Introduction to PDEs, Fall 2022

Homework 9, Due Dec 8

Name:_______________

1. We know that the Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0, \\ 0, & x < 0 \end{cases} \tag{0.1}$$

has the Dirac-delta function $\delta(x)$ be its weak derivative. As I mentioned in class you might find in some textbooks that a Heaviside function is defined otherwise such as

$$H(x) = \begin{cases} 1 & x > 0, \\ \frac{1}{2} \text{ (or any other number)} & x = 0, \\ 0, & x < 0. \end{cases} \tag{0.2}$$

However, according to Lebesgue's theory, the value of a function at a single point (or a zero measure set) does not affect its properties in general and two functions that are equal almost everywhere are considered to be identical. Accordingly, the two forms of $H(x)$ are identical while we shall take the former in our course. Similarly, the weak derivative of a function is unique up to a measure zero, that being said, if $f(x)$ is a weak derivative of $F(x)$, then $g(x)$ is also a weak derivative, if $f(x)$ and $g(x)$ only differ on a zero measure set. This applies further.

A so-called bump function is given as $B(x) = xH(x)$. Show by definition that the weak derivative of $B(x)$ is $H(x)$.

2. Find the weak derivative of $F(x)$, denoted by $f(x)$

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases} \tag{0.3}$$

3. It is necessary to point out that in the definition of a weak derivative, some textbooks require that both F and f are L^1_{loc} (here "loc" means being locally integrable in the sense that it is integrable over any compact subset of Ω). Let $\Omega = (0, 2)$ and define

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 2, & 1 \le x < 2. \end{cases} \tag{0.4}$$

Show that $F' = f$ does not exist in the weak sense by the following contradiction argument: suppose that the weak derivative f exists, show that for any test function $\phi(x) \in C^1_0(0, 2)$ (some textbooks use C^∞_c, where "c" denotes compact) we have

$$\int_0^2 f\phi dx = \int_0^1 \phi dx + \phi(1).$$

Now choose a sequence of test functions $\phi_m(x)$ satisfying

$$0 \le \phi_m(x) \le 1, \phi_m(1) = 1, \phi_m(x) \to 0, \forall x \ne 1, m \to \infty$$

and then obtain a contradiction from the identity above.

4. Assume that $F_n(x)$ converges to $F(x)$ weakly, and let $f_n(x)$ and $f(x)$ be their weak derivatives respectively. Prove that $f_n(x)$ also converges to $f(x)$ weakly.

5. One can easily generalize the second–order operator to higher dimension, the Laplace operator Δ over $\Omega \subset \mathbb{R}^n$, $n \geq 1$.

   (a) We say that f is radially symmetric if $f(x) = f(r)$, $r = |x| := \sqrt{\sum_{i=1}^n x_i^2}$. Prove that

   $$\Delta f(r) = f''(r) + \frac{n-1}{r} f'(r),$$

   where the prime denotes a derivative taken with respect to r.

   (b) Denote that $G(r) := \frac{1}{2\pi} \ln r$ for $n = 2$. We shall show that $\Delta G = \delta(r)$. For this moment, let us consider its regularization over $2D$ of the form

   $$G_\epsilon(r) = \frac{1}{2\pi} \ln(r + \epsilon), \epsilon > 0.$$

   Show that $\Delta G_\epsilon(r)$ converges to $\delta(x)$ in distribution as $\epsilon \to 0^+$. Hint: you can either apply Lebesgue's dominated convergence theorem, or use ϵ–δ language. Make sure you have checked all the conditions when applying the former one.

   (c) Denote $G(r) := -\frac{1}{4\pi r^2}$ for $n = 3$. Mimic (b) by finding an approximation G_ϵ and then show that this approximation ΔG_ϵ convergence to $\delta(x)$ in distribution.