Agoh–Giuga conjecture

In number theory the **Agoh–Giuga conjecture** on the Bernoulli numbers B_k postulates that p is a prime number if and only if

$$pB_{p-1} \equiv -1 \pmod{p}.$$

It is named after Takashi Agoh and Giuseppe Giuga.

Equivalent formulation

The conjecture as stated above is due to Takashi Agoh (1990); an equivalent formulation is due to Giuseppe Giuga, from 1950, to the effect that p is prime if and only if

$$1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \pmod{p}$$

which may also be written as

$$\sum_{i=1}^{p-1} i^{p-1} \equiv -1 \pmod{p}.$$

It is trivial to show that p being prime is sufficient for the second equivalence to hold, since if p is prime, Fermat's little theorem states that

$$a^{p-1} \equiv 1 \pmod{p}$$

for $a = 1, 2, \dots, p-1$, and the equivalence follows, since $p - 1 \equiv -1 \pmod{p}$.

Status

The statement is still a conjecture since it has not yet been proven that if a number n is not prime (that is, n is composite), then the formula does not hold. It has been shown that a composite number n satisfies the formula if and only if it is both a Carmichael number and a Giuga number, and that if such a number exists, it has at least 13,800 digits (Borwein, Borwein, Borwein, Girgensohn 1996). Laerte Sorini, finally, in a work of 2001 showed that a possible counterexample should be a number n greater than 10^{36067} which represents the limit suggested by Bedocchi for the demonstration technique specified by Giuga to his own conjecture.

Relation to Wilson's theorem

The Agoh–Giuga conjecture bears a similarity to Wilson's theorem, which has been proven to be true. Wilson's theorem states that a number p is prime if and only if

$$(p-1)! \equiv -1 \pmod{p},$$

which may also be written as

$$\prod_{i=1}^{p-1} i \equiv -1 \pmod{p}.$$

For an odd prime p we have

$$\prod_{i=1}^{p-1} i^{p-1} \equiv (-1)^{p-1} \equiv 1 \pmod{p},$$

and for p=2 we have

$$\prod_{i=1}^{p-1} i^{p-1} \equiv (-1)^{p-1} \equiv 1 \pmod{p}.$$

So, the truth of the Agoh–Giuga conjecture combined with Wilson's theorem would give: a number *p* is prime if and only if

$$\sum_{i=1}^{p-1} i^{p-1} \equiv -1 \pmod{p}$$

and

$$\prod_{i=1}^{p-1} i^{p-1} \equiv 1 \pmod{p}.$$

See also

- Bernoulli number § Arithmetical properties of the Bernoulli numbers

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