Introduction to PDEs, Fall 2022

Homework 8 Solutions

Name:________________

1. We have defined the Dirac-delta function $\delta(x)$ in 1D in class.

   (1) generalize the definition to ND, $N \geq 2$.

   (2) give an example of the nascent delta function as above;

   Solution 1. *(1) all requirements in the definition remain the same except that $\mathbb{R}$ becomes $\mathbb{R}^N$; (2) one can simply extend the hat function to ND.*

2. For multi-dimensional domain in $\mathbb{R}^N$, $N \geq 2$, with $x = (x_1, x_2, ..., x_N)$, the analog heat kernel is

$$G(x,t) = \frac{1}{(4\pi Dt)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4Dt}},$$

   where $\xi = 0$ is chosen. Show that

   (i) $G_t = D\Delta G$;

   (ii) $\int_{\mathbb{R}^N} G(x,t)dx = 1$.

   Solution 2. *Similar as before, I assume that the students can verify these facts through straightforward calculations and I skip the details here.*

3. Show the following facts for $\delta(x)$:

   (1). $\delta(x) = \delta(-x)$;

   (2). $\delta(kx) = \frac{\delta(x)}{|k|}$, where k is a non-zero constant;

   (3). $\int_{\mathbb{R}} f(x)\delta(x - x_0)dx = f(x_0)$; $\delta(x - x_0)$ is occasionally written as $\delta_{x_0}(x)$;

   (4). Let $f(x)$ be continuous except for a jump–discontinuity at 0. Show that

$$\frac{f(0^-) + f(0^+)}{2} = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

   Remark: For (1) and (2), you are indeed asked to show that they are equal in the distribution sense, but not point wisely.

   Solution 3. *(1). I would like to mention that, whenever checking identities involving $\delta(x)$, it is necessary to check that it holds both points wisely and in the distribution sense. First of all, it is easy to check that $\delta(x) = \delta(-x)$ point wisely (which is only a formal identity), hence I skip it here. To show that they are the same in the distribution sense, we choose an arbitrary continuous function $\phi(x)$ and have that*

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)dx = \phi(0) = \phi(-0) = \int_{-\infty}^{\infty} \phi(-x)\delta(x)dx = \int_{-\infty}^{\infty} \phi(\xi)\delta(-\xi)d\xi,$$

   where we denote $\xi := -x$ in the last identity. Therefore we have the desired identity. The verification of (2) and (3) follows the same approach and I skip it here.

   (4). There are several ways that one can obtain this result and here is one of them. Let us introduce

$$g(x) = \frac{f(x) + f(-x)}{2}, \forall x \in \mathbb{R};$$

if we further define

$$g(0) := \frac{f(0^+) + f(0^-)}{2},$$

then it is easy to see that $g(x)$ *is continuous over* $\mathbb{R}^n$*. Therefore, by the definition of a Dirac–delta function, we have that*

$$\int_{\mathbb{R}} g(x)\delta(x)dx = g(0) = \frac{f(0^+) + f(0^-)}{2};$$

on the other hand, we have that

$$\begin{aligned}
&\int_{\mathbb{R}} g(x)\delta(x)dx = \frac{1}{2}\int_{\mathbb{R}} (f(x) + f(-x))\delta(x)dx \\
=&\frac{1}{2}\int_{\mathbb{R}} f(x)\delta(x)dx + \frac{1}{2}\int_{\mathbb{R}} f(-x)\delta(x)dx \\
=&\int_{\mathbb{R}} f(x)\delta(x)dx,
\end{aligned}$$

where the last identity follows from a change of variable as above. Another approach of the same spirit is to introduce $F(x) = 2g(x)$ *and define* $F(0)$ *as it should be. This should give a method that is intuitively not straightforward, but rigorous.*

4. Define the following sequence of functions from $(-1, 1) \to \mathbb{R}$

$$f_n(x) = \begin{cases} -n, \ x \in (-\frac{1}{n}, 0), \\ n, \ x \in (0, \frac{1}{n}), \\ 0, \ \text{elsewhere}. \end{cases} \tag{0.1}$$

Find the distribution limit of $f_n(x)$. Prove your claim.

Solution 4. *First of all, it is easy to see that* $f_n(x) \to 0$ *point wisely, therefore it is natural to guess that as* $n \to \infty$ *the distribution limit of* $f_n(x)$ *is* $f(x) \equiv 0$*. Indeed, this is a good guess and in order to prove this it is equivalent to showing that*

$$\int_{\mathbb{R}} f_n(x)\phi(x)dx \to \int_{\mathbb{R}} f(x)\phi(x)dx = 0$$

for any continuous and bounded function $\phi(x)$ *as* $n \to \infty$*. Indeed, we have*

$$\int_{\mathbb{R}} f_n(x)\phi(x)dx = \phi(\xi_n)\int_{\mathbb{R}} f_n(x)dx = \phi(\xi_n)\Big(\int_{-\frac{1}{n}}^{0}(-n)dx + \int_0^{\frac{1}{n}} ndx\Big) = 0 = \int_{\mathbb{R}} f(x)\phi(x)dx,$$

where $\xi_n \in [-\frac{1}{n}, \frac{1}{n}]$ *in light of Mean Value Theorem. This shows that* $f \equiv 0$ *is the distribution limit of* $f_n(x)$ *by definition. I would like to mention that, sometimes weak convergence is treated as convergence in distribution, in the sense that they are almost the same except that one requires the test function* ϕ *to be in* L^q *(the dual space), while the other to be continuous and bounded, or* $L^\infty \cap C^0$*. This was noted in class.*

5. Let us revisit Lebesgue's dominated convergence theorem. It states that: if $f_n(x) \to f(x)$ pointwisely/a.e., and there exists an integrable function $g(x)$ such that $|f_n(x)| \leq g(x)$ pointwisely/a.e., then one has that

$$\lim_{n\to\infty}\int_{\Omega} f_n(x)dx = \int_{\Omega} f(x)dx,$$

or equivalently $\|f_n - f\|_{L^1(\Omega)} \to 0$. Here $g(x)$ means that $g \in L^1(\Omega)$. Use this fact to prove its generalized version: if all the statements above hold, except that $g \in L^p(\Omega)$, $p \in (1, \infty)$, then we have that $\|f_n - f\|_{L^p(\Omega)} \to 0$. I wish this gives you some motivation for further (strong) convergence on the $(G * f)(x, t) \to f(x)$ that we proved in class.

Solution 5. *This is obvious. Since $f_n \to f$ pointwisely, one has the $|f_n - f|^p \to 0$ pointwisely. On the other hand,*

$$|f_n - f|^p \leq 2^p |g|^p,$$

therefore, applying Lebesgue's dominated convergence theorem, in light of the fact that the dominating function $|g|^p$ is integrable, gives that

$$\int_\Omega |f_n - f|^p dx \to 0,$$

hence $\|f_n - f\|_{L^p(\Omega)} \to 0$.

6. We know that the solution $u(x,t)$ to the following Cauchy problem

$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty, \infty), t > 0, \\ u(x,0) = \phi(x), & x \in (-\infty, \infty), \end{cases} \tag{0.2}$$

is given by

$$u(x,t) = \int_{\mathbb{R}} \phi(\xi) G(x,t;\xi) d\xi,$$

with

$$G(x,t;\xi) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-\xi)^2}{4Dt}} {}^*.$$

Suppose that $\phi(x) \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ (i.e., continuous and bounded).

(a) Prove that $u(x,t) \in C_x(\mathbb{R})$, i.e., continuous with respect to x. Hint: you can either use $\epsilon - \delta$ language or show that $u(x_n, t) \to u(x,t)$ as $x_n \to x$ for each fixed (x,t). Similarly, you can continue to prove that $u \in C_x^k(\mathbb{R})$ for any $k \in \mathbb{N}^+$, hence C_x^∞, while you can skip this part;

(b) Indeed, as you may have seen from your proof, the continuity condition on initial data $\phi(x)$ above is not required, i.e., $u(x,t) \in C^\infty(\mathbb{R} \times [\epsilon, \infty))$ for any $\epsilon > 0$, if $\phi(x) \in L^\infty(\mathbb{R})$ (even if it has jump or discontinuity). To illustrate this, let us assume $\phi(x) = 1$ for $x \in (-1,1)$ and $\phi(x) = 0$ elsewhere, therefore it has jumps at $x = \pm 1$. Choose $D = 1$, then use MATLAB to plot the integral solution $u(x,t)$ above for time $t = 0.001$, $t = 0.01$, $t = 0.1$ and $t = 1$ in the same coordinate–choose the integration limit to be $(-M, M)$ for some M large enough so it approximates the exact solution. One shall see that $u(x,t)$ becomes smooth at $x = \pm 1$ for any small time $t > 0$, though two jumps are present in the initial data; this is the so-called smoothing or regularizing effect of diffusion.

Solution 6. *(a). I shall take the second approach here, i.e., by showing that $|u_n(x) - u(x)| \to 0$ if $|x_n - x| \to 0$ ($n \to \infty$) with (x,t) fixed. To this end, we observe that for each fixed pair $(x,t) \in \mathbb{R} \times \mathbb{R}^+$*

$$\begin{aligned} |u(x_n,t) - u(x,t)| =& \Big| \int_{\mathbb{R}^n} \phi(\xi) \Big(G(x_n,t,\xi) - G(x,t,\xi) \Big) d\xi \Big| \\ \leq& \int_{\mathbb{R}^n} |\phi(\xi)| \Big| G(x_n,t,\xi) - G(x,t,\xi) \Big| d\xi \\ \leq& \|\phi(\xi)\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Big| G(x_n,t,\xi) - G(x,t,\xi) \Big| d\xi. \end{aligned}$$

It is easy to see that, in the last integral, the integrand $|G(x_n,t,\xi) - G(x,t,\xi)|$ converges to zero point wisely, therefore to obtain the convergence of the integral to zero through Lebesgue's dominated convergence theorem, one only needs to that the integrand is bounded by an integrable function or a constant. The latter case is impossible since it has singularities at both x_n and x. To show the former, one can apply the Mean Value Theorem to have that

$$|G(x_n,t,\xi) - G(x,t,\xi)| = G_x(\tilde{x},t,\xi)(x_n - x),$$

*I would like to point out that there should be no confusion on the notation here, where $G(x-\xi;t)$ was adopted in class. It is a matter of the integration variable and the parameter.

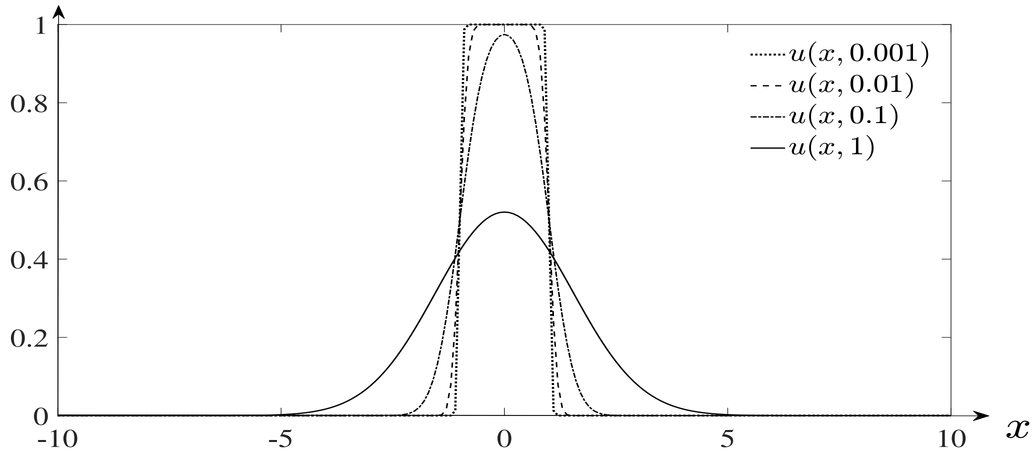

Figure 1: Smoothing effect of diffusion with diffusion rate chosen $D = 1$. The initial condition is a characteristic function supported on $[-1, 1]$. We see that the singularities at $x \pm 1$ is smeared out immediately after even a tiny time $t = 0.001$. Indeed, $u(x,t)$ is C^∞ smooth in x for each $t > 0$ as one can prove.

for some $\tilde{x} \in (x-1, x+1)$, for n large enough, show that G_x is absolutely integrable. I leave this to the student to verify. An alternative way is to follow the approach in the class by breaking $\mathbb{R}^n$ into two regions (three indeed) as follows: it is easy to know that, for any $\epsilon > 0$, one can choose M large enough such that

$$\int_{I_M} \Big|G(x_n,t,\xi) - G(x,t,\xi)\Big| d\xi < \frac{\epsilon}{2},$$

where I denote for notational simplicity the outer region to be

$$I_M := \{|\xi| \geq M\}.$$

Note that I_M may depend on x and t, while it can be uniform in x_n as long as n is large enough; on the other hand, in this inner region, we have that, for n sufficiently large $G(x_n,t,\xi) < G(x,t,\xi) + 1$ hence

$$|G(x_n,t,\xi) - G(x,t,\xi)| < 2G(x,t,\xi) + 1,$$

which is, though not bounded in $\mathbb{R}^n \backslash I_M$, absolutely integrable over the inner region. Therefore, since the integrand converges to zero point wisely, one can apply the dominated convergence theorem to obtain that for n sufficiently large

$$\int_{\mathbb{R}^n \backslash I_M} \Big|G(x_n,t,\xi) - G(x,t,\xi)\Big| d\xi < \frac{\epsilon}{2}.$$

I would like to remark that this is why sometimes it is necessary or friendly to students/beginners to split $\mathbb{R}^n$ in this form.

(b). See figure 1.

7. Consider the following problem over the half-plane

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t > 0, \\ u(x,0) = 0, & x \in (0,\infty), \\ u(0,t) = N_0, & t > 0, \end{cases} \tag{0.3}$$

where N_0 is a point heating source at the endpoint.

(1) Solve for $u(x,t)$ in terms of an integral.

(2) Set $D = N_0 = 1$. Plot your solve for $t = 0.001$, 0.01, 0.1 and 1. The graphs should match the physical description of the problem.

Solution 7. *Let us first consider its counterpart over $\mathbb{R}$ in the following form*

$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty,\infty), t>0, \\ u(x,0)=0, & x \in (-\infty,\infty), \\ u(0,t)=\phi(x), & t>0, \end{cases} \tag{0.4}$$

with

$$\phi(x) = \begin{cases} 0, & x>0, \\ \Phi(x), & x<0, \end{cases}$$

$\Phi(x)$ to be chosen. We observe that the solution to (0.4) solves (0.6) except the boundary condition. Therefore we shall choose $\Phi(x)$ to this end. Note that the solution to (0.4) is

$$u(x,t) = \int_{-\infty}^{\infty} \phi(\xi)G(x,t;\xi)d\xi = \int_{-\infty}^{0} \Phi(\xi)G(x,t;\xi)d\xi.$$

Coping it with the initial condition $u(x,0)=N_0$ gives us

$$N_0 = \int_{-\infty}^{0} \Phi(\xi)G(0,t;\xi)d\xi.$$

There are various choices of $\Phi(x)$ through which one can achieve this identity and the simplest one is a constant $\Phi \equiv K$. In this case, using the fact that

$$\int_{-\infty}^{0} G(0,t,\xi)d\xi = \frac{1}{2}$$

gives us that $\Phi(x) \equiv K = 2N_0$. Therefore we have that

$$u(x,t) = 2N_0 \int_{-\infty}^{0} G(x,t;\xi)d\xi$$

or an equivalent form

$$u(x,t) = 2N_0 \int_{0}^{\infty} G(x,t;-\xi)d\xi = \frac{N_0}{\sqrt{4Dt}} \int_{0}^{\infty} e^{-\frac{|x+\xi|^2}{4\pi Dt}} d\xi.$$

Again, I would like to remark that here the choice of such Φ is apparently not unique, however different Φ gives rise to different integral, while in the end, $u(x,t)$ end up the same since it is unique (though we have not proved the uniqueness for the Cauchy's problem). This is very similar to what we convert inhomogeneous boundary conditions into homogeneous ones, one has infinitely many choices of $w(x,t)$, and different ones give rise to different problems, while eventually ending up with the same solution for the original problem.

Even though the approach above may seem awkward or not natural to some of you, it is wrong to solve this problem by the kernel of half-plane

$$G^*(x,t;\xi) = G(x,t;\xi) - G(x,t;-\xi)$$

since the boundary condition is inhomogeneous. Some of your peer students have redone the whole problem from the beginning, that being said, first consider the problem over $(0,L)$

$$\begin{cases} u_t = Du_{xx}, & x \in (0,L), t>0, \\ u(x,0)=0, & x \in (0,L), \\ u(0,t)=N_0, & t>0, \end{cases} \tag{0.5}$$

work on it by converting the inhomogeneous boundary condition into a homogeneous one, and then send L to infinity, which I have no problem with. In this spirit, one can simply denote $v(x,t) := u(x,t)-N_0$, then $v(x,t)$ satisfies

$$\begin{cases} v_t = Dv_{xx}, & x \in (0,\infty), t>0, \\ v(x,0)=-N_0, & x \in (0,\infty), \\ v(0,t)=0, & t>0, \end{cases} \tag{0.6}$$

therefore by the formula for the half-line problem, we have that

$$v(x,t) = -N_0 \int_0^{\infty} G(x,t;\xi) - G(x,t;-\xi)d\xi$$

hence

$$u(x,t) = N_0 - N_0 \int_0^{\infty} G(x,t;\xi) - G(x,t;-\xi)d\xi,$$

which, in light of the identity

$$1 - \int_0^{\infty} G(x,t;\xi)d\xi = 1 - \int_{-\infty}^{0} G(x,t;-\xi)d\xi = \int_0^{\infty} G(x,t;-\xi)d\xi,$$

also gives rise to the desired expression of $u(x,t)$ as above.

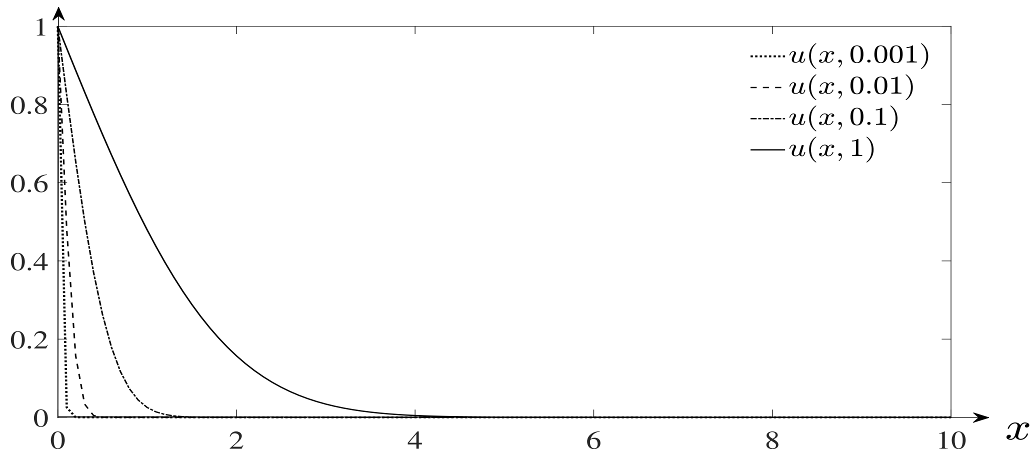

Figure 2: Solution $u(x,t)$ at different times. One observes that the DBC acts as a source that keeps pumping the heat into the region, hence for each location the temperature is increasing with respect to time as one can observe.

8. Use physical interpretations (or design some mental experiments) to obtain the solution to the following problems by using $u(x,t)$ in (0.2) without solving them as you have done before (note that you already know the results according to lecture and the last HW)

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t > 0, \\ u(x,0) = \phi(x), & x \in (0,\infty), \\ u(0,t) = 0, & t > 0, \end{cases} \tag{0.7}$$

and

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t > 0, \\ u(x,0) = \phi(x), & x \in (0,\infty), \\ u_x(0,t) = 0, & t > 0. \end{cases} \tag{0.8}$$

Hint: while G presents a unit heating source, $-G$ presents a unit cooling source.

Solution 8. *When putting a unit of thermal energy (heat) at location ξ at time $t = 0$, the distribution $G(x,t;\xi)$ is the temperature at space-time location (x,t). However, if one puts another cooling point source at $-\xi$ at the same time, it is not hard to see, by the symmetry of G, that the resulting temperature is $G^*(x,t;\chi) = G(x,t;\xi) - G(x,t;-\xi)$ and it is always zero at location $x = 0$. Therefore, for general initial data $\phi(x)$, the resulting temperature subject to zero boundary condition at $x = 0$ is the convolution of G^* with ϕ. Similarly, for (0.8), one can put the two unit thermal energies at $x = \pm\xi$ respectively, resulting in heat kernel $\tilde{G} = G(x,t;\xi) + G(x,t;-\xi)$, which satisfies the Neumann boundary condition, and then the convolution of $\tilde{G}$ with ϕ is the solution for (0.8). For (0.7), one can perform a similar except with a heating and cooling resource at each size.*

9. This problem is to give you a flavor of how PDE is connected and applied to finance. To be specific, we will obtain the price of a European option explicitly by solving the classical 1D heat equation, leading to the seminal Black-Scholes formula.

In mathematical finance, the Black-Scholes or Black-Scholes-Merton model is a PDE that describes the price evolution of a European call or European put under the Black-Scholes model. Let us denote S_t as the price of the underlying risky asset at time t, typically a stock (which is regarded as an independent variable to which investigator adjust their investing strategy) and μ as the risk–free interest rate, then the Black-Scholes model assumes that the price of S follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t{}^{\dagger},$$

where W_t is Brownian motion (a continuous stochastic process the increments of which are Gaussian, i.e., $W_{t+s} - W_s \sim N(0,s)$ for each $t, s > 0$, one can simply it treat it formally as a randomly increasing or decreasing process as time varies). $\sigma > 0$ is the magnitude of such randomness, so this term measures the risk (of investing in this stock) and is called *volatility* in finance (think of this as the variance of a random variable in statistics). It is not hard to imagine that S is now a stochastic process, i.e., at each fixed time S is a random variable with random event ω hidden. Financial time-series data indicate that the volatility can depend on a stock price S and other factors in practice (e.g., time-delay, term structure, component structure, volatility smile, etc., if you are aware), however, for the sake of mathematical simplicity (all models are wrong), it is assumed that σ is a constant in the B–S model (well, this is not useful, to be honest with you). According to the model, the equation above gives rise to the following PDE

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, & S > 0, 0 < t < T, \\ V(S,T) = \phi(S), & S > 0, \end{cases} \tag{0.9}$$

where $T > 0$ is a pre-given time, the so-called strike time. V is the option value (as the dependent value) (the word *option* in finance means the right to buy or sell an asset at a predetermined price, or the so-called strike price K, before or on the strike time T. Both K and T are specified in the contract.) You need to know some basic stochastic calculus or Itô calculus in order to derive this PDE, which is out of the scope of this homework or this course. Anyhow, let us just focus on cooking up this PDE at this moment. It is necessary to point out that instead of an initial condition, (0.9) is coupled with a terminal condition. This is because the diffusion rate becomes $-\frac{1}{2}\sigma^2 S^2$ if you want to write it as a heat equation, therefore there is no contradiction to the principle that diffusion rate can not be negative.

It is the purpose of this homework to show that (0.9) can be transformed into the Cauchy problem of the heat equation and then be solved explicitly. To this end, do the followings:

(a). Introduce the new variables

$$S = Ke^x, t = T - \frac{\tau}{\sigma^2/2},$$

where the constant K is the strike price. Let $v(x,\tau) = V(S,t)$. Show that

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \Big(\frac{2r}{\sigma^2} - 1\Big)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v.$$

(b). Introduce

$$u(x,\tau) = e^{ax+b\tau}v(x,\tau).$$

Choose constants a and b such that $u(x,\tau)$ satisfies

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{0.10}$$

(c) A European option may be exercised only at the expiration date T of the option, and in particular, the call option of a European option is defined mathematically by letting

$$\phi(S) = \max\{S - K, 0\}.$$

†Here the subindex t, like it or not, merely means a notation, not the partial derivative.

Solve the classical heat equation (0.10) under this terminal condition. Write your solution $u(x,\tau)$ in terms of integrals.

(d). In terms of the solutions in (c) and the transformations. Show that the solution $V(S,t)$ to (0.9) is

$$V = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln S/K + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln S/K + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and N is the cumulative normal distribution that

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy.$$

(e) Suppose that $S = 200$, $K = 210$, $r = 2\%$ is the annual interest rate, $\sigma = 0.58$ and the expiration date is in two months. Find the call value V. In order to make use of this formula, you need to make sure that all the parameters are on the same scale.

Solution 9. *(a). Since $S = Ke^x, t = T - \frac{\tau}{\sigma^2/2}$, we have that $\tau = (T-t)\frac{\sigma^2}{2}, x = \ln\frac{S}{K}$ and therefore*

$$\frac{\partial\tau}{\partial t} = -\frac{\sigma^2}{2}, \frac{\partial x}{\partial S} = \frac{1}{S},$$

$$\frac{\partial V}{\partial t} = \frac{\partial\tau}{\partial t}\frac{\partial v}{\partial\tau} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau}, \frac{\partial V}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial v}{\partial x} = \frac{1}{S}\frac{\partial v}{\partial x}$$

and

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \left(-\frac{1}{S^2}\right)\frac{\partial v}{\partial x} + \frac{1}{S}\frac{\partial x}{\partial S}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(-\frac{1}{S^2}\right)\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}.$$

Substituting these identities into the PDE leads us to

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau} + \frac{\sigma^2}{2}S^2\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS\frac{1}{S}\frac{\partial v}{\partial x} - rv = 0,$$

which implies

$$\frac{\partial v}{\partial\tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v.$$

(b). Let $u(x,t) = e^{ax+b\tau}v(x,t)$ for some constants a and b to be determined. Then

$$\frac{\partial u}{\partial\tau} = \left(bv + \frac{\partial v}{\partial\tau}\right)e^{ax+b\tau}$$

and

$$\frac{\partial u}{\partial x} = \left(av + \frac{\partial v}{\partial x}\right)e^{ax+b\tau}, \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 v}{\partial x^2} + 2a\frac{\partial v}{\partial x} + a^2 v\right)e^{ax+b\tau}.$$

Therefore, we have that

$$\begin{aligned}\frac{\partial u}{\partial\tau} - \frac{\partial^2 u}{\partial x^2} &= \left(bv + \frac{\partial v}{\partial\tau} - \frac{\partial^2 v}{\partial x^2} - 2a\frac{\partial v}{\partial x} - a^2 v\right)e^{ax+b\tau} \\ &= \left(\frac{\partial v}{\partial\tau} - \frac{\partial^2 v}{\partial x^2} - 2a\frac{\partial v}{\partial x} + (b-a^2)v\right)e^{ax+b\tau}\end{aligned}$$

Now we choose a and b such that

$$\begin{cases} 2a = k-1 \\ b - a^2 = k \end{cases}$$

with $k = \frac{2r}{\sigma^2}$, i.e.,

$$a = \frac{k-1}{2} = \frac{2r - \sigma^2}{2\sigma^2}, b = \frac{(k+1)^2}{4} = \frac{(2r+\sigma^2)^2}{4\sigma^4}.$$

Then we readily see that $u(x,\tau)$ satisfies

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

(c). $\tau = 0$ for $t = T$, then the terminal condition becomes

$$V(S,T) = v(x,0) = \phi(S) = \phi(Ke^x) = \max\{Ke^x - K, 0\} = K(e^x - 1)^+$$

Thus we can obtain $u(x,0)$ is

$$u(x,0) = e^{ax}v(x,0) = e^{\frac{k-1}{2}x}v(x,0) = K(e^{\frac{(k+1)x}{2}} - e^{\frac{(k-1)x}{2}})^+ := \Phi(x),$$

where $k = \frac{2r}{\sigma^2}$ as given above.

On the other hand, we know that the solution to the heat equation takes the integral form

$$u(x,\tau) = G * \Phi(x,\tau) = \int_{\mathbb{R}} G(x-\xi,\tau)\Phi(\xi)d\xi,$$

where $G(x-\xi,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-\xi)^2}{4\tau}}$ and $\Phi(\xi) = K(e^{\frac{(k+1)\xi}{2}} - e^{\frac{(k-1)\xi}{2}})^+$. To be precise, we can rewrite $u(x,\tau)$ as

$$\begin{aligned} u(x,\tau) =& \frac{K}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}(e^{\frac{(k+1)\xi}{2}} - e^{\frac{(k-1)\xi}{2}})^+ e^{-\frac{(x-\xi)^2}{4\tau}}d\xi \\ =& \frac{K}{\sqrt{4\pi\tau}}\int_0^\infty (e^{\frac{(k+1)\xi}{2}} - e^{\frac{(k-1)\xi}{2}})e^{-\frac{(x-\xi)^2}{4\tau}}d\xi, \end{aligned}$$

because

$$\Phi(\xi) = \begin{cases} K(e^{\frac{(k+1)\xi}{2}} - e^{\frac{(k-1)\xi}{2}}), & \xi > 0, \\ 0, & \mathit{otherwise}. \end{cases}$$

(d). Now we change the variable ξ as $\xi = x + \sqrt{2\tau}\eta$ and obtain

$$\begin{aligned} u(x,\tau) =& \frac{K}{\sqrt{4\pi\tau}}\int_0^\infty (e^{\frac{(k+1)\xi}{2}} - e^{\frac{(k-1)\xi}{2}})e^{-\frac{(x-\xi)^2}{4\tau}}d\xi \\ =& \frac{K}{\sqrt{4\pi\tau}}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty (e^{\frac{(k+1)(x+\sqrt{2\tau}\eta)}{2}} - e^{\frac{(k-1)(x+\sqrt{2\tau}\eta)}{2}})e^{-\frac{\eta^2}{2}}\sqrt{2\tau}d\eta \\ =& \frac{K}{\sqrt{2\pi}}e^{\frac{(k+1)x}{2}+\frac{1}{4}\tau(k+1)^2}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k+1))^2}{2}}d\eta \\ &- \frac{K}{\sqrt{2\pi}}e^{\frac{(k-1)x}{2}+\frac{1}{4}\tau(k-1)^2}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k-1))^2}{2}}d\eta. \end{aligned}$$

For the first term, we denote $y = \eta - \frac{1}{2}\sqrt{2\tau}(k+1)$ and have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k+1))^2}{2}}d\eta =& \frac{1}{2\pi}\int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}\sqrt{2\tau}(k+1)}^\infty e^{-\frac{y^2}{2}}dy = \frac{1}{2\pi}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k+1)} e^{-\frac{y^2}{2}}dy \\ =& N\Big(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}\sqrt{2\tau}(k+1)\Big) = N(d_1), \end{aligned}$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}\sqrt{2\tau}(k+1) = \frac{\ln\frac{S}{K} + (\frac{2r}{\sigma^2}+1)\frac{\sigma^2}{2}(T-t)}{\sqrt{2\frac{\sigma^2}{2}(T-t)}} = \frac{\ln S/K + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Similarly, we let $y = \eta - \frac{1}{2}\sqrt{2\tau}(k-1)$ and have that

$$\frac{1}{2\pi}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k-1))^2}{2}}d\eta = \frac{1}{2\pi}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k-1)} e^{-\frac{y^2}{2}}dy$$
$$= N\Big(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}\sqrt{2\tau}(k-1)\Big) = N(d_2),$$

where

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}\sqrt{2\tau}(k-1) = \frac{\ln\frac{S}{K} + (\frac{2r}{\sigma^2}-1)\frac{\sigma^2}{2}(T-t)}{\sqrt{2\frac{\sigma^2}{2}(T-t)}} = \frac{\ln S/K + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Finally, using $u = e^{ax+b\tau}v$ leads us to the desired

$$\begin{aligned} V =& e^{-ax-b\tau}u = e^{-\frac{k-1}{2}x-\frac{(k+1)^2}{4}\tau}\Big(Ke^{\frac{(k+1)x}{2}+\frac{1}{4}\tau(k+1)^2}N(d_1) - Ke^{\frac{(k-1)x}{2}+\frac{1}{4}\tau(k-1)^2}N(d_2)\Big) \\ =& Ke^{x}N(d_1) - Ke^{k\tau}N(d_2) \\ =& SN(d_1) - Ke^{r(T-t)}N(d_2), \end{aligned}$$

where

$$d_1 = \frac{\ln S/K + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln S/K + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and N is the cumulative normal distribution that

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

(e). Choosing $S = 200, K = 210, r = 0.02, \sigma = 0.58, T = \frac{2}{12}$ and $t = 0$, we can calculate

$$\begin{aligned} d_1 &= \frac{\ln S/K + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx -0.0736, \\ d_2 &= \frac{\ln S/K + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx -0.3104, \\ N(d_1) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{y^2}{2}}dy \approx 0.470671, \\ N(d_2) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}dy \approx 0.378141, \end{aligned}$$

and eventually evaluate that $V = SN(d_1) - Ke^{-r(T-t)}N(d_2) \approx 14.988848$.

10. Let us consider the multi-dimensional heat equation in $\mathbb{R}^n$, $n \geq 2$

$$\begin{cases} u_t = D\Delta u, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{0.11}$$

i) use your gut feeling to write down the solution of (0.11) without solving it. Hint: what is the fundamental solution in nD?

ii) To solve (0.11), let us consider the *dilation scaling* with constants α and β

$$u(x,t) := \frac{1}{t^\alpha}v\big(\frac{x}{t^\beta}\big), x \in \mathbb{R}^n, t > 0.$$

Denote $y := \frac{x}{t^\beta}$. Find the PDE for $v(y)$ and show that $\beta = \frac{1}{2}$;

iii) Let $w(|y|) = v(y)$. Find the PDE for $w(|y|)$ and show that $\alpha = \frac{n}{2}$;

iv) Suppose that (for physical consideration) both w and w' converge to zero as $|y| \to \infty$, Show that $w(r) = C_0 e^{-\frac{r^2}{4}}$, where C_0 is a positive constant to be determined such that the total mass/integral of w is unit. Compare it with your intuition.

Solution 10. *i) We mimic the 1D scenario by putting a unit heat source at location ξ, then the resulting temperature profile is the fundamental solution of the heat equation in high dimension which takes the following form (we just guess)*

$$G(x,t;\xi) = \frac{1}{\sqrt{(4\pi Dt)^n}} e^{\frac{-|x-\xi|^2}{4Dt}}.$$

The appearance of n is due to the restriction that the total mass (heat) of G must be 1. You can either perform straightforward calculations or conduct a mental experiment as follows: integrating $e^{\frac{-|x-\xi|^2}{4Dt}}$ over x_1 in $\mathbb{R}$ gives $\sqrt{4\pi Dt}$, over x_2 $\mathbb{R}$ gives another $\sqrt{4\pi Dt}$, ..., hence n is needed in the denominator.

ii) By straightforward calculations, we find that

$$\alpha t^{-\alpha+1} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0.$$

This equation only holds without having (anything) blow up if $\beta = \frac{1}{2}$. Then the PDE reduces to

$$\alpha v(y) + \frac{1}{2} y \cdot \nabla v(y) + \Delta v(y) = 0.$$

iii) If we further restrict v to be radial (i.e., radially symmetric), then one can find by straightforward calculations that

$$\alpha w(r) + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0.$$

If one converts the ODE for rw', then $\alpha = \frac{n}{2}$ is necessary and one finds that $(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0$, which thereupon implies

$$r^{n-1} w' + \frac{1}{2} r^n w = C.$$

This constant C must be zero since we always need w and w' to decay to zero in infinity. Finally, $w' = -\frac{r}{2} w$ and $w = C_0 e^{\frac{r^2}{4}}$ with C_0 being the same constant as above that normalize the integral.

iv)