Introduction to PDEs, Fall 2022

Test 1 Make-up

For 2 hours and 35 minutes

Name(Print):______________

Student No:______________

Signature:______________

There are 10 problems, 10 points each, 100 points in total.

Show `details` to get full credits. Make your justifications `clear` and `direct`.

Leave the following table blank

Score Table		
Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	Total score	

1. Let us consider a similar scenario in a 2D lattice with mesh size $\Delta x = \Delta y$. Let $u(x, y, t)$ be the number of particles at location $(x, y) \in \mathbb{R}^2$ and time $t > 0$. Suppose that each particle, at the next time $t + \Delta$, moves northwards, southwards, westwards or eastwards with probability $\frac{1}{4}$, i.e., $p((x, y) \to (x \pm \Delta x, y), t) = p((x, y) \to (x, y \pm \Delta y), t) = \frac{1}{4}$. Assume that $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+ > 0$. Derive the PDE for $u(x, y, t)$.

2. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$.

   (i) Write down the definitions of Dirichlet, Neumann, Robin boundary conditions.

   (ii) What are the physical interpretations of DBC and NBC if $u(x, t)$ represents the temperature?

   (iii) What are the biological interpretations of DBC and NBC if $u(x, t)$ represents the population density?

3. Consider the following initial boundary value problem

$$\begin{cases} u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \gamma(x), & x \in \partial\Omega, t > 0. \end{cases} \tag{1}$$

Use the energy method to prove the uniqueness of (1).

4. Solve the following eigen-value problem for the eigen–pair $(X_n(x), \lambda_n)$

$$\begin{cases} X''(x) + \lambda X(x) = 0, \quad x \in (0, L), \\ X'(0) = X'(L) = 0. \end{cases} \tag{2}$$

5. 1) Write down the definition of $L^2(\Omega)$ and L^2-norm;

   2) Write down the definition that functions f and g are orthogonal in $L^2(\Omega)$;

   3) Consider the following problem

$$\begin{cases} \Delta w + \lambda w = 0, & x \in \Omega, \\ \alpha \frac{\partial u}{\partial \mathbf{n}} + \beta u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

   where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $\alpha^2 + \beta^2 \neq 0$. Prove that w_m and w_n, corresponding to λ_m and λ_n respectively, are orthogonal in $L^2(\Omega)$, whenever $\lambda_m \neq \lambda_n$.

6. Consider the following IBVP

$$\begin{cases} u_t = Du_{xx}, & x \in (0, L), t > 0, \\ u(x, 0) = \phi(x) \geq, \not\equiv 0, & x \in (0, L), \\ u(x, t) = 0, & x = 0, L, t > 0. \end{cases} \tag{4}$$

i) Without solving (4), describe the behavior of its solution in the long time, i.e., as $t \to \infty$. Draw $u(x, t)$ for several time, say $t = 1, 2, 5, 10...$ to illustrate your results;

ii) We know that solution to (4) is unique. Find this solution in terms of infinite series.

...continue working here if needed

7. Consider the following problem over a 2D square $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \Delta u = 0, & x \in (0,1) \times (0,1) \\ u_x(0,y) = u_x(1,y) = 0, & y \in (0,1), \\ u(x,0) = 0, u(x,1) = x. & \end{cases} \tag{5}$$

Find $u(x,y)$ in terms of infinite series by starting with $u(x,y) = X(x)Y(y)$.

8. Let us consider the following problem with heating/cooling resources under DBC

$$\begin{cases} u_t = u_{xx} + e^{-t} \sin 2x, & x \in (0,1), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,1), \\ u = 0, & x = 0, 1, t \in \mathbb{R}^+. \end{cases} \tag{6}$$

Find the solution to (6) in terms of infinite series.

...continue working here if needed

9. Consider

$$\begin{cases} u_t = Du_{xx}, & x \in (0, L), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0, L), \\ u(0,t) = \mu_1(t), u(L,t) = \mu_2(t), & t \in \mathbb{R}^+. \end{cases} \tag{7}$$

Solving for $u(x,t)$ in terms of infinite series;

...continue working here if needed

10. Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 1$. Suppose that $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

1) Write down the definitions of strong convergence and weak convergence of f_n to f in $L^p(\Omega)$;

2) prove that strong convergence in L^p implies weak convergence;

3) give a counter-example that weak convergence does not imply strong convergence. Proof is **NOT** needed.