THE ASYMPTOTIC IN WARING'S PROBLEM OVER FUNCTION FIELDS VIA SINGULAR SETS IN THE CIRCLE METHOD

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Abstract. We give results on the asymptotic in Waring's problem over function fields that are stronger than the results obtained over the integers using the main conjecture in Vinogradov's mean value theorem. Similar estimates apply to Manin's conjecture for Fermat hypersurfaces over function fields. Following an idea of Pugin, rather than applying analytic methods to estimate the minor arcs, we treat them as complete exponential sums over finite fields and apply results of Katz, which bound the sum in terms of the dimension of a certain singular locus, which we estimate by tangent space calculations.

1. Introduction

The asymptotic in Waring's problem is the formula

$$\#\{\mathbf{a} \in (\mathbb{N})^s \mid \sum_{i=1}^s a_i^k = n\} = (1+o(1)) n^{s/k-1} \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \prod_p \lim_{r\to\infty} \frac{\#\{\mathbf{b} \in (\mathbb{Z}/p^r)^s \mid \sum_{i=1}^s b_i^k \equiv n \bmod p^r\}}{p^{r(s-1)}}$$

where the boldface variables $\mathbf{a}$ represent vectors $a_1, \dots, a_s$ and the term $\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}$ is the local factor at ∞. Here $o(1)$ should go to 0 as n goes to ∞ with fixed s, k.

This was obtained by Hardy and Littlewood for s sufficiently large with respect to k. Further research on the asymptotic has focused on proving it for s as small as possible in terms of k, with work of Vaughan and Wooley, culminating in the result of Bourgain [1, Theorem 11] proving the asymptotic for $s \geq k^2 + k - \max_{t \leq k} \lceil t \frac{k-t-1}{k-t+1} \rceil$, using the breakthrough work of Bourgain, Demeter, and Guth [2] proving the main conjecture of Vinogradov's mean value theorem. (More precisely, [1, Theorem 11] only gives the best lower bound for $k > 12$. For $4 \leq k \leq 12$, the best bound is given by a different formula and obtained from a conditional result of Wooley [10, Theorem 4.1] rendered unconditional by [2]. For $k = 3$ the result is known for $s \geq 8$ due to earlier work of Vaughan [9].)

We now describe an analogous problem for polynomials over finite fields. Let $\mathbb{F}_q$ be a finite field and $\mathbb{F}_q[T]$ the ring of polynomials in one variable over $\mathbb{F}_q$. A *prime* in $\mathbb{F}_q[T]$ is a monic irreducible polynomial. In this case, there is no good analogue of the positivity of the a_i that ensures the equation $\sum_{i=1}^s a_i^k = n$ has finitely many solutions. Instead, it is most natural to restrict attention to polynomials a_i of degree at most e. This gives the following problem.

Problem 1.1. Fix a finite field $\mathbb{F}_q$ and a positive integer k. Obtain, for s as small as possible, for all natural numbers e and polynomials $f \in \mathbb{F}_q[T]$ of degree $\leq ke$, the estimate

$$(1) \qquad \#\{\mathbf{a} \in (\mathbb{F}_q[T])^s \mid \deg a_i \leq e, \sum_{i=1}^{s} a_i^k = f\} = (1+o(1))q^{e(s-k)+s-1}\ell_\infty(f) \prod_{\substack{\pi \in \mathbb{F}_q[T] \\ \text{prime}}} \ell_\pi(f)$$

where

$$\ell_\infty(f) = \lim_{r \to \infty} \frac{\#\{\mathbf{b} \in (\mathbb{F}_q[u]/u^r)^s \mid \sum_{i=1}^{s} b_i^k \equiv u^{ke} f(u^{-1}) \bmod u^r\}}{q^{r(s-1)}}$$

$$\ell_\pi(f) = \lim_{r \to \infty} \frac{\#\{\mathbf{b} \in (\mathbb{F}_q[T]/\pi^r)^s \mid \sum_{i=1}^{s} b_i^k \equiv f \bmod \pi^r\}}{q^{r(s-1)\deg \pi}}$$

and $o(1)$ goes to 0 as e goes to ∞ for fixed q, s, k.

Here the term ℓ_∞ is the local factor at ∞ and ℓ_π is the local factor at the prime π. In comparing these bounds, note that the expression $q^{\deg \pi}$ is comparable to p and q^e is comparable to $n^{1/k}$ so $q^{e(s-k)}$ is comparable to $n^{s/k-1}$. The q^{s-1} factor could be folded into the local factor at ∞ if desired.

Prior work [7, 12] has often considered the case where e is "as small as possible", i.e. $e = \lceil \frac{\deg f}{k} \rceil$ or $e = \frac{\deg f}{k} + 1$ in certain exceptional cases. However, here we do not make that restriction. Removing the restriction matters little, except for the fact that it allows a direct application to Manin's conjecture, because shrinking e tends to make the problem more difficult.

Let p be the characteristic of $\mathbb{F}_q$.

Yamagishi [12, Theorem 1.4] obtained (1) in the case $k < p$ when $s \geq 2k^2 - 2\lfloor (\log k)/\log(2) \rfloor$, as well as $s \geq 86$ when $k = 7$ and $s \geq 2k^2 - 11$ when $k \geq 8$, generalizing earlier work of Kubota [6]. Yamagishi [12, Theorem 1.3] also obtained more complex bounds in the $k > p$ case. These bounds were obtained before [2] and could likely be improved using the function field analogue [11, Corollary 17.3] but this does not seem to have been done in the literature.

We obtain (1) with an improved (linear) dependence of s on k, at the cost of assuming that both q and p are somewhat large. (Note that [12, Theorem 1.3] gives linear dependence of s on k in the special case where k is a power of p plus 1, by taking advantage of the exceptional behavior of pth powers in characteristic p which makes the setting fundamentally different from the $p > k$ one.)

To describe the conditions under which we can obtain (1), we first need to define a polygon: Let $\Delta_{k,p}$ be the convex hull of the set of points $(i,j) \in \mathbb{N}^2$ such that $\gcd(i,j,p) = 1$ but $p \mid (k-1)i - j$. For $\gamma \in \mathbb{R}$, let $\gamma\Delta_{k,p}$ be the dilation $\{\gamma v \mid v \in \Delta_{k,p}\}$ of $\Delta_{k,p}$ by γ. Let $\gamma_{k,p}$ be the maximum value of γ such that $(1, \frac{k-2}{2}) \in \gamma\Delta_{k,p}$.

Theorem 1.2. *Fix a finite field $\mathbb{F}_q$ of characteristic p and positive integers k and s such that $2 \leq k < p$. For all natural numbers e and polynomials $f \in \mathbb{F}_q[T]$ of degree $\leq ke$, we have (1) as soon as*

$$q > (d+1)^{2(k-1)}$$

and

$$s > \max\left(\frac{2(k-\gamma_{k,p}-2\frac{\log(k+1)}{\log q})}{1-\gamma_{k,p}-2\frac{\log(k+1)}{\log q}}, \frac{2k}{1-2(k-1)\frac{\log(k+1)}{\log q}}\right).$$

For q sufficiently large, it suffices to have (taking $\lim_{q\to\infty}$ in the lower bound for s)

$$s > \max\left(\frac{2(k-\gamma_{k,p})}{1-\gamma_{k,p}}, 2k\right) = \frac{2(k-\gamma_{k,p})}{1-\gamma_{k,p}}$$

since $\gamma_{k,p} \in (0,1)$ so $\frac{k-\gamma_{k,p}}{1-\gamma_{k,p}} > k$.

We have (by Lemma 3.9 below)

$$\gamma_{k,p} \le \frac{k-2}{2k-2}\left(1+\frac{k}{p}\right)$$

so we obtain (1) when q is sufficiently large (depending on s,k) and

$$s > \frac{2(k(2k-2)p-(k-2)(k+p))}{(2k-2)p-(k-2)(k+p)} = \frac{2(2k^2-3k+2)p-2k(k-2)}{k(p+2-k)}$$

which for p sufficiently large (depending on s,k) simplifies to $s > 4k-6+\frac{4}{k}$ which since s is an integer is equivalent to $s \ge 8$ for $k=3$, $s \ge 12$ for $k=4$, and $s \ge 4k-5$ for all higher k. For $k > 3$ this improves on the best known bounds in in the integer setting, e.g. for $k=4$ [10, Theorem 4.1] requires $s \ge 15$ but we require $s \ge 12$, while for $k=5$ [10, Theorem 4.1] requires $s \ge 23$ but we require $s \ge 15$.

In fact, our result gives a power saving, but because the exact power saving has a complicated dependence on s,k,q,p we defer its statement to Theorem 3.16.

We also have an analogue for a general function field. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. Denote the set of closed points of C, or, equivalently, places of the function field $\mathbb{F}_q(C)$, by $|C|$. For v a closed point of C, let $\mathcal{O}_{C_v}$ be the corresponding local ring with uniformizer π_v.

Problem 1.3. Fix a finite field $\mathbb{F}_q$ and a positive integer k. Obtain, for s as small as possible, for all natural numbers e, line bundles L on C of degree e, and sections $f \in H^0(C, L^k)$, the estimate

$$\#\{\mathbf{a} \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^k = f\} = (1+o(1))q^{e(s-k)+(s-1)(1-g)} \prod_{v\in|C|} \ell_v(f) \tag{2}$$

where

$$\ell_v(f) = \lim_{r\to\infty} \frac{\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^s \mid \sum_{i=1}^s b_i^k \equiv f \bmod \pi^r\}}{q^{r(s-1)\deg v}}$$

and f is interpreted as an element of $\mathcal{O}_{C_v}$ by dividing by the kth power of an arbitrary local generator of L, and $o(1)$ goes to 0 as e goes to ∞ for fixed q,s,k,C.

Here the local generator of L at v is well-defined up to multiplication by a unit in $\mathcal{O}_{C_v}$, so dividing f by its kth power leaves f well-defined up to multiplication by the kth power of a unit, which does not change the count $\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^s \mid \sum_{i=1}^s b_i^k \equiv f \bmod \pi^r\}$.

Theorem 1.4. *Fix a finite field $\mathbb{F}_q$ of characteristic p and positive integers k and s such that $2 \leq k < p$. For all natural numbers e and polynomials $f \in \mathbb{F}_q[T]$ of degree $\leq ke$, we have (2) as soon as*

$$q > (d+1)^{2(k-1)}$$

and

$$s > \max\left(\frac{2(k-\gamma_{k,p} - 2\frac{\log(k+1)}{\log q})}{1-\gamma_{k,p} - 2\frac{\log(k+1)}{\log q}}, \frac{2k}{1-2(k-1)\frac{\log(k+1)}{\log q}} \right).$$

Since the conditions for an asymptotic are identical to the case of Theorem 1.2, the same considerations apply as to when the asymptotic holds for q or p sufficiently large.

Theorem 1.2 is obtained by setting $C = \mathbb{P}^1$. In this case, the unique line bundle of degree e is $\mathcal{O}(e)$, whose sections are the polynomials in T of degree $\leq e$. Correspondingly, the sections of $L^k = \mathcal{O}(ke)$ are the polynomials of degree $\leq ke$. The closed points of $\mathbb{P}^1$ consist of the closed points of $\mathbb{A}^1$, which are in one-to-one correspondence with the primes $\pi \in \mathbb{F}_q[T]$, together with the point at ∞. The local ring at π is $\mathbb{F}_q[T]_{(\pi)}$, with uniformizer π, and we have $\mathbb{F}_q[T]_{(\pi)}/\pi^r = \mathbb{F}_q[T]/\pi^r$. The local ring at ∞ is $\mathbb{F}_q[u]_{(u)}$ where $u = T^{-1}$ so $T = u^{-1}$. A local generator of L at ∞ is given by $T^e = u^{-e}$ and dividing f by the kth power of this local generator is equivalent to multiplying by u^{ke}.

The application to Manin's conjecture for Fermat hypersurfaces is described by the following theorem.

Theorem 1.5. *Fix a finite field $\mathbb{F}_q$ of characteristic p and positive integers n and d such that $2 \leq d < p$. Let X be the hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the equation $\sum_{i=0}^n x_i^d = 0$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. Assume that*

$$q > (d+1)^{2(d-1)}$$

and

$$n+1 > \max\left(\frac{2(d-\gamma_{d,p} - 2\frac{\log(d+1)}{\log q})}{1-\gamma_{d,p} - 2\frac{\log(d+1)}{\log q}}, \frac{2d}{1-2(d-1)\frac{\log(d+1)}{\log q}} \right).$$

Then

$$\{f \colon C \to X \mid \textit{degree } e\} = (1+o(1))\frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\prod_{v \in |C|}\left((1-q^{-\deg v})\frac{\#X(\mathbb{F}_{q^{\deg v}})}{q^{(n-1)\deg v}} \right).$$

1.1. **Outline of proof.** Our method of proof begins with the standard approach via the circle method, obtaining a sum of exponential sums, and dividing these into major arcs and minor arcs. For the major arcs, our treatment is similar to prior work in the circle method. For the minor arcs, we take a different approach. Instead of Weyl differencing, efficient congruencing, or any other analytic method, we treat these sums geometrically. The sums are over s-tuples of polynomials of degree $\leq e$ in $\mathbb{F}_q[T]$. One can view the set of polynomials of degree $\leq e$ as an $\mathbb{F}_q[T]$-analgoue of an interval around 0, and therefore the sums as $\mathbb{F}_q[T]$-analogues of short exponential sums. However, this set of polynomials is also a vector space of dimension $e+1$ over $\mathbb{F}_q$, so the sums are over a $s(e+1)$-dimensional vector space over $\mathbb{F}_q$, and can also be viewed as complete exponential sums associated to a polynomial function on this vector space. We then apply a classical estimate of Katz for complete exponential sums in high dimension in terms of the dimension of the singular locus of the vanishing set of this polynomial (more precisely, the vanishing set of its leading term, but the polynomial is homogeneous and thus equal to its leading term). The lower the dimension of the singular locus, the stronger this estimate is.

An identical approach up to this point was taken by Pugin in the closely related problem of counting $\mathbb{F}_q(T)$-points on the Fermat hypersurface. However, the step where the dimension is calculated [8, Lemma 2.5.1] contains a gap: It is claimed that the dimension of the inverse image under the diagonal morphism $\mathbb{A}^{r+1} \to (\mathbb{A}^{r+1})^{d-1}$ of a closed set is at most $\frac{1}{d-1}$ times the dimension of the closed set, which is not true in general, for example if the closed set is itself the diagonal. (This is true if the closed set is a product of $d-1$ closed subsets of $\mathbb{A}^{r+1}$ but that condition does not hold here.)

Instead, we describe the singular locus explicitly as a set of polynomials a whose $k-1$st power is congruent modulo a certain modulus (associated to a closed set Z) to a constant times a low-degree polynomial c. To bound the dimension of this set, we first stratify by the joint root multiplicities of a and c. We estimate the dimension of the intersection of the singular locus with any stratum by bounding the dimension of its tangent space at an arbitrary point. The reason this leads to a good estimate is that, when describing the tangent space to the solution set of any equation, we consider solutions to the derivative of the equation. We may as well take the logarithmic derivative, and doing this simplifies the equation greatly: The $k-1$st power becomes a multiplication by $k-1$. Bounding the dimension of the space of solutions to this simplified equation is considerably easier. The greatest difficulty occurs in certain cases where the polynomial has repeated roots, which the stratification assists us in handling.

The method does not currently apply to Manin's conjecture for hypersurfaces other than the Fermat hypersurface or to representability by general homogeneous polynomials. In these settings, we can describe the singular locus explicitly, but it is not clear how to bound its dimension. In the final section, we give an approach to bound the dimension of the singular locus in terms of dimensions of moduli spaces of curves on a certain blowup of projective space.

Let X be a smooth hypersurface in $\mathbb{P}^n$ defined by a polynomial F of degree d coprime to the characteristic of the base field. Let ∇F be the tuple of polynomials $\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}$. Since X is smooth and the degree d is coprime to the characteristic, the tuple of polynomials ∇F has no common zeroes except 0. Since ∇F is an $n+1$-tuple of polynomials in $n+1$ variables with no common zeroes except 0, ∇F defines a map $\mathbb{P}^n \to \mathbb{P}^n$. Let Y be the blowup of $\mathbb{P}^n \times \mathbb{P}^n$ along the graph of this map. Let E be the exceptional divisor of this blowup. Since the graph of ∇F is a smooth codimension n subset isomorphic to $\mathbb{P}^n$, E is isomorphic to a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^n$.

We give in Proposition 5.1 below a bound for the dimension of the relevant singular locus (which we describe in more detail in §5) in terms of the dimensions of moduli spaces of maps $C \to Y$ whose image is not entirely contained in the exceptional divisor E. This gives a case where upper bounds on the dimension of the moduli space of maps to one variety (equivalently, upper bounds on the number of $\mathbb{F}_q(C)$-rational points of the variety that are uniform in q) can give precise asymptotics for the number of $\mathbb{F}_q(C)$ points on a different variety. Such results were known before (for instance, the approach to Waring's problem via the Vinogradov mean value theorem exactly involves turning upper bounds into asymptotics) but these could only apply to hypersurfaces of a special form like the Fermat hypersurface, while this approach potentially applies to an arbitrary hypersurface. However, since it is not clear how to understand the dimension of the moduli space of curves on this blowup, the approach is only speculative for now.

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2. The circle method: setup and major arcs

Throughout the proof we fix a finite field $\mathbb{F}_q$, a smooth geometrically irreducible curve C over $\mathbb{F}_q$, positive integers s, k, e, a line bundle L of degree e on C, and $f \in H^0(C, L^k)$. We also fix an additive character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Implicit constants in big O notation will be uniform in these parameters unless specified otherwise.

We assume throughout that $e > 2g-2$, and $e > 0$ if $g = 0$, which implies that $\dim H^0(C,L) = e+1-g$ and $\dim H^0(C, L^k) = ke+1-g$.

For $\alpha \in H^0(C, L^k)^\vee$, let

$$S_1(\alpha) = \sum_{a \in H^0(C,L)} \psi(\alpha(a^k)). \tag{3}$$

The first step of the circle method is provided by the following lemma.

Lemma 2.1. *We have*

$$\#\{\mathbf{a} \in H^0(C,L)^s \mid \sum_{i=1}^s a_i^k = f\} = \frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} S_1(\alpha)^s \overline{\psi(\alpha(f))}. \tag{4}$$

Proof. We have

$$\frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} S_1(\alpha)^s \overline{\psi(\alpha(f))} = \frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} \overline{\psi(\alpha(f))} \Big(\sum_{a \in H^0(C,L)} \psi(\alpha(a^k)) \Big)^s$$

$$= \frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} \overline{\psi(\alpha(f))} \sum_{\mathbf{a} \in H^0(C,L)^s} \prod_{i=1}^s \psi(\alpha(a_i^k))$$

$$= \frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} \sum_{\mathbf{a} \in H^0(C,L)^s} \prod_{i=1}^s \psi(\alpha(\sum_{i=1}^s a_i^k - f))$$

$$= \sum_{\mathbf{a} \in H^0(C,L)^s} \frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} \psi(\alpha(\sum_{i=1}^s a_i^k - f))$$

and

$$\frac{1}{q^{ke+1-g}} \sum_{\alpha \in H^0(C,L^k)^\vee} \psi(\alpha(\sum_{i=1}^s a_i^k - f)) = \begin{cases} 1 & \text{if } \sum_{i=1}^s a_i^k = f \\ 0 & \text{otherwise} \end{cases}$$

which gives (4). □

In this section, we begin by analyzing the different α appearing in the sum of (4). We break the α into major arcs and minor arcs. We then analyze the major arcs, explicitly evaluating the sums $S_1(\alpha)$ for α in the major arcs, and then extracting the main term from the sum over α in the major arcs, up to an error term which we bound.

We say $\alpha \in H^0(C, L^k)^\vee$ *factors through* a closed subscheme $Z \subset C$ if α factors through the restriction map $H^0(C, L^k) \to H^0(Z, L^k)$. If $ke - \deg Z > 2g-2$ then the restriction map is surjective so the induced linear form $H^0(Z, L^k) \to \mathbb{F}_q$ is unique. In this case we will refer to the linear form $H^0(Z, L^k) \to \mathbb{F}_q$ by $\overline{\alpha}$, or, if the dependence on Z is relevant, by $\overline{\alpha}_Z$. We say $\overline{\alpha} \in H^0(Z, L^k)^\vee$ is *nondegenerate* if $\overline{\alpha}$ does not factor through $H^0(Z', L^k)$ for any proper closed subscheme Z' of Z.

The analogue of Dirichlet's approximation in this context is [3, Lemma 9] which states that any $\alpha \in H^0(C, L^k)$ factors through some closed subscheme $Z \subset C$ of degree $\leq \frac{ke}{2} + 1$. Furthermore, the closed subscheme $\subset C$ of minimal degree through which α factors is unique as long as it has degree $< \frac{ke}{2} - g + 1$ [3, Lemma 10]. We let $\deg \alpha$ be the minimum degree of a closed subscheme Z through which α factors. Then $\deg \alpha$ is an integer betwen 0 and $\frac{ke}{2} + 1$. If Z is a closed subscheme of minimal degree through which α factors, then $\overline{\alpha}$ is nondegenerate.

We will consider α to lie in the major arcs if $\deg(\alpha) \leq e - 2g + 1$, and otherwise to lie in the minor arcs.

For $\overline{\alpha} \in H^0(Z, L^k)^\vee$, let

$$S_Z(\overline{\alpha}) = \frac{1}{q^{\deg Z}} \sum_{\overline{a} \in H^0(Z,L)} \psi(\overline{\alpha}(\overline{a}^k)).$$

We now, in Lemma 2.2, evaluate $S_1(\alpha)$ for α in the major arcs in terms of $S_Z(\overline{\alpha})$, then, in Lemma 2.3, prove a multiplicativity property of $S_Z(\overline{\alpha})$, allowing us to reduce the study of $S_Z(\overline{\alpha})$ to the case of Z supported at a single closed point, then, in Lemma 2.4, bound $S_Z(\overline{\alpha})$ for Z supported at a single closed point, and, in Lemma 2.5, relate $S_Z(\overline{\alpha})$ for Z supported at a single closed point to the local terms $\ell_v(f)$. These results enable us to, in Lemma 2.6, express the main term of Theorem 1.4 as a sum of $S_Z(\overline{\alpha})$, so that we can, in Lemma 2.7, evaluate the number of representations of f as equal to the main term plus two error terms, one of which we immediately handle in Lemma 2.8 and the other we will handle in Section 3.

Lemma 2.2. *Let $\alpha \in H^0(C, L^k)^\vee$ factor through a closed subscheme Z of C of degree $\leq e - 2g + 1$. Then*

$$S_1(\alpha) = q^{e-g+1} S_Z(\overline{\alpha}). \tag{5}$$

Proof. The line bundle $L(-Z)$ has degree $e - \deg Z \geq 2g-1$ and thus $H^1(C, L(-Z))$ vanishes. The short exact sequence $0 \to L(-Z) \to L \to L\mid_Z \to 0$ induces a long exact sequence

$$\cdots \to H^0(C, L) \to H^0(Z, L) \to H^1(C, L(-Z)) \to \ldots.$$

Combining these, we conclude that $H^0(C,L) \to H^0(Z,L)$ is surjective. It follows that each $\overline{a} \in H^0(Z,L)$ has exactly

$$\frac{\#H^0(C,L)}{\#H^0(Z,L)} = \frac{q^{\dim H^0(C,L)}}{q^{\dim H^0(Z,L)}} = \frac{q^{e-g+1}}{q^{\deg Z}}$$

preimages $a \in H^0(C,L)$. For any of these preimages, the factorization of α through Z implies that $\psi(\alpha(a^k)) = \psi(\overline{\alpha}(\overline{a}^k))$. So the sum in (3) reduces to

$$\frac{q^{e-g+1}}{q^{\deg Z}} \sum_{\overline{a} \in H^0(Z,L)} \psi(\overline{\alpha}(\overline{a}^k)) = q^{e-g+1} S_Z(\overline{\alpha}). \qquad \square$$

The next lemma gives the multiplicativity properties of the sums S_Z.

Lemma 2.3. *Let Z be a disjoint union of subschemes Z_1 and Z_2 and fix $\overline{\alpha} \in H^0(Z,L^k)^\vee$. Let $\overline{\alpha}_{Z_i}$ be the restriction of α to $H^0(Z_i,L^k)$. Then we have*

$$S_Z(\overline{\alpha}) = S_{Z_1}(\overline{\alpha}_{Z_1}) S_{Z_2}(\overline{\alpha}_{Z_2})$$

and $\overline{\alpha}(f) = \overline{\alpha}_{Z_1}(f)\overline{\alpha}_{Z_2}(f)$. Furthermore, $\overline{\alpha}$ is nondegenerate if and only if $\overline{\alpha}_{Z_1}$ and $\overline{\alpha}_{Z_2}$ are nondegenerate.

Proof. The first two claims follow immediately from the "Chinese remainder theorem" splitting $H^0(Z,L^k) = H^0(Z_1,L^k) \times H^0(Z_2,L^k)$.

For the last claim, if $\overline{\alpha}$ factors through some proper subscheme Z' then $\overline{\alpha}_{Z_1}$ factors through $Z_1 \cap Z'$ and $\overline{\alpha}_{Z_2}$ factors through $Z_2 \cap Z'$ and at least one of these is proper. Conversely, if $\overline{\alpha}_{Z_1}$ factors through a proper subscheme Z_1' then $\overline{\alpha}$ factors through $Z_1' \cup Z_2$, which is proper, and similarly with $\overline{\alpha}_{Z_2}$. $\square$

Using Lemma 2.3, we can reduce the calculation of S_Z to Z supported at a single closed point. To that end, for $v \in C$ a closed point and m a nonnegative integer, let $m[v]$ be the unique closed subscheme of C of length m supported at v. The notation is chosen since we will sometimes identify closed subschemes with their divisors.

Lemma 2.4. *Let $v \in C$ be a closed point, m a nonnegative integer, and $\overline{\alpha} \in H^0(m[v],L^k)^\vee$ a nondegenerate linear form. Assume k is relatively prime to p.*

If $m \not\equiv 1 \mod k$ then

$$S_{m[v]}(\overline{\alpha}) = \frac{1}{q^{\lceil \frac{m}{k} \rceil \deg v}}.$$

If $m \equiv 1 \mod k$ then

$$\left| S_{m[v]}(\overline{\alpha}) \right| \leq \frac{k-1}{q^{\left(\frac{m-1}{k} + \frac{1}{2}\right) \deg v}}.$$

Proof. We have

$$S_{m[v]}(\overline{\alpha}) = \frac{1}{q^{m \deg v}} \sum_{\overline{a} \in H^0(m[v],L)} \psi(\overline{\alpha}(\overline{a}^k)).$$

We may fix a local generator of L at v, giving an isomorphism between $H^0(m[v],L)$ and $H^0(m[v],\mathcal{O}_{m[v]})$ and inducing an isomorphism beween $H^0(m[v],L^k)$ and $H^0(m[v],\mathcal{O}_{m[v]})$. Furthermore we may write $H^0(m[v],\mathcal{O}_{m[v]})$ as $\mathcal{O}_{C_v}/\pi^m$ where $\mathcal{O}_{C_v}$ is the local ring of C at v

and π is a uniformizer. We may thus express $\overline{\alpha}$ as a nondegenerate linear form on $\mathcal{O}_{C_v}/\pi^m$ and obtain

$$S_{m[v]}(\overline{\alpha}) = \frac{1}{q^{m \deg v}} \sum_{\overline{a} \in \mathcal{O}_{C_v}/\pi^m} \psi(\overline{\alpha}(\overline{a}^k)).$$

We first handle the case $m = 0$, which is trivial, and $m = 1$, where $\mathcal{O}_{C_v}/\pi$ is a finite field $\mathbb{F}_{q^{\deg v}}$ and the sum $\sum_{\overline{a} \in \mathcal{O}_{C_v}/\pi} \psi(\overline{\alpha}(\overline{a}^k))$ is a Gauss sum of an additive character composed with the kth power map over that finite field. The bound

$$\Big| \sum_{\overline{a} \in \mathcal{O}_{C_v}/\pi} \psi(\overline{\alpha}(\overline{a}^k)) \Big| \leq (k-1) q^{\frac{\deg v}{2}}$$

then follows from the classical bound for Gauss sums, and implies the $m = 1$ case.

For $m \geq 2$ we fix a uniformizer π of the local ring at v and split the sum into $\overline{a}$ that are multiples of π and those that are not, i.e.

$$\sum_{\overline{a} \in \mathcal{O}_{C_v}/\pi^m} \psi(\overline{\alpha}(\overline{a}^k)) = \sum_{\substack{\overline{a} \in \mathcal{O}_{C_v}/\pi^m \\ \pi \nmid \overline{a}}} \psi(\overline{\alpha}(\overline{a}^k)) + \sum_{\substack{\overline{a} \in \mathcal{O}_{C_v}/\pi^m \\ \pi \mid \overline{a}}} \psi(\overline{\alpha}(\overline{a}^k)).$$

The first term vanishes for $m \geq 2$ by a stationary phase argument: For $b \in \mathbb{F}_{q^{\deg v}}$ we have $(\overline{a} + \pi^{m-1} b)^k = \overline{a}^k + k\overline{a}^{k-1}\pi^{m-1} b$ since $(\pi^{m-1})^2 = \pi^{2m-2}$ is divisible by π^m and hence vanishes in $\mathcal{O}_{C_v}/\pi^m$. Thus

$$\psi(\overline{\alpha}((\overline{a} + \pi^{m-1}b)^k)) = \psi(\overline{\alpha}(\overline{a}^k + k\overline{a}^{k-1}\pi^{m-1}b)) = \psi(\overline{\alpha}(\overline{a}^k))\psi(\overline{\alpha}(k\overline{a}^{k-1}\pi^{m-1}b)).$$

Because k and $\overline{a}$ are nonzero modulo π, as b runs over $\mathbb{F}_{q^{\deg v}}$, the product $k\overline{a}^{k-1}\pi^{m-1}b$ must run over all multiples of π^{m-1} in $\mathcal{O}_{C_v}/\pi^m$. Since α is nondegenerate, this implies the $\mathbb{F}_q$-linear map $b \mapsto \overline{\alpha}(k\overline{a}^{k-1}\pi^{m-1}b)$ is nonzero and hence surjective. So there exists some b such that $\psi(\overline{\alpha}(k\overline{a}^{k-1}\pi^{m-1}b)) \neq 1$. Furthermore, the choice of b depends only on $\overline{a}$ mod π. The change of variables $\overline{a} \mapsto \overline{a} + \pi^{m-1}b$ cancels the contributions of all $\overline{a}$ in a single nonzero residue class mod π. It follows that

$$\sum_{\substack{\overline{a} \in \mathcal{O}_{C_v}/\pi^m \\ \pi \nmid \overline{a}}} \psi(\overline{\alpha}(\overline{a}^k)) = 0.$$

For $2 \leq m \leq k$, if $\pi \mid \overline{a}$ then $\pi^m \mid \overline{a}^k$ so that

$$\sum_{\substack{\overline{a} \in \mathcal{O}_{C_v}/\pi^m \\ \pi \mid \overline{a}}} \psi(\overline{\alpha}(\overline{a}^k)) = \sum_{\substack{\overline{a} \in \mathcal{O}_{C_v}/\pi^m \\ \pi \mid \overline{a}}} 1 = q^{\deg v (m-1)}$$

and thus

$$S_{m[v]}(\overline{\alpha}) = \frac{1}{q^{\deg v}},$$

handling this case.

Finally if $m > k$ then we can define $\overline{\alpha}' \in (\mathcal{O}_{C_v}/\pi^{m-k})^\vee$ by the rule $\overline{\alpha}'(c) = \overline{\alpha}(\pi^k c)$. Then $\overline{\alpha}'$ is nondegenerate since $\overline{\alpha}$ is and

$$\sum_{\substack{\overline{a}\in\mathcal{O}_{C_v}/\pi^m \\ \pi|\overline{a}}} \psi(\overline{\alpha}(\overline{a}^k)) = \sum_{\substack{\overline{a}\in\mathcal{O}_{C_v}/\pi^m \\ \pi|\overline{a}}} \psi(\overline{\alpha}'((\overline{a}/\pi)^k)) = q^{\deg v(k-1)} \sum_{\overline{a}'\in\mathcal{O}_{C_v}/\pi^{m-k}} \psi(\overline{\alpha}'(a'^k))$$

since each residue class $a' \in \mathcal{O}_{C_v}/\pi^{m-k}$ may be expressed as $\overline{a}/\pi$ for exactly $q^{\deg v(k-1)}$ residue classes $\overline{a} \in \mathcal{O}_{C_v}/\pi^m$, all divisible by π, those being the $q^{\deg v(k-1)}$ solutions to the congruence $\overline{a} = \pi\overline{a}' \mod \pi^{m-k+1}$. This gives

$$S_{m[v]}(\overline{\alpha}) = \frac{1}{q^{\deg v}} S_{(m-k)[v]}(\overline{\alpha'})$$

which handles all the cases $m > k$ by induction on m. □

The next lemmas show that plugging the right-hand side of (5) into a formula similar to (4) gives the desired main term.

Lemma 2.5. *We have*

$$\ell_v(f) = \sum_{m=0}^{\infty} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \textit{nondegenerate}}} S_{m[v]}(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}.$$

Proof. It suffices to check that

$$\sum_{m=0}^{r} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \text{nondegenerate}}} S_{m[v]}(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))} = \frac{\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^s \mid \sum_{i=1}^s b_i^k \equiv f \bmod \pi^r\}}{q^{r(s-1)\deg v}} \tag{6}$$

as taking the limits of both sides as r goes to ∞ gives the stated equality. We now verify this. (Compare [3, Lemma 11] which is the same argument in a slightly different setting.)

We first check that

$$\sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} S_{m[v]}(\overline{\beta})^s \overline{\psi(\overline{\beta}(f))} = \frac{\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^s \mid \sum_{i=1}^s b_i^k \equiv f \bmod \pi^r\}}{q^{r(s-1)\deg v}}. \tag{7}$$

To do this we expand

$$\begin{aligned}
\sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} S_{m[v]}(\overline{\beta})^s \overline{\psi(\beta(f))} &= \sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} \left(\frac{1}{q^{r\deg v}} \sum_{\overline{a}\in H^0(r[v],L)} \psi(\overline{\beta}(\overline{a}^k)) \right)^s \overline{\psi(\beta(f))} \\
&= \frac{1}{q^{rs\deg v}} \sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} \overline{\psi(\beta(f))} \sum_{\overline{\mathbf{a}}\in H^0(r[v],L)^s} \prod_{i=1}^s \psi(\overline{\beta}(\overline{a}_i^k)) \\
&= \frac{1}{q^{rs\deg v}} \sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} \sum_{\overline{\mathbf{a}}\in H^0(r[v],L)^s} \psi(\overline{\beta}(\sum_{i=1}^s \overline{a}_i^k - f)) \\
&= \frac{1}{q^{rs\deg v}} \sum_{\overline{\mathbf{a}}\in H^0(r[v],L)^s} \sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} \psi(\overline{\beta}(\sum_{i=1}^s \overline{a}_i^k - f))
\end{aligned}$$

and

$$\sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} \psi(\overline{\beta}(\sum_{i=1}^{s}\overline{a}_i^k - f)) = \begin{cases} q^{r\deg v} & \text{if } \sum_{i=1}^s \overline{a}_i^k = f \\ 0 & \text{otherwise} \end{cases}$$

which gives

$$\sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} S_{m[v]}(\overline{\beta})^s \overline{\psi(\beta(f))} = \frac{\#\{\mathbf{a}\in (H^0(r[V],L^k))^s \mid \sum_{i=1}^s a_i^k = f\}}{q^{r(s-1)\deg v}}$$

and then fixing a local generator of L to obtain a bijection between $H^0(r[v],L^k))$ to and $\mathcal{O}_{C_v}/\pi_v^r$ gives (7).

We next observe that for $\overline{\beta}\in H^0(r[v],L^k)^\vee$, if $\overline{\beta}$ factors through $m[v]$ then $\overline{\beta}$ factors through $(m+1)[v]$. There hence exists a unique m such that $\overline{\beta}$ factors through $m[v]$ but not through $(m-1)[v]$. Furthermore, since every proper closed subscheme of $m[v]$ has the form $m'[v]$ for some $m'<m$, the factorization of $\overline{\beta}$ through $m[v]$ is a nondegenerate form $\overline{\alpha}\in H^0(m[v],L^k)^\vee$. For this $\overline{\alpha}$ we trivially have $\overline{\alpha}(f)=\overline{\beta}(f)$. We also have

$$S_{m[v]}(\overline{\alpha}) = S_{r[v]}(\overline{\beta})$$

since the natural map $H^0(r[v],L^k)\to H^0(m[v],L^k)$ is surjective with fibers of size $q^{(r-m)\deg v} = \frac{q^{\deg r[v]}}{q^{\deg m[v]}}$.

Finally, each $\overline{\alpha}$ composes with the projection $H^0(r[v],L^k)\to H^0(m[v],L^k)$ to give a linear form $\overline{\beta}\in H^0(r[v],L^k)^\vee$, so each $\overline{\alpha}$ arises from exactly one such β. Combining all these observations, we obtain

$$\sum_{m=0}^{r} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \text{nondegenerate}}} S_{m[v]}(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))} = \sum_{\overline{\beta}\in H^0(r[v],L^k)^\vee} S_{m[v]}(\overline{\beta})^s \overline{\psi(\overline{\beta}(f))} \tag{8}$$

Combining (7) and (8), we obtain (6). □

Lemma 2.6. *Assume $s\geq 5$ and k is coprime to p. Then we have*

$$\frac{q^{s(e+1-g)}}{q^{ke+1-g}} \sum_{Z\subset C \text{ closed}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}$$
$$= q^{e(s-k)+(s-1)(1-g)} \prod_{v\in|C|} \ell_v(f).$$

Proof. Since $\frac{q^{s(e+1-g)}}{q^{ke+1-g}} = q^{e(s-k)+(s-1)(1-g)}$, these terms can be ignored.

Each closed subscheme Z can be written uniquely as a disjoint union of subschemes of the form $m[v]$ for closed points v and positive integers m. This and Lemma 2.3 gives the factorization

$$\sum_{Z\subset C\text{ closed}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))} = \prod_{v\in|C|} \sum_{m=0}^{\infty} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \text{nondegenerate}}} S_{m[v]}(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}.$$

Here Lemma 2.4 implies, since $s \geq 5$, that the product of sums is absolutely convergent and so the sum is absolutely convergent and thus the formal manipulation is analytically valid. The result then follows from Lemma 2.5. $\square$

Lemma 2.7. *Assume $s \geq 5$, k is coprime to p, and either $k < 2$ or $k = 2$ and $g > 0$. We have*

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^k = f\} - q^{e(s-k)+(s-1)(1-g)} \prod_{v\in|C|} \ell_v(f)$$

$$(9)\quad = \frac{1}{q^{ke+1-g}} \sum_{\substack{\alpha\in H^0(C,L^k)^\vee \\ \deg \alpha > e-2g+1}} S_1(\alpha)^s \overline{\psi(\overline{\alpha}(f))} - \frac{q^{s(e-g+1)}}{q^{ke+1-g}} \sum_{\substack{Z\subset C \textit{ closed} \\ \deg Z > e-2g+1}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \textit{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}.$$

Proof. We apply (4) to obtain

$$\#\{a_1,\dots,a_s \in H^0(C,L) \mid \sum_{i=1}^{s} a_i^k = f\}$$

$$= \frac{1}{q^{ke+1-g}} \sum_{\substack{\alpha\in H^0(C,L^k)^\vee \\ \deg \alpha \leq e-2g+1}} S_1(\alpha)^s \overline{\psi(\alpha(f))} + \frac{1}{q^{ke+1-g}} \sum_{\substack{\alpha\in H^0(C,L^k)^\vee \\ \deg \alpha > e-2g+1}} S_1(\alpha)^s \overline{\psi(\alpha(f))}.$$

We have $\frac{ke}{2} - g + 1 = e - 2g + 1 + \frac{(k-2)e}{2} + g$ which since $e > 0$ and either $k > 2$ or $g > 0$ implies that $e - 2g + 1 < \frac{ke}{2} - 2g + 1$. Thus if $\deg \alpha \leq e - 2g + 1$ then there exists a unique closed subscheme Z of minimal degree through which α factors, and $\overline{\alpha}$ is certainly nondegenerate. Conversely, if Z is a closed subscheme of degree $\leq e - 2g + 1$ and $\overline{\alpha} \in H^0(Z,L^k)^\vee$ is nondegenerate then $\alpha\colon H^0(C,L^k) \to H^0(Z,L^k) \to \mathbb{F}_q$ has degree $\leq e - 2g + 1$ and, by [3, Lemma 10], Z is the minimal subscheme through which α factors. Thus

$$\frac{1}{q^{ke+1-g}} \sum_{\substack{\alpha\in H^0(C,L^k)^\vee \\ \deg \alpha \leq e-2g+1}} S_1(\alpha)^s \overline{\psi(\alpha(f))}$$

$$= \frac{q^{s(e-g+1)}}{q^{ke+1-g}} \sum_{\substack{Z\subset C \text{ closed} \\ \deg Z \leq e-2g+1}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}$$

$$= \frac{q^{s(e-g+1)}}{q^{ke+1-g}} \sum_{Z\subset C \text{ closed}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))} - \frac{q^{s(e-g+1)}}{q^{ke+1-g}} \sum_{\substack{Z\subset C \text{ closed} \\ \deg Z > e-2g+1}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}.$$

Plugging in Lemma 2.6 then gives the desired statement. $\square$

To prove Theorem 1.4, it follows from Lemma 2.7 that it suffices to give an upper bound for the two sums in (9). The first part consists of the minor arcs and will be handled in the next section. The second part will be handled in the next lemma.

Lemma 2.8. *Assume $k \geq 2$ and k is coprime to p. For each $\delta < \frac{s-\max(k,3)-1}{k}$ we have*

$$\sum_{\substack{Z\subset C \text{ closed} \\ \deg Z > e-2g+1}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\alpha(f))} = O_{s,k,\delta}((1+q^{-\frac{1}{2}})^{O_{s,k}(g)} q^{-\delta(e-2g+2)}) = O_{s,k,\delta,g}(q^{-\delta(e-2g+2)}).$$

Proof. It suffices to bound

$$\sum_{\substack{Z\subset C \text{ closed} \\ \deg Z > e-2g+1}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} |S_Z(\overline{\alpha})|^s$$

which is the sum over $d > e-2g+1$ of the coefficient of u^d in

$$\sum_{Z\subset C \text{ closed}} u^{\deg Z} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} |S_Z(\overline{\alpha})|^s.$$

The sum of the coefficients of u^d in this power series for $d > e-2g+1$ is at most $q^{-\delta(e-2g+2)}$ times the value of this power series at $u = q^\delta$. So it suffices to show that the value of the power series at $u = q^\delta$ is $O_{s,k,\delta}((1+q^{-\frac{1}{2}})^{O_{s,k}(g)})$. (The weaker $O_{s,k,\delta,g}(q^{-\delta(e-2g+2)})$ claim is immediate since $(1+q^{-\frac{1}{2}})^{O_{s,k}(g)} = O_{s,k,\delta,g}(1)$.) Our first few bounds will use only that u is a positive real, but after this we will use the assumption $\delta < \frac{s-\max(k,3)-1}{k}$.

By Lemma 2.3 we have

$$\sum_{Z\subset C \text{ closed}} u^{\deg Z} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^k)^\vee \\ \text{nondegenerate}}} |S_Z(\overline{\alpha})|^s = \prod_{v\in|C|} \sum_{m=0}^{\infty} u^{m\deg v} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \text{nondegenerate}}} \left|S_{m[v]}(\overline{\alpha})\right|^s.$$

By Lemma 2.4 we have

$$\sum_{m=0}^{\infty} u^{m\deg v} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \text{nondegenerate}}} \left|S_{m[v]}(\overline{\alpha})\right|^s$$

$$\leq \sum_{m=0}^{\infty} u^{m\deg v} q^{m\deg v} \begin{cases} \frac{1}{q^{s\lceil \frac{m}{k}\rceil \deg v}} & \text{if } m \not\equiv 1 \mod k \\ \frac{(k-1)^s}{q^{s\left(\frac{m-1}{k}+\frac{1}{2}\right)\deg v}} & \text{if } m \equiv 1 \mod k \end{cases}$$

$$\leq \frac{1}{(1-u^{\deg v} q^{\deg v - \frac{s}{2}\deg v})^{(k-1)^s}} \prod_{j=2}^{k} \frac{1}{1-u^{j\deg v} q^{j\deg v - s\deg v}}$$

so that

$$\prod_{v\in|C|} \sum_{m=0}^{\infty} u^{m\deg v} \sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee \\ \text{nondegenerate}}} \left|S_{m[v]}(\overline{\alpha})\right|^s$$

$$\leq \prod_{v\in|C|} \left(\frac{1}{(1-u^{\deg v} q^{\deg v - \frac{s}{2}\deg v})^{(k-1)^s}} \prod_{j=2}^{k} \frac{1}{1-u^{j\deg v} q^{j\deg v - s\deg v}} \right)$$

$$\leq \zeta_C(uq^{1-\frac{s}{2}})^{(k-1)^s}\prod_{j=1}^{k-1}\zeta_C(u^jq^{j-s}) \leq \frac{(1+uq^{\frac{3}{2}-\frac{s}{2}})^{2g(k-1)^s}}{(1-uq^{1-\frac{s}{2}})^{(k-1)^s}(1-uq^{2-\frac{s}{2}})^{(k-1)^s}}\prod_{j=2}^{k}\frac{(1+u^jq^{j+\frac{1}{2}-s})^{2g}}{(1-u^jq^{j-s})(1-u^jq^{j+1-s})}.$$

The assumption $\delta < \frac{s-\max(k,3)-1}{k}$ implies that $\delta < \frac{s-k-1}{k}$ so that $j\delta < s-j-1$ for all $j \leq k$ and also that $\delta < \frac{s-4}{k} \leq \frac{s-4}{2} = \frac{s}{2}-2$. Combining these, we see that all the terms in the denominator have the form $(1-q^f)$ with $f<0$ depending on s,k,δ and so are lower bounded by $(1-2^f)$ which depends only on s,k,δ. Similarly, the terms in the numerator are bounded by $1+q^{-1/2}$ and the number of terms appearing is $2g((k-1)^s+(k-1)) = O_{s,k}(g)$. This gives the desired bound $O_{s,k,\delta}((1+q^{-\frac{1}{2}})^{O_{s,k}(g)}q^{-\delta(e-2g+2)})$ for the value of the power series. $\square$

Finally, we prove a lower bound on the main term (equivalently, upper bound on the inverse of the main term) that will be needed for Theorem 1.4:

Lemma 2.9. *Assume that k is coprime to p, $k \geq 2$, $s > k+1$, $s \geq 5$, and $q > (k-1)^4$. We have*

$$\Big(\prod_{v\in|C|}\ell_v(f)\Big)^{-1} = O_{s,k,g}(1).$$

The bound $q > (k-1)^4$ is not optimal, but certainly one must take $q > (k-1)^2$ since if q is a perfect square, $k = \sqrt{q}+1$, and v is a place of degree 1, then $\ell_v(f) = 0$ for f whose restriction to that place does not lie in the subfield $\mathbb{F}_{\sqrt{q}}$.

Proof. By Lemma 2.5 and 2.4 we have

$$\ell_v(f) = \sum_{m=0}^{\infty}\sum_{\substack{\overline{\alpha}\in H^0(m[v],L^k)^\vee\\ \text{nondegenerate}}} S_{m[v]}(\overline{\alpha})^s\overline{\psi(\alpha(f))}$$

$$\geq 1 - \sum_{\substack{m>0\\ m\not\equiv 1 \mod k}}(q^{m\deg v}-q^{(m-1)\deg v})\frac{1}{q^{s\lceil\frac{m}{k}\rceil\deg v}} - \sum_{\substack{m>0\\ m\equiv 1 \mod k}}(q^{m\deg v}-q^{(m-1)\deg v})\frac{(k-1)^s}{q^{s\left(\frac{m-1}{k}+\frac{1}{2}\right)\deg v}}$$

$$= 1 - \frac{1}{1-q^{(k-s)\deg v}}(q^{\deg v}-1)\Big(\frac{(k-1)^s}{q^{\frac{s}{2}\deg v}} + \sum_{m=2}^{k}\frac{q^{(m-1)\deg v}}{q^{s\deg v}}\Big).$$

We observe that $\frac{1}{1-q^{(k-s)\deg v}}(q^{\deg v}-1) \leq q^{\deg v}$ and that $\sum_{m=2}^{k}\frac{q^{(m-1)\deg v}}{q^{s\deg v}} = O(q^{(k-1-s)\deg v})$ so that

$$\ell_v(f) = 1 - O_{s,k}(q^{(1-\frac{s}{2})\deg v} + q^{(k-s)\deg v}) = 1 - O_{s,k}(q^{-\frac{3}{2}\deg v}).$$

Thus as long as $\ell_v(f)$ is bounded away from 0 we have

$$\ell_v(f)^{-1} = 1 + O_{s,k}(q^{-\frac{3}{2}\deg v}).$$

The product of this over v is bounded by $\zeta_C(q^{-3/2})^{O_{s,k}(1)} = O_{s,k,g}(1)$ so it remains to check $\ell_v(f)$ is bounded away from 0.

To do this, we bound $\sum_{m=2}^{k}\frac{q^{(m-1)\deg v}}{q^{s\deg v}}$ more precisely by $\frac{1}{1-q^{-\deg v}}q^{(k-1-s)\deg v} \leq \frac{1}{q^{\deg v}(q^{\deg v}-1)}$.

On the other hand we have $(k-1)^s \leq q^{\frac{s}{4}}$ so $\frac{(k-1)^s}{q^{\frac{s}{2}\deg v}} \leq \frac{1}{q^{\frac{s}{4}\deg v}}$. Thus

$$\ell_v(f) \geq 1 - \frac{1}{1-q^{(k-s)\deg v}}(q^{\deg v}-1)\Big(\frac{(k-1)^s}{q^{\frac{s}{2}\deg v}} + \sum_{m=2}^{k} \frac{q^{(m-1)\deg v}}{q^{s\deg v}}\Big)$$
$$\geq 1 - q^{\deg v}\Big(\frac{1}{q^{\frac{s}{4}\deg v}} + \frac{1}{q^{\deg v}(q^{\deg v}-1)}\Big) \geq 1 - \Big(\frac{1}{q^{\frac{1}{4}\deg v}} + \frac{1}{q^{\deg v}-1}\Big) > 0$$

as long as $q^{\deg v} > 4.3$, which happens as long as $q > 4.3$, which follows from the assumption $q > (k-1)^4$ if $k \geq 3$. If $k = 2$, this does not follow, but we have $(k-1)^s = 1$ so we can replace the last line by

$$\geq 1 - q^{\deg v}\Big(\frac{1}{q^{\frac{s}{2}\deg v}} + \frac{1}{q^{\deg v}(q^{\deg v}-1)}\Big) \geq 1 - \Big(\frac{1}{q^{\frac{3}{2}\deg v}} + \frac{1}{q^{\deg v}-1}\Big) > 0$$

which holds as long as $q^{\deg v} > 4$ which happens because k is prime to p and so $q^{\deg v} \geq q \geq p \geq 3$. ∎

Finally, we remark briefly on how to handle the case $k = 2, g = 0$ dropped in Lemma 2.7. This case was essentially handled in [12, Corollary 1.2], but without an explicit power savings error term. We explain the same argument in our notation here. Since this case is easier than the other cases, we will be brief.

Lemma 2.10. *Assume $s \geq 5$, $k = 2$ is coprime to p, and $g = 0$. For each $\delta < \frac{s-4}{2}$ we have*

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^2 = f\} - q^{e(s-2)+(s-1)} \prod_{v\in|C|} \ell_v(f) = O_{s,\delta}(q^{-\delta(e+1)}).$$

Proof. We fix a degree one point ∞ of $C \cong \mathbb{P}^1$ and claim that when $k = 2, g = 0$ each α factors through a unique closed subscheme Z such that $\deg Z \cup \{\infty\} = e+1$, where union of subschemes corresponds to intersection of ideals. This existence can be obtained from classical Dirichlet approximation [6, Lemma 3], or from observing that it suffices to check that the restriction of α from $H^0(C,L^2)$ to $H^0(C,L^2(-\infty))$ factors through a subscheme Z' of degree $\leq e$ and take $Z = Z' + [\infty]$ (where sum of subschemes corresponds to multiplication of ideals), and then following the proof of [3, Lemma 9]. The uniqueness follows from [3, Lemma 10] (or can also be obtained classically).

The existence and uniqueness immediately gives the following analogue of Lemma 2.7, where the minor arc term has disappeared and the other term is adjusted slightly:

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^2 = f\} - q^{e(s-2)+(s-1)} \prod_{v\in|C|} \ell_v(f)$$
$$= -\frac{q^{s(e+1)}}{q^{2e+1}} \sum_{\substack{Z\subset C \text{ closed}\\ \deg Z\cup\{\infty\}>e+1}} \sum_{\substack{\overline{\alpha}\in H^0(Z,L^2)^\vee\\ \text{nondegenerate}}} S_Z(\overline{\alpha})^s \overline{\psi(\overline{\alpha}(f))}.$$

We can follow the proof of Lemma 2.8 to bound this adjusted error term by $O(q^{-\delta(e+1)})$ for $\delta < \frac{s-4}{2}$, by considering the sum of the coefficients of u^d for $d > e$ in an Euler product whose term for $v \neq \infty$ is unchanged and whose term for $v = \infty$ is $\sum_{m=0}^{\infty} u^{\max(m-1,0)} \sum_{\substack{\overline{\alpha}\in H^0(m[\infty],L^2)^\vee\\ \text{nondegenerate}}} \left|S_{m[\infty]}(\overline{\alpha})\right|^s$.

Applying Lemma 2.4 bounds this local factor by $\frac{1}{1-u^{\deg v}q^{\deg v-\frac{s}{2}\deg v}}$ for $v\neq\infty$ or $1+\frac{q^{1-\frac{s}{2}}}{1-uq^{1-\frac{s}{2}}}$ for $v=\infty$. The product of this local bound is

$$\prod_{v\in|\mathbb{P}^1\setminus\{\infty\}|}\frac{1}{1-u^{\deg v}q^{\deg v-\frac{s}{2}\deg v}}\times\left(1+\frac{q^{1-\frac{s}{2}}}{1-uq^{1-\frac{s}{2}}}\right)=\frac{1}{1-uq^{2-\frac{s}{2}}}\left(1+\frac{q^{1-\frac{s}{2}}}{1-uq^{1-\frac{s}{2}}}\right).$$

For $u=q^{\delta}$ with $\delta<\frac{s-4}{2}$, all terms in both denominators are bounded away from 0 by a constant depending only on s,δ and $q^{1-\frac{s}{2}}\leq 1$ so this expression is $O_{s,\delta}(1)$, giving the desired bound as in Lemma 2.8. □

3. Minor arcs via singular exponential sums

For the minor arcs, it is crucial that we think of $H^0(C,L)$ as an algebraic variety, whose R-points are $H^0(C,L)\otimes_{\mathbb{F}_q}R$ for any $\mathbb{F}_q$-algebra R. In particular, this is an affine space.

For $\alpha\in H^0(C,L^k)^\vee$, let $\operatorname{Sing}_\alpha=\{a\in H^0(C,L)\mid \alpha(a^{k-1}b)=0 \text{ for all } b\in H^0(C,L)\}$, which is a closed subscheme of $H^0(C,L)$ cut out by polynomials.

Lemma 3.1. *Assume k is coprime to p. Then*

$$|S(\alpha)|\leq 3(k+1)^{e+1-g}q^{\frac{e+1-g+\dim\operatorname{Sing}_\alpha}{2}}.$$

Proof. This can be obtained from results of Katz [4], more specifically by a slight modification of the proof of [4, Theorem 4]. To begin with, we explain Katz's setup and the assumptions of [4, Theorem 4] and explain why they hold in our setting. Specifically, Katz' field k is our $\mathbb{F}_q$. Katz's setup requires an integer N, an integer $r\geq 1$, and a list of r positive integers $D_1,\ldots,D_r$. Inside $\mathbb{P}^N$, he considers a closed subscheme X defined by a set of r homogeneous equations of degrees $D_1,\ldots,D_r$. For our case $N=e+2-g$, $r=1$, $D_1=1$, and $X=\mathbb{P}^{e+1-g}$ defined by a single linear form. (The linear form is only included to satisfy the assumption $r\geq 1$ of Katz's setup, which seems not to be strictly necessary, but including it leads to no significant loss.) The dimension of X is $n=e+1-g$. Katz's hypotheses (H1) and (H1') on X are trivially satisfied. Next Katz considers a linear form $L\in H^0(X,\mathcal{O}(1))$, an integer d, and $H\in H^0(X,\mathcal{O}(d))$. We take L an arbitrary linear form so that the locus where L is nonvanishing is $\mathbb{A}^{e+1-g}$. We identify this affine space with $H^0(C,L)$. On $H^0(C,L)$, $\alpha(a^k)$ is a homogeneous polynomial function of degree k. A polynomial function of degree k on affine space gives by homogenization a section of $\mathcal{O}(k)$ on projective space. We take $d=k$ and H to be this section.

Next Katz makes the assumption (H2) that the scheme-theoretic intersection $X\cap L\cap H$ has dimension $n-2$. Here $X\cap L$ is the projective space $\mathbb{P}^{e-g}=\mathbb{P}^{n-1}$, so it suffices to check that H is nontrivial when restricted to this projective space, i.e. the leading coefficient of the polynomial function $\alpha(a^k)$ is nonzero. Since $\alpha(a^k)$ is homogeneous, this happens if and only if $\alpha(a^k)$ is not identically zero. We will check shortly that if $\alpha(a^k)$ is identically zero then $\operatorname{Sing}_\alpha$ is all of $H^0(C,L)$ so $\dim\operatorname{Sing}_\alpha=e+1-g$ and the stated bound follows from the trivial bound, so we can ignore this case and hence assume (H2).

Finally Katz defines δ to be the dimension of the singular locus of $X\cap L\cap H$ and ϵ to be the dimension of the singular locus of $X\cap L$. Since $X\cap L$ is just projective space, it is smooth, so $\epsilon=-1$. Since $X\cap L\cap H$ is the hypersurface in $\mathbb{P}^{e-g}$ defined by the leading term of the polynomial $\alpha(a^k)$, its singular locus is the set of nonzero vectors up to scaling where that leading term and its derivative in each direction vanish. Since $\alpha(a^k)$ is homogeneous of degree k, this is simply the set of nonzero vectors up to scaling where $\alpha(a^k)$ and its derivative in each direction vanish. We claim that Sing_α is the set of vectors where $\alpha(a^k)$ and its derivative in each direction vanish. Since modding out by scaling reduces the dimension by 1, this will imply that $\delta = \dim \mathrm{Sing}_\alpha - 1$. To check this, note that the derivative of $\alpha(a^k)$ in the direction given by a vector b is

$$\frac{d}{du}\alpha((a+ub)^k) = \alpha(\frac{d}{du}(a+ub)^k) = \alpha(ka^{k-1}b) = k\alpha(a^{k-1}b).$$

Since $p \nmid k$, this is zero for all b if and only if $\alpha(a^{k-1}b) = 0$ for all b. Furthermore, if $\alpha(a^{k-1}b) = 0$ for all b then clearly $\alpha(a^k) = 0$, verifying the characterization of Sing_α. This characterization also implies that if $\alpha(a^k)$ is identically zero, so that its derivatives are identically zero, then Sing_α is the whole space, which we claimed earlier.

Then our setup satisfies all the assumptions of [4, Theorem 4], together with the additional hypothesis $\epsilon \leq \delta$ of part (1) of that theorem.

This gives a bound for the exponential sum from [4, Theorem 4]

$$|S(\alpha)| \leq (4\sup(1+D_1,\ldots,1+D_r,d)+5)^{N+r} q^{\frac{n+1+\delta}{2}}$$

which plugging in our fixed values of N, r, D_1, n, δ gives

$$|S(\alpha)| \leq (4k+5)^{e+3-g} q^{\frac{e+1-g+\dim \mathrm{Sing}_\alpha}{2}}.$$

However, the constant $(4\sup(1+D_1,\ldots,1+D_r,d)+5)^{N+r}$ arises in the proof as a bound for the total degree of the L-function of the exponential sum. (This step of the argument is given in [4, Two paragraphs after the statement of Theorem 4 on p. 879], with the proof [4, p. 892] devoted to bounding the zeroes and poles of the L-function).

However, later work of Katz may be used to bound the total degree of the L-function by $3(k+1)^{e+1-g}$. Indeed [5, Theorem 10] states that for a polynomial f of degree at most d in N variables over a finite field F of characteristic p, the sum $\sigma_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)})$ of the dimensions of the compactly supported cohomology groups $H^i_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)})$ is at most $3(d+1)^N$. Applying this to $f = \alpha(a^k)$ we will again take $d = k$ but now take $N = e+1-g$, the dimension of the relevant affine space. The Lefschetz fixed point formula shows that the L-function of the exponential sum $\sum_{x \in \mathbb{A}^N(F)} \psi(f(x))$ is

$$\prod_i \det\bigl(1 - u\,\mathrm{Frob}_F, H^i_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)})\bigr)^{(-1)^i}.$$

Since $\det\bigl(1 - u\,\mathrm{Frob}_F, H^i_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)})\bigr)$ is a polynomial in u of degree $\dim H^i_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)})$, this implies that the L-function has total degree at most

$$\sum_i \dim H^i_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)}) = \sigma_c(\mathbb{A}^N_{\overline{F}}, \mathcal{L}_{\psi(f)}) \leq 3(d+1)^N = 3(k+1)^{e+1-g}$$

which gives the bound

$$|S(\alpha)| \leq 3(k+1)^{e+1-g} q^{\frac{e+1-g+\dim \mathrm{Sing}_\alpha}{2}}.$$ □

Lemma 3.2. *For each $\overline{\alpha} \in H^0(Z, L^k)^\vee$, there is a unique $\tilde{\alpha} \in H^0(Z, K_C(Z) \otimes L^{-k})$ such that $\overline{\alpha}(f)$ is given by the sum over points $v \in Z$ of the residue of $f\tilde{\alpha} \in H^0(Z, K_C(-Z))$ at v.*

If $\overline{\alpha}$ is nondegenerate then $\tilde{\alpha}$ is invertible on Z.

Proof. It suffices to check that the pairing

$$H^0(Z, L^k) \times H^0(Z, K_C(Z) \otimes L^{-k}) \to \mathbb{F}_q$$

given by the sum over $v \in |Z|$ of the residue of the product at v is a perfect pairing. Since both sides can be expressed as a direct sum over points of v, and the pairing of the summand of $H(Z, L^k)$ associated to v_1 with the summand of $H^0(Z, K_C(Z) \otimes L^{-k})$ associated to v_2 vanishes unless $v_1 = v_2$, it suffices to check the pairing is perfect upon restriction to the summands associated to v for each point v, i.e. it suffices to assume $v = Z^m$.

In this case, we work in the local ring $\mathcal{O}_{C_v}$ with uniformizer π_v. We can fix a local trivialization of L at v. This lets us write $H^0(m[v], L) = \mathcal{O}_{C_v}/\pi_v^m$ which has basis $1, \pi_v, \dots, \pi_v^{m-1}$. Similarly, $H^0(m[v], K_C(m[v]) \otimes L^{-k}) = \pi_v^{-m}\mathcal{O}_{C_v} d\pi_v / \mathcal{O}_{C_v} d\pi_v$ which has basis $\pi_v^{-1} d\pi_v, \dots, \pi_v^{-m} d\pi_v$. The residue of the product $\pi_v^i \cdot \pi_v^{-j} d\pi_v$ is 1 if $j = i + 1$ and 0 if $j > i + 1$, which implies the pairing is perfect.

Finally, $\tilde{\alpha}$ fails to be invertible if and only if, restricted to the summand associated to some $v \in |Z|$, the coefficient of $\pi_v^{-m} d\pi_v$ in $\tilde{\alpha}$ is zero. By the above calculation of the residue pairing, the coefficient is the residue of $\pi_v^{m-1}\tilde{\alpha}$ and thus is $\overline{\alpha}(\pi_v^{m-1})$. If this vanishes, then $\overline{\alpha}$ factors through the subscheme Z' which is identical except its multiplicity of v is one less, and hence is not nondegenerate. □

Lemma 3.3. *Assume α factors through Z and fix $\tilde{\alpha}$ as in Lemma 3.2. Then we have $a \in \mathrm{Sing}_\alpha$ if and only if there exists $c \in H^0(C, K_C(Z) \otimes L^{-1})$ such that $c\,|_Z = \tilde{\alpha} a^{k-1}$.*

Proof. Suppose such a c exists. Then for $b \in H^0(C, L)$, $\alpha(a^{k-1}b)$ is the sum over $v \in Z$ of the residue of $\tilde{\alpha} a^{k-1} b$. Since $\tilde{\alpha} a^{k-1} b$ and cb agree as sections of $K_C(Z)$ restricted to Z, they have the same residue at each $v \in Z$, and hence $\alpha(a^{k-1}b)$ is the sum over $v \in C$ of the residue of cb. Now cb is a global section of $H^0(C, K_C(Z))$ and thus cb is a meromorphic differential form with poles only in Z. By the residue theorem, the sum of the residues of cb over the points of Z vanishes. This verifies "if".

Next consider the short exact sequence $L(-Z) \to L \to L\,|_Z$ which induces a long exact sequence in cohomology $H^0(C, L(-Z)) \to H^0(C, L) \to H^0(Z, L) \to H^1(C, L(-Z)) \to H^1(C, L)$. Hence the space of linear forms on $H^0(Z, L)$ that vanish on $H^0(C, L)$ is dual to the kernel of $H^1(C, L(-Z)) \to H^1(C, L)$, thus by Serre duality isomorphic to the cokernel of $H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C(Z) \otimes L^{-1})$. In particular, these spaces have the same dimension. Every element of $H^0(C, K_C(Z) \otimes L^{-1})$ defines via the residue pairing a linear form on $H^0(Z, L)$ that vanishes on $H^0(C, L)$ and two elements define the same linear form if and only if their restriction to Z vanishes, i.e. if and only if their difference lies in $H^0(C, K_C \otimes L^{-1})$, so the residue pairing gives an injection from the cokernel to the space of linear forms on $H^0(Z, L)$ vanishing on $H^0(C, L)$, which must therefore be a surjection. Thus, if the linear form $b \mapsto \alpha(a^{k-1}b) = \mathrm{res}(\tilde{\alpha} a^{k-1} b)$ vanishes on $b \in H^0(C, L)$, then there must exist such a c, verifying "only if". □

If c as in Lemma 3.3 exists, it is necessarily unique, since two choices of c differ by an element of $H^0(C, K_C \otimes L^{-1})$ which vanishes as $e > 2g-2$. Thus letting

$$\mathrm{Sing}_{\tilde{\alpha},Z} = \{a \in H^0(C,L), c \in H^0(C, K_C(Z) \otimes L^{-1}) \mid c\,|_Z = \tilde{\alpha} a^{k-1}\}$$

we have

$$\dim \mathrm{Sing}_\alpha = \mathrm{Sing}_{\tilde{\alpha},Z}\,. \tag{10}$$

(In fact, one can check that these two spaces are isomorphic, but this is not needed for us.)

The same argument shows that for $(a,c) \in \mathrm{Sing}_{\tilde{\alpha},Z}$ with $a = 0$ that $c\,|_Z = 0$ so $c = 0$. We thus split $\mathrm{Sing}_{\tilde{\alpha},Z}$ into a closed set where $c = 0$ and an open set where $a, c \neq 0$. We can further stratify the open set where $a, c \neq 0$ as follows: to each point which is a vanishing point of either a or c, we associate a pair of nonnegative integers (i,j), not both 0, where i is the order of vanishing of a at that point and j is the order of vanishing of c at that point. A pair a, c then provides a function μ from pairs of nonnegative integers not both 0 to nonnegative integers where $p(i,j)$ is the total degee of points associated to the pair (i,j). We have $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)} \mu(i,j)i = e$ and $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)} \mu(i,j)j = 2g-2+\deg Z - e$. The set of all points $(a,c) \in \mathrm{Sing}^{a,c\neq 0}_{\tilde{\alpha},Z}$ providing a given function μ then forms a locally closed subset $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$. We then have

$$\mathrm{Sing}_{\tilde{\alpha},Z} = \mathrm{Sing}^{c=0}_{\tilde{\alpha},Z} \cup \mathrm{Sing}^{a,c\neq 0}_{\tilde{\alpha},Z} \tag{11}$$

and

$$\mathrm{Sing}^{a,c\neq 0}_{\tilde{\alpha},Z} = \bigcup_{\substack{\mu\colon \mathbb{N}^2\setminus\{(0,0)\}\to\mathbb{N} \\ \sum_{i,j}\mu(i,j)i = e \\ \sum_{i,}\mu(i,j)j = 2g-2+\deg Z - e}} \mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}\,. \tag{12}$$

We estimate the dimension of each locally closed subset in this decomposition separately.

Lemma 3.4. *Let Z be a closed subset and fix $\tilde{\alpha} \in H^0(Z, K_C(Z) \otimes L^{-k})$ invertible. For $v \in |Z|$, let m_v be the multiplicity of v in Z. Then*

$$\dim \mathrm{Sing}^{c=0}_{\tilde{\alpha},Z} \leq \max\Big(e - g + 1 - \sum_{v\in|Z|} \left\lceil \frac{m_v}{k-1} \right\rceil \deg v, \frac{e + 1 - \sum_{v\in|Z|} \lceil \frac{m_v}{k-1} \rceil \deg v}{2}, 0\Big).$$

Proof. Plugging in $c = 0$ to the definition of $\mathrm{Sing}_{\tilde{\alpha},Z}$, we see that

$$\mathrm{Sing}^{c=0}_{\tilde{\alpha},Z} = \{a \in H^0(C,L) \mid a^{k-1}\tilde{\alpha} = 0\} = \{a \in H^0(C,L) \mid a^{k-1}\,|_Z = 0\}$$

since $\tilde{\alpha}$ is invertible. We have $a^{k-1} \mid Z = 0$ if and only if a^{k-1} vanishes to order at least m_v at each point $v \in |Z|$, in other words if a vanishes to order at least $\lceil \frac{m_v}{k-1} \rceil$ at each point $v \in |Z|$. This happens if and only if a is a global section of $L(-\sum_{v\in|Z|} \lceil \frac{m_v}{k-1} \rceil m[v])$, which is a line bundle of degree $e - \sum_{v\in|Z|} \lceil \frac{m_v}{k} \rceil \deg v$. The result is then given by the fact that a line bundle of degree d has a space of global sections of dimension at most $\max(g+1-d, \frac{d+1}{2}, 0)$ since its space of global sections has dimension at most $\max(g+1-d, 0)$ unless it is special and $\frac{d+1}{2}$ if it is special by Clifford's theorem. □

The loci $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$ are inverse images of certain locally closed subsets of $\mathrm{Sym}^e(C)\times\mathrm{Sym}^{2g-2+\deg Z-e}(C)$. We will first describe the tangent space of $\mathrm{Sym}^n C$. Using that, we will describe the tangent spaces of these subsets, and finally the tangent spaces of their inverse images $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$, which will ultimately allow us to bound the dimension of $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$.

$\mathrm{Sym}^n C$ can be equivalently expressed as the Hilbert scheme parameterizing length n closed subschemes of C. The tangent space to the Hilbert scheme of a projective scheme X at an ideal sheaf I is canonically identified with $\mathrm{Hom}(I, \mathcal{O}_X/I)$, with the map given explicitly on a family of ideals, in other words an ideal on a product $X\times S$ that is flat over S, by lifting elements of the ideal I parameterized by a point $x\in S$ to elements of the family of ideals over a neighborhood of x and then modding out by I to produce an element of the tensor product of $\mathcal{O}_X/I$ with the maximal ideal of x. This gives a map from the tangent space, which is the dual of the maximal ideal modulo the maximal ideal squared, to $\mathrm{Hom}(I,\mathcal{O}_X/I)$. In the case of a curve C, at a point corresponding to an effective divisor $D\subset C$ of degree n, the ideal $I=\mathcal{O}_C(-D)$ so the tangent space is identified with

$$\mathrm{Hom}(\mathcal{O}_C(-D),\mathcal{O}_C/\mathcal{O}_C(-D)) = \mathrm{Hom}(\mathcal{O}_C(-D),\mathcal{O}_D) = H^0(D,\mathcal{O}_C(D)).$$

By construction, this identification is local on D, i.e. if D splits as a disjoint union of divisors D_1, D_2 of degrees n_1, n_2 then this identification is compatible with the natural map $\mathrm{Sym}^{n_1} C\times \mathrm{Sym}^{n_2} C\to \mathrm{Sym}^n C$ and the natural isomorphism $H^0(D,\mathcal{O}_C(D))\cong H^0(D_1,\mathcal{O}_C(D_1))\oplus H^0(D_2,\mathcal{O}_C(D_2))$.

Also by construction, for L a line bundle of degree n, the derivative at a section f of the natural map $H^0(C,L)\setminus 0\to \mathrm{Sym}^n(C)$ that sends a section to its vanishing locus sends a tangent vector ∂f to $\frac{\partial f}{f}\in H^0(D,\mathcal{O}_C(D))$, since the tangent vector corresponds to the family of ideals generated by $f+\epsilon\partial f$ and thus the induced element of $\mathrm{Hom}(\mathcal{O}_C(-D),\mathcal{O}_C/\mathcal{O}_C(-D))$ sends f to ∂f.

For $(i,j)\in\mathbb{N}^2\setminus\{(0,0)\}$ let $w(i,j)$ be the greatest power of p dividing both i and j. In other words, $w(i,j)$ is the p-adic valuation of $\gcd(i,j)$. For a natural number w, let C_w be the unique smooth projective curve with a totally inseparable map of degree p^w from C, in other words the unique curve whose function field is the field of p^wth powers of elements of $\mathbb{F}_q(C)$. This can be expressed also as the pullback of C under either the wth power of the arithmetic Frobenius or the wth power of the geometric Frobenius, but it is both difficult and unnecessary to remember which.

Lemma 3.5. *Let μ be a function from $\mathbb{N}^2\setminus\{(0,0)\}$ to $\mathbb{N}$ such that $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)}\mu(i,j)i=e$ and $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)}\mu(i,j)j=2g-2+\deg Z-e$. The map*

$$\prod_{(i,j)\in\mathbb{N}^2\setminus(0,0)} C^{\mu(i,j)}\to \mathrm{Sym}^e(C)\times\mathrm{Sym}^{2g-2+\deg Z-e}(C)$$

that sends a tuple $(x_{i,j,t})_{1\le t\le\mu(i,j)}$ to the pair of divisors

$$\sum_{(i,j)\in\mathbb{N}^2\setminus\{(0,0)\}}\sum_{t=1}^{\mu(i,j)} i[x_{i,j,t}],\ \sum_{(i,j)\in\mathbb{N}^2\setminus\{(0,0)\}}\sum_{t=1}^{\mu(i,j)} j[x_{i,j,t}]$$

factors through

$$\prod_{(i,j)\in\mathbb{N}^2\setminus(0,0)} C_{w(i,j)}^{\mu(i,j)}.$$

Restricted to the locus where the $x_{i,j,t}$ are distinct, the map

$$\prod_{(i,j)\in\mathbb{N}^2\setminus(0,0)} C_w^{\mu(i,j)} \to \operatorname{Sym}^e(C)\times\operatorname{Sym}^{2g-2+\deg Z-e}(C)$$

sends a tangent vector expressed as a tuple $\partial x_{i,j,t}$ of tangent vectors on $C_{w(i,j)}$ to the element of $H^0(D_1,\mathcal{O}_C(-D_1))\times H^0(D_2,\mathcal{O}_C(-D_2))$ that near the point $x_{i,j,t}$ with uniformizer $\pi_{x_{i,j,t}}$ takes the value

$$\Big(\frac{i}{p^{w(i,j)}}\frac{\langle \partial x_{i,j,t}, \pi_{x_{i,j,t}}^{p^{w(i,j)}}\rangle}{\pi_{x_{i,j,t}}^{p^{w(i,j)}}}, \frac{j}{p^{w(i,j)}}\frac{\langle \partial x_{i,j,t}, \pi_{x_{i,j,t}}^{p^{w(i,j)}}\rangle}{\pi_{x_{i,j,t}}^{p^{w(i,j)}}}\Big) \tag{13}$$

where $\langle,\rangle$ denotes the natural pairing between tangent vectors of $C_{w(i,j)}$ and elements of the maximal ideal of $C_{w(i,j)}$, noting that $\pi^{p^{w(i,j)}}$ lies in this maximal ideal.

Finally, restricted to the locus where the $x_{i,j,t}$ are distinct, this map induces an isomorphism of the tangent space of $\prod_{(i,j)\in\mathbb{N}^2\setminus(0,0)} C_w^{\mu(i,j)}$ to the tangent space of its image.

Proof. We handle each claim in turn.

For the factorization statement, by the existence of various multiplication maps $\operatorname{Sym}^{n_1} C\times \operatorname{Sym}^{n_2} C\to\operatorname{Sym}^{n_1+n_2} C$, it suffices to prove that the map $C\to\operatorname{Sym}^i C\times\operatorname{Sym}^j C$ sending x to $(i[x],j[x])$ factors through $C_{w(i,j)}$. Since by definition i and j are divisible by $p^{w(i,j)}$, it suffices to check that the map $C\to\operatorname{Sym}^{p^w} C$ sending x to $p^w[x]$ factors through C_w. The factorization of a map from a curve through a totally inseparable map of degree p^w may be checked étale-locally. Since étale-locally C is isomorphic to $\mathbb{A}^1$, we may work in $\mathbb{A}^1$, where this map sends a point x to the vanishing locus of the polynomial $(T-x)^{p^w}=T^{p^w}-x^{p^w}$ which may be expressed only in terms of x^{p^w} and thus factors through the unique totally inseparable map of degree p^w.

To calculate the induced map on tangent spaces, we may use the compatibility of the identification of the tangent space with $H^0(D,\mathcal{O}_C(D))$ with the multiplication maps $\operatorname{Sym}^{n_1} C\times\operatorname{Sym}^{n_2} C\to\operatorname{Sym}^{n_1+n_2} C$. Because of this it suffices to prove that the map $C_{w(i,j)}\to\operatorname{Sym}^i C\times\operatorname{Sym}^j C$ described in the previous paragraph sends a tangent vector $\partial x_{i,j,t}$ to (13). The ideal-theoretic description of the identification makes it clear that it is étale-local. (The non-existence of the Hilbert scheme of an arbitrary scheme is not problematic since we can define the tangent space of a non-representable functor). For compactness of notation write $w=w(i,j)$. The uniformizer $\pi_{x_{i,j,}}$ gives an étale map to $\mathbb{A}^1$, and working étale-locally, this map sends x^{p^w} to

$$((T-x)^i,(T-x)^j)=((T^{p^w}-x^{p^w})^{\frac{i}{p^w}},(T^{p^w}-x^{p^w})^{\frac{j}{p^w}}).$$

Recalling the identification of tangent spaces sends the family of ideals generated by a family of functions f to $\frac{\partial f}{f}$, we may calculate the induced map on tangent spaces by taking the generator of the family of ideals, differentiating with respect to x^{p^w}, and then dividing by the generator, obtaining

$$\Big(\frac{-\frac{i}{p^w}(T^{p^w}-x^{p^w})^{\frac{i}{p^w}-1}dx^{p^w}}{(T^{p^w}-x^{p^w})^{\frac{i}{p^w}}},\frac{-\frac{j}{pw}(T^{p^w}-x^{p^w})^{\frac{j}{pw}-1}dx^{p^w}}{(T^{p^w-x^{p^w}})^{\frac{j}{p^w}}}\Big)$$

$$=\Big(-\frac{i}{p^w}\frac{dx^{p^w}}{T^{p^w}-x^{p^w}},\frac{i}{p^w}\frac{dx^{p^w}}{T^{p^w}-x^{p^w}}\Big).$$

Pulling $T^{p^w}-x^{p^w}$ back along the étale-local map $\pi_{i,j,t}$ gives $\pi_{i,j,t}^{p^w}$ and pulling dx^{p^w} back gives $\langle \partial x_{i,j,t}, \pi_{i,j,t}^{p^w}\rangle$. This gives (13).

Finally, to check that this map gives an isomorphism on tangent spaces onto its image, it suffices to check it is a map between smooth varieties which is injective on tangent spaces and open onto its image. The fact that both varieties are smooth is straightforward.

The fact that it is injective on tangent spaces follows from the explicit formula for the induced map on tangent spaces: Since the tangent space of the source and target both split as a product over $x_{i,j,t}$, with the map compatible with that splitting, it suffices to check the restriction to a given $x_{i,j,t}$ is injective. Since $w(i,j)$ is the highest power of p dividing both i and j, at least one of $\frac{i}{p^{w(i,j)}}$ and $\frac{j}{p^{w(i,j)}}$ must be coprime to p. Hence it suffices to check that for $\partial x_{i,j,t}$ nonzero, $\frac{\langle \partial x_{i,j,t}, \pi_{x_{i,j,t}}^{p^{w(i,j)}}\rangle}{\pi_{x_{i,j,t}}^{p^{w(i,j)}}}$ is nonzero, which is clear since $\pi_{x_{i,j,t}}^{p^{w(i,j)}}$ is a uniformizer of $C_{w(i,j)}$ at $x_{i,j,t}$ so that $\langle \partial x_{i,j,t}, \pi_{x_{i,j,t}}^{p^{w(i,j)}}\rangle \neq 0$ for $\partial x_{i,j,t}\neq 0$ and $\frac{1}{\pi_{x_{i,j,t}}^{p^{w(i,j)}}}$ is nonzero modulo $\mathcal{O}_C$ and thus nonzero as an element of $H^0(i[x_{i,j,t}],\mathcal{O}_C(-i[x_{i,j,t}]))$.

To check that the morphism is open onto its image, we first calculate its image, which consists of pairs of divisors (D_1,D_2) on C such that the number of geometric points whose multiplicity in D_1 is i and whose multiplicity in D_2 is j is exactly $\mu(i,j)$. To check openness, we need to check that for a family of pairs of divisors satisfying the conditions and a labeling over one point in the family of their common vanishing points by triples (i,j,t), we can extend that labeling to nearby points in the family. This follows from the existence of a specialization map from geometric points of C over the generic point of the family to geometric points of C over the special point. □

Lemma 3.6. *For μ a function from pairs of nonnegative integers to nonnegative integers satisfying $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)}\mu(i,j)i=e$ and $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)}\mu(i,j)j=2g-2+\deg Z-e$, we have*

$$\dim \mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}\leq g+1+\sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\}\\ p\mid\frac{(k-1)i-j}{\gcd(i,j)}}}\mu(i,j).$$

Proof. Write $V(f)$ for the set of vanishing points of a section f. We begin by further stratifying $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$ into strata $\mathrm{Sing}^{\mu,r}_{\tilde{\alpha},Z}$ consisting of points (a,c) of $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$ such that $(V(a)\cup V(c))\cap Z$ is r. On each component of each stratum, the set of points of Z where a or c vanishes must be constant. It suffices to prove the same upper bound for the dimension of each stratum, and hence suffices to prove the same upper bound for the dimension of each stratum at each point. Fix now a point (a,c) of $\mathrm{Sing}^{\mu,r}_{\tilde{\alpha},Z}$. The tangent space of $\mathrm{Sing}_{\tilde{\alpha},Z}$ at

(a,c) consists of pairs of $\partial a \in H^0(C,L)$ and $\partial c \in H^0(C, K_C(Z)\otimes L^{-1})$ such that

$$\partial c\mid_Z = (k-1)\tilde{\alpha} a^{k-2}\partial a.$$

Because $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$ is the inverse image of the image in $\mathrm{Sym}^e(C)\times \mathrm{Sym}^{2g-2+\deg Z-e}(C)$ of the map discussed in Lemma 3.5, the tangent space of $\mathrm{Sing}^{\mu}_{\tilde{\alpha},Z}$ is the subset of pairs (a,c) satisfying the additional condition that $(\frac{\partial a}{a}, \frac{\partial c}{c})$ lies in the tangent space of that image. By Lemma 3.5, this means that at each geometric point x at which a vanishes to multiplicity i and j vanishes to multiplicity c, the pair $(\frac{\partial a}{a}, \frac{\partial c}{c})$ must be a scalar multiple of the pair

$$\Big(\frac{i}{p^{w(i,j)}}\frac{1}{\pi_{x_{i,j,t}}^{p^{w(i,j)}}}, \frac{j}{p^{w(i,j)}}\frac{1}{\pi_{x_{i,j,t}}^{p^{w(i,j)}}}\Big). \tag{14}$$

Finally, because on each component of $\mathrm{Sing}^{\mu,r}_{\tilde{\alpha},Z}$ the vanishing points of a and c in Z are fixed, if the vector lies in the tangent space of $\mathrm{Sing}^{\mu,r}_{\tilde{\alpha},Z}$ then at each geometric point of Z to which a vanishes to multiplicity i and j vanishes to multiplicity c, the scalar must vanish and so the pair $(\frac{\partial a}{a}, \frac{\partial c}{c})$ cannot have a pole at these points.

Since $c = \tilde{\alpha}a^{k-1}$ on restriction to Z, we have

$$\frac{\partial c}{c}c = \partial c = (k-1)\tilde{\alpha}a^{k-2}\partial a = (k-1)\frac{\partial a}{a}\tilde{\alpha}a^{k-1} = (k-1)\frac{\partial a}{a}c \tag{15}$$

on restriction to Z.

Fix now a geometric point x of Z, with multiplicity m. Let d be the order of vanishing of c at x. Then $\frac{\partial c}{c} - (k-1)\frac{\partial a}{a}$ vanishes to order at least $m-\min(d,m)$ at x by (15). Thus, viewed as a section of $K_C(Z)$,

$$ac\Big(\frac{\partial c}{c} - (k-1)\frac{\partial a}{a}\Big) = a\partial c - (k-1)c\partial a \tag{16}$$

vanishes to order at most m at x since the order of vanishing of c at least cancels the $-\min(d,m)$ term. Applying this for all x, we see that (16) is in fact a section of $K_C(Z-Z) = K_C$. Since the space of sections of K_C is g-dimensional, (16) must vanish on a subspace of the tangent space of codimension at most g. It suffices to show this subspace has dimension at most $1+\sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\}\\ p\mid\frac{(k-1)i-j}{\gcd(i,j}}}\mu(i,j)$.

On this subspace, we have the equation

$$\frac{\partial c}{c} = (k-1)\frac{\partial a}{a} \tag{17}$$

of meromorphic functions on C. Now we use the fact that, at each vanishing point of a or c, $(\frac{\partial a}{a}, \frac{\partial c}{c})$ must be a scalar multiple of (14). This forces the scalar to be 0 unless

$$(k-1)\frac{i}{p^{w(i,j)}} - \frac{j}{p^{w(i,j)}} = 0$$

which happens if and only if $p \mid \frac{(k-1)i-j}{\gcd(i,j)}$ since the greatest power of p dividing $\gcd(i,j)$ is $p^{w(i,j)}$. Hence the number of scalars that are possibly nonvanishing is the number of points satisfying this, i.e. $\sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\}\\ p\mid\frac{(k-1)i-j}{\gcd(i,j)}}}\mu(i,j)$. If all these scalars vanish then $\frac{\partial a}{a}$ and $\frac{\partial c}{c}$ have poles nowhere and hence are scalars, and the equation (17) forces them to lie in a one-dimensional space. This gives the desired bound for the total dimension. $\square$

We now recall the polygon $\Delta_{k,p}$ mentioned in the introduction, i.e. the convex hull of the set of points $\begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{N}^2$ such that $\gcd(i,j,p) = 1$ but $p \mid (k-1)i - j$. We also recall that $\gamma_{k,p}$ is the maximum value of γ such that $\begin{pmatrix} 1 \\ \frac{k-2}{2} \end{pmatrix} \in \gamma\Delta_{k,p}$. We begin with some preparatory lemmas on this polygon and $\gamma_{k,p}$ before using Lemma 3.6 to relate $\dim \operatorname{Sing}_{\tilde{\alpha},Z}$ to $\gamma_{k,p}$.

Lemma 3.7. *For* $\begin{pmatrix} x \\ y \end{pmatrix} \in \Delta_{k,p}$, *we have* $\begin{pmatrix} x+a \\ y+b \end{pmatrix} \in \Delta_{k,p}$ *whenever* $a, b \geq 0$.

Proof. This follows immediately from the fact that if $\begin{pmatrix} i \\ j \end{pmatrix}$ satisfy $\gcd(i,j,p) = 1$ but $p \mid (k-1)i - j$ then so do $\begin{pmatrix} i+p \\ j \end{pmatrix}$ and $\begin{pmatrix} i \\ j+p \end{pmatrix}$, together with the definition of convex hull. $\square$

The next two results give upper and lower bounds on $\gamma_{k,p}$.

Lemma 3.8. *We have* $\gamma_{k,p} \geq \frac{k-2}{2k-2}$.

Proof. $\Delta_{k,p}$ certainly contains the point $\begin{pmatrix} 1 \\ k-1 \end{pmatrix}$. By Lemma 3.7, $\Delta_{k,p}$ contains all points $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x \geq 1$ and $y \geq k-1$. In particular this includes $\begin{pmatrix} \frac{2k-2}{k-2} \\ k-1 \end{pmatrix} = \frac{2k-2}{k-2} \begin{pmatrix} 1 \\ \frac{k-2}{2} \end{pmatrix}$, showing that $\begin{pmatrix} 1 \\ \frac{k-2}{2} \end{pmatrix} \in \frac{k-2}{2k-2}\Delta_{k,p}$ and hence $\gamma_{k,p} \geq \frac{k-2}{2k-2}$. $\square$

Lemma 3.9. *If* $p > k-1$ *we have*

$$\gamma_{k,p} \leq \frac{k-2}{2k-2}\left(1 + \frac{k}{p}\right).$$

Proof. If $p \mid (k-1)i - j$ but $\gcd(i,j,p) = 1$ then we cannot have $j = 0$ as this implies $p \mid (k-1)i$ and thus $p \mid i$ so $\gcd(i,j,p) = p$, and we cannot have $i = 0$ as this implies $p \mid j$ so $\gcd(i,j,p) = p$. Since $p \mid (k-1)i - j$, we have either $(k-1)i - j \geq p$ or $(k-1)i - j \leq 0$. In the first case we have $j \geq 1$ so $i \geq \frac{p+1}{k-1}$ and in the second case we have $i \geq 1$ so $j \geq k-1$. Hence in either case we have

$$(k-1)(k-2)i + (p+2-k)j \geq p(k-1) \tag{18}$$

and thus (18) is satisfied for each $\begin{pmatrix} i \\ j \end{pmatrix} \in \Delta_{k,p}$.

In particular if $\begin{pmatrix} 1 \\ \frac{k-2}{2} \end{pmatrix} \in \gamma\Delta_{k,p}$ this implies

$$(k-1)(k-2) + (p+2-k)\frac{k-2}{2} \geq \gamma p(k-1)$$

so

$$\gamma \le \frac{2(k-1)(k-2) + (p+2-k)(k-2)}{2p(k-1)} = \frac{k-2}{2k-2}\left(1 + \frac{k}{p}\right).$$

Since this holds for all such γ, we have $\gamma_{k,p} \le \frac{k-2}{2k-2}\left(1+\frac{k}{p}\right)$. ☐

Lemma 3.10. *We have*

$$\dim \operatorname{Sing}^{a,c\neq 0}_{\tilde{\alpha},Z} \le g + 1 + \max\left\{\lambda \mid \begin{pmatrix} e \\ 2g-2+\deg Z - e\end{pmatrix} \in \lambda\Delta_{k,p}\right\}.$$

Proof. In view of (12) and Lemma 3.6, it suffices to prove that for μ a function from pairs of nonnegative integers to nonnegative integers satisfying $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)} \mu(i,j)i = e$ and $\sum_{(i,j)\in\mathbb{N}^2\setminus(0,0)} \mu(i,j)j = 2g-2+\deg Z - e$ we have

$$\max\left\{\lambda \mid \begin{pmatrix} e \\ 2g-2+\deg Z - e\end{pmatrix} \in \lambda\Delta_{k,p}\right\} \ge \sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\} \\ p \mid \frac{(k-1)i-j}{\gcd(i,j)}}} \mu(i,j). \tag{19}$$

Let

$$\lambda = \sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\} \\ p \mid \frac{(k-1)i-j}{\gcd(i,j)}}} \mu(i,j)$$

and then we have

$$\begin{pmatrix} e \\ 2g-2+\deg Z - e\end{pmatrix} = \sum_{(i,j)\in\mathbb{N}^2\setminus\{0,0\}} \mu(i,j)\begin{pmatrix} i \\ j\end{pmatrix}$$

$$= \lambda \sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\} \\ p \mid \frac{(k-1)i-j}{\gcd(i,j)}}} \frac{\mu(i,j)}{\lambda}\begin{pmatrix} i \\ j\end{pmatrix} + \sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\} \\ p \nmid \frac{(k-1)i-j}{\gcd(i,j)}}} \mu(i,j)\begin{pmatrix} i \\ j\end{pmatrix}.$$

Now the first term $\lambda \sum_{\substack{(i,j)\in\mathbb{N}^2\setminus\{0,0\} \\ p \mid \frac{(k-1)i-j}{\gcd(i,j)}}} \frac{\mu(i,j)}{\lambda}\begin{pmatrix} i \\ j\end{pmatrix}$ is λ times a convex combination of the vectors $\begin{pmatrix} i \\ j\end{pmatrix}$ which are all either in $\Delta_{k,p}$ by definition or are positive integer multiples of points of $\Delta_{k,p}$ and thus lie in $\Delta_{k,p}$ by Lemma 3.7. Hence the first term lies in $\lambda\Delta_{k,p}$. Since the second term is a vector with nonnegative entries, we have

$$\begin{pmatrix} e \\ 2g-2+\deg Z - e\end{pmatrix} \in \lambda\Delta_{k,p}$$

by Lemma 3.7, which implies (19). ☐

Lemma 3.11. *For Z the minimum closed subscheme through which α factors, we have*

$$\dim \operatorname{Sing}^{a,c\neq 0}_{\tilde{\alpha},Z} \le g + 1 + \left(e + \frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}.$$

where the expression $\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}$ *is understood to vanish if* $k=2$*, since* $\gamma_{k,p}=0$ *in that case, even though the denominator vanishes.*

Proof. First assume $k>2$. We observe that $\deg Z \leq \frac{ke}{2}+1$ so by Lemma 3.7 we have

$$\max\left\{\lambda \mid \begin{pmatrix} e \\ 2g-2+\deg Z - e\end{pmatrix} \in \lambda\Delta_{k,p}\right\} \leq \max\left\{\lambda \mid \begin{pmatrix} e \\ 2g-2+\frac{ke}{2}+1-e\end{pmatrix} \in \lambda\Delta_{k,p}\right\}$$

$$\leq \max\left\{\lambda \mid \begin{pmatrix} e+\frac{2\max(2g-1,0)}{k-2} \\ \max(2g-1,0)+\frac{(k-2)e}{2}\end{pmatrix} \in \lambda\Delta_{k,p}\right\} = (e+\frac{2\max(2g-1,0)}{k-2})\gamma_{k,p}.$$

The result then follows from Lemma 3.4.

In the $k=2$ case, we must prove a sharper bound. In fact we will prove this bound for $\dim \mathrm{Sing}_\alpha$ directly, after observing that $\mathrm{Sing}^{a,c\neq 0}_{\tilde\alpha,Z}$ is nonempty only if $H^0(C,K_C(Z)\otimes L^{-1})$ is nonzero which requires $\deg Z \geq e+2-2g$. The minimality of $\deg Z$ implies $\deg Z \leq \frac{ke+1}{2} = e+\frac{1}{2}$ and hence $\deg Z \leq e$ since $\deg Z$ is an integer.

By Lemma 3.3 we have $a \in \mathrm{Sing}_\alpha$ if and only if there exists $c \in H^0(C,K_C(Z)\otimes L^{-1})$ such that $c\,|_Z = \tilde\alpha a$. Since $\tilde\alpha$ is invertible, the restriction of a to $H^0(Z,L)$ is uniquely determined by c. The choices for a given a fixed restriction to $H^0(Z,L)$ are a torsor for $H^0(C,L(-Z))$. Thus

$$\dim \mathrm{Sing}_\alpha \leq \dim H^0(C,K_C(Z)\otimes L^{-1}) + \dim H^0(C,L(-Z))$$
$$= \dim H^0(C,L(-Z)) + \dim H^1(C,L(-Z)) \leq g+1$$

by Clifford's theorem since $\deg L(-Z) = e - \deg Z$ lies in $[0,2g-2]$. □

Lemma 3.12. *For* Z *the minimum closed subscheme through which* α *factors, we have*

$$\dim \mathrm{Sing}_\alpha \leq \max\left(e-g+1-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil \deg v,\ g+1+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}\right).$$

Proof. By (10) and (11), we have

$$\mathrm{Sing}_\alpha = \dim \mathrm{Sing}_{\tilde\alpha,Z} \leq \max(\dim \mathrm{Sing}^{c=0}_{\tilde\alpha,Z}, \dim \mathrm{Sing}^{a,c\neq 0}_{\tilde\alpha,Z}).$$

It follows from Lemma 3.4 that

$$\dim \mathrm{Sing}^{c=0}_{\tilde\alpha,Z} \leq \max\left(e-g+1-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil \deg v, g\right).$$

where $g \leq g+1+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}$. Combining this with Lemma 3.10, we obtain the bound. □

Lemma 3.13. *For each positive* $\delta < \frac{\frac{s}{2}-k}{k-1}$ *we have*

$$\sum_{\substack{Z\subset C\\ \deg Z > e-2g+1}} q^{\deg Z - s\frac{\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil \deg v}{2}} \leq O_{s,k,\delta}((1+q^{-1/2})^{O_k(g)}q^{-\delta(e-2g+2)}) \leq O_{s,k,\delta,g}(q^{-\delta(e-2g+2)}).$$

Proof. For $u = q^\delta$ we have

$$\sum_{\substack{Z\subset C\\ \deg Z>e-2g+1}} q^{\deg Z - s\frac{\sum_{v\in|Z|}\left\lceil \frac{m_v}{k-1}\right\rceil \deg v}{2}} \leq q^{-\delta(e-2g+2)} \sum_{Z\subset C} u^{\deg Z} q^{\deg Z - s\frac{\sum_{v\in|Z|}\left\lceil \frac{m_v}{k-1}\right\rceil \deg v}{2}}$$

so it suffices to prove

$$\sum_{Z\subset C} u^{\deg Z} q^{\deg Z - s\frac{\sum_{v\in|Z|}\left\lceil \frac{m_v}{k-1}\right\rceil \deg v}{2}} = O_{s,k,\delta}((1+q^{-1/2})^{O_{s,k}(g)}).$$

(The second inequality in the statement follows from $(1+q^{-\frac{1}{2}})^{O_{s,k}(g)} = O_{s,k,\delta,g}(1)$.) We have an Euler product expansion

$$\sum_{Z\subset C} u^{\deg Z} q^{\deg Z - s\frac{\sum_{v\in|Z|}\left\lceil \frac{m_v}{k-1}\right\rceil \deg v}{2}} = \prod_{v\in|C|}\sum_{m=0}^{\infty} (uq)^{m\deg v} q^{-\frac{s\lceil\frac{m}{k-1}\rceil \deg v}{2}}$$

$$\leq \prod_{v\in|C|}\prod_{j=1}^{k-1} \frac{1}{1-(uq)^{j\deg v} q^{-\frac{s\deg v}{2}}}$$

$$= \prod_{j=1}^{k-1} \zeta_C(u^j q^{j-\frac{s}{2}}) \leq \prod_{j=1}^{k-1} \frac{(1+u^j q^{j-\frac{s-1}{2}})^{2g}}{(1-u^j q^{j-\frac{s}{2}})(1-u^j q^{j+1-\frac{s}{2}})}.$$

If $\delta < \frac{\frac{s}{2}-k}{k-1}$ then $j\delta + j + 1 < \frac{s}{2}$ for all $j \leq k-1$ so all the terms in the denominator have the form $(1-q^f)$ with $f<0$ depending on s,k,δ and so are lower bounded by $(1-2^f)$ which depends only on s,k,δ. Similarly, the terms in the numerator are bounded by $1+q^{-1/2}$ and the number of terms appearing is $2g(k-1) = O_k(g)$. □

Lemma 3.14. *Assume k is coprime to p. For all positive $\delta < \frac{\frac{s}{2}-k}{k-1}$ we have*

$$\sum_{\substack{\alpha\in H^0(C,L^k)^\vee\\ \deg\alpha>e-2g+1}} |S_1(\alpha)|^s$$

$$\leq kq^{(k+1)e+2-2g} 3^{s-2}(k+1)^{(s-2)(e+1-g)} q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}$$
$$+O_{s,k,\delta}((k+1)^{s(e+1-g)} q^{s(e+1-g)} (1+q^{-1/2})^{O_k(g)} q^{-\delta(e-2g+2)}).$$

Proof. Let α factor minimally through a closed subscheme Z. Then by Lemma 3.1 and Lemma 3.12 we have

$$|S_1(\alpha)| \leq 3(k+1)^{(e+1-g)} q^{\frac{e+1-g+\dim \mathrm{Sing}_\alpha}{2}}$$

$$\leq 3(k+1)^{(e+1-g)} q^{\frac{e+1-g+\max\left(e-g+1-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil \deg v,\, g+1+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}\right)}{2}}$$

$$= \max\left(3(k+1)^{(e+1-g)} q^{\frac{2e+2-2g-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil \deg v,}{2}}, 3(k+1)^{(e+1-g)} q^{\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}\right)$$

and thus

$$|S_1(\alpha)|^s \leq \max\left(3^s((k+1)^{s(e+1-g)}q^{s\frac{2e+2-2g-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil\deg v,}{2}}, 3^{s-2}(k+1)^{(s-2)(e+1-g)}q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}|S_1(\alpha)|^2\right)$$

$$\leq 3^s(k+1)^{s(e+1-g)}q^{s\frac{2e+2-2g-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil\deg v,}{2}}+3^{s-2}(k+1)^{(s-2)(e+1-g)}q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}|S_1(\alpha)|^2$$

so that (choosing for each α a minimal closed subscheme Z)

$$\sum_{\substack{\alpha\in H^0(C,L^k)^\vee\\ \deg\alpha>e-2g+1}}|S_1(\alpha)|^s$$

$$\leq \sum_{\substack{\alpha\in H^0(C,L^k)^\vee\\ \deg\alpha>e-2g+1}}(3^s(k+1)^{s(e+1-g)}q^{s\frac{2e+2-2g-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil\deg v,}{2}}$$

$$+\sum_{\substack{\alpha\in H^0(C,L^k)^\vee\\ \deg\alpha>e-2g+1}}3^{s-2}(k+1)^{(s-2)(e+1-g)}q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}|S_1(\alpha)|^2.$$

For the second term we use the Plancherel formula estimate

$$\sum_{\alpha\in H^0(C,L^k)^\vee}|S_1(\alpha)|^2 = q^{ke+1-g}\left|\{a_1,a_2\in H^0(C,L)\mid a_1^k=a_2^k\}\right|\leq kq^{(k+1)e+2-2g}$$

to obtain

$$\sum_{\substack{\alpha\in H^0(C,L^k)^\vee\\ \deg\alpha>e-2g+1}}3^{s-2}(k+1)^{(s-2)(e+1-g)}q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}|S_1(\alpha)|^2.$$

$$\leq \sum_{\alpha\in H^0(C,L^k)^\vee}3^{s-2}(k+1)^{(s-2)(e+1-g)}q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}|S_1(\alpha)|^2$$

$$\leq kq^{(k+1)e+2-2g}3^{s-2}(k+1)^{(s-2)(e+1-g)}q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}.$$

For the first term we observe that there are at most $q^{\deg Z}$ choices of α for each subscheme Z to obtain

$$\sum_{\substack{\alpha\in H^0(C,L^k)^\vee\\ \deg\alpha>e-2g+1}}3^s(k+1)^{s(e+1-g)}q^{s\frac{2e+2-2g-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil\deg v,}{2}}$$

$$\leq 3^s(k+1)^{s(e+1-g)}\sum_{\substack{Z\subset C\\ \deg Z>e-2g+1}}q^{\deg Z+s\frac{2e+2-2g-\sum_{v\in|Z|}\left\lceil\frac{m_v}{k-1}\right\rceil\deg v,}{2}}$$

$$\leq 3^s(k+1)^{s(e+1-g)}q^{s(e+1-g)}O_{s,k,\delta}((1+q^{-1/2})^{O_k(g)}q^{-\delta(e-2g+2)})$$

for all $\delta<\frac{\frac{s}{2}-k}{k-1}$ by Lemma 3.13. We can then absorb 3^s into the big O. $\square$

We first state a version of the main theorem with a complicated estimate that preserves as much uniformity as possible in the variables q, g. We will then state a simpler version that drops uniformity in q, g and clarifies when obtain an asymptotic as $e \to \infty$ with other parameters fixed.

Theorem 3.15. *Assume $k \geq 2$, and $s > 2k$. Fix $\delta < \frac{\frac{s}{2}-k}{k-1}$. For any finite field $\mathbb{F}_q$ of characteristic $p \nmid k$, curve C of genus g over $\mathbb{F}_q$, and natural number $e > 2g-2$ we have*

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^k = f\} - q^{e(s-k)+(s-1)(1-g)} \prod_{v \in |C|} \ell_v(f)$$

$$= O_{s,k}(q^{e+1-g}(k+1)^{(s-2)(e-g)} q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}})$$

$$+O_{s,k,\delta}((k+1)^{s(e-g)} \frac{q^{s(e-g+1)}}{q^{ke+1-g}} (1+q^{-1/2})^{O_{s,k}(g)} q^{-\delta(e-2g+2)})$$

Proof. Note first that $\delta < \frac{\frac{s}{2}-k}{k-1}$ implies $\delta < \frac{s-\max(k,3)-1}{k}$ since if $k=2$ we have

$$\frac{\frac{s}{2}-k}{k-1} = \frac{s}{2} - 2 = \frac{s-4}{2} = \frac{s-\max(k,3)-1}{k}$$

and if $k > 2$ we have $k/2 < k-1$ and thus

$$k(s/2-k) = ks/2 - k^2 < (k-1)s - k^2 + 1 = (k-1)(s-k-1).$$

In the case $k = 2, g = 0$, the stated error term is worse than the error term in Lemma 2.10 and we can conclude immediately. Otherwise, combining Lemmas 2.7 (noting $s > 2k \geq 4$ so $s \geq 5$), 2.8, and 3.14 we obtain for $\delta < \frac{\frac{s}{2}-k}{k-1}$.

$$\left|\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^k = f\} - q^{e(s-k)+(s-1)(1-g)} \prod_{v \in |C|} \ell_v(f)\right|$$

$$\leq kq^{e+1-g} 3^{s-2} (k+1)^{(s-2)(e+1-g)} q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}}$$

$$+O_{s,k,\delta}((k+1)^{s(e+1-g)} \frac{q^{s(e-g+1)}}{q^{ke+1-g}} (1+q^{-1/2})^{O_k(g)} q^{-\delta(e-2g+2)})$$

$$+\frac{q^{s(e-g+1)}}{q^{ke+1-g}} O_{s,k,\delta}((1+q^{-1/2})^{O_{k,s}(g)} q^{-\delta(e-2g+2)}).$$

For the first term, absorbing k and $3^{s-2}(k+1)^{(s-2)}$ into the constant in the big O, we obtain the error term $O_{s,k}(q^{e+1-g}(k+1)^{(s-2)(e-g)} q^{(s-2)\frac{e+2+\left(e+\frac{2\max(2g-1,0)}{k-2}\right)\gamma_{k,p}}{2}})$.

The last two terms can be combined into the single term $O_{s,k,\delta}((k+1)^{s(e-g)} \frac{q^{s(e-g+1)}}{q^{ke+1-g}}(1+q^{-1/2})^{O_{s,k}(g)} q^{-\delta(e-2g+2)})$. ∎

Theorem 3.16. *Assume $k \geq 2$, and $s > 2k$. Fix a finite field $\mathbb{F}_q$ of characteristic $p > k$ and curve C of genus g over $\mathbb{F}_q$. Fix θ such that*

$$\theta < \frac{\frac{s}{2} - k}{k-1} - s\frac{\log(k+1)}{\log q}$$

and

$$\theta \leq s - k - (s-2)\left(\frac{1+\gamma_{k,p}}{2} + \frac{\log(k+1)}{\log q}\right) - 1.$$

Then for any number $e > 2g-2$ we have

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^s a_i^k = f\} - q^{e(s-k)+(s-1)(1-g)} \prod_{v \in |C|} \ell_v(f)$$

$$= O_{s,k,q,g,\theta}(q^{(s-k-\theta)e}).$$

Such a $\theta > 0$ exists if and only if

$$q > (k+1)^{2(k-1)}$$

and

$$s > \max\left(\frac{2(k-\gamma_{k,p} - 2\frac{\log(k+1)}{\log q})}{1-\gamma_{k,p} - 2\frac{\log(k+1)}{\log q}}, \frac{2k}{1-2(k-1)\frac{\log(k+1)}{\log q}}\right).$$

Proof. We choose $\delta = \theta + s\frac{\log(k+1)}{\log q}$ and the first assumed upper bound on θ gives $\delta < \frac{\frac{s}{2}-k}{k-1}$. Taking Theorem 3.15 and dropping every term that depends only on q and g, we obtain

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^s a_i^k = f\} - q^{e(s-k)+(s-1)(1-g)} \prod_{v \in |C|} \ell_v(f)$$

$$= O_{s,k}(q^e(k+1)^{(s-2)e}q^{(s-2)\frac{1+\gamma_{k,p}}{2}e}) + O_{s,k,,g,q,\theta}((k+1)^{se}q^{(s-k-\delta)e}).$$

With this choice of δ we have $(k+1)^{se}q^{(s-k-\delta)e} = q^{(s-k-\theta)e}$. Our second assumption on θ is equivalent to $q^e(k+1)^{(s-2)e}q^{(s-2)\frac{1+\gamma_{k,p}}{2}e} \leq q^{(s-k-\theta)e}$. So both terms are $O_{s,k,q,g,\theta}(q^{(s-k-\theta)e})$, as desired.

A positive θ exists if and only if both upper bounds for θ are positive. These upper bounds are each affine functions of s and take negative values respectively at $s=0$ and $s=2$, so for $s > 2k > 2$ they can only be positive if the slope in s is positive. The slope in s is respectively $\frac{1}{2(k-1)} - \frac{\log(k+1)}{\log q}$ for the first bound and $\frac{1-\gamma_{k,p}}{2} - \frac{\log(k+1)}{\log q}$ for the second bound. The positivity of the slopes is equivalent to the lower bound

$$q > \max((k+1)^{2(k-1)}, (k+1)^{\frac{2}{1-\gamma_{k,p}}})$$

but since $p > k$ we have $\gamma_{k,p} \leq \frac{k-2}{2k-2}\left(1+\frac{k}{p}\right) \leq \frac{k-2}{k-1}$ so $\frac{2}{1-\gamma_{k,p}} \leq 2(k-1)$ and hence the maximum is always equal to $(k+1)^{2(k-1)}$, which is the stated lower bound on q.

Given positive slopes, we can solve for the minimum value of s where each upper bound on θ is positive. This gives the stated lower bounds on s. $\square$

Proof of Theorems 1.2 and 1.4. Since Theorem 1.2 is the special case of Theorem 1.4 where $C = \mathbb{P}^1$, we focus on the proof of Theorem 1.4.

We apply Theorem 3.16 to obtain

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^k = f\} - q^{e(s-k)+(s-1)(1-g)} \prod_{v \in |C|} \ell_v(f)$$
$$= O_{s,k,q,g,\delta}(q^{(s-k-\theta)e})$$

for some $\theta > 0$, since the conditions in Theorem 3.16 for $\theta > 0$ to exist are exactly the conditions in Theorem 1.4. The condition $k \geq 2$ and $p > k$ of Theorem 3.16 are also assumed in Theorem 1.4. The condition $s > 2k$ of Theorem 3.16 follows from the condition

$$s > \max\left(\frac{2(k - \gamma_{k,p} - 2\frac{\log(k+1)}{\log q})}{1 - \gamma_{k,p} - 2\frac{\log(k+1)}{\log q}}, \frac{2k}{1 - 2(k-1)\frac{\log(k+1)}{\log q}} \right)$$

since $1 - 2(k-1)\frac{\log(k+1)}{\log q} < 1$. The condition $e > 2g - 2$ can be dropped since the small e case can be handled by increasing the implicit constant.

To obtain the desired asymptotic

$$\#\{a \in H^0(C,L)^s \mid \sum_{i=1}^{s} a_i^k = f\} = (1 + o(1)) q^{e(s-k)+(s-1)(1-g)} \prod_{v \in |C|} \ell_v(f)$$

it suffices to show that $\prod_{v \in |C|} \ell_v(f)^{-1} = O(1)$ so that the main term has size $\gg q^{e(s-k)}$. This is accomplished by Lemma 2.9 as long as $k \geq 2, s > k+1, s \geq 5$, and $q > (k-1)^4$. All these conditions follow from our assumptions: $k \geq 2$ is simply assumed itself, while $s > k+1$ and $s \geq 5$ follow from the earlier checked $s > 2k$, and $q > (k-1)^4$ is trivial if $k = 2$ and otherwise follows from $q > (k+1)^{2(k-1)}$ which since $k \geq 3$ implies $q > (k+1)^4 > (k-1)^4$. $\square$

4. Manin's conjecture for Fermat hypersurfaces

Let X be the hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the equation $\sum_{i=0}^n x_i^d = 0$. The goal of this section is to prove Theorem 1.5, estimating $\#\{f : C \to X \mid \text{degree } e\}$ for a smooth projective curve C of genus g over $\mathbb{F}_q$.

It turns out that this can be expressed as a sum over effective divisors D on C. This sum involves the Möbius function of a divisor D: We define the Möbius function $\mu(D)$ to equal 0 if D has multiplicity > 1 at any point and otherwise to equal (-1) raised to a power equal to the number of closed points in D.

Lemma 4.1. *Fix a finite field $\mathbb{F}_q$ and positive integers n and d. Let X be the hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the equation $\sum_{i=0}^n x_i^d = 0$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. For a nonnegative integer e we have*

$$\#\{f : C \to X \mid \textit{degree } e\} = \frac{1}{q-1} \sum_{D \textit{ effective}} \mu(D) \sum_{\substack{L \textit{ on } C \\ \textit{degree } e - \deg(D)}} \left(\#\{\mathbf{a} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^{n} a_i^d = 0\} - 1 \right).$$

Proof. Every map of degree e from C to X defines a line bundle of degree e on C, by pulling back $\mathcal{O}(1)$, and an $n+1$-tuple of sections of this line bundle, whose dth powers sum to zero, and which don't all vanish at the same point. The sections are well-defined up to the action of the automorphisms $\mathbb{F}_q^\times$ of the line bundle. Conversely, given a line bundle of degree e and $n+1$ sections satisfying these two conditions, we obtain a map $C \to X$. This implies

$$\#\{f\colon C \to X \mid \text{degree } e\} = \frac{1}{q-1} \sum_{\substack{L \text{ on } C \\ \text{degree } e}} \#\{\mathbf{x} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n x_i^d = 0,\ \text{no common zero}\}.$$

The condition that $x_0, \dots, x_n$ have no common zero is the condition that for each closed point v of C the x_i do not all vanish at v. This condition can be detected by Möbius inversion, equivalently, inclusion-exclusion, as an alternating sum over divisors of C. We must first remove the tuples which are all zero since otherwise the sum over D would be infinite.

$$\frac{1}{q-1} \sum_{\substack{L \text{ on } C \\ \text{degree } e}} \#\{\mathbf{x} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n x_i^d = 0,\ \text{no common zero}\}$$

$$= \frac{1}{q-1} \sum_{\substack{L \text{ on } C \\ \text{degree } e}} \sum_{D \text{ effective}} \mu(D) \#\{\mathbf{x} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n x_i^d = 0, \text{not all 0, vanishing at each point of } D\}$$

$$= \frac{1}{q-1} \sum_{\substack{L \text{ on } C \\ \text{degree } e}} \sum_{D \text{ effective}} \mu(D) \#\{\mathbf{a} \in H^0(C,L(-D))^{n+1} \mid \sum_{i=0}^n a_i^d = 0,\ \text{not all } 0\}$$

$$= \frac{1}{q-1} \sum_{\substack{L \text{ on } C \\ \text{degree } e}} \sum_{D \text{ effective}} \mu(D) \big(\#\{\mathbf{a} \in H^0(C,L(-D))^{n+1} \mid \sum_{i=0}^n a_i^d = 0\} - 1\big)$$

$$= \frac{1}{q-1} \sum_{D \text{ effective}} \mu(D) \sum_{\substack{L \text{ on } C \\ \text{degree } e - \deg(D)}} \big(\#\{\mathbf{a} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\} - 1\big).$$

□

To obtain Theorem 1.5, we will plug Theorem 3.16 into Lemma 4.1 to estimate the individual terms

$$\#\{\mathbf{a} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}.$$

The main term of Theorem 3.16, in these variables, is $q^{(e - \deg D)(n+1-d) + n(1-g)} \prod_{v \in |C|} \ell_v(0)$. Summing this main term turns out to give the main term of Theorem 1.5.

Lemma 4.2. *Fix a finite field $\mathbb{F}_q$ and positive integers $n > 3$ and d. Let X be the hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the equation $\sum_{i=0}^n x_i^d = 0$. Let C be a smooth projective geometrically* *irreducible curve of genus g over $\mathbb{F}_q$. For a nonnegative integer e we have*

$$\frac{1}{q-1}\sum_{D \text{ effective}} \mu(D) \sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}} q^{(e-\deg D)(n+1-d)+n(1-g)} \prod_{v\in|C|} \ell_v(0)$$
$$= \frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\prod_{v\in|C|}\Big((1-q^{-\deg v})\frac{\#X(\mathbb{F}_{q^{\deg v}})}{q^{(n-1)\deg v}}\Big).$$

Proof. We have

$$\frac{1}{q-1}\sum_{D \text{ effective}} \mu(D) \sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}} q^{(e-\deg D)(n+1-d)+n(1-g)} \prod_{v\in|C|} \ell_v(f)$$
$$= \frac{1}{q-1}\sum_{D \text{ effective}} \mu(D)\#\,\mathrm{Pic}^0(C) q^{(e-\deg D)(n+1-d)+n(1-g)} \prod_{v\in|C|} \ell_v(f)$$
$$= \frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\sum_{D \text{ effective}} \mu(D) q^{-\deg D(n+1-d)} \prod_{v\in|C|} \ell_v(f)$$
$$= \frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\prod_{v\in|C|}(1-q^{-\deg v(n+1-d)}) \prod_{v\in|C|} \ell_v(f).$$

Now for $v \in |C|$ we have

$$(1-q^{-\deg v(n+1-d)})\ell_v(f)$$
$$= (1-q^{-\deg v(n+1-d)})\lim_{r\to\infty}\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi^r\}}{q^{rn\deg v}}$$
$$= \lim_{r\to\infty}\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r\}}{q^{rn\deg v}} - q^{-\deg v(n+1-d)}\lim_{r\to\infty}\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r\}}{q^{rn\deg v}}$$
$$= \lim_{r\to\infty}\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r\}}{q^{rn\deg v}} - q^{-\deg v(n+1-d)}\lim_{r\to\infty}\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^{r-d})^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^{r-d}\}}{q^{(r-d)n\deg v}}$$
$$= \lim_{r\to\infty}\Big(\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r\}}{q^{rn\deg v}} - q^{-\deg v(n+1-d)}\frac{\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^{r-d})^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^{r-d}\}}{q^{(r-d)n\deg v}}\Big).$$

For $\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1}$ such that $\sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r$, if $b_0,\dots,b_n$ are all divisible by π then $\frac{b_0}{\pi},\dots,\frac{b_n}{\pi}$ are all well-defined in $\mathcal{O}_{C_v}/\pi_v^{r-1}$ and satisfy $\sum_{i=0}^n \left(\frac{b_i}{\pi}\right)^d = 0 \bmod \pi_v^{r-d}$. Modding these out by π_v^{r-d}, they give a tuple in O_{C_v}/π_v^{r-d} of solutions to $\sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^{r-d}$, and each solution to that equation defines $q^{(d-1)(n+1)\deg v}$ solutions to $\sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r$. This gives

$$\#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r\}$$
$$= \#\{\mathbf{b}\in(\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r, \pi \nmid \gcd(b_0,\dots,b_n)\}$$

$$+\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r, \pi \mid \gcd(b_0,\dots,b_n)\}$$

$$= \#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r, \pi \nmid \gcd(b_0,\dots,b_n)\}$$

$$+q^{(d-1)(n+1)\deg v}\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^{r-d})^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^{r-d}\}$$

Since

$$(d-1)(n+1) - rn = -(n+1-d) - (r-d)n,$$

plugging this in, the second term cancels, and we obtain

$$(1-q^{-\deg v(n+1-d)})\ell_v(f)$$

$$= \lim_{r\to\infty} \frac{\#\{\mathbf{b} \in (\mathcal{O}_{C_v}/\pi_v^r)^{n+1} \mid \sum_{i=0}^n b_i^d \equiv 0 \bmod \pi_v^r, \pi \nmid \gcd(b_0,\dots,b_n)\}}{q^{rn\deg v}}$$

$$= \frac{\#\{b_0,\dots,b_n \in \mathbb{F}_{q^{\deg v}} \mid \sum_{i=0}^n b_i^d = 0\}}{q^{n\deg v}} = (1-q^{-\deg v})\frac{\#X(\mathbb{F}_{q^{\deg v}})}{q^{(n-1)\deg v}}$$

since the limit is attained already at $r = 1$ by Hensel's lemma.

Plugging this in gives

$$\frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\prod_{v\in|C|}\Big((1-q^{-\deg v})\frac{\#X(\mathbb{F}_{q^{\deg v}})}{q^{(n-1)\deg v}}\Big). \qquad \square$$

However, a subtlety is that Theorem 3.16 may only be applied to estimate

$$\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}} \#\{\mathbf{a} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}$$

if $e - \deg(D) > 2g-2$. If $e - \deg(D) \in [0, 2g-2]$, we will instead use a "trivial bound" based on upper bounding $\#\{\mathbf{a} \in H^0(C,L)^{n+1}\}$ using Clifford's theorem. If $e - \deg(D) < 0$ then $H^0(C,L)$ consists only of the zero vector so $\#\{\mathbf{a} \in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\} = 1$ and these terms cancel. Breaking up into different ranges, we will obtain an estimate with several different error terms, and we will then check in turn that each error term is smaller than the main term.

Lemma 4.3. *Fix a finite field $\mathbb{F}_q$ of characteristic p and positive integers n and d such that $2 \le d < p$. Let X be the hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the equation $\sum_{i=0}^n x_i^d = 0$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. Assume that*

$$q > (d+1)^{2(d-1)}$$

and

$$n+1 > \max\left(\frac{2(d-\gamma_{d,p} - 2\frac{\log(d+1)}{\log q})}{1-\gamma_{d,p} - 2\frac{\log(d+1)}{\log q}}, \frac{2d}{1-2(d-1)\frac{\log(d+1)}{\log q}}\right).$$

Then there exists $\theta > 0$ such that we have

$$\Big|\#\{f\colon C \to X \mid \textit{degree } e\} - \frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\prod_{v\in|C|}\Big((1-q^{-\deg v})\frac{\#X(\mathbb{F}_{q^{\deg v}})}{q^{(n-1)\deg v}}\Big)\Big| \leq$$

$$O_{n,d,q,g,\theta}\Big(\frac{\#\mathrm{Pic}^0(C)}{q-1}\sum_{\substack{D \textit{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)q^{(n+1-d-\theta)(e-\deg D)}\Big)$$

$$+\frac{\#\mathrm{Pic}^0(C)}{q-1}\sum_{\substack{D \textit{ effective}\\ e+2-2g\leq \deg D\leq e}} q^{(n+1)\frac{e-\deg D+2}{2}} + \Big|\frac{\#\mathrm{Pic}^0(C)}{q-1}\sum_{\substack{D \textit{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\Big|$$

$$+\Big|\frac{\#\mathrm{Pic}^0(C)}{q-1}\sum_{\substack{D \textit{ effective}\\ \deg D\geq e+\min(2-2g,1)}}\mu(D)q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)\Big|.$$

Proof. We apply Lemma 4.1. If $e > \deg(D)$ then for L of degree $e - \deg(D) < 0$ we have $H^0(C, L) = \{0\}$ so $\{\mathbf{a} \in H^0(C, L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\} = \{0^{n+1}\}$ has cardinality 1. Thus

$$\frac{1}{q-1}\sum_{D \text{ effective}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\Big(\#\{\mathbf{a}\in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}-1\Big)$$

$$(20)\qquad = \frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\Big(\#\{\mathbf{a}\in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}-1\Big)$$

$$+\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ e+2-2g\leq \deg D\leq e}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\Big(\#\{\mathbf{a}\in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}-1\Big)$$

as the terms with $e > \deg D$ vanish. For the second term of (20), we observe that
(21)

$$0 \leq \#\{\mathbf{a}\in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}-1 \leq \#\{\mathbf{a}\in H^0(C,L)^{n+1}\} = q^{(n+1)\dim H^0(C,L)} \leq q^{(n+1)\frac{e-\deg D+2}{2}}$$

by Clifford's theorem. For the first term of (20), we first split off the -1, getting

$$\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\Big(\#\{\mathbf{a}\in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}-1\Big)$$

$$(22)\qquad = \frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\#\{\mathbf{a}\in H^0(C,L)^{n+1} \mid \sum_{i=0}^n a_i^d = 0\}$$

$$-\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+2-2g}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}1$$

Applying Theorem 3.16 gives

$$\begin{aligned}
&\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\#\{\mathbf{a}\in H^0(C,L)^{n+1}\mid \sum_{i=0}^n a_i^d=0\}\\
&=\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}\Big(q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)+O_{n,d,q,g,\theta}(q^{(n+1-d-\theta)(e-\deg D)})\Big)\\
&=\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)\\
&+O_{n,d,q,g,\theta}\Big(\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}q^{(n+1-d-\theta)(e-\deg D)}\Big)
\end{aligned}\tag{23}$$

Adding terms back in and applying Lemma 4.2 gives

$$\begin{aligned}
&\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)\\
&=\frac{1}{q-1}\sum_{D \text{ effective}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)\\
&-\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D\geq e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)\\
&=\frac{q^{e(n+1-d)+n(1-g)}\#\mathrm{Pic}^0(C)}{q-1}\prod_{v\in|C|}\Big((1-q^{-\deg v})\frac{\#X(\mathbb{F}_{q^{\deg v}})}{q^{(n-1)\deg v}}\Big)\\
&-\frac{1}{q-1}\sum_{\substack{D \text{ effective}\\ \deg D\geq e+\min(2-2g,1)}}\mu(D)\sum_{\substack{L \text{ on } C\\ \text{degree } e-\deg(D)}}q^{(e-\deg D)(n+1-d)+n(1-g)}\prod_{v\in|C|}\ell_v(0)
\end{aligned}\tag{24}$$

Combining Lemma 4.1, (20), (21), (22), (23), and (24), we obtain the statement, after observing that in each case the term summed over L is independent of L and thus we may replace the sum over L with the length $\#\,\mathrm{Pic}^0(C)$ of the sum. □

Lemma 4.4. *Fix a finite field $\mathbb{F}_q$ and positive integers n,d with $n>d$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. Let $\theta>0$ be a real number. For a nonnegative integer e we have*

$$\sum_{\substack{D \text{ effective}\\ \deg D<e+\min(2-2g,1)}}\mu(D)q^{(n+1-d-\theta)(e-\deg D)}=o_{n,d,q,g,\theta}(q^{(n+1-d)e}).$$

Proof. Since $n > d$ and $\theta > 0$, we can always replace θ with a smaller value such that $\theta < n - d$ and $\theta > 0$. Since this can only grow the left-hand side, it suffices to handle the case $\theta < n - d$. We have

$$\sum_{\substack{D \text{ effective} \\ \deg D < \min(2-2g,1)}} q^{(n+1-d-\theta)(e-\deg D)} \leq \sum_{D \text{ effective}} q^{(n+1-d-\theta)(e-\deg D)}$$

$$= q^{(n+1-d-\theta)e} \prod_{v \in C} \frac{1}{1 - q^{-(n+1+d-\theta)\deg v}} = q^{(n+1-d-\theta)e} \zeta_C(q^{-(n+1+d-\theta)})$$

which is $\ll q^{(n+1-d-\theta)e}$ and thus is $o(q^{(n+1-d)})$ since $\theta < n - d$ ensures $\zeta_C(q^{n+1+d-\theta}) = O_{n,d,q,g,\theta}(1)$. ☐

Lemma 4.5. *Fix a finite field $\mathbb{F}_q$ and positive integers n, d with $n > d$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. We have*

$$\sum_{\substack{D \text{ effective} \\ e+2-2g \leq \deg D \leq e}} q^{(n+1)\frac{e-\deg D+2}{2}} = o_{n,q,g}(q^{(n+1-d)e}).$$

Proof. Since $e - \deg D \in [0, 2g-2]$ is bounded, the term $q^{(n+1)\frac{e-\deg D+2}{2}}$ is $O_{n,g,q}(1)$. So it suffices to show that

$$\sum_{\substack{D \text{ effective} \\ e+2-2g \leq \deg D \leq e}} 1 = o_{n,q,g}(q^{(n+1-d)e}).$$

But the left-hand side is bounded by the number of effective divisors of degree at most e, which by a zeta function argument is $O_{q,g}(q^e)$, and hence is $o_{q,g}(q^{(n+1-d)e})$ since $n > d$. ☐

Lemma 4.6. *Fix a finite field $\mathbb{F}_q$ and positive integers n, d with $n > d$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. We have*

$$\sum_{\substack{D \text{ effective} \\ \deg D < e+2-2g}} \mu(D) = o_{q,g}(q^{(n+1-d)e}).$$

Proof. The left-hand side is bounded by the number of effective divisors of degree at most $e + 2 - 2g$, which by a zeta function argument is $O_{q,g}(q^{e+2-2g}) = O_{q,g}(q^e)$, and hence is $o_{q,g}(q^{(n+1-d)e})$ since $n > d$. ☐

Lemma 4.7. *Fix a finite field $\mathbb{F}_q$ and positive integers n, d with $n > d$. Let C be a smooth projective geometrically irreducible curve of genus g over $\mathbb{F}_q$. We have*

$$\frac{1}{q-1} \sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} \mu(D) q^{(e-\deg D)(n+1-d)+n(1-g)} \prod_{v \in |C|} \ell_v(0) = o_{n,d,q,g}(q^{(n+1-d)e}).$$

Proof. The $\prod_{v \in |C|} \ell_v(0)$ and $q^{n(1-g)}$ factors are $O_{n,d,q,g}(1)$ and can be ignored. So it suffices to prove

$$\sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} \mu(D) q^{(e-\deg D)(n+1-d)} = o_{n,d,q,g}(q^{(n+1-d)e})$$

which dividing both sides by $q^{e(n+1-d)}$ is equivaelnt to

$$\sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} \mu(D) q^{-\deg D(n+1-d)} = o_{n,d,q,g}(1).$$

We have

$$\left| \sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} \mu(D) q^{-\deg D(n+1-d)} \right| \leq \sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} |\mu(D)| \, q^{-\deg D(n+1-d)} \leq \sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} q^{-\deg D(n+1-d)}$$

so it suffices to prove that $\sum_{\substack{D \text{ effective} \\ \deg D \geq e+2-2g}} q^{-\deg D(n+1-d)} = o(1)$, which is equivalent to the convergence of the sum $\sum_{D \text{ effective}} q^{-\deg D(n+1-d)} = \zeta_C(q^{-(n+1-d)})$, which indeed converges as $n > d$ so $n+1-d > 1$. $\square$

Proof of Theorem 1.5. This follows upon combining Lemmas 4.3, 4.4, 4.5, 4.6, and 4.7, obsevring in each case that $\frac{\# \operatorname{Pic}^0(C)}{q-1} = O_{q,g}(1)$ and can be ignored. $\square$

5. The singular locus in the arbitrary hypersurface case

Let F be a polynomial of degree d in $n+1$ variables $x_0, \dots, x_n$ whose vanishing locus in $\mathbb{P}^n$ is a smooth hypersurface X. Assume that the characteristic of $\mathbb{F}_q$ does not divide d. We consider in this section what happens if we apply similar techniques to those in the remainder of the paper to count maps from C to X of degree e or to count tuples $a_0, \dots, a_n$ in $H^0(C, L)$ such that $F(a_0, \dots, a_n)$ takes any fixed value.

In this case, for $\alpha \in H^0(C, L^d)^\vee$, the relevant exponential sum is

$$S^F(\alpha) = \sum_{a_0, \dots, a_n \in H^0(C,L)} \psi(\alpha(F(a_0, \dots, a_n)),$$

To estimate this sum, the analogue of Lemma 3.1 concerns the singular locus

$$\operatorname{Sing}^F_\alpha = \{a_0, \dots, a_n \in H^0(C, L) \mid \alpha(b \frac{\partial F}{\partial x_i}(a_0, \dots, a_n)) = 0 \text{ for all } i \in \{0, \dots, n\}, b \in H^0(C, L)\},$$

which is a closed subscheme of $H^0(C, L)^{n+1}$ for a line bundle L of degree e.

It turns out to be more natural to estimate the dimension of $\operatorname{Sing}^F_\alpha$ "on average" over different α, i.e. to understand

$$\operatorname{Sing}^{F,m} = \{Z \subset C, \overline{\alpha} \in H^0(Z, L)^\vee, a_0, \dots, a_n \in H^0(C, L) \mid \deg Z = m, \overline{\alpha} \text{ nondegenerate}, (a_0, \dots, a_n) \in \operatorname{Sing}^F_\alpha\}.$$

We now recall notation from the introduction. Let ∇F be the tuple of polynomials $\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}$. Since ∇F is an $n+1$-tuple of polynomials in $n+1$ variables with no common zeroes except 0, ∇F defines a map $\mathbb{P}^n \to \mathbb{P}^n$. Let Y be the blowup of $\mathbb{P}^n \times \mathbb{P}^n$ along the graph of this map. Let E be the exceptional divisor of this blowup.

We will relate $\operatorname{Sing}^{F,m}$ to moduli spaces of maps from C to Y. The degree of a map $C \to Y$ can be expressed as a triple of nonnegative integers: in order, the degree of the induced map to the first $\mathbb{P}^n$ (the source of the map ∇F), the degree of the induced map to the second $\mathbb{P}^n$ (the target of the map ∇F), and the degree of intersection with the exceptional divisor. We let $\operatorname{Mor}_{i_1,i_2,i_3}(C, Y)$ be the moduli space parameterizing morphisms $C \to X$ whose degree is the triple (i_1, i_2, i_3), and we let $\mathrm{Mor}'_{i_1,i_2,i_3}(C,Y)$ be the open subspace parameterizing only those maps whose image is not entirely contained in the exceptional divisor E.

The goal of this subsection is to prove the following proposition.

Proposition 5.1. *For n, d, e, m natural numbers satisfying*

$$n+1 \geq 2d,$$
$$e > \max(4g - 4\frac{d-1}{d-2} + \frac{2d}{d-2}, \frac{2((n+3)d-4)}{(n+1-2d)(d-2)})$$

and

$$m \leq \frac{de}{2} + 1,$$

we have

$$\dim \mathrm{Sing}^{F,m} \leq \max\Big(\max_{\substack{0 \leq j_1 \leq e \\ 0 \leq j_2 \leq m+2g-2-e \\ j_3 \leq m \\ j_3 \leq j_2 \\ j_3 \leq (d-1)j_1}} \dim \mathrm{Mor}'_{e-j_1, m+2g-2-e-j_2, m-j_3}(C,Y) + 2 + j_1 + j_2,$$

$$m + \lceil \frac{m}{d-1} \rceil + (n+1)(e+1-g-\lceil \frac{m}{d-1} \rceil), m + \frac{de-m-2g+2}{d-1} + (n+1)(e+1-g-\frac{de-m-2g+2}{d-1}), 2g-1+2m\Big).$$

We note that the expected dimension of $\mathrm{Mor}'_{i_1,i_2,i_3}(C,Y)$ is $(n+1)i_1 + (n+1)i_2 - (n-1)i_3 - 2n(g-1)$ since the anticanonical divisor is $n+1$ times the hyperplane class of the first $\mathbb{P}^n$ plus $n+1$ times the hyperplane class of the second $\mathbb{P}^n$ minus $(n-1)$ times the exceptional divisor and the dimension is $2n$. If the true dimension is equal to the expected dimension, then the maximum over j_1, j_2, j_3 is attained for $j_1 = j_2 = j_3 = 0$ at a dimension bound of

$$(n+1)(m+2g-2) - (n-1)m - 2n(g-1) = 2m + 2g - 2$$

which is dominated by the other terms.

Remark 5.2. To make use of Proposition 5.1, we could use the bound on $\dim \mathrm{Sing}^{F,m}$ to bound Sing^F_α for typical α, and obtain a bound for $S^F(\alpha)$. To obtain interesting arithmetic consequences, we need the sum over α of $S^F(\alpha)$ to be dominated by $S^F(0)$. In this remark, we will explain under what conditions that might be possible.

The analogue of Lemma 3.1 will give a bound for $S^F(\alpha)$ of

$$q^{\frac{(n+1)(e+1-g)+\dim \mathrm{Sing}^F_\alpha + 1}{2}}$$

times a Betti number bound factor. If we stratify the space of possible linear forms α of degree m by $\dim \mathrm{Sing}^F_\alpha$, then a codimension c stratum will consist of α with $\dim \mathrm{Sing}^F_\alpha \leq \dim \mathrm{Sing}^{F,m} + c$. The number of points in the codimension c stratum will be q^{2m-c} times a Betti number bound factor, so the total contribution of this stratum is

$$q^{2m-c} q^{\frac{(n+1)(e+1-g)+\dim \mathrm{Sing}^{F,m} + c + 1}{2}}$$

which is maximized for $c = 0$ with a value of

$$q^{2m + \frac{(n+1)(e+1-g)+\dim \mathrm{Sing}^{F,m} + c + 1}{2}}.$$

Since $S^F(0) = q^{(n+1)(e+1-g)}$, a bound for the sum over α of $S^F(\alpha)$ of the form $q^{(n+1)(e+1-g)-\frac{e\delta}{2}}$ times a Betti number factor would suffice for the main term to dominate the error term as long as the Betti number bound is exponential and q is sufficiently large, which are reasonable assumptions to make. This requires

$$\dim \mathrm{Sing}^{F,m} \overset{?}{<} (n+1)(e+1-g) - 4m - e\delta + O(1) \text{ for all } m \in (e+1-2g, \frac{de}{2}+1]. \tag{25}$$

A linear programming calculation (Corollary A.1) shows that we obtain (25) for e sufficiently large as long as $n > 5d - 5 + (d-1)\delta$ and for all tuples i_1, i_2, i_3 of nonnegative integers such that $i_1 + i_2 \le i_3 + 2g - 2$ and $d(i_2 + 1 - 2g) \le (d-2)(i_3 - 1)$ we have

(26)

$$\dim \mathrm{Mor}'_{i_1,i_2,i_3} \le O(1) + \begin{cases} (n+2-\delta)i_1 + i_2 - 5i_3 & \text{if } i_3 \le \frac{di_1}{2} + 1 \\ (n+2-\frac{5d}{2}-\delta)i_1 + i_2 & \text{if } i_3 > \frac{di_1}{2} + 1 \text{ and } \frac{(d-2)i_1}{2} + 2g - 1 \ge i_2 \\ i_1 + \frac{2(n+1-\delta)-4d-2}{d-2} i_2 & \text{if } \frac{(d-2)i_1}{2} + 2g - 1 < i_2 \end{cases}.$$

Depending on n, (26) may be considerably weaker than the claim that $\dim \mathrm{Mor}'_{i_1,i_2,i_3}(C,Y) = (n+1)i_1 + (n+1)i_2 - (n-1)i_3 - 2n(g-1)$, but it is not clear when we can establish this weaker statement.

Even with strong assumptions on $\dim \mathrm{Mor}'_{i_1,i_2,i_3}(C,Y)$, the lower bound on n is worse than in the Fermat case because the estimate on the ℓ^2 norm of $S_1(\alpha)$ that is used in the Fermat case does not have an analogue in the general case. It might be possible to rectify this by understanding the dimensions of the individual $\dim \mathrm{Sing}^F_\alpha$, showing that in fact in low codimension strata the dimensions are smaller than expected from $\dim \mathrm{Sing}^{F,m}$.

We now begin the proof of Proposition 5.1. We will keep the assumptions of Proposition 5.1 on n, d, m, e throughout. We first introduce a modified form of the singular locus. Let

$$\widetilde{\mathrm{Sing}}^{F,m} = \{Z \subset C, \tilde\alpha \in H^0(Z, K_C(Z) \otimes L^{-d}), a_0, \dots, a_n \in H^0(C, L), c_0, \dots, c_n \in H^0(C, K_C(Z) \otimes L^{-1}) \mid$$
$$\deg Z = m, \tilde\alpha \text{ invertible}, c_i |_Z = \tilde\alpha \frac{\partial F}{\partial x_i}(a_0, \dots, a_n) \text{ for all } i\}$$

Lemma 5.3. *We have*
$$\dim \mathrm{Sing}^{F,m} = \dim \widetilde{\mathrm{Sing}}^{F,m}.$$

Proof. Lemma 3.2 and an argument identical to Lemma 3.3 together imply that each nondegenerate $\overline{\alpha} \in H^0(Z, L^d)^\vee$ is associated to an invertible $\tilde\alpha \in H^0(Z, K_C(Z) \otimes L^{-d})$ and we have $(a_0, \dots, a_n) \in \mathrm{Sing}^F_\alpha$ if and only if there exist $c_0, \dots, c_n \in H^0(C, K_C(Z) \otimes L^{-1})$ such that $c_i |_Z = \tilde\alpha \frac{\partial F}{\partial x_i}(a_0, \dots, a_n)$ for all i from 0 to n. □

We divide $\widetilde{\mathrm{Sing}}^{F,m}$ into various locally closed subsets and bound the dimension of each one.

Lemma 5.4. *The dimension of the locus in* $\widetilde{\mathrm{Sing}}^{F,m}$ *where* $c_0, \dots, c_n = 0$ *is*
$$m + \lceil \frac{m}{d-1} \rceil + (n+1)(e+1-g-\lceil \frac{m}{d-1} \rceil).$$

Proof. On this locus, the condition $c_i \mid_Z = \tilde{\alpha}\frac{\partial F}{\partial x_i}(a_0, \dots, a_n)$ simply forces $\frac{\partial F}{\partial x_i}(a_0, \dots, a_n) \mid_Z = 0$. In particular, the condition is independent of the choice of $\tilde{\alpha}$. Furthermore, since F defines a smooth hypersurface, we have $\frac{\partial F}{\partial x_i}(a_0, \dots, a_n) \mid_Z = 0$ for all i if and only if $a_0, \dots, a_n$ all vanish at each point of Z to order at least $\frac{1}{d-1}$ times the multiplicity of that point in Z. The "if" is relatively clear and the "only if" follows from the fact that if $a_0, \dots, a_n$ do not all vanish on a point then $\frac{\partial F}{\partial x_i}(a_0, \dots, a_n)$ cannot vanish at that point for all i. It follows that the dimension of the locus in $\widetilde{\text{Sing}}^{F,m}$ where $c_0, \dots, c_n = 0$ is

$$m + \dim\{Z \subset C, a_0, \dots, a_n \in H^0(C, L) \mid \deg Z = m, \gcd(a_0, \dots, a_n)^{d-1} \mid_Z = 0\}.$$

For the tuple $a_0, \dots, a_n$ all zero the space of possible Z has degree m. For any other tuple $a_0, \dots, a_n$ there are finitely many possible Z, and such a Z exists if and only if $\gcd(a_0, \dots, a_n)$ has degree at least $\frac{m}{d-1}$. Hence the dimension of the locus in $\widetilde{\text{Sing}}^{F,m}$ where $c_0, \dots, c_n = 0$ is

$$\max(2m, m + \dim\{a_0, \dots, a_n \in H^0(C, L) \mid \deg \gcd(a_0, \dots, a_n) \geq \frac{m}{d-1}\}).$$

To obtain a tuple $a_0, \dots, a_n$ whose gcd has degree at least $\frac{m}{d-1}$, we can choose a divisor D of degree $\lceil \frac{m}{d-1} \rceil$ and then choose $n+1$ sections of $H^0(C, L(-D))$. Every such tuple arises this way from at least one D. The dimension of the space of divisors of degree $\lceil \frac{m}{d-1} \rceil$ is $\lceil \frac{m}{d-1} \rceil$ and each such divisor has a space of global sections of dimension at most $e + 1 - g - \lceil \frac{m}{d-1} \rceil$ since we have

$$e - \lceil \frac{m}{d-1} \rceil \geq e - \lceil \frac{\frac{de}{2} + 1}{d-1} \rceil \geq e - \frac{\frac{de}{2} + 1}{d-1} - 1 = e\frac{d-2}{2d-2} - \frac{d}{d-1} > 2g - 2 \tag{27}$$

by assumption on e. So the total dimension of the locus in $\widetilde{\text{Sing}}^{F,m}$ where $c_0, \dots, c_n = 0$ is

$$\max(2m, m + \lceil \frac{m}{d-1} \rceil + (n+1)(e + 1 - g - \lceil \frac{m}{d-1} \rceil)).$$

We furthermore have

$$(n+1)(e + 1 - g - \lceil \frac{m}{d-1} \rceil) > (n+1)(e\frac{d-2}{2d-2} - \frac{d}{d-1} + 1 - g) > \frac{n+1}{2}(e\frac{d-2}{2d-2} - \frac{d}{d-1})$$

while

$$m - \lceil \frac{m}{d-1} \rceil \leq \frac{m(d-2)}{d-1} \leq e\frac{d(d-2)}{2d-2} + \frac{d-2}{d-1}$$

and by assumption on e we have

$$e\frac{d(d-2)}{2d-2} + \frac{d-2}{d-1} < \frac{n+1}{2}(e\frac{d-2}{2d-2} - \frac{d}{d-1})$$

so the maximum is always

$$m + \lceil \frac{m}{d-1} \rceil + (n+1)(e + 1 - g - \lceil \frac{m}{d-1} \rceil).$$ □

On the complementary locus where some $c_i \neq 0$, we must have $c_i \mid_Z \neq 0$, which forces us to have some $a_i \neq 0$. Then $a_0, \dots, a_n$ define a map $C \to \mathbb{P}^n$ of degree $e - j_1$, where j_1 is the total degree of the common vanishing locus of $a_0, \dots, a_n$. Similarly $c_0, \dots, c_n$ define a map $C \to \mathbb{P}^n$ of degree $2g - 2 + m - e - j_2$, where j_2 is the total degree of the common vanishing locus of $c_0, \dots, c_n$. Combining these maps, we obtain a map $C \to \mathbb{P}^n \times \mathbb{P}^n$. There are two possibilities: either the image of this map is contained in the graph of ∇F, or not.

Lemma 5.5. *The total dimension of the locus in* $\widetilde{\mathrm{Sing}}^{F,m}$ *where some* $c_i \neq 0$ *and the induced map* $C \to \mathbb{P}^n \times \mathbb{P}^n$ *has image in the graph of* ∇F *is at most*

$$\max_{\substack{0 \leq j_1 \leq e \\ 0 \leq j_2 \leq 2g-2+m-e \\ 2g-2+m-e-j_2=(d-1)(e-j_1)}} m + j_1 + j_2 + (n+1)(\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0)).$$

Proof. If the image of the map $C \to \mathbb{P}^n \times \mathbb{P}^n$ is contained in the graph of ∇F, then the map $C \to \mathbb{P}^n$ defined by $c_0, \dots, c_n$ is obtained as the composition of the map $C \to \mathbb{P}^n$ defined by $a_0, \dots, a_n$ with ∇F. In particular, this forces

$$2g - 2 + m - e - j_2 = (d-1)(e - j_1).$$

To choose a point in this locus, we first choose the common vanishing locus of the $a_0, \dots, a_n$, a divisor D on C of degree j_1. We then choose the $a_0, \dots, a_n$ as sections of $H^0(C, L(-D))$, nowhere all vanishing. Applying ∇F, we get a tuple of sections of $H^0(C, L^{d-1}(-(d-1)D))$, nowhere all vanishing, and then multiply them all by a divisor of degree j_2 to obtain $c_0, \dots, c_n$. The space of choices of Z and $\tilde{\alpha}$ compatible with a given $a_0, \dots, a_n, c_0, \dots, c_n$ has dimension at most m: For each point v of Z outside the common vanishing locus of the a_i, the equation $c_i \mid_Z = \tilde{\alpha}\frac{\partial F}{\partial x_i}(a_0, \dots, a_n)$ determines the restriction of $\tilde{\alpha}$ to v. The dimension of the space of valid choices of $\tilde{\alpha}$ for a given Z is thus at most the total multiplicity of the points of Z that are also vanishing points of a_i. However, the dimension of the space of divisors of degree m where the total multiplicity of a given finite set is r is $m - r$, so the total dimension is at most m regardless of the choice of total multiplicity.

Adding together this dimension m, the dimension j_1 of the space of divisors of degree j_1, the dimension j_2 of the space of divisors of degree j_2, and $n+1$ times the maximum possible dimension $\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0)$ of the space of sections of $H^0(C, L(-D))$, we obtain the stated formula. ∎

Lemma 5.6. *We have*

$$\max_{\substack{0 \leq j_1 \leq e \\ 0 \leq j_2 \leq 2g-2+m-e \\ 2g-2+m-e-j_2=(d-1)(e-j_1)}} m + j_1 + j_2 + (n+1)(\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0))$$

$$\leq \max(m + \lceil \frac{m}{d-1} \rceil + (n+1)(e+1-g-\lceil \frac{m}{d-1} \rceil), m + \frac{de-m-2g+2}{d-1} + (n+1)(e+1-g-\frac{de-m-2g+2}{d-1}), 2g-1+2m).$$

Proof. Since $n+1 \geq 2d$, increasing j_1 by 1 and j_2 by $d-1$ always reduces the expression

$$m + j_1 + j_2 + (n+1)(\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0)) \tag{28}$$

unless $e - j_1 \leq -2$ and $\max(e+1-g-j_1, \frac{e+1-j_1}{2}, 0) = 0$ already, in which case increasing j_1 by 1 and j_2 by $d-1$ increases the expression (28). Hence the maximum value of (28) is always attained at either the minimum value of j_1 or the maximum value of j_1. For the minimum value, the two constraints are $0 \leq j_1$ and $0 \leq j_2 = 2g-2+m-e-(d-1)(e-j_1)$, in other words $de - m - 2g + 2 \leq (d-1)j_1$. The second bound is always stricter since $m \leq \frac{de}{2} + 1$ so

$$de - m - 2g + 2 \geq \frac{de}{2} - 2g + 1 > -1$$

since $e \geq 2g - 2$ and $d \geq 2$. Hence the minimum value is j_1 is $\lceil \frac{de-m-2g+2}{d-1} \rceil$. We can simplify by plugging in $j_1 = \frac{de-m-2g+2}{d-1}$ which gives a slightly worse bound.

The maximum value of j_1 is e, which forces $j_2 = 2g - 2 + m - e$. Hence the maximum over j_1, j_2 can be bounded by

$$\max_{(j_1,j_2)=(\frac{de-m-2g+2}{d-1},0) \text{ or } (e,2g-2+m-e)} m + j_1 + j_2 + (n+1)\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0).$$

We can simplify this expression. In the case $j_1 = \frac{de-m-2g+2}{d-1}$, $j_2 = 0$, we can assume the maximum value of (28) is attained at the minimum value of j_1. Thus $\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0) > 0$. If in addition $j_1 > \lceil \frac{m}{d-1} \rceil$ then in any case the expression $m + j_1 + (n+1)(\max(e+1-g-j_1, \frac{e+2-j_1}{2}))$ is bounded by $m + \lceil \frac{m}{d-1} \rceil + (n+1)(e+1-g-\lceil \frac{m}{d-1} \rceil)$ since the expression $m + j + (n+1)(\max(e+1-g-j_1, \frac{e+2-j}{2}))$ decreases as a function of j and from (27) we have $e - \lceil \frac{m}{d-1} \rceil > 2g-2$ so for $j = \lceil \frac{m}{d-1} \rfloor$ the maximum is given by $e+1-g-\lceil \frac{m}{d-1} \rceil$. If we make the opposite assumption that $j_1 \leq \lceil \frac{m}{d-1} \rceil$, which by (27) implies $j_1 < e+2-2g$, then we have $\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0)) = e+1-g-j_1$. These cases give the first two terms in the statement.

In the case $(j_1, j_2) = (e, 2g-2+m-e)$ we have $(\max(e+1-g-j_1, \frac{e+2-j_1}{2}, 0) = \max(1-g, 1, 0) = 1$ so the value of (28) is $2g - 1 + 2m$. This gives the last term in the statement. □

Lemma 5.7. *The dimension of the locus in* $\widetilde{\mathrm{Sing}}^{F,m}$ *where some* $c_i \neq 0$ *and the map* $C \to \mathbb{P}^n \times \mathbb{P}^n$ *is not contained in the graph of* ∇F *is at most*

$$\max_{\substack{0 \leq j_1 \leq e \\ 0 \leq j_2 \leq m+2g-2-e \\ j_3 \leq m \\ j_3 \leq j_2 \\ j_3 \leq (d-1)j_1}} \dim \mathrm{Mor}'_{e-j_1, m+2g-2-e-j_2, m-j_3}(C, Y) + 2 + j_1 + j_2.$$

Proof. If the image of the map $C \to \mathbb{P}^n \times \mathbb{P}^n$ is not contained in the graph of ∇F, then it lifts uniquely to a map $f: C \to Y$ by the valuative criterion of properness applied to the blowup $Y \to \mathbb{P}^n \times \mathbb{P}^n$. The degree of this map is $(e - j_1, 2g - 2 + m - e - j_2, m - j_3)$ for some integer j_3, where $m - j_3$ is the degree of $f^{-1}(E)$, i.e. the length as a scheme of the inverse image of the graph of ∇F under $C \to \mathbb{P}^n \times \mathbb{P}^n$. Given such a map f, $(a_0, \dots, a_n)$ are determined up to scaling by f and the choice of a divisor D_1 of degree j_1 on which $a_0, \dots, a_n$ all vanish, and $(c_0, \dots, c_n)$ are determined up to scaling by f and the choice of divisor D_2 of degree j_2 on which $c_0, \dots, c_n$ all vanish. The scaling factors add 2 to the dimension. Not all divisors work, as we must have the inverse image of the hyperplane class of the first $\mathbb{P}^n$ plus D_1 agree with the class of L, and a similar criterion involving D_2, but forgetting this restriction still gives a valid upper bound for the dimension.

The divisor Z is, in this case, tightly constrained by the map. Let v be a point in the support of Z with uniformizer π. Let o_1 be the multiplicity of v in D_1, o_2 be the multiplicity of v in D_2, w be the multiplicity of v in $f^{-1}(E)$, and z be the multiplicity of v in Z. Then we can check that:

$$z \leq w + \min((d-1)o_1, o_2)) \tag{29}$$

since $\frac{\partial F}{\partial x_i}(a_0, \dots, a_n)$ all vanish to order $(d-1)o_1$, $(c_0, \dots, c_n)$ all vanish to order o_2, the first w nonvanishing coefficients in their π-adic expansions all agree up to a common scalar, and their remaining coefficients do not agree, so in total at most $w + \min((d-1)o_1, o_2))$ coefficients agree. Furthermore, the dimension of the space of choices for $\tilde{\alpha}$, restricted to the subscheme $z[v]$, is at most $\min((d-1)o_1, o_2)$ since if $o_2 < (d-1)o_1$ then a choice for $\tilde{\alpha}$ only exists if $z \leq o_2$, in which case the dimension of the space of choices is z, and otherwise the equation $\tilde{\alpha}\frac{\partial F}{\partial x_i}(a_0, \dots, a_n) = c_i \mid_Z$ for some i such that $\frac{\partial F}{\partial x_i}(a_0, \dots, a_n)$ vanishes to order exactly $(d-1)o_1$ has a space of solutions of dimension at most $(d-1)o_1$.

In particular, every point in the support of Z must either lie in $f^{-1}(E)$ or lie in both D_1 and D_2, so there are certainly at most finitely many possible choices for Z. The dimension of the space of total choices for α is at most $j_1 + j_2$ minus the number of distinct points in the support of $D_1 \cup D_2$, since the local contribution to $\deg D_1 + \deg D_2$ minus the number of points in the support is $o_1 + o_2 - 1 \geq \min((d-1)o_1, o_2)$ unless $o_1 = o_2 = 0$ in which case the local contribution is 0. Since the dimension of the space of pairs of divisors of degrees j_1, j_2 supported in a set of size r is at most r, the dimension of the space of choices of D_1, D_2, Z, α is at most $j_1 + j_2$, giving that the dimension of the locus in $\widetilde{\text{Sing}}^{F,m}$ corresponding to a given map $f \colon C \to Y$ of multidegree $e - j_1, m + 2g - 2 - j_2, m - j_3$ is at most $2 + j_1 + j_2$.

However, summing (29) over all points gives the constraint

$$m \leq m - j_3 + \min((d-1)j_1, j_2)$$

which must be satisfied if any Z exists. This constraint implies $j_3 \leq (d-1)j_1$ and $j_3 \leq d_2$. $\square$

Proof of Proposition 5.1. By Lemma 5.3 it suffices to bound the dimension of $\widetilde{\text{Sing}}^{F,m}$. We divide $\widetilde{\text{Sing}}^{F,m}$ into three loci, based on whether some $c_i \neq 0$, and, if so, whether the image of the map $C \to \mathbb{P}^n \times \mathbb{P}^n$ is contained in the graph of ∇F, and its dimension is the maximal dimension of each locus. These dimensions are bounded in Lemmas 5.4, 5.5, and 5.7, with the expression of Lemma 5.5 simplified in Lemma 5.6. Combining the expressions of Lemmas 5.4, 5.6, and 5.7, we obtain the statement of Proposition 5.1. $\square$

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Appendix A. Linear programming for the arbitrary hypersurface case

Corollary A.1. *The bound (25) is satisfied for $\delta > 0$ as long as $n > 5d - 5 + (d-1)\delta$, e is sufficiently large depending on n, g, d, δ and for all tuples i_1, i_2, i_3 of nonnegative integers such that $i_1 + i_2 \le i_3 + 2g - 2$ and $d(i_2 + 1 - 2g) \le (d-2)(i_3 - 1)$ we have*

$$\dim \mathrm{Mor}'_{i_1,i_2,i_3} \le O(1) + \begin{cases} (n+2-\delta)i_1 + i_2 - 5i_3 & \text{if } i_3 \le \frac{di_1}{2} + 1 \\ (n+2-\frac{5d}{2}-\delta)i_1 + i_2 & \text{if } i_3 > \frac{di_1}{2} + 1 \text{ and } \frac{(d-2)i_1}{2} + 2g - 1 \ge i_2 \\ i_1 + \frac{2(n+1-\delta)-4d-2}{d-2} i_2 & \text{if } \frac{(d-2)i_1}{2} + 2g - 1 < i_2 \end{cases}. \tag{30}$$

Proof. In view of Proposition 5.1, it suffices to show that the right hand side of Proposition 5.1 is bounded by the right-hand side of (25). In other words, we must establish that

$$\dim \mathrm{Mor}'_{e-j_1,m+2g-2-e-j_2,m-j_3}(C,Y) < (n+1)(e+1-g) - 4m - 2 - j_1 - j_2 - e\delta + O(1) \tag{31}$$

for all $m \in (e+1-2g, \frac{de}{2}+1]$ and j_1, j_2, j_3 satisfying $0 \le j_1 \le e$, $0 \le j_2 \le m + 2g - 2 - e$, $j_3 \le m$, $j_3 \le j_2$, $j_3 \le (d-1)j_1$ and also that

$$m + \lceil \frac{m}{d-1} \rceil + (n+1)(e+1-g-\lceil \frac{m}{d-1} \rceil) < (n+1)(e+1-g) - 4m - e\delta \tag{32}$$

$$m + \frac{de-m-2g+2}{d-1} + (n+1)(e+1-g-\frac{de-m-2g+2}{d-1}) < (n+1)(e+1-g) - 4m - e\delta \tag{33}$$

$$2g - 1 + 2m < (n+1)(e+1-g) - 2m - e\delta \tag{34}$$

in each case for $m \in (e+1-2g, \frac{de}{2}+1]$, where we have dropped the $O(1)$ as unnecessary in the last three equations.

We handle these in reverse order. For (34), the left hand side grows and the right hand side shrinks as m grows, so it suffices to handle the case $m = \frac{de}{2}+1$, where the desired bound is

$$2g + 1 + de < (n+1)(e+1-g) - 2de - 4 - e\delta$$

or equivalently

$$2g + 4 + (n+1)g + n < (n+1-3d-\delta)e$$

which is satisfied for e sufficiently large as long as $n+1 > 3d+\delta$, which is weaker than our assumption.

For (33), we first simplify by cancelling terms, obtaining

$$5m + e\delta < n\frac{de - m - 2g + 2}{d-1}.$$

We again observe that the left-hand side grows and the right-hand side shrinks when m grows, and thus may substitute $m = \frac{de}{2}+1$, giving

$$5\frac{de}{2} + 5 + e\delta < n\frac{\frac{de}{2} - 2g + 1}{d-1}$$

or equivalently

$$5 + n\frac{2g-1}{d-1} < \left(n\frac{d}{2(d-1)} - \frac{5d}{2} - \delta\right) e$$

which is satisfied for e sufficiently large as long as $n\frac{d}{2(d-1)} > \frac{5d}{2} + \delta$, which is equivalent to $n > 5d - 5 + \delta\frac{2(d-1)}{d}$, which is weaker than our assumption.

For (32), we first simplify by canceling terms, obtaining

$$5m + e\delta < n\lceil\frac{m}{d-1}\rceil$$

for which it suffices to have

$$5m + e\delta < n\frac{m}{d-1}.$$

Since our assumption implies $n > 5d-5$, the right hand side grows faster than the left hand side with m, so it suffices to handle the case $m = e+2-2g$, which gives

$$5e + 10 - 10g + e\delta < n\frac{e+2-2g}{d-1}$$

or equivalently

$$10 - 10g + n\frac{2g-2}{d-1} < (\frac{n}{d-1} - 5 - \delta)e$$

which is satisfied for e sufficiently large as long as $n > 5d-5+(d-1)\delta$, which we assumed.

We finally consider the most difficult equation (31). We fix e, m, j_1, j_2, j_3 satisfying the hypotheses

$$0 \leq j_1 \leq e, 0 \leq j_2 \leq m+2g-2-e, j_3 \leq m, j_3 \leq j_2, j_3 \leq (d-1)j_1, e \geq 0, e+2-2g \leq m, m \leq \frac{de}{2}+1$$

and set $i_1 = e - j_1, i_2 = m + 2g - 2 - e - j_2, i_3 = m - j_3$. We have

$$i_3 + 2g - 2 - i_1 - i_2 = j_1 + j_2 - j_3 \geq j_1 \geq 0$$

so we always have $i_1 + i_2 \leq i_3 + 2g - 2$.

We also have

$$i_2 + 1 - 2g = m - e - j_2 - 1 \leq m - e - 1 \leq \frac{de}{2} + 1 - e - 1 = \frac{d-2}{2}e$$
$$= \frac{d-2}{2}(i_3 - i_2 + 2g - 2 + j_3 - j_2) \leq \frac{d-2}{2}(i_3 - i_2 + 2g - 2)$$

which gives $d(i_2+1-2g) \leq (d-2)(i_3-1)$.

Using the equations $e = i_1 + j_1$, $m = i_3 + j_3$, $j_2 = m + 2g - 2 - e - i_2 = i_3 + j_3 + 2g - 2 - i_1 - j_1 - i_2 = i_3 - i_1 - i_2 + j_3 - j_1 + O(1)$ we obtain

$$\begin{aligned}(n+1)(e+1-g) - 4m - 2 - j_1 - j_2 - e\delta &= O(1) + (n+1-\delta)e - 4m - j_1 - j_2 \\ &= O(1) + (n+1-\delta)(i_1+j_1) - 4i_3 - 4j_3 - j_1 - i_3 + i_1 + i_2 - j_3 + j_1 \\ &= O(1) + (n+2-\delta)i_1 + i_2 - 5i_3 + (n+1-\delta)j_1 - 5j_3 \\ &\geq O(1) + (n+2-\delta)i_1 + i_2 - 5i_3 + (n+1-\delta)j_1 - 5(d-1)j_1 \\ &\geq O(1) + (n+2-\delta)i_1 + i_2 - 5i_3\end{aligned}$$

where in the last two lines we use that $j_3 \leq (d-1)j_1$ and $j_1 \geq 0$ while $n+1-\delta > 5(d-1)$, which is weaker than our assumption. This verifies (31) in the first case of (30).

Next, using $j_2 = m + 2g - 2 - e - i_2 = m - e - i_2 + O(1)$ and $m \leq \frac{de}{2} + 1$ and $e = i_1 + j_1$ we obtain

$$(n+1)(e+1-g) - 4m - 2 - j_1 - j_2 - e\delta = O(1) + (n+1-\delta)e - 4m - j_1 - j_2 = O(1) + (n+2-\delta)e - 5m - j_1 + i_2$$

$$\geq O(1) + (n+2-\frac{5d}{2}-\delta)e - j_1 + i_2 = O(1) + (n+2-\frac{5d}{2}-\delta)i_1 + (n+1-\frac{5d}{2}-\delta)j_1 + i_2$$

$$\geq O(1) + (n+2-\frac{5d}{2}-\delta)i_1 + i_2$$

since $j_1 \geq 0$ and $n+1 \geq \frac{5d}{2} + \delta$. This verifies (31) in the second case of (30).

In the third case, we set $\tilde{m} = \frac{de}{2} + 1 - m$ so that we have $\tilde{m} \geq 0$ and observe that

$$i_2 + j_2 + 1 - 2g = m - 1 - e = \frac{de}{2} - \tilde{m} - e = \frac{(d-2)e}{2} - \tilde{m}$$

so that

$$e = \frac{2}{d-2}(i_2 + j_2 + \tilde{m} + 1 - 2g) = \frac{2}{d-2}(i_2 + j_2 + \tilde{m}) + O(1)$$

and thus

$$j_1 = e - i_1 = \frac{2}{d-2}(i_2 + j_2 + \tilde{m}) - i_1 + O(1)$$

which gives

$$(n+1)(e+1-g) - 4m - 2 - j_1 - j_2 - e\delta = O(1) + (n+1-\delta)e - 4m - j_1 - j_2$$

$$= O(1) + \frac{2}{d-2}(n+1-\delta)(i_2+j_2+\tilde{m}) - \frac{4d}{d-2}(i_2+j_2+\tilde{m}) + 4\tilde{m} - \frac{2}{d-2}(i_2+j_2+\tilde{m}) + i_1 - j_2$$

$$= O(1) + i_1 + \frac{2(n+1-\delta)-4d-2}{d-2}i_2 + \left(\frac{2(n+1-\delta)-4d-2}{d-2} - 1\right)j_2 + \left(\frac{2(n+1-\delta)-4d-2}{d-2} + 4\right)\tilde{m}.$$

We have $\frac{2(n+1-\delta)-4d-2}{d-2} - 1 > 0$ since $2(n+1-\delta) > 4d + 2 + (d-2) = 5d$ as this is weaker than our assumption. Thus we also have $\frac{2(n+1-\delta)-4d-2}{d-2} + 4 > 0$. Since $j_2 \geq 0$ and $\tilde{m} \geq 0$, these terms may be dropped, and we obtain

$$(n+1)(e+1-g) - 4m - 2 - j_1 - j_2 - e\delta \geq O(1) + i_1 + \frac{2(n+1-\delta)-4d-2}{d-2}i_2.$$

This verifies (31) in the third case of (30). $\square$

One can further check that (30) is sharp, in the sense that for each triple of nonnegative integers i_1, i_2, i_3 with $i_1 + i_2 \leq i_3 + 2g - 2$ there exist values of m, e, j_1, j_2, j_3 satisfying all the inequalities where the needed dimension bound in (31) in fact equals the assumed dimension bound in (30). We never need to check the inequalities $j_1 \leq e, j_2 \leq m + 2g - 2 - e, j_3 \leq m$ as these are equivalent to $i_1, i_2, i_3 \geq 0$ which are assumed anyways. Furthermore $e + 2 - 2g \leq m$ follows from $0 \leq j_2 \leq m + 2g - 2 - e$ and so does not need to be checked separately, and the same is true for $e \geq 0$ following from $0 \leq j_1 \leq e$. The remaining inequalities that need to be checked are $0 \leq j_1, 0 \leq j_2, j_3 \leq j_2, j_3 \leq (d-1)j_1$, and $m \leq \frac{de}{2} + 1$.

In the first case, we take $e = i_1$, $m = i_3$, $j_1 = 0$, $j_2 = i_3 + 2g - 2 - i_1 - i_2$, $j_3 = 0$. The assumption on i_1, i_2, i_3 implies $j_2 \geq 0$, from which it is easy to see that all the inequalities are satisfied except possibly $m \leq \frac{de}{2} + 1$. The inequality $m \leq \frac{de}{2} + 1$ expands to $i_3 \leq \frac{di_1}{2} + 1$ which is assumed in the first case.

In the second case, we take $e = i_1$, $m = \frac{de}{2} + 1 = \frac{di_1}{2} + 1$, $j_1 = 0$, $j_2 = \frac{(d-2)i_1}{2} + 2g - 1 - i_2$, $j_3 = \frac{di_2}{2} + 1 - i_3$. The assumption $i_3 > \frac{di_1}{2} + 1$ implies $j_3 < 0$ and the assumption $\frac{(d-2)i_1}{2} + 2g - 1 \geq i_2$ implies $j_2 \geq 0$. This verifies all the inequalities.

In the third case, we take $e = \frac{2}{d-2}(i_2 + 1 - 2g)$, $m = \frac{de}{2} + 1 = \frac{d}{d-2}(i_2 + 1 - 2g) + 1$, $j_1 = \frac{2}{d-2}(i_2 + 1 - 2g) - i_1$, $j_2 = 0$, $j_3 = \frac{d}{d-2}(i_2 + 1 - 2g) + 1 - i_3$. The assumption $\frac{(d-2)i_1}{2} + 2g - 1 < i_2$ gives $j_1 > 0$. The assumption $d(i_2 + 1 - 2g) \leq d(i_3 - 1)$ gives $j_3 \leq 0$. This verifies all the inequalities.