Introduction to PDEs, Fall 2022

Homework 6 Solutions

Name:________________

1. In measure theory, there are two additional convergence manners for $f_n \to f$: convergence in measure (called convergence in probability) and convergence almost everywhere (convergence almost surely). This should give you a flavor that probability is a measure and vice versa. Convergence in measure states that for each fixed $\varepsilon > 0$
$$m\big(\{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}\big) \to 0, \text{ as } n \to \infty,$$
and convergence almost everywhere means that the measure of the non-convergence region is zero, i.e., for each fixed $\varepsilon > 0$
$$m\big(\{x \in \Omega : \lim_{n\to\infty} |f_n(x) - f(x)| \geq \varepsilon\}\big) = 0.$$
i) what are the relationships between these two convergence manners? Prove your claims or give a counter-example.

   ii) what are their relationships between strong convergence (convergence in L^2 for instance)? Prove your claims or give a counter-example.

   I would like to point out that convergence is global behavior in strong contrast to pointwise convergence since all the points are involved in the convergence limit.

   Solution 1. *There facts are from your undergraduate course, and I assume that each of you know these statements and their proofs. I present them here to give you a taste how they apply to/in PDEs.*

2. It is known that strong convergence implies weak convergence, while not the converse. One counter-example we mentioned in class is $f_n(x) := \sin nx$ over $(0, 2\pi)$.

   (i) Prove that $\sin nx \nrightarrow 0$ in $L^2((0, 2\pi))$.

   (ii) Prove that $\sin nx \rightharpoonup 0$ weakly by showing
$$\int_0^{2\pi} g(x) \sin nx dx \to 0 = \Big(\int_0^{2\pi} g(x) 0 dx\Big), \forall g \in L^2((0, 2\pi)).$$
If suffices even if $g \in L^1$. Hint: Riemann–Lebesgue lemma.

   Solution 2. *(i). It is very easy to show that $\|\sin nx\|_{L^2((0,2\pi))} \nrightarrow 0$ by straightforward calculations, hence the strong convergence is impossible. (What is the value?)*

   (ii). Applying Riemann–Lebesgue lemma gives the desired limit. I already presented partial approach in class and I assume/need that you know how to prove this lemma in detail. One student, if you all remember, mentioned that you learnt this in Calculus. It is possible but I still doubt it as the L^2-norm was not introduced by then. Wish me wrong.

3. We recall that $f_n(x) \rightharpoonup f(x)$ weakly in L^p (resp. convergence in distribution) if for any $\phi \in L^q$ (resp. continuous and bounded), which is its conjugate space with $\frac{1}{p} + \frac{1}{q} = 1$, we have that
$$\int_\Omega f_n \phi dx \to \int_\Omega f \phi dx.$$
Here we see that for any g in L^q
$$< \cdot, g >= \int_\Omega \cdot g$$
defines a bounded linear functional for L^p. Then we also call L^q *the dual space* of L^p since any element in L^q defines a functional for L^q.

(i) Another type of convergence that you may see sometimes is $\|f_n\|_p \to \|f\|_p$, which merely states the convergence of a sequence of real numbers. Prove that if $f_n \to f$ in L^p, then $\|f_n\|_p \to \|f\|_p$ (Use Minkowski triangle inequality); however the opposite statement is not necessarily true. Give a counter-example and show it;

(ii) We have proved that strong convergence in L^p implies the weak convergence by Holder's inequality, however, the opposite statement is not necessarily true. For example, prove that $\sin nx$ converges to zero weakly, but not strongly in L^p. Hint: Riemann–Lebesgue lemma;

(iii) Prove that, if $f_n \rightharpoonup f$ weakly and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ strongly.

Solution 3. *Minkowski's triangle inequality states that, $\forall f, g \in L^p$, $p \in (1,\infty)$, we always have that $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$. Then it is not hard to obtain that*

$$\big|\|f_n\|_{L^p} - \|f\|_{L^p}\big| \leq \|f_n - f\|_{L^p} \to 0,$$

which proves the desired claim. You can fill in the details yourself. I skip presenting a counter-example here, however, here is a hint on how you can construct such a counter-example: think of f_n and f as points in a plane and their norms measure the distance from the origin, therefore $\|f_n - f\|_{L^p} \to 0$ means that the distance between f_n and f goes to zero, while $\|f_n\|_{L^p} \to \|f\|_{L^p}$ merely means that the distance of f_n to the origin converges to that of f. Now it is not hard to surmise that the former implies the latter, but not the other way. I assume that you can find a counter-example.

Finally, we shall prove that though each condition in does not, while both conditions, imply the strong convergence. To prove this, let us divide our discussions into the following cases:

case 1: $p = 2$. Then the conclusion is straightforward following Cauchy–Schwarz inequality. I assume that you have no problem proving this case;

case 2: $p > 2$. We first see that for any $z \in \mathbb{R}$

$$|z+1|^p \geq c|z|^p + pz + 1,$$

where c is a positive constant independent of z. (In order to prove this fact, we just need to show that $\frac{|z+1|^p - pz - 1}{|z|^p}$ has a positive lower bounded c over $\mathbb{R}$). Now we can let $z = \frac{f_n - f}{f}$ in this inequality, multiply it by $|f|^p$ and then integrate the new one over Ω to obtain

$$\int_\Omega |f_n|^p dx \geq \int_\Omega |f|^p dx + p\int_\Omega |f|^{p-2} f(f_n - f)dx + c\int_\Omega |f_n - f|^p dx.$$

Since $f_n \rightharpoonup f$ weakly, we see that the second integral on the right hand size of the equality converges to zero (think of $|f|^{p-2}f$ as a test function). On the other hand, we have that $\int_\Omega |f_n|^p dx \to \int_\Omega |f|^p dx$ thanks to the strong convergence, therefore we must have

$$\int_\Omega |f_n - f|^p dx \to 0,$$

which implies the strong convergence.

case 3: $p \in (1,2)$. The proof of this part is a little bit tricky. Similar as above, we can show (by straightforward calculations) that $\forall z \in \mathbb{R}$

$$\begin{aligned} |z+1| \geq & c|z|^p + pz + 1, \ \textit{if } |z| \geq 1, \\ |z+1| \geq & c|z|^2 + pz + 1, \ \textit{if } |z| \geq 1. \end{aligned}$$

In order to apply these inequalities, we shall choose $z = \frac{f_n - f}{f}$. Denote

$$\Omega_n := \{x \in \Omega; |z| \geq 1\},$$

then we can have by the same calculations as above that

$$\begin{aligned} \int_\Omega |f_n|^p dx &= \int_{\Omega \setminus \Omega_n} |f_n|^p dx + \int_{\Omega_n} |f_n|^p dx \\ &= \int_{\Omega \setminus \Omega_n} |z+1|^p |f|^p dx + \int_{\Omega_n} |z+1|^p |f|^p dx \\ &\geq \int_{\Omega \setminus \Omega_n} (c|z|^2 + pz + 1)|f|^p dx + \int_{\Omega_n} (c|z|^p + pz + 1)|f|^p dx, \end{aligned}$$

which implies, in light of the formula of z, that

$$\int_{\Omega\setminus\Omega_n}(f_n-f)^2|f|^{p-2}dx+\int_{\Omega_n}|f_n-f|^p dx\to 0.$$

Both integrals are nonnegative, hence both should converge to zero

$$\int_{\Omega\setminus\Omega_n}(f_n-f)^2|f|^{p-2}dx\to 0,\int_{\Omega_n}|f_n-f|^p dx\to 0.$$

In particular, we only need to show that

$$\int_{\Omega\setminus\Omega_n}|f_n-f|^p dx\to 0.$$

For this purpose, we shall apply Holder's inequality or Schwarz's inequality as following. Note that $|f_n-f|<|f|$ in $\Omega\setminus\Omega_n$, then we have that

$$\begin{aligned}\int_{\Omega\setminus\Omega_n}|f_n-f|^p dx\le&\int_{\Omega\setminus\Omega_n}|f|^{p-1}|f_n-f|dx\\ \le&\Big(\int_{\Omega\setminus\Omega_n}|f|^p\Big)^{\frac{1}{2}}\Big(\int_{\Omega\setminus\Omega_n}|f|^{p-2}|f_n-f|^2dx\Big)^{\frac{1}{2}}\\ \le&\Big(\int_{\Omega}|f|^p\Big)^{\frac{1}{2}}\Big(\int_{\Omega\setminus\Omega_n}|f|^{p-2}|f_n-f|^2dx\Big)^{\frac{1}{2}}\to 0,\end{aligned}$$

which is the desired claim and the proof completes.