Introduction to PDEs, Fall 2022

Homework 9 Solutions

Name:________________

1. We know that the Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0, \\ 0, & x < 0 \end{cases} \tag{0.1}$$

has the Dirac-delta function $\delta(x)$ be its weak derivative. As I mentioned in class you might find in some textbooks that a Heaviside function is defined otherwise such as

$$H(x) = \begin{cases} 1 & x > 0, \\ \frac{1}{2} \text{ (or any other number)} & x = 0, \\ 0, & x < 0. \end{cases} \tag{0.2}$$

However, according to Lebesgue's theory, the value of a function at a single point (or a zero measure set) does not affect its properties in general and two functions that are equal almost everywhere are considered to be identical. Accordingly, the two forms of $H(x)$ are identical while we shall take the former in our course. Similarly, the weak derivative of a function is unique up to a measure zero, that being said, if $f(x)$ is a weak derivative of $F(x)$, then $g(x)$ is also a weak derivative, if $f(x)$ and $g(x)$ only differ on a zero measure set. This applies further.

A so-called bump function is given as $B(x) = xH(x)$. Show by definition that the weak derivative of $B(x)$ is $H(x)$.

Solution 1. *Let $v(x) \in L^1_{loc}(\mathbb{R}^1)$ be a weak derivative of $R(x)$, then we have from the definition of the weak derivative that, for M being large (or, you can just work on $(-\infty, \infty)$)*

$$\int_{-M}^{M} R(x)\phi'(x)dx = -\int_{-M}^{M} v(x)\phi(x)dx$$

for all $\phi(x) \in C^1_0(-M, M)$, i.e., $\phi(x) \in C^1(-M, M)$ and $\phi(x) = 0$ for $|x| > M$. Note that we usually send $M = \infty$, and the test function in C_0^∞, however, this alternative just gives you the impression that they are equivalent in the definition. Then we have from integration by parts that

$$\int_{-M}^{M} R(x)\phi'(x)dx = \int_{0}^{M} x d\phi(x) = -\int_{0}^{M} \phi(x)dx = -\int_{-M}^{M} H(x)\phi(x)dx,$$

therefore $H(x)$ is a weak derivative of $R(x)$. Finally, I want to remark that the weak derivative is unique in the sense of measure zero, i.e., out of a region of zero measure.

2. Find the weak derivative of $F(x)$, denoted by $f(x)$

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases} \tag{0.3}$$

Solution 2. *Formally we see that the weak derivative of $F(x)$, denoted by $f(x)$, is*

$$f(x) = \begin{cases} 1, & 0 < x \le 1, \\ 0, & 1 \le x < 2. \end{cases} \tag{0.4}$$

To prove this by definition, we choose any $\phi \in C_c^\infty(0,2)$ *and can easily find that*

$$\int_0^2 F\phi' dx = \int_0^1 F\phi' dx + \int_1^2 F\phi' dx = -\int_0^1 \phi dx = \int_0^2 f\phi dx,$$

with f *given above. This is done. Note that function* $F(x)$ *is define, in this example, over* $(0,2)$, *therefore the test function must be compactly supported over* $(0,2)$, *not* $(-\infty,\infty)$ *any more. I wish this logic is not too difficult to follow.*

3. It is necessary to point out that in the definition of a weak derivative, some textbooks require that both F and f are L^1_{loc} (here "loc" means being locally integrable in the sense that it is integrable over any compact subset of Ω). Let $\Omega = (0,2)$ and define

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 2, & 1 \le x < 2. \end{cases} \tag{0.5}$$

Show that $F' = f$ does not exist in the weak sense by the following contradiction argument: suppose that the weak derivative f exists, show that for any test function $\phi(x) \in C_0^1(0,2)$ (some textbooks use C_c^∞, where "c" denotes compact) we have

$$\int_0^2 f\phi dx = \int_0^1 \phi dx + \phi(1).$$

Now choose a sequence of test functions $\phi_m(x)$ satisfying

$$0 \le \phi_m(x) \le 1, \phi_m(1) = 1, \phi_m(x) \to 0, \forall x \ne 1, m \to \infty$$

and then obtain a contradiction from the identity above.

Solution 3. *We argue by contradiction and assume that the weak derivative exists. Then choose a sequence of test functions* $\{\phi_m\}_{m=1}^\infty$ *above, and then we have*

$$1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \Big(\int_0^2 f\phi_m dx - \int_0^1 \phi_m dx \Big) = 0,$$

which is a contradiction.

4. Assume that $F_n(x)$ converges to $F(x)$ weakly, and let $f_n(x)$ and $f(x)$ be their weak derivatives respectively. Prove that $f_n(x)$ also converges to $f(x)$ weakly.

Solution 4. *Since* $f_n(x)$ *is the weak derivative of* $F_n(x)$, *we have that for any test function* $\phi \in C_c(\Omega)$ *that*

$$-\int_\Omega \phi'(x) F_n(x) dx = \int_\Omega \phi(x) f_n(x) dx; \tag{0.6}$$

similarly, we have that

$$-\int_\Omega \phi'(x) F(x) dx = \int_\Omega \phi(x) f(x) dx; \tag{0.7}$$

Since $F_n \to F$ *weakly (in* $L^p(\Omega)$ *for instance), we have that for any function* ψ *in its dual space (the space of all its bounded linear functionals), we have that*

$$\int_\Omega F_n(x)\psi(x) dx \to \int_\Omega F(x)\psi(x) dx;$$

there is an advanced theory in functional analysis that C_c^∞ *function is always in the dual space of probably all the function spaces we work on, therefore we can choose* $\psi = \phi'$ *and obtain from (0.6) and (0.7) that*

$$\int_\Omega f_n(x)\psi(x) dx \to \int_\Omega f(x)\psi(x) dx,$$

and this implies that $f_n \to f$ *weakly.*

5. One can easily generalize the second–order operator to higher dimension, the Laplace operator Δ over $\Omega \subset \mathbb{R}^n$, $n \geq 1$.
   (a) We say that f is radially symmetric if $f(x) = f(r)$, $r = |x| := \sqrt{\sum_{i=1}^n x_i^2}$. Prove that
   $$\Delta f(r) = f''(r) + \frac{n-1}{r} f'(r),$$
   where the prime denotes a derivative taken with respect to r.
   (b) Denote that $G(r) := \frac{1}{2\pi} \ln r$ for $n = 2$. We shall show that $\Delta G = \delta(r)$. For this moment, let us consider its regularization over $2D$ of the form
   $$G_\epsilon(r) = \frac{1}{2\pi} \ln(r + \epsilon), \epsilon > 0.$$
   Show that $\Delta G_\epsilon(r)$ converges to $\delta(x)$ in distribution as $\epsilon \to 0^+$. Hint: you can either apply Lebesgue's dominated convergence theorem, or use ϵ–δ language. Make sure you have checked all the conditions when applying the former one.
   (c) Denote $G(r) := -\frac{1}{4\pi r}$ for $n = 3$. Mimic (b) by finding an approximation G_ϵ and then show that this approximation ΔG_ϵ convergence to $\delta(x)$ in distribution.

Solution 5. *(a) We show this by straightforward calculations. First of all, we have that* $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$. *Then by chain rule, we have*
$$\frac{\partial f(r)}{\partial x_i} = \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\partial f(r)}{\partial r}\frac{x_i}{r};$$
moreover, we have
$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial r}\frac{x_i}{r}\Big) = \frac{\partial^2 f}{\partial r^2}\big(\frac{x_i}{r}\big)^2 + \frac{\partial f}{\partial r}\frac{\partial}{\partial x_i}\big(\frac{x_i}{r}\big);$$
finally, using the fact that
$$\frac{\partial}{\partial x_i}\big(\frac{x_i}{r}\big) = \frac{1}{r} - \frac{x_i^2}{r^3}$$
leads us to the expected identity.

(b) By the identity in (a), we have that
$$\Delta G_\epsilon(r) = G_\epsilon''(r) + \frac{1}{r}G_\epsilon'(r) = \frac{1}{2\pi}\Big(\frac{1}{r+\epsilon} + \frac{1}{r}\cdot\frac{-1}{(r+\epsilon)^2}\Big) = \delta_\epsilon(x) := \frac{\epsilon}{2\pi(r+\epsilon)^2},$$
and we shall show $\delta_\epsilon(x)$ converges to $\delta(x)$ is distribution. Note that here its distribution limit $\delta(x)$ satisfies all the properties except that it is multi-dimensional.

To this end, we first see that formally $\delta_\epsilon(x) \to \infty$ if $x = 0$, $\to 0$ if $x \neq 0$. Next, we have that
$$\int_{\mathbb{R}^2} \delta_\epsilon(x)dx = \int_{\mathbb{R}^2} \frac{\epsilon}{2\pi(r+\epsilon)^2}dx = \int_0^\infty \frac{\epsilon r}{(r+\epsilon)^2}dr = -\frac{\epsilon}{r+\epsilon}\Big|_0^\infty = 1.$$
Now, we only need to show that for any text function $\phi(x) \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$
$$\int_{\mathbb{R}^2} \delta_\epsilon(x)\phi(x)dx \to \phi(0),$$
or equivalently
$$\int_{\mathbb{R}^2} \frac{\epsilon}{2\pi(r+\epsilon)^2}\phi(x)dx \to \phi(0).$$
To show this, we observe that for any δ small, one can choose R large enough such that
$$\int_{\mathbb{R}^2 \backslash B_0(M)} \frac{\epsilon}{2\pi(r+\epsilon)^2}\phi(x)dx = \delta;$$
on the other hand, one has from the dominated convergence theorem that as $\epsilon \to 0$
$$\int_{B_0(M)} \frac{\epsilon}{2\pi(r+\epsilon)^2}\phi(x)dx = \phi(x_\epsilon)\int_{B_0(M)} \frac{\epsilon}{2\pi(r+\epsilon)^2}dx \leq \phi(x_\epsilon)(1-\delta) \to \phi(0),$$
since δ is arbitrary. I would like to mention that one can also apply the standard ϵ-δ to prove this.

(c) I skip the proof here. It follows from straightforward calculations as above.