Introduction to PDEs, Fall 2022

Homework 2 Solutions

Name:______________

1. Consider the following reaction-diffusion equation with Robin boundary condition

$$\begin{cases} u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial \mathbf{n}} = \gamma, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.1}$$

Use the energy method to:

(i) prove the uniqueness of (0.1) when $\alpha = 0$ and $\beta \neq 0$;

(ii) prove the uniqueness when both α and β are not zero. You might need to discuss the signs of α and β.

Solution 1. *I shall only deal with (ii) with (i) can be treated by the same arguments. To show the uniqueness of (0.1), it is equivalent to show that the following homogeneous IBVP admits only zero solution,*

$$\begin{cases} w_t = D\Delta w, & x \in \Omega, t > 0 \\ w(x,0) = 0, & x \in \Omega, \\ \alpha w + \beta \frac{\partial w}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.2}$$

To this end, we define an energy functional for (0.2) as

$$E(t) = \int_\Omega w^2(x,t)dx, t \geq 0,$$

and we can easily see that $E(0) = 0$. Now we differentiate $E(t)$ with respect to time t and collect that

$$E'(t) = 2D\Big(-\frac{\beta}{\alpha}\int_{\partial\Omega} w(x,t)^2 dS - \int_\Omega |\nabla w(x,t)|^2 dx\Big) = -2D\int_\Omega |\nabla w(x,t)|^2 dx \leq 0, \tag{0.3}$$

then it follows that $E'(t) \leq 0$ hence the solution to (0.2) with RBC is unique if $\frac{\beta}{\alpha} > 0$; however, if $\frac{\beta}{\alpha} < 0$, we can not determine the sign of $E(t)$, therefore we can not determine the uniqueness for (0.2) with the RBC in this case.

Remark: *The claim that we can not determine the uniqueness here does not necessarily mean that there is no uniqueness to this problem. It merely means that this particular energy functional can not be used to prove the uniqueness if the solution is unique at all.*

2. The so-called energy

$$E(t) = \int_\Omega w^2(x,t)dx$$

might not be necessary a physical energy such as kinetic energy or potential energy. But the term "energy" is used since it has many similarities as a physical energy, for example, is always positive, is increasing if temperature u increases. It is better called energy-functional as we did in class (i.e., the function of functions).

(i) let us define

$$E(t) := \int_\Omega w^4(x,t)dx.$$

Use this new energy-functional to prove the uniqueness to (0.1) with $\alpha = 1$ and $\beta = 0$.

(iii) can you use $E(t) := \int_\Omega w^3(x,t)dx$ for this purpose?

Solution 2. *(i) Again, to prove the uniqueness of (0.1), it is sufficient to show that the following system has only zero solution*

$$\begin{cases} w_t = D\Delta w, & x \in \Omega, t > 0, \\ w(x,0) = 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.4}$$

In light of the PDE and boundary condition, we have from straightforward calculations that

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_\Omega w^4(x,t)dx = 4\int_\Omega w^3 \frac{\partial w}{\partial t}dx \\ &= 4D\int_\Omega w^3 \Delta w dx = 4D\Big(\int_{\partial\Omega} w^3 \frac{\partial w}{\partial \boldsymbol{n}}dS - 3\int_\Omega w^2 |\nabla w|^2 dx\Big) \\ &= -12D\int_\Omega w^2|\nabla w|^2 dx \leq 0, \end{aligned}$$

therefore $E(t)$ is monotone deceasing in time which, together with the fact $E(0) = 0$, implies that $E(t) = 0$ for all $t > 0$. Therefore we must have that $w(x,t) \equiv 0$ as expected.

(ii) There are several reasons one can not use this newly defined energy to prove the uniqueness. First of all, it is unknown the monotonicity of $E(t)$; one is also not able to show that $E(t)$ is always positive/non-negative. However, failing to do so does not necessarily that the absence of uniqueness; it merely means that employing this particular choice does not suffice. In practical (research) problems, one shall see many problems can be solved provided with such functionals, which are usually very difficult to obtain or do not exist at all.

3. In general, one can not apply energy methods to problems with $f = f(x,t,u)$; many problems have indeed more than one solutions. However, it works for problems with special reaction term f (note that f is called reaction in general because heat produces through *chemical reactions*; and therefore you may see in elsewhere that the heat equations is also called reaction-diffusion equation).

(i). Use energy method to prove uniqueness to the following problem

$$\begin{cases} u_t = D\Delta u - u, & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \gamma, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.5}$$

(ii). Do the same for the following problem with $f = f(u)$ dependent only on u

$$\begin{cases} u_t = D\Delta u + f(u), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \gamma, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.6}$$

For what conditions (on f) do you have uniqueness of (0.6)? Hint: according to intermediate value theorem, $f(u_1) - f(u_2) = f'(u_1 + \theta(u_2 - u_1))(u_1 - u_2)$ for some $\theta \in [0,1]$. Remark: you should see that it also works even if $f = f(x,t,u)$, while I skip x and t with loss of generality.

Solution 3. *(i) Similar as above, to prove the uniqueness of (0.5), it suffices to show that the following problem admits only the trivial solution $w(x,t) \equiv 0$, for all $(x,t) \in \Omega \times \mathbb{R}^+$,*

$$\begin{cases} w_t = D\Delta w - w, & x \in \Omega, t \in \mathbb{R}^+, \\ u(x,0) = 0, & x \in \Omega, \\ w(x,t) = \gamma, & x \in \partial\Omega, t \in \mathbb{R}^+. \end{cases} \tag{0.7}$$

Endow (0.1) with the energy

$$E(w) := \int_\Omega w^2(x,t)dx.$$

One obtains from the PDE and the integration by parts that

$$\frac{d}{dt}E(t) = 2\int_\Omega w w_t dx = -2\int_\Omega (D|\nabla w|^2 + w^2)dx \le 0,$$

hence $E(t)$ is non-increasing and $E(t) \le 0$. Moreover, the facts that $E(t) \ge 0$ and $E(0) = 0$ imply that $E(t) = 0$, $\forall t \in \mathbb{R}^+$, and the uniqueness follows.

(ii). For this nonlinear problem, we shall prove that the following problem has only the trivial solution

$$\begin{cases} w_t = D\Delta w + f(u_1) - f(u_2), & x \in \Omega, t \in \mathbb{R}^+, \\ u(x,0) = 0, & x \in \Omega, \\ w(x,t) = \gamma, & x \in \partial\Omega, t \in \mathbb{R}^+, \end{cases} \tag{0.8}$$

assuming that u_1 and u_2 are two solutions with $w := u_1 - u_2$. Recall that the intermediate value theorem states that there exists $\theta \in [0,1]$ such that $f(u_1) - f(u_2) = f'(u^*)(u_1 - u_2)$, $u^* = \theta u_1 + (1-\theta)u_2$. Then by the same calculations above, we have that

$$\frac{d}{dt}E(t) = 2\int_\Omega w w_t dx = -2\int_\Omega (D|\nabla w|^2 + \int_\Omega f'(u^*)w^2)dx.$$

If $f'(u) \le 0$ for any $u \in \mathbb{R}$, we conclude that $E'(t) \le 0$ and the uniqueness follows as above.

4. Energy method can also be applied to problems of other various forms. For example, consider the following Initial Boundary Value Problem (a wave equation, you do not need to know anything about it now)

$$\begin{cases} u_{tt} = D\Delta u + f(x,t), & x \in \Omega, t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u_t(x,0) = h(x), & x \in \partial\Omega, t \in \mathbb{R}^+. \end{cases} \tag{0.9}$$

Use the energy-functional

$$E(t) := \frac{1}{2}\int_\Omega w_t^2(x,t) + |\nabla w(x,t)|^2 dx$$

to prove the uniqueness of (0.9). You need to justify each step of your arguments rigorously.

Solution 4. *Since this PDE is nonlinear, to show the uniqueness, it is sufficient to show that the following problem has only zero solution*

$$\begin{cases} w_{tt} = D\Delta w, & x \in \Omega, t \in \mathbb{R}^+, \\ w(x,0) = 0, & x \in \Omega, \\ w_t(x,t) = 0, & x \in \partial\Omega. \end{cases} \tag{0.10}$$

To this end, we compute the rate of change of $E(t)$ to be

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_\Omega w_t w_{tt} + D\nabla w_t \cdot \nabla w dx - (\textit{chain rule}) \\ &= \int_\Omega w_t(w_{tt} - D\Delta w)dx - (\textit{PDE; the boundary integral disappears due to the zero BC}) \\ &= 0, \end{aligned}$$

where I skip the detailed calculations in between. Therefore $E(t) = E(0) = 0$ and this implies that $w(x,t) \equiv 0$ for $(x,t) \in \Omega \times (0,\infty)$.

5. We shall see in coming lectures that heat equation can not be solved back in time (i.e., only the case $t \in \mathbb{R}^+$, or some finite lower bounded, can be well investigated). This effect is called the *ill-posedness* of heat equation backwards in time. However, backward heat equation has uniqueness, surprisingly or not.

To show this, let us take the 1-D heat equation over $\Omega = (0, L)$ with DBC for example. Similar as in class, it is equivalent to prove that the following problem has only zero solution $w(x,t) \equiv 0$ over $(0, L) \times (-\infty, 0)$

$$\begin{cases} w_t = D w_{xx}, & x \in (0,L), t \in \mathbb{R}^-, \\ w(x,0) = 0, & x \in (0,L), \\ w(x,t) = 0, & x = 0, L, t \in \mathbb{R}^-, \end{cases} \tag{0.11}$$

and furthermore it is sufficient to prove that $E(t) = 0$ for all $t \in \mathbb{R}^-$, with $E(t) := \int_\Omega w^2(x,t)dx$. It is not easy to see that the fact $\frac{dE(t)}{dt} \leq 0$ leads to no contradiction since $E(t) \geq 0$ for $t \in \mathbb{R}^-$, which is totally reasonable.

Let us argue by contradiction as follows. If not, say $E(t_0) > 0$ for some $t_0 > 0$. Then by the continuity, we can always find $t_1 \in (t_0, 0]$ such that $E(t) > 0$ in (t_0, t_1) and $E(t_1) = 0$ (draw a graph to for yourself; such t_1 exists since at least $E(0) = 0$).

(i) Find $E'(t)$ and $E''(t)$;

(ii) Prove the Cauchy–Schwartz inequality

$$\Big| \int_0^L f(x)g(x)dx \Big| \leq \Big(\int_0^L f^2(x)dx \Big)^2 \Big(\int_0^L g^2(x)dx \Big)^2;$$

this is also referred to as Holder's inequality with $p = 2$. Hint: $p(r) = \int_0^L (f(x) + rg(x))^2 dx$ is always nonnegative, $\forall r \in \mathbb{R}$;

(iii) Prove that $(E')^2 \leq EE''$, $\forall t \in [t_0, t_1)$;

(iv) Prove that $(\ln E)'' \geq 0$, $\forall t \in [t_0, t_1)$, and then use this to derive a contraction to the fact that $E(t_1) = 0$.

Solution 5. *(i) One has as before that*

$$E'(t) = -2D \int_\Omega |\nabla w|^2 dx,$$

and $E''(t) = \int_\Omega (2ww_t)_t dx = 2\int_\Omega (w_t)^2 dx + 2\int_\Omega ww_{tt}dx$, *which, together with the PDE, implies that*

$$E''(t) = \int_\Omega (2ww_t)_t dx = 2D \int_\Omega (\Delta w)^2 dx + 2 \int_\Omega w \Delta w_t dx.$$

Now, we employ Green's second identity with DBC $w = 0$ *and again the PDE to obtain*

$$E''(t) = \int_\Omega (2ww_t)_t dx = 2D \int_\Omega (\Delta w)^2 dx + 2 \int_\Omega \Delta w w_t dx = 4D \int_\Omega (\Delta w)^2 dx.$$

(ii). It is easy to see that the quadratic function

$$p(r) = \int_0^L (f(x) + rg(x))^2 dx = (\int_0^L g^2 dx) r^2 + 2(\int_0^L fg dx) r + (\int_0^L f^2 dx)$$

is always non–negative, therefore its discriminant is non-positive, and this implies the desired Cauchy–Schwarz inequality.

(iii) By C–S inequality, one easily observes that $(E')^2 \leq EE''$, $\forall t \in [t_0, t_1)$;

(iv) Now, since $(\ln E(t))'' = \frac{E''E - (E')^2}{E^2} \geq 0$ *for* $t \in (t_0, t_1)$, *we have that* $\ln E(t)$ *is convex(non-concave) in this interval. This is impossible since* $E(t) \equiv 0$ *for all* $t \geq t_1$ *(draw a graph of* $E(t)$ *yourself). The uniqueness follows.*

6. Use Green's first identity to show that

$$\int_\Omega f\Delta g - \Delta f g dx = \int_{\partial\Omega} f \frac{\partial g}{\partial \mathbf{n}} - \frac{\partial f}{\partial \mathbf{n}} g dS. \tag{0.12}$$

(0.12) is called Green's second identity. What is (0.12) when $\Omega = (a, b)$?

Solution 6. *Let us recall the Green's first identity that*

$$\int_{\Omega} f\Delta g dx + \int_{\Omega} \nabla f \cdot \nabla g dx = \int_{\partial\Omega} f\frac{\partial g}{\partial \boldsymbol{n}} dS, \tag{0.13}$$

which switching f and g in (0.13) gives us

$$\int_{\Omega} g\Delta f dx + \int_{\Omega} \nabla g \cdot \nabla f dx = \int_{\partial\Omega} g\frac{\partial f}{\partial \boldsymbol{n}} dS. \tag{0.14}$$

Then subtracting (0.14) from (0.13) readily gives us (0.12).
When $\Omega = (a, b)$, (0.12) becomes

$$\int_a^b (fg'' - f''g)dx = (fg' - f'g)|_a^b.$$

This is called Newton's second identity?