_3 &=& 0 \end{array}$$

Conclusion: the c's are all zero, and so the set is linearly independent.

(b) The second set is dependent.

$$1 \cdot (\vec{u} - \vec{v}) + 1 \cdot (\vec{v} - \vec{w}) + 1 \cdot (\vec{w} - \vec{u}) = \vec{0}$$

Beyond that, the two statements are unrelated in the sense that any of the first set could be either independent or dependent. For instance, in $\mathbb{R}^3$, we can have that the first is independent while the second is not

$$\{\vec{u}, \vec{v}, \vec{w}\} = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\} \qquad \{\vec{u} - \vec{v}, \vec{v} - \vec{w}, \vec{w} - \vec{u}\} = \{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\}$$

or that both are dependent.

$$\{\vec{u}, \vec{v}, \vec{w}\} = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\} \qquad \{\vec{u} - \vec{v}, \vec{v} - \vec{w}, \vec{w} - \vec{u}\} = \{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\}$$

Two.II.1.33 Yes; here is one.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\}$$

Two.II.1.36 **(a)** The intersection of two linearly independent sets $S\cap T$ must be linearly independent as it is a subset of the linearly independent set S (as well as the linearly independent set T also, of course).

(b) The complement of a linearly independent set is linearly dependent as it contains the zero vector.

(c) We must produce an example. One, in $\mathbb{R}^2$, is

$$S=\{\begin{pmatrix}1\\0\end{pmatrix}\}\quad\text{and}\quad T=\{\begin{pmatrix}2\\0\end{pmatrix}\}$$

since the linear dependence of $S_1\cup S_2$ is easily seen.

(d) The union of two linearly independent sets $S\cup T$ is linearly independent if and only if their spans have a trivial intersection $[S]\cap[T]=\{\vec{0}\}$. To prove that, assume that S and T are linearly independent subsets of some vector space.

For the 'only if' direction, assume that the intersection of the spans is trivial $[S]\cap[T]=\{\vec{0}\}$. Consider the set $S\cup T$. Any linear relationship $c_1\vec{s}_1+\cdots+c_n\vec{s}_n+d_1\vec{t}_1+\cdots+d_m\vec{t}_m=\vec{0}$ gives $c_1\vec{s}_1+\cdots+c_n\vec{s}_n=-d_1\vec{t}_1-\cdots-d_m\vec{t}_m$. The left side of that equation sums to a vector in $[S]$, and the right side is a vector in $[T]$. Therefore, since the intersection of the spans is trivial, both sides equal the zero vector. Because S is linearly independent, all of the c's are zero. Because T is linearly independent, all of the d's are zero. Thus, the original linear relationship among members of $S\cup T$ only holds if all of the coefficients are zero. That shows that $S\cup T$ is linearly independent.

For the 'if' half we can make the same argument in reverse. If the union $S\cup T$ is linearly independent, that is, if the only solution to $c_1\vec{s}_1+\cdots+c_n\vec{s}_n+d_1\vec{t}_1+\cdots+d_m\vec{t}_m=\vec{0}$ is the trivial solution $c_1=0,\ldots,d_m=0$, then any vector $\vec{v}$ in the intersection of the spans $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n=-d_1\vec{t}_1-\cdots=d_m\vec{t}_m$ must be the zero vector because each scalar is zero.

Two.II.1.37 **(a)** We do induction on the number of vectors in the finite set S.

The base case is that S has no elements. In this case S is linearly independent and there is nothing to check — a subset of S that has the same span as S is S itself.

For the inductive step assume that the theorem is true for all sets of size $n=0$, $n=1$, …, $n=k$ in order to prove that it holds when S has $n=k+1$ elements. If the $k+1$-element set $S=\{\vec{s}_0,\ldots,\vec{s}_k\}$ is linearly independent then the theorem is trivial, so assume that it is dependent. By Corollary 1.17 there is an $\vec{s}_i$ that is a linear combination of other vectors in S. Define $S_1=S-\{\vec{s}_i\}$ and note that S_1 has the same span as S by Lemma 1.1. The set S_1 has k elements and so the inductive hypothesis applies to give that it has a linearly independent subset with the same span. That subset of S_1 is the desired subset of S.

(b) Here is a sketch of the argument. The induction argument details have been left out.

If the finite set S is empty then there is nothing to prove. If $S=\{\vec{0}\}$ then the empty subset will do.

Otherwise, take some nonzero vector $\vec{s}_1\in S$ and define $S_1=\{\vec{s}_1\}$. If $[S_1]=[S]$ then this proof is finished by noting that S_1 is linearly independent.

If not, then there is a nonzero vector $\vec{s}_2\in S-[S_1]$ (if every $\vec{s}\in S$ is in $[S_1]$ then $[S_1]=[S]$). Define $S_2=S_1\cup\{\vec{s}_2\}$. If $[S_2]=[S]$ then this proof is finished by using Theorem 1.17 to show that S_2 is linearly independent.

Repeat the last paragraph until a set with a big enough span appears. That must eventually happen because S is finite, and $[S]$ will be reached at worst when every vector from S has been used.

Two.II.1.39 Recall that two vectors from $\mathbb{R}^n$ are perpendicular if and only if their dot product is zero.

(a) Assume that $\vec{v}$ and $\vec{w}$ are perpendicular nonzero vectors in $\mathbb{R}^n$, with $n>1$. With the linear relationship $c\vec{v}+d\vec{w}=\vec{0}$, apply $\vec{v}$ to both sides to conclude that $c\cdot\|\vec{v}\|^2+d\cdot 0=0$. Because $\vec{v}\neq\vec{0}$ we have that $c=0$. A similar application of $\vec{w}$ shows that $d=0$.

(b) Two vectors in $\mathbb{R}^1$ are perpendicular if and only if at least one of them is zero.

We define $\mathbb{R}^0$ to be a trivial space, and so both $\vec{v}$ and $\vec{w}$ are the zero vector.

(c) The right generalization is to look at a set $\{\vec{v}_1,\ldots,\vec{v}_n\}\subseteq\mathbb{R}^k$ of vectors that are *mutually orthogonal* (also called *pairwise perpendicular*): if $i\neq j$ then $\vec{v}_i$ is perpendicular to $\vec{v}_j$. Mimicing the proof of the first item above shows that such a set of nonzero vectors is linearly independent.

Subsection Two.III.1: Basis

Two.III.1.16 By Theorem 1.12, each is a basis if and only if each vector in the space can be given in a unique way as a linear combination of the given vectors.

(a) Yes this is a basis. The relation

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}3\\2\\1\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives

$$\left(\begin{array}{ccc|c}1&3&0&x\\2&2&0&y\\3&1&1&z\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\xrightarrow{-2\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&3&0&x\\0&-4&0&-2x+y\\0&0&1&x-2y+z\end{array}\right)$$

which has the unique solution $c_3=x-2y+z$, $c_2=x/2-y/4$, and $c_1=-x/2+3y/4$.

(b) This is not a basis. Setting it up as in the prior item

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

gives a linear system whose solution

$$\left(\begin{array}{cc|c}1&3&x\\2&2&y\\3&1&z\end{array}\right)\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\xrightarrow{-2\rho_2+\rho_3}\left(\begin{array}{cc|c}1&3&x\\0&-4&-2x+y\\0&0&x-2y+z\end{array}\right)$$

is possible if and only if the three-tall vector's components x, y, and z satisfy $x-2y+z=0$. For instance, we can find the coefficients c_1 and c_2 that work when $x=1$, $y=1$, and $z=1$. However, there are no c's that work for $x=1$, $y=1$, and $z=2$. Thus this is not a basis; it does not span the space.

(c) Yes, this is a basis. Setting up the relationship leads to this reduction

$$\left(\begin{array}{ccc|c}0&1&2&x\\2&1&5&y\\-1&1&0&z\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_3}\xrightarrow{2\rho_1+\rho_2}\xrightarrow{-(1/3)\rho_2+\rho_3}\left(\begin{array}{ccc|c}-1&1&0&z\\0&3&5&y+2z\\0&0&1/3&x-y/3-2z/3\end{array}\right)$$

which has a unique solution for each triple of components x, y, and z.

(d) No, this is not a basis. The reduction

$$\left(\begin{array}{ccc|c}0&1&1&x\\2&1&3&y\\-1&1&0&z\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_3}\xrightarrow{2\rho_1+\rho_2}\xrightarrow{(-1/3)\rho_2+\rho_3}\left(\begin{array}{ccc|c}-1&1&0&z\\0&3&3&y+2z\\0&0&0&x-y/3-2z/3\end{array}\right)$$

which does not have a solution for each triple x, y, and z. Instead, the span of the given set includes only those three-tall vectors where $x=y/3+2z/3$.

Two.III.1.17 **(a)** We solve

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

with

$$\left(\begin{array}{cc|c}1&-1&1\\1&1&2\end{array}\right)\xrightarrow{-\rho_1+\rho_2}\left(\begin{array}{cc|c}1&-1&1\\0&2&1\end{array}\right)$$

and conclude that $c_2=1/2$ and so $c_1=3/2$. Thus, the representation is this.

$$\mathrm{Rep}_B(\begin{pmatrix}1\\2\end{pmatrix})=\begin{pmatrix}3/2\\1/2\end{pmatrix}_B$$

(b) The relationship $c_1\cdot(1)+c_2\cdot(1+x)+c_3\cdot(1+x+x^2)+c_4\cdot(1+x+x^2+x^3)=x^2+x^3$ is easily solved by eye to give that $c_4=1$, $c_3=0$, $c_2=-1$, and $c_1=0$.

$$\mathrm{Rep}_D(x^2+x^3)=\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}_D$$

(c) $\text{Rep}_{\mathcal{E}_4}(\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}_{\mathcal{E}_4}$

Two.III.1.20 There are many bases. This is a natural one.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

Two.III.1.21 For each item, many answers are possible.

(a) One way to proceed is to parametrize by expressing the a_2 as a combination of the other two $a_2 = 2a_1 + a_0$. Then $a_2x^2 + a_1x + a_0$ is $(2a_1 + a_0)x^2 + a_1x + a_0$ and

$$\{(2a_1 + a_0)x^2 + a_1x + a_0 \bigm| a_1, a_0 \in \mathbb{R}\} = \{a_1 \cdot (2x^2 + x) + a_0 \cdot (x^2 + 1) \bigm| a_1, a_0 \in \mathbb{R}\}$$

suggests $\langle 2x^2 + x, x^2 + 1\rangle$. This only shows that it spans, but checking that it is linearly independent is routine.

(b) Paramatrize $\{\begin{pmatrix} a & b & c \end{pmatrix} \bigm| a + b = 0\}$ to get $\{\begin{pmatrix} -b & b & c \end{pmatrix} \bigm| b, c \in \mathbb{R}\}$, which suggests using the sequence $\langle \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\rangle$. We've shown that it spans, and checking that it is linearly independent is easy.

(c) Rewriting

$$\{\begin{pmatrix} a & b \\ 0 & 2b \end{pmatrix} \bigm| a, b \in \mathbb{R}\} = \{a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \bigm| a, b \in \mathbb{R}\}$$

suggests this for the basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \rangle$$

Two.III.1.23 **(a)** Asking which $a_0 + a_1x + a_2x^2$ can be expressed as $c_1 \cdot (1 + x) + c_2 \cdot (1 + 2x)$ gives rise to three linear equations, describing the coefficients of x^2, x, and the constants.

$$\begin{aligned} c_1 + c_2 &= a_0 \\ c_1 + 2c_2 &= a_1 \\ 0 &= a_2 \end{aligned}$$

Gauss' method with back-substitution shows, provided that $a_2 = 0$, that $c_2 = -a_0 + a_1$ and $c_1 = 2a_0 - a_1$. Thus, with $a_2 = 0$, that we can compute appropriate c_1 and c_2 for any a_0 and a_1. So the span is the entire set of linear polynomials $\{a_0 + a_1x \bigm| a_0, a_1 \in \mathbb{R}\}$. Paramatrizing that set $\{a_0 \cdot 1 + a_1 \cdot x \bigm| a_0, a_1 \in \mathbb{R}\}$ suggests a basis $\langle 1, x\rangle$ (we've shown that it spans; checking linear independence is easy).

(b) With

$$a_0 + a_1x + a_2x^2 = c_1 \cdot (2 - 2x) + c_2 \cdot (3 + 4x^2) = (2c_1 + 3c_2) + (-2c_1)x + (4c_2)x^2$$

we get this system.

$$\begin{array}{rcl} 2c_1 + 3c_2 = a_0 & & 2c_1 + 3c_2 = a_0 \\ -2c_1 = a_1 & \xrightarrow{\rho_1+\rho_2} \xrightarrow{(-4/3)\rho_2+\rho_3} & 3c_2 = a_0 + a_1 \\ 4c_2 = a_2 & & 0 = (-4/3)a_0 - (4/3)a_1 + a_2 \end{array}$$

Thus, the only quadratic polynomials $a_0 + a_1x + a_2x^2$ with associated c's are the ones such that $0 = (-4/3)a_0 - (4/3)a_1 + a_2$. Hence the span is $\{(-a_1 + (3/4)a_2) + a_1x + a_2x^2 \bigm| a_1, a_2 \in \mathbb{R}\}$. Paramatrizing gives $\{a_1 \cdot (-1 + x) + a_2 \cdot ((3/4) + x^2) \bigm| a_1, a_2 \in \mathbb{R}\}$, which suggests $\langle -1 + x, (3/4) + x^2\rangle$ (checking that it is linearly independent is routine).

Two.III.1.24 **(a)** The subspace is $\{a_0 + a_1x + a_2x^2 + a_3x^3 \bigm| a_0 + 7a_1 + 49a_2 + 343a_3 = 0\}$. Rewriting $a_0 = -7a_1 - 49a_2 - 343a_3$ gives $\{(-7a_1 - 49a_2 - 343a_3) + a_1x + a_2x^2 + a_3x^3 \bigm| a_1, a_2, a_3 \in \mathbb{R}\}$, which, on breaking out the parameters, suggests $\langle -7 + x, -49 + x^2, -343 + x^3\rangle$ for the basis (it is easily verified).

(b) The given subspace is the collection of cubics $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$ and $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$. Gauss' method

$$\begin{array}{rcl} a_0 + 7a_1 + 49a_2 + 343a_3 = 0 & \xrightarrow{-\rho_1+\rho_2} & a_0 + 7a_1 + 49a_2 + 343a_3 = 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 = 0 & & -2a_1 - 24a_2 - 218a_3 = 0 \end{array}$$

gives that $a_1 = -12a_2 - 109a_3$ and that $a_0 = 35a_2 + 420a_3$. Rewriting $(35a_2 + 420a_3) + (-12a_2 - 109a_3)x + a_2x^2 + a_3x^3$ as $a_2 \cdot (35 - 12x + x^2) + a_3 \cdot (420 - 109x + x^3)$ suggests this for a basis $\langle 35 - 12x + x^2, 420 - 109x + x^3\rangle$. The above shows that it spans the space. Checking it is linearly independent is routine. (*Comment.* A worthwhile check is to verify that both polynomials in the basis have both seven and five as roots.)

(c) Here there are three conditions on the cubics, that $a_0 + 7a_1 + 49a_2 + 343a_3 = 0$, that $a_0 + 5a_1 + 25a_2 + 125a_3 = 0$, and that $a_0 + 3a_1 + 9a_2 + 27a_3 = 0$. Gauss' method

$$\begin{array}{rcl} a_0 + 7a_1 + 49a_2 + 343a_3 &=& 0 \\ a_0 + 5a_1 + 25a_2 + 125a_3 &=& 0 \\ a_0 + 3a_1 + 9a_2 + 27a_3 &=& 0 \end{array} \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \begin{array}{rcl} a_0 + 7a_1 + 49a_2 + 343a_3 &=& 0 \\ -2a_1 - 24a_2 - 218a_3 &=& 0 \\ 8a_2 + 120a_3 &=& 0 \end{array}$$

yields the single free variable a_3, with $a_2 = -15a_3$, $a_1 = 71a_3$, and $a_0 = -105a_3$. The paramatrization is this.

$$\{(-105a_3) + (71a_3)x + (-15a_3)x^2 + (a_3)x^3 \mid a_3 \in \mathbb{R}\} = \{a_3 \cdot (-105 + 71x - 15x^2 + x^3) \mid a_3 \in \mathbb{R}\}$$

Therefore, a natural candidate for the basis is $\langle -105 + 71x - 15x^2 + x^3 \rangle$. It spans the space by the work above. It is clearly linearly independent because it is a one-element set (with that single element not the zero object of the space). Thus, any cubic through the three points $(7, 0)$, $(5, 0)$, and $(3, 0)$ is a multiple of this one. (*Comment.* As in the prior question, a worthwhile check is to verify that plugging seven, five, and three into this polynomial yields zero each time.)

(d) This is the trivial subspace of $\mathcal{P}_3$. Thus, the basis is empty $\langle\rangle$.

Remark. The polynomial in the third item could alternatively have been derived by multiplying out $(x - 7)(x - 5)(x - 3)$.

Two.III.1.27 **(a)** To show that it is linearly independent, note that $d_1(c_1\vec{\beta}_1) + d_2(c_2\vec{\beta}_2) + d_3(c_3\vec{\beta}_3) = \vec{0}$ gives that $(d_1c_1)\vec{\beta}_1 + (d_2c_2)\vec{\beta}_2 + (d_3c_3)\vec{\beta}_3 = \vec{0}$, which in turn implies that each d_ic_i is zero. But with $c_i \neq 0$ that means that each d_i is zero. Showing that it spans the space is much the same; because $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ is a basis, and so spans the space, we can for any $\vec{v}$ write $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$, and then $\vec{v} = (d_1/c_1)(c_1\vec{\beta}_1) + (d_2/c_2)(c_2\vec{\beta}_2) + (d_3/c_3)(c_3\vec{\beta}_3)$.

If any of the scalars are zero then the result is not a basis, because it is not linearly independent.

(b) Showing that $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3 \rangle$ is linearly independent is easy. To show that it spans the space, assume that $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + d_3\vec{\beta}_3$. Then, we can represent the same $\vec{v}$ with respect to $\langle 2\vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3 \rangle$ in this way $\vec{v} = (1/2)(d_1 - d_2 - d_3)(2\vec{\beta}_1) + d_2(\vec{\beta}_1 + \vec{\beta}_2) + d_3(\vec{\beta}_1 + \vec{\beta}_3)$.

Two.III.1.29 To show that each scalar is zero, simply subtract $c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k - c_{k+1}\vec{\beta}_{k+1} - \cdots - c_n\vec{\beta}_n = \vec{0}$. The obvious generalization is that in any equation involving only the $\vec{\beta}$'s, and in which each $\vec{\beta}$ appears only once, each scalar is zero. For instance, an equation with a combination of the even-indexed basis vectors (i.e., $\vec{\beta}_2$, $\vec{\beta}_4$, etc.) on the right and the odd-indexed basis vectors on the left also gives the conclusion that all of the coefficients are zero.

Two.III.1.32 **(a)** Describing the vector space as

$$\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

suggests this for a basis.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$$

Verification is easy.

(b) This is one possible basis.

$$\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rangle$$

(c) As in the prior two questions, we can form a basis from two kinds of matrices. First are the matrices with a single one on the diagonal and all other entries zero (there are n of those matrices). Second are the matrices with two opposed off-diagonal entries are ones and all other entries are zeros. (That is, all entries in M are zero except that $m_{i,j}$ and $m_{j,i}$ are one.)

Two.III.1.33 **(a)** Any four vectors from $\mathbb{R}^3$ are linearly related because the vector equation

$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + c_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} + c_4 \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives rise to a linear system

$$\begin{array}{rcl} x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 &=& 0 \\ y_1c_1 + y_2c_2 + y_3c_3 + y_4c_4 &=& 0 \\ z_1c_1 + z_2c_2 + z_3c_3 + z_4c_4 &=& 0 \end{array}$$

that is homogeneous (and so has a solution) and has four unknowns but only three equations, and therefore has nontrivial solutions. (Of course, this argument applies to any subset of $\mathbb{R}^3$ with four or more vectors.)

(b) Given $x_1, \ldots, z_2$,
$$S = \{ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \}$$
to decide which vectors
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
are in the span of S, set up
$$c_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
and row reduce the resulting system.
$$\begin{aligned} x_1c_1 + x_2c_2 &= x \\ y_1c_1 + y_2c_2 &= y \\ z_1c_1 + z_2c_2 &= z \end{aligned}$$
There are two variables c_1 and c_2 but three equations, so when Gauss' method finishes, on the bottom row there will be some relationship of the form $0 = m_1x + m_2y + m_3z$. Hence, vectors in the span of the two-element set S must satisfy some restriction. Hence the span is not all of $\mathbb{R}^3$.

Subsection Two.III.2: Dimension

Two.III.2.14 One basis is $\langle 1, x, x^2 \rangle$, and so the dimension is three.

Two.III.2.16 For this space
$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \} = \{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \cdots + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \}$$
this is a natural basis.
$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$
The dimension is four.

Two.III.2.18 The bases for these spaces are developed in the answer set of the prior subsection.

(a) One basis is $\langle -7 + x, -49 + x^2, -343 + x^3 \rangle$. The dimension is three.
(b) One basis is $\langle 35 - 12x + x^2, 420 - 109x + x^3 \rangle$ so the dimension is two.
(c) A basis is $\{ -105 + 71x - 15x^2 + x^3 \}$. The dimension is one.
(d) This is the trivial subspace of $\mathcal{P}_3$ and so the basis is empty. The dimension is zero.

Two.III.2.22 In a four-dimensional space a set of four vectors is linearly independent if and only if it spans the space. The form of these vectors makes linear independence easy to show (look at the equation of fourth components, then at the equation of third components, etc.).

Two.III.2.24 **(a)** One **(b)** Two **(c)** n

Two.III.2.31 Let B_U be a basis for U and let B_W be a basis for W. The set $B_U \cup B_W$ is linearly dependent as it is a six member subset of the five-dimensional space $\mathbb{R}^5$. Thus some member of B_W is in the span of B_U, and thus $U \cap W$ is more than just the trivial space $\{\vec{0}\}$.

Generalization: if U, W are subspaces of a vector space of dimension n and if $\dim(U) + \dim(W) > n$ then they have a nontrivial intersection.

Two.III.2.33 **(a)** A basis for U is a linearly independent set in W and so can be expanded via Corollary 2.10 to a basis for W. The second basis has at least as many members as the first.
(b) One direction is clear: if $V = W$ then they have the same dimension. For the converse, let B_U be a basis for U. It is a linearly independent subset of W and so can be expanded to a basis for W. If $\dim(U) = \dim(W)$ then this basis for W has no more members than does B_U and so equals B_U. Since U and W have the same bases, they are equal.

(c) Let W be the space of finite-degree polynomials and let U be the subspace of polynomails that have only even-powered terms $\{a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n} \mid a_0, \ldots, a_n \in \mathbb{R}\}$. Both spaces have infinite dimension, but U is a proper subspace.

Subsection Two.III.3: Vector Spaces and Linear Systems

Two.III.3.17 **(a)** Yes. To see if there are c_1 and c_2 such that $c_1 \cdot \begin{pmatrix}2 & 1\end{pmatrix} + c_2 \cdot \begin{pmatrix}3 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}$ we solve

$$\begin{aligned} 2c_1 + 3c_2 &= 1 \\ c_1 + c_2 &= 0 \end{aligned}$$

and get $c_1 = -1$ and $c_2 = 1$. Thus the vector is in the row space.

(b) No. The equation $c_1 \begin{pmatrix}0 & 1 & 3\end{pmatrix} + c_2 \begin{pmatrix}-1 & 0 & 1\end{pmatrix} + c_3 \begin{pmatrix}-1 & 2 & 7\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}$ has no solution.

$$\left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 7 & 1 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \xrightarrow{-3\rho_1+\rho_2} \xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Thus, the vector is not in the row space.

Two.III.3.18 **(a)** No. To see if there are $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \begin{pmatrix}1 \\ 1\end{pmatrix} + c_2 \begin{pmatrix}1 \\ 1\end{pmatrix} = \begin{pmatrix}1 \\ 3\end{pmatrix}$$

we can use Gauss' method on the resulting linear system.

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + c_2 &= 3 \end{aligned} \xrightarrow{-\rho_1+\rho_2} \begin{aligned} c_1 + c_2 &= 1 \\ 0 &= 2 \end{aligned}$$

There is no solution and so the vector is not in the column space.

(b) Yes. From this relationship

$$c_1 \begin{pmatrix}1 \\ 2 \\ 1\end{pmatrix} + c_2 \begin{pmatrix}3 \\ 0 \\ -3\end{pmatrix} + c_3 \begin{pmatrix}1 \\ 4 \\ 3\end{pmatrix} = \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix}$$

we get a linear system that, when Gauss' method is applied,

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 2 & 0 & 4 & 0 \\ 1 & -3 & -3 & 0 \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & -6 & 2 & -2 \\ 0 & 0 & -6 & 1 \end{array}\right)$$

yields a solution. Thus, the vector is in the column space.

Two.III.3.19 A routine Gaussian reduction

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & 1 \end{pmatrix} \xrightarrow[-(1/2)\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3} \xrightarrow{-\rho_2+\rho_3} \xrightarrow{-\rho_3+\rho_4} \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -11/2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

suggests this basis $\langle \begin{pmatrix}2 & 0 & 3 & 4\end{pmatrix}, \begin{pmatrix}0 & 1 & 1 & -1\end{pmatrix}, \begin{pmatrix}0 & 0 & -11/2 & -3\end{pmatrix} \rangle$.

Another, perhaps more convenient procedure, is to swap rows first,

$$\xrightarrow{\rho_1 \leftrightarrow \rho_4} \xrightarrow[-2\rho_1+\rho_4]{-3\rho_1+\rho_3} \xrightarrow{-\rho_2+\rho_3} \xrightarrow{-\rho_3+\rho_4} \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

leading to the basis $\langle \begin{pmatrix}1 & 0 & -4 & -1\end{pmatrix}, \begin{pmatrix}0 & 1 & 1 & -1\end{pmatrix}, \begin{pmatrix}0 & 0 & 11 & 6\end{pmatrix} \rangle$.

Two.III.3.20 **(a)** This reduction

$$\xrightarrow[-(1/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2} \xrightarrow{-(1/3)\rho_2+\rho_3} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -3/2 & 1/2 \\ 0 & 0 & 4/3 \end{pmatrix}$$

shows that the row rank, and hence the rank, is three.

(b) Inspection of the columns shows that that the others are multiples of the first (inspection of the rows shows the same thing). Thus the rank is one.

Alternatively, the reduction

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{pmatrix} \xrightarrow[2\rho_1+\rho_3]{-3\rho_1+\rho_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

shows the same thing.

(c) This calculation

$$\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 1 \\ 6 & 4 & 3 \end{pmatrix} \xrightarrow[-6\rho_1+\rho_3]{-5\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -14 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

shows that the rank is two.

(d) The rank is zero.

Two.III.3.21 **(a)** This reduction

$$\begin{pmatrix} 1 & 3 \\ -1 & 3 \\ 1 & 4 \\ 2 & 1 \end{pmatrix} \xrightarrow[\substack{-\rho_1+\rho_3 \\ -2\rho_1+\rho_4}]{\rho_1+\rho_2} \xrightarrow[(5/6)\rho_2+\rho_4]{-(1/6)\rho_2+\rho_3} \begin{pmatrix} 1 & 3 \\ 0 & 6 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

gives $\langle \begin{pmatrix} 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 6 \end{pmatrix} \rangle$.

(b) Transposing and reducing

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 1 & -3 & -3 \end{pmatrix} \xrightarrow[-\rho_1+\rho_3]{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & -5 & -4 \end{pmatrix} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

and then transposing back gives this basis.

$$\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ -4 \end{pmatrix} \rangle$$

(c) Notice first that the surrounding space is given as $\mathcal{P}_3$, not $\mathcal{P}_2$. Then, taking the first polynomial $1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$ to be "the same" as the row vector $\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$, etc., leads to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 2 & -1 & 0 \end{pmatrix} \xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which yields the basis $\langle 1 + x, -x - x^2 \rangle$.

(d) Here "the same" gives

$$\begin{pmatrix} 1 & 0 & 1 & 3 & 1 & -1 \\ 1 & 0 & 3 & 2 & 1 & 4 \\ -1 & 0 & 5 & -1 & -1 & -9 \end{pmatrix} \xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{2\rho_2+\rho_3} \begin{pmatrix} 1 & 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

leading to this basis.

$$\langle \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 5 \end{pmatrix} \rangle$$

Two.III.3.23 If $a \neq 0$ then a choice of $d = (c/a)b$ will make the second row be a multiple of the first, specifically, c/a times the first. If $a = 0$ and $b = 0$ then a choice of $d = 1$ will ensure that the second row is nonzero. If $a = 0$ and $b \neq 0$ and $c = 0$ then any choice for d will do, since the matrix will automatically have rank one (even with the choice of $d = 0$). Finally, if $a = 0$ and $b \neq 0$ and $c \neq 0$ then no choice for d will suffice because the matrix is sure to have rank two.

Two.III.3.26 The set of columns must be dependent because the rank of the matrix is at most five while there are nine columns.

Two.III.3.29 It is a subspace because it is the column space of the matrix

$$\begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

of coefficients. To find a basis for the column space,

$$\{c_1 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

we take the three vectors from the spanning set, transpose, reduce,

$$\begin{pmatrix}3&1&2\\2&0&2\\4&-1&5\end{pmatrix}\xrightarrow[-(4/3)\rho_1+\rho_3]{-(2/3)\rho_1+\rho_2}\xrightarrow{-(7/2)\rho_2+\rho_3}\begin{pmatrix}3&1&2\\0&-2/3&2/3\\0&0&0\end{pmatrix}$$

and transpose back to get this.

$$\langle\begin{pmatrix}3\\1\\2\end{pmatrix},\begin{pmatrix}0\\-2/3\\2/3\end{pmatrix}\rangle$$

Two.III.3.31 **(a)** These reductions give different bases.

$$\begin{pmatrix}1&2&0\\1&2&1\end{pmatrix}\xrightarrow{-\rho_1+\rho_2}\begin{pmatrix}1&2&0\\0&0&1\end{pmatrix}\qquad\begin{pmatrix}1&2&0\\1&2&1\end{pmatrix}\xrightarrow{-\rho_1+\rho_2}\xrightarrow{2\rho_2}\begin{pmatrix}1&2&0\\0&0&2\end{pmatrix}$$

(b) An easy example is this.

$$\begin{pmatrix}1&2&1\\3&1&4\end{pmatrix}\qquad\begin{pmatrix}1&2&1\\3&1&4\\0&0&0\end{pmatrix}$$

This is a less simplistic example.

$$\begin{pmatrix}1&2&1\\3&1&4\end{pmatrix}\qquad\begin{pmatrix}1&2&1\\3&1&4\\2&4&2\\4&3&5\end{pmatrix}$$

(c) Assume that A and B are matrices with equal row spaces. Construct a matrix C with the rows of A above the rows of B, and another matrix D with the rows of B above the rows of A.

$$C=\begin{pmatrix}A\\B\end{pmatrix}\qquad D=\begin{pmatrix}B\\A\end{pmatrix}$$

Observe that C and D are row-equivalent (via a sequence of row-swaps) and so Gauss-Jordan reduce to the same reduced echelon form matrix.

Because the row spaces are equal, the rows of B are linear combinations of the rows of A so Gauss-Jordan reduction on C simply turns the rows of B to zero rows and thus the nonzero rows of C are just the nonzero rows obtained by Gauss-Jordan reducing A. The same can be said for the matrix D — Gauss-Jordan reduction on D gives the same non-zero rows as are produced by reduction on B alone. Therefore, A yields the same nonzero rows as C, which yields the same nonzero rows as D, which yields the same nonzero rows as B.

Two.III.3.34 Because the rows of a matrix A are turned into the columns of A^{trans} the dimension of the row space of A equals the dimension of the column space of A^{trans}. But the dimension of the row space of A is the rank of A and the dimension of the column space of A^{trans} is the rank of A^{trans}. Thus the two ranks are equal.

Two.III.3.36 No. Here, Gauss' method does not change the column space.

$$\begin{pmatrix}1&0\\3&1\end{pmatrix}\xrightarrow{-3\rho_1+\rho_2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Two.III.3.40 Clearly $\text{rank}(A)=\text{rank}(-A)$ as Gauss' method allows us to multiply all rows of a matrix by -1. In the same way, when $k\neq 0$ we have $\text{rank}(A)=\text{rank}(kA)$.

Addition is more interesting. The rank of a sum can be smaller than the rank of the summands.

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}-1&-2\\-3&-4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

The rank of a sum can be bigger than the rank of the summands.

$$\begin{pmatrix}1&2\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\3&4\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

But there is an upper bound (other than the size of the matrices). In general, $\text{rank}(A+B)\leq\text{rank}(A)+\text{rank}(B)$.

To prove this, note that Gaussian elimination can be performed on $A+B$ in either of two ways: we can first add A to B and then apply the appropriate sequence of reduction steps

$$(A+B)\xrightarrow{\text{step}_1}\cdots\xrightarrow{\text{step}_k}\text{echelon form}$$

or we can get the same results by performing step_1 through step_k separately on A and B, and then adding. The largest rank that we can end with in the second case is clearly the sum of the ranks. (The matrices above give examples of both possibilities, $\text{rank}(A + B) < \text{rank}(A) + \text{rank}(B)$ and $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$, happening.)

Subsection Two.III.4: Combining Subspaces

Two.III.4.20 With each of these we can apply Lemma 4.15.

(a) Yes. The plane is the sum of this W_1 and W_2 because for any scalars a and b
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ 0 \end{pmatrix} + \begin{pmatrix} b \\ b \end{pmatrix}$$
shows that the general vector is a sum of vectors from the two parts. And, these two subspaces are (different) lines through the origin, and so have a trivial intersection.

(b) Yes. To see that any vector in the plane is a combination of vectors from these parts, consider this relationship.
$$\begin{pmatrix} a \\ b \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1.1 \end{pmatrix}$$
We could now simply note that the set
$$\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1.1 \end{pmatrix}\}$$
is a basis for the space (because it is clearly linearly independent, and has size two in $\mathbb{R}^2$), and thus ther is one and only one solution to the above equation, implying that all decompositions are unique. Alternatively, we can solve
$$\begin{array}{rl} c_1 + c_2 = a \\ c_1 + 1.1c_2 = b \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1 + c_2 = & a \\ 0.1c_2 = & -a+b \end{array}$$
to get that $c_2 = 10(-a + b)$ and $c_1 = 11a - 10b$, and so we have
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11a-10b \\ 11a-10b \end{pmatrix} + \begin{pmatrix} -10a+10b \\ 1.1\cdot(-10a+10b) \end{pmatrix}$$
as required. As with the prior answer, each of the two subspaces is a line through the origin, and their intersection is trivial.

(c) Yes. Each vector in the plane is a sum in this way
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
and the intersection of the two subspaces is trivial.

(d) No. The intersection is not trivial.

(e) No. These are not subspaces.

Two.III.4.21 With each of these we can use Lemma 4.15.

(a) Any vector in $\mathbb{R}^3$ can be decomposed as this sum.
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$
And, the intersection of the xy-plane and the z-axis is the trivial subspace.

(b) Any vector in $\mathbb{R}^3$ can be decomposed as
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-z \\ y-z \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ z \\ z \end{pmatrix}$$
and the intersection of the two spaces is trivial.

Two.III.4.23 To show that they are subspaces is routine. We will argue they are complements with Lemma 4.15. The intersection $\mathcal{E} \cap \mathcal{O}$ is trivial because the only polynomial satisfying both conditions $p(-x) = p(x)$ and $p(-x) = -p(x)$ is the zero polynomial. To see that the entire space is the sum of the subspaces $\mathcal{E} + \mathcal{O} = \mathcal{P}_n$, note that the polynomials $p_0(x) = 1$, $p_2(x) = x^2$, $p_4(x) = x^4$, etc., are in $\mathcal{E}$ and also note that the polynomials $p_1(x) = x$, $p_3(x) = x^3$, etc., are in $\mathcal{O}$. Hence any member of $\mathcal{P}_n$ is a combination of members of $\mathcal{E}$ and $\mathcal{O}$.

Two.III.4.25 Clearly each is a subspace. The bases $B_i = \langle x^i \rangle$ for the subspaces, when concatenated, form a basis for the whole space.

Two.III.4.31 Of course, the zero vector is in all of the subspaces, so the intersection contains at least that one vector.. By the definition of direct sum the set $\{W_1, \ldots, W_k\}$ is independent and so no nonzero vector of W_i is a multiple of a member of W_j, when $i \neq j$. In particular, no nonzero vector from W_i equals a member of W_j.

Two.III.4.33 Yes. For any subspace of a vector space we can take any basis $\langle \vec{\omega}_1, \ldots, \vec{\omega}_k \rangle$ for that subspace and extend it to a basis $\langle \vec{\omega}_1, \ldots, \vec{\omega}_k, \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$ for the whole space. Then the complemen of the original subspace has this for a basis: $\langle \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$.

Two.III.4.34 **(a)** It must. Any member of $W_1 + W_2$ can be written $\vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$. As S_1 spans W_1, the vector $\vec{w}_1$ is a combination of members of S_1. Similarly $\vec{w}_2$ is a combination of members of S_2.

(b) An easy way to see that it can be linearly independent is to take each to be the empty set. On the other hand, in the space $\mathbb{R}^1$, if $W_1 = \mathbb{R}^1$ and $W_2 = \mathbb{R}^1$ and $S_1 = \{1\}$ and $S_2 = \{2\}$, then their union $S_1 \cup S_2$ is not independent.

Two.III.4.40 **(a)** The set

$$\{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = 0 \text{ for all } x \in \mathbb{R}\}$$

is easily seen to be the y-axis.

(b) The yz-plane.

(c) The z-axis.

(d) Assume that U is a subspace of some $\mathbb{R}^n$. Because $U^\perp$ contains the zero vector, since that vector is perpendicular to everything, we need only show that the orthocomplement is closed under linear combinations of two elements. If $\vec{w}_1, \vec{w}_2 \in U^\perp$ then $\vec{w}_1 \cdot \vec{u} = 0$ and $\vec{w}_2 \cdot \vec{u} = 0$ for all $\vec{u} \in U$. Thus $(c_1\vec{w}_1 + c_2\vec{w}_2) \cdot \vec{u} = c_1(\vec{w}_1 \cdot \vec{u}) + c_2(\vec{w}_2 \cdot \vec{u}) = 0$ for all $\vec{u} \in U$ and so $U^\perp$ is closed under linear combinations.

(e) The only vector orthogonal to itself is the zero vector.

(f) This is immediate.

(g) To prove that the dimensions add, it suffices by Corollary 4.13 and Lemma 4.15 to show that $U \cap U^\perp$ is the trivial subspace $\{\vec{0}\}$. But this is one of the prior items in this problem.

Two.III.4.41 Yes. The left-to-right implication is Corollary 4.13. For the other direction, assume that $\dim(V) = \dim(W_1) + \cdots + \dim(W_k)$. Let $B_1, \ldots, B_k$ be bases for $W_1, \ldots, W_k$. As V is the sum of the subspaces, any $\vec{v} \in V$ can be written $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$ and expressing each $\vec{w}_i$ as a combination of vectors from the associated basis B_i shows that the concatenation $B_1 {}^\frown \cdots {}^\frown B_k$ spans V. Now, that concatenation has $\dim(W_1) + \cdots + \dim(W_k)$ members, and so it is a spanning set of size $\dim(V)$. The concatenation is therefore a basis for V. Thus V is the direct sum.

Topic: Fields

Topic: Crystals

Topic: Dimensional Analysis

Chapter Three: Maps Between Spaces

Subsection Three.I.1: Definition and Examples

Three.I.1.10 **(a)** Call the map f.

$$\begin{pmatrix}a & b\end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix}a\\ b\end{pmatrix}$$

It is one-to-one because if f sends two members of the domain to the same image, that is, if $f\left(\begin{pmatrix}a & b\end{pmatrix}\right) = f\left(\begin{pmatrix}c & d\end{pmatrix}\right)$, then the definition of f gives that

$$\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}c\\ d\end{pmatrix}$$

and since column vectors are equal only if they have equal components, we have that $a = c$ and that $b = d$. Thus, if f maps two row vectors from the domain to the same column vector then the two row vectors are equal: $\begin{pmatrix}a & b\end{pmatrix} = \begin{pmatrix}c & d\end{pmatrix}$.

To show that f is onto we must show that any member of the codomain $\mathbb{R}^2$ is the image under f of some row vector. That's easy;

$$\begin{pmatrix}x\\ y\end{pmatrix}$$

is $f\left(\begin{pmatrix}x & y\end{pmatrix}\right)$.

The computation for preservation of addition is this.

$$f\left(\begin{pmatrix}a & b\end{pmatrix} + \begin{pmatrix}c & d\end{pmatrix}\right) = f\left(\begin{pmatrix}a+c & b+d\end{pmatrix}\right) = \begin{pmatrix}a+c\\ b+d\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} + \begin{pmatrix}c\\ d\end{pmatrix} = f\left(\begin{pmatrix}a & b\end{pmatrix}\right) + f\left(\begin{pmatrix}c & d\end{pmatrix}\right)$$

The computation for preservation of scalar multiplication is similar.

$$f\left(r\cdot\begin{pmatrix}a & b\end{pmatrix}\right) = f\left(\begin{pmatrix}ra & rb\end{pmatrix}\right) = \begin{pmatrix}ra\\ rb\end{pmatrix} = r\cdot\begin{pmatrix}a\\ b\end{pmatrix} = r\cdot f\left(\begin{pmatrix}a & b\end{pmatrix}\right)$$

(b) Denote the map from Example 1.2 by f. To show that it is one-to-one, assume that $f(a_0+a_1x+a_2x^2) = f(b_0+b_1x+b_2x^2)$. Then by the definition of the function,

$$\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix} = \begin{pmatrix}b_0\\ b_1\\ b_2\end{pmatrix}$$

and so $a_0 = b_0$ and $a_1 = b_1$ and $a_2 = b_2$. Thus $a_0+a_1x+a_2x^2 = b_0+b_1x+b_2x^2$, and consequently f is one-to-one.

The function f is onto because there is a polynomial sent to

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

by f, namely, $a+bx+cx^2$.

As for structure, this shows that f preserves addition

$$\begin{aligned}
f\left((a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)\right) &= f\left((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2\right)\\
&= \begin{pmatrix}a_0+b_0\\ a_1+b_1\\ a_2+b_2\end{pmatrix}\\
&= \begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix} + \begin{pmatrix}b_0\\ b_1\\ b_2\end{pmatrix}\\
&= f(a_0+a_1x+a_2x^2) + f(b_0+b_1x+b_2x^2)
\end{aligned}$$

and this shows

$$f(\,r(a_0+a_1x+a_2x^2)\,)=f(\,(ra_0)+(ra_1)x+(ra_2)x^2\,)$$
$$=\begin{pmatrix}ra_0\\ra_1\\ra_2\end{pmatrix}$$
$$=r\cdot\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}$$
$$=r\,f(a_0+a_1x+a_2x^2)$$

that it preserves scalar multiplication.

Three.I.1.11 These are the images.
(a) $\begin{pmatrix}5\\-2\end{pmatrix}$ **(b)** $\begin{pmatrix}0\\2\end{pmatrix}$ **(c)** $\begin{pmatrix}-1\\1\end{pmatrix}$

To prove that f is one-to-one, assume that it maps two linear polynomials to the same image $f(a_1+b_1x)=f(a_2+b_2x)$. Then

$$\begin{pmatrix}a_1-b_1\\b_1\end{pmatrix}=\begin{pmatrix}a_2-b_2\\b_2\end{pmatrix}$$

and so, since column vectors are equal only when their components are equal, $b_1=b_2$ and $a_1=a_2$. That shows that the two linear polynomials are equal, and so f is one-to-one.

To show that f is onto, note that

$$\begin{pmatrix}s\\t\end{pmatrix}$$

is the image of $(s-t)+tx$.

To check that f preserves structure, we can use item (2) of Lemma 1.9.

$$f\left(c_1\cdot(a_1+b_1x)+c_2\cdot(a_2+b_2x)\right)=f\left((c_1a_1+c_2a_2)+(c_1b_1+c_2b_2)x\right)$$
$$=\begin{pmatrix}(c_1a_1+c_2a_2)-(c_1b_1+c_2b_2)\\c_1b_1+c_2b_2\end{pmatrix}$$
$$=c_1\cdot\begin{pmatrix}a_1-b_1\\b_1\end{pmatrix}+c_2\cdot\begin{pmatrix}a_2-b_2\\b_2\end{pmatrix}$$
$$=c_1\cdot f(a_1+b_1x)+c_2\cdot f(a_2+b_2x)$$

Three.I.1.13 **(a)** No; this map is not one-to-one. In particular, the matrix of all zeroes is mapped to the same image as the matrix of all ones.

(b) Yes, this is an isomorphism.

It is one-to-one:

$$\text{if } f(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})=f(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}) \text{ then } \begin{pmatrix}a_1+b_1+c_1+d_1\\a_1+b_1+c_1\\a_1+b_1\\a_1\end{pmatrix}=\begin{pmatrix}a_2+b_2+c_2+d_2\\a_2+b_2+c_2\\a_2+b_2\\a_2\end{pmatrix}$$

gives that $a_1=a_2$, and that $b_1=b_2$, and that $c_1=c_2$, and that $d_1=d_2$.

It is onto, since this shows

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=f(\begin{pmatrix}w&z-w\\y-z&x-y\end{pmatrix})$$

that any four-tall vector is the image of a 2×2 matrix.

Finally, it preserves combinations

$$
\begin{aligned}
f(\,r_1\cdot\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\,)&=f(\begin{pmatrix}r_1a_1+r_2a_2&r_1b_1+r_2b_2\\r_1c_1+r_2c_2&r_1d_1+r_2d_2\end{pmatrix})\\
&=\begin{pmatrix}r_1a_1+\cdots+r_2d_2\\r_1a_1+\cdots+r_2c_2\\r_1a_1+\cdots+r_2b_2\\r_1a_1+r_2a_2\end{pmatrix}\\
&=r_1\cdot\begin{pmatrix}a_1+\cdots+d_1\\a_1+\cdots+c_1\\a_1+b_1\\a_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2+\cdots+d_2\\a_2+\cdots+c_2\\a_2+b_2\\a_2\end{pmatrix}\\
&=r_1\cdot f(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})+r_2\cdot f(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})
\end{aligned}
$$

and so item (2) of Lemma 1.9 shows that it preserves structure.

(c) Yes, it is an isomorphism.

To show that it is one-to-one, we suppose that two members of the domain have the same image under f.

$$f(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})=f(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})$$

This gives, by the definition of f, that $c_1+(d_1+c_1)x+(b_1+a_1)x^2+a_1x^3=c_2+(d_2+c_2)x+(b_2+a_2)x^2+a_2x^3$ and then the fact that polynomials are equal only when their coefficients are equal gives a set of linear equations

$$
\begin{aligned}
c_1&=c_2\\
d_1+c_1&=d_2+c_2\\
b_1+a_1&=b_2+a_2\\
a_1&=a_2
\end{aligned}
$$

that has only the solution $a_1=a_2$, $b_1=b_2$, $c_1=c_2$, and $d_1=d_2$.

To show that f is onto, we note that $p+qx+rx^2+sx^3$ is the image under f of this matrix.

$$\begin{pmatrix}s&r-s\\p&q-p\end{pmatrix}$$

We can check that f preserves structure by using item (2) of Lemma 1.9.

$$
\begin{aligned}
f(r_1\cdot\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})&=f(\begin{pmatrix}r_1a_1+r_2a_2&r_1b_1+r_2b_2\\r_1c_1+r_2c_2&r_1d_1+r_2d_2\end{pmatrix})\\
&=(r_1c_1+r_2c_2)+(r_1d_1+r_2d_2+r_1c_1+r_2c_2)x\\
&\qquad+(r_1b_1+r_2b_2+r_1a_1+r_2a_2)x^2+(r_1a_1+r_2a_2)x^3\\
&=r_1\cdot\big(c_1+(d_1+c_1)x+(b_1+a_1)x^2+a_1x^3\big)\\
&\qquad+r_2\cdot\big(c_2+(d_2+c_2)x+(b_2+a_2)x^2+a_2x^3\big)\\
&=r_1\cdot f(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})+r_2\cdot f(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})
\end{aligned}
$$

(d) No, this map does not preserve structure. For instance, it does not send the zero matrix to the zero polynomial.

Three.I.1.15 Many maps are possible. Here are two.

$$\begin{pmatrix}a&b\end{pmatrix}\mapsto\begin{pmatrix}b\\a\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}a&b\end{pmatrix}\mapsto\begin{pmatrix}2a\\b\end{pmatrix}$$

The verifications are straightforward adaptations of the others above.

Three.I.1.17 The space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because it is not a subset of $\mathbb{R}^3$. The two-tall vectors in $\mathbb{R}^2$ are not members of $\mathbb{R}^3$.

The natural isomorphism $\iota\colon\mathbb{R}^2\to\mathbb{R}^3$ (called the *injection* map) is this.

$$\begin{pmatrix}x\\y\end{pmatrix}\stackrel{\iota}{\longmapsto}\begin{pmatrix}x\\y\\0\end{pmatrix}$$

This map is one-to-one because

$$f(\begin{pmatrix}x_1\\y_1\end{pmatrix})=f(\begin{pmatrix}x_2\\y_2\end{pmatrix})\quad\text{implies}\quad\begin{pmatrix}x_1\\y_1\\0\end{pmatrix}=\begin{pmatrix}x_2\\y_2\\0\end{pmatrix}$$

which in turn implies that $x_1=x_2$ and $y_1=y_2$, and therefore the initial two two-tall vectors are equal.

Because

$$\begin{pmatrix}x\\y\\0\end{pmatrix}=f(\begin{pmatrix}x\\y\end{pmatrix})$$

this map is onto the xy-plane.

To show that this map preserves structure, we will use item (2) of Lemma 1.9 and show

$$\begin{aligned}f(c_1\cdot\begin{pmatrix}x_1\\y_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\end{pmatrix})=f(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\end{pmatrix})&=\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\0\end{pmatrix}\\&=c_1\cdot\begin{pmatrix}x_1\\y_1\\0\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\0\end{pmatrix}=c_1\cdot f(\begin{pmatrix}x_1\\y_1\end{pmatrix})+c_2\cdot f(\begin{pmatrix}x_2\\y_2\end{pmatrix})\end{aligned}$$

that it preserves combinations of two vectors.

Three.I.1.19 When k is the product $k=mn$, here is an isomorphism.

$$\begin{pmatrix}r_1&r_2&\ldots&\\&\vdots&&\\&&\ldots&r_{m\cdot n}\end{pmatrix}\mapsto\begin{pmatrix}r_1\\r_2\\\vdots\\r_{m\cdot n}\end{pmatrix}$$

Checking that this is an isomorphism is easy.

Three.I.1.26 In each item, following item (2) of Lemma 1.9, we show that the map preserves structure by showing that the it preserves linear combinations of two members of the domain.

(a) The identity map is clearly one-to-one and onto. For linear combinations the check is easy.

$$\mathrm{id}(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2)=c_1\vec{v}_1+c_2\vec{v}_2=c_1\cdot\mathrm{id}(\vec{v}_1)+c_2\cdot\mathrm{id}(\vec{v}_2)$$

(b) The inverse of a correspondence is also a correspondence (as stated in the appendix), so we need only check that the inverse preserves linear combinations. Assume that $\vec{w}_1=f(\vec{v}_1)$ (so $f^{-1}(\vec{w}_1)=\vec{v}_1$) and assume that $\vec{w}_2=f(\vec{v}_2)$.

$$\begin{aligned}f^{-1}(c_1\cdot\vec{w}_1+c_2\cdot\vec{w}_2)&=f^{-1}\big(c_1\cdot f(\vec{v}_1)+c_2\cdot f(\vec{v}_2)\big)\\&=f^{-1}\big(f\big(c_1\vec{v}_1+c_2\vec{v}_2)\big)\\&=c_1\vec{v}_1+c_2\vec{v}_2\\&=c_1\cdot f^{-1}(\vec{w}_1)+c_2\cdot f^{-1}(\vec{w}_2)\end{aligned}$$

(c) The composition of two correspondences is a correspondence (as stated in the appendix), so we need only check that the composition map preserves linear combinations.

$$\begin{aligned}g\circ f\,\big(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2\big)&=g\big(f(c_1\vec{v}_1+c_2\vec{v}_2)\big)\\&=g\big(c_1\cdot f(\vec{v}_1)+c_2\cdot f(\vec{v}_2)\big)\\&=c_1\cdot g\big(f(\vec{v}_1))+c_2\cdot g(f(\vec{v}_2)\big)\\&=c_1\cdot g\circ f\,(\vec{v}_1)+c_2\cdot g\circ f\,(\vec{v}_2)\end{aligned}$$

Three.I.1.29 **(a)** This map is one-to-one because if $d_s(\vec{v}_1)=d_s(\vec{v}_2)$ then by definition of the map, $s\cdot\vec{v}_1=s\cdot\vec{v}_2$ and so $\vec{v}_1=\vec{v}_2$, as s is nonzero. This map is onto as any $\vec{w}\in\mathbb{R}^2$ is the image of $\vec{v}=(1/s)\cdot\vec{w}$ (again, note that s is nonzero). (Another way to see that this map is a correspondence is to observe that it has an inverse: the inverse of d_s is $d_{1/s}$.)

To finish, note that this map preserves linear combinations

$$d_s(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2)=s(c_1\vec{v}_1+c_2\vec{v}_2)=c_1s\vec{v}_1+c_2s\vec{v}_2=c_1\cdot d_s(\vec{v}_1)+c_2\cdot d_s(\vec{v}_2)$$

and therefore is an isomorphism.

(b) As in the prior item, we can show that the map t_θ is a correspondence by noting that it has an inverse, $t_{-\theta}$.

That the map preserves structure is geometrically easy to see. For instance, adding two vectors and then rotating them has the same effect as rotating first and then adding. For an algebraic argument, consider polar coordinates: the map t_θ sends the vector with endpoint (r,ϕ) to the vector with endpoint $(r,\phi+\theta)$. Then the familiar trigonometric formulas $\cos(\phi+\theta)=\cos\phi\,\cos\theta-\sin\phi\,\sin\theta$ and $\sin(\phi+\theta)=\sin\phi\,\cos\theta+\cos\phi\,\sin\theta$ show how to express the map's action in the usual rectangular coordinate system.

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}r\cos\phi\\r\sin\phi\end{pmatrix}\overset{t_\theta}{\longmapsto}\begin{pmatrix}r\cos(\phi+\theta)\\r\sin(\phi+\theta)\end{pmatrix}=\begin{pmatrix}x\cos\theta-y\sin\theta\\x\sin\theta+y\cos\theta\end{pmatrix}$$

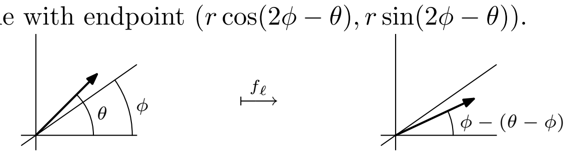
Now the calculation for preservation of addition is routine.

$$\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}\overset{t_\theta}{\longmapsto}\begin{pmatrix}(x_1+x_2)\cos\theta-(y_1+y_2)\sin\theta\\(x_1+x_2)\sin\theta+(y_1+y_2)\cos\theta\end{pmatrix}=\begin{pmatrix}x_1\cos\theta-y_1\sin\theta\\x_1\sin\theta+y_1\cos\theta\end{pmatrix}+\begin{pmatrix}x_2\cos\theta-y_2\sin\theta\\x_2\sin\theta+y_2\cos\theta\end{pmatrix}$$

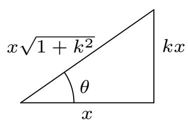
The calculation for preservation of scalar multiplication is similar.

(c) This map is a correspondence because it has an inverse (namely, itself).

As in the last item, that the reflection map preserves structure is geometrically easy to see: adding vectors and then reflecting gives the same result as reflecting first and then adding, for instance. For an algebraic proof, suppose that the line ℓ has slope k (the case of a line with undefined slope can be done as a separate, but easy, case). We can follow the hint and use polar coordinates: where the line ℓ forms an angle of ϕ with the x-axis, the action of f_ℓ is to send the vector with endpoint $(r\cos\theta,r\sin\theta)$ to the one with endpoint $(r\cos(2\phi-\theta),r\sin(2\phi-\theta))$.



To convert to rectangular coordinates, we will use some trigonometric formulas, as we did in the prior item. First observe that $\cos\phi$ and $\sin\phi$ can be determined from the slope k of the line. This picture



gives that $\cos\phi=1/\sqrt{1+k^2}$ and $\sin\phi=k/\sqrt{1+k^2}$. Now,

$$\begin{aligned}\cos(2\phi-\theta)&=\cos(2\phi)\,\cos\theta+\sin(2\phi)\,\sin\theta\\&=\bigl(\cos^2\phi-\sin^2\phi\bigr)\,\cos\theta+(2\sin\phi\cos\phi)\,\sin\theta\\&=\left((\frac{1}{\sqrt{1+k^2}})^2-(\frac{k}{\sqrt{1+k^2}})^2\right)\cos\theta+\left(2\frac{k}{\sqrt{1+k^2}}\frac{1}{\sqrt{1+k^2}}\right)\sin\theta\\&=\left(\frac{1-k^2}{1+k^2}\right)\cos\theta+\left(\frac{2k}{1+k^2}\right)\sin\theta\end{aligned}$$

and thus the first component of the image vector is this.

$$r\cdot\cos(2\phi-\theta)=\frac{1-k^2}{1+k^2}\cdot x+\frac{2k}{1+k^2}\cdot y$$

A similar calculation shows that the second component of the image vector is this.

$$r\cdot\sin(2\phi-\theta)=\frac{2k}{1+k^2}\cdot x-\frac{1-k^2}{1+k^2}\cdot y$$

With this algebraic description of the action of f_ℓ

$$\begin{pmatrix}x\\y\end{pmatrix}\overset{f_\ell}{\longmapsto}\begin{pmatrix}(1-k^2/1+k^2)\cdot x+(2k/1+k^2)\cdot y\\(2k/1+k^2)\cdot x-(1-k^2/1+k^2)\cdot y\end{pmatrix}$$

checking that it preserves structure is routine.

Subsection Three.I.2: Dimension Characterizes Isomorphism

Three.I.2.8 Each pair of spaces is isomorphic if and only if the two have the same dimension. We can, when there is an isomorphism, state a map, but it isn't strictly necessary.

(a) No, they have different dimensions.
(b) No, they have different dimensions.
(c) Yes, they have the same dimension. One isomorphism is this.

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \mapsto \begin{pmatrix} a \\ \vdots \\ f \end{pmatrix}$$

(d) Yes, they have the same dimension. This is an isomorphism.

$$a + bx + \cdots + fx^5 \mapsto \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

(e) Yes, both have dimension $2k$.

Three.I.2.9 **(a)** $\mathrm{Rep}_B(3-2x) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Three.I.2.10 They have different dimensions.

Three.I.2.11 Yes, both are mn-dimensional.

Three.I.2.12 Yes, any two (nondegenerate) planes are both two-dimensional vector spaces.

Three.I.2.16 Where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$, the inverse is this.

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \mapsto c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$$

Three.I.2.17 All three spaces have dimension equal to the rank of the matrix.

Three.I.2.23 Yes.

Assume that V is a vector space with basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and that W is another vector space such that the map $f\colon B \to W$ is a correspondence. Consider the extension $\hat{f}\colon V \to W$ of f.

$$\hat{f}(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n).$$

The map $\hat{f}$ is an isomorphism.

First, $\hat{f}$ is well-defined because every member of V has one and only one representation as a linear combination of elements of B.

Second, $\hat{f}$ is one-to-one because every member of W has only one representation as a linear combination of elements of $\langle f(\vec{\beta}_1), \ldots, f(\vec{\beta}_n) \rangle$. That map $\hat{f}$ is onto because every member of W has at least one representation as a linear combination of members of $\langle f(\vec{\beta}_1), \ldots, f(\vec{\beta}_n) \rangle$.

Finally, preservation of structure is routine to check. For instance, here is the preservation of addition calculation.

$$\begin{aligned} \hat{f}(\,(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + (d_1\vec{\beta}_1 + \cdots + d_n\vec{\beta}_n)\,) &= \hat{f}(\,(c_1+d_1)\vec{\beta}_1 + \cdots + (c_n+d_n)\vec{\beta}_n\,) \\ &= (c_1+d_1)f(\vec{\beta}_1) + \cdots + (c_n+d_n)f(\vec{\beta}_n) \\ &= c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) + d_1 f(\vec{\beta}_1) + \cdots + d_n f(\vec{\beta}_n) \\ &= \hat{f}(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + +\hat{f}(d_1\vec{\beta}_1 + \cdots + d_n\vec{\beta}_n). \end{aligned}$$

Preservation of scalar multiplication is similar.

Subsection Three.II.1: Definition

Three.II.1.17 **(a)** Yes. The verification is straightforward.

$$\begin{aligned}
h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}) &= h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix}) \\
&= \begin{pmatrix}c_1x_1+c_2x_2\\c_1x_1+c_2x_2+c_1y_1+c_2y_2+c_1z_1+c_2z_2\end{pmatrix} \\
&= c_1\cdot\begin{pmatrix}x_1\\x_1+y_1+z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\c_2+y_2+z_2\end{pmatrix} \\
&= c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})
\end{aligned}$$

(b) Yes. The verification is easy.

$$\begin{aligned}
h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}) &= h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix}) \\
&= \begin{pmatrix}0\\0\end{pmatrix} \\
&= c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})
\end{aligned}$$

(c) No. An example of an addition that is not respected is this.

$$h(\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix})=\begin{pmatrix}1\\1\end{pmatrix}\neq h(\begin{pmatrix}0\\0\\0\end{pmatrix})+h(\begin{pmatrix}0\\0\\0\end{pmatrix})$$

(d) Yes. The verification is straightforward.

$$\begin{aligned}
h(c_1\cdot\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}) &= h(\begin{pmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{pmatrix}) \\
&= \begin{pmatrix}2(c_1x_1+c_2x_2)+(c_1y_1+c_2y_2)\\3(c_1y_1+c_2y_2)-4(c_1z_1+c_2z_2)\end{pmatrix} \\
&= c_1\cdot\begin{pmatrix}2x_1+y_1\\3y_1-4z_1\end{pmatrix}+c_2\cdot\begin{pmatrix}2x_2+y_2\\3y_2-4z_2\end{pmatrix} \\
&= c_1\cdot h(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+c_2\cdot h(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})
\end{aligned}$$

Three.II.1.18 For each, we must either check that linear combinations are preserved, or give an example of a linear combination that is not.

(a) Yes. The check that it preserves combinations is routine.

$$\begin{aligned}
h(r_1\cdot\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}) &= h(\begin{pmatrix}r_1a_1+r_2a_2&r_1b_1+r_2b_2\\r_1c_1+r_2c_2&r_1d_1+r_2d_2\end{pmatrix}) \\
&= (r_1a_1+r_2a_2)+(r_1d_1+r_2d_2) \\
&= r_1(a_1+d_1)+r_2(a_2+d_2) \\
&= r_1\cdot h(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})+r_2\cdot h(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})
\end{aligned}$$

(b) No. For instance, not preserved is multiplication by the scalar 2.

$$h(2\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix})=h(\begin{pmatrix}2&0\\0&2\end{pmatrix})=4\quad\text{while}\quad 2\cdot h(\begin{pmatrix}1&0\\0&1\end{pmatrix})=2\cdot 1=2$$

(c) Yes. This is the check that it preserves combinations of two members of the domain.

$$\begin{aligned}
h(r_1\cdot\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+r_2\cdot\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}) &= h(\begin{pmatrix}r_1a_1+r_2a_2&r_1b_1+r_2b_2\\r_1c_1+r_2c_2&r_1d_1+r_2d_2\end{pmatrix}) \\
&= 2(r_1a_1+r_2a_2)+3(r_1b_1+r_2b_2)+(r_1c_1+r_2c_2)-(r_1d_1+r_2d_2) \\
&= r_1(2a_1+3b_1+c_1-d_1)+r_2(2a_2+3b_2+c_2-d_2) \\
&= r_1\cdot h(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix})+r_2\cdot h(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix})
\end{aligned}$$

(d) No. An example of a combination that is not preserved is this.

$$h(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix})=h(\begin{pmatrix}2&0\\0&0\end{pmatrix})=4\quad\text{while}\quad h(\begin{pmatrix}1&0\\0&0\end{pmatrix})+h(\begin{pmatrix}1&0\\0&0\end{pmatrix})=1+1=2$$

Three.II.1.19 The check that each is a homomorphisms is routine. Here is the check for the differentiation map.

$$\begin{aligned}
&\frac{d}{dx}(r\cdot(a_0+a_1x+a_2x^2+a_3x^3)+s\cdot(b_0+b_1x+b_2x^2+b_3x^3)) \\
&\qquad= \frac{d}{dx}((ra_0+sb_0)+(ra_1+sb_1)x+(ra_2+sb_2)x^2+(ra_3+sb_3)x^3) \\
&\qquad= (ra_1+sb_1)+2(ra_2+sb_2)x+3(ra_3+sb_3)x^2 \\
&\qquad= r\cdot(a_1+2a_2x+3a_3x^2)+s\cdot(b_1+2b_2x+3b_3x^2) \\
&\qquad= r\cdot\frac{d}{dx}(a_0+a_1x+a_2x^2+a_3x^3)+s\cdot\frac{d}{dx}(b_0+b_1x+b_2x^2+b_3x^3)
\end{aligned}$$

(An alternate proof is to simply note that this is a property of differentiation that is familar from calculus.)

These two maps are not inverses as this composition does not act as the identity map on this element of the domain.

$$1\in\mathcal{P}_3\overset{d/dx}{\longmapsto}0\in\mathcal{P}_2\overset{\int}{\longmapsto}0\in\mathcal{P}_3$$

Three.II.1.23 **(a)** This map does not preserve structure since $f(1+1)=3$, while $f(1)+f(1)=2$.
(b) The check is routine.

$$\begin{aligned}
f(r_1\cdot\begin{pmatrix}x_1\\y_1\end{pmatrix}+r_2\cdot\begin{pmatrix}x_2\\y_2\end{pmatrix}) &= f(\begin{pmatrix}r_1x_1+r_2x_2\\r_1y_1+r_2y_2\end{pmatrix}) \\
&= (r_1x_1+r_2x_2)+2(r_1y_1+r_2y_2) \\
&= r_1\cdot(x_1+2y_1)+r_2\cdot(x_2+2y_2) \\
&= r_1\cdot f(\begin{pmatrix}x_1\\y_1\end{pmatrix})+r_2\cdot f(\begin{pmatrix}x_2\\y_2\end{pmatrix})
\end{aligned}$$

Three.II.1.24 Yes. Where $h\colon V\to W$ is linear, $h(\vec{u}-\vec{v})=h(\vec{u}+(-1)\cdot\vec{v})=h(\vec{u})+(-1)\cdot h(\vec{v})=h(\vec{u})-h(\vec{v})$.

Three.II.1.26 That it is a homomorphism follows from the familiar rules that the logarithm of a product is the sum of the logarithms $\ln(ab)=\ln(a)+\ln(b)$ and that the logarithm of a power is the multiple of the logarithm $\ln(a^r)=r\ln(a)$. This map is an isomorphism because it has an inverse, namely, the exponential map, so it is a correspondence, and therefore it is an isomorphism.

Three.II.1.27 Where $\hat{x}=x/2$ and $\hat{y}=y/3$, the image set is

$$\{\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\mid\frac{(2\hat{x})^2}{4}+\frac{(3\hat{y})^2}{9}=1\}=\{\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\mid\hat{x}^2+\hat{y}^2=1\}$$

the unit circle in the $\hat{x}\hat{y}$-plane.

Three.II.1.28 The circumference function $r\mapsto 2\pi r$ is linear. Thus we have $2\pi\cdot(r_{\text{earth}}+6)-2\pi\cdot(r_{\text{earth}})=12\pi$. Observe that it takes the same amount of extra rope to raise the circle from tightly wound around a basketball to six feet above that basketball as it does to raise it from tightly wound around the earth to six feet above the earth.

Three.II.1.29 Verifying that it is linear is routine.

$$\begin{aligned}
h(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) &= h(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{pmatrix}) \\
&= 3(c_1x_1 + c_2x_2) - (c_1y_1 + c_2y_2) - (c_1z_1 + c_2z_2) \\
&= c_1 \cdot (3x_1 - y_1 - z_1) + c_2 \cdot (3x_2 - y_2 - z_2) \\
&= c_1 \cdot h(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + c_2 \cdot h(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix})
\end{aligned}$$

The natural guess at a generalization is that for any fixed $\vec{k} \in \mathbb{R}^3$ the map $\vec{v} \mapsto \vec{v} \bullet \vec{k}$ is linear. This statement is true. It follows from properties of the dot product we have seen earlier: $(\vec{v}+\vec{u})\bullet\vec{k} = \vec{v}\bullet\vec{k}+\vec{u}\bullet\vec{k}$ and $(r\vec{v})\bullet\vec{k} = r(\vec{v}\bullet\vec{k})$. (The natural guess at a generalization of this generalization, that the map from $\mathbb{R}^n$ to $\mathbb{R}$ whose action consists of taking the dot product of its argument with a fixed vector $\vec{k} \in \mathbb{R}^n$ is linear, is also true.)

Three.II.1.33 **(a)** Yes. The set of $\vec{w}\,$'s cannot be linearly independent if the set of $\vec{v}\,$'s is linearly dependent because any nontrivial relationship in the domain $\vec{0}_V = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ would give a nontrivial relationship in the range $h(\vec{0}_V) = \vec{0}_W = h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n) = c_1\vec{w} + \cdots + c_n\vec{w}_n$.

(b) Not necessarily. For instance, the transformation of $\mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

sends this linearly independent set in the domain to a linearly dependent image.

$$\{\vec{v}_1, \vec{v}_2\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \;\mapsto\; \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\} = \{\vec{w}_1, \vec{w}_2\}$$

(c) Not necessarily. An example is the projection map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

and this set that does not span the domain but maps to a set that does span the codomain.

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\} \stackrel{\pi}{\longmapsto} \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

(d) Not necessarily. For instance, the injection map $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$ sends the standard basis $\mathcal{E}_2$ for the domain to a set that does not span the codomain. (*Remark.* However, the set of $\vec{w}$'s does span the range. A proof is easy.)

Three.II.1.36 **(a)** For $\vec{v}_0, \vec{v}_1 \in \mathbb{R}^n$, the line through $\vec{v}_0$ with direction $\vec{v}_1$ is the set $\{\vec{v}_0 + t\cdot\vec{v}_1 \mid t \in \mathbb{R}\}$. The image under h of that line $\{h(\vec{v}_0 + t\cdot\vec{v}_1) \mid t \in \mathbb{R}\} = \{h(\vec{v}_0) + t\cdot h(\vec{v}_1) \mid t \in \mathbb{R}\}$ is the line through $h(\vec{v}_0)$ with direction $h(\vec{v}_1)$. If $h(\vec{v}_1)$ is the zero vector then this line is degenerate.

(b) A k-dimensional linear surface in $\mathbb{R}^n$ maps to a (possibly degenerate) k-dimensional linear surface in $\mathbb{R}^m$. The proof is just like that the one for the line.

Subsection Three.II.2: Rangespace and Nullspace

Three.II.2.22 First, to answer whether a polynomial is in the nullspace, we have to consider it as a member of the domain $\mathcal{P}_3$. To answer whether it is in the rangespace, we consider it as a member of the codomain $\mathcal{P}_4$. That is, for $p(x) = x^4$, the question of whether it is in the rangespace is sensible but the question of whether it is in the nullspace is not because it is not even in the domain.

(a) The polynomial $x^3 \in \mathcal{P}_3$ is not in the nullspace because $h(x^3) = x^4$ is not the zero polynomial in $\mathcal{P}_4$. The polynomial $x^3 \in \mathcal{P}_4$ is in the rangespace because $x^2 \in \mathcal{P}_3$ is mapped by h to x^3.

(b) The answer to both questions is, "Yes, because $h(0) = 0$." The polynomial $0 \in \mathcal{P}_3$ is in the nullspace because it is mapped by h to the zero polynomial in $\mathcal{P}_4$. The polynomial $0 \in \mathcal{P}_4$ is in the rangespace because it is the image, under h, of $0 \in \mathcal{P}_3$.

(c) The polynomial $7 \in \mathcal{P}_3$ is not in the nullspace because $h(7) = 7x$ is not the zero polynomial in $\mathcal{P}_4$. The polynomial $x^3 \in \mathcal{P}_4$ is not in the rangespace because there is no member of the domain that when multiplied by x gives the constant polynomial $p(x) = 7$.

(d) The polynomial $12x - 0.5x^3 \in \mathcal{P}_3$ is not in the nullspace because $h(12x - 0.5x^3) = 12x^2 - 0.5x^4$. The polynomial $12x - 0.5x^3 \in \mathcal{P}_4$ is in the rangespace because it is the image of $12 - 0.5x^2$.

(e) The polynomial $1 + 3x^2 - x^3 \in \mathcal{P}_3$ is not in the nullspace because $h(1 + 3x^2 - x^3) = x + 3x^3 - x^4$. The polynomial $1 + 3x^2 - x^3 \in \mathcal{P}_4$ is not in the rangespace because of the constant term.

Three.II.2.23 **(a)** The nullspace is

$$\mathscr{N}(h) = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \bigm| a + ax + ax^2 + 0x^3 = 0 + 0x + 0x^2 + 0x^3\} = \{\begin{pmatrix} 0 \\ b \end{pmatrix} \bigm| b \in \mathbb{R}\}$$

while the rangespace is

$$\mathscr{R}(h) = \{a + ax + ax^2 \in \mathcal{P}_3 \bigm| a, b \in \mathbb{R}\} = \{a \cdot (1 + x + x^2) \bigm| a \in \mathbb{R}\}$$

and so the nullity is one and the rank is one.

(b) The nullspace is this.

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigm| a + d = 0\} = \{\begin{pmatrix} -d & b \\ c & d \end{pmatrix} \bigm| b, c, d \in \mathbb{R}\}$$

The rangespace

$$\mathscr{R}(h)\{a + d \bigm| a, b, c, d \in \mathbb{R}\}$$

is all of $\mathbb{R}$ (we can get any real number by taking d to be 0 and taking a to be the desired number). Thus, the nullity is three and the rank is one.

(c) The nullspace is

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigm| a + b + c = 0 \text{ and } d = 0\} = \{\begin{pmatrix} -b - c & b \\ c & 0 \end{pmatrix} \bigm| b, c \in \mathbb{R}\}$$

while the rangespace is $\mathscr{R}(h) = \{r + sx^2 \bigm| r, s \in \mathbb{R}\}$. Thus, the nullity is two and the rank is two.

(d) The nullspace is all of $\mathbb{R}^3$ so the nullity is three. The rangespace is the trivial subspace of $\mathbb{R}^4$ so the rank is zero.

Three.II.2.24 For each, use the result that the rank plus the nullity equals the dimension of the domain.

(a) 0 **(b)** 3 **(c)** 3 **(d)** 0

Three.II.2.25 Because

$$\frac{d}{dx}\,(a_0 + a_1x + \cdots + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

we have this.

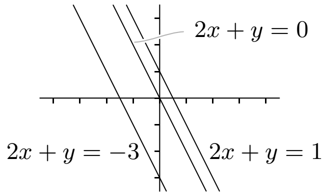
$$\begin{aligned}
\mathscr{N}(\frac{d}{dx}) &= \{a_0 + \cdots + a_nx^n \bigm| a_1 + 2a_2x + \cdots + na_nx^{n-1} = 0 + 0x + \cdots + 0x^{n-1}\} \\
&= \{a_0 + \cdots + a_nx^n \bigm| a_1 = 0, \text{ and } a_2 = 0, \ldots, a_n = 0\} \\
&= \{a_0 + 0x + 0x^2 + \cdots + 0x^n \bigm| a_0 \in \mathbb{R}\}
\end{aligned}$$

In the same way,

$$\mathscr{N}(\frac{d^k}{dx^k}) = \{a_0 + a_1x + \cdots + a_nx^n \bigm| a_0, \ldots, a_{k-1} \in \mathbb{R}\}$$

for $k \leq n$.

Three.II.2.28 All inverse images are lines with slope -2.



Three.II.2.29 These are the inverses.

(a) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_1x + (a_2/2)x^2 + (a_3/3)x^3$

(b) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_2x + a_1x^2 + a_3x^3$

(c) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_3 + a_0x + a_1x^2 + a_2x^3$

(d) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3$

For instance, for the second one, the map given in the question sends $0+1x+2x^2+3x^3 \mapsto 0+2x+1x^2+3x^3$ and then the inverse above sends $0+2x+1x^2+3x^3 \mapsto 0+1x+2x^2+3x^3$. So this map is actually self-inverse.

Three.II.2.33 The nullspace is this.

$$\begin{aligned}\{a_0+a_1x+\cdots+a_nx^n \bigm| a_0(1)+\frac{a_1}{2}(1^2)+\cdots+\frac{a_n}{n+1}(1^{n+1})=0\}\\ =\{a_0+a_1x+\cdots+a_nx^n \bigm| a_0+(a_1/2)+\cdots+(a_{n+1}/n+1)=0\}\end{aligned}$$

Thus the nullity is n.

Three.II.2.38 Yes. For the transformation of $\mathbb{R}^2$ given by

$$\begin{pmatrix}x\\y\end{pmatrix}\stackrel{h}{\longmapsto}\begin{pmatrix}0\\x\end{pmatrix}$$

we have this.

$$\mathscr{N}(h)=\{\begin{pmatrix}0\\y\end{pmatrix} \bigm| y\in\mathbb{R}\}=\mathscr{R}(h)$$

Remark. We will see more of this in the fifth chapter.

Three.II.2.40 **(a)** We will show that the two sets are equal $h^{-1}(\vec{w})=\{\vec{v}+\vec{n} \bigm| \vec{n}\in\mathscr{N}(h)\}$ by mutual inclusion. For the $\{\vec{v}+\vec{n} \bigm| \vec{n}\in\mathscr{N}(h)\}\subseteq h^{-1}(\vec{w})$ direction, just note that $h(\vec{v}+\vec{n})=h(\vec{v})+h(\vec{n})$ equals $\vec{w}$, and so any member of the first set is a member of the second. For the $h^{-1}(\vec{w})\subseteq\{\vec{v}+\vec{n} \bigm| \vec{n}\in\mathscr{N}(h)\}$ direction, consider $\vec{u}\in h^{-1}(\vec{w})$. Because h is linear, $h(\vec{u})=h(\vec{v})$ implies that $h(\vec{u}-\vec{v})=\vec{0}$. We can write $\vec{u}-\vec{v}$ as $\vec{n}$, and then we have that $\vec{u}\in\{\vec{v}+\vec{n} \bigm| \vec{n}\in\mathscr{N}(h)\}$, as desired, because $\vec{u}=\vec{v}+(\vec{u}-\vec{v})$.

(b) This check is routine.

(c) This is immediate.

(d) For the linearity check, briefly, where c,d are scalars and $\vec{x},\vec{y}\in\mathbb{R}^n$ have components $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$, we have this.

$$\begin{aligned}h(c\cdot\vec{x}+d\cdot\vec{y})&=\begin{pmatrix}a_{1,1}(cx_1+dy_1)+\cdots+a_{1,n}(cx_n+dy_n)\\ \vdots\\ a_{m,1}(cx_1+dy_1)+\cdots+a_{m,n}(cx_n+dy_n)\end{pmatrix}\\ &=\begin{pmatrix}a_{1,1}cx_1+\cdots+a_{1,n}cx_n\\ \vdots\\ a_{m,1}cx_1+\cdots+a_{m,n}cx_n\end{pmatrix}+\begin{pmatrix}a_{1,1}dy_1+\cdots+a_{1,n}dy_n\\ \vdots\\ a_{m,1}dy_1+\cdots+a_{m,n}dy_n\end{pmatrix}\\ &=c\cdot h(\vec{x})+d\cdot h(\vec{y})\end{aligned}$$

The appropriate conclusion is that General = Particular + Homogeneous.

(e) Each power of the derivative is linear because of the rules

$$\frac{d^k}{dx^k}(f(x)+g(x))=\frac{d^k}{dx^k}f(x)+\frac{d^k}{dx^k}g(x)\quad\text{and}\quad\frac{d^k}{dx^k}rf(x)=r\frac{d^k}{dx^k}f(x)$$

from calculus. Thus the given map is a linear transformation of the space because any linear combination of linear maps is also a linear map by Lemma 1.16. The appropriate conclusion is General = Particular + Homogeneous, where the associated homogeneous differential equation has a constant of 0.

Subsection Three.III.1: Representing Linear Maps with Matrices

Three.III.1.11 **(a)** $\begin{pmatrix}1\cdot2+3\cdot1+1\cdot0\\ 0\cdot2+(-1)\cdot1+2\cdot0\\ 1\cdot2+1\cdot1+0\cdot0\end{pmatrix}=\begin{pmatrix}5\\-1\\3\end{pmatrix}$ **(b)** Not defined. **(c)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$

Three.III.1.13 Matrix-vector multiplication gives rise to a linear system.

$$\begin{aligned}2x+y+z&=8\\ y+3z&=4\\ x-y+2z&=4\end{aligned}$$

Gaussian reduction shows that $z=1$, $y=1$, and $x=3$.

Three.III.1.14 Here are two ways to get the answer.

First, obviously $1-3x+2x^2 = 1\cdot 1 - 3\cdot x + 2\cdot x^2$, and so we can apply the general property of preservation of combinations to get $h(1-3x+2x^2) = h(1\cdot 1-3\cdot x+2\cdot x^2) = 1\cdot h(1)-3\cdot h(x)+2\cdot h(x^2) = 1\cdot(1+x)-3\cdot(1+2x)+2\cdot(x-x^3) = -2-3x-2x^3$.

The other way uses the computation scheme developed in this subsection. Because we know where these elements of the space go, we consider this basis $B=\langle 1,x,x^2\rangle$ for the domain. Arbitrarily, we can take $D=\langle 1,x,x^2,x^3\rangle$ as a basis for the codomain. With those choices, we have that

$$\mathrm{Rep}_{B,D}(h)=\begin{pmatrix}1&1&0\\1&2&1\\0&0&0\\0&0&-1\end{pmatrix}_{B,D}$$

and, as

$$\mathrm{Rep}_B(1-3x+2x^2)=\begin{pmatrix}1\\-3\\2\end{pmatrix}_B$$

the matrix-vector multiplication calculation gives this.

$$\mathrm{Rep}_D(h(1-3x+2x^2))=\begin{pmatrix}1&1&0\\1&2&1\\0&0&0\\0&0&-1\end{pmatrix}_{B,D}\begin{pmatrix}1\\-3\\2\end{pmatrix}_B=\begin{pmatrix}-2\\-3\\0\\-2\end{pmatrix}_D$$

Thus, $h(1-3x+2x^2) = -2\cdot 1-3\cdot x+0\cdot x^2-2\cdot x^3 = -2-3x-2x^3$, as above.

Three.III.1.15 Again, as recalled in the subsection, with respect to $\mathcal{E}_i$, a column vector represents itself.

(a) To represent h with respect to $\mathcal{E}_2,\mathcal{E}_3$ we take the images of the basis vectors from the domain, and represent them with respect to the basis for the codomain.

$$\mathrm{Rep}_{\mathcal{E}_3}(\,h(\vec{e}_1)\,)=\mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}2\\2\\0\end{pmatrix})=\begin{pmatrix}2\\2\\0\end{pmatrix}\qquad \mathrm{Rep}_{\mathcal{E}_3}(\,h(\vec{e}_2)\,)=\mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\1\\-1\end{pmatrix})=\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

These are adjoined to make the matrix.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_3}(h)=\begin{pmatrix}2&0\\2&1\\0&-1\end{pmatrix}$$

(b) For any $\vec{v}$ in the domain $\mathbb{R}^2$,

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{v})=\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix}v_1\\v_2\end{pmatrix})=\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

and so

$$\mathrm{Rep}_{\mathcal{E}_3}(\,h(\vec{v})\,)=\begin{pmatrix}2&0\\2&1\\0&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}2v_1\\2v_1+v_2\\-v_2\end{pmatrix}$$

is the desired representation.

Three.III.1.16 **(a)** We must first find the image of each vector from the domain's basis, and then represent that image with respect to the codomain's basis.

$$\mathrm{Rep}_B(\frac{d\,1}{dx})=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\frac{d\,x}{dx})=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\frac{d\,x^2}{dx})=\begin{pmatrix}0\\2\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\frac{d\,x^3}{dx})=\begin{pmatrix}0\\0\\3\\0\end{pmatrix}$$

Those representations are then adjoined to make the matrix representing the map.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx})=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}$$

(b) Proceeding as in the prior item, we represent the images of the domain's basis vectors

$$\mathrm{Rep}_B(\frac{d\,1}{dx})=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\frac{d\,x}{dx})=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\frac{d\,x^2}{dx})=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\frac{d\,x^3}{dx})=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

and adjoin to make the matrix.

$$\text{Rep}_{B,D}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Three.III.1.17 For each, we must find the image of each of the domain's basis vectors, represent each image with respect to the codomain's basis, and then adjoin those representations to get the matrix.

(a) The basis vectors from the domain have these images

$$1 \mapsto 0 \quad x \mapsto 1 \quad x^2 \mapsto 2x \quad \ldots$$

and these images are represented with respect to the codomain's basis in this way.

$$\text{Rep}_B(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \\ \end{pmatrix} \quad \text{Rep}_B(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \\ \end{pmatrix} \quad \text{Rep}_B(2x) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ \\ \end{pmatrix} \quad \ldots \quad \text{Rep}_B(nx^{n-1}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n \\ 0 \end{pmatrix}$$

The matrix

$$\text{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

has $n+1$ rows and columns.

(b) Once the images under this map of the domain's basis vectors are determined

$$1 \mapsto x \quad x \mapsto x^2/2 \quad x^2 \mapsto x^3/3 \quad \ldots$$

then they can be represented with respect to the codomain's basis

$$\text{Rep}_{B_{n+1}}(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \\ \end{pmatrix} \quad \text{Rep}_{B_{n+1}}(x^2/2) = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ \vdots \\ \\ \end{pmatrix} \quad \ldots \quad \text{Rep}_{B_{n+1}}(x^{n+1}/(n+1)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1/(n+1) \end{pmatrix}$$

and put together to make the matrix.

$$\text{Rep}_{B_n,B_{n+1}}(\int) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1/2 & \ldots & 0 & 0 \\ & \vdots & & & \\ 0 & 0 & \ldots & 0 & 1/(n+1) \end{pmatrix}$$

(c) The images of the basis vectors of the domain are

$$1 \mapsto 1 \quad x \mapsto 1/2 \quad x^2 \mapsto 1/3 \quad \ldots$$

and they are represented with respect to the codomain's basis as

$$\text{Rep}_{\mathcal{E}_1}(1) = 1 \quad \text{Rep}_{\mathcal{E}_1}(1/2) = 1/2 \quad \ldots$$

so the matrix is

$$\text{Rep}_{B,\mathcal{E}_1}(\int) = \begin{pmatrix} 1 & 1/2 & \cdots & 1/n & 1/(n+1) \end{pmatrix}$$

(this is an $1\times(n+1)$ matrix).

(d) Here, the images of the domain's basis vectors are

$$1 \mapsto 1 \quad x \mapsto 3 \quad x^2 \mapsto 9 \quad \ldots$$

and they are represented in the codomain as

$$\text{Rep}_{\mathcal{E}_1}(1) = 1 \quad \text{Rep}_{\mathcal{E}_1}(3) = 3 \quad \text{Rep}_{\mathcal{E}_1}(9) = 9 \quad \ldots$$

and so the matrix is this.

$$\text{Rep}_{B,\mathcal{E}_1}(\int_0^1) = \begin{pmatrix} 1 & 3 & 9 & \cdots & 3^n \end{pmatrix}$$

(e) The images of the basis vectors from the domain are

$1 \mapsto 1 \quad x \mapsto x+1 = 1+x \quad x^2 \mapsto (x+1)^2 = 1+2x+x^2 \quad x^3 \mapsto (x+1)^3 = 1+3x+3x^2+x^3 \quad \ldots$

which are represented as

$$\mathrm{Rep}_B(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathrm{Rep}_B(1+x) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathrm{Rep}_B(1+2x+x^2) = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots$$

The resulting matrix

$$\mathrm{Rep}_{B,B}(\int) = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & 3 & \ldots & \binom{n}{2} \\ 0 & 0 & 1 & 3 & \ldots & \binom{n}{3} \\ & \vdots & & & & \\ 0 & 0 & 0 & & \ldots & 1 \end{pmatrix}$$

is *Pascal's triangle* (recall that $\binom{n}{r}$ is the number of ways to choose r things, without order and without repetition, from a set of size n).

Three.III.1.22 Call the map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$.

(a) To represent this map with respect to the standard bases, we must find, and then represent, the images of the vectors $\vec{e}_1$ and $\vec{e}_2$ from the domain's basis. The image of $\vec{e}_1$ is given.

One way to find the image of $\vec{e}_2$ is by eye—we can see this.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

A more systemmatic way to find the image of $\vec{e}_2$ is to use the given information to represent the transformation, and then use that representation to determine the image. Taking this for a basis,

$$C = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$$

the given information says this.

$$\mathrm{Rep}_{C,\mathcal{E}_2}(t) \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

As

$$\mathrm{Rep}_C(\vec{e}_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}_C$$

we have that

$$\mathrm{Rep}_{\mathcal{E}_2}(t(\vec{e}_2)) = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}_{C,\mathcal{E}_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_C = \begin{pmatrix} 3 \\ 0 \end{pmatrix}_{\mathcal{E}_2}$$

and consequently we know that $t(\vec{e}_2) = 3 \cdot \vec{e}_1$ (since, with respect to the standard basis, this vector is represented by itself). Therefore, this is the representation of t with respect to $\mathcal{E}_2, \mathcal{E}_2$.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) = \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}_{\mathcal{E}_2,\mathcal{E}_2}$$

(b) To use the matrix developed in the prior item, note that

$$\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix} 0 \\ 5 \end{pmatrix}) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}_{\mathcal{E}_2}$$

and so we have this is the representation, with respect to the codomain's basis, of the image of the given vector.

$$\mathrm{Rep}_{\mathcal{E}_2}(t(\begin{pmatrix} 0 \\ 5 \end{pmatrix})) = \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}_{\mathcal{E}_2,\mathcal{E}_2} \begin{pmatrix} 0 \\ 5 \end{pmatrix}_{\mathcal{E}_2} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}_{\mathcal{E}_2}$$

Because the codomain's basis is the standard one, and so vectors in the codomain are represented by themselves, we have this.

$$t(\begin{pmatrix} 0 \\ 5 \end{pmatrix}) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

(c) We first find the image of each member of B, and then represent those images with respect to D. For the first step, we can use the matrix developed earlier.

$$\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix}1\\-1\end{pmatrix})=\begin{pmatrix}-1&3\\0&0\end{pmatrix}_{\mathcal{E}_2,\mathcal{E}_2}\begin{pmatrix}1\\-1\end{pmatrix}_{\mathcal{E}_2}=\begin{pmatrix}-4\\0\end{pmatrix}_{\mathcal{E}_2}\quad\text{so}\quad t(\begin{pmatrix}1\\-1\end{pmatrix})=\begin{pmatrix}-4\\0\end{pmatrix}$$

Actually, for the second member of B there is no need to apply the matrix because the problem statement gives its image.

$$t(\begin{pmatrix}1\\1\end{pmatrix})=\begin{pmatrix}2\\0\end{pmatrix}$$

Now representing those images with respect to D is routine.

$$\mathrm{Rep}_D(\begin{pmatrix}-4\\0\end{pmatrix})=\begin{pmatrix}-1\\2\end{pmatrix}_D\quad\text{and}\quad\mathrm{Rep}_D(\begin{pmatrix}2\\0\end{pmatrix})=\begin{pmatrix}1/2\\-1\end{pmatrix}_D$$

Thus, the matrix is this.

$$\mathrm{Rep}_{B,D}(t)=\begin{pmatrix}-1&1/2\\2&-1\end{pmatrix}_{B,D}$$

(d) We know the images of the members of the domain's basis from the prior item.

$$t(\begin{pmatrix}1\\-1\end{pmatrix})=\begin{pmatrix}-4\\0\end{pmatrix}\qquad t(\begin{pmatrix}1\\1\end{pmatrix})=\begin{pmatrix}2\\0\end{pmatrix}$$

We can compute the representation of those images with respect to the codomain's basis.

$$\mathrm{Rep}_B(\begin{pmatrix}-4\\0\end{pmatrix})=\begin{pmatrix}-2\\-2\end{pmatrix}_B\quad\text{and}\quad\mathrm{Rep}_B(\begin{pmatrix}2\\0\end{pmatrix})=\begin{pmatrix}1\\1\end{pmatrix}_B$$

Thus this is the matrix.

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}-2&1\\-2&1\end{pmatrix}_{B,B}$$

Three.III.1.25 **(a)** The images of the basis vectors for the domain are $\cos x\overset{d/dx}{\longmapsto}-\sin x$ and $\sin x\overset{d/dx}{\longmapsto}\cos x$. Representing those with respect to the codomain's basis (again, B) and adjoining the representations gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx})=\begin{pmatrix}0&1\\-1&0\end{pmatrix}_{B,B}$$

(b) The images of the vectors in the domain's basis are $e^x\overset{d/dx}{\longmapsto}e^x$ and $e^{2x}\overset{d/dx}{\longmapsto}2e^{2x}$. Representing with respect to the codomain's basis and adjoining gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx})=\begin{pmatrix}1&0\\0&2\end{pmatrix}_{B,B}$$

(c) The images of the members of the domain's basis are $1\overset{d/dx}{\longmapsto}0$, $x\overset{d/dx}{\longmapsto}1$, $e^x\overset{d/dx}{\longmapsto}e^x$, and $xe^x\overset{d/dx}{\longmapsto}e^x+xe^x$. Representing these images with respect to B and adjoining gives this matrix.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx})=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}_{B,B}$$

Three.III.1.27 Yes, for two reasons.

First, the two maps h and $\hat{h}$ need not have the same domain and codomain. For instance,

$$\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

represents a map $h\colon\mathbb{R}^2\to\mathbb{R}^2$ with respect to the standard bases that sends

$$\begin{pmatrix}1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\3\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0\\1\end{pmatrix}\mapsto\begin{pmatrix}2\\4\end{pmatrix}$$

and also represents a $\hat{h}\colon\mathcal{P}_1\to\mathbb{R}^2$ with respect to $\langle 1,x\rangle$ and $\mathcal{E}_2$ that acts in this way.

$$1\mapsto\begin{pmatrix}1\\3\end{pmatrix}\quad\text{and}\quad x\mapsto\begin{pmatrix}2\\4\end{pmatrix}$$

The second reason is that, even if the domain and codomain of h and $\hat{h}$ coincide, different bases produce different maps. An example is the 2×2 identity matrix

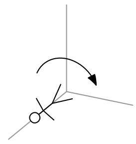
$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

which represents the identity map on $\mathbb{R}^2$ with respect to $\mathcal{E}_2,\mathcal{E}_2$. However, with respect to $\mathcal{E}_2$ for the domain but the basis $D=\langle\vec{e}_2,\vec{e}_1\rangle$ for the codomain, the same matrix I represents the map that swaps the first and second components

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}y\\x\end{pmatrix}$$

(that is, reflection about the line $y=x$).

Three.III.1.29 **(a)** The picture is this.



The images of the vectors from the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\0\\0\end{pmatrix}\qquad\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\\cos\theta\\-\sin\theta\end{pmatrix}\qquad\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\\sin\theta\\\cos\theta\end{pmatrix}$$

are represented with respect to the codomain's basis (again, $\mathcal{E}_3$) by themselves, so adjoining the representations to make the matrix gives this.

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(r_\theta)=\begin{pmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{pmatrix}$$

(b) The picture is similar to the one in the prior answer. The images of the vectors from the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}\cos\theta\\0\\\sin\theta\end{pmatrix}\qquad\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}0\\1\\0\end{pmatrix}\qquad\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}-\sin\theta\\0\\\cos\theta\end{pmatrix}$$

are represented with respect to the codomain's basis $\mathcal{E}_3$ by themselves, so this is the matrix.

$$\begin{pmatrix}\cos\theta&0&-\sin\theta\\0&1&0\\\sin\theta&0&\cos\theta\end{pmatrix}$$

(c) To a person standing up, with the vertical z-axis, a rotation of the xy-plane that is clockwise proceeds from the positive y-axis to the positive x-axis. That is, it rotates opposite to the direction in Example 1.8. The images of the vectors from the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\mapsto\begin{pmatrix}\cos\theta\\-\sin\theta\\0\end{pmatrix}\qquad\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}\sin\theta\\\cos\theta\\0\end{pmatrix}\qquad\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}$$

are represented with respect to $\mathcal{E}_3$ by themselves, so the matrix is this.

$$\begin{pmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}$$

(d) $\begin{pmatrix}\cos\theta&\sin\theta&0&0\\-\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

Subsection Three.III.2: Any Matrix Represents a Linear Map

Three.III.2.9 **(a)** Yes; we are asking if there are scalars c_1 and c_2 such that

$$c_1\begin{pmatrix}2\\2\end{pmatrix}+c_2\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}1\\-3\end{pmatrix}$$

which gives rise to a linear system

$$\begin{aligned}2c_1+c_2&=1\\2c_1+5c_2&=-3\end{aligned}\;\xrightarrow{-\rho_1+\rho_2}\;\begin{aligned}2c_1+c_2&=1\\4c_2&=-4\end{aligned}$$

and Gauss' method produces $c_2=-1$ and $c_1=1$. That is, there is indeed such a pair of scalars and so the vector is indeed in the column space of the matrix.

(b) No; we are asking if there are scalars c_1 and c_2 such that

$$c_1\begin{pmatrix}4\\2\end{pmatrix}+c_2\begin{pmatrix}-8\\-4\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

and one way to proceed is to consider the resulting linear system

$$\begin{aligned}4c_1-8c_2&=0\\2c_1-4c_2&=1\end{aligned}$$

that is easily seen to have no solution. Another way to proceed is to note that any linear combination of the columns on the left has a second component half as big as its first component, but the vector on the right does not meet that criterion.

(c) Yes; we can simply observe that the vector is the first column minus the second. Or, failing that, setting up the relationship among the columns

$$c_1\begin{pmatrix}1\\1\\-1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\\-1\end{pmatrix}+c_3\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$$

and considering the resulting linear system

$$\begin{array}{rl}c_1-c_2+c_3&=2\\c_1+c_2-c_3&=0\\-c_1-c_2+c_3&=0\end{array}\;\xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2}\;\begin{array}{rl}c_1-c_2+c_3&=2\\2c_2-2c_3&=-2\\-2c_2+2c_3&=2\end{array}\;\xrightarrow{\rho_2+\rho_3}\;\begin{array}{rl}c_1-c_2+c_3&=2\\2c_2-2c_3&=-2\\0&=0\end{array}$$

gives the additional information (beyond that there is at least one solution) that there are infinitely many solutions. Paramatizing gives $c_2=-1+c_3$ and $c_1=1$, and so taking c_3 to be zero gives a particular solution of $c_1=1$, $c_2=-1$, and $c_3=0$ (which is, of course, the observation made at the start).

Three.III.2.10 As described in the subsection, with respect to the standard bases, representations are transparent, and so, for instance, the first matrix describes this map.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{E}_3}\mapsto\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{E}_2}=\begin{pmatrix}1\\0\end{pmatrix}\qquad\begin{pmatrix}0\\1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\1\end{pmatrix}\qquad\begin{pmatrix}0\\0\\1\end{pmatrix}\mapsto\begin{pmatrix}3\\4\end{pmatrix}$$

So, for this first one, we are asking whether thare are scalars such that

$$c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}+c_3\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

that is, whether the vector is in the column space of the matrix.

(a) Yes. We can get this conclusion by setting up the resulting linear system and applying Gauss' method, as usual. Another way to get it is to note by inspection of the equation of columns that taking $c_3=3/4$, and $c_1=-5/4$, and $c_2=0$ will do. Still a third way to get this conclusion is to note that the rank of the matrix is two, which equals the dimension of the codomain, and so the map is onto — the range is all of $\mathbb{R}^2$ and in particular includes the given vector.

(b) No; note that all of the columns in the matrix have a second component that is twice the first, while the vector does not. Alternatively, the column space of the matrix is

$$\{c_1\begin{pmatrix}2\\4\end{pmatrix}+c_2\begin{pmatrix}0\\0\end{pmatrix}+c_3\begin{pmatrix}3\\6\end{pmatrix}\mid c_1,c_2,c_3\in\mathbb{R}\}=\{c\begin{pmatrix}1\\2\end{pmatrix}\mid c\in\mathbb{R}\}$$

(which is the fact already noted, but was arrived at by calculation rather than inspiration), and the given vector is not in this set.

Three.III.2.11 **(a)** The first member of the basis

$$\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}_B$$

is mapped to

$$\begin{pmatrix}1/2\\-1/2\end{pmatrix}_D$$

which is this member of the codomain.

$$\frac{1}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

(b) The second member of the basis is mapped

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}_B\mapsto\begin{pmatrix}(1/2\\1/2\end{pmatrix}_D$$

to this member of the codomain.

$$\frac{1}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

(c) Because the map that the matrix represents is the identity map on the basis, it must be the identity on all members of the domain. We can come to the same conclusion in another way by considering

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}_B$$

which is mapped to

$$\begin{pmatrix}(x+y)/2\\(x-y)/2\end{pmatrix}_D$$

which represents this member of $\mathbb{R}^2$.

$$\frac{x+y}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}+\frac{x-y}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

Three.III.2.13 This is the action of the map (writing B for the basis of $\mathcal{P}_2$).

$$\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{E}_3}\mapsto\begin{pmatrix}1\\0\\1\end{pmatrix}_B=1+x\qquad\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}_{\mathcal{E}_3}\mapsto\begin{pmatrix}3\\1\\0\end{pmatrix}_B=4+x^2\qquad\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}_{\mathcal{E}_3}\mapsto\begin{pmatrix}0\\0\\1\end{pmatrix}_B=x$$

We can thus decide if $1+2x$ is in the range of the map by looking for scalars c_1, c_2, and c_3 such that

$$c_1\cdot(1)+c_2\cdot(1+x^2)+c_3\cdot(x)=1+2x$$

and obviously $c_1=1$, $c_2=0$, and $c_3=1$ suffice. Thus it is in the range.

Three.III.2.15 No, the rangespaces may differ. Example 2.2 shows this.

Three.III.2.16 Recall that the represention map

$$V\overset{\mathrm{Rep}_B}{\longmapsto}\mathbb{R}^n$$

is an isomorphism. Thus, its inverse map $\mathrm{Rep}_B^{-1}\colon\mathbb{R}^n\to V$ is also an isomorphism. The desired transformation of $\mathbb{R}^n$ is then this composition.

$$\mathbb{R}^n\overset{\mathrm{Rep}_B^{-1}}{\longmapsto}V\overset{\mathrm{Rep}_D}{\longmapsto}\mathbb{R}^n$$

Because a composition of isomorphisms is also an isomorphism, this map $\mathrm{Rep}_D\circ\mathrm{Rep}_B^{-1}$ is an isomorphism.

Three.III.2.18 This is immediate from Corollary 2.6.

Subsection Three.IV.1: Sums and Scalar Products

Three.IV.1.7 **(a)** $\begin{pmatrix}7&0&6\\9&1&6\end{pmatrix}$ **(b)** $\begin{pmatrix}12&-6&-6\\6&12&18\end{pmatrix}$ **(c)** $\begin{pmatrix}4&2\\0&6\end{pmatrix}$ **(d)** $\begin{pmatrix}-1&28\\2&1\end{pmatrix}$

(e) Not defined.

Three.IV.1.9 First, each of these properties is easy to check in an entry-by-entry way. For example, writing

$$G=\begin{pmatrix}g_{1,1}&\ldots&g_{1,n}\\\vdots&&\vdots\\g_{m,1}&\ldots&g_{m,n}\end{pmatrix}\qquad H=\begin{pmatrix}h_{1,1}&\ldots&h_{1,n}\\\vdots&&\vdots\\h_{m,1}&\ldots&h_{m,n}\end{pmatrix}$$

then, by definition we have

$$G+H=\begin{pmatrix}g_{1,1}+h_{1,1}&\ldots&g_{1,n}+h_{1,n}\\\vdots&&\vdots\\g_{m,1}+h_{m,1}&\ldots&g_{m,n}+h_{m,n}\end{pmatrix}\qquad H+G=\begin{pmatrix}h_{1,1}+g_{1,1}&\ldots&h_{1,n}+g_{1,n}\\\vdots&&\vdots\\h_{m,1}+g_{m,1}&\ldots&h_{m,n}+g_{m,n}\end{pmatrix}$$

and the two are equal since their entries are equal $g_{i,j}+h_{i,j}=h_{i,j}+g_{i,j}$. That is, each of these is easy to check by using Definition 1.3 alone.

However, each property is also easy to understand in terms of the represented maps, by applying Theorem 1.5 as well as the definition.

(a) The two maps $g+h$ and $h+g$ are equal because $g(\vec{v})+h(\vec{v}) = h(\vec{v})+g(\vec{v})$, as addition is commutative in any vector space. Because the maps are the same, they must have the same representative.
(b) As with the prior answer, except that here we apply that vector space addition is associative.
(c) As before, except that here we note that $g(\vec{v}) + z(\vec{v}) = g(\vec{v}) + \vec{0} = g(\vec{v})$.
(d) Apply that $0 \cdot g(\vec{v}) = \vec{0} = z(\vec{v})$.
(e) Apply that $(r + s) \cdot g(\vec{v}) = r \cdot g(\vec{v}) + s \cdot g(\vec{v})$.
(f) Apply the prior two items with $r = 1$ and $s = -1$.
(g) Apply that $r \cdot (g(\vec{v}) + h(\vec{v})) = r \cdot g(\vec{v}) + r \cdot h(\vec{v})$.
(h) Apply that $(rs) \cdot g(\vec{v}) = r \cdot (s \cdot g(\vec{v}))$.

Three.IV.1.11 Fix bases B and D for V and W, and consider $\mathrm{Rep}_{B,D}\colon \mathcal{L}(V,W) \to \mathcal{M}_{m\times n}$ associating each linear map with the matrix representing that map $h \mapsto \mathrm{Rep}_{B,D}(h)$. From the prior section we know that (under fixed bases) the matrices correspond to linear maps, so the representation map is one-to-one and onto. That it preserves linear operations is Theorem 1.5.

Three.IV.1.12 Fix bases and represent the transformations with 2×2 matrices. The space of matrices $\mathcal{M}_{2\times 2}$ has dimension four, and hence the above six-element set is linearly dependent. By the prior exercise that extends to a dependence of maps. (The misleading part is only that there are six transformations, not five, so that we have more than we need to give the existence of the dependence.)

Three.IV.1.15 (a) For $H + H^{\mathrm{trans}}$, the i,j entry is $h_{i,j} + h_{j,i}$ and the j,i entry of is $h_{j,i} + h_{i,j}$. The two are equal and thus $H + H^{\mathrm{trans}}$ is symmetric.

Every symmetric matrix does have that form, since it can be written $H = (1/2) \cdot (H + H^{\mathrm{trans}})$.

(b) The set of symmetric matrices is nonempty as it contains the zero matrix. Clearly a scalar multiple of a symmetric matrix is symmetric. A sum $H+G$ of two symmetric matrices is symmetric because $h_{i,j} + g_{i,j} = h_{j,i} + g_{j,i}$ (since $h_{i,j} = h_{j,i}$ and $g_{i,j} = g_{j,i}$). Thus the subset is nonempty and closed under the inherited operations, and so it is a subspace.

Three.IV.1.16 (a) Scalar multiplication leaves the rank of a matrix unchanged except that multiplication by zero leaves the matrix with rank zero. (This follows from the first theorem of the book, that multiplying a row by a nonzero scalar doesn't change the solution set of the associated linear system.)

(b) A sum of rank n matrices can have rank less than n. For instance, for any matrix H, the sum $H + (-1) \cdot H$ has rank zero.

A sum of rank n matrices can have rank greater than n. Here are rank one matrices that sum to a rank two matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Subsection Three.IV.2: Matrix Multiplication

Three.IV.2.14 (a) $\begin{pmatrix} 0 & 15.5 \\ 0 & -19 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & -1 \\ 17 & -1 & -1 \end{pmatrix}$ (c) Not defined. (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Three.IV.2.15 (a) $\begin{pmatrix} 1 & -2 \\ 10 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ -36 & 34 \end{pmatrix}$ (c) $\begin{pmatrix} -18 & 17 \\ -24 & 16 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -18 & 17 \\ -24 & 16 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ -36 & 34 \end{pmatrix}$

Three.IV.2.17 (a) 2×1 (b) 1×1 (c) Not defined. (d) 2×2

Three.IV.2.18 We have

$$\begin{aligned} h_{1,1} \cdot (g_{1,1}y_1 + g_{1,2}y_2) + h_{1,2} \cdot (g_{2,1}y_1 + g_{2,2}y_2) + h_{1,3} \cdot (g_{3,1}y_1 + g_{3,2}y_2) &= d_1 \\ h_{2,1} \cdot (g_{1,1}y_1 + g_{1,2}y_2) + h_{2,2} \cdot (g_{2,1}y_1 + g_{2,2}y_2) + h_{2,3} \cdot (g_{3,1}y_1 + g_{3,2}y_2) &= d_2 \end{aligned}$$

which, after expanding and regrouping about the y's yields this.

$$\begin{aligned} (h_{1,1}g_{1,1} + h_{1,2}g_{2,1} + h_{1,3}g_{3,1})y_1 + (h_{1,1}g_{1,2} + h_{1,2}g_{2,2} + h_{1,3}g_{3,2})y_2 &= d_1 \\ (h_{2,1}g_{1,1} + h_{2,2}g_{2,1} + h_{2,3}g_{3,1})y_1 + (h_{2,1}g_{1,2} + h_{2,2}g_{2,2} + h_{2,3}g_{3,2})y_2 &= d_2 \end{aligned}$$

The starting system, and the system used for the substitutions, can be expressed in matrix language.

$$\begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = H \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \qquad \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \\ g_{3,1} & g_{3,2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = G \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

With this, the substitution is $\vec{d} = H\vec{x} = H(G\vec{y}) = (HG)\vec{y}$.

Three.IV.2.20 The action of d/dx on B is $1 \mapsto 0$, $x \mapsto 1$, $x^2 \mapsto 2x$, ... and so this is its $(n+1)\times(n+1)$ matrix representation.

$$\mathrm{Rep}_{B,B}(\frac{d}{dx}) = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & n \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

The product of this matrix with itself is defined because the matrix is square.

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & n \\ 0 & 0 & 0 & & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 2 & 0 & & 0 \\ 0 & 0 & 0 & 6 & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & & & n(n-1) \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{pmatrix}$$

The map so represented is the composition

$$p \xmapsto{\frac{d}{dx}} \frac{d\,p}{dx} \xmapsto{\frac{d}{dx}} \frac{d^2\,p}{dx^2}$$

which is the second derivative operation.

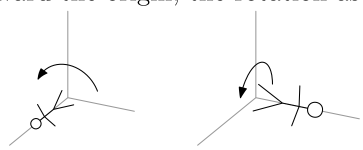
Three.IV.2.23 Each follows easily from the associated map fact. For instance, p applications of the transformation h, following q applications, is simply $p+q$ applications.

Three.IV.2.24 Although these can be done by going through the indices, they are best understood in terms of the represented maps. That is, fix spaces and bases so that the matrices represent linear maps f, g, h.

(a) Yes; we have both $r \cdot (g \circ h)\,(\vec{v}) = r \cdot g(\,h(\vec{v})\,) = (r \cdot g) \circ h\,(\vec{v})$ and $g \circ (r \cdot h)\,(\vec{v}) = g(\,r \cdot h(\vec{v})\,) = r \cdot g(h(\vec{v})) = r \cdot (g \circ h)\,(\vec{v})$ (the second equality holds because of the linearity of g).

(b) Both answers are yes. First, $f\circ(rg+sh)$ and $r\cdot(f\circ g)+s\cdot(f\circ h)$ both send $\vec{v}$ to $r\cdot f(g(\vec{v}))+s\cdot f(h(\vec{v}))$; the calculation is as in the prior item (using the linearity of f for the first one). For the other, $(rf+sg)\circ h$ and $r\cdot(f\circ h)+s\cdot(g\circ h)$ both send $\vec{v}$ to $r\cdot f(h(\vec{v}))+s\cdot g(h(\vec{v}))$.

Three.IV.2.26 Consider $r_x, r_y\colon \mathbb{R}^3 \to \mathbb{R}^3$ rotating all vectors $\pi/2$ radians counterclockwise about the x and y axes (counterclockwise in the sense that a person whose head is at $\vec{e}_1$ or $\vec{e}_2$ and whose feet are at the origin sees, when looking toward the origin, the rotation as counterclockwise).



Rotating r_x first and then r_y is different than rotating r_y first and then r_x. In particular, $r_x(\vec{e}_3) = -\vec{e}_2$ so $r_y \circ r_x(\vec{e}_3) = -\vec{e}_2$, while $r_y(\vec{e}_3) = \vec{e}_1$ so $r_x \circ r_y(\vec{e}_3) = \vec{e}_1$, and hence the maps do not commute.

Three.IV.2.30 **(a)** Either of these.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\pi_x} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \xmapsto{\pi_y} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\pi_y} \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \xmapsto{\pi_x} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) The composition is the fifth derivative map d^5/dx^5 on the space of fourth-degree polynomials.

(c) With respect to the natural bases,

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\pi_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\pi_y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and their product (in either order) is the zero matrix.

(d) Where $B = \langle 1, x, x^2, x^3, x^4 \rangle$,

$$\mathrm{Rep}_{B,B}(\frac{d^2}{dx^2}) = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Rep}_{B,B}(\frac{d^3}{dx^3}) = \begin{pmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and their product (in either order) is the zero matrix.

Three.IV.2.32 Because the identity map acts on the basis B as $\vec{\beta}_1 \mapsto \vec{\beta}_1, \ldots, \vec{\beta}_n \mapsto \vec{\beta}_n$, the representation is this.

$$\begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$

The second part of the question is obvious from Theorem 2.6.

Subsection Three.IV.3: Mechanics of Matrix Multiplication

Three.IV.3.23 **(a)** The second matrix has its first row multiplied by 3 and its second row multiplied by 0.

$$\begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix}$$

(b) The second matrix has its first row multiplied by 4 and its second row multiplied by 2.

$$\begin{pmatrix} 4 & 8 \\ 6 & 8 \end{pmatrix}$$

(c) The second matrix undergoes the pivot operation of replacing the second row with -2 times the first row added to the second.

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

(d) The first matrix undergoes the column operation of: the second column is replaced by -1 times the first column plus the second.

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

(e) The first matrix has its columns swapped.

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

Three.IV.3.24 **(a)** The incidence matrix is this (e.g, the first row shows that there is only one connection including Burlington, the road to Winooski).

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) Because these are two-way roads, any road connecting city i to city j gives a connection between city j and city i.

(c) The square of the incidence matrix tells how cities are connected by trips involving two roads.

Three.IV.3.25 The pay due each person appears in the matrix product of the two arrays.

Three.IV.3.27 The set of diagonal matrices is nonempty as the zero matrix is diagonal. Clearly it is closed under scalar multiples and sums. Therefore it is a subspace. The dimension is n; here is a basis.

$$\{\begin{pmatrix} 1 & 0 & \ldots & \\ 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & \ldots & \\ 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}\}$$

Three.IV.3.31 A permutation matrix has a single one in each row and column, and all its other entries are zeroes. Fix such a matrix. Suppose that the i-th row has its one in its j-th column. Then no other row has its one in the j-th column; every other row has a zero in the j-th column. Thus the dot product of the i-th row and any other row is zero.

The i-th row of the product is made up of the dot products of the i-th row of the matrix and the columns of the transpose. By the last paragraph, all such dot products are zero except for the i-th one, which is one.

Three.IV.3.35 The i-th row of GH is made up of the dot products of the i-th row of G with the columns of H. The dot product of a zero row with a column is zero.

It works for columns if stated correctly: if H has a column of zeros then GH (if defined) has a column of zeros. The proof is easy.

Three.IV.3.38 *Chapter Five gives a less computational reason — the trace of a matrix is the second coefficient in its characteristic polynomial — but for now we can use indices.* We have

$$\begin{aligned}\operatorname{trace}(GH) &= (g_{1,1}h_{1,1} + g_{1,2}h_{2,1} + \cdots + g_{1,n}h_{n,1}) \\ &\quad + (g_{2,1}h_{1,2} + g_{2,2}h_{2,2} + \cdots + g_{2,n}h_{n,2}) \\ &\quad + \cdots + (g_{n,1}h_{1,n} + g_{n,2}h_{2,n} + \cdots + g_{n,n}h_{n,n})\end{aligned}$$

while

$$\begin{aligned}\operatorname{trace}(HG) &= (h_{1,1}g_{1,1} + h_{1,2}g_{2,1} + \cdots + h_{1,n}g_{n,1}) \\ &\quad + (h_{2,1}g_{1,2} + h_{2,2}g_{2,2} + \cdots + h_{2,n}g_{n,2}) \\ &\quad + \cdots + (h_{n,1}g_{1,n} + h_{n,2}g_{2,n} + \cdots + h_{n,n}g_{n,n})\end{aligned}$$

and the two are equal.

Three.IV.3.39 A matrix is upper triangular if and only if its i,j entry is zero whenever $i > j$. Thus, if G, H are upper triangular then $h_{i,j}$ and $g_{i,j}$ are zero when $i > j$. An entry in the product $p_{i,j} = g_{i,1}h_{1,j} + \cdots + g_{i,n}h_{n,j}$ is zero unless at least some of the terms are nonzero, that is, unless for at least some of the summands $g_{i,r}h_{r,j}$ both $i \leq r$ and $r \leq j$. Of course, if $i > j$ this cannot happen and so the product of two upper triangular matrices is upper triangular. (A similar argument works for lower triangular matrices.)

Three.IV.3.41 Matrices representing (say, with respect to $\mathcal{E}_2, \mathcal{E}_2 \subset \mathbb{R}^2$) the maps that send

$$\vec{\beta}_1 \stackrel{h}{\longmapsto} \vec{\beta}_1 \quad \vec{\beta}_2 \stackrel{h}{\longmapsto} \vec{0}$$

and

$$\vec{\beta}_1 \stackrel{g}{\longmapsto} \vec{\beta}_2 \quad \vec{\beta}_2 \stackrel{g}{\longmapsto} \vec{0}$$

will do.

Subsection Three.IV.4: Inverses

Three.IV.4.14 **(a)** Yes, it has an inverse: $ad - bc = 2 \cdot 1 - 1 \cdot (-1) \neq 0$. **(b)** Yes. **(c)** No.

Three.IV.4.15 **(a)** $\dfrac{1}{2 \cdot 1 - 1 \cdot (-1)} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \dfrac{1}{3} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix}$

(b) $\dfrac{1}{0 \cdot (-3) - 4 \cdot 1} \cdot \begin{pmatrix} -3 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 1 \\ 1/4 & 0 \end{pmatrix}$

(c) The prior question shows that no inverse exists.

Three.IV.4.16 **(a)** The reduction is routine.

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \xrightarrow[(1/2)\rho_2]{(1/3)\rho_1} \left(\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/2 \end{array}\right) \xrightarrow{-(1/3)\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & -1/6 \\ 0 & 1 & 0 & 1/2 \end{array}\right)$$

This answer agrees with the answer from the check.

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{-1} = \frac{1}{3 \cdot 2 - 0 \cdot 1} \cdot \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \frac{1}{6} \cdot \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$$

(b) This reduction is easy.

$$\left(\begin{array}{cc|cc} 2 & 1/2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-(3/2)\rho_1+\rho_2} \left(\begin{array}{cc|cc} 2 & 1/2 & 1 & 0 \\ 0 & 1/4 & -3/2 & 1 \end{array}\right)$$

$$\xrightarrow[4\rho_2]{(1/2)\rho_1} \left(\begin{array}{cc|cc} 1 & 1/4 & 1/2 & 0 \\ 0 & 1 & -6 & 4 \end{array}\right) \xrightarrow{-(1/4)\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -6 & 4 \end{array}\right)$$

The check agrees.

$$\frac{1}{2\cdot 1-3\cdot(1/2)}\cdot\begin{pmatrix} 1 & -1/2 \\ -3 & 2 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 & -1/2 \\ -3 & 2 \end{pmatrix}$$

(c) Trying the Gauss-Jordan reduction

$$\left(\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{(1/2)\rho_1+\rho_2} \left(\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{array}\right)$$

shows that the left side won't reduce to the identity, so no inverse exists. The check $ad-bc=2\cdot 2-(-4)\cdot(-1)=0$ agrees.

(d) This produces an inverse.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\rho_1+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[-\rho_3]{(1/2)\rho_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \xrightarrow[-3\rho_3+\rho_1]{-2\rho_3+\rho_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 4 & -3 & 3 \\ 0 & 1 & 0 & 2 & -3/2 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3/2 & 1 \\ 0 & 1 & 0 & 2 & -3/2 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

(e) This is one way to do the reduction.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\rho_3\leftrightarrow\rho_1} \left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 2 & 3 & -2 & 0 & 0 & 1 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 7 & 1 & 1/2 & 0 \end{array}\right) \xrightarrow[\substack{-(1/2)\rho_2 \\ (1/7)\rho_3}]{(1/2)\rho_1} \left(\begin{array}{ccc|ccc} 1 & 3/2 & -1 & 0 & 0 & 1/2 \\ 0 & 1 & -2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{array}\right)$$

$$\xrightarrow[\rho_3+\rho_1]{2\rho_3+\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 1/7 & 1/14 & 1/2 \\ 0 & 1 & 0 & 2/7 & -5/14 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{array}\right) \xrightarrow{-(3/2)\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/7 & 17/28 & 1/2 \\ 0 & 1 & 0 & 2/7 & -5/14 & 0 \\ 0 & 0 & 1 & 1/7 & 1/14 & 0 \end{array}\right)$$

(f) There is no inverse.

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ 4 & -2 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2} \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -9/2 & -1/2 & 1 & 0 \\ 0 & -6 & -9 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -9/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right)$$

As a check, note that the third column of the starting matrix is $3/2$ times the second, and so it is indeed singular and therefore has no inverse.

Three.IV.4.17 We can use Corollary 4.12.

$$\frac{1}{1\cdot 5-2\cdot 3}\cdot\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$

Three.IV.4.19 Yes: $T^k(T^{-1})^k=(TT\cdots T)\cdot(T^{-1}T^{-1}\cdots T^{-1})=T^{k-1}(TT^{-1})(T^{-1})^{k-1}=\cdots=I$.

Three.IV.4.26 The associativity of matrix multiplication gives on the one hand $H^{-1}(HG)=H^{-1}Z=Z$, and on the other that $H^{-1}(HG)=(H^{-1}H)G=IG=G$.

Three.IV.4.28 Checking that when $I-T$ is multiplied on both sides by that expression (assuming that T^4 is the zero matrix) then the result is the identity matrix is easy. The obvious generalization is that if T^n is the zero matrix then $(I-T)^{-1}=I+T+T^2+\cdots+T^{n-1}$; the check again is easy.

Three.IV.4.29 The powers of the matrix are formed by taking the powers of the diagonal entries. That is, D^2 is all zeros except for diagonal entries of ${d_{1,1}}^2$, ${d_{2,2}}^2$, etc. This suggests defining D^0 to be the identity matrix.

Three.IV.4.34 For the answer to the items making up the first half, see Exercise 30. For the proof in the second half, assume that A is a zero divisor so there is a nonzero matrix B with $AB = Z$ (or else $BA = Z$; this case is similar), If A is invertible then $A^{-1}(AB) = (A^{-1}A)B = IB = B$ but also $A^{-1}(AB) = A^{-1}Z = Z$, contradicting that B is nonzero.

Subsection Three.V.1: Changing Representations of Vectors

Three.V.1.6 For the matrix to change bases from D to $\mathcal{E}_2$ we need that $\text{Rep}_{\mathcal{E}_2}(\text{id}(\vec{\delta}_1)) = \text{Rep}_{\mathcal{E}_2}(\vec{\delta}_1)$ and that $\text{Rep}_{\mathcal{E}_2}(\text{id}(\vec{\delta}_2)) = \text{Rep}_{\mathcal{E}_2}(\vec{\delta}_2)$. Of course, the representation of a vector in $\mathbb{R}^2$ with respect to the standard basis is easy.

$$\text{Rep}_{\mathcal{E}_2}(\vec{\delta}_1) = \begin{pmatrix}2\\1\end{pmatrix} \qquad \text{Rep}_{\mathcal{E}_2}(\vec{\delta}_2) = \begin{pmatrix}-2\\4\end{pmatrix}$$

Concatenating those two together to make the columns of the change of basis matrix gives this.

$$\text{Rep}_{D,\mathcal{E}_2}(\text{id}) = \begin{pmatrix}2&-2\\1&4\end{pmatrix}$$

The change of basis matrix in the other direction can be gotten by calculating $\text{Rep}_D(\text{id}(\vec{e}_1)) = \text{Rep}_D(\vec{e}_1)$ and $\text{Rep}_D(\text{id}(\vec{e}_2)) = \text{Rep}_D(\vec{e}_2)$ (this job is routine) or it can be found by taking the inverse of the above matrix. Because of the formula for the inverse of a 2×2 matrix, this is easy.

$$\text{Rep}_{\mathcal{E}_2,D}(\text{id}) = \frac{1}{10}\cdot\begin{pmatrix}4&2\\-1&2\end{pmatrix} = \begin{pmatrix}4/10&2/10\\-1/10&2/10\end{pmatrix}$$

Three.V.1.7 In each case, the columns $\text{Rep}_D(\text{id}(\vec{\beta}_1)) = \text{Rep}_D(\vec{\beta}_1)$ and $\text{Rep}_D(\text{id}(\vec{\beta}_2)) = \text{Rep}_D(\vec{\beta}_2)$ are concatenated to make the change of basis matrix $\text{Rep}_{B,D}(\text{id})$.

(a) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ **(b)** $\begin{pmatrix}2&-1/2\\-1&1/2\end{pmatrix}$ **(c)** $\begin{pmatrix}1&1\\2&4\end{pmatrix}$ **(d)** $\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$

Three.V.1.9 The columns vector representations $\text{Rep}_D(\text{id}(\vec{\beta}_1)) = \text{Rep}_D(\vec{\beta}_1)$, and $\text{Rep}_D(\text{id}(\vec{\beta}_2)) = \text{Rep}_D(\vec{\beta}_2)$, and $\text{Rep}_D(\text{id}(\vec{\beta}_3)) = \text{Rep}_D(\vec{\beta}_3)$ make the change of basis matrix $\text{Rep}_{B,D}(\text{id})$.

(a) $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ **(b)** $\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}$ **(c)** $\begin{pmatrix}1&-1&1/2\\1&1&-1/2\\0&2&0\end{pmatrix}$

E.g., for the first column of the first matrix, $1 = 0\cdot x^2 + 1\cdot 1 + 0\cdot x$.

Three.V.1.10 A matrix changes bases if and only if it is nonsingular.

(a) This matrix is nonsingular and so changes bases. Finding to what basis $\mathcal{E}_2$ is changed means finding D such that

$$\text{Rep}_{\mathcal{E}_2,D}(\text{id}) = \begin{pmatrix}5&0\\0&4\end{pmatrix}$$

and by the definition of how a matrix represents a linear map, we have this.

$$\text{Rep}_D(\text{id}(\vec{e}_1)) = \text{Rep}_D(\vec{e}_1) = \begin{pmatrix}5\\0\end{pmatrix} \qquad \text{Rep}_D(\text{id}(\vec{e}_2)) = \text{Rep}_D(\vec{e}_2) = \begin{pmatrix}0\\4\end{pmatrix}$$

Where

$$D = \langle\begin{pmatrix}x_1\\y_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\end{pmatrix}\rangle$$

we can either solve the system

$$\begin{pmatrix}1\\0\end{pmatrix} = 5\begin{pmatrix}x_1\\y_1\end{pmatrix} + 0\begin{pmatrix}x_2\\y_1\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} = 0\begin{pmatrix}x_1\\y_1\end{pmatrix} + 4\begin{pmatrix}x_2\\y_1\end{pmatrix}$$

or else just spot the answer (thinking of the proof of Lemma 1.4).

$$D = \langle\begin{pmatrix}1/5\\0\end{pmatrix}, \begin{pmatrix}0\\1/4\end{pmatrix}\rangle$$

(b) Yes, this matrix is nonsingular and so changes bases. To calculate D, we proceed as above with

$$D = \langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle$$

to solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 3 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + 1 \begin{pmatrix} x_2 \\ y_1 \end{pmatrix}$$

and get this.

$$D = \langle \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle$$

(c) No, this matrix does not change bases because it is nonsingular.

(d) Yes, this matrix changes bases because it is nonsingular. The calculation of the changed-to basis is as above.

$$D = \langle \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \rangle$$

Three.V.1.14 The appropriately-sized identity matrix.

Three.V.1.17 Taking H as a change of basis matrix $H = \mathrm{Rep}_{B,\mathcal{E}_n}(\mathrm{id})$, its columns are

$$\begin{pmatrix} h_{1,i} \\ \vdots \\ h_{n,i} \end{pmatrix} = \mathrm{Rep}_{\mathcal{E}_n}(\mathrm{id}(\vec{\beta}_i)) = \mathrm{Rep}_{\mathcal{E}_n}(\vec{\beta}_i)$$

and, because representations with respect to the standard basis are transparent, we have this.

$$\begin{pmatrix} h_{1,i} \\ \vdots \\ h_{n,i} \end{pmatrix} = \vec{\beta}_i$$

That is, the basis is the one composed of the columns of H.

Three.V.1.18 **(a)** We can change the starting vector representation to the ending one through a sequence of row operations. The proof tells us what how the bases change. We start by swapping the first and second rows of the representation with respect to B to get a representation with resepect to a new basis B_1.

$$\mathrm{Rep}_{B_1}(1 - x + 3x^2 - x^3) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}_{B_1} \qquad B_1 = \langle 1 - x, 1 + x, x^2 + x^3, x^2 - x^3 \rangle$$

We next add -2 times the third row of the vector representation to the fourth row.

$$\mathrm{Rep}_{B_3}(1 - x + 3x^2 - x^3) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{B_2} \qquad B_2 = \langle 1 - x, 1 + x, 3x^2 - x^3, x^2 - x^3 \rangle$$

(The third element of B_2 is the third element of B_1 minus -2 times the fourth element of B_1.) Now we can finish by doubling the third row.

$$\mathrm{Rep}_{D}(1 - x + 3x^2 - x^3) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}_{D} \qquad D = \langle 1 - x, 1 + x, (3x^2 - x^3)/2, x^2 - x^3 \rangle$$

(b) Here are three different approaches to stating such a result. The first is the assertion: where V is a vector space with basis B and $\vec{v} \in V$ is nonzero, for any nonzero column vector $\vec{z}$ (whose number of components equals the dimension of V) there is a change of basis matrix M such that $M \cdot \mathrm{Rep}_B(\vec{v}) = \vec{z}$. The second possible statement: for any (n-dimensional) vector space V and any nonzero vector $\vec{v} \in V$, where $\vec{z}_1, \vec{z}_2 \in \mathbb{R}^n$ are nonzero, there are bases $B, D \subset V$ such that $\mathrm{Rep}_B(\vec{v}) = \vec{z}_1$ and $\mathrm{Rep}_D(\vec{v}) = \vec{z}_2$. The third is: for any nonzero $\vec{v}$ member of any vector space (of dimension n) and any nonzero column vector (with n components) there is a basis such that $\vec{v}$ is represented with respect to that basis by that column vector.

The first and second statements follow easily from the third. The first follows because the third statement gives a basis D such that $\mathrm{Rep}_D(\vec{v}) = \vec{z}$ and then $\mathrm{Rep}_{B,D}(\mathrm{id})$ is the desired M. The second follows from the third because it is just a doubled application of it.

A way to prove the third is as in the answer to the first part of this question. Here is a sketch. Represent $\vec{v}$ with respect to any basis B with a column vector $\vec{z}_1$. This column vector must have a nonzero component because $\vec{v}$ is a nonzero vector. Use that component in a sequence of row operations to convert $\vec{z}_1$ to $\vec{z}$. (This sketch could be filled out as an induction argument on the dimension of V.)

Three.V.1.20 A change of basis matrix is nonsingular and thus has rank equal to the number of its columns. Therefore its set of columns is a linearly independent subset of size n in $\mathbb{R}^n$ and it is thus a basis. The answer to the second half is also 'yes'; all implications in the prior sentence reverse (that is, all of the 'if ... then ...' parts of the prior sentence convert to 'if and only if' parts).

Three.V.1.21 In response to the first half of the question, there are infinitely many such matrices. One of them represents with respect to $\mathcal{E}_2$ the transformation of $\mathbb{R}^2$ with this action.

$$\begin{pmatrix}1\\0\end{pmatrix}\mapsto\begin{pmatrix}4\\0\end{pmatrix}\qquad\begin{pmatrix}0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\-1/3\end{pmatrix}$$

The problem of specifying two distinct input/output pairs is a bit trickier. The fact that matrices have a linear action precludes some possibilities.

(a) Yes, there is such a matrix. These conditions

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\qquad\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$$

can be solved

$$\begin{array}{rcrcl} a+3b & & & = & 1\\ & & c+3d & = & 1\\ 2a-\ b & & & = & -1\\ & & 2c-\ d & = & -1 \end{array}$$

to give this matrix.

$$\begin{pmatrix}-2/7&3/7\\-2/7&3/7\end{pmatrix}$$

(b) No, because

$$2\cdot\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}\quad\text{but}\quad 2\cdot\begin{pmatrix}1\\1\end{pmatrix}\neq\begin{pmatrix}-1\\-1\end{pmatrix}$$

no linear action can produce this effect.

(c) A sufficient condition is that $\{\vec{v}_1,\vec{v}_2\}$ be linearly independent, but that's not a necessary condition. A necessary and sufficient condition is that any linear dependences among the starting vectors appear also among the ending vectors. That is,

$$c_1\vec{v}_1+c_2\vec{v}_2=\vec{0}\quad\text{implies}\quad c_1\vec{w}_1+c_2\vec{w}_2=\vec{0}.$$

The proof of this condition is routine.

Subsection Three.V.2: Changing Map Representations

Three.V.2.10 **(a)** Yes, each has rank two.
(b) Yes, they have the same rank.
(c) No, they have different ranks.

Three.V.2.11 We need only decide what the rank of each is.

(a) $\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}$ **(b)** $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$

Three.V.2.13 Where H and $\hat{H}$ are $m{\times}n$, the matrix P is $m{\times}m$ while Q is $n{\times}n$.

Three.V.2.14 Any $n{\times}n$ matrix is nonsingular if and only if it has rank n, that is, by Theorem 2.6, if and only if it is matrix equivalent to the $n{\times}n$ matrix whose diagonal is all ones.

Three.V.2.15 If $PAQ=I$ then $QPAQ=Q$, so $QPA=I$, and so $QP=A^{-1}$.

Three.V.2.16 By the definition following Example 2.2, a matrix M is diagonalizable if it represents $M = \mathrm{Rep}_{B,D}(t)$ a transformation with the property that there is some basis $\hat{B}$ such that $\mathrm{Rep}_{\hat{B},\hat{B}}(t)$ is a diagonal matrix — the starting and ending bases must be equal. But Theorem 2.6 says only that there are $\hat{B}$ and $\hat{D}$ such that we can change to a representation $\mathrm{Rep}_{\hat{B},\hat{D}}(t)$ and get a diagonal matrix. We have no reason to suspect that we could pick the two $\hat{B}$ and $\hat{D}$ so that they are equal.

Three.V.2.19 For reflexivity, to show that any matrix is matrix equivalent to itself, take P and Q to be identity matrices. For symmetry, if $H_1 = PH_2Q$ then $H_2 = P^{-1}H_1Q^{-1}$ (inverses exist because P and Q are nonsingular). Finally, for transitivity, assume that $H_1 = P_2H_2Q_2$ and that $H_2 = P_3H_3Q_3$. Then substitution gives $H_1 = P_2(P_3H_3Q_3)Q_2 = (P_2P_3)H_3(Q_3Q_2)$. A product of nonsingular matrices is nonsingular (we've shown that the product of invertible matrices is invertible; in fact, we've shown how to calculate the inverse) and so H_1 is therefore matrix equivalent to H_3.

Three.V.2.20 By Theorem 2.6, a zero matrix is alone in its class because it is the only $m \times n$ of rank zero. No other matrix is alone in its class; any nonzero scalar product of a matrix has the same rank as that matrix.

Three.V.2.27 **(a)** The definition is suggested by the appropriate arrow diagram.

$$\begin{array}{ccc} V_{\text{w.r.t. } B_1} & \xrightarrow[T]{t} & V_{\text{w.r.t. } B_1} \\ \mathrm{id}\downarrow & & \mathrm{id}\downarrow \\ V_{\text{w.r.t. } B_2} & \xrightarrow[\hat{T}]{t} & V_{\text{w.r.t. } B_2} \end{array}$$

Call matrices $T, \hat{T}$ *similar* if there is a nonsingular matrix P such that $\hat{T} = P^{-1}TP$.

(b) Take P^{-1} to be P and take P to be Q.

(c) *This is as in Exercise 19.* Reflexivity is obvious: $T = I^{-1}TI$. Symmetry is also easy: $\hat{T} = P^{-1}TP$ implies that $T = P\hat{T}P^{-1}$ (multiply the first equation from the right by P^{-1} and from the left by P). For transitivity, assume that $T_1 = {P_2}^{-1}T_2P_2$ and that $T_2 = {P_3}^{-1}T_3P_3$. Then $T_1 = {P_2}^{-1}({P_3}^{-1}T_3P_3)P_2 = ({P_2}^{-1}{P_3}^{-1})T_3(P_3P_2)$ and we are finished on noting that P_3P_2 is an invertible matrix with inverse ${P_2}^{-1}{P_3}^{-1}$.

(d) Assume that $\hat{T} = P^{-1}TP$. For the squares: $\hat{T}^2 = (P^{-1}TP)(P^{-1}TP) = P^{-1}T(PP^{-1})TP = P^{-1}T^2P$. Higher powers follow by induction.

(e) These two are matrix equivalent but their squares are not matrix equivalent.

$$\begin{pmatrix}1&0\\0&0\end{pmatrix} \qquad \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

By the prior item, matrix similarity and matrix equivalence are thus different.

Subsection Three.VI.1: Orthogonal Projection Into a Line

Three.VI.1.7 Each is a straightforward application of the formula from Definition 1.1.

(a) $$\frac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-2\end{pmatrix}}{\begin{pmatrix}3\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\end{pmatrix}}\cdot\begin{pmatrix}3\\-2\end{pmatrix} = \frac{4}{13}\cdot\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}12/13\\-8/13\end{pmatrix}$$

(b) $$\frac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\0\end{pmatrix}}{\begin{pmatrix}3\\0\end{pmatrix}\cdot\begin{pmatrix}3\\0\end{pmatrix}}\cdot\begin{pmatrix}3\\0\end{pmatrix} = \frac{2}{3}\cdot\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

(c) $$\frac{\begin{pmatrix}1\\1\\4\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}}{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac{-1}{6}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}-1/6\\-1/3\\1/6\end{pmatrix}$$

(d) $$\frac{\begin{pmatrix}1\\1\\4\end{pmatrix}\cdot\begin{pmatrix}3\\3\\12\end{pmatrix}}{\begin{pmatrix}3\\3\\12\end{pmatrix}\cdot\begin{pmatrix}3\\3\\12\end{pmatrix}}\cdot\begin{pmatrix}3\\3\\12\end{pmatrix} = \frac{1}{3}\cdot\begin{pmatrix}3\\3\\12\end{pmatrix} = \begin{pmatrix}1\\1\\4\end{pmatrix}$$

Three.VI.1.8 **(a)** $\dfrac{\begin{pmatrix}2\\-1\\4\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\-3\end{pmatrix}}{\begin{pmatrix}-3\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\-3\end{pmatrix}}\cdot\begin{pmatrix}-3\\1\\-3\end{pmatrix}=\dfrac{-19}{19}\cdot\begin{pmatrix}-3\\1\\-3\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}$

(b) Writing the line as $\{c\cdot\begin{pmatrix}1\\3\end{pmatrix}\mid c\in\mathbb{R}\}$ gives this projection.

$$\frac{\begin{pmatrix}-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\3\end{pmatrix}}\cdot\begin{pmatrix}1\\3\end{pmatrix}=\frac{-4}{10}\cdot\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}-2/5\\-6/5\end{pmatrix}$$

Three.VI.1.10 **(a)** $\dfrac{\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}}{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}}\cdot\begin{pmatrix}3\\1\end{pmatrix}=\dfrac{1}{2}\cdot\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}3/2\\1/2\end{pmatrix}$ **(b)** $\dfrac{\begin{pmatrix}0\\4\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}}{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}}\cdot\begin{pmatrix}3\\1\end{pmatrix}=\dfrac{2}{5}\cdot\begin{pmatrix}3\\1\end{pmatrix}=$ $\begin{pmatrix}6/5\\2/5\end{pmatrix}$

In general the projection is this.

$$\frac{\begin{pmatrix}x_1\\x_2\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}}{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix}}\cdot\begin{pmatrix}3\\1\end{pmatrix}=\frac{3x_1+x_2}{10}\cdot\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}(9x_1+3x_2)/10\\(3x_1+x_2)/10\end{pmatrix}$$

The appropriate matrix is this.

$$\begin{pmatrix}9/10&3/10\\3/10&1/10\end{pmatrix}$$

Three.VI.1.15 The proof is simply a calculation.

$$\|\frac{\vec v\cdot\vec s}{\vec s\cdot\vec s}\cdot\vec s\,\|=|\frac{\vec v\cdot\vec s}{\vec s\cdot\vec s}|\cdot\|\vec s\,\|=\frac{|\vec v\cdot\vec s\,|}{\|\vec s\,\|^2}\cdot\|\vec s\,\|=\frac{|\vec v\cdot\vec s\,|}{\|\vec s\,\|}$$

Three.VI.1.18 The Cauchy-Schwartz inequality $|\vec v\cdot\vec s\,|\le\|\vec v\,\|\cdot\|\vec s\,\|$ gives that this fraction

$$\|\frac{\vec v\cdot\vec s}{\vec s\cdot\vec s}\cdot\vec s\,\|=|\frac{\vec v\cdot\vec s}{\vec s\cdot\vec s}|\cdot\|\vec s\,\|=\frac{|\vec v\cdot\vec s\,|}{\|\vec s\,\|^2}\cdot\|\vec s\,\|=\frac{|\vec v\cdot\vec s\,|}{\|\vec s\,\|}$$

when divided by $\|\vec v\,\|$ is less than or equal to one. That is, $\|\vec v\,\|$ is larger than or equal to the fraction.

Three.VI.1.19 Write $c\vec s$ for $\vec q$, and calculate: $(\vec v\cdot c\vec s/c\vec s\cdot c\vec s\,)\cdot c\vec s=(\vec v\cdot\vec s/\vec s\cdot\vec s\,)\cdot\vec s$.

Three.VI.1.20 **(a)** Fixing

$$\vec s=\begin{pmatrix}1\\1\end{pmatrix}$$

as the vector whose span is the line, the formula gives this action,

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\frac{\begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\frac{x+y}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}(x+y)/2\\(x+y)/2\end{pmatrix}$$

which is the effect of this matrix.

$$\begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix}$$

(b) Rotating the entire plane $\pi/4$ radians clockwise brings the $y=x$ line to lie on the x-axis. Now projecting and then rotating back has the desired effect.

Subsection Three.VI.2: Gram-Schmidt Orthogonalization

Three.VI.2.10 **(a)**

$$\vec{\kappa}_1=\begin{pmatrix}2\\2\\2\end{pmatrix}$$

$$\vec{\kappa}_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}-\mbox{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}1\\0\\-1\end{pmatrix})=\begin{pmatrix}1\\0\\-1\end{pmatrix}-\frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}{\begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}-\frac{0}{12}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\vec{\kappa}_3=\begin{pmatrix}0\\3\\1\end{pmatrix}-\mbox{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}0\\3\\1\end{pmatrix})-\mbox{proj}_{[\vec{\kappa}_2]}(\begin{pmatrix}0\\3\\1\end{pmatrix})=\begin{pmatrix}0\\3\\1\end{pmatrix}-\frac{\begin{pmatrix}0\\3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}{\begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}-\frac{\begin{pmatrix}0\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$=\begin{pmatrix}0\\3\\1\end{pmatrix}-\frac{8}{12}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}-\frac{-1}{2}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}-5/6\\5/3\\-5/6\end{pmatrix}$$

(b)

$$\vec{\kappa}_1=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\vec{\kappa}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}-\mbox{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}0\\1\\0\end{pmatrix})=\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{-1}{2}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$$

$$\vec{\kappa}_3=\begin{pmatrix}2\\3\\1\end{pmatrix}-\mbox{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}2\\3\\1\end{pmatrix})-\mbox{proj}_{[\vec{\kappa}_2]}(\begin{pmatrix}2\\3\\1\end{pmatrix})$$

$$=\begin{pmatrix}2\\3\\1\end{pmatrix}-\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}-\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}}{\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$$

$$=\begin{pmatrix}2\\3\\1\end{pmatrix}-\frac{-1}{2}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}-\frac{5/2}{1/2}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

The corresponding orthonormal bases for the two parts of this question are these.

$$\langle\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix},\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix},\begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}\rangle\qquad\langle\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix},\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\rangle$$

Three.VI.2.11 The given space can be paramatrized in this way.

$$\{\begin{pmatrix}x\\y\\z\end{pmatrix}\bigm| x=y-z\}=\{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot y+\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot z\bigm| y,z\in\mathbb{R}\}$$

So we take the basis

$$\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix} \rangle$$

apply the Gram-Schmidt process

$$\vec{\kappa}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\vec{\kappa}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \mathrm{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}-1\\0\\1\end{pmatrix}) = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \frac{-1}{2}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\1\end{pmatrix}$$

and then normalize.

$$\langle \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}, \begin{pmatrix}-1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix} \rangle$$

Three.VI.2.14 The process leaves the basis unchanged.

Three.VI.2.17 **(a)** The representation can be done by eye.

$$\begin{pmatrix}2\\3\end{pmatrix} = 3\cdot\begin{pmatrix}1\\1\end{pmatrix} + (-1)\cdot\begin{pmatrix}1\\0\end{pmatrix} \qquad \mathrm{Rep}_B(\vec{v}\,) = \begin{pmatrix}3\\-1\end{pmatrix}_B$$

The two projections are also easy.

$$\mathrm{proj}_{[\vec{\beta}_1]}(\begin{pmatrix}2\\3\end{pmatrix}) = \frac{\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \frac{5}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix} \qquad \mathrm{proj}_{[\vec{\beta}_2]}(\begin{pmatrix}2\\3\end{pmatrix}) = \frac{\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{2}{1}\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

(b) As above, the representation can be done by eye

$$\begin{pmatrix}2\\3\end{pmatrix} = (5/2)\cdot\begin{pmatrix}1\\1\end{pmatrix} + (-1/2)\cdot\begin{pmatrix}1\\-1\end{pmatrix}$$

and the two projections are easy.

$$\mathrm{proj}_{[\vec{\beta}_1]}(\begin{pmatrix}2\\3\end{pmatrix}) = \frac{\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \frac{5}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix} \qquad \mathrm{proj}_{[\vec{\beta}_2]}(\begin{pmatrix}2\\3\end{pmatrix}) = \frac{\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}}\cdot\begin{pmatrix}1\\-1\end{pmatrix} = \frac{-1}{2}\cdot\begin{pmatrix}1\\-1\end{pmatrix}$$

Note the recurrence of the $5/2$ and the $-1/2$.

(c) Represent $\vec{v}$ with respect to the basis

$$\mathrm{Rep}_K(\vec{v}\,) = \begin{pmatrix}r_1\\\vdots\\r_k\end{pmatrix}$$

so that $\vec{v} = r_1\vec{\kappa}_1 + \cdots + r_k\vec{\kappa}_k$. To determine r_i, take the dot product of both sides with $\vec{\kappa}_i$.

$$\vec{v}\cdot\vec{\kappa}_i = (r_1\vec{\kappa}_1 + \cdots + r_k\vec{\kappa}_k)\cdot\vec{\kappa}_i = r_1\cdot 0 + \cdots + r_i\cdot(\vec{\kappa}_i\cdot\vec{\kappa}_i) + \cdots + r_k\cdot 0$$

Solving for r_i yields the desired coefficient.

(d) This is a restatement of the prior item.

Subsection Three.VI.3: Projection Into a Subspace

Three.VI.3.10 **(a)** When bases for the subspaces

$$B_M = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle \qquad B_N = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rangle$$

are concatenated

$$B = B_M \frown B_N = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \rangle$$

and the given vector is represented

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

then the answer comes from retaining the M part and dropping the N part.

$$\text{proj}_{M,N}(\begin{pmatrix} 3 \\ -2 \end{pmatrix}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) When the bases

$$B_M = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \qquad B_N \langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle$$

are concatenated, and the vector is represented,

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (4/3) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1/3) \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

then retaining only the M part gives this answer.

$$\text{proj}_{M,N}(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) = \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}$$

(c) With these bases

$$B_M = \langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle \qquad B_N = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

the representation with respect to the concatenation is this.

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and so the projection is this.

$$\text{proj}_{M,N}(\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

Three.VI.3.11 As in Example 3.5, we can simplify the calculation by just finding the space of vectors perpendicular to all the the vectors in M's basis.

(a) Paramatrizing to get

$$M = \{c \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

gives that

$$M^\perp \{\begin{pmatrix} u \\ v \end{pmatrix} \mid 0 = \begin{pmatrix} u \\ v \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \end{pmatrix}\} = \{\begin{pmatrix} u \\ v \end{pmatrix} \mid 0 = -u + v\}$$

Paramatrizing the one-equation linear system gives this description.

$$M^\perp = \{k \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid k \in \mathbb{R}\}$$

(b) As in the answer to the prior part, M can be described as a span

$$M = \{c \cdot \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\} \qquad B_M = \langle \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \rangle$$

and then $M^\perp$ is the set of vectors perpendicular to the one vector in this basis.

$$M^\perp = \{\begin{pmatrix} u \\ v \end{pmatrix} \mid (3/2) \cdot u + 1 \cdot v = 0\} = \{k \cdot \begin{pmatrix} -2/3 \\ 1 \end{pmatrix} \mid k \in \mathbb{R}\}$$

(c) Paramatrizing the linear requirement in the description of M gives this basis.

$$M=\{c\cdot\begin{pmatrix}1\\1\end{pmatrix}\mid c\in\mathbb{R}\}\qquad B_M=\langle\begin{pmatrix}1\\1\end{pmatrix}\rangle$$

Now, $M^{\perp}$ is the set of vectors perpendicular to (the one vector in) B_M.

$$M^{\perp}=\{\begin{pmatrix}u\\v\end{pmatrix}\mid u+v=0\}=\{k\cdot\begin{pmatrix}-1\\1\end{pmatrix}\mid k\in\mathbb{R}\}$$

(By the way, this answer checks with the first item in this question.)

(d) Every vector in the space is perpendicular to the zero vector so $M^{\perp}=\mathbb{R}^n$.

(e) The appropriate description and basis for M are routine.

$$M=\{y\cdot\begin{pmatrix}0\\1\end{pmatrix}\mid y\in\mathbb{R}\}\qquad B_M=\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle$$

Then

$$M^{\perp}=\{\begin{pmatrix}u\\v\end{pmatrix}\mid 0\cdot u+1\cdot v=0\}=\{k\cdot\begin{pmatrix}1\\0\end{pmatrix}\mid k\in\mathbb{R}\}$$

and so (y-axis)$^{\perp}=x$-axis.

(f) The description of M is easy to find by paramatrizing.

$$M=\{c\cdot\begin{pmatrix}3\\1\\0\end{pmatrix}+d\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\mid c,d\in\mathbb{R}\}\qquad B_M=\langle\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\rangle$$

Finding $M^{\perp}$ here just requires solving a linear system with two equations

$$\begin{array}{rl}3u+v&=0\\ u+w&=0\end{array}\;\xrightarrow{-(1/3)\rho_1+\rho_2}\;\begin{array}{rl}3u+v&=0\\ -(1/3)v+w&=0\end{array}$$

and paramatrizing.

$$M^{\perp}=\{k\cdot\begin{pmatrix}-1\\3\\1\end{pmatrix}\mid k\in\mathbb{R}\}$$

(g) Here, M is one-dimensional

$$M=\{c\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}\mid c\in\mathbb{R}\}\qquad B_M=\langle\begin{pmatrix}0\\-1\\1\end{pmatrix}\rangle$$

and as a result, $M^{\perp}$ is two-dimensional.

$$M^{\perp}=\{\begin{pmatrix}u\\v\\w\end{pmatrix}\mid 0\cdot u-1\cdot v+1\cdot w=0\}=\{j\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+k\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}\mid j,k\in\mathbb{R}\}$$

Three.VI.3.13 **(a)** Paramatrizing gives this.

$$M=\{c\cdot\begin{pmatrix}-1\\1\end{pmatrix}\mid c\in\mathbb{R}\}$$

For the first way, we take the vector spanning the line M to be

$$\vec{s}=\begin{pmatrix}-1\\1\end{pmatrix}$$

and the Definition 1.1 formula gives this.

$$\mathrm{proj}_{[\vec{s}\,]}(\begin{pmatrix}1\\-3\end{pmatrix})=\frac{\begin{pmatrix}1\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\end{pmatrix}}{\begin{pmatrix}-1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\end{pmatrix}}\cdot\begin{pmatrix}-1\\1\end{pmatrix}=\frac{-4}{2}\cdot\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2\\-2\end{pmatrix}$$

For the second way, we fix

$$B_M=\langle\begin{pmatrix}-1\\1\end{pmatrix}\rangle$$

and so (as in Example 3.5 and 3.6, we can just find the vectors perpendicular to all of the members of the basis)

$$M^{\perp}=\{\begin{pmatrix}u\\v\end{pmatrix}\mid -1\cdot u+1\cdot v=0\}=\{k\cdot\begin{pmatrix}1\\1\end{pmatrix}\mid k\in\mathbb{R}\}\qquad B_{M^{\perp}}=\langle\begin{pmatrix}1\\1\end{pmatrix}\rangle$$

and representing the vector with respect to the concatenation gives this.

$$\begin{pmatrix}1\\-3\end{pmatrix} = -2\cdot\begin{pmatrix}-1\\1\end{pmatrix} - 1\cdot\begin{pmatrix}1\\1\end{pmatrix}$$

Keeping the M part yields the answer.

$$\mbox{proj}_{M,M^\perp}(\begin{pmatrix}1\\-3\end{pmatrix}) = \begin{pmatrix}2\\-2\end{pmatrix}$$

The third part is also a simple calculation (there is a 1×1 matrix in the middle, and the inverse of it is also 1×1)

$$\begin{aligned}A\left(A^{\mathrm{trans}}A\right)^{-1}A^{\mathrm{trans}} &= \begin{pmatrix}-1\\1\end{pmatrix}\left(\begin{pmatrix}-1&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}\right)^{-1}\begin{pmatrix}-1&1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}2\end{pmatrix}^{-1}\begin{pmatrix}-1&1\end{pmatrix}\\ &= \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}1/2\end{pmatrix}\begin{pmatrix}-1&1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-1/2&1/2\end{pmatrix} = \begin{pmatrix}1/2&-1/2\\-1/2&1/2\end{pmatrix}\end{aligned}$$

which of course gives the same answer.

$$\mbox{proj}_{M}(\begin{pmatrix}1\\-3\end{pmatrix}) = \begin{pmatrix}1/2&-1/2\\-1/2&1/2\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$

(b) Paramatrization gives this.

$$M = \{c\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} \mid c\in\mathbb{R}\}$$

With that, the formula for the first way gives this.

$$\frac{\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}}{\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \frac{2}{2}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

To proceed by the second method we find $M^\perp$,

$$M^\perp = \{\begin{pmatrix}u\\v\\w\end{pmatrix} \mid -u+w=0\} = \{j\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} + k\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} \mid j,k\in\mathbb{R}\}$$

find the representation of the given vector with respect to the concatenation of the bases B_M and $B_{M^\perp}$

$$\begin{pmatrix}0\\1\\2\end{pmatrix} = 1\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} + 1\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}$$

and retain only the M part.

$$\mbox{proj}_{M}(\begin{pmatrix}0\\1\\2\end{pmatrix}) = 1\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Finally, for the third method, the matrix calculation

$$\begin{aligned}A\left(A^{\mathrm{trans}}A\right)^{-1}A^{\mathrm{trans}} &= \begin{pmatrix}-1\\0\\1\end{pmatrix}\left(\begin{pmatrix}-1&0&1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)^{-1}\begin{pmatrix}-1&0&1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}\begin{pmatrix}2\end{pmatrix}^{-1}\begin{pmatrix}-1&0&1\end{pmatrix}\\ &= \begin{pmatrix}-1\\0\\1\end{pmatrix}\begin{pmatrix}1/2\end{pmatrix}\begin{pmatrix}-1&0&1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}\begin{pmatrix}-1/2&0&1/2\end{pmatrix} = \begin{pmatrix}1/2&0&-1/2\\0&0&0\\-1/2&0&1/2\end{pmatrix}\end{aligned}$$

followed by matrix-vector multiplication

$$\mbox{proj}_{M}(\begin{pmatrix}0\\1\\2\end{pmatrix}) \begin{pmatrix}1/2&0&-1/2\\0&0&0\\-1/2&0&1/2\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

gives the answer.

Three.VI.3.18 If $V = M \oplus N$ then every vector can be decomposed uniquely as $\vec{v} = \vec{m} + \vec{n}$. For all $\vec{v}$ the map p gives $p(\vec{v}) = \vec{m}$ if and only if $\vec{v} - p(\vec{v}) = \vec{n}$, as required.

Three.VI.3.19 Let $\vec{v}$ be perpendicular to every $\vec{w} \in S$. Then $\vec{v} \cdot (c_1\vec{w}_1 + \cdots + c_n\vec{w}_n) = \vec{v} \cdot (c_1\vec{w}_1) + \cdots + \vec{v} \cdot (c_n \cdot \vec{w}_n) = c_1(\vec{v} \cdot \vec{w}_1) + \cdots + c_n(\vec{v} \cdot \vec{w}_n) = c_1 \cdot 0 + \cdots + c_n \cdot 0 = 0$.

Three.VI.3.22 The two must be equal, even only under the seemingly weaker condition that they yield the same result on all orthogonal projections. Consider the subspace M spanned by the set $\{\vec{v}_1, \vec{v}_2\}$. Since each is in M, the orthogonal projection of $\vec{v}_1$ into M is $\vec{v}_1$ and the orthogonal projection of $\vec{v}_2$ into M is $\vec{v}_2$. For their projections into M to be equal, they must be equal.

Three.VI.3.23 **(a)** We will show that the sets are mutually inclusive, $M \subseteq (M^\perp)^\perp$ and $(M^\perp)^\perp \subseteq M$. For the first, if $\vec{m} \in M$ then by the definition of the perp operation, $\vec{m}$ is perpendicular to every $\vec{v} \in M^\perp$, and therefore (again by the definition of the perp operation) $\vec{m} \in (M^\perp)^\perp$. For the other direction, consider $\vec{v} \in (M^\perp)^\perp$. Lemma 3.7's proof shows that $\mathbb{R}^n = M \oplus M^\perp$ and that we can give an orthogonal basis for the space $\langle \vec{\kappa}_1, \ldots, \vec{\kappa}_k, \vec{\kappa}_{k+1}, \ldots, \vec{\kappa}_n \rangle$ such that the first half $\langle \vec{\kappa}_1, \ldots, \vec{\kappa}_k \rangle$ is a basis for M and the second half is a basis for $M^\perp$. The proof also checks that each vector in the space is the sum of its orthogonal projections onto the lines spanned by these basis vectors.

$$\vec{v} = \text{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \text{proj}_{[\vec{\kappa}_n]}(\vec{v}\,)$$

Because $\vec{v} \in (M^\perp)^\perp$, it is perpendicular to every vector in $M^\perp$, and so the projections in the second half are all zero. Thus $\vec{v} = \text{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \text{proj}_{[\vec{\kappa}_k]}(\vec{v}\,)$, which is a linear combination of vectors from M, and so $\vec{v} \in M$. (*Remark.* Here is a slicker way to do the second half: write the space both as $M \oplus M^\perp$ and as $M^\perp \oplus (M^\perp)^\perp$. Because the first half showed that $M \subseteq (M^\perp)^\perp$ and the prior sentence shows that the dimension of the two subspaces M and $(M^\perp)^\perp$ are equal, we can conclude that M equals $(M^\perp)^\perp$.)

(b) Because $M \subseteq N$, any $\vec{v}$ that is perpendicular to every vector in N is also perpendicular to every vector in M. But that sentence simply says that $N^\perp \subseteq M^\perp$.

(c) We will again show that the sets are equal by mutual inclusion. The first direction is easy; any $\vec{v}$ perpendicular to every vector in $M + N = \{\vec{m} + \vec{n} \mid \vec{m} \in M,\, \vec{n} \in N\}$ is perpendicular to every vector of the form $\vec{m} + \vec{0}$ (that is, every vector in M) and every vector of the form $\vec{0} + \vec{n}$ (every vector in N), and so $(M + N)^\perp \subseteq M^\perp \cap N^\perp$. The second direction is also routine; any vector $\vec{v} \in M^\perp \cap N^\perp$ is perpendicular to any vector of the form $c\vec{m} + d\vec{n}$ because $\vec{v} \cdot (c\vec{m} + d\vec{n}) = c \cdot (\vec{v} \cdot \vec{m}) + d \cdot (\vec{v} \cdot \vec{n}) = c \cdot 0 + d \cdot 0 = 0$.

Three.VI.3.24 **(a)** The representation of

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \stackrel{f}{\longmapsto} 1v_1 + 2v_2 + 3v_3$$

is this.

$$\text{Rep}_{\mathcal{E}_3,\mathcal{E}_1}(f) = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

By the definition of f

$$\mathscr{N}(f) = \{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid 1v_1 + 2v_2 + 3v_3 = 0\} = \{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\}$$

and this second description exactly says this.

$$\mathscr{N}(f)^\perp = [\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\}]$$

(b) The generalization is that for any $f\colon \mathbb{R}^n \to \mathbb{R}$ there is a vector $\vec{h}$ so that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \stackrel{f}{\longmapsto} h_1v_1 + \cdots + h_nv_n$$

and $\vec{h} \in \mathscr{N}(f)^\perp$. We can prove this by, as in the prior item, representing f with respect to the standard bases and taking $\vec{h}$ to be the column vector gotten by transposing the one row of that matrix representation.

(c) Of course,

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_2}(f)=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$$

and so the nullspace is this set.

$$\mathscr{N}(f)\{\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\mid\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\}$$

That description makes clear that

$$\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\in\mathscr{N}(f)^{\perp}$$

and since $\mathscr{N}(f)^{\perp}$ is a subspace of $\mathbb{R}^n$, the span of the two vectors is a subspace of the perp of the nullspace. To see that this containment is an equality, take

$$M=[\{\begin{pmatrix}1\\2\\3\end{pmatrix}\}]\qquad N=[\{\begin{pmatrix}4\\5\\6\end{pmatrix}\}]$$

in the third item of Exercise 23, as suggested in the hint.

(d) As above, generalizing from the specific case is easy: for any $f\colon\mathbb{R}^n\to\mathbb{R}^m$ the matrix H representing the map with respect to the standard bases describes the action

$$\begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix}\stackrel{f}{\longmapsto}\begin{pmatrix}h_{1,1}v_1+h_{1,2}v_2+\cdots+h_{1,n}v_n\\\vdots\\h_{m,1}v_1+h_{m,2}v_2+\cdots+h_{m,n}v_n\end{pmatrix}$$

and the description of the nullspace gives that on transposing the m rows of H

$$\vec{h}_1=\begin{pmatrix}h_{1,1}\\h_{1,2}\\\vdots\\h_{1,n}\end{pmatrix},\ldots\vec{h}_m=\begin{pmatrix}h_{m,1}\\h_{m,2}\\\vdots\\h_{m,n}\end{pmatrix}$$

we have $\mathscr{N}(f)^{\perp}=[\{\vec{h}_1,\ldots,\vec{h}_m\}]$. (In [Strang 93], this space is described as the transpose of the row space of H.)

Topic: Line of Best Fit

Topic: Geometry of Linear Maps

Topic: Markov Chains

Topic: Orthonormal Matrices

Chapter Four: Determinants

Subsection Four.I.1: Exploration

Four.I.1.1 **(a)** 4 **(b)** 3 **(c)** -12

Four.I.1.3 For the first, apply the formula in this section, note that any term with a d, g, or h is zero, and simplify. Lower-triangular matrices work the same way.

Four.I.1.4 **(a)** Nonsingular, the determinant is -1.
(b) Nonsingular, the determinant is -1.
(c) Singular, the determinant is 0.

Four.I.1.6 **(a)** $\det(B)=\det(A)$ via $-2\rho_1+\rho_2$
(b) $\det(B)=-\det(A)$ via $\rho_2\leftrightarrow\rho_3$
(c) $\det(B)=(1/2)\cdot\det(A)$ via $(1/2)\rho_2$

Four.I.1.8 This equation

$$0=\det(\begin{pmatrix}12-x&4\\8&8-x\end{pmatrix})=64-20x+x^2=(x-16)(x-4)$$

has roots $x=16$ and $x=4$.

Four.I.1.11 **(a)** The comparison with the formula given in the preamble to this section is easy.
(b) While it holds for 2×2 matrices

$$\left(\begin{array}{cc|c}h_{1,1}&h_{1,2}&h_{1,1}\\h_{2,1}&h_{2,2}&h_{2,1}\end{array}\right)\begin{aligned}[t]&=h_{1,1}h_{2,2}+h_{1,2}h_{2,1}\\&\quad-h_{2,1}h_{1,2}-h_{2,2}h_{1,1}\\&=h_{1,1}h_{2,2}-h_{1,2}h_{2,1}\end{aligned}$$

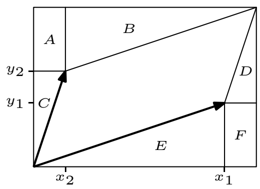
it does not hold for 4×4 matrices. An example is that this matrix is singular because the second and third rows are equal

$$\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\-1&0&0&1\end{pmatrix}$$

but following the scheme of the mnemonic does not give zero.

$$\left(\begin{array}{cccc|ccc}1&0&0&1&1&0&0\\0&1&1&0&0&1&1\\0&1&1&0&0&1&1\\-1&0&0&1&-1&0&0\end{array}\right)\begin{aligned}&=1+0+0+0\\&\quad-(-1)-0-0-0\end{aligned}$$

Four.I.1.14 One way is to count these areas



by taking the area of the entire rectangle and subtracting the area of A the upper-left rectangle, B the upper-middle triangle, D the upper-right triangle, C the lower-left triangle, E the lower-middle triangle, and F the lower-right rectangle $(x_1+x_2)(y_1+y_2)-x_2y_1-(1/2)x_1y_1-(1/2)x_2y_2-(1/2)x_2y_2-(1/2)x_1y_1-x_2y_1$. Simplification gives the determinant formula.

This determinant is the negative of the one above; the formula distinguishes whether the second column is counterclockwise from the first.

Four.I.1.16 No. Recall that constants come out one row at a time.

$$\det(\begin{pmatrix}2&4\\2&6\end{pmatrix})=2\cdot\det(\begin{pmatrix}1&2\\2&6\end{pmatrix})=2\cdot 2\cdot\det(\begin{pmatrix}1&2\\1&3\end{pmatrix})$$

This contradicts linearity (here we didn't need S, i.e., we can take S to be the zero matrix).

Subsection Four.I.2: Properties of Determinants

Four.I.2.7 **(a)** $\begin{vmatrix}3&1&2\\3&1&0\\0&1&4\end{vmatrix}=\begin{vmatrix}3&1&2\\0&0&-2\\0&1&4\end{vmatrix}=-\begin{vmatrix}3&1&2\\0&1&4\\0&0&-2\end{vmatrix}=6$

(b) $\begin{vmatrix}1&0&0&1\\2&1&1&0\\-1&0&1&0\\1&1&1&0\end{vmatrix}=\begin{vmatrix}1&0&0&1\\0&1&1&-2\\0&0&1&1\\0&1&1&-1\end{vmatrix}=\begin{vmatrix}1&0&0&1\\0&1&1&-2\\0&0&1&1\\0&0&0&1\end{vmatrix}=1$

Four.I.2.10 **(a)** Property (2) of the definition of determinants applies via the swap $\rho_1 \leftrightarrow \rho_3$.

$$\begin{vmatrix}h_{3,1}&h_{3,2}&h_{3,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\h_{1,1}&h_{1,2}&h_{1,3}\end{vmatrix}=-\begin{vmatrix}h_{1,1}&h_{1,2}&h_{1,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\h_{3,1}&h_{3,2}&h_{3,3}\end{vmatrix}$$

(b) Property (3) applies.

$$\begin{vmatrix}-h_{1,1}&-h_{1,2}&-h_{1,3}\\-2h_{2,1}&-2h_{2,2}&-2h_{2,3}\\-3h_{3,1}&-3h_{3,2}&-3h_{3,3}\end{vmatrix}=(-1)\cdot(-2)\cdot(-3)\cdot\begin{vmatrix}h_{1,1}&h_{1,2}&h_{1,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\h_{3,1}&h_{3,2}&h_{3,3}\end{vmatrix}=(-6)\cdot\begin{vmatrix}h_{1,1}&h_{1,2}&h_{1,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\h_{3,1}&h_{3,2}&h_{3,3}\end{vmatrix}$$

(c)

$$\begin{aligned}\begin{vmatrix}h_{1,1}+h_{3,1}&h_{1,2}+h_{3,2}&h_{1,3}+h_{3,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\5h_{3,1}&5h_{3,2}&5h_{3,3}\end{vmatrix}&=5\cdot\begin{vmatrix}h_{1,1}+h_{3,1}&h_{1,2}+h_{3,2}&h_{1,3}+h_{3,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\h_{3,1}&h_{3,2}&h_{3,3}\end{vmatrix}\\&=5\cdot\begin{vmatrix}h_{1,1}&h_{1,2}&h_{1,3}\\h_{2,1}&h_{2,2}&h_{2,3}\\h_{3,1}&h_{3,2}&h_{3,3}\end{vmatrix}\end{aligned}$$

Four.I.2.11 A diagonal matrix is in echelon form, so the determinant is the product down the diagonal.

Four.I.2.13 Pivoting by adding the second row to the first gives a matrix whose first row is $x+y+z$ times its third row.

Four.I.2.16 This one

$$A=B=\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

is easy to check.

$$|A+B|=\begin{vmatrix}2&4\\6&8\end{vmatrix}=-8\qquad |A|+|B|=-2-2=-4$$

By the way, this also gives an example where scalar multiplication is not preserved $|2\cdot A|\neq 2\cdot|A|$.

Four.I.2.22 A matrix with only rational entries can be reduced with Gauss' method to an echelon form matrix using only rational arithmetic. Thus the entries on the diagonal must be rationals, and so the product down the diagonal is rational.

Subsection Four.I.3: The Permutation Expansion

Four.I.3.14 **(a)** This matrix is singular.

$$\begin{aligned}\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}&=(1)(5)(9)\,|P_{\phi_1}|+(1)(6)(8)\,|P_{\phi_2}|+(2)(4)(9)\,|P_{\phi_3}|\\&\quad+(2)(6)(7)\,|P_{\phi_4}|+(3)(4)(8)\,|P_{\phi_5}|+(7)(5)(3)\,|P_{\phi_6}|\\&=0\end{aligned}$$

(b) This matrix is nonsingular.

$$\begin{vmatrix}2&2&1\\3&-1&0\\-2&0&5\end{vmatrix} = (2)(-1)(5)\,|P_{\phi_1}| + (2)(0)(0)\,|P_{\phi_2}| + (2)(3)(5)\,|P_{\phi_3}| \\ \quad + (2)(0)(-2)\,|P_{\phi_4}| + (1)(3)(0)\,|P_{\phi_5}| + (-2)(-1)(1)\,|P_{\phi_6}| \\ = -42$$

Four.I.3.15 **(a)** Gauss' method gives this

$$\begin{vmatrix}2&1\\3&1\end{vmatrix} = \begin{vmatrix}2&1\\0&-1/2\end{vmatrix} = -1$$

and permutation expansion gives this.

$$\begin{vmatrix}2&1\\3&1\end{vmatrix} = \begin{vmatrix}2&0\\0&1\end{vmatrix} + \begin{vmatrix}0&1\\3&0\end{vmatrix} = (2)(1)\begin{vmatrix}1&0\\0&1\end{vmatrix} + (1)(3)\begin{vmatrix}0&1\\1&0\end{vmatrix} = -1$$

(b) Gauss' method gives this

$$\begin{vmatrix}0&1&4\\0&2&3\\1&5&1\end{vmatrix} = -\begin{vmatrix}1&5&1\\0&2&3\\0&1&4\end{vmatrix} = -\begin{vmatrix}1&5&1\\0&2&3\\0&0&5/2\end{vmatrix} = -5$$

and the permutation expansion gives this.

$$\begin{vmatrix}0&1&4\\0&2&3\\1&5&1\end{vmatrix} = (0)(2)(1)\,|P_{\phi_1}| + (0)(3)(5)\,|P_{\phi_2}| + (1)(0)(1)\,|P_{\phi_3}| \\ \quad + (1)(3)(1)\,|P_{\phi_4}| + (4)(0)(5)\,|P_{\phi_5}| + (1)(2)(0)\,|P_{\phi_6}| \\ = -5$$

Four.I.3.16 Following Example 3.6 gives this.

$$\begin{aligned}\begin{vmatrix}t_{1,1}&t_{1,2}&t_{1,3}\\t_{2,1}&t_{2,2}&t_{2,3}\\t_{3,1}&t_{3,2}&t_{3,3}\end{vmatrix} &= t_{1,1}t_{2,2}t_{3,3}\,|P_{\phi_1}| + t_{1,1}t_{2,3}t_{3,2}\,|P_{\phi_2}| \\ &\quad + t_{1,2}t_{2,1}t_{3,3}\,|P_{\phi_3}| + t_{1,2}t_{2,3}t_{3,1}\,|P_{\phi_4}| \\ &\quad + t_{1,3}t_{2,1}t_{3,2}\,|P_{\phi_5}| + t_{1,3}t_{2,2}t_{3,1}\,|P_{\phi_6}| \\ &= t_{1,1}t_{2,2}t_{3,3}(+1) + t_{1,1}t_{2,3}t_{3,2}(-1) \\ &\quad + t_{1,2}t_{2,1}t_{3,3}(-1) + t_{1,2}t_{2,3}t_{3,1}(+1) \\ &\quad + t_{1,3}t_{2,1}t_{3,2}(+1) + t_{1,3}t_{2,2}t_{3,1}(-1)\end{aligned}$$

Four.I.3.23 An $n\times n$ matrix with a nonzero determinant has rank n so its columns form a basis for $\mathbb{R}^n$.

Four.I.3.28 Showing that no placement of three zeros suffices is routine. Four zeroes does suffice; put them all in the same row or column.

Four.I.3.29 The $n = 3$ case shows what to do. The pivot operations of $-x_1\rho_2 + \rho_3$ and $-x_1\rho_1 + \rho_2$ give this.

$$\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\x_1^2&x_2^2&x_3^2\end{vmatrix} = \begin{vmatrix}1&1&1\\x_1&x_2&x_3\\0&(-x_1+x_2)x_2&(-x_1+x_3)x_3\end{vmatrix} = \begin{vmatrix}1&1&1\\0&-x_1+x_2&-x_1+x_3\\0&(-x_1+x_2)x_2&(-x_1+x_3)x_3\end{vmatrix}$$

Then the pivot operation of $x_2\rho_2 + \rho_3$ gives the desired result.

$$= \begin{vmatrix}1&1&1\\0&-x_1+x_2&-x_1+x_3\\0&0&(-x_1+x_3)(-x_2+x_3)\end{vmatrix} = (x_2-x_1)(x_3-x_1)(x_3-x_2)$$

Four.I.3.31 The $n = 3$ case shows what happens.

$$|T - rI| = \begin{vmatrix}t_{1,1}-x&t_{1,2}&t_{1,3}\\t_{2,1}&t_{2,2}-x&t_{2,3}\\t_{3,1}&t_{3,2}&t_{3,3}-x\end{vmatrix}$$

Each term in the permutation expansion has three factors drawn from entries in the matrix (e.g., $(t_{1,1} - x)(t_{2,2} - x)(t_{3,3} - x)$ and $(t_{1,1} - x)(t_{2,3})(t_{3,2})$), and so the determinant is expressible as a polynomial in x of degree 3. Such a polynomial has at most 3 roots.

In general, the permutation expansion shows that the determinant can be written as a sum of terms, each with n factors, giving a polynomial of degree n. A polynomial of degree n has at most n roots.

Subsection Four.I.4: Determinants Exist

Four.I.4.11 Each of these is easy to check.

(a)

permutation	ϕ_1	ϕ_2
inverse	ϕ_1	ϕ_2

(b)

permutation	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
inverse	ϕ_1	ϕ_2	ϕ_3	ϕ_5	ϕ_4	ϕ_6

Four.I.4.12 **(a)** $\operatorname{sgn}(\phi_1) = +1$, $\operatorname{sgn}(\phi_2) = -1$
(b) $\operatorname{sgn}(\phi_1) = +1$, $\operatorname{sgn}(\phi_2) = -1$, $\operatorname{sgn}(\phi_3) = -1$, $\operatorname{sgn}(\phi_4) = +1$, $\operatorname{sgn}(\phi_5) = +1$, $\operatorname{sgn}(\phi_6) = -1$

Four.I.4.16 This does not say that m is the least number of swaps to produce an identity, nor does it say that m is the most. It instead says that there is a way to swap to the identity in exactly m steps.

Let ι_j be the first row that is inverted with respect to a prior row and let ι_k be the first row giving that inversion. We have this interval of rows.

$$\begin{pmatrix} \vdots \\ \iota_k \\ \iota_{r_1} \\ \vdots \\ \iota_{r_s} \\ \iota_j \\ \vdots \end{pmatrix} \qquad j < k < r_1 < \cdots < r_s$$

Swap.

$$\begin{pmatrix} \vdots \\ \iota_j \\ \iota_{r_1} \\ \vdots \\ \iota_{r_s} \\ \iota_k \\ \vdots \end{pmatrix}$$

The second matrix has one fewer inversion because there is one fewer inversion in the interval (s vs. $s+1$) and inversions involving rows outside the interval are not affected.

Proceed in this way, at each step reducing the number of inversions by one with each row swap. When no inversions remain the result is the identity.

The contrast with Corollary 4.6 is that the statement of this exercise is a 'there exists' statement: there exists a way to swap to the identity in exactly m steps. But the corollary is a 'for all' statement: for all ways to swap to the identity, the parity (evenness or oddness) is the same.

Four.I.4.17 **(a)** First, $g(\phi_1)$ is the product of the single factor $2-1$ and so $g(\phi_1) = 1$. Second, $g(\phi_2)$ is the product of the single factor $1-2$ and so $g(\phi_2) = -1$.

(b)

permutation ϕ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
$g(\phi)$	2	-2	-2	2	2	-2

(c) Note that $\phi(j) - \phi(i)$ is negative if and only if $\iota_{\phi(j)}$ and $\iota_{\phi(i)}$ are in an inversion of their usual order.

Subsection Four.II.1: Determinants as Size Functions

Four.II.1.9 Solving

$$c_1 \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

gives the unique solution $c_3 = 11/57$, $c_2 = -40/57$ and $c_1 = 99/57$. Because $c_1 > 1$, the vector is not in the box.

Four.II.1.10 Move the parallelepiped to start at the origin, so that it becomes the box formed by

$$\langle\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\rangle$$

and now the absolute value of this determinant is easily computed as 3.

$$\begin{vmatrix}3&2\\0&1\end{vmatrix}=3$$

Four.II.1.11 **(a)** 3 **(b)** 9 **(c)** 1/9

Four.II.1.12 Express each transformation with respect to the standard bases and find the determinant.

(a) 6 **(b)** -1 **(c)** -5

Four.II.1.16 That picture is drawn to mislead. The picture on the left is not the box formed by two vectors. If we slide it to the origin then it becomes the box formed by this sequence.

$$\langle\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\rangle$$

Then the image under the action of the matrix is the box formed by this sequence.

$$\langle\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\rangle$$

which has an area of 4.

Four.II.1.17 Yes to both. For instance, the first is $|TS|=|T|\cdot|S|=|S|\cdot|T|=|ST|$.

Four.II.1.19 $\begin{vmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{vmatrix}=1$

Four.II.1.20 No, for instance the determinant of

$$T=\begin{pmatrix}2&0\\0&1/2\end{pmatrix}$$

is 1 so it preserves areas, but the vector $T\vec{e}_1$ has length 2.

Four.II.1.21 It is zero.

Four.II.1.22 Two of the three sides of the triangle are formed by these vectors.

$$\begin{pmatrix}2\\2\\2\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}\qquad\begin{pmatrix}3\\-1\\4\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}2\\-3\\3\end{pmatrix}$$

One way to find the area of this triangle is to produce a length-one vector orthogonal to these two. From these two relations

$$\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad\begin{pmatrix}2\\-3\\3\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

we get a system

$$\begin{array}{rcl}x+z&=&0\\2x-3y+3z&=&0\end{array}\xrightarrow{-2\rho_1+\rho_2}\begin{array}{rcl}x+z&=&0\\-3y+z&=&0\end{array}$$

with this solution set.

$$\{\begin{pmatrix}-1\\1/3\\1\end{pmatrix}z\mid z\in\mathbb{R}\},$$

A solution of length one is this.

$$\frac{1}{\sqrt{19/9}}\begin{pmatrix}-1\\1/3\\1\end{pmatrix}$$

Thus the area of the triangle is the absolute value of this determinant.

$$\begin{vmatrix}1&2&-3/\sqrt{19}\\0&-3&1/\sqrt{19}\\1&3&3/\sqrt{19}\end{vmatrix}=-12/\sqrt{19}$$

Four.II.1.23 **(a)** Because the image of a linearly dependent set is linearly dependent, if the vectors forming S make a linearly dependent set, so that $|S|=0$, then the vectors forming $t(S)$ make a linearly dependent set, so that $|TS|=0$, and in this case the equation holds.

(b) We must check that if $T \xrightarrow{k\rho_i+\rho_j} \hat{T}$ then $d(T) = |TS|/|S| = |\hat{T}S|/|S| = d(\hat{T})$. We can do this by checking that pivoting first and then multiplying to get $\hat{T}S$ gives the same result as multiplying first to get TS and then pivoting (because the determinant $|TS|$ is unaffected by the pivot so we'll then have that $|\hat{T}S| = |TS|$ and hence that $d(\hat{T}) = d(T)$). This check runs: after adding k times row i of TS to row j of TS, the j, p entry is $(kt_{i,1} + t_{j,1})s_{1,p} + \cdots + (kt_{i,r} + t_{j,r})s_{r,p}$, which is the j, p entry of $\hat{T}S$.

(c) For the second property, we need only check that swapping $T \xrightarrow{\rho_i \leftrightarrow \rho_j} \hat{T}$ and then multiplying to get $\hat{T}S$ gives the same result as multiplying T by S first and then swapping (because, as the determinant $|TS|$ changes sign on the row swap, we'll then have $|\hat{T}S| = -|TS|$, and so $d(\hat{T}) = -d(T)$). This ckeck runs just like the one for the first property.

For the third property, we need only show that performing $T \xrightarrow{k\rho_i} \hat{T}$ and then computing $\hat{T}S$ gives the same result as first computing TS and then performing the scalar multiplication (as the determinant $|TS|$ is rescaled by k, we'll have $|\hat{T}S| = k|TS|$ and so $d(\hat{T}) = k\, d(T)$). Here too, the argument runs just as above.

The fourth property, that if T is I then the result is 1, is obvious.

(d) Determinant functions are unique, so $|TS|/|S| = d(T) = |T|$, and so $|TS| = |T||S|$.

Four.II.1.26 If $H = P^{-1}GP$ then $|H| = |P^{-1}||G||P| = |P^{-1}||P||G| = |P^{-1}P||G| = |G|$.

Four.II.1.29 **(a)** An algebraic check is easy.

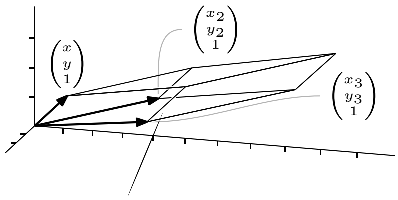
$$0 = xy_2 + x_2y_3 + x_3y - x_3y_2 - xy_3 - x_2y = x \cdot (y_2 - y_3) + y \cdot (x_3 - x_2) + x_2y_3 - x_3y_2$$

simplifies to the familiar form

$$y = x \cdot (x_3 - x_2)/(y_3 - y_2) + (x_2y_3 - x_3y_2)/(y_3 - y_2)$$

(the $y_3 - y_2 = 0$ case is easily handled).

For geometric insight, this picture shows that the box formed by the three vectors. Note that all three vectors end in the $z = 1$ plane. Below the two vectors on the right is the line through (x_2, y_2) and (x_3, y_3).



The box will have a nonzero volume unless the triangle formed by the ends of the three is degenerate. That only happens (assuming that $(x_2, y_3) \neq (x_3, y_3)$) if (x, y) lies on the line through the other two.

(b) *This is how the answer was given in the cited source.* The altitude through (x_1, y_1) of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is found in the usual way from the normal form of the above:

$$\frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Another step shows the area of the triangle to be

$$\frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

This exposition reveals the *modus operandi* more clearly than the usual proof of showing a collection of terms to be identitical with the determinant.

(c) *This is how the answer was given in the cited source.* Let

$$D = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

then the area of the triangle is $(1/2)|D|$. Now if the coordinates are all integers, then D is an integer.

Subsection Four.III.1: Laplace's Expansion

Four.III.1.13 **(a)** $(-1)^{2+3}\begin{vmatrix}1&0\\0&2\end{vmatrix}=-2$ **(b)** $(-1)^{3+2}\begin{vmatrix}1&2\\-1&3\end{vmatrix}=-5$ **(c)** $(-1)^{4}\begin{vmatrix}-1&1\\0&2\end{vmatrix}=-2$

Four.III.1.14 **(a)** $3\cdot(+1)\begin{vmatrix}2&2\\3&0\end{vmatrix}+0\cdot(-1)\begin{vmatrix}1&2\\-1&0\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1&2\\-1&3\end{vmatrix}=-13$

(b) $1\cdot(-1)\begin{vmatrix}0&1\\3&0\end{vmatrix}+2\cdot(+1)\begin{vmatrix}3&1\\-1&0\end{vmatrix}+2\cdot(-1)\begin{vmatrix}3&0\\-1&3\end{vmatrix}=-13$

(c) $1\cdot(+1)\begin{vmatrix}1&2\\-1&3\end{vmatrix}+2\cdot(-1)\begin{vmatrix}3&0\\-1&3\end{vmatrix}+0\cdot(+1)\begin{vmatrix}3&0\\1&2\end{vmatrix}=-13$

Four.III.1.16 **(a)**
$$\begin{pmatrix}T_{1,1}&T_{2,1}&T_{3,1}\\T_{1,2}&T_{2,2}&T_{3,2}\\T_{1,3}&T_{2,3}&T_{3,3}\end{pmatrix}=\begin{pmatrix}\begin{vmatrix}0&2\\0&1\end{vmatrix}&-\begin{vmatrix}1&4\\0&1\end{vmatrix}&\begin{vmatrix}1&4\\0&2\end{vmatrix}\\-\begin{vmatrix}-1&2\\1&1\end{vmatrix}&\begin{vmatrix}2&4\\1&1\end{vmatrix}&-\begin{vmatrix}2&4\\-1&2\end{vmatrix}\\\begin{vmatrix}-1&0\\1&0\end{vmatrix}&-\begin{vmatrix}2&1\\1&0\end{vmatrix}&\begin{vmatrix}2&1\\-1&0\end{vmatrix}\end{pmatrix}=\begin{pmatrix}0&-1&2\\3&-2&-8\\0&1&1\end{pmatrix}$$

(b) The minors are 1×1: $\begin{pmatrix}T_{1,1}&T_{2,1}\\T_{1,2}&T_{2,2}\end{pmatrix}=\begin{pmatrix}|4|&-|-1|\\-|2|&|3|\end{pmatrix}=\begin{pmatrix}4&1\\-2&3\end{pmatrix}$.

(c) $\begin{pmatrix}0&-1\\-5&1\end{pmatrix}$

(d)
$$\begin{pmatrix}T_{1,1}&T_{2,1}&T_{3,1}\\T_{1,2}&T_{2,2}&T_{3,2}\\T_{1,3}&T_{2,3}&T_{3,3}\end{pmatrix}=\begin{pmatrix}\begin{vmatrix}0&3\\8&9\end{vmatrix}&-\begin{vmatrix}4&3\\8&9\end{vmatrix}&\begin{vmatrix}4&3\\0&3\end{vmatrix}\\-\begin{vmatrix}-1&3\\1&9\end{vmatrix}&\begin{vmatrix}1&3\\1&9\end{vmatrix}&-\begin{vmatrix}1&3\\-1&3\end{vmatrix}\\\begin{vmatrix}-1&0\\1&8\end{vmatrix}&-\begin{vmatrix}1&4\\1&8\end{vmatrix}&\begin{vmatrix}1&4\\-1&0\end{vmatrix}\end{pmatrix}=\begin{pmatrix}-24&-12&12\\12&6&-6\\-8&-4&4\end{pmatrix}$$

Four.III.1.17 **(a)** $(1/3)\cdot\begin{pmatrix}0&-1&2\\3&-2&-8\\0&1&1\end{pmatrix}=\begin{pmatrix}0&-1/3&2/3\\1&-2/3&-8/3\\0&1/3&1/3\end{pmatrix}$

(b) $(1/14)\cdot\begin{pmatrix}4&1\\-2&3\end{pmatrix}=\begin{pmatrix}2/7&1/14\\-1/7&3/14\end{pmatrix}$

(c) $(1/-5)\cdot\begin{pmatrix}0&-1\\-5&1\end{pmatrix}=\begin{pmatrix}0&1/5\\1&-1/5\end{pmatrix}$

(d) The matrix has a zero determinant, and so has no inverse.

Four.III.1.19 The determinant
$$\begin{vmatrix}a&b\\c&d\end{vmatrix}$$
expanded on the first row gives $a\cdot(+1)|d|+b\cdot(-1)|c|=ad-bc$ (note the two 1×1 minors).

Four.III.1.20 The determinant of
$$\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$$
is this.
$$a\cdot\begin{vmatrix}e&f\\h&i\end{vmatrix}-b\cdot\begin{vmatrix}d&f\\g&i\end{vmatrix}+c\cdot\begin{vmatrix}d&e\\g&h\end{vmatrix}=a(ei-fh)-b(di-fg)+c(dh-eg)$$

Four.III.1.21 **(a)** $\begin{pmatrix}T_{1,1}&T_{2,1}\\T_{1,2}&T_{2,2}\end{pmatrix}=\begin{pmatrix}|t_{2,2}|&-|t_{1,2}|\\-|t_{2,1}|&|t_{1,1}|\end{pmatrix}=\begin{pmatrix}t_{2,2}&-t_{1,2}\\-t_{2,1}&t_{1,1}\end{pmatrix}$

(b) $(1/t_{1,1}t_{2,2}-t_{1,2}t_{2,1})\cdot\begin{pmatrix}t_{2,2}&-t_{1,2}\\-t_{2,1}&t_{1,1}\end{pmatrix}$

Four.III.1.22 No. Here is a determinant whose value
$$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=1$$
dosn't equal the result of expanding down the diagonal.
$$1\cdot(+1)\begin{vmatrix}1&0\\0&1\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1&0\\0&1\end{vmatrix}+1\cdot(+1)\begin{vmatrix}1&0\\0&1\end{vmatrix}=3$$

Four.III.1.24 Just note that if $S = T^{\text{trans}}$ then the cofactor $S_{j,i}$ equals the cofactor $T_{i,j}$ because $(-1)^{j+i} = (-1)^{i+j}$ and because the minors are the transposes of each other (and the determinant of a transpose equals the determinant of the matrix).

Topic: Cramer's Rule

Topic: Speed of Calculating Determinants

Topic: Projective Geometry

Chapter Five: Similarity

Subsection Five.II.1: Definition and Examples

Five.II.1.5 **(a)** Because the matrix (2) is 1×1, the matrices P and P^{-1} are also 1×1 and so where $P=(p)$ the inverse is $P^{-1}=(1/p)$. Thus $P(2)P^{-1}=(p)(2)(1/p)=(2)$.
(b) Yes: recall that scalar multiples can be brought out of a matrix $P(cI)P^{-1}=cPIP^{-1}=cI$. By the way, the zero and identity matrices are the special cases $c=0$ and $c=1$.
(c) No, as this example shows.

$$\begin{pmatrix}1&-2\\-1&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-3\end{pmatrix}\begin{pmatrix}-1&-2\\-1&-1\end{pmatrix}=\begin{pmatrix}-5&-4\\2&1\end{pmatrix}$$

Five.II.1.8 One possible choice of the bases is

$$B=\langle\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\rangle \qquad D=\mathcal{E}_2=\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\rangle$$

(this B is suggested by the map description). To find the matrix $T=\mathrm{Rep}_{B,B}(t)$, solve the relations

$$c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix} \qquad \hat{c}_1\begin{pmatrix}1\\2\end{pmatrix}+\hat{c}_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$$

to get $c_1=1$, $c_2=-2$, $\hat{c}_1=1/3$ and $\hat{c}_2=4/3$.

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}1&1/3\\-2&4/3\end{pmatrix}$$

Finding $\mathrm{Rep}_{D,D}(t)$ involves a bit more computation. We first find $t(\vec{e}_1)$. The relation

$$c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

gives $c_1=1/3$ and $c_2=-2/3$, and so

$$\mathrm{Rep}_B(\vec{e}_1)=\begin{pmatrix}1/3\\-2/3\end{pmatrix}_B$$

making

$$\mathrm{Rep}_B(t(\vec{e}_1))=\begin{pmatrix}1&1/3\\-2&4/3\end{pmatrix}_{B,B}\begin{pmatrix}1/3\\-2/3\end{pmatrix}_B=\begin{pmatrix}1/9\\-14/9\end{pmatrix}_B$$

and hence t acts on the first basis vector $\vec{e}_1$ in this way.

$$t(\vec{e}_1)=(1/9)\cdot\begin{pmatrix}1\\2\end{pmatrix}-(14/9)\cdot\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}5/3\\-4/3\end{pmatrix}$$

The computation for $t(\vec{e}_2)$ is similar. The relation

$$c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

gives $c_1=1/3$ and $c_2=1/3$, so

$$\mathrm{Rep}_B(\vec{e}_1)=\begin{pmatrix}1/3\\1/3\end{pmatrix}_B$$

making

$$\mathrm{Rep}_B(t(\vec{e}_1))=\begin{pmatrix}1&1/3\\-2&4/3\end{pmatrix}_{B,B}\begin{pmatrix}1/3\\1/3\end{pmatrix}_B=\begin{pmatrix}4/9\\-2/9\end{pmatrix}_B$$

and hence t acts on the second basis vector $\vec{e}_2$ in this way.

$$t(\vec{e}_2)=(4/9)\cdot\begin{pmatrix}1\\2\end{pmatrix}-(2/9)\cdot\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}2/3\\2/3\end{pmatrix}$$

Therefore

$$\mathrm{Rep}_{D,D}(t)=\begin{pmatrix}5/3&2/3\\-4/3&2/3\end{pmatrix}$$

and these are the change of basis matrices.

$$P = \mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \qquad P^{-1} = \bigl(\mathrm{Rep}_{B,D}(\mathrm{id})\bigr)^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{pmatrix}$$

The check of these computations is routine.

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/3 \\ -2 & 4/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 5/3 & 2/3 \\ -4/3 & 2/3 \end{pmatrix}$$

Five.II.1.10 No. If $A = PBP^{-1}$ then $A^2 = (PBP^{-1})(PBP^{-1}) = PB^2P^{-1}$.

Five.II.1.11 Matrix similarity is a special case of matrix equivalence (if matrices are similar then they are matrix equivalent) and matrix equivalence preserves nonsingularity.

Five.II.1.12 A matrix is similar to itself; take P to be the identity matrix: $IPI^{-1} = IPI = P$.

If T is similar to S then $T = PSP^{-1}$ and so $P^{-1}TP = S$. Rewrite this as $S = (P^{-1})T(P^{-1})^{-1}$ to conclude that S is similar to T.

If T is similar to S and S is similar to U then $T = PSP^{-1}$ and $S = QUQ^{-1}$. Then $T = PQUQ^{-1}P^{-1} = (PQ)U(PQ)^{-1}$, showing that T is similar to U.

Five.II.1.17 The k-th powers are similar because, where each matrix represents the map t, the k-th powers represent t^k, the composition of k-many t's. (For instance, if $T = reptB, B$ then $T^2 = \mathrm{Rep}_{B,B}(t \circ t)$.)

Restated more computationally, if $T = PSP^{-1}$ then $T^2 = (PSP^{-1})(PSP^{-1}) = PS^2P^{-1}$. Induction extends that to all powers.

For the $k \leq 0$ case, suppose that S is invertible and that $T = PSP^{-1}$. Note that T is invertible: $T^{-1} = (PSP^{-1})^{-1} = PS^{-1}P^{-1}$, and that same equation shows that T^{-1} is similar to S^{-1}. Other negative powers are now given by the first paragraph.

Five.II.1.18 In conceptual terms, both represent $p(t)$ for some transformation t. In computational terms, we have this.

$$\begin{aligned} p(T) &= c_n(PSP^{-1})^n + \cdots + c_1(PSP^{-1}) + c_0I \\ &= c_nPS^nP^{-1} + \cdots + c_1PSP^{-1} + c_0I \\ &= Pc_nS^nP^{-1} + \cdots + Pc_1SP^{-1} + Pc_0P^{-1} \\ &= P(c_nS^n + \cdots + c_1S + c_0)P^{-1} \end{aligned}$$

Subsection Five.II.2: Diagonalizability

Five.II.2.6 Because the basis vectors are chosen arbitrarily, many different answers are possible. However, here is one way to go; to diagonalize

$$T = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

take it as the representation of a transformation with respect to the standard basis $T = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ and look for $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ such that

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

that is, such that $t(\vec{\beta}_1) = \lambda_1$ and $t(\vec{\beta}_2) = \lambda_2$.

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \vec{\beta}_1 = \lambda_1 \cdot \vec{\beta}_1 \qquad \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \vec{\beta}_2 = \lambda_2 \cdot \vec{\beta}_2$$

We are looking for scalars x such that this equation

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has solutions b_1 and b_2, which are not both zero. Rewrite that as a linear system

$$\begin{aligned} (4 - x) \cdot b_1 + \quad -2 \cdot b_2 &= 0 \\ 1 \cdot b_1 + (1 - x) \cdot b_2 &= 0 \end{aligned}$$

If $x=4$ then the first equation gives that $b_2=0$, and then the second equation gives that $b_1=0$. The case where both b's are zero is disallowed so we can assume that $x\neq 4$.

$$\xrightarrow{(-1/(4-x))\rho_1+\rho_2}\quad\begin{aligned}(4-x)\cdot b_1+\qquad\qquad -2\cdot b_2&=0\\ ((x^2-5x+6)/(4-x))\cdot b_2&=0\end{aligned}$$

Consider the bottom equation. If $b_2=0$ then the first equation gives $b_1=0$ or $x=4$. The $b_1=b_2=0$ case is disallowed. The other possibility for the bottom equation is that the numerator of the fraction $x^2-5x+6=(x-2)(x-3)$ is zero. The $x=2$ case gives a first equation of $2b_1-2b_2=0$, and so associated with $x=2$ we have vectors whose first and second components are equal:

$$\vec{\beta}_1=\begin{pmatrix}1\\1\end{pmatrix}\qquad(\text{so }\begin{pmatrix}4&-2\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=2\cdot\begin{pmatrix}1\\1\end{pmatrix},\text{ and }\lambda_1=2).$$

If $x=3$ then the first equation is $b_1-2b_2=0$ and so the associated vectors are those whose first component is twice their second:

$$\vec{\beta}_2=\begin{pmatrix}2\\1\end{pmatrix}\qquad(\text{so }\begin{pmatrix}4&-2\\1&1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=3\cdot\begin{pmatrix}2\\1\end{pmatrix},\text{ and so }\lambda_2=3).$$

This picture

$$\begin{array}{ccc}\mathbb{R}^2_{\text{w.r.t. }\mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{R}^2_{\text{w.r.t. }\mathcal{E}_2}\\ \text{id}\big\downarrow & & \text{id}\big\downarrow\\ \mathbb{R}^2_{\text{w.r.t. }B} & \xrightarrow[D]{t} & \mathbb{R}^2_{\text{w.r.t. }B}\end{array}$$

shows how to get the diagonalization.

$$\begin{pmatrix}2&0\\0&3\end{pmatrix}=\begin{pmatrix}1&2\\1&1\end{pmatrix}^{-1}\begin{pmatrix}4&-2\\1&1\end{pmatrix}\begin{pmatrix}1&2\\1&1\end{pmatrix}$$

Comment. This equation matches the $T=PSP^{-1}$ definition under this renaming.

$$T=\begin{pmatrix}2&0\\0&3\end{pmatrix}\quad P=\begin{pmatrix}1&2\\1&1\end{pmatrix}^{-1}\quad p^{-1}=\begin{pmatrix}1&2\\1&1\end{pmatrix}\quad S=\begin{pmatrix}4&-2\\1&1\end{pmatrix}$$

Five.II.2.8 For any integer p,

$$\begin{pmatrix}d_1&0&\\0&\ddots&\\&&d_n\end{pmatrix}^p=\begin{pmatrix}d_1^p&0&\\0&\ddots&\\&&d_n^p\end{pmatrix}.$$

Five.II.2.11 To check that the inverse of a diagonal matrix is the diagonal matrix of the inverses, just multiply.

$$\begin{pmatrix}a_{1,1}&0&&\\0&a_{2,2}&&\\&&\ddots&\\&&&a_{n,n}\end{pmatrix}\begin{pmatrix}1/a_{1,1}&0&&\\0&1/a_{2,2}&&\\&&\ddots&\\&&&1/a_{n,n}\end{pmatrix}$$

(Showing that it is a left inverse is just as easy.)

If a diagonal entry is zero then the diagonal matrix is singular; it has a zero determinant.

Five.II.2.15 **(a)** $\begin{pmatrix}1&1\\0&-1\end{pmatrix}^{-1}\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$

(b) $\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Five.II.2.17 If

$$P\begin{pmatrix}1&c\\0&1\end{pmatrix}P^{-1}=\begin{pmatrix}a&0\\0&b\end{pmatrix}$$

then

$$P\begin{pmatrix}1&c\\0&1\end{pmatrix}=\begin{pmatrix}a&0\\0&b\end{pmatrix}P$$

so

$$\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}1&c\\0&1\end{pmatrix}=\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix}$$

$$\begin{pmatrix}p&cp+q\\r&cr+s\end{pmatrix}=\begin{pmatrix}ap&aq\\br&bs\end{pmatrix}$$

The $1,1$ entries show that $a = 1$ and the $1,2$ entries then show that $pc = 0$. Since $c \neq 0$ this means that $p = 0$. The $2,1$ entries show that $b = 1$ and the $2,2$ entries then show that $rc = 0$. Since $c \neq 0$ this means that $r = 0$. But if both p and r are 0 then P is not invertible.

Subsection Five.II.3: Eigenvalues and Eigenvectors

Five.II.3.21 **(a)** The characteristic equation is $(3-x)(-1-x) = 0$. Its roots, the eigenvalues, are $\lambda_1 = 3$ and $\lambda_2 = -1$. For the eigenvectors we consider this equation.

$$\begin{pmatrix} 3-x & 0 \\ 8 & -1-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the eigenvector associated with $\lambda_1 = 3$, we consider the resulting linear system.

$$\begin{aligned} 0 \cdot b_1 + 0 \cdot b_2 &= 0 \\ 8 \cdot b_1 + -4 \cdot b_2 &= 0 \end{aligned}$$

The eigenspace is the set of vectors whose second component is twice the first component.

$$\{\begin{pmatrix} b_2/2 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} b_2/2 \\ b_2 \end{pmatrix} = 3 \cdot \begin{pmatrix} b_2/2 \\ b_2 \end{pmatrix}$$

(Here, the parameter is b_2 only because that is the variable that is free in the above system.) Hence, this is an eigenvector associated with the eigenvalue 3.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Finding an eigenvector associated with $\lambda_2 = -1$ is similar. This system

$$\begin{aligned} 4 \cdot b_1 + 0 \cdot b_2 &= 0 \\ 8 \cdot b_1 + 0 \cdot b_2 &= 0 \end{aligned}$$

leads to the set of vectors whose first component is zero.

$$\{\begin{pmatrix} 0 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ b_2 \end{pmatrix} = -1 \cdot \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$$

And so this is an eigenvector associated with λ_2.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) The characteristic equation is

$$0 = \begin{vmatrix} 3-x & 2 \\ -1 & -x \end{vmatrix} = x^2 - 3x + 2 = (x-2)(x-1)$$

and so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$. To find eigenvectors, consider this system.

$$\begin{aligned} (3-x) \cdot b_1 + 2 \cdot b_2 &= 0 \\ -1 \cdot b_1 - x \cdot b_2 &= 0 \end{aligned}$$

For $\lambda_1 = 2$ we get

$$\begin{aligned} 1 \cdot b_1 + 2 \cdot b_2 &= 0 \\ -1 \cdot b_1 - 2 \cdot b_2 &= 0 \end{aligned}$$

leading to this eigenspace and eigenvector.

$$\{\begin{pmatrix} -2b_2 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1$ the system is

$$\begin{aligned} 2 \cdot b_1 + 2 \cdot b_2 &= 0 \\ -1 \cdot b_1 - 1 \cdot b_2 &= 0 \end{aligned}$$

leading to this.

$$\{\begin{pmatrix} -b_2 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C}\} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Five.II.3.24 **(a)** The characteristic equation is

$$0=\begin{vmatrix}3-x&-2&0\\-2&3-x&0\\0&0&5-x\end{vmatrix}=x^3-11x^2+35x-25=(x-1)(x-5)^2$$

and so the eigenvalues are $\lambda_1=1$ and also the repeated eigenvalue $\lambda_2=5$. To find eigenvectors, consider this system.

$$\begin{aligned}(3-x)\cdot b_1-\quad 2\cdot b_2\qquad&=0\\-2\cdot b_1+(3-x)\cdot b_2\qquad&=0\\(5-x)\cdot b_3&=0\end{aligned}$$

For $\lambda_1=1$ we get

$$\begin{aligned}2\cdot b_1-2\cdot b_2\qquad&=0\\-2\cdot b_1+2\cdot b_2\qquad&=0\\4\cdot b_3&=0\end{aligned}$$

leading to this eigenspace and eigenvector.

$$\{\begin{pmatrix}b_2\\b_2\\0\end{pmatrix}\bigm| b_2\in\mathbb{C}\}\qquad\begin{pmatrix}1\\1\\0\end{pmatrix}$$

For $\lambda_2=1$ the system is

$$\begin{aligned}-2\cdot b_1-2\cdot b_2\qquad&=0\\-2\cdot b_1-2\cdot b_2\qquad&=0\\0\cdot b_3&=0\end{aligned}$$

leading to this.

$$\{\begin{pmatrix}-b_2\\b_2\\0\end{pmatrix}+\begin{pmatrix}0\\0\\b_3\end{pmatrix}\bigm| b_2,b_3\in\mathbb{C}\}\qquad\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$$

(b) The characteristic equation is

$$0=\begin{vmatrix}-x&1&0\\0&-x&1\\4&-17&8-x\end{vmatrix}=-x^3+8x^2-17x+4=-1\cdot(x-4)(x^2-4x+1)$$

and the eigenvalues are $\lambda_1=4$ and (by using the quadratic equation) $\lambda_2=2+\sqrt{3}$ and $\lambda_3=2-\sqrt{3}$. To find eigenvectors, consider this system.

$$\begin{aligned}-x\cdot b_1+\quad b_2\qquad\qquad&=0\\-x\cdot b_2+\qquad b_3&=0\\4\cdot b_1-\ 17\cdot b_2+(8-x)\cdot b_3&=0\end{aligned}$$

Substituting $x=\lambda_1=4$ gives the system

$$\begin{aligned}-4\cdot b_1+\quad b_2\qquad\quad&=0\\-4\cdot b_2+\quad b_3&=0\\4\cdot b_1-\ 17\cdot b_2+4\cdot b_3&=0\end{aligned}\xrightarrow{\rho_1+\rho_3}\begin{aligned}-4\cdot b_1+\quad b_2\qquad\quad&=0\\-4\cdot b_2+\quad b_3&=0\\-16\cdot b_2+4\cdot b_3&=0\end{aligned}\xrightarrow{-4\rho_2+\rho_3}\begin{aligned}-4\cdot b_1+\quad b_2\qquad&=0\\-4\cdot b_2+b_3&=0\\0&=0\end{aligned}$$

leading to this eigenspace and eigenvector.

$$V_4=\{\begin{pmatrix}(1/16)\cdot b_3\\(1/4)\cdot b_3\\b_3\end{pmatrix}\bigm| b_2\in\mathbb{C}\}\qquad\begin{pmatrix}1\\4\\16\end{pmatrix}$$

Substituting $x=\lambda_2=2+\sqrt{3}$ gives the system

$$\begin{aligned}(-2-\sqrt{3})\cdot b_1+\qquad b_2\qquad\qquad&=0\\(-2-\sqrt{3})\cdot b_2+\qquad b_3&=0\\4\cdot b_1-\qquad 17\cdot b_2+(6-\sqrt{3})\cdot b_3&=0\end{aligned}$$

$$\xrightarrow{(-4/(-2-\sqrt{3}))\rho_1+\rho_3}\begin{aligned}(-2-\sqrt{3})\cdot b_1+\qquad b_2\qquad\qquad&=0\\(-2-\sqrt{3})\cdot b_2+\qquad b_3&=0\\+(-9-4\sqrt{3})\cdot b_2+(6-\sqrt{3})\cdot b_3&=0\end{aligned}$$

(the middle coefficient in the third equation equals the number $(-4/(-2-\sqrt{3}))-17$; find a common denominator of $-2-\sqrt{3}$ and then rationalize the denominator by multiplying the top and bottom of the frsction by $-2+\sqrt{3}$)

$$\xrightarrow{((9+4\sqrt{3})/(-2-\sqrt{3}))\rho_2+\rho_3}\begin{aligned}(-2-\sqrt{3})\cdot b_1+\qquad b_2\qquad&=0\\(-2-\sqrt{3})\cdot b_2+b_3&=0\\0&=0\end{aligned}$$

which leads to this eigenspace and eigenvector.

$$V_{2+\sqrt{3}} = \{\begin{pmatrix} (1/(2+\sqrt{3})^2)\cdot b_3 \\ (1/(2+\sqrt{3}))\cdot b_3 \\ b_3 \end{pmatrix} \bigm| b_3\in\mathbb{C}\} \qquad \begin{pmatrix} (1/(2+\sqrt{3})^2) \\ (1/(2+\sqrt{3})) \\ 1 \end{pmatrix}$$

Finally, substituting $x=\lambda_3=2-\sqrt{3}$ gives the system

$$\begin{array}{rcrcrl} (-2+\sqrt{3})\cdot b_1 & + & b_2 & & & =0 \\ & & (-2+\sqrt{3})\cdot b_2 & + & b_3 & =0 \\ 4\cdot b_1 & - & 17\cdot b_2 & + & (6+\sqrt{3})\cdot b_3 & =0 \end{array}$$

$$\xrightarrow{(-4/(-2+\sqrt{3}))\rho_1+\rho_3} \begin{array}{rcrcrl} (-2+\sqrt{3})\cdot b_1 & + & b_2 & & & =0 \\ & & (-2+\sqrt{3})\cdot b_2 & + & b_3 & =0 \\ & & (-9+4\sqrt{3})\cdot b_2 & + & (6+\sqrt{3})\cdot b_3 & =0 \end{array}$$

$$\xrightarrow{((9-4\sqrt{3})/(-2+\sqrt{3}))\rho_2+\rho_3} \begin{array}{rcrcrl} (-2+\sqrt{3})\cdot b_1 & + & b_2 & & & =0 \\ & & (-2+\sqrt{3})\cdot b_2 & + & b_3 & =0 \\ & & & & 0 & =0 \end{array}$$

which gives this eigenspace and eigenvector.

$$V_{2-\sqrt{3}} = \{\begin{pmatrix} (1/(2+\sqrt{3})^2)\cdot b_3 \\ (1/(2-\sqrt{3}))\cdot b_3 \\ b_3 \end{pmatrix} \bigm| b_3\in\mathbb{C}\} \qquad \begin{pmatrix} (1/(-2+\sqrt{3})^2) \\ (1/(-2+\sqrt{3})) \\ 1 \end{pmatrix}$$

Five.II.3.25 With respect to the natural basis $B=\langle 1,x,x^2\rangle$ the matrix representation is this.

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}$$

Thus the characteristic equation

$$0=\begin{pmatrix} 5-x & 6 & 2 \\ 0 & -1-x & -8 \\ 1 & 0 & -2-x \end{pmatrix} = (5-x)(-1-x)(-2-x)-48-2\cdot(-1-x)$$

is $0=-x^3+2x^2+15x-36=-1\cdot(x+4)(x-3)^2$. To find the associated eigenvectors, consider this system.

$$\begin{array}{rcrcrl} (5-x)\cdot b_1 & + & 6\cdot b_2 & + & 2\cdot b_3 & =0 \\ & & (-1-x)\cdot b_2 & - & 8\cdot b_3 & =0 \\ b_1 & & & + & (-2-x)\cdot b_3 & =0 \end{array}$$

Plugging in $x=\lambda_1=4$ gives

$$\begin{array}{rcrcrl} b_1 & + & 6\cdot b_2 & + & 2\cdot b_3 & =0 \\ & & -5\cdot b_2 & - & 8\cdot b_3 & =0 \\ b_1 & & & - & 6\cdot b_3 & =0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcrcrl} b_1 & + & 6\cdot b_2 & + & 2\cdot b_3 & =0 \\ & & -5\cdot b_2 & - & 8\cdot b_3 & =0 \\ & & -6\cdot b_2 & - & 8\cdot b_3 & =0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcrcrl} b_1 & + & 6\cdot b_2 & + & 2\cdot b_3 & =0 \\ & & -5\cdot b_2 & - & 8\cdot b_3 & =0 \\ & & -6\cdot b_2 & - & 8\cdot b_3 & =0 \end{array}$$

Five.II.3.27 Fix the natural basis $B=\langle 1,x,x^2,x^3\rangle$. The map's action is $1\mapsto 0$, $x\mapsto 1$, $x^2\mapsto 2x$, and $x^3\mapsto 3x^2$ and its representation is easy to compute.

$$T=\mathrm{Rep}_{B,B}(d/dx)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{B,B}$$

We find the eigenvalues with this computation.

$$0=|T-xI|=\begin{vmatrix} -x & 1 & 0 & 0 \\ 0 & -x & 2 & 0 \\ 0 & 0 & -x & 3 \\ 0 & 0 & 0 & -x \end{vmatrix}=x^4$$

Thus the map has the single eigenvalue $\lambda=0$. To find the associated eigenvectors, we solve

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{B,B} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_B = 0\cdot\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_B \qquad\Longrightarrow\qquad b_2=0,\ b_3=0,\ b_4=0$$

to get this eigenspace.

$$\{\begin{pmatrix}b_1\\0\\0\\0\end{pmatrix}_B \bigm| b_1\in\mathbb{C}\} = \{b_1+0\cdot x+0\cdot x^2+0\cdot x^3 \bigm| b_1\in\mathbb{C}\} = \{b_1 \bigm| b_1\in\mathbb{C}\}$$

Five.II.3.29 Just expand the determinant of $T-xI$.

$$\begin{vmatrix}a-x & c\\ b & d-x\end{vmatrix} = (a-x)(d-x)-bc = x^2+(-a-d)\cdot x+(ad-bc)$$

Five.II.3.31 **(a)** Yes, use $\lambda=1$ and the identity map.
(b) Yes, use the transformation that multiplies by λ.

Five.II.3.32 If $t(\vec{v})=\lambda\cdot\vec{v}$ then $\vec{v}\mapsto\vec{0}$ under the map $t-\lambda\cdot\mathrm{id}$.

Five.II.3.34 Consider an eigenspace V_λ. Any $\vec{w}\in V_\lambda$ is the image $\vec{w}=\lambda\cdot\vec{v}$ of some $\vec{v}\in V_\lambda$ (namely, $\vec{v}=(1/\lambda)\cdot\vec{w}$). Thus, on V_λ (which is a nontrivial subspace) the action of t^{-1} is $t^{-1}(\vec{w})=\vec{v}=(1/\lambda)\cdot\vec{w}$, and so $1/\lambda$ is an eigenvalue of t^{-1}.

Five.II.3.35 **(a)** We have $(cT+dI)\vec{v}=cT\vec{v}+dI\vec{v}=c\lambda\vec{v}+d\vec{v}=(c\lambda+d)\cdot\vec{v}$.
(b) Suppose that $S=PTP^{-1}$ is diagonal. Then $P(cT+dI)P^{-1}=P(cT)P^{-1}+P(dI)P^{-1}=cPTP^{-1}+dI=cS+dI$ is also diagonal.

Five.II.3.36 The scalar λ is an eigenvalue if and only if the transformation $t-\lambda\mathrm{id}$ is singular. A transformation is singular if and only if it is not an isomorphism (that is, a transformation is an isomorphism if and only if it is nonsingular).

Subsection Five.III.1: Self-Composition

Subsection Five.III.2: Strings

Five.III.2.17 Three. It is at least three because $\ell^2(\,(1,1,1)\,)=(0,0,1)\neq\vec{0}$. It is at most three because $(x,y,z)\mapsto(0,x,y)\mapsto(0,0,x)\mapsto(0,0,0)$.

Five.III.2.18 **(a)** The domain has dimension four. The map's action is that any vector in the space $c_1\cdot\vec{\beta}_1+c_2\cdot\vec{\beta}_2+c_3\cdot\vec{\beta}_3+c_4\cdot\vec{\beta}_4$ is sent to $c_1\cdot\vec{\beta}_2+c_2\cdot\vec{0}+c_3\cdot\vec{\beta}_4+c_4\cdot\vec{0}=c_1\cdot\vec{\beta}_3+c_3\cdot\vec{\beta}_4$. The first application of the map sends two basis vectors $\vec{\beta}_2$ and $\vec{\beta}_4$ to zero, and therefore the nullspace has dimension two and the rangespace has dimension two. With a second application, all four basis vectors are sent to zero and so the nullspace of the second power has dimension four while the rangespace of the second power has dimension zero. Thus the index of nilpotency is two. This is the canonical form.

$$\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$$

(b) The dimension of the domain of this map is six. For the first power the dimension of the nullspace is four and the dimension of the rangespace is two. For the second power the dimension of the nullspace is five and the dimension of the rangespace is one. Then the third iteration results in a nullspace of dimension six and a rangespace of dimension zero. The index of nilpotency is three, and this is the canonical form.

$$\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

(c) The dimension of the domain is three, and the index of nilpotency is three. The first power's null space has dimension one and its range space has dimension two. The second power's null space has dimension two and its range space has dimension one. Finally, the third power's null space has dimension three and its range space has dimension zero. Here is the canonical form matrix.

$$\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix}$$

Five.III.2.20 The table os calculations

$$\begin{array}{c|cc}
p & N^p & \mathscr{N}(N^p) \\ \hline
1 & \begin{pmatrix} 0&1&1&0&1\\ 0&0&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} & \{\begin{pmatrix} r\\ u\\ -u-v\\ u\\ v \end{pmatrix} \bigm| r,u,v\in\mathbb{C}\} \\
2 & \begin{pmatrix} 0&0&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} & \{\begin{pmatrix} r\\ s\\ -u-v\\ u\\ v \end{pmatrix} \bigm| r,s,u,v\in\mathbb{C}\} \\
2 & \text{–}\textit{zero matrix}\text{–} & \mathbb{C}^5
\end{array}$$

gives these requirements of the string basis: three basis vectors are sent directly to zero, one more basis vector is sent to zero by a second application, and the final basis vector is sent to zero by a third application. Thus, the string basis has this form.

$$\begin{array}{l}
\vec\beta_1 \mapsto \vec\beta_2 \mapsto \vec\beta_3 \mapsto \vec 0 \\
\vec\beta_4 \mapsto \vec 0 \\
\vec\beta_5 \mapsto \vec 0
\end{array}$$

From that the canonical form is immediate.

$$\begin{pmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

Five.III.2.21 **(a)** The canonical form has a 3×3 block and a 2×2 block

$$\left(\begin{array}{ccc|cc} 0&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&1&0 \end{array}\right)$$

corresponding to the length three string and the length two string in the basis.

(b) Assume that N is the representation of the underlying map with respect to the standard basis. Let B be the basis to which we will change. By the similarity diagram

$$\begin{array}{ccc}
\mathbb{C}^2_{\text{w.r.t. }\mathcal{E}_2} & \xrightarrow[N]{n} & \mathbb{C}^2_{\text{w.r.t. }\mathcal{E}_2} \\
{\scriptstyle\text{id}}\big\downarrow{\scriptstyle P} & & {\scriptstyle\text{id}}\big\downarrow{\scriptstyle P} \\
C^2_{\text{w.r.t. }B} & \xrightarrow{n} & \mathbb{C}^2_{\text{w.r.t. }B}
\end{array}$$

we have that the canonical form matrix is PNP^{-1} where

$$P^{-1} = \text{Rep}_{B,\mathcal{E}_5}(\text{id}) = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&1&0\\ 1&0&1&0&0\\ 0&0&1&1&1\\ 0&0&0&0&1 \end{pmatrix}$$

and P is the inverse of that.

$$P = \text{Rep}_{\mathcal{E}_5,B}(\text{id}) = (P^{-1})^{-1} = \begin{pmatrix} 1&0&0&0&0\\ -1&1&1&-1&1\\ -1&0&1&0&0\\ 1&0&-1&1&-1\\ 0&0&0&0&1 \end{pmatrix}$$

(c) The calculation to check this is routine.

Five.III.2.22 **(a)** The calculation

$$\begin{array}{c|cc} p & N^p & \mathscr{N}(N^p) \\ \hline 1 & \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix} & \{\begin{pmatrix} u \\ u \end{pmatrix} \mid u \in \mathbb{C}\} \\ 2 & \text{–}\textit{zero matrix}\text{–} & \mathbb{C}^2 \end{array}$$

shows that any map represented by the matrix must act on the string basis in this way

$$\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$$

because the nullspace after one application has dimension one and exactly one basis vector, $\vec{\beta}_2$, is sent to zero. Therefore, this representation with respect to $\langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ is the canonical form.

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(b) The calculation here is similar to the prior one.

$$\begin{array}{c|cc} p & N^p & \mathscr{N}(N^p) \\ \hline 1 & \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} & \{\begin{pmatrix} u \\ v \\ v \end{pmatrix} \mid u, v \in \mathbb{C}\} \\ 2 & \text{–}\textit{zero matrix}\text{–} & \mathbb{C}^3 \end{array}$$

The table shows that the string basis is of the form

$$\begin{array}{l} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} \\ \vec{\beta}_3 \mapsto \vec{0} \end{array}$$

because the nullspace after one application of the map has dimension two — $\vec{\beta}_2$ and $\vec{\beta}_3$ are both sent to zero — and one more iteration results in the additional vector being brought to zero.

(c) The calculation

$$\begin{array}{c|cc} p & N^p & \mathscr{N}(N^p) \\ \hline 1 & \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} & \{\begin{pmatrix} u \\ 0 \\ -u \end{pmatrix} \mid u \in \mathbb{C}\} \\ 2 & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} & \{\begin{pmatrix} u \\ v \\ -u \end{pmatrix} \mid u, v \in \mathbb{C}\} \\ 3 & \text{–}\textit{zero matrix}\text{–} & \mathbb{C}^3 \end{array}$$

shows that any map represented by this basis must act on a string basis in this way.

$$\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$$

Therefore, this is the canonical form.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Five.III.2.25 Observe that a canonical form nilpotent matrix has only zero eigenvalues; e.g., the determinant of this lower-triangular matrix

$$\begin{pmatrix} -x & 0 & 0 \\ 1 & -x & 0 \\ 0 & 1 & -x \end{pmatrix}$$

is $(-x)^3$, the only root of which is zero. But similar matrices have the same eigenvalues and every nilpotent matrix is similar to one in canonical form.

Another way to see this is to observe that a nilpotent matrix sends all vectors to zero after some number of iterations, but that conflicts with an action on an eigenspace $\vec{v} \mapsto \lambda\vec{v}$ unless λ is zero.

Five.III.2.28 **(a)** Any member $\vec{w}$ of the span can be written as a linear combination $\vec{w} = c_0 \cdot \vec{v} + c_1 \cdot t(\vec{v}) + \cdots + c_{k-1} \cdot t^{k-1}(\vec{v})$. But then, by the linearity of the map, $t(\vec{w}) = c_0 \cdot t(\vec{v}) + c_1 \cdot t^2(\vec{v}) + \cdots + c_{k-2} \cdot t^{k-1}(\vec{v}) + c_{k-1} \cdot \vec{0}$ is also in the span.

(b) The operation in the prior item, when iterated k times, will result in a linear combination of zeros.

(c) If $\vec{v} = \vec{0}$ then the set is empty and so is linearly independent by definition. Otherwise write $c_1\vec{v} + \cdots + c_{k-1}t^{k-1}(\vec{v}) = \vec{0}$ and apply t^{k-1} to both sides. The right side gives $\vec{0}$ while the left side gives $c_1 t^{k-1}(\vec{v})$; conclude that $c_1 = 0$. Continue in this way by applying t^{k-2} to both sides, etc.

(d) Of course, t acts on the span by acting on this basis as a single, k-long, t-string.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{pmatrix}$$

Five.III.2.34 Let $t\colon V \to V$ be the transformation. If $\operatorname{rank}(t) = \operatorname{nullity}(t)$ then the equation $\operatorname{rank}(t) + \operatorname{nullity}(t) = \dim(V)$ shows that $\dim(V)$ is even.

Subsection Five.IV.1: Polynomials of Maps and Matrices

Five.IV.1.13 For each, the minimal polynomial must have a leading coefficient of 1 and Theorem 1.8, the Cayley-Hamilton Theorem, says that the minimal polynomial must contain the same linear factors as the characteristic polynomial, although possibly of lower degree but not of zero degree.

(a) The possibilities are $m_1(x) = x - 3$, $m_2(x) = (x-3)^2$, $m_3(x) = (x-3)^3$, and $m_4(x) = (x-3)^4$. Note that the 8 has been dropped because a minimal polynomial must have a leading coefficient of one. The first is a degree one polynomial, the second is degree two, the third is degree three, and the fourth is degree four.

(b) The possibilities are $m_1(x) = (x+1)(x-4)$, $m_2(x) = (x+1)^2(x-4)$, and $m_3(x) = (x+1)^3(x-4)$. The first is a quadratic polynomial, that is, it has degree two. The second has degree three, and the third has degree four.

(c) We have $m_1(x) = (x-2)(x-5)$, $m_2(x) = (x-2)^2(x-5)$, $m_3(x) = (x-2)(x-5)^2$, and $m_4(x) = (x-2)^2(x-5)^2$. They are polynomials of degree two, three, three, and four.

(d) The possiblities are $m_1(x) = (x+3)(x-1)(x-2)$, $m_2(x) = (x+3)^2(x-1)(x-2)$, $m_3(x) = (x+3)(x-1)(x-2)^2$, and $m_4(x) = (x+3)^2(x-1)(x-2)^2$. The degree of m_1 is three, the degree of m_2 is four, the degree of m_3 is four, and the degree of m_4 is five.

Five.IV.1.14 In each case we will use the method of Example 1.12.

(a) Because T is triangular, $T - xI$ is also triangular

$$T - xI = \begin{pmatrix} 3-x & 0 & 0 \\ 1 & 3-x & 0 \\ 0 & 0 & 4-x \end{pmatrix}$$

the characteristic polynomial is easy $c(x) = |T - xI| = (3-x)^2(4-x) = -1\cdot(x-3)^2(x-4)$. There are only two possibilities for the minimal polynomial, $m_1(x) = (x-3)(x-4)$ and $m_2(x) = (x-3)^2(x-4)$. (Note that the characteristic polynomial has a negative sign but the minimal polynomial does not since it must have a leading coefficient of one). Because $m_1(T)$ is not the zero matrix

$$(T-3I)(T-4I) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the minimal polynomial is $m(x) = m_2(x)$.

$$(T-3I)^2(T-4I) = (T-3I)\cdot\big((T-3I)(T-4I)\big) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) As in the prior item, the fact that the matrix is triangular makes computation of the characteristic polynomial easy.

$$c(x) = |T - xI| = \begin{vmatrix} 3-x & 0 & 0 \\ 1 & 3-x & 0 \\ 0 & 0 & 3-x \end{vmatrix} = (3-x)^3 = -1\cdot(x-3)^3$$

There are three possibilities for the minimal polynomial $m_1(x) = (x-3)$, $m_2(x) = (x-3)^2$, and $m_3(x) = (x-3)^3$. We settle the question by computing $m_1(T)$

$$T - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $m_2(T)$.

$$(T-3I)^2 = \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

Because $m_2(T)$ is the zero matrix, $m_2(x)$ is the minimal polynomial.

(c) Again, the matrix is triangular.

$$c(x) = |T-xI| = \begin{vmatrix} 3-x&0&0\\ 1&3-x&0\\ 0&1&3-x \end{vmatrix} = (3-x)^3 = -1\cdot(x-3)^3$$

Again, there are three possibilities for the minimal polynomial $m_1(x)=(x-3)$, $m_2(x)=(x-3)^2$, and $m_3(x)=(x-3)^3$. We compute $m_1(T)$

$$T-3I = \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix}$$

and $m_2(T)$

$$(T-3I)^2 = \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}$$

and $m_3(T)$.

$$(T-3I)^3 = (T-3I)^2(T-3I) = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

Therefore, the minimal polynomial is $m(x)=m_3(x)=(x-3)^3$.

(d) This case is also triangular, here upper triangular.

$$c(x) = |T-xI| = \begin{vmatrix} 2-x&0&1\\ 0&6-x&2\\ 0&0&2-x \end{vmatrix} = (2-x)^2(6-x) = -(x-2)^2(x-6)$$

There are two possibilities for the minimal polynomial, $m_1(x)=(x-2)(x-6)$ and $m_2(x)=(x-2)^2(x-6)$. Computation shows that the minimal polynomial isn't $m_1(x)$.

$$(T-2I)(T-6I) = \begin{pmatrix} 0&0&1\\ 0&4&2\\ 0&0&0 \end{pmatrix}\begin{pmatrix} -4&0&1\\ 0&0&2\\ 0&0&-4 \end{pmatrix} = \begin{pmatrix} 0&0&-4\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

It therefore must be that $m(x)=m_2(x)=(x-2)^2(x-6)$. Here is a verification.

$$(T-2I)^2(T-6I) = (T-2I)\cdot\bigl((T-2I)(T-6I)\bigr) = \begin{pmatrix} 0&0&1\\ 0&4&2\\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&-4\\ 0&0&0\\ 0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

(e) The characteristic polynomial is

$$c(x) = |T-xI| = \begin{vmatrix} 2-x&2&1\\ 0&6-x&2\\ 0&0&2-x \end{vmatrix} = (2-x)^2(6-x) = -(x-2)^2(x-6)$$

and there are two possibilities for the minimal polynomial, $m_1(x)=(x-2)(x-6)$ and $m_2(x)=(x-2)^2(x-6)$. Checking the first one

$$(T-2I)(T-6I) = \begin{pmatrix} 0&2&1\\ 0&4&2\\ 0&0&0 \end{pmatrix}\begin{pmatrix} -4&2&1\\ 0&0&2\\ 0&0&-4 \end{pmatrix} = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

shows that the minimal polynomial is $m(x)=m_1(x)=(x-2)(x-6)$.

(f) The characteristic polynomial is this.

$$c(x) = |T-xI| = \begin{vmatrix} -1-x&4&0&0&0\\ 0&3-x&0&0&0\\ 0&-4&-1-x&0&0\\ 3&-9&-4&2-x&-1\\ 1&5&4&1&4-x \end{vmatrix} = (x-3)^3(x+1)^2$$

There are a number of possibilities for the minimal polynomial, listed here by ascending degree: $m_1(x) = (x-3)(x+1)$, $m_1(x) = (x-3)^2(x+1)$, $m_1(x) = (x-3)(x+1)^2$, $m_1(x) = (x-3)^3(x+1)$, $m_1(x) = (x-3)^2(x+1)^2$, and $m_1(x) = (x-3)^3(x+1)^2$. The first one doesn't pan out

$$(T-3I)(T+1I) = \begin{pmatrix} -4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -4 & 0 & 0 \\ 3 & -9 & -4 & -1 & -1 \\ 1 & 5 & 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 3 & -9 & -4 & 3 & -1 \\ 1 & 5 & 4 & 1 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & -4 & -4 \\ 4 & 4 & 0 & 4 & 4 \end{pmatrix}$$

but the second one does.

$$(T-3I)^2(T+1I) = (T-3I)\big((T-3I)(T+1I)\big)$$
$$= \begin{pmatrix} -4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -4 & 0 & 0 \\ 3 & -9 & -4 & -1 & -1 \\ 1 & 5 & 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & -4 & -4 \\ 4 & 4 & 0 & 4 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The minimal polynomial is $m(x) = (x-3)^2(x+1)$.

Five.IV.1.16 We know that $\mathcal{P}_n$ is a dimension $n+1$ space and that the differentiation operator is nilpotent of index $n+1$ (for instance, taking $n = 3$, $\mathcal{P}_3 = \{c_3x^3 + c_2x^2 + c_1x + c_0 \mid c_3, \ldots, c_0 \in \mathbb{C}\}$ and the fourth derivative of a cubic is the zero polynomial). Represent this operator using the canonical form for nilpotent transformations.

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & & 0 \\ 1 & 0 & 0 & & & 0 \\ 0 & 1 & 0 & & & \\ & & & \ddots & & \\ 0 & 0 & 0 & & 1 & 0 \end{pmatrix}$$

This is an $(n+1)\times(n+1)$ matrix with an easy characteristic polynomial, $c(x) = x^{n+1}$. (*Remark:* this matrix is $\mathrm{Rep}_{B,B}(d/dx)$ where $B = \langle x^n, nx^{n-1}, n(n-1)x^{n-2}, \ldots, n!\rangle$.) To find the minimal polynomial as in Example 1.12 we consider the powers of $T - 0I = T$. But, of course, the first power of T that is the zero matrix is the power $n+1$. So the minimal polynomial is also x^{n+1}.

Five.IV.1.17 Call the matrix T and suppose that it is $n\times n$. Because T is triangular, and so $T - xI$ is triangular, the characteristic polynomial is $c(x) = (x-\lambda)^n$. To see that the minimal polynomial is the same, consider $T - \lambda I$.

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & & \\ & & \ddots & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

Recognize it as the canonical form for a transformation that is nilpotent of degree n; the power $(T-\lambda I)^j$ is zero first when j is n.

Five.IV.1.23 By the Cayley-Hamilton theorem the degree of the minimal polynomial is less than or equal to the degree of the characteristic polynomial, n. Example 1.6 shows that n can happen.

Five.IV.1.24 Suppose that t's only eigenvalue is zero. Then the characteristic polynomial of t is x^n. Because t satisfies its characteristic polynomial, it is a nilpotent map.

Five.IV.1.26 The polynomial can be read geometrically to say "a 60° rotation minus two rotations of 30° equals the identity."

Five.IV.1.28 This subsection starts with the observation that the powers of a linear transformation cannot climb forever without a "repeat", that is, that for some power n there is a linear relationship $c_n \cdot t^n + \cdots + c_1 \cdot t + c_0 \cdot \mathrm{id} = z$ where z is the zero transformation. The definition of projection is that for such a map one linear relationship is quadratic, $t^2 - t = z$. To finish, we need only consider whether this relationship might not be minimal, that is, are there projections for which the minimal polynomial is constant or linear?

For the minimal polynomial to be constant, the map would have to satisfy that $c_0 \cdot \mathrm{id} = z$, where $c_0 = 1$ since the leading coefficient of a minimal polynomial is 1. This is only satisfied by the zero transformation on a trivial space. This is indeed a projection, but not a very interesting one.

For the minimal polynomial of a transformation to be linear would give $c_1 \cdot t + c_0 \cdot \mathrm{id} = z$ where $c_1 = 1$. This equation gives $t = -c_0 \cdot \mathrm{id}$. Coupling it with the requirement that $t^2 = t$ gives $t^2 = (-c_0)^2 \cdot \mathrm{id} = -c_0 \cdot \mathrm{id}$, which gives that $c_0 = 0$ and t is the zero transformation or that $c_0 = 1$ and t is the identity.

Thus, except in the cases where the projection is a zero map or an identity map, the minimal polynomial is $m(x) = x^2 - x$.

Five.IV.1.33 **(a)** For the inductive step, assume that Lemma 1.7 is true for polynomials of degree $i, \ldots, k-1$ and consider a polynomial $f(x)$ of degree k. Factor $f(x) = k(x-\lambda_1)^{q_1} \cdots (x-\lambda_\ell)^{q_\ell}$ and let $k(x-\lambda_1)^{q_1-1} \cdots (x-\lambda_\ell)^{q_\ell}$ be $c_{n-1}x^{n-1} + \cdots + c_1x + c_0$. Substitute:

$$\begin{aligned}
k(t-\lambda_1)^{q_1} \circ \cdots \circ (t-\lambda_\ell)^{q_\ell}(\vec{v}) &= (t-\lambda_1) \circ (t-\lambda_1)^{q_1} \circ \cdots \circ (t-\lambda_\ell)^{q_\ell}(\vec{v}) \\
&= (t-\lambda_1)\,(c_{n-1}t^{n-1}(\vec{v}) + \cdots + c_0\vec{v}) \\
&= f(t)(\vec{v})
\end{aligned}$$

(the second equality follows from the inductive hypothesis and the third from the linearity of t).

(b) One example is to consider the squaring map $s\colon \mathbb{R} \to \mathbb{R}$ given by $s(x) = x^2$. It is nonlinear. The action defined by the polynomial $f(t) = t^2 - 1$ changes s to $f(s) = s^2 - 1$, which is this map.

$$x \stackrel{s^2-1}{\longmapsto} s \circ s(x) - 1 = x^4 - 1$$

Observe that this map differs from the map $(s-1) \circ (s+1)$; for instance, the first map takes $x = 5$ to 624 while the second one takes $x = 5$ to 675.

Subsection Five.IV.2: Jordan Canonical Form

Five.IV.2.19 **(a)** The transformation $t - 3$ is nilpotent (that is, $\mathscr{N}_\infty(t-3)$ is the entire space) and it acts on a string basis via two strings, $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{\beta}_4 \mapsto \vec{0}$ and $\vec{\beta}_5 \mapsto \vec{0}$. Consequently, $t - 3$ can be represented in this canonical form.

$$N_3 = \begin{pmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \end{pmatrix}$$

and therefore T is similar to this this canonical form matrix.

$$J_3 = N_3 + 3I = \begin{pmatrix} 3&0&0&0&0 \\ 1&3&0&0&0 \\ 0&1&3&0&0 \\ 0&0&1&3&0 \\ 0&0&0&0&3 \end{pmatrix}$$

(b) The restriction of the transformation $s + 1$ is nilpotent on the subspace $\mathscr{N}_\infty(s+1)$, and the action on a string basis is given as $\vec{\beta}_1 \mapsto \vec{0}$. The restriction of the transformation $s - 2$ is nilpotent on the subspace $\mathscr{N}_\infty(s-2)$, having the action on a string basis of $\vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$ and $\vec{\beta}_4 \mapsto \vec{\beta}_5 \mapsto \vec{0}$. Consequently the Jordan form is this

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

(note that the blocks are arranged with the least eigenvalue first).

Five.IV.2.21 The general procedure is to factor the characteristic polynomial $c(x) = (x-\lambda_1)^{p_1}(x-\lambda_2)^{p_2}\cdots$ to get the eigenvalues λ_1, λ_2, etc. Then, for each λ_i we find a string basis for the action of the transformation $t-\lambda_i$ when restricted to $\mathscr{N}_\infty(t-\lambda_i)$, by computing the powers of the matrix $T-\lambda_i I$ and finding the associated null spaces, until these null spaces settle down (do not change), at which point we have the generalized null space. The dimensions of those null spaces (the nullities) tell us the action of $t-\lambda_i$ on a string basis for the generalized null space, and so we can write the pattern of subdiagonal ones to have N_{λ_i}. From this matrix, the Jordan block J_{λ_i} associated with λ_i is immediate $J_{\lambda_i} = N_{\lambda_i} + \lambda_i I$. Finally, after we have done this for each eigenvalue, we put them together into the canonical form.

(a) The characteristic polynomial of this matrix is $c(x) = (-10-x)(10-x)+100 = x^2$, so it has only the single eigenvalue $\lambda = 0$.

power p	$(T+0\cdot I)^p$	$\mathscr{N}((t-0)^p)$	*nullity*
1	$\begin{pmatrix} -10 & 4 \\ -25 & 10 \end{pmatrix}$	$\{\begin{pmatrix} 2y/5 \\ y \end{pmatrix} \mid y\in\mathbb{C}\}$	1
2	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathbb{C}^2$	2

(Thus, this transformation is nilpotent: $\mathscr{N}_\infty(t-0)$ is the entire space). From the nullities we know that t's action on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$. This is the canonical form matrix for the action of $t-0$ on $\mathscr{N}_\infty(t-0) = \mathbb{C}^2$

$$N_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and this is the Jordan form of the matrix.

$$J_0 = N_0 + 0\cdot I = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Note that if a matrix is nilpotent then its canonical form equals its Jordan form.

We can find such a string basis using the techniques of the prior section.

$$B = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -10 \\ -25 \end{pmatrix} \rangle$$

The first basis vector has been taken so that it is in the null space of t^2 but is not in the null space of t. The second basis vector is the image of the first under t.

(b) The characteristic polynomial of this matrix is $c(x) = (x+1)^2$, so it is a single-eigenvalue matrix. (That is, the generalized null space of $t+1$ is the entire space.) We have

$$\mathscr{N}(t+1) = \{\begin{pmatrix} 2y/3 \\ y \end{pmatrix} \mid y\in\mathbb{C}\} \qquad \mathscr{N}((t+1)^2) = \mathbb{C}^2$$

and so the action of $t+1$ on an associated string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$. Thus,

$$N_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the Jordan form of T is

$$J_{-1} = N_{-1} + -1\cdot I = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

and choosing vectors from the above null spaces gives this string basis (many other choices are possible).

$$B = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 9 \end{pmatrix} \rangle$$

(c) The characteristic polynomial $c(x) = (1-x)(4-x)^2 = -1\cdot(x-1)(x-4)^2$ has two roots and they are the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

We handle the two eigenvalues separately. For λ_1, the calculation of the powers of $T-1I$ yields

$$\mathscr{N}(t-1) = \{\begin{pmatrix}0\\y\\0\end{pmatrix} \mid y \in \mathbb{C}\}$$

and the null space of $(t-1)^2$ is the same. Thus this set is the generalized null space $\mathscr{N}_\infty(t-1)$. The nullities show that the action of the restriction of $t-1$ to the generalized null space on a string basis is $\vec{\beta}_1 \mapsto \vec{0}$.

A similar calculation for $\lambda_2 = 4$ gives these null spaces.

$$\mathscr{N}(t-4) = \{\begin{pmatrix}0\\z\\z\end{pmatrix} \mid z \in \mathbb{C}\} \qquad \mathscr{N}((t-4)^2) = \{\begin{pmatrix}y-z\\y\\z\end{pmatrix} \mid y,z \in \mathbb{C}\}$$

(The null space of $(t-4)^3$ is the same, as it must be because the power of the term associated with $\lambda_2 = 4$ in the characteristic polynomial is two, and so the restriction of $t-2$ to the generalized null space $\mathscr{N}_\infty(t-2)$ is nilpotent of index at most two — it takes at most two applications of $t-2$ for the null space to settle down.) The pattern of how the nullities rise tells us that the action of $t-4$ on an associated string basis for $\mathscr{N}_\infty(t-4)$ is $\vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$.

Putting the information for the two eigenvalues together gives the Jordan form of the transformation t.

$$\begin{pmatrix}1&0&0\\0&4&0\\0&1&4\end{pmatrix}$$

We can take elements of the nullspaces to get an appropriate basis.

$$B = B_1 ^\frown B_4 = \langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\5\\5\end{pmatrix} \rangle$$

(d) The characteristic polynomial is $c(x) = (-2-x)(4-x)^2 = -1\cdot(x+2)(x-4)^2$.

For the eigenvalue λ_{-2}, calculation of the powers of $T+2I$ yields this.

$$\mathscr{N}(t+2) = \{\begin{pmatrix}z\\z\\z\end{pmatrix} \mid z \in \mathbb{C}\}$$

The null space of $(t+2)^2$ is the same, and so this is the generalized null space $\mathscr{N}_\infty(t+2)$. Thus the action of the restriction of $t+2$ to $\mathscr{N}_\infty(t+2)$ on an associated string basis is $\vec{\beta}_1 \mapsto \vec{0}$.

For $\lambda_2 = 4$, computing the powers of $T-4I$ yields

$$\mathscr{N}(t-4) = \{\begin{pmatrix}z\\-z\\z\end{pmatrix} \mid z \in \mathbb{C}\} \qquad \mathscr{N}((t-4)^2) = \{\begin{pmatrix}x\\-z\\z\end{pmatrix} \mid x,z \in \mathbb{C}\}$$

and so the action of $t-4$ on a string basis for $\mathscr{N}_\infty(t-4)$ is $\vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$.

Therefore the Jordan form is

$$\begin{pmatrix}-2&0&0\\0&4&0\\0&1&4\end{pmatrix}$$

and a suitable basis is this.

$$B = B_{-2} ^\frown B_4 = \langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix} \rangle$$

(e) The characteristic polynomial of this matrix is $c(x) = (2-x)^3 = -1\cdot(x-2)^3$. This matrix has only a single eigenvalue, $\lambda = 2$. By finding the powers of $T-2I$ we have

$$\mathscr{N}(t-2) = \{\begin{pmatrix}-y\\y\\0\end{pmatrix} \mid y \in \mathbb{C}\} \qquad \mathscr{N}((t-2)^2) = \{\begin{pmatrix}-y-(1/2)z\\y\\z\end{pmatrix} \mid y,z \in \mathbb{C}\} \qquad \mathscr{N}((t-2)^3) = \mathbb{C}^3$$

and so the action of $t-2$ on an associated string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$. The Jordan form is this

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

and one choice of basis is this.

$$B = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -9 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \rangle$$

(f) The characteristic polynomial $c(x) = (1-x)^3 = -(x-1)^3$ has only a single root, so the matrix has only a single eigenvalue $\lambda = 1$. Finding the powers of $T - 1I$ and calculating the null spaces

$$\mathscr{N}(t-1) = \{ \begin{pmatrix} -2y+z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{C} \} \qquad \mathscr{N}((t-1)^2) = \mathbb{C}^3$$

shows that the action of the nilpotent map $t-1$ on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$ and $\vec{\beta}_3 \mapsto \vec{0}$. Therefore the Jordan form is

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and an appropriate basis (a string basis associated with $t-1$) is this.

$$B = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

(g) The characteristic polynomial is a bit large for by-hand calculation, but just manageable $c(x) = x^4 - 24x^3 + 216x^2 - 864x + 1296 = (x-6)^4$. This is a single-eigenvalue map, so the transformation $t-6$ is nilpotent. The null spaces

$$\mathscr{N}(t-6) = \{ \begin{pmatrix} -z-w \\ -z-w \\ z \\ w \end{pmatrix} \mid z, w \in \mathbb{C} \} \quad \mathscr{N}((t-6)^2) = \{ \begin{pmatrix} x \\ -z-w \\ z \\ w \end{pmatrix} \mid x, z, w \in \mathbb{C} \} \quad \mathscr{N}((t-6)^3) = \mathbb{C}^4$$

and the nullities show that the action of $t-6$ on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$ and $\vec{\beta}_4 \mapsto \vec{0}$. The Jordan form is

$$\begin{pmatrix} 6 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

and finding a suitable string basis is routine.

$$B = \langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -6 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \rangle$$

Five.IV.2.22 There are two eigenvalues, $\lambda_1 = -2$ and $\lambda_2 = 1$. The restriction of $t+2$ to $\mathscr{N}_\infty(t+2)$ could have either of these actions on an associated string basis.

$$\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} \qquad \begin{matrix} \vec{\beta}_1 \mapsto \vec{0} \\ \vec{\beta}_2 \mapsto \vec{0} \end{matrix}$$

The restriction of $t-1$ to $\mathscr{N}_\infty(t-1)$ could have either of these actions on an associated string basis.

$$\vec{\beta}_3 \mapsto \vec{\beta}_4 \mapsto \vec{0} \qquad \begin{matrix} \vec{\beta}_3 \mapsto \vec{0} \\ \vec{\beta}_4 \mapsto \vec{0} \end{matrix}$$

In combination, that makes four possible Jordan forms, the two first actions, the second and first, the first and second, and the two second actions.

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Five.IV.2.24 The action of $t+1$ on a string basis for $\mathscr{N}_\infty(t+1)$ must be $\vec{\beta}_1 \mapsto \vec{0}$. Because of the power of $x-2$ in the minimal polynomial, a string basis for $t-2$ has length two and so the action of $t-2$ on $\mathscr{N}_\infty(t-2)$ must be of this form.

$$\begin{aligned}&\vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}\\ &\vec{\beta}_4 \mapsto \vec{0}\end{aligned}$$

Therefore there is only one Jordan form that is possible.

$$\begin{pmatrix}-1&0&0&0\\0&2&0&0\\0&1&2&0\\0&0&0&2\end{pmatrix}$$

Five.IV.2.26 **(a)** The characteristic polynomial is $c(x)=x(x-1)$. For $\lambda_1=0$ we have

$$\mathscr{N}(t-0)=\{\begin{pmatrix}-y\\y\end{pmatrix} \mid y\in\mathbb{C}\}$$

(of course, the null space of t^2 is the same). For $\lambda_2=1$,

$$\mathscr{N}(t-1)=\{\begin{pmatrix}x\\0\end{pmatrix} \mid x\in\mathbb{C}\}$$

(and the null space of $(t-1)^2$ is the same). We can take this basis

$$B=\langle\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\rangle$$

to get the diagonalization.

$$\begin{pmatrix}1&1\\-1&0\end{pmatrix}^{-1}\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\-1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

(b) The characteristic polynomial is $c(x)=x^2-1=(x+1)(x-1)$. For $\lambda_1=-1$,

$$\mathscr{N}(t+1)=\{\begin{pmatrix}-y\\y\end{pmatrix} \mid y\in\mathbb{C}\}$$

and the null space of $(t+1)^2$ is the same. For $\lambda_2=1$

$$\mathscr{N}(t-1)=\{\begin{pmatrix}y\\y\end{pmatrix} \mid y\in\mathbb{C}\}$$

and the null space of $(t-1)^2$ is the same. We can take this basis

$$B=\langle\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\rangle$$

to get a diagonalization.

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

Five.IV.2.27 The transformation $d/dx\colon\mathcal{P}_3\to\mathcal{P}_3$ is nilpotent. Its action on $B=\langle x^3,3x^2,6x,6\rangle$ is $x^3\mapsto 3x^2\mapsto 6x\mapsto 6\mapsto 0$. Its Jordan form is its canonical form as a nilpotent matrix.

$$J=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

Five.IV.2.28 Yes. Each has the characteristic polynomial $(x+1)^2$. Calculations of the powers of $T_1+1\cdot I$ and $T_2+1\cdot I$ gives these two.

$$\mathscr{N}(t_1+1)=\{\begin{pmatrix}y/2\\y\end{pmatrix} \mid y\in\mathbb{C}\}\qquad \mathscr{N}(t_2+1)=\{\begin{pmatrix}0\\y\end{pmatrix} \mid y\in\mathbb{C}\}$$

(Of course, for each the null space of the square is the entire space.) The way that the nullities rise shows that each is similar to this Jordan form matrix

$$\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$$

and they are therefore similar to each other.

Five.IV.2.31 Jordan form is unique. A diagonal matrix is in Jordan form. Thus the Jordan form of a diagonalizable matrix is its diagonalization. If the minimal polynomial has factors to some power higher than one then the Jordan form has subdiagonal 1's, and so is not diagonal.

Five.IV.2.37 Yes, the intersection of t invariant subspaces is t invariant. Assume that M and N are t invariant. If $\vec{v} \in M \cap N$ then $t(\vec{v}) \in M$ by the invariance of M and $t(\vec{v}) \in N$ by the invariance of N.

Of course, the union of two subspaces need not be a subspace (remember that the x- and y-axes are subspaces of the plane $\mathbb{R}^2$ but the union of the two axes fails to be closed under vector addition, for instance it does not contain $\vec{e}_1+\vec{e}_2$.) However, the union of invariant subsets is an invariant subset; if $\vec{v} \in M \cup N$ then $\vec{v} \in M$ or $\vec{v} \in N$ so $t(\vec{v}) \in M$ or $t(\vec{v}) \in N$.

No, the complement of an invariant subspace need not be invariant. Consider the subspace

$$\{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{C}\}$$

of $\mathbb{C}^2$ under the zero transformation.

Yes, the sum of two invariant subspaces is invariant. The check is easy.

Topic: Method of Powers

Topic: Stable Populations

Topic: Linear Recurrences