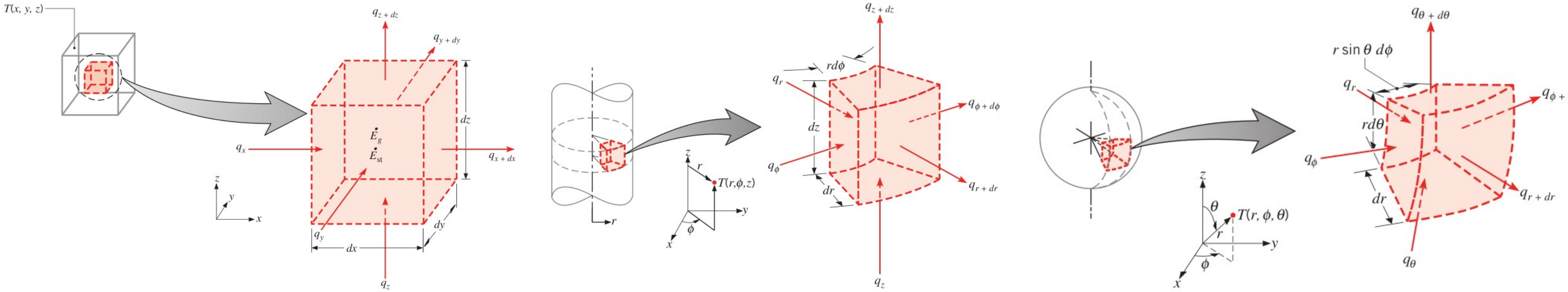
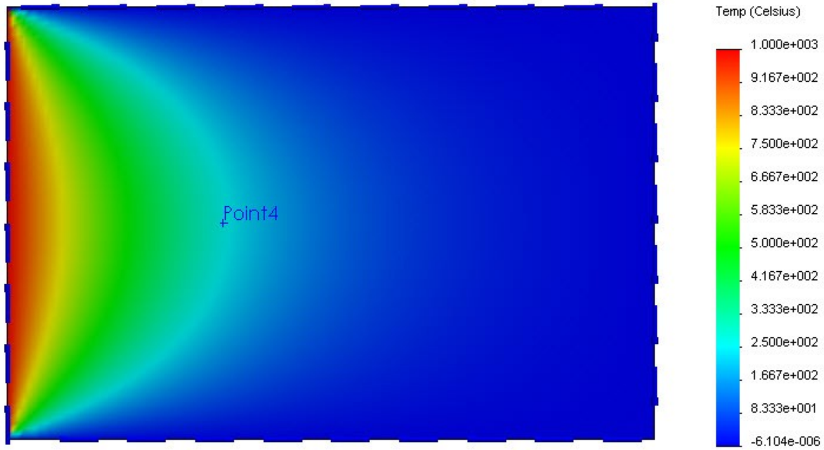
Heat Diffusion Equation



Heat Diffusion Equation

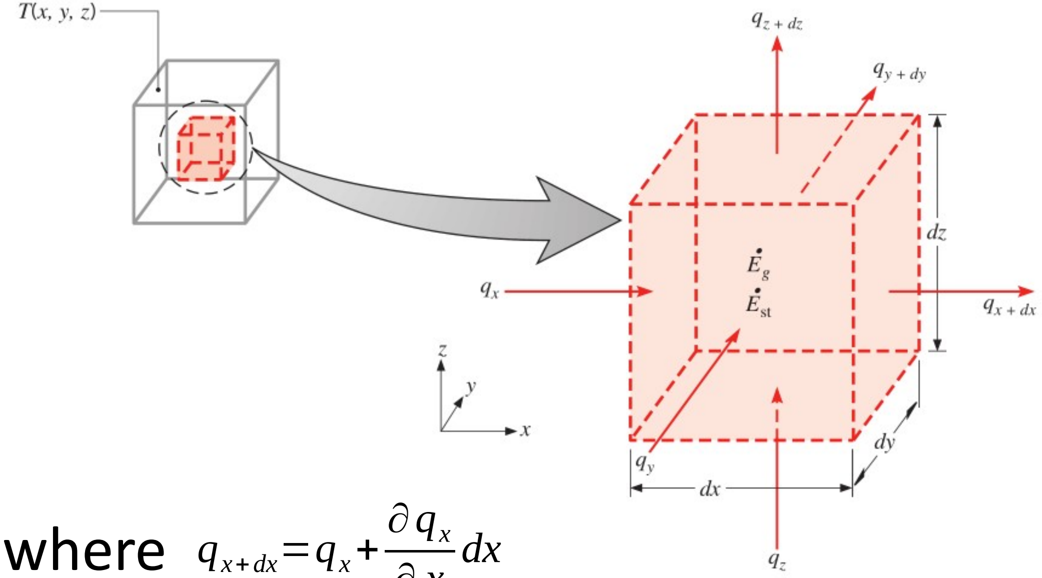
- Goal is to describe the temperature in any particular point within a substance.
- This is known as the temperature distribution.
- Once the distribution is known, the conduction heat flux may be determined at any particular point.
- First lets consider a homogeneous substance that is described using Cartesian coordinates.
- The temperature distribution would then be

$$T(x, y, z, t)$$



Heat Diffusion Equation

- Looking at a differential sized portion of the material



where $$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

- The $\dot{E}_g$ term refers to the rate of thermal energy generation (e.g. resistance heating).
- The $\dot{E}_{st}$ term refers to the energy storage ability of the material; this is related to the thermal capacity of the substance.

Heat Diffusion Equation

- 1^{st} Law may then be applied.
- Then some math is done.
- Which yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

- This is known as the heat diffusion equation for Cartesian coordinates.
- It is a partial differential equation.
- It governs the temperature distribution within a material.

Some Simplified Forms

- If the thermal conductivity k is constant

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- If the system is steady state

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

- If the heat transfer is 1D

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}=\rho\, c_p\frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\dot{q}=\rho\, c_p\frac{\partial T}{\partial t}$$

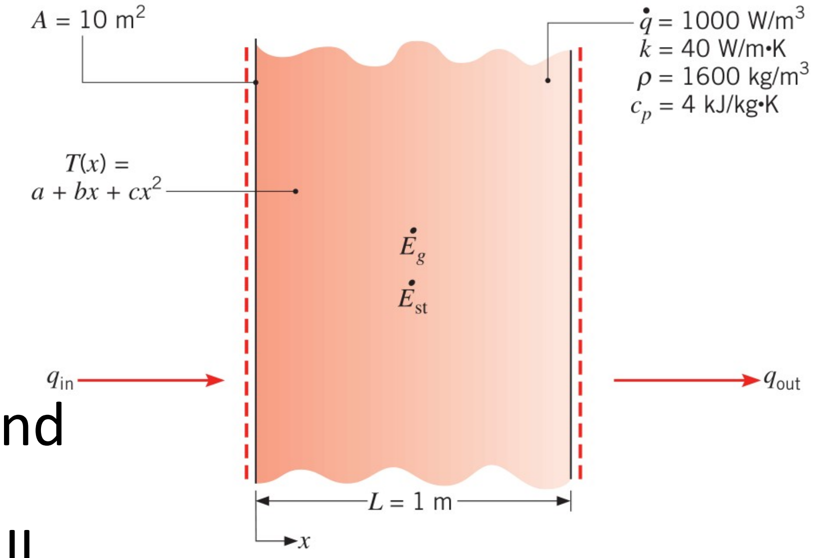
$$\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho\, c_p\frac{\partial T}{\partial t}$$

Example 1

The temperature distribution across a wall $1\ m$ thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in $°C$ and x is in meters, while a, b, and c are as shown to the right. A uniform heat generation of $\dot{q} = 1000\ W/m^3$, is present in the wall of area $10\ m^2$ having properties as shown. Determine the rate of heat transfer entering and leaving the wall at $x=0\ m$ and $x=1\ m$, respectively. Determine the rate of change of energy storage in the wall. Determine the time rate of temperature change at $x=0$, 0.25, and $0.5\ m$.



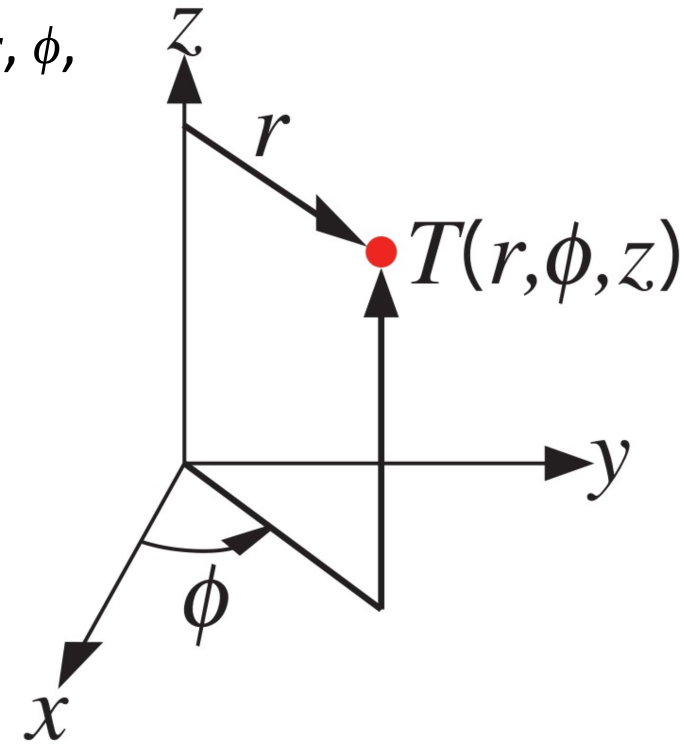
$$a = 900\ °C$$

$$b = -300\ \frac{°C}{m}$$

$$c = -50\ \frac{°C}{m^2}$$

Cylindrical Coordinate System

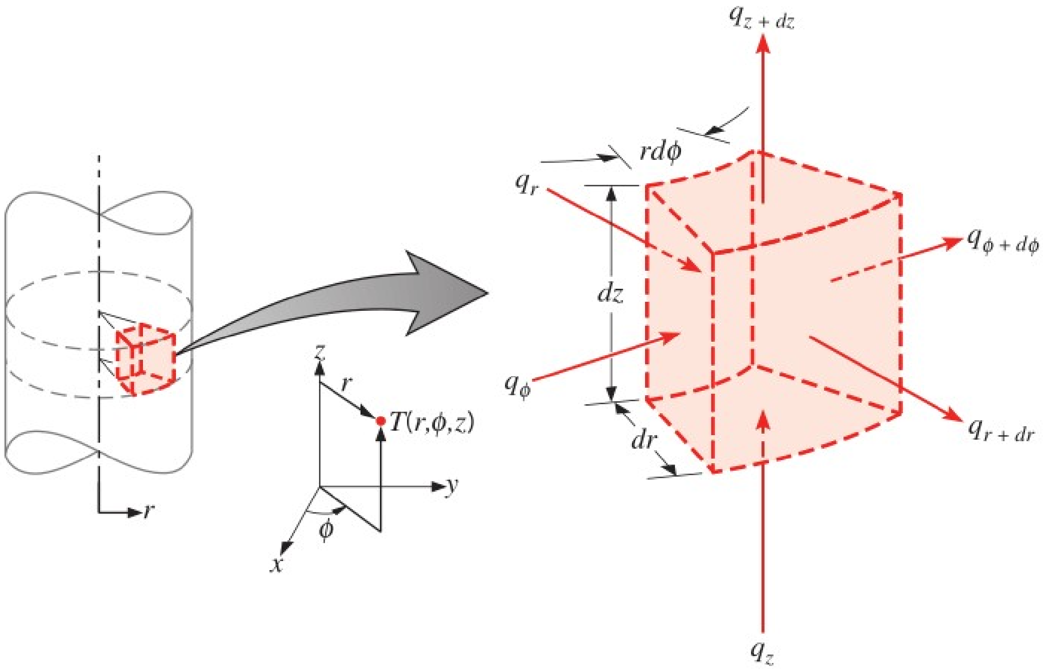
- An alternative coordinate system
- Instead of x, y, and z like when using Cartesian coordinates, cylindrical coordinate system use r, ϕ, and z.
- r is a radius.
- ϕ is an angle.
- z is a height.
- The advantage is when cylindrical objects are analyzed.



Heat Diffusion Equation for Cylindrical Systems

- Using the cylindrical coordinate system, a similar approach may be used to derive the heat diffusion equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$



Some Simplified Forms

- If the thermal conductivity k is constant

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- If the system is steady state

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

- If the heat transfer is 1D

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Example 2

Uniform internal heat generation of $5\times10^7\ W/m^3$ is occurring in a cylindrical nuclear reactor fuel rod of $50\ mm$ diameter, and under steady state conditions the temperature distribution is of the form

$$T(r)=a+br^2$$

where T is in $°C$ and r in meters, while a and b are as indicate below. The fuel rod properties are shown below.

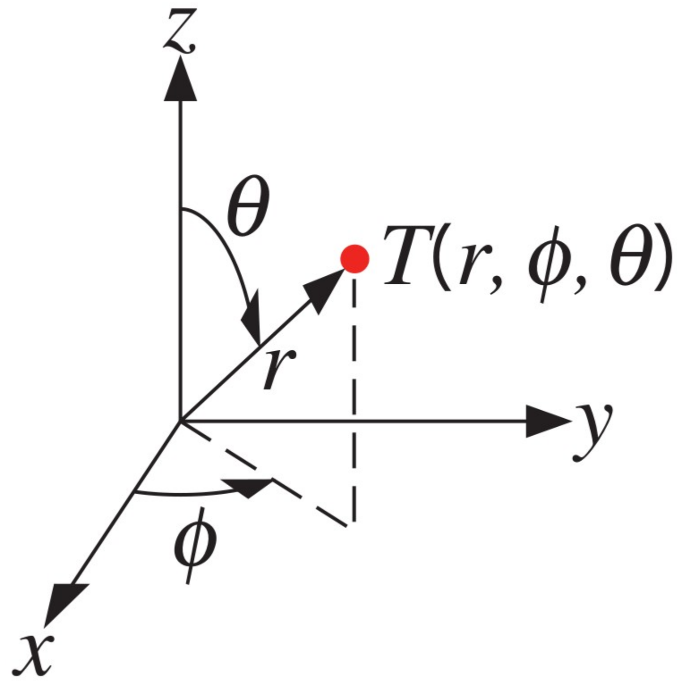
(a) What is the rate of heat transfer per unit length of the rod at $r=0\ mm$ (the center line) and at $r=25\ mm$ (the surface)?

(b) If the reactor power level is suddenly increased to $10^8\ W/m^3$, what is the initial time rate of temperature change at the center line and rod surface?

$$a=800\,°C \quad b=-4.167\times10^5\frac{°C}{m^2} \quad k=30\frac{W}{m\,K} \quad \rho=1100\frac{kg}{m^3} \quad c_p=800\frac{J}{kg\,K}$$

Spherical Coordinate System

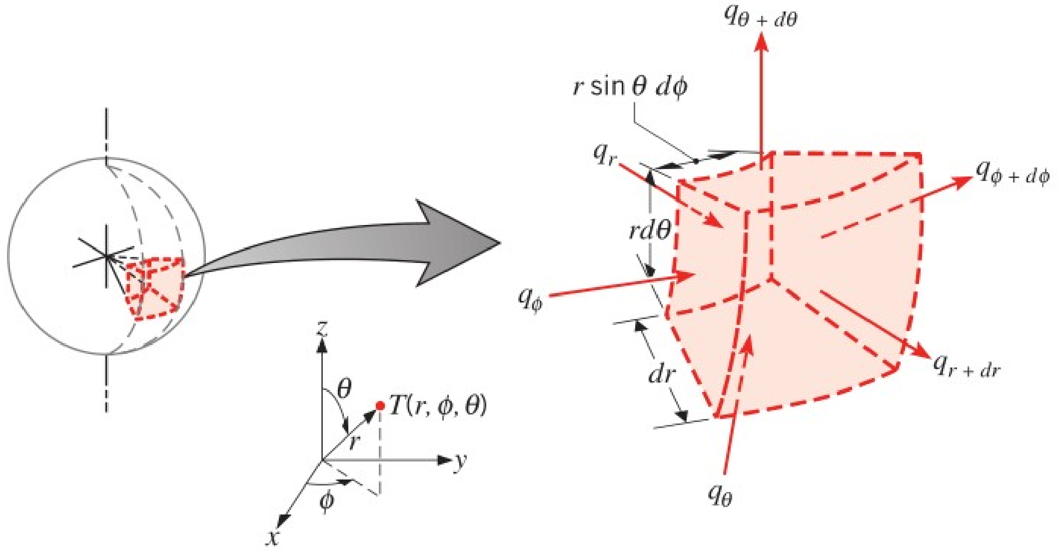
- Another alternative coordinate system
- Instead of x, y, and z like when using Cartesian coordinates, cylindrical coordinate system use r, ϕ, and θ.
- r is a radius.
- ϕ is an angle within the x-y plane.
- θ is an angle from the z axis.
- The advantage is when spherical objects are analyzed.



Heat Diffusion Equation for Spherical Systems

- Using the spherical coordinate system, a similar approach may be used to derive the heat diffusion equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(k r^2 \frac{\partial T}{\partial r}\right)+\frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p \frac{\partial T}{\partial t}$$



Some Simplified Forms

- If the thermal conductivity k is constant

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- If the system is steady state

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=0$$

- If the heat transfer is 1D

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

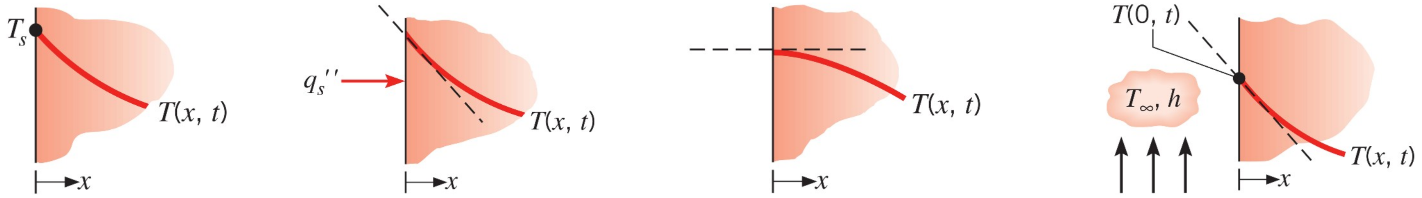
$$\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Boundary and Initial Conditions

- Boundary and initial conditions are constraints placed on a system that dictate how a particular location (as in the case of boundary conditions) or instant of time (as in the case of initial conditions) behaves.

- There are several common boundary conditions:
  - Constant surface temperature $T(0,t)=T_s$
  - Constant surface heat flux $-k\left.\frac{\partial T}{\partial x}\right|_{x=0}=q_s''$
  - Adiabatic surface $\left.\frac{\partial T}{\partial x}\right|_{x=0}=0$
  - Convection surface condition $-k\left.\frac{\partial T}{\partial x}\right|_{x=0}=h(T_\infty-T(0,t))$



Example 3

A long copper bar of rectangular cross section, whose width w is much greater than its thickness L, is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink, T_o. Suddenly, an electric current is passed through the bar and a stream of air of temperature T_∞ is passed over the top surface, while the bottom surface continues to be maintained at T_o. Determine the boundary and initial conditions that could be used to solve this arrangement for the temperature distribution as a function of both position and time.