Introduction to PDEs, Fall 2022

Homework 10 Solutions

Name:________________

1. Let us denote
$$F(x) := \frac{1}{2}\int_{-\infty}^{\infty} |x-y|\sin y dy.$$
Show that $\frac{d^2F(x)}{dx^2} = \sin x$. (Remark: this fact is not intuitively simple)

   Solution 1. *To find the fundamental solutions of $\mathcal{L}$, it is sufficient to find $G(x)$ such that $\mathcal{L}G = \delta(x)$. On the other hand, we already know from class that the weak derivative of $H(x)$ is $\delta(x)$, therefore we have that $\frac{dG(x)}{dx} = H(x)$, in the weak sense.*

   It seems necessary to mention that, if f is the weak derivative of F, so is it a weak derivative for $F+c$ for any constant c. In this spirit, we see that $H(x)+c$ also admits $\delta(x)$ as the weak derivative, hence $xH(x)+cx$ is the fundamental solution of $\frac{d^2}{dx^2}$ for any c. In particular, choosing $c = \frac{1}{2}$, with $xH(x) + \frac{x}{2}$, implies that $\frac{1}{2}|x|$ is a fundamental solution of $\mathcal{L}$, therefore we have
$$F(x) = \frac{1}{2}\int_{-\infty}^{\infty} |x-y|\sin y dy.$$
   as desired.

   Remark: One was able to show this by straightforward calculation as in class or by using Fourier transform as in HW 10. An analog is the following simple fact that, if $G(x)$ is a fundamental solution of $\mathcal{L}$, so is $G(x)+V(x)$ for any $V(x)$ such that $\mathcal{L}V(x) = 0$.

2. 1) Show that $F(x;c) = xH(x) + cx$, $\forall c \in \mathbb{R}$, is a fundamental solution of $\mathcal{L} = \frac{d^2}{dx^2}$;

   2) Choose several different c and plot the figure of $F(x;c)$ to give yourself some intuition. Remark: Any fundamental solution of $\mathcal{L}$ must be of this form for some c. Some students approached me after the class about the $\mathcal{L}$. Here it is merely a notation for this linear operator.

   3) Find a general formula for the fundamental solution G for $\mathcal{L}$. Hint: solve $\mathcal{G} = 0$ for $x < 0$ and $x > 0$ by avoiding the singularity point first. Then apply the integral condition for the Delta function.

   Solution 2. *1) This can be verified easily by definition and I skip the details.*

   2) when $c = -\frac{1}{2}$, one collects the absolute value function;

   3) one can easily find that $G(x)$ must be a piece-wise linear function, and equate the slopes by matching the integral condition.

3. Suppose that u is a harmonic function in a plane disk $B_2(0) \subset \mathbb{R}^2$, i.e., centered at the origin with radius 2, and $u = 3\cos 2\theta + 1$ for $r = 2$. Calculate the value of u at the origin without finding the solution u.

   Solution 3. *From the mean value property, we have that*
$$u(0) = \frac{1}{4\pi}\int_{\partial B_0(2)} udS,$$
   where dS represents the differential for a line integral. It is easy to know that $dS = rd\theta$ for the circle with radius r, hence for $\partial B_0(2)$ we have that
$$u(0) = \frac{1}{2\pi}\int_{\partial B_0(2)} (3\cos 2\theta + 1)d\theta = \frac{2\pi}{2\pi} = 1.$$

4. Let us recall from the Green's second identity that

$$\int_{\Omega} u\Delta G - \Delta u G dx = \int_{\partial\Omega} u\frac{\partial G}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}} G dS^{*}.$$

I want to remind you that in multi-variate calculus, one typically requires that both u and G are at least twice differentiable for this identity to hold. However, now that you understand the weak derivative, the Laplacian Δ can be treated in the weak sense without ruining this equality, hence the smoothness of u and G are no longer required in the classical sense.

Note that G is not unique for $\Delta G(\mathbf{x}) = \delta(\mathbf{x})$ to hold since $\Delta(G + \tilde{G}) = \delta(\mathbf{x})$ if $\Delta\tilde{G} \equiv 0$.

Let us consider the following problem

$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

1) show that for any G^* such that $\Delta G^* = \delta(\mathbf{x})$, we have that for any $x_0 \in \Omega$

$$u(\mathbf{x}_0) = \int_{\Omega} f G^* d\mathbf{x} + \int_{\partial\Omega} g\frac{\partial G^*}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}} G^* dS. \tag{0.1}$$

You should write explicitly in this formula as, e.g., $f(\mathbf{x})G^*(\mathbf{x}_0 - \mathbf{x})$,...

2) In (0.1), we note that $\frac{\partial u}{\partial \mathbf{n}}$ is not known, therefore one might want to choose $G^* = 0$ on $\partial\Omega$ such that this surface integral disappears. However, this is only doable for special geometries.

Let us consider Ω the upper half plane $\mathbb{R}^2_+ : \{\mathbf{x} = (x, y) \in \mathbb{R}^2 | x \in (-\infty, \infty), y \in (0, \infty)\}$. Find $G^*(\mathbf{x})$ such that $\Delta G(\mathbf{x}) = 0$ in $\mathbb{R}^2_+$ and $G(\mathbf{x})$ on $\partial\mathbb{R}^2_+$ (i.e., the x-axis. Indeed, the term for $|x| \to \infty$ disappear.) Hint: $G^*(\mathbf{x}; \mathbf{x}_0) = G(\mathbf{x}; \mathbf{x}_0) + \tilde{G}(\mathbf{x}; \mathbf{x}_0)$ as suggested earlier. Choose $\tilde{G}$ such that $G^* \equiv 0$ on the boundary.

Solution 4. *1) This is already there as since $\Delta G = \delta(x)$;*

2) For any $\boldsymbol{x}_0 = (x_0, y_0)$, we choose its mirror symmetry about the x-axis as $\boldsymbol{x}_0^* = (x_0, -y_0)$. Then we denote $G^*(\boldsymbol{x}; \boldsymbol{x}_0) = \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0| - \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0^*| = \frac{1}{2\pi}\ln\frac{|\boldsymbol{x}-\boldsymbol{x}_0|}{|\boldsymbol{x}-\boldsymbol{x}_0^*|}$, then G^* satisfies the desired properties and we collect from (0.1) that

$$u(\boldsymbol{x}_0) = \int_{\Omega} f G^* d\boldsymbol{x} + \int_{\partial\Omega} g\frac{\partial G^*}{\partial \boldsymbol{n}} dS;$$

in particular, when $f \equiv 0$ (i.e., u is harmonic) and Ω is the upper half plane $\mathbb{R}^2_+$, we have that

$$u(x_0, y_0) = \int_{\partial\Omega} g\frac{\partial G^*}{\partial \boldsymbol{n}} dS = \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{g(x)}{(x - x_0)^2 + y^2} dx.$$

Note that I skip the calculations for the surface integral.

5. Find the harmonic function u over $\mathbb{R}^2_+$ such that

$$\Delta u = 0, x \in (-\infty, \infty), y \in (0, \infty),$$

subject to the boundary condition

$$u(x, 0) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Then plot $u(x, y)$ over $\mathbb{R}^2_+$ to illustrate your solution.

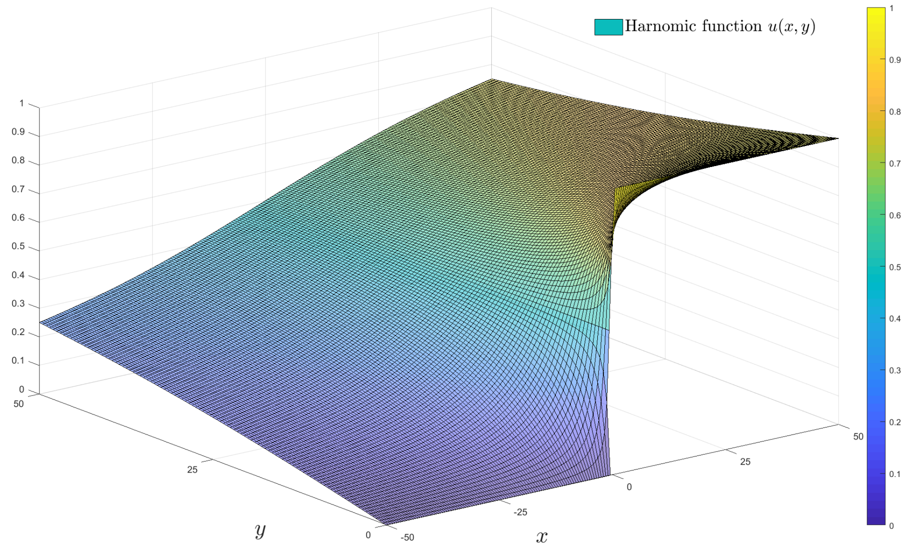
Solution 5. *According to the integral presentation given above, we know that*

$$u(x_0, y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{\phi(x)}{(x - x_0)^2 + y_0^2} dx = \frac{y_0}{\pi}\int_{0}^{\infty} \frac{1}{(x - x_0)^2 + y_0^2} dx,$$

which can be simplified, through the fact that $(\arctan x)' = \frac{1}{1+x^2}$, as

$$u(x_0, y_0) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan\frac{x_0}{y_0}\Big).$$

*I switched the order so one collects $u(x_0)$ without the negative sign.



6. Let u be a radially symmetric function such $u = u(r)$, $r = |\mathbf{x}| = \sqrt{\sum x_i^2}$, $\mathbf{x} := (x_1, x_2, \dots, x_n)$. Prove that $\frac{\partial u(r)}{\partial \mathbf{n}} = \frac{\partial u(r)}{\partial r}$, where $\mathbf{n}$ is the unit outer normal derivative.

   Solution 6. *Skipped.*