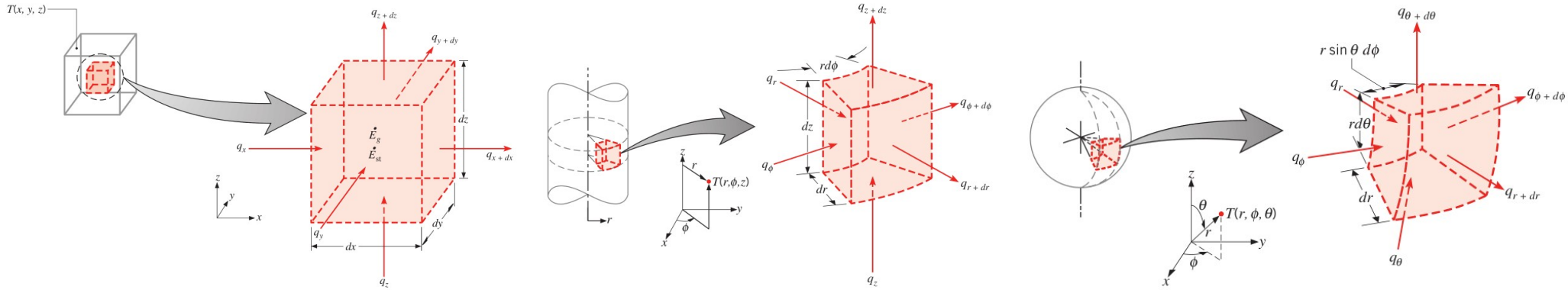


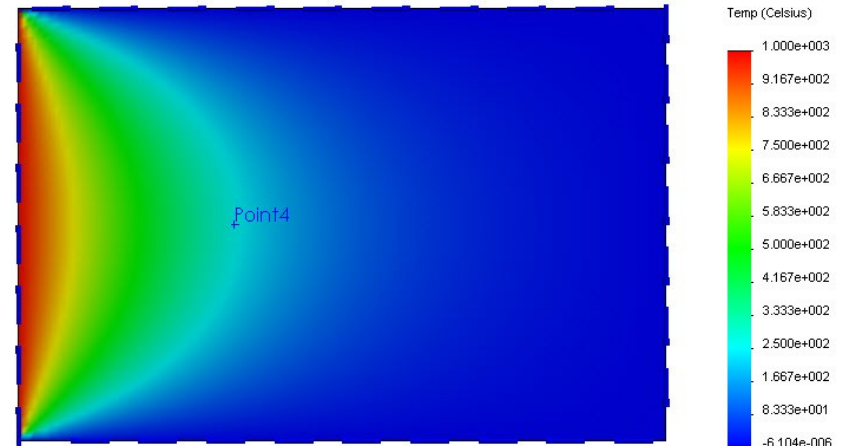
Heat Diffusion Equation



Heat Diffusion Equation

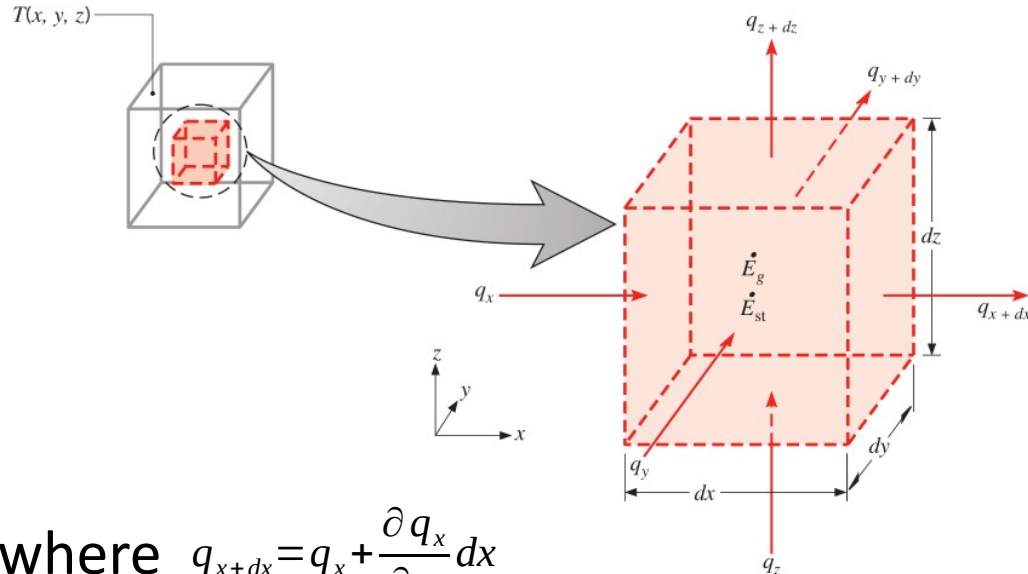
- Goal is to describe the temperature in any particular point within a substance.
- This is known as the temperature distribution.
- Once the distribution is known, the conduction heat flux may be determined at any particular point.
- First lets consider a homogeneous substance that is described using Cartesian coordinates.
- The temperature distribution would then be

$$T(x, y, z, t)$$



Heat Diffusion Equation

- Looking at a differential sized portion of the material



where

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

- The \dot{E}_g term refers to the rate of thermal energy generation (e.g. resistance heating).
- The \dot{E}_{st} term refers to the energy storage ability of the material; this is related to the thermal capacity of the substance.

Heat Diffusion Equation

- 1st Law may then be applied.
- Then some math is done.

- Which yields

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

- This is known as the heat diffusion equation for Cartesian coordinates.
- It is a partial differential equation.
- It governs the temperature distribution within a material.

Some Simplified Forms

- If the thermal conductivity k is constant

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

- If the system is steady state

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = 0$$

- If the heat transfer is 1D

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

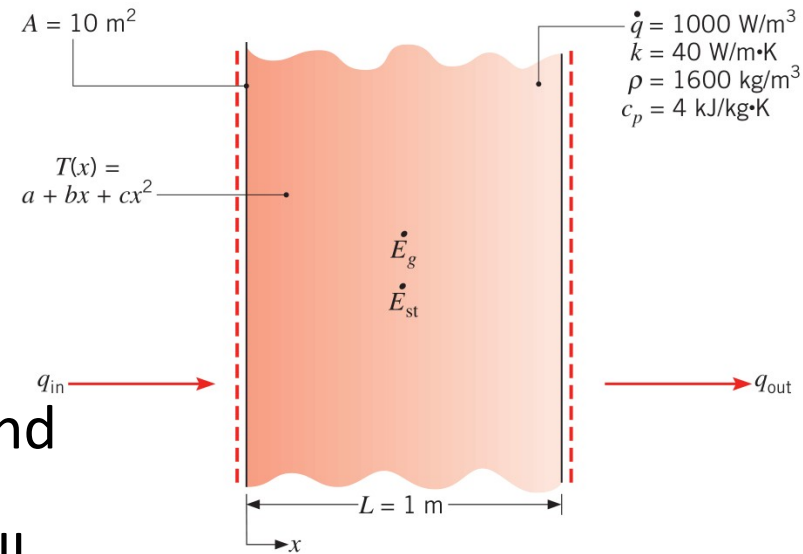
Example 1

The temperature distribution across a wall 1 m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in $^{\circ}\text{C}$ and x is in meters, while a , b , and c are as shown to the right. A uniform heat generation of $\dot{q} = 1000\text{ W/m}^3$, is present in the wall of area 10 m^2 having properties as shown.

Determine the rate of heat transfer entering and leaving the wall at $x=0\text{ m}$ and $x=1\text{ m}$, respectively. Determine the rate of change of energy storage in the wall. Determine the time rate of temperature change at $x=0, 0.25$, and 0.5 m .



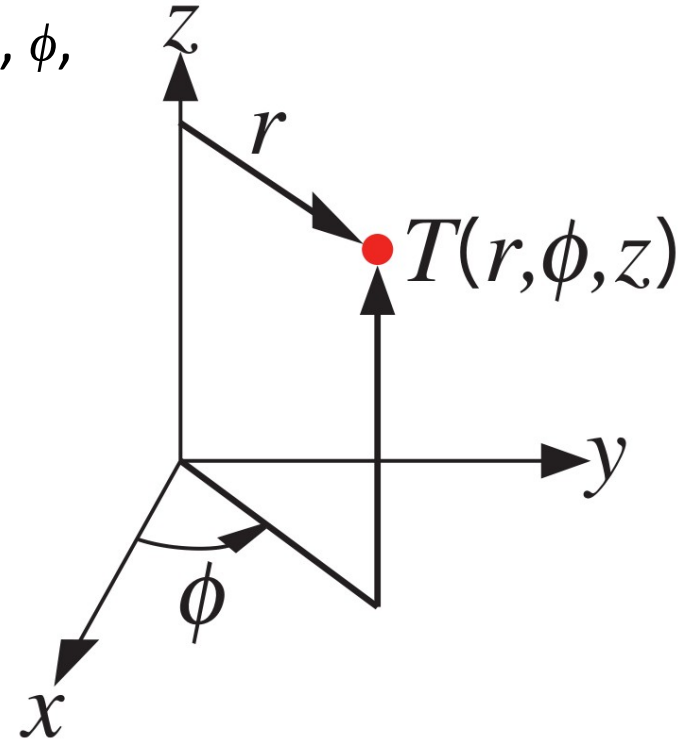
$$a = 900\text{ }^{\circ}\text{C}$$

$$b = -300\frac{^{\circ}\text{C}}{\text{m}}$$

$$c = -50\frac{^{\circ}\text{C}}{\text{m}^2}$$

Cylindrical Coordinate System

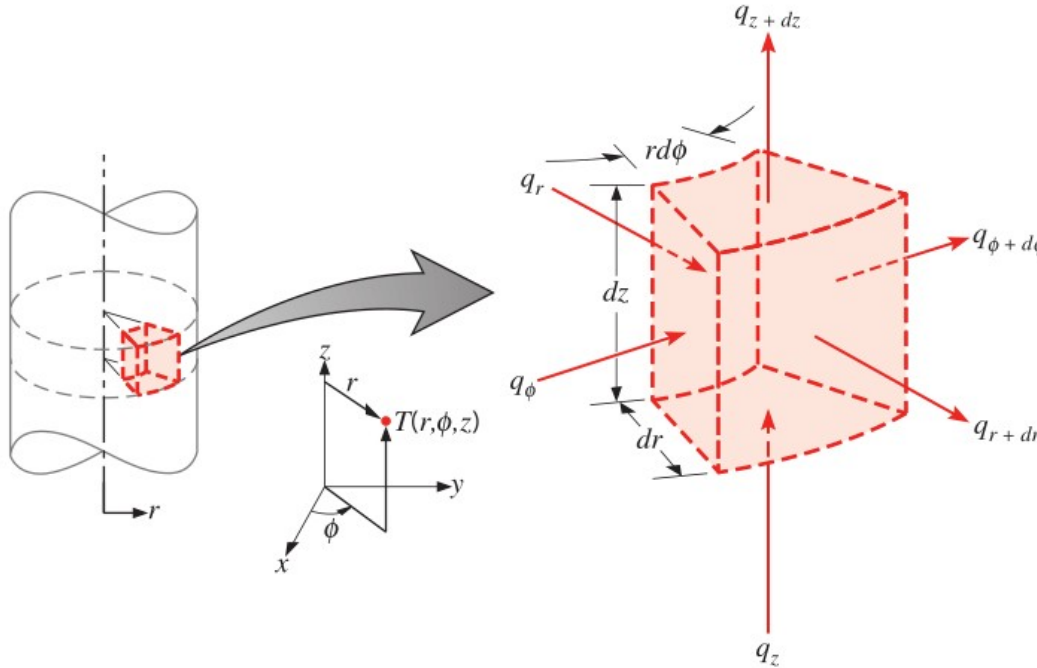
- An alternative coordinate system
- Instead of x , y , and z like when using Cartesian coordinates, cylindrical coordinate system use r , ϕ , and z .
- r is a radius.
- ϕ is an angle.
- z is a height.
- The advantage is when cylindrical objects are analyzed.



Heat Diffusion Equation for Cylindrical Systems

- Using the cylindrical coordinate system, a similar approach may be used to derive the heat diffusion equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$



Some Simplified Forms

- If the thermal conductivity k is constant

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

- If the system is steady state

$$\frac{1}{r} \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = 0$$

- If the heat transfer is 1D

$$\frac{1}{r} \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Example 2

Uniform internal heat generation of $5 \times 10^7 \text{ W/m}^3$ is occurring in a cylindrical nuclear reactor fuel rod of 50 mm diameter, and under steady state conditions the temperature distribution is of the form

$$T(r) = a + br^2$$

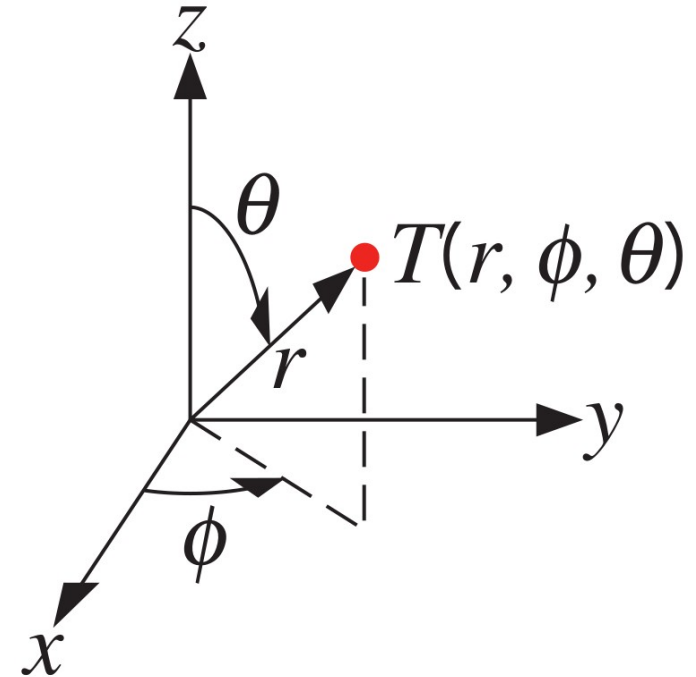
where T is in $^{\circ}\text{C}$ and r in meters, while a and b are as indicate below. The fuel rod properties are shown below.

- (a) What is the rate of heat transfer per unit length of the rod at $r=0 \text{ mm}$ (the center line) and at $r=25 \text{ mm}$ (the surface)?
- (b) If the reactor power level is suddenly increased to 10^8 W/m^3 , what is the initial time rate of temperature change at the center line and rod surface?

$$a = 800^{\circ}\text{C} \quad b = -4.167 \times 10^5 \frac{^{\circ}\text{C}}{\text{m}^2} \quad k = 30 \frac{\text{W}}{\text{m K}} \quad \rho = 1100 \frac{\text{kg}}{\text{m}^3} \quad c_p = 800 \frac{\text{J}}{\text{kg K}}$$

Spherical Coordinate System

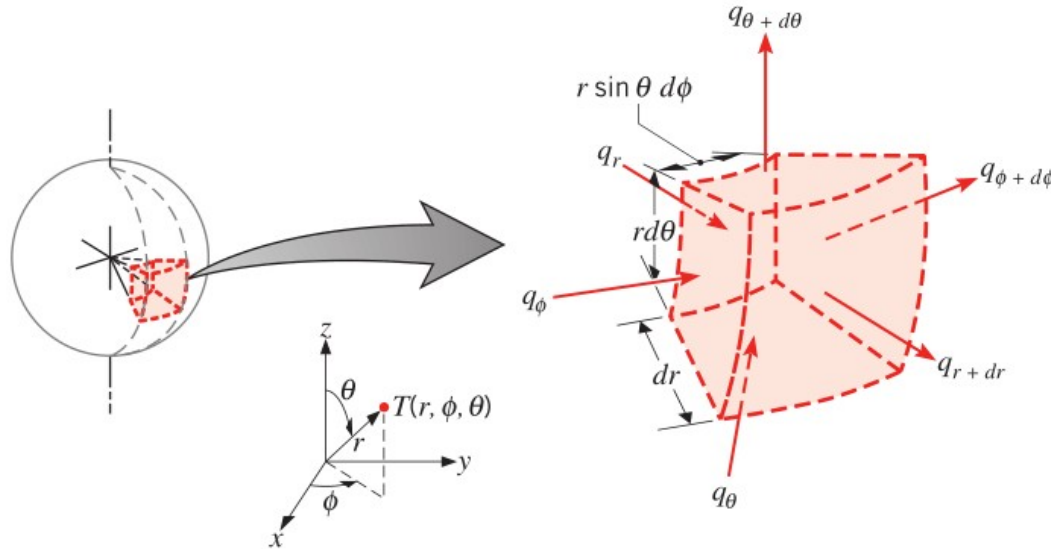
- Another alternative coordinate system
- Instead of x , y , and z like when using Cartesian coordinates, cylindrical coordinate system use r , ϕ , and θ .
- r is a radius.
- ϕ is an angle within the x - y plane.
- θ is an angle from the z axis.
- The advantage is when spherical objects are analyzed.



Heat Diffusion Equation for Spherical Systems

- Using the spherical coordinate system, a similar approach may be used to derive the heat diffusion equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(k r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(k \sin \theta \frac{\partial T}{\partial \theta} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$



Some Simplified Forms

- If the thermal conductivity k is constant

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

- If the system is steady state

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(k r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(k \sin \theta \frac{\partial T}{\partial \theta} \right) + \dot{q} = 0$$

- If the heat transfer is 1D

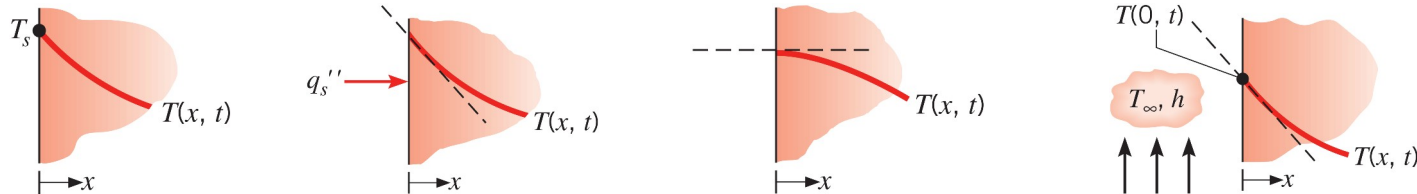
$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(k r^2 \frac{\partial T}{\partial r} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

$$\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(k \sin \theta \frac{\partial T}{\partial \theta} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Boundary and Initial Conditions

- Boundary and initial conditions are constraints placed on a system that dictate how a particular location (as in the case of boundary conditions) or instant of time (as in the case of initial conditions) behaves.
- There are several common boundary conditions:
 - Constant surface temperature $T(0, t) = T_s$
 - Constant surface heat flux $-k \frac{\partial T}{\partial x} \Big|_{x=0} = q_s''$
 - Adiabatic surface $\frac{\partial T}{\partial x} \Big|_{x=0} = 0$
 - Convection surface condition $-k \frac{\partial T}{\partial x} \Big|_{x=0} = h(T_\infty - T(0, t))$



Example 3

A long copper bar of rectangular cross section, whose width w is much greater than its thickness L , is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink, T_o . Suddenly, an electric current is passed through the bar and a stream of air of temperature T_∞ is passed over the top surface, while the bottom surface continues to be maintained at T_o . Determine the boundary and initial conditions that could be used to solve this arrangement for the temperature distribution as a function of both position and time.

