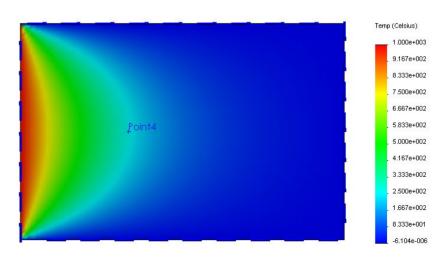
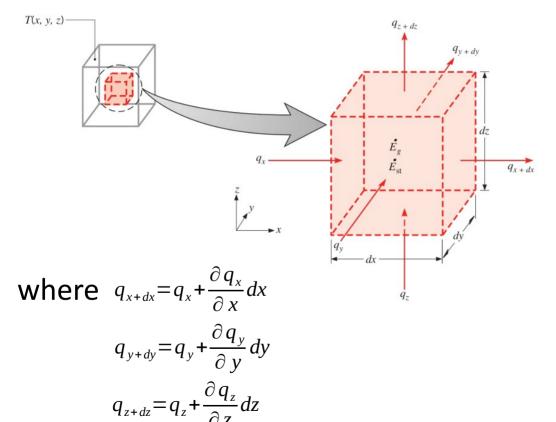


- Goal is to describe the temperature in any particular point within a substance.
- This is known as the temperature distribution.
- Once the distribution is known, the conduction heat flux may be determined at any particular point.
- First lets consider a homogeneous substance that is described using Cartesian coordinates.
- The temperature distribution would then be



Looking at a differential sized portion of the material



- The  $\dot{E}_g$  term refers to the rate of thermal energy generation (e.g. resistance heating).
- The  $\dot{E}_{st}$  term refers to the energy storage ability of the material; this is related to the thermal capacity of the substance.

- 1st Law may then be applied.
- Then some math is done.
- Which yields

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

- This is known as the heat diffusion equation for Cartesian coordinates.
- It is a partial differential equation.
- It governs the temperature distribution within a material.

# **Some Simplified Forms**

• If the thermal conductivity *k* is constant

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

• If the system is steady state

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{q} = 0$$

If the heat transfer is 1D

$$\frac{\partial}{\partial x} (k \frac{\partial T}{\partial x}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial y} (k \frac{\partial T}{\partial y}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

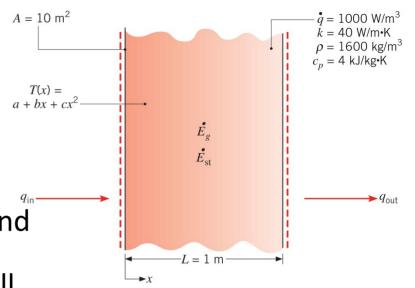
$$\frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

# **Example 1**

The temperature distribution across a wall  $1 m_{a+bx+cx^2}^{T(x)}$  thick at a certain instant of time is given as

$$T(x)=a+bx+cx^2$$

where T is in  ${}^{\circ}C$  and x is in meters, while a, b, and c are as shown to the right. A uniform heat generation of  $q = 1000 \ W/m^3$ , is present in the wall of area  $10 \ m^2$  having properties as shown. Determine the rate of heat transfer entering and leaving the wall at  $x=0 \ m$  and  $x=1 \ m$ , respectively. Determine the rate of change of energy storage in the wall. Determine the time rate of temperature change at x=0, 0.25, and  $0.5 \ m$ .



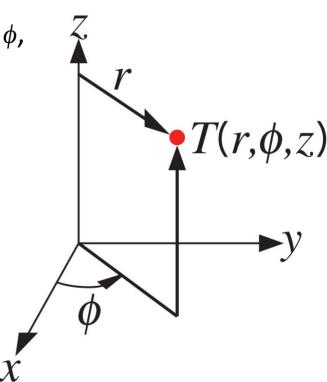
$$a=900 \,^{\circ}C$$

$$b=-300 \,^{\circ}C \over m$$

$$c = -50 \frac{^{\circ}C}{m^2}$$

### **Cylindrical Coordinate System**

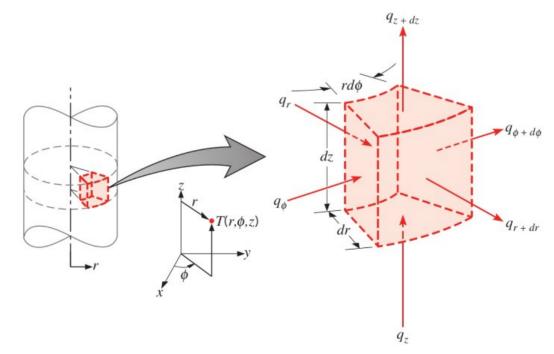
- An alternative coordinate system
- Instead of x, y, and z like when using Cartesian coordinates, cylindrical coordinate system use r,  $\phi$ , and z.
- r is a radius.
- $\phi$  is an angle.
- z is a height.
- The advantage is when cylindrical objects are analyzed.



#### **Heat Diffusion Equation for Cylindrical Systems**

 Using the cylindrical coordinate system, a similar approach may be used to derive the heat diffusion equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$



# **Some Simplified Forms**

• If the thermal conductivity k is constant

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

If the system is steady state

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = 0$$

If the heat transfer is 1D

$$\frac{1}{r} \frac{\partial}{\partial r} (kr \frac{\partial T}{\partial r}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{1}{r^2} \frac{\partial}{\partial \phi} (k \frac{\partial T}{\partial \phi}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

# Example 2

Uniform internal heat generation of  $5 \times 10^7$  W/m<sup>3</sup> is occurring in a cylindrical nuclear reactor fuel rod of 50 mm diameter, and under steady state conditions the temperature distribution is of the form

$$T(r)=a+br^2$$

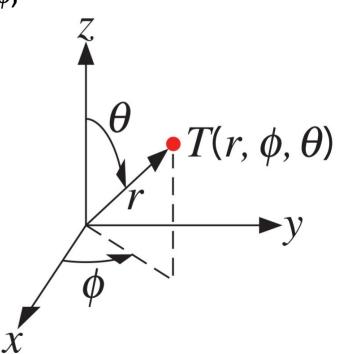
where T is in  ${}^{\circ}C$  and r in meters, while a and b are as indicate below. The fuel rod properties are shown below.

- (a) What is the rate of heat transfer per unit length of the rod at  $r=0 \, mm$  (the center line) and at  $r=25 \, mm$  (the surface)?
- (b) If the reactor power level is suddenly increased to  $10^8$  W/m³, what is the initial time rate of temperature change at the center line and rod surface?

$$a = 800 \,^{\circ}C$$
  $b = -4.167 \times 10^{5} \,^{\circ}C \over m^{2}$   $k = 30 \, \frac{W}{m \, K}$   $\rho = 1100 \, \frac{kg}{m^{3}}$   $c_{p} = 800 \, \frac{J}{kg \, K}$ 

### **Spherical Coordinate System**

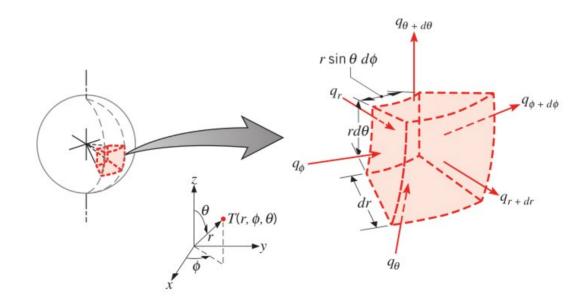
- Another alternative coordinate system
- Instead of x, y, and z like when using Cartesian coordinates, cylindrical coordinate system use r,  $\phi$ , and  $\theta$ .
- r is a radius.
- $\phi$  is an angle within the x-y plane.
- $\theta$  is an angle from the z axis.
- The advantage is when spherical objects are analyzed.



#### **Heat Diffusion Equation for Spherical Systems**

 Using the spherical coordinate system, a similar approach may be used to derive the heat diffusion equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(k\,r^2\frac{\partial T}{\partial r}) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}(k\,\frac{\partial T}{\partial \phi}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(k\sin\theta\,\frac{\partial T}{\partial \theta}) + \dot{q} = \rho\,c_p\frac{\partial T}{\partial t}$$



# **Some Simplified Forms**

• If the thermal conductivity k is constant

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

If the system is steady state

$$\frac{1}{r^2} \frac{\partial}{\partial r} (kr^2 \frac{\partial T}{\partial r}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} (k \frac{\partial T}{\partial \phi}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (k \sin \theta \frac{\partial T}{\partial \theta}) + \dot{q} = 0$$

• If the heat transfer is 1D

$$\frac{1}{r^{2}} \frac{\partial}{\partial r} (kr^{2} \frac{\partial T}{\partial r}) + \dot{q} = \rho c_{p} \frac{\partial T}{\partial t}$$

$$\frac{1}{r^{2} \sin^{2} \theta} \frac{\partial}{\partial \phi} (k \frac{\partial T}{\partial \phi}) + \dot{q} = \rho c_{p} \frac{\partial T}{\partial t}$$

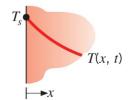
$$\frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial \theta} (k \sin \theta \frac{\partial T}{\partial \theta}) + \dot{q} = \rho c_{p} \frac{\partial T}{\partial t}$$

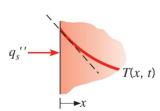
#### **Boundary and Initial Conditions**

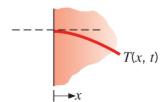
- Boundary and initial conditions are constraints placed on a system that dictate how a particular location (as in the case of boundary conditions) or instant of time (as in the case of initial conditions) behaves.
- There are several common boundary conditions:
  - Constant surface temperature  $T(0,t)=T_s$

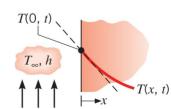
  - Constant surface temperature

     Constant surface heat flux  $-k \frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$  Adiabatic surface  $\frac{\partial T}{\partial x}\Big|_{x=0} = 0$  Convection surface condition  $-k \frac{\partial T}{\partial x}\Big|_{x=0} = h(T_{\infty} T(0, t))$









# Example 3

A long copper bar of rectangular cross section, whose width w is much greater than its thickness L, is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink,  $T_{o}$ . Suddenly, an electric current is passed through the bar and a stream of air of temperature  $T_{\infty}$  is passed over the top surface, while the bottom surface continues to be maintained at  $T_{o}$ . Determine the boundary and initial conditions that could be used to solve this arrangement for the temperature distribution as a function of both position and time. Copper bar  $(k, \alpha)$ T(x, y, z, t)

Heat sink  $T_o$