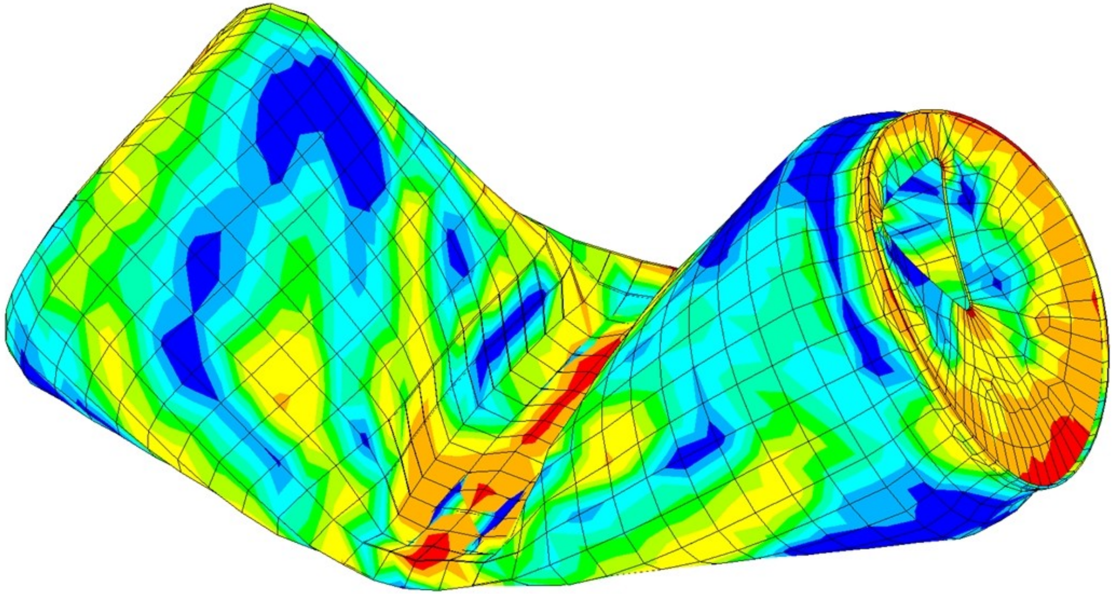
Finite Element Analysis

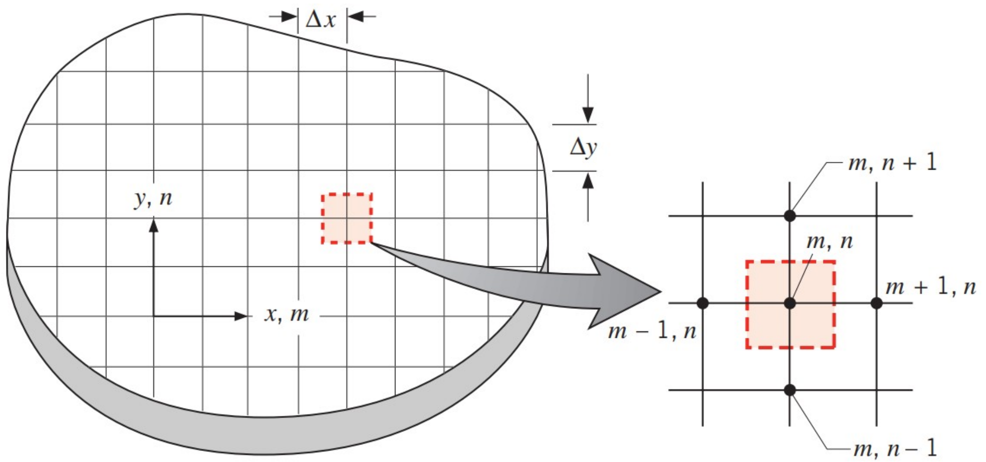


Finite Element Analysis

- Up until this point, the governing equations have been used to determine the temperature distribution $T(x)$ as a continuous function.
- Finite Element Analysis (FEA) forgoes this in an effort to simplify things.
- With FEA, the temperature distribution is discretized (i.e., broken into small or "finite" pieces).
- In doing so, an approximation of the temperature distribution is created.
- If the pieces are small enough, the approximation can be good enough.

Finite Difference Method

- FEA is a broad term that encompasses many different methods.
- One method for heat transfer is the Finite Difference Method (FDM).
- Instead of finding the T at any given point, FDM finds an average T at predefined points.
- Points are known as nodal points, or simply nodes.
- Collection of nodes is known as nodal network, grid, or mesh.
- Notice m, and n notation.



FDM – The Math

- Consider HDE of an interior node, in a 2D isotropic body with no heat generation under steady state conditions.

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=0$$

- The thermal conductivity can then be factored out, leaving

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

- The second order partial at node m, n for x may be approximated by

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n}\approx\frac{\left.\frac{\partial T}{\partial x}\right|_{m+\frac{1}{2},n}-\left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2},n}}{\Delta x}$$

FDM – The Math (cont.)

- The first order partials may also be approximated by

$$\left.\frac{\partial T}{\partial x}\right|_{m+\frac{1}{2},n} \approx \frac{T|_{m+1,n} - T|_{m,n}}{\Delta x} \qquad \left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2},n} \approx \frac{T|_{m,n} - T|_{m-1,n}}{\Delta x}$$

- Combining the two into the previous expression shows

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

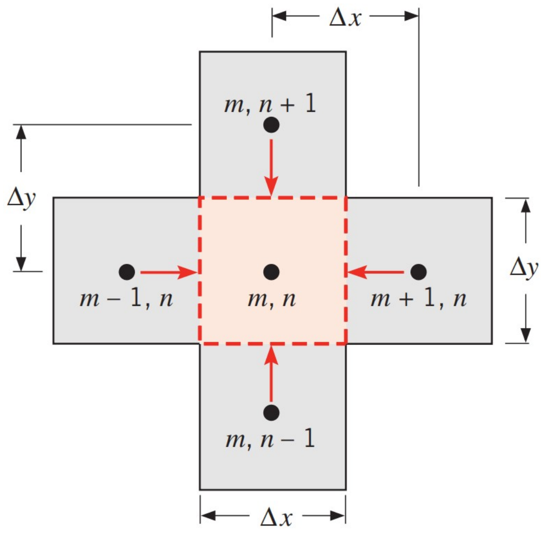
- A similar process may be repeated for the y direction yielding

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

- If $\Delta x = \Delta y$, then HDE at node m, n approximately becomes

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

The Energy Balance Method



- FDM works well for a very special case:
  - Interior node
  - Isotropic
  - No heat generation
  - Steady State
- The Energy Balance Method (EBM) tries to address some of these limitations.
- EBM applies an energy balance to a finite square or cube.
- The EBM can then accommodate heat generation, convection conditions, constant heat flux conditions, etc.

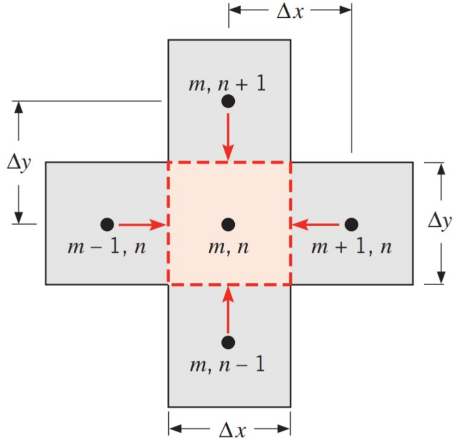
EBM – The Math

- First assume all energy is entering the finite element control volume

$$E_{in} + E_g = 0$$

- For a 2D system as seen, the energy balance is

$$\sum_{i=1}^{4} q_{(i)\rightarrow(m,n)} + \dot{q}\left(\Delta x \cdot \Delta y \cdot 1\right) = 0$$



- Since all sides pertain to conduction, the q terms become

$$q_{(m-1,n)\rightarrow(m,n)} = k\left(\Delta y \cdot 1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1)\rightarrow(m,n)} = k\left(\Delta x \cdot 1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$q_{(m+1,n)\rightarrow(m,n)} = k\left(\Delta y \cdot 1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n-1)\rightarrow(m,n)} = k\left(\Delta x \cdot 1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

EBM – The Math (cont.)

- If $\Delta x = \Delta y$, the four q expressions may be combined in the energy balance yielding

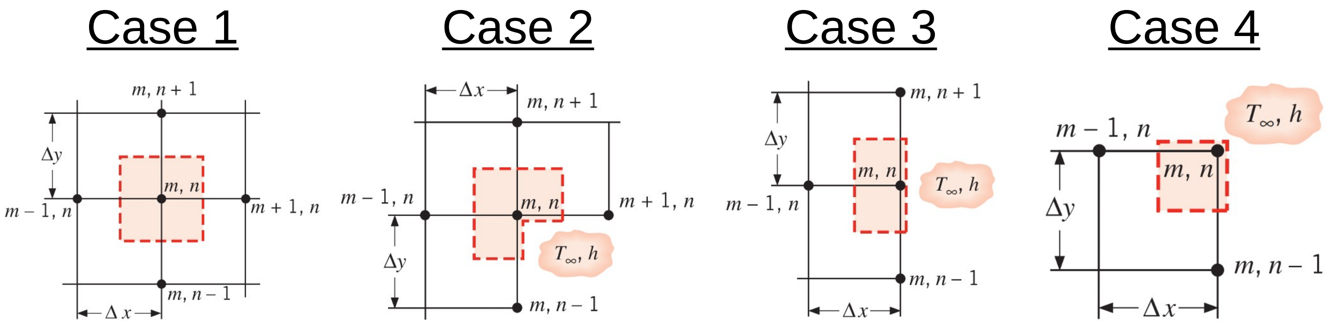
$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q} \cdot 1 \cdot (\Delta x)^2}{k} - 4T_{m,n} = 0$$

- This expression accounts for heat generation unlike the FDM.
- Other cases may be analyzed using this method
  - Node at an internal corner with convection
  - Node at a plane surface with convection
  - Node at an external corner with convection

EDM Case Summary

Note: $\Delta x = \Delta y$ and no heat generation

- Case 1: $T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$
- Case 2: $2\left(T_{m-1,n} + T_{m,n+1}\right) + \left(T_{m+1,n} + T_{m,n-1}\right) + 2\frac{h\Delta x}{k}T_\infty - 2\left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$
- Case 3: $\left(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right) + 2\frac{h\Delta x}{k}T_\infty - 2\left(2 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$
- Case 4: $\left(T_{m,n-1} + T_{m-1,n}\right) + 2\frac{h\Delta x}{k}T_\infty - 2\left(1 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$
- If adiabatic instead of convection, let $h=0$



Example 1

Using the energy balance method, derive the finite-difference equation for the (m, n) nodal point located on a plane, insulated surface of a medium with uniform heat generation.

Solving the Mesh

- For a particular nodal mesh, there are $m \times n$ nodes.
- Thus, there is a $m \times n$ system of equations that must be solved
  - Very hard to do by hand
- A computer is well suited for this purpose if the system of equations can be inputted in a consistent manner.

Solving the Mesh (cont.)

- Generalizing the system of equations shows

$$
\begin{array}{ccccc}
a_{11}T_1 & +a_{12}T_2 & \cdots & +a_{1N}T_N = & C_1 \\
a_{21}T_1 & +a_{22}T_2 & \cdots & +a_{2N}T_N = & C_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{N1}T_1 & +a_{N2}T_2 & \cdots & +a_{NN}T_N = & C_N
\end{array}
$$

  where the a values are the coefficients of the temperature terms, the C values are known as the forcing values, and $N = m \cdot n$.

- The forcing values contain quantities like Δx, k, h, T_∞, etc.

Solving the Mesh (cont.)

- The system of equations written in this form may be concisely written as

$$AT=C$$

where

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \qquad T=\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} \qquad C=\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

- This notation is known as a matrix equation.
- To solve, linear algebra may be used.

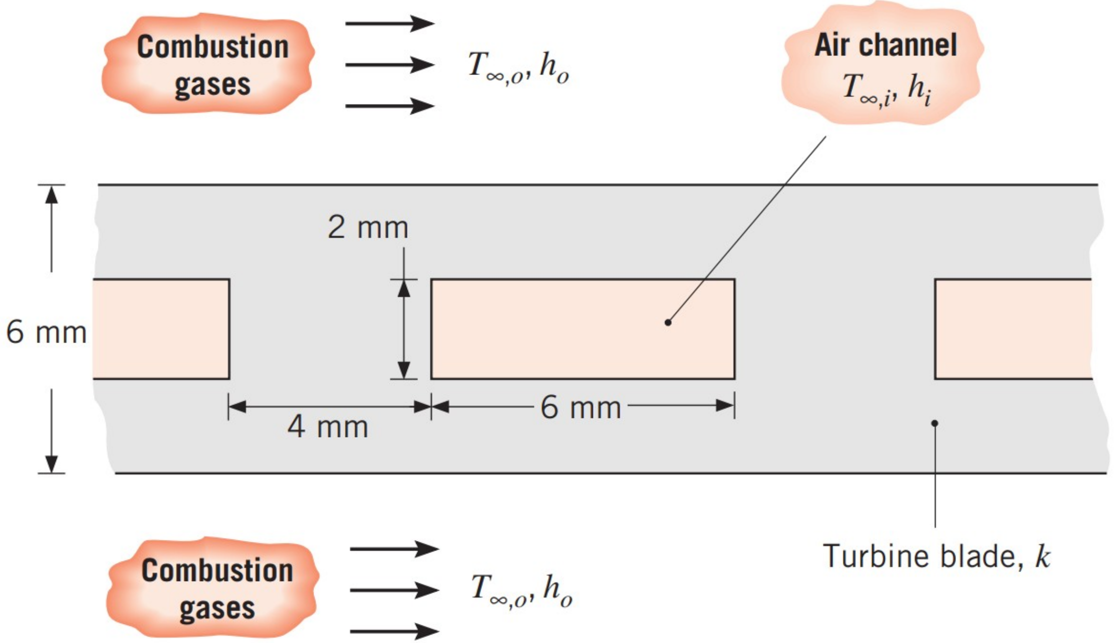
$$T=A^{-1}C$$

Example 2

A major objective in advancing gas turbine engine technologies is to increase the temperature limit associated with operation of the gas turbine blades. This limit determines the permissible turbine gas inlet temperature, which, in turn, strongly influences overall system performance. In addition to fabricating turbine blades from special, high-temperature, high-strength superalloys, it is common to use internal cooling by machining flow channels within the blades and routing air through the channels. We wish to assess the effect of such a scheme by approximating the blade as a rectangular solid in which rectangular channels are machined. The blade, which has a thermal conductivity of $k=25\ W/mK$, is $6\ mm$ thick, and each channel has a $2\ mm \times 6\ mm$ rectangular cross section, with a $4\ mm$ spacing between adjoining channels.

Example 2 (cont.)

Under operating conditions for which $h_o = 1000\ W/m^2K$, $T_{\infty,o} = 1700\ K$, $h_i = 200\ W/m^2K$, and $T_{\infty,i} = 400\ K$, determine the temperature field in the turbine blade. At what location is the temperature a maximum?

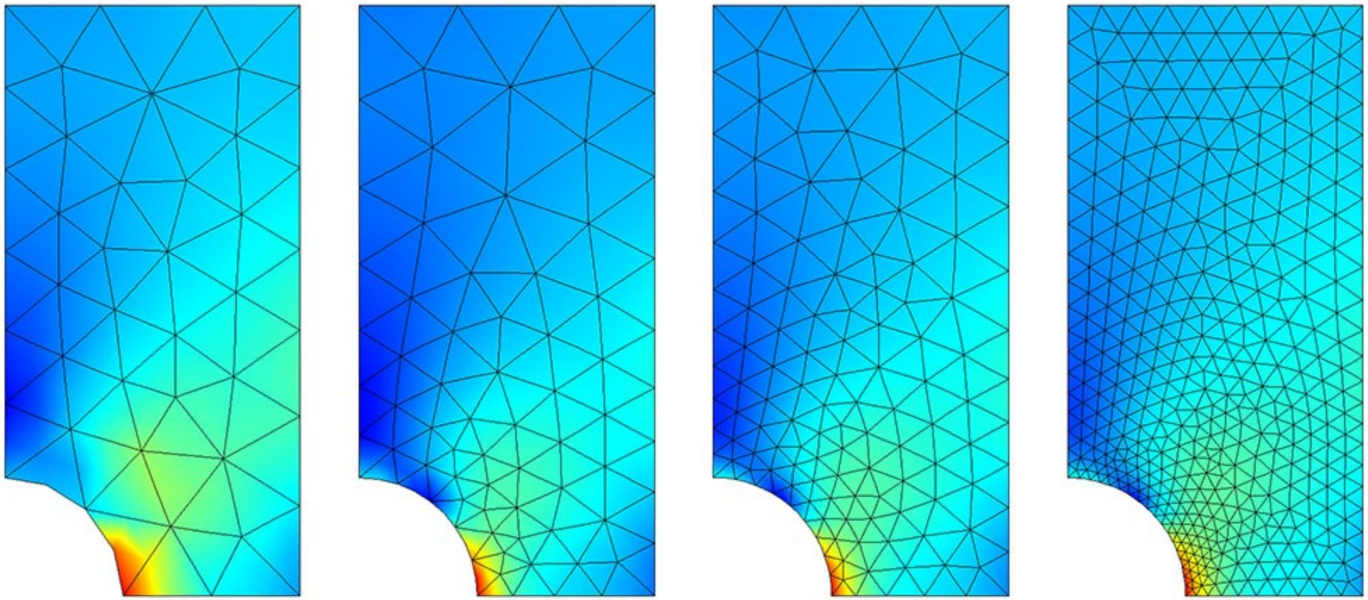


Practical FEA

- As seen from the previous example problem, it is not very practical to perform FEA by hand.
- Computers are well suited for this purpose though.
- There are many different FEA packages
  - SolidWorks
  - Ansys
  - LISA-FEA
- Regardless of the software used, it is important to find a mesh independent result.

Mesh Independent Results

- The goal of a mesh study (a.k.a. mesh refinement study) is to obtain the same results regardless of the mesh used.
- This is done by refining the mesh or increasing the mesh into smaller and smaller elements.



Example 3

Repeat example problem 2, but use LISA-FEA to determine the maximum temperature in the turbine blade. Perform a mesh refinement study to achieve mesh independent results.

https://youtu.be/M16BdeW-p8o