Introduction to PDEs, Fall 2022

Homework 10, Due Dec 15

Name:________________

1. Let us denote
$$F(x) := \frac{1}{2}\int_{-\infty}^{\infty} |x-y|\sin y dy.$$
Show that $\frac{d^2F(x)}{dx^2} = \sin x$. (Remark: this fact is not intuitively simple)

2. 1) Show that $F(x;c) = xH(x) + cx$, $\forall c \in \mathbb{R}$, is a fundamental solution of $\mathcal{L} = \frac{d^2}{dx^2}$;

   2) Choose several different c and plot the figure of $F(x;c)$ to give yourself some intuition. Remark: Any fundamental solution of $\mathcal{L}$ must be of this form for some c. Some students approached me after the class about the $\mathcal{L}$. Here it is merely a notation for this linear operator.

   3) Find a general formula for the fundamental solution G for $\mathcal{L}$. Hint: solve $\mathcal{G} = 0$ for $x < 0$ and $x > 0$ by avoiding the singularity point first. Then apply the integral condition for the Delta function.

3. Suppose that u is a harmonic function in a plane disk $B_2(0) \subset \mathbb{R}^2$, i.e., centered at the origin with radius 2, and $u = 3\cos 2\theta + 1$ for $r = 2$. Calculate the value of u at the origin without finding the solution u.

4. Let us recall from the Green's second identity that
$$\int_\Omega u\Delta G - \Delta u G dx = \int_{\partial\Omega} u\frac{\partial G}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}} G dS^{*}.$$
I want to remind you that in multi-variate calculus, one typically requires that both u and G are at least twice differentiable for this identity to hold. However, now that you understand the weak derivative, the Laplacian Δ can be treated in the weak sense without ruining this equality, hence the smoothness of u and G are no longer required in the classical sense.

   Note that G is not unique for $\Delta G(\mathbf{x}) = \delta(\mathbf{x})$ to hold since $\Delta(G + \tilde{G}) = \delta(\mathbf{x})$ if $\Delta\tilde{G} \equiv 0$.

   Let us consider the following problem
$$\begin{cases} \Delta u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$
   1) show that for any G^* such that $\Delta G^* = \delta(\mathbf{x})$, we have that for any $x_0 \in \Omega$
$$u(\mathbf{x}_0) = \int_\Omega fG^* d\mathbf{x} + \int_{\partial\Omega} g\frac{\partial G^*}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}} G^* dS. \tag{0.1}$$
   You should write explicitly in this formula as, e.g., $f(\mathbf{x})G^*(\mathbf{x}_0 - \mathbf{x})$,...

   2) In (0.1), we note that $\frac{\partial u}{\partial \mathbf{n}}$ is not known, therefore one might want to choose $G^* = 0$ on $\partial\Omega$ such that this surface integral disappears. However, this is only doable for special geometries.

   Let us consider Ω the upper half plane $\mathbb{R}^2_+ : \{\mathbf{x} = (x,y) \in \mathbb{R}^2 | x \in (-\infty,\infty), y \in (0,\infty)\}$. Find $G^*(\mathbf{x})$ such that $\Delta G(\mathbf{x}) = 0$ in $\mathbb{R}^2_+$ and $G(\mathbf{x})$ on $\partial\mathbb{R}^2_+$ (i.e., the x-axis. Indeed, the term for $|x| \to \infty$ disappear.) Hint: $G^*(\mathbf{x};\mathbf{x}_0) = G(\mathbf{x};\mathbf{x}_0) + \tilde{G}(\mathbf{x};\mathbf{x}_0)$ as suggested earlier. Choose $\tilde{G}$ such that $G^* \equiv 0$ on the boundary.

*I switched the order so one collects $u(x_0)$ without the negative sign.

5. Find the harmonic function u over $\mathbb{R}^2_+$ such that

$$\Delta u = 0, x \in (-\infty, \infty), y \in (0, \infty),$$

subject to the boundary condition

$$u(x, 0) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Then plot $u(x, y)$ over $\mathbb{R}^2_+$ to illustrate your solution.

6. Let u be a radially symmetric function such $u = u(r)$, $r = |\mathbf{x}| = \sqrt{\sum x_i^2}$, $\mathbf{x} := (x_1, x_2, \ldots, x_n)$. Prove that $\frac{\partial u(r)}{\partial \mathbf{n}} = \frac{\partial u(r)}{\partial r}$, where $\mathbf{n}$ is the unit outer normal derivative.