Introduction to PDEs, Fall 2022

Homework 1 Solutions

Name:______________

1. Let us consider the discrete model of random walk and assume for simplicity of computation that $\Delta x = \Delta t = 1$. Then we know that the discrete equation over the whole lattice is

$$u(x, t+1) = \frac{1}{2}\big(u(x-1,t) + u(x+1,t)\big), x = \pm 1, \pm 2, \pm 3, ..., t = 0, 1, 2, ... \tag{0.1}$$

Suppose that initially, we put a number of $(4-x^2)^+$ particles at location x, where "+" here denotes the positive part. That is, the initial data are given such that $u(x,0) = 4 - x^2$ for $|x| \leq 2$ and $u(x,0) \equiv 0$ for $|x| \geq 2$.

Let us consider $u(\pm 3, t)$ at time $t = 2$ for example. It is easy to see that we need the values of $u(\pm 4, 1)$, which further require the values of $u(\pm 5, 0)$. Similarly, one traces back to $u(\pm 6, 0)$ in order to evaluate $u(\pm 3, 3)$. Some of you may have recognized the mechanism/scheme as a binomial tree, I wish this gives you some intuition that the particles, which move locally to their neighboring sites at the next time step, will eventually spread out the whole region (we will see later in this course that the speed is infinite if $\Delta x \to 0^+$). Or you can imagine that u is the number of infected living organisms and the disease will eventually dominate the whole space if it spreads out randomly.

(i) plot $u(x,t)$ for $x = \pm 3, \pm 2, \pm 1, 0$ at time $t = 0, 1, ..., 6$, and connect the neighbouring dots with straight lines;

(ii) now set $\Delta x = \Delta t = 0.5$ and plot $u(x,t)$ for $x = \pm 3, \pm 2.5, \pm 2, \pm 1.5, \pm 1, 0$ at time $t = 0, 1, ..., 6$. I suggest you use MATLAB or other computational software for the calculations. Now you see, the discrete problem is, to be frank, simple but computationally annoying; do the same for $\Delta x = \Delta t = 0.01$. It becomes tedious with your bare hands but not if using a computer program. 1) That is why it makes well sense to study the continuous case, which approximates the discrete ones when Δx and Δt are small; 2) now you are learning the finite difference method solving the heat equation without realizing it;

(iii) now let us go back to the same problem with $\Delta x = \Delta t = 1$ and the same initial condition $u(x,0) = (4-x^2)^+$ but now over the finite interval $(-5,5)$. We are well set for $u(x,1)$ at all x except $x = \pm 5$, because $x = \pm 6$ is not considered in this finite interval. Therefore, we need to set specific conditions for $u(\pm 5, t)$ for any time t in order to calculate $u(\pm 4, t+1)$, so on and so forth. This condition is called the boundary condition and it must be set for any PDE over a finite interval. One of the types is the so-called Dirichlet boundary condition (you have seen this already) and we set that $u(\pm 5, t) = 0$ (or some other constant) for any $t > 0$. Suppose that $u(\pm 5, t) = 0$ for any $t \geq 0$. plot $u(x,t)$ for $x = \pm 5, ..., \pm 1, 0$ at time $t = 0, 1, ..., 6$, and connect the neighbouring dots with straight lines;

(iv) do the same as in (iii) with $u(\pm 5, t) = 2$ for any $t > 0$;

(v) do the same as in (iii) with $u(-5,t) = 1$ and $u(5,t) = 3$ for any $t > 0$. Now you can see that different boundary conditions can have different effects on the solution's behavior;

(vi) do the same as in (iii) with $u(-5,t) = 1$ and $u(5,t) = 3$ for any $t > 0$, but $u_0(x) = (x^2-4)^+$. Now you can see that different initial conditions can have different effects on the solution's behavior;

Note: you can always, for your entertainment but no need to show me, choose Δx and Δt to another size, say 10^{-3}, to see how the discrete solution approaches the continuum solution;

Solution 1. *While I plotted these graphs using MATLAB, you are free to use any programming languages that you prefer. However, I would like to note each one of you should know at least one of the basic computational techniques/skills to avoid brawling with your bare hands.*

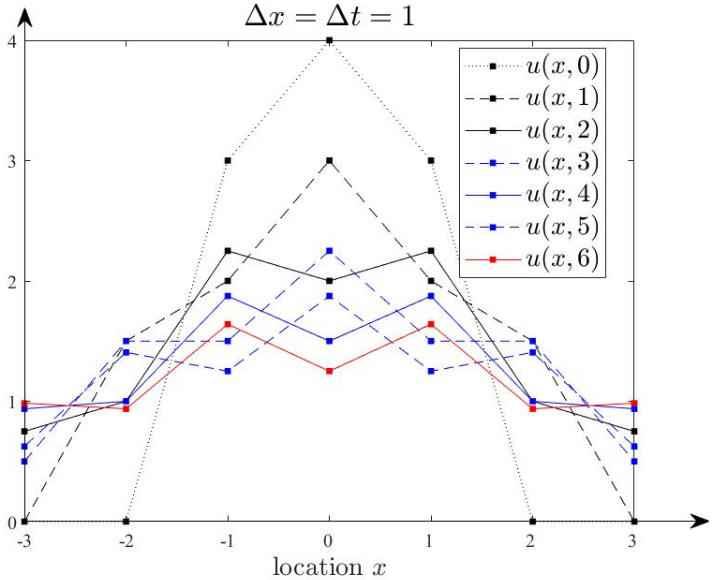

Figure 1: Plot in (i) of Problem 1

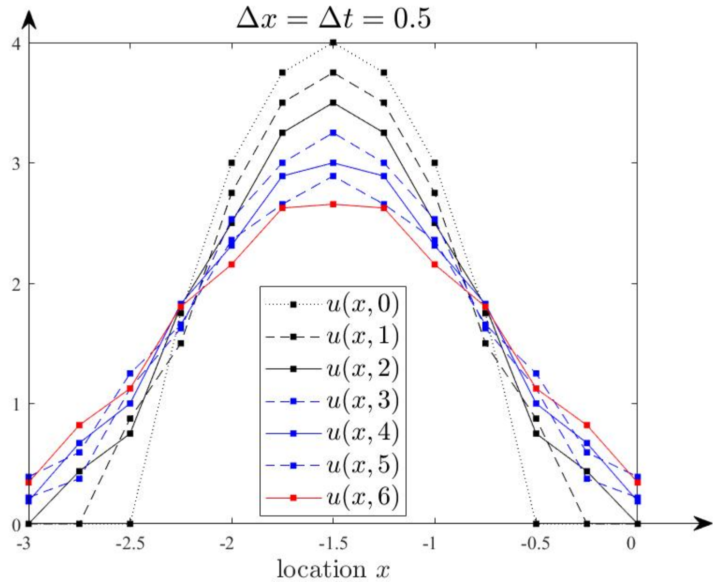

Figure 2: Plot in (ii) of Problem 1

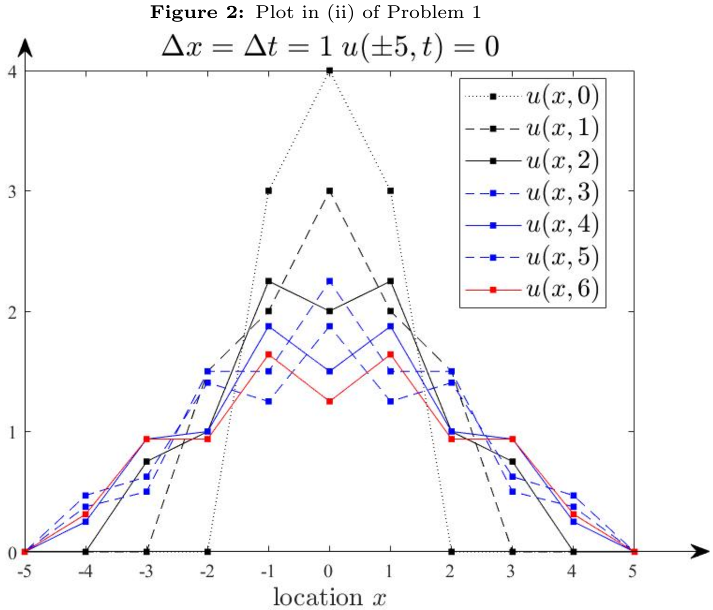

Figure 3: Plot in (iii) of Problem 1

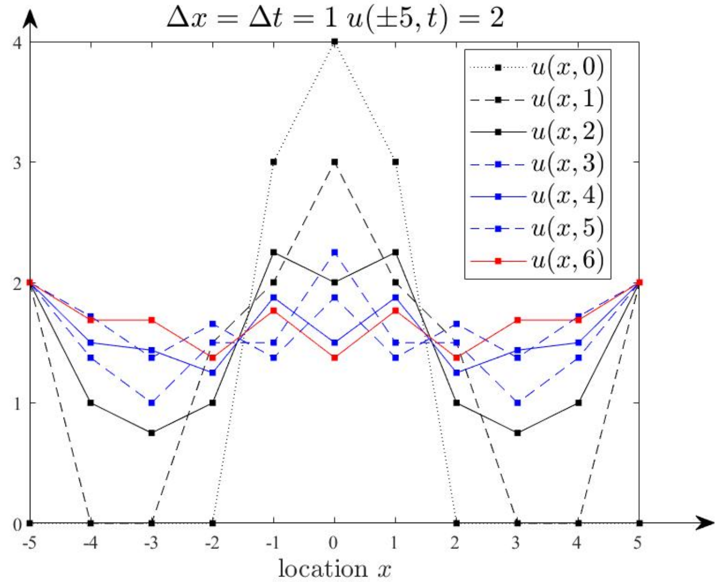

Figure 4: Plot in (iv) of Problem 1

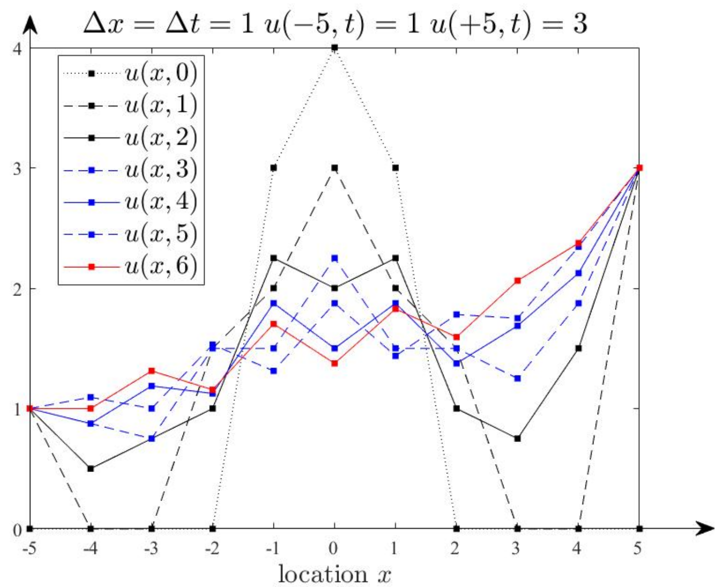

Figure 5: Plot in (v) of Problem 1

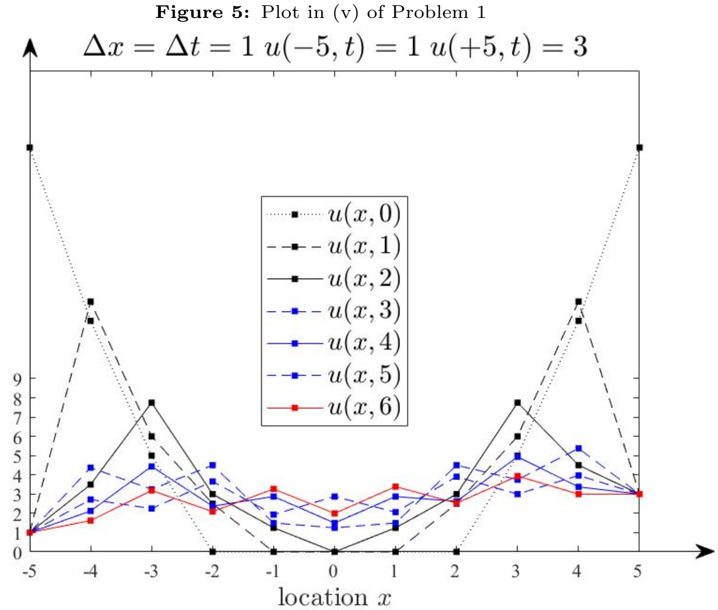

Figure 6: Plot in (vi) of Problem 1

2. Now let us go back to our baby example in 1D lattice/grid: each particle at x either moves to $x-\Delta x$ or $x+\Delta x$ with probability $\frac{1}{2}$, at any time t. As I mentioned in class, some students may have concern that it is unfair or unrealistic to assume that each particle only moves to $x \pm \Delta x$ at the next time step, and it is possible that, for example, the particle may move to $x \pm \Delta x$ with probability $\frac{1}{4}$, and move to $x \pm 2\Delta x$ with a small probability, say, $\frac{1}{8}$, and move to $x \pm 3\Delta x$ with an even smaller probability, say, $\frac{1}{16}$, and to $x \pm 4\Delta x$, $x \pm 5\Delta x$...so on and so forth, with all the probabilities adding up to 1. I do not have much to disagree with this possibility, however, this problem is designed to show that this case also leads to the classical heat equation, with merely a variation of the diffusion rate. For the simplicity of our mathematical analysis, and without losing our generality, let us assume that each particle can only move to $x \pm \Delta x$, $x \pm 2\Delta x$, with probability

$$p(x \to x \pm \Delta x, t) = \frac{\alpha}{2}, p(x \to x \pm 2\Delta x, t) = \frac{1-\alpha}{2},$$

hence $p(x \to x \pm 3\Delta x, t) = p(x \to x \pm 4\Delta x, t) = ... = 0$. Derive the PDE for $u(x,t)$ by the microscopic approach.

Solution 2. *First of all, the difference equation is*

$$u(x, t+\Delta t) = \frac{\alpha}{2}\big(u(x+\Delta x, t) + u(x-\Delta x, t)\big) + \frac{1-\alpha}{2}\big(u(x+2\Delta x, t) + u(x-2\Delta x, t)\big),$$

and by expanding their Taylor series up to the order of Δx^2 and Δ we have

$$u_t \Delta t + O(\Delta t^2) + \Big(\frac{\alpha}{2}\Delta x^2 + 2(1-\alpha)\Delta^2\Big)u_{xx} + O(\Delta x^3),$$

therefore the PDE is $u_t = \frac{4-3\alpha}{2} D u_{xx}$.

3. Suppose that $u(x,t)$ moves to $x \pm \Delta x$ at any time t with a probability p which depends location x but not time, i.e.,

$$p(x \to x \pm \Delta x, t) = \rho(x)$$

Put $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+$. Derive the PDE for $u(x,t)$. What if the probability also depends on time, i.e.,

$$p(x \to x \pm \Delta x, t) = \rho(x,t)$$

or

$$p(x \to x \pm \Delta x, t) = \rho(x, t+\Delta t).$$

Solution 3. *First of all, we consider the case when $p(x \to x \pm \Delta x, t) = \rho(x)$, therefore $p(x \pm \Delta x \to x, t) = \rho(x \pm \Delta x)$ and $p(x \to x, t) = 1 - 2\rho(x)$. By the same arguments as above*

$$\begin{aligned} u(x, t+\Delta t) =& u(x+\Delta x, t)p(x+\Delta x \to x, t) + u(x-\Delta x, t)p(x-\Delta x \to x, t) + u(x,t)p(x \to x, t) \\ =& u(x+\Delta x, t)\rho(x+\Delta x) + u(x-\Delta x, t)\rho(x-\Delta x) + u(x,t)(1-2\rho(x)). \end{aligned} \tag{0.2}$$

You can expand both $u(x \pm \Delta x, t)$ and $\rho(x \pm \Delta x)$ to simplify (0.2); however, it seems much easier if we put $w(x \pm \Delta x, t) := u(x \pm \Delta x, t)\rho(x \pm \Delta x)$. The RHS of (0.2) then becomes

$$w(x+\Delta x, t) + w(x-\Delta x, t) + u(x,t)(1-2\rho(x)) = \frac{\partial^2 w}{\partial x^2}\Delta x^2 + u(x,t) + O(\Delta x^4)$$

and equating it with the LHS, with $\frac{\Delta x^2}{\Delta t} \to 0^+$, we arrive at the PDE: $u_t = D(\rho(x)u(x,t))_{xx}$.

Second of all, if the probability also depends on time $p(x \to x \pm \Delta x, t) = \rho(x,t)$. RHS of (0.2) does not change since we do not need to expand them with respect to time t, therefore we should still have the same PDE as above $u_t = D(\rho(x)u(x,t))_{xx}$.

However, if the transition probability depends on time as $p(x \to x \pm \Delta x, t) = \rho(x, t+\Delta t)$, then (0.2) should be

$$\begin{aligned} u(x, t+\Delta t) =& u(x+\Delta x, t)p(x+\Delta x \to x, t) + u(x-\Delta x, t)p(x-\Delta x \to x, t) + u(x,t)p(x \to x, t) \\ =& u(x+\Delta x, t)\rho(x+\Delta x, t+\Delta t) + u(x-\Delta x, t)\rho(x-\Delta x, t+\Delta t) \\ &+ u(x,t)(1-2\rho(x, t+\Delta t)) \end{aligned} \tag{0.3}$$

and we need to work on (0.3) in a bit details. You can just expand each terms and then multiply them out, however we can put $w(x \pm \Delta x, t) := u(x \pm \Delta x, t)\rho(x \pm \Delta x)$ as above and rewrite (0.3) into

$$\begin{aligned} u(x,t+\Delta t) =& u(x+\Delta x,t)\rho(x+\Delta x,t+\Delta t) + u(x-\Delta x,t)\rho(x-\Delta x,t+\Delta t) \\ &+ u(x,t)(1-2\rho(x,t+\Delta t)) \\ =& w(x+\Delta x,t) + w(x-\Delta x,t) + u(x,t)(1-2\rho(x,t+\Delta t)) \\ &+ u(x+\Delta x,t)\Big(\rho(x+\Delta x,t+\Delta t) - \rho(x+\Delta x,t)\Big) \\ &+ u(x-\Delta x,t)\Big(\rho(x-\Delta x,t+\Delta t) - \rho(x-\Delta x,t)\Big) \\ =& 2w(x,t) + w_{xx}(x,t)\Delta x^2 + O(\Delta x^4) + u(x,t)(1-2\rho(x,t+\Delta t)) \\ &+ u(x+\Delta x,t)\Big(\rho_t(x+\Delta x,t)\Delta t + O(\Delta t^2)\Big) \\ &+ u(x-\Delta x,t)\Big(\rho_t(x-\Delta x,t)\Delta t + O(\Delta t^2)\Big). \end{aligned} \tag{0.4}$$

Since we shall send both Δx and Δt to zero with $\frac{\Delta x^2}{\Delta t} = D > 0$, therefore we shall only need to keep the residual terms (or the so–called remainders) up to order of Δx^2 and Δt. By keeping on expanding in (0.4), we collect

$$\begin{aligned} u(x,t+\Delta t) =& 2w(x,t) + w_{xx}(x,t)\Delta x^2 + O(\Delta x^4) + u(x,t)(1-2\rho(x,t) - 2\rho_t(x,t)\Delta t + O(\Delta t^2)) \\ &+ u(x+\Delta x,t)\rho_t(x+\Delta x,t)\Delta t + u(x-\Delta x,t)\rho_t(x-\Delta x,t)\Delta t + O(\Delta t^2) \\ =& u(x,t) + w_{xx}(x,t)\Delta x^2 - 2u(x,t)\rho_t(x,t)\Delta t + O(\Delta x^4) + O(\Delta t^2), \end{aligned} \tag{0.5}$$

where we used the notation $w := u\rho$ to derive the last identity. Finally, we divide BHS of (0.5) by Δt and send it to zero with $D = \frac{\Delta x^2}{\Delta t} > 0$, then we can collect the PDE $u_t + 2u\rho_t = D(u\rho)_{xx}$. For example, if ρ is independent of time t, we have $\rho_t = 0$ hence this PDE reduces to the one above.

4. Now change the probability above as arrival–dependent, i.e.,

$$p(x \to x \pm \Delta x, t) = \rho(x \pm \Delta x).$$

Derive the PDE for $u(x,t)$. Is the PDE the same as the one above? Compare them and state your observation.

Solution 4. *When the probability is arrival–dependent, we then have $p(x \pm \Delta x \to x, t) = \rho(x)$ and $p(x \to x, t) = 1 - \rho(x+\Delta x) - \rho(x-\Delta x)$, therefore, similar as above, we can derive the difference equation*

$$\begin{aligned} u(x,t+\Delta t) =& u(x+\Delta x,t)p(x+\Delta x \to x,t) + u(x-\Delta x,t)p(x-\Delta x \to x,t) + u(x,t)p(x \to x,t) \\ =& u(x+\Delta x,t)\rho(x) + u(x-\Delta x,t)\rho(x) + u(x,t)(1-\rho(x+\Delta x) - \rho(x-\Delta x)). \end{aligned} \tag{0.6}$$

It is not difficult to simplify (0.6) and we should arrive at the PDE: $u_t = D(\rho u_{xx} - u\rho_{xx})$.

5. Let us consider a similar scenario in a 2D lattice with mesh size $\Delta x = \Delta y$. Let $u(x,y,t)$ be the number of particles at location $(x,y) \in \mathbb{R}^2$ and time $t > 0$. Suppose that each particle, at the next time $t + \Delta$, moves northwards, southwards, westwards or eastwards with probability $\frac{1}{4}$, i.e., $p((x,y) \to (x \pm \Delta x, y), t) = p((x,y) \to (x, y \pm \Delta y), t) = \frac{1}{4}$. Assume that $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+ > 0$. Derive the PDE for $u(x,y,t)$. Hint: use Taylor expansion for multivariate functions.

Solution 5. *We have that*

$$u(x,y,t+\Delta t) = \frac{1}{4}u(x-\Delta x,y,t) + \frac{1}{4}u(x+\Delta x,y,t) + \frac{1}{4}u(x,y-\Delta y,t) + \frac{1}{4}u(x,y+\Delta y,t).$$

Now we need to apply the Taylor expansion for multi–variable functions:

$$f(x+\Delta x, y+\Delta y) = f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y + \frac{1}{2}\Big(f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2\Big) + H.O.T,$$

where H.O.T means higher order terms.

Note that the partial derivative terms are cancelled for the difference equations above. Since $\Delta x = \Delta y$ *and* $D = \frac{\Delta x^2}{\Delta t}$*, we have the PDE:* $u_t = \frac{D}{4}(u_{xx} + u_{yy})$*; now we know that it is convenient to write this PDE as* $u_t = \frac{D}{4}\Delta u$*, and it is a multi–dimensional heat equation.*

6. Do the same as above for $\Delta x = 2\Delta y$.

   Solution 6. *The arguments here are the same as above. Let* $D = \frac{\Delta x^2}{\Delta t}$*, then we have* $\frac{\Delta y^2}{\Delta t} = \frac{D}{4}$*. The PDE is:* $u_t = \frac{D}{4}(u_{xx} + \frac{1}{4}u_{yy})$*. Remark: We can also call this as a heat equation, but it is an isotropic heat equation, i.e., describing the temperature within a heterogeneous medium. The diffusion rate of* x *direction is 4 times that of the* y *direction, since for a fixed time period* Δt*, the particles moves twice distance in* x *direction than that in* y*. This means that, in a plain language, the particle has a tendency to move (randomly) in the* x *direction 4 times than that in the* y *direction. I wish this also gives you more intuition on the physical interpretations of the diffusion rate.*

7. (only for motivated students) Let us assume that each particle can also move across the diagonal such that $p((x, y) \to (x \pm \Delta x, y), t) = p((x, y) \to (x, y \pm \Delta y), t) = \alpha$ and $p((x, y) \to (x \pm \Delta x, y \pm \Delta y), t) = 1/4 - \alpha$, $\alpha \in (0, 1/4)$.

   Solution 7. *Skipped. The main idea is that one still obtain the classical heat equation, thought with a slightly different diffusion rate.*

8. Let us now consider a bar or rod in $\mathbb{R}^3$ which is inhomogeneous in x–direction but homogeneous in both y and z directions. We assume that the density $\rho(x)$ ($gram/cm.^3$), the cross-sectional area $A(x)$ (cm^2), the specific heat capacity $c(x)$ ($calorie/gram.degree$) as well as the thermal conductivity $\kappa(x)$ ($calorie/cm.degree.second$) are now functions of x but are constant throughout any particular cross-section. Therefore, the temperature $u(x, t)$ is a constant for any particular cross-section. Suppose that there is no heat or cooling resource within the bar. Show that the equation of such heat flow is

$$c(x)\rho(x)A(x)\frac{\partial u}{\partial t} = \kappa(x)A(x)\frac{\partial^2 u}{\partial x^2} + \Big(\kappa(x)A'(x) + A(x)\kappa'(x)\Big)\frac{\partial u}{\partial x}, \tag{0.7}$$

   where $'$ denotes the derivative taken concerning x. You need to present your justifications in a logically well-ordered way. Remark: now the arbitrary domain Ω is a cylinder and its outer normal can be explicitly calculated.

   Solution 8. *I would like to mention that you can work it out similar to what we did in the lecture: consider any section of the bar from* $x = a$ *to* $x = b$*, with* a *and* b *being arbitrary. Then it is easy to know that the total thermal energy within this region is*

$$E(t) = \int_a^b c(x)\rho(x)A(x)u(x, t)dx$$

   and the rate of change of the energy is

$$\frac{dE(t)}{dt} = \frac{d}{dt}\int_a^b c(x)\rho(x)A(x)u(x, t)dx = \int_a^b c(x)\rho(x)A(x)\frac{\partial u(x, t)}{\partial t}dx.$$

   On the other hand, by the conservation of thermal energy, we know that the rate of change must equals those through the end points (here are the two cross-sections at a *and* b*), which takes the form*

$$\kappa(x)A(x)\frac{\partial u(b, t)}{\partial x} - \kappa(x)A(x)\frac{\partial u(a, t)}{\partial x} = \int_a^b \frac{\partial}{\partial x}\Big(\kappa(x)A(x)\frac{\partial u(x, t)}{\partial x}dx\Big),$$

   where the last identity follows from the fundamental theorem of calculus. Equating the equations above, we have

$$\int_a^b c(x)\rho(x)A(x)\frac{\partial u(x, t)}{\partial t}dx = \int_a^b \frac{\partial}{\partial x}\Big(\kappa(x)A(x)\frac{\partial u(x, t)}{\partial x}dx\Big)$$

hold for any a and b. Therefore, we can show that, through the same contradiction argument, $u(x,t)$ satisfies the following PDE

$$c(x)\rho(x)A(x)\frac{\partial u(x,t)}{\partial t}=\frac{\partial}{\partial x}\Big(\kappa(x)A(x)\frac{\partial u(x,t)}{\partial x}\Big),$$

or (0.7). Both hand sides have the unit

$$\frac{calorie}{gram\times degree}\times\frac{gram}{cm^3}\times cm^2\times\frac{degree}{second}=\frac{calorie}{cm\times second}$$

9. A steady-state of a time-dependent equation means that the temperature at any point is independent of time. Find the steady state of a heat equation over $\Omega=(0,L)$ with the left end temperature fixed at u_0 and the right end fixed at u_1. Remark: you might help you 'see' what the flux rate is proportional to the 'gradient'.

   Solution 9. *For heat equation $u_t=Du_{xx}$, its steady state is a time–independent solution, i.e., $u_t=0$ hence $u=u(x)$ and accordingly one has that $u_{xx}=0$, which in light of the end points conditions $u(0)=u_0$ and $u(L)=u_1$, gives us that*

   $$u(x)=(1-\frac{x}{L})u_0+\frac{x}{L}u_1$$

   is the steady state. It is easy to see that it is a straight line with slope $\frac{u_1-u_0}{L}$. Therefore, when L is sufficiently small, one readily observes that the how fast it varies spatially is the negative gradient (slope).

10. Denote $u_n(x,t):=e^{-D(\frac{n\pi}{L})^2t}\sin\frac{n\pi x}{L}$.

    i) show that for each $N<\infty$, the series $\sum_{n=1}^N c_nu_n(x,t)$ is a solution to the following baby heat equation

    $$\frac{\partial u}{\partial t}=D\frac{\partial^2u}{\partial x^2},$$

    where c_n are constants. Remark: the series is a solution for $N=\infty$ as long it converges and we will discuss it later in this course;

    ii) assume that $L=\pi$ and $D=0.05$. Use MATLAB or other software to plot $u_1(x,t)$ over $x\in(0,\pi)$ with $t=0$, $t=0.5$, $t=2$ in the same coordinate. Do the same for $D=0.1$ and $D=1$. What are your observations and explain them intuitively;

    iii) again, for $D=1$, use MATLAB or other software to plot $1u_1(x,t)+0.5u_2(x,t)$ over $x\in(0,\pi)$ with $t=0$, $t=0.5$, $t=2$ in the same coordinate;

    Solution 10. *We find that $\partial_tu_n=-D(\frac{n\pi}{L})^2u_n$ and $\partial_{xx}u_n=-(\frac{n\pi}{L})^2u_n$, therefore, u_n satisfies the PDE. There are several observations we have from these plots. First of all, for each D, $u_1(x,t)$ converges to zero as t increases; the profile of such convergence takes the sine function. This is sometimes called the mode of the pattern (think about blowing a balloon and the shape/pattern how the balloon grows is the mode). In the last plot, we investigate the effect of diffusion size D on the decay rate of $u_1(x,2)$. It demonstrates that a larger D results in a faster decay to zero of $u_1(x,2)$; indeed this is true, not only for $t=2$ but also any time $t>0$. For example, when $D=0.01$, $u_1(x,2)$ has a profile of sine function, however when $D=5$, $u_1(x,2)$ is almost zero. I wish this problem gives you some pictures of how the diffusion rate affects the evolution of solutions of heat equations. Generally speaking, for a single heat equation or reaction-diffusion in general, if the diffusion rate is sufficiently large, then the solution $u(x,t)$ always converges to a constant solution (depending on the initial data and boundary condition). This is called the stabilizing effect of the diffusion rate. I would like to elaborate a little bit here that, for multidimensional heat equation over a bounded domain on the other hand, usually one can not write a clean formula as $u_n(x,t)$ above (unless for some very geometries such as a rectangle or a disk), then the how fast the solutions decay depends not only on the diffusion rate but also the shape of the domain (or the principal eigenvalue of Laplace operator in general). However, the stabilizing effect is no longer true when it comes to a system of reaction-diffusion equations (i.e., when there are more than two equations), and*

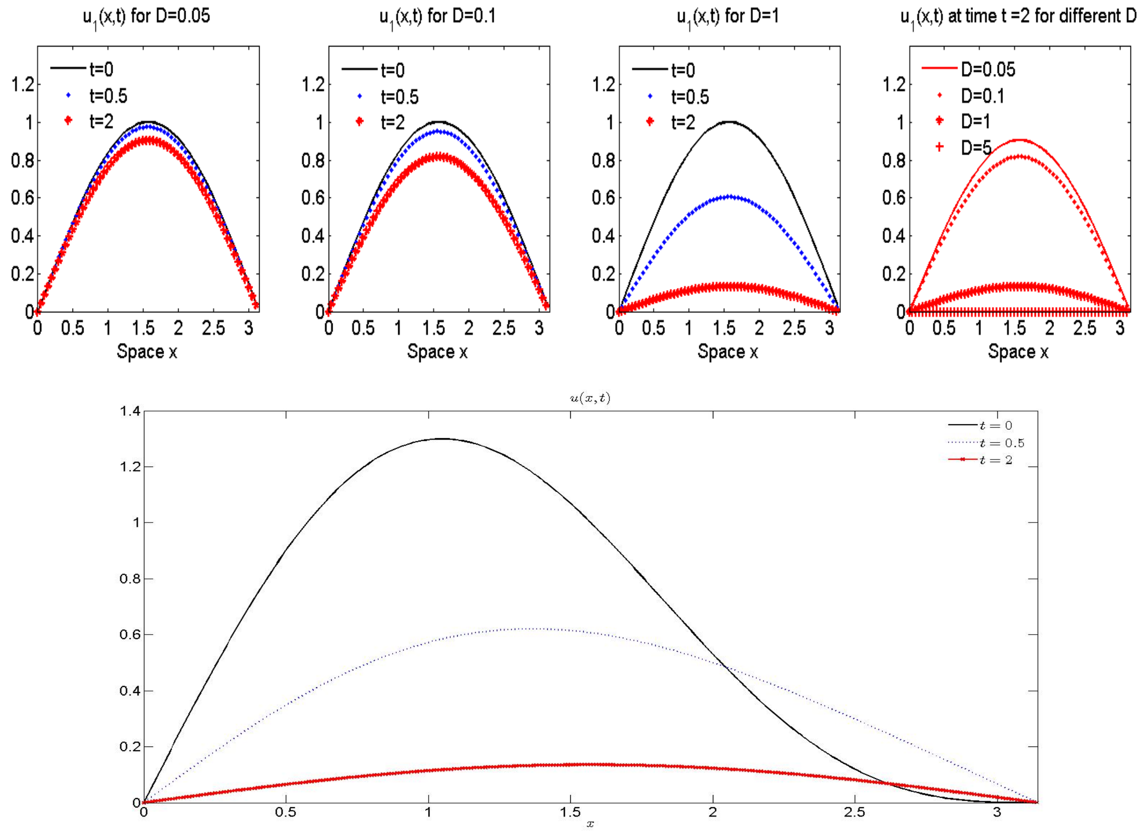

Figure 7: **Top**: The first three plots show the evolution of $u(x,t)$ at time $t = 0, 0.5$ and 2 for each fixed diffusion rate D; the last plot shows the effect of diffusion rate D on the profile of $u(x,t)$ for time $t = 2$. **Bottom:** The plots of $1u_1(x,t) + 0.5u_2(x,t)$ at the specified times.

sometimes the system may admit a solution that converges to nonconstant solutions even when the diffusion rate(s) is(are) large. This phenomenon is called Turing's instability and such nonconstant stationary solutions are called Turing's patterns. This was first discovered by and named after the British mathematician Alan Turing, who is widely considered to be the father of theoretical computer science and artificial intelligence. Since then, mathematicians have developed reaction-diffusion models to study the emergence of patterns on animals such as stripes of zebra, dots on fish, etc., concerning the chemical reaction and diffusion of two substances, called activator and inhibitor. The study of Turing's patterns has emerged as one of the most extensively studied topics over the past few decades. Very recently, mathematical models of this fashion have been proposed and studied to investigate the well-observed hotspots in urban criminal activities. We might come back to these topics (reaction-diffusion systems) if time allows.

11. Let us consider the following generalized heat equation for $m \geq 1$

$$u_t = \Delta(u^m), \mathbf{x} \in \mathbb{R}^N, t \in \mathbb{R}^+, \tag{0.8}$$

which reduces to the classical heat equation when $m = 1$, and to Boussinesq's equation when $m = 2$. Note that one can rewrite $u_t = \nabla \cdot (mu^{m-1}\nabla u)$, hence the diffusion rate is recognized as mu^{m-1}. This equation was proposed in the study of ideal gas flowing isentropically in a homogeneous medium or the flow of fluid through porous media (such as oil through the soil), where the law, instead of Fourier's law of constant diffusivity, $\mathbf{J} = -u^{m-1}\nabla u$ is usually observed. You do not need to know why this particular form is chosen, but it is not surprising to imagine that the flow of dye in water behaves differently from that of the oil in the soil.

A fundamental solution to the problem in 1D was obtained in the 1950s by Russian mathematician

Barenblatt, where in $\mathbb{R}^N$, $N \geq 1$, one has

$$u(\mathbf{x}, t) = t^{-\alpha}\Big(C - \kappa|\mathbf{x}|^2 t^{-2\beta}\Big)_+^{\frac{1}{m-1}}, \tag{0.9}$$

where

$$(f)_+ := \max\{f, 0\}, \alpha := \frac{N}{(m-1)N+2}, \beta := \frac{\alpha}{N}, \kappa = \frac{\alpha(m-1)}{2mN}$$

and C is any positive constant.

(1) Prove that (0.9) satisfies the PME (0.8);

(2) In 1D ($n = 1$), choose $m = 2$ and $C = 10$, and then plot $u(x, t)$ for $t = 1$, 2, 5 and 10.

Solution 11. *(1) can be verified by straightforward calculations;*

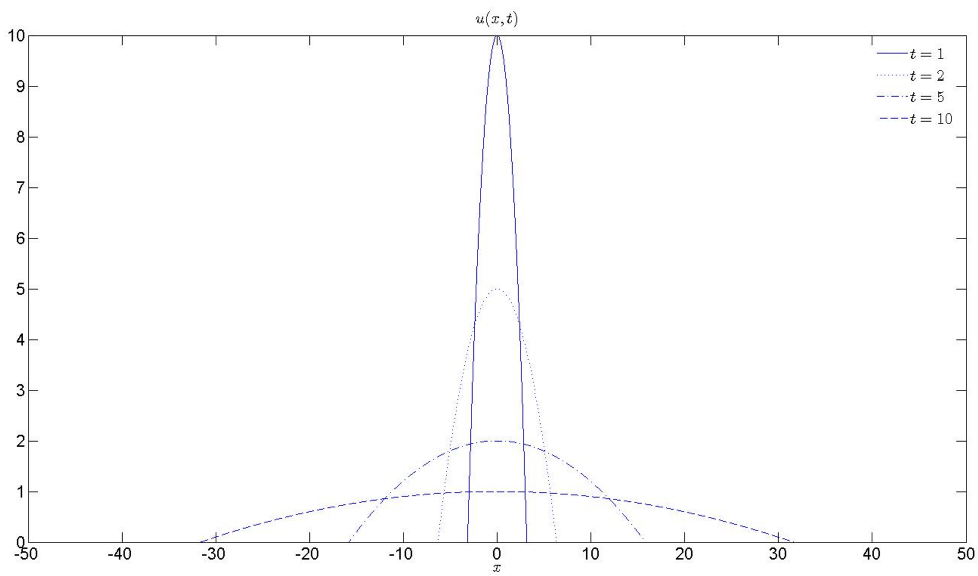

Figure 8: The solutions at the specified times with the given coefficients. I might have chosen the values for α, β, κ in the problem, but you are free to choose your own set of parameters. One observes from these plots that the solution *travels* a finite distance over a finite time, and this is called the finite speed of propagation. We shall see that the heat equation has an infinite speed of propagation later in this course.

12. Go to review on the following topics: gradient; directional derivative; multivariate integral; surface integral; divergence theorem; No need to turn in your review.

Solution 12. *I assume that you have done so.*