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HW5 Written: Counting

1 There are $\binom{7}{5}=21$ total strings. Since $\binom{5}{3}=10$ of these strings contain exactly 1 e, we have $21-\frac{10}{2}=16$ total subsets after accounting for multiplicity. 10 of these have two e's, so there are $10\cdot\frac{5!}{2!}+6\cdot 5!=1{,}320$ ways to rearrange these subsets.

2 First choose two different card numbers, which can be done $\binom{13}{2}$ ways. For both of these cards, we pick 2 of the 4 suits: $\binom{4}{2}\binom{4}{2}$ ways. Now there are 44 remaining cards to choose from, so $\binom{13}{2}\binom{4}{2}^2 44=123{,}552$ ways.

Requiring the pairs to be the same color results in just 1 way to pick the second card in each pair, so there are $\binom{13}{2}44=3{,}432$ ways.

3 Put the two best players on separate teams. Then there are 11 left. There are $\binom{11}{6}\binom{5}{5}$ ways these can be separated into two teams of 6 and 7. To account for the teams being distinguishable, we multiply by two. So, we have $2\cdot\binom{11}{6}=924$ ways.

4 Case 1: Violinist does not play a song for the fighting couple. Then, there are $\binom{16+6-1}{5}$ ways to play 16 indistinguishable songs to 6 distinguishable couples.

Case 2: If the violinist plays 1 song for the fighting couple, there are $\binom{15+6-1}{5}$ ways to play the songs for the other couples.

Thus, we have $\binom{21}{5}+\binom{20}{5}=35{,}853$ total ways

5 The values 1, 2 form a two-node BST, which can be made two ways: .

Let the number of two-node BSTs be a_2. The number of 0 and 1 node BSTs are then $a_0=a_1=1$. The two-node BST $\{1,2\}$ must be the left child of 3. The three-node BST from $\{10,11,12\}$ must be the right child of 9, and there are $a_3=a_2a_0+a_1a_1+a_0a_2=5$ arrangements for these, since we can fix a root and then count the ways we can make all two-node BSTs from it (2 left, 1 left 1 right, 2 right).

Then, the five-node BST $\{4,5,6,7,8\}$ is the left child of 9, and can be arranged $a_5 = a_4a_0 + a_3a_1 + a_2a_2 + a_1a_3 + a_0a_4 = (5+2+2+5) + 5 + 4 + 5 + (5+2+2+5) = 43$ ways.
So, there are $43 \cdot 5 \cdot 2 = 430$ possible BSTs.

6 Using my knowledge from math407, we can use Stirling numbers of the second kind, which represent the number of ways n distinguishable objects can go into k indistinguishable, non-empty boxes.

Case 1: No break: There are 4 nurses, so

$$S(10,4) = \frac{1}{4!}\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^n$$

$$= 34{,}105$$

Case 2: Break: There are 3 nurses, so

$$S(10,3) = \frac{1}{3!}\sum_{i=0}^{3}(-1)^i\binom{3}{i}(3-i)^n$$

$$= 9{,}330$$

So, our total answer is $S(10,4) + S(10,3) = 43{,}435$ combinations.

Probability

1 There are $\binom{21}{13}$ ways to have 13 students answer one question, and 21^{13} ways for the Professor to distribute his questions. Thus, we have $\binom{21}{13}/21^{13} \approx 1.3173 \cdot 10^{-12}$.

2 There are $5 \cdot 4$ ways to choose the first two even digits, and 5 ways to choose the last odd digit. There are $3 \cdot 2$ ways for the other digits to be both even, $2 \cdot 4 \cdot 3$ ways to have 1 odd and 1 even, and $4 \cdot 3$ ways for both to be odd. Thus, the probability of any number meeting this criteria is $(5\cdot4\cdot3\cdot2\cdot5 + 2\cdot5\cdot4\cdot5\cdot3\cdot4 + 5\cdot4\cdot5\cdot4\cdot3)/100{,}000$
$\Rightarrow p = 0.042$. The probability of 7 out of 10 meeting the criteria is then $\binom{10}{7}(0.042)^7(1-0.042)^3 \approx 2.4323 \cdot 10^{-8}$

P(a die shows 4 or more)

3 First, find $P(A), P(B), P(AB)$:

$P(A) = \binom{3}{2}(\frac{1}{2})^2(\frac{1}{2}) + \binom{3}{3}(\frac{1}{2})^3 = 1/2$

$P(B) = 6/216 = 1/36$

$P(AB) = 3/216 = 1/72$, since the only possibilities are $\{444, 555, 666\}$

Now, observe that $P(A)P(B) = 1/72 = P(AB)$, so A and B are independent.

4 There are 10 choices for a low card $(A, 2, \cdots, 10)$. For each straight, there are 4^5 arrangements for the suits of each card. Then, the probability of a straight is $P(\text{straight}) = 10 \cdot 4^5 / \binom{52}{5} = 0.0039$. So, the expected number of hands before getting a straight is the expected value of the geometric distribution with $p = 0.0039$, which is $\frac{1}{p} \approx 253.8047$

5 We can use Bayes' rule: Let A be the event that the super star plays, and let B be the event that the team wins 3/5 games. Then, our desired probability is $P(A|B)$. We have $P(A) = 0.65$, $P(B|A) = \binom{5}{3} 0.75^3 \cdot 0.25^2 \approx 0.2637$, and $P(B|\bar{A}) = \binom{5}{3} 0.40^3 \cdot 0.60^2 \approx 0.2304$. Thus, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{0.2637 \cdot 0.65}{0.2637 \cdot 0.65 + 0.2304 \cdot 0.35} \approx 0.68$$