Introduction to PDEs, Fall 2022

Homework 7 Solutions

Name:________________

1. In class, we arrived at an integral of the following form when evaluating $G_L^{\pm}$

$$I(c) = \int_0^\infty e^{-w^2 b} \cos(wc) dw,$$

where b and c are constants.

(i) (Optional) Evaluate this integral through integration by parts or any method you know;

(ii). An alternative approach is to solve an ODE that $I(c)$ satisfies. Show that $I(c)$ satisfies

$$\frac{dI(c)}{dc} = -\frac{c}{2b} I(c);$$

(iii). Show that $I(0) = \sqrt{\frac{\pi}{4b}}$ and solve the ODE in (ii) to find $I(c)$.

Solution 1. *(i) Skipped.*

(ii). Differentiate $I(c)$ w.r.t t and we have

$$\frac{dI(c)}{dc} = \frac{d}{dc} \int_0^\infty e^{-w^2 b} \cos(wc) dw = \int_0^\infty e^{-w^2 b} \Big(\frac{d}{dc} \cos(wc) \Big) dw$$
$$= - \int_0^\infty w e^{-w^2 b} \sin(wc) dw.$$

Meanwhile, we can evaluate RHS of the equation

$$-\frac{c}{2b} I(c) = -\frac{c}{2b} \int_0^\infty e^{-w^2 b} \cos(wc) dw = -\frac{1}{2b} \int_0^\infty e^{-w^2 b} d \sin(wc)$$
$$= -\frac{1}{2b} e^{-w^2 b} \sin(wc) \Big|_0^\infty + \frac{1}{2b} \int_0^\infty \sin(wc) d e^{-w^2 b} = -\int_0^\infty w e^{-w^2 b} \sin(wc) dw.$$

Therefore, $I(c)$ satisfies

$$\frac{dI(c)}{dc} = -\frac{c}{2b} I(c).$$

(iii). In order to evaluate the value of $I(0)$, we calculate

$$I^2(0) = \Big(\int_0^\infty e^{-w^2 b} dw \Big)^2 = \int_0^\infty e^{-x^2 b} dx \cdot \int_0^\infty e^{-y^2 b} dy$$
$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2) b} dx dy$$

Let $x = r \sin\theta$ and $y = r\cos\theta$. Then we can have that

$$I^2(0) = \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty e^{-r^2 b} r dr = \frac{\pi}{2} (-\frac{1}{2b}) e^{-r^2 b} \Big|_0^\infty = \frac{\pi}{4b}.$$

Of course $I(0) > 0$, we can get $I(0) = \sqrt{\frac{\pi}{4b}}$. From (i), we can write the equation in form of

$$\frac{dI(c)}{I(c)} = -\frac{c}{2b} dc.$$

Integrating it over $(0,t)$*, we have that*

$$\ln I(c)\Big|_0^t = -\frac{c^2}{4b}\Big|_0^t = -\frac{t^2}{4b}$$

i.e., $I(t) = I(0)e^{-\frac{t^2}{4b}}$*, which is*

$$I(c) = \sqrt{\frac{\pi}{4b}}e^{-\frac{c^2}{4b}}.$$

2. We know from class that the solution to the following problem

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,\infty), \\ u(0,t) = u(\infty,t) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{0.1}$$

is given in the following form*

$$u(x,t) = \int_0^\infty \big(G^-(\xi;x,t) - G^+(\xi;x,t)\big)\phi(\xi)d\xi.$$

Note that this integral above can be evaluated symbolically. Choose $D = 1$ and the initial data to be $\phi(x) \equiv 1$ for $x \in (0,1) \cup (2,3)$ and $\phi(x) \equiv 0$ otherwise. Plot the solution of (0.1) at times $t = 10^{-4}, 10^{-3}, 0.1, 0.5, 1$ and 5. Note that this integral over $(0,\infty)$ must be truncated over $(0,L)$ for L large. Choose your own L. (You should know how to choose such L up to certain accuracy by now).

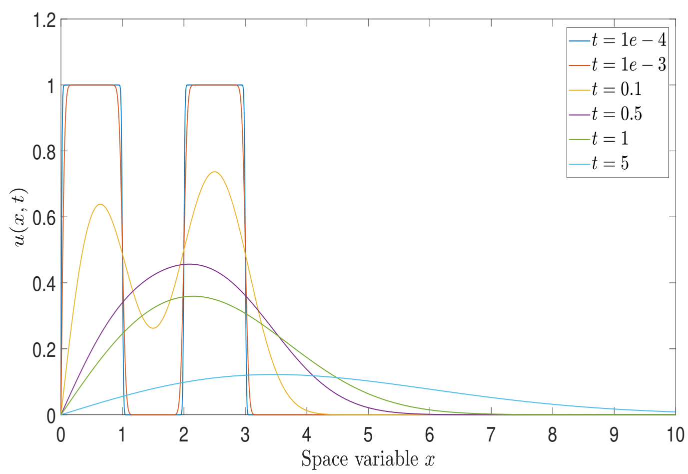


Figure 1: Problem-2

Solution 2.

3. Let us consider the following IBVP over half line $(0,\infty)$ with Neumann boundary condition

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,\infty), \\ u_x(0,t) = 0, & t \in (0,\infty), \end{cases} \tag{0.2}$$

Similar as in class, tackle this problem by first solving its counterpart in $(0,L)$ and then sending $L \to \infty$. Hint: the suggested solution is

$$u(x,t) = \int_0^\infty \Big(G(\xi;x,t) + G(x,t;-\xi)\Big)\phi(\xi)d\xi.$$

*Throughout this homework, and probably the whole course, $G(\xi;x,t)$ is the heat kernel and it is explicitly given by

$$G(\xi;x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{|x-\xi|^2}{4Dt}}.$$

Solution 3. *First of all, we consider the corresponding problem over* $(0, L)$

$$\begin{cases} u_t = Du_{xx}, & x \in (0, L), t > 0, \\ u(x, 0) = \phi(x), & x \in (0, L), \\ u_x(0, t) = 0, & x \in (0, L). \end{cases} \tag{0.3}$$

We already know that the solution to (0.3) in infinite series takes the form

$$u(x, t) = \int_0^L \phi(\xi) G_L(x, t; \xi) d\xi,$$

with

$$G_L(x, t; \xi) := \frac{2}{L} \sum_{n=0}^{\infty} e^{-D(\frac{n\pi}{L})^2 t} \cos \frac{n\pi x}{L} \cos \frac{n\pi \xi}{L},$$

which, in light of the new notation $\Delta w := \frac{\pi}{L}$ *becomes*

$$G_L(x, t; \xi) := \frac{2}{\pi} \sum_{n=0}^{\infty} e^{-D(n\Delta w)^2 t} \cos(n\Delta w x) \cos(n\Delta w \xi) \Delta w.$$

Thanks to the formula $\cos a \cos b = \frac{1}{2}\big(\cos(a + b) + \cos(a - b)\big)$*, we have that*

$$G_L(x, t; \xi) := \frac{1}{\pi} \sum_{n=0}^{\infty} e^{-D(n\Delta w)^2 t} \Big(\cos n\Delta w(x + \xi) + \cos n\Delta w(x - \xi) \Big) \Delta w.$$

Now I skip the rest details and assume that you have no problem showing that it converges to $G(x, t; \xi) + G(x, t; -\xi)$*,* G *the heat kernel. I would like to mention that we will see from a physical interpretation why this problem must have its solution in this form.*

4. Let us consider the following Cauchy problem

$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty, \infty), t \in \mathbb{R}^+, \\ u(x, 0) = \phi(x), & x \in (-\infty, \infty). \end{cases} \tag{0.4}$$

We can approximate the solution to this problem by first solving its counterpart in $(-L, L)$, which has been in a previous homework, and then sending $L \to \infty$.

Consider

$$\begin{cases} u_t = Du_{xx}, & x \in (-L, L), t \in \mathbb{R}^+, \\ u(x, 0) = \phi(x), & x \in (-L, L), \\ u(-L, t) = u(L, t) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{0.5}$$

(i). write the solution to (0.5) in terms of infinite series; you just present your final results, no need to show the details here;

(ii). write the series above into an integral and then evaluate this integral by sending $L \to \infty$. Suggested answer:

$$u(x, t) = \int_{\mathbb{R}} G(\xi; x, t) \phi(\xi) d\xi, \tag{0.6}$$

We shall see several important applications of solution (0.6) in the future.

Solution 4. *(i) we know from a previous HW that the corresponding eigenfunctions are* $\sin \frac{n\pi(x+L)}{2L}$ *and therefore we can write the solutions into the following form*

$$u(x, t) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi(x + L)}{2L}.$$

For further reference, I shall include details that obtain the coefficients of the series. Substituting it into the PDE gives us

$$\sum_{n=1}^{\infty} C_n'(t) \sin \frac{n\pi(x + L)}{2L} = -D \sum_{n=1}^{\infty} C_n(t) \Big(\frac{n\pi}{2L}\Big)^2 \sin \frac{n\pi(x + L)}{2L},$$

which further implies due to the orthogonality of the eigenfunctions that

$$C_n'(t) = -D\Big(\frac{n\pi}{2L}\Big)^2 C_n(t), n = 1, 2, ...$$

Solving this ODE gives rise to

$$C_n(t) = C_n(0)e^{-D(\frac{n\pi}{2L})^2 t}$$

hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n(0)e^{-D(\frac{n\pi}{2L})^2 t} \sin\frac{n\pi(x+L)}{2L}.$$

To decipher the initial condition, we have that

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} C_n(0)\sin\frac{n\pi(x+L)}{2L},$$

which implies

$$C_n(0) = \frac{1}{L}\int_{-L}^{L}\phi(x)\sin\frac{n\pi(x+L)}{2L}dx,$$

where we have applied the fact that

$$\int_{-L}^{L}\sin^2\frac{n\pi(x+L)}{2L}dx = L.$$

Finally, the solution, in terms of an infinite series, is

$$u(x,t) = \int_{-L}^{L}\phi(\xi)G_L(x,t;\xi)d\xi,$$

where

$$G_L(x,t;\xi) := \frac{1}{L}\sum_{n=1}^{\infty} e^{-D(\frac{n\pi}{2L})^2 t}\sin\frac{n\pi(x+L)}{2L}\sin\frac{n\pi(\xi+L)}{2L}$$

or for later application

$$G_L(x,t;\xi) = \frac{1}{L}\sum_{n=1}^{\infty} e^{-D(\frac{n\pi}{2L})^2 t}\sin\Big(\frac{n\pi x}{2L}+\frac{n\pi}{2}\Big)\sin\Big(\frac{n\pi\xi}{2L}+\frac{n\pi}{2}\Big). \tag{0.7}$$

(ii) now we proceed to investigate the limit of $G_L(x,t;\xi)$ as $L\to\infty$. Similar as in class, let us denote $\Delta w := \frac{\pi}{2L}$, then we can rewrite $G_L(x,t;\xi)$ in (0.7) into

$$G_L(x,t;\xi) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-D(n\Delta w)^2 t}\sin\Big(n\Delta w x+\frac{n\pi}{2}\Big)\sin\Big(n\Delta w\xi+\frac{n\pi}{2}\Big)\Delta w,$$

and we shall investigate $G_L(x,t;\xi)$ as $L\to\infty$.
First of all, in light of the compound-angle formula

$$\sin a\sin b = \frac{1}{2}\big(\cos(a-b)-\cos(a+b)\big),$$

we have that

$$\begin{aligned}
G_L(x,t;\xi) =& \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-D(n\Delta w)^2 t}\Big(\cos n\Delta w(x-\xi) - \cos(n\Delta w(x+\xi)+n\pi)\Big)\Delta w \\
=& \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-D(n\Delta w)^2 t}\cos n\Delta w(x-\xi)\Delta w \\
& - \frac{1}{\pi}\sum_{n=1}^{\infty}\Big(e^{-D(n\Delta w)^2 t}\cos(n\Delta w(x+\xi)+n\pi)\Big)\Delta w \\
=& \overbrace{\frac{1}{\pi}\sum_{n=1}^{\infty} e^{-D(n\Delta w)^2 t}\cos n\Delta w(x-\xi)\Delta w}^{I} - \overbrace{\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n e^{-D(n\Delta w)^2 t}\cos n\Delta w(x+\xi)\Delta w}^{J}
\end{aligned} \tag{0.8}$$

Note that for the first identity in (0.8) one does not always have

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty}a_n + \sum_{n=1}^{\infty}b_n,$$

however it is true when both $\{a_n\}$ and $\{b_n\}$ are convergent, as they are here.

According to our lecture, one finds that as $\Delta w \to 0$

$$I \to \frac{1}{\pi}\frac{\sqrt{\pi}}{\sqrt{4Dt}}e^{-\frac{|x-\xi|^2}{4Dt}} = G(x,t;\xi),$$

which is the heat kernel. Now we shall show that $J \to 0$ as $\Delta w \to 0$. Note that J has alternating terms, then similar as above, we can split it into two parts with $n = 2k-1$ and $2k$ as

$$J = \frac{1}{\pi}\sum_{k=1}^{\infty} -e^{-D((2k-1)\Delta w)^2 t}\cos((2k-1)\Delta w(x+\xi))\Delta w + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-D(2k\Delta w)^2 t}\cos 2k\Delta w(x+\xi)\Delta w$$

or

$$J = \frac{1}{\pi}\sum_{k=1}^{\infty}\Big(e^{-D(2k\Delta w)^2 t}\cos 2k\Delta w(x+\xi) - e^{-D((2k-1)\Delta w)^2 t}\cos((2k-1)\Delta w(x+\xi))\Big)\Delta w.$$

Starting from here, I surmise there are various labor-saving methods that you may apply to show that $J \to 0$ as $\Delta w \to 0$ and I would be more than happy to know if you can propose an easier solution (let me know if you do have one). However, I shall simply use brutal force as follows. Note that the function J takes the form $f(2k\Delta w)g(2k\Delta w) - f((2k-1)\Delta w)g((2k-1)\Delta w)$, the natural way is to rewrite it into $f(2k\Delta w)g(2k\Delta w) - f((2k-1)\Delta w)g(2k\Delta w) + f((2k-1)\Delta w)g(2k\Delta w) - f((2k-1)\Delta w)g((2k-1)\Delta w)$ for easier cooking. In this spirit, one has that

$$\begin{aligned}
J =& \overbrace{\frac{1}{\pi}\sum_{k=1}^{\infty} e^{-D(2k\Delta w)^2 t}\Big(\cos 2k\Delta w(x+\xi) - \cos((2k-1)\Delta w(x+\xi))\Big)\Delta w}^{J_1} \\
&+ \overbrace{\frac{1}{\pi}\sum_{k=1}^{\infty}\Big(e^{-D(2k\Delta w)^2 t} - e^{-D((2k-1)\Delta w)^2 t}\Big)\Big(\cos((2k-1)\Delta w(x+\xi))\Big)\Delta w}^{J_2}
\end{aligned} \tag{0.9}$$

To cook up J_1, we use the compound–angle formula $\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$ to have that

$$J_1 = \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-D(2k\Delta w)^2 t}(-2)\sin\Delta w(x+\xi)\sin\big((2k-1)\Delta w(x+\xi)\big)\Delta w;$$

note that for x and ξ fixed, $\sin \Delta w(x+\xi) = O(\Delta w)$ *as* $\Delta w \to 0$*, therefore we have from the fact* $|\sin\big((2k-1)\Delta w(x+\xi)\big)| \le 1$ *that*

$$|J_1| \le O(\Delta w) \overbrace{\sum_{k=1}^{\infty} e^{-D(2k\Delta w)^2 t}\Delta w}^{J_{11}},$$

with $\frac{2}{\pi}$ *embedded into* $O(\Delta w)$*. It is easy to see that*

$$J_{11} \to \int_{-\infty}^{\infty} e^{-D(2w)^2 t} dw < \infty,$$

therefore $J_1 \to 0$ *as* $\Delta w \to 0$*.*

To estimate J_2*, we first have from Mean Value Theorem that, for some* $k^* \in [k-\frac{1}{2}, k]$ *one has*

$$e^{-D(2k\Delta w)^2 t} - e^{-D((2k-1)\Delta w)^2 t} = e^{-D(2k^*\Delta w)^2 t}(Dt)(\Delta w)((4k-1)\Delta w).$$

Why? It holds because $e^{-a^2} - e^{-b^2} = e^{-c^2}(b^2 - a^2)$ *for some* $c \in [a,b]$*. Therefore one has that*

$$|J_2| \le O(\Delta w)\frac{1}{\pi}\sum_{k=1}^{\infty} e^{-D(2k^*\Delta w)^2 t}((4k-1)\Delta w)\Delta w,$$

which again converges to zero, since this infinite series is finite (either you can find it to be approximate of an integral or show directly that it is bounded). This verifies that $J \to 0$ *as* $\Delta w \to 0$*. Finally, we finish the proof and collect (0.6) as expected.*

5. The heat kernel $G(\xi; x, t)$ is sometimes called fundamental solution of heat equation

$$G(\xi; x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-\xi)^2}{4Dt}}.$$

Prove that

(i) $|\frac{\partial G}{\partial x}| \to 0$ as $|x| \to \infty$ for each t and ξ. Prove the same for $\frac{\partial^m G}{\partial x^m}$ for each $m \in \mathbb{N}^+$;

(ii) $G_t = DG_{xx}$, $x \in \mathbb{R}, t \in \mathbb{R}^+$;

(iii) $\int_{\mathbb{R}} G(\xi; x, t) dx = 1$.

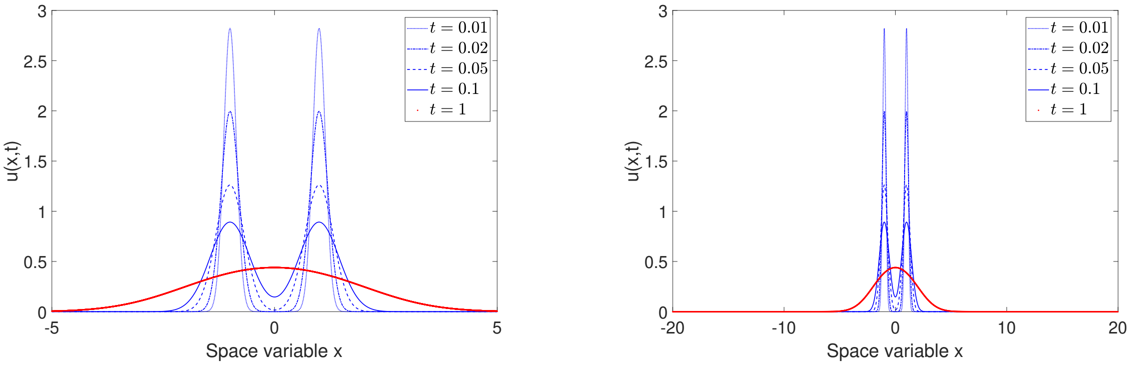
Remark: I would like to note that we write the kernel $G(\xi; x, t)$ and $G(\xi; x, t)$ interchangeably. The former is to highlight the eventual solution as a function of x and t, whereas the latter is to focus on treating ξ as the integration variable whenever applicable.

Solution 5. *All the conclusions above can be verified through straightforward calculations and I skip typing them here. I would like to point out that in (i), one should treat both t and x as fixed parameters.*

6. To give yourself some physical intuitions on the heat kernel, let us consider the following situation in $\mathbb{R}$: put two separate units of thermal heat at locations $\xi = -1$ and $\xi = 1$ respectively at time $t = 0$. Suppose that the temperature $u(x,t)$ satisfies the heat equation with diffusion rate $D = 1$, then it is given by the following explicit form

$$u(x,t) = G(x,t;-1) + G(x,t;1) = \frac{1}{\sqrt{4\pi t}}\Big(e^{-\frac{(x+1)^2}{4t}} + e^{-\frac{(x-1)^2}{4t}}\Big).$$

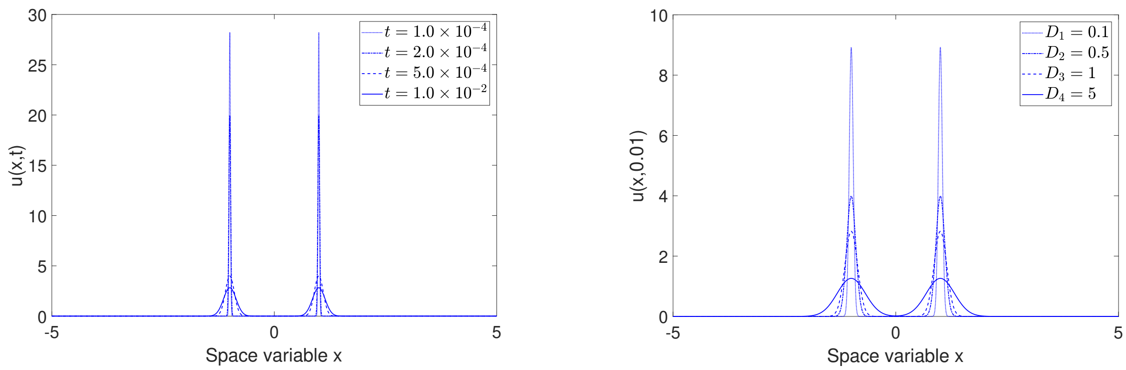
Plot $u(x,t)$ over $x \in (-5,5)$ with $t = 0.01, 0.02, 0.05$, 0.1 and 1 *on the same coordinate* in $(-R,R)$ (if R is large, then it approximates the whole line) to illustrate your results–please use different colors and/or line styles for better effects. We will know more about physical intuition in the future; indeed you should already have an intuition about: i) the evolution of thermal energy; ii) the connection between diffusion and Brownian motion or normal distribution.)



The evolution of the compounded heat kernel

The evolution zoomed out in $(-20, 20)$

Figure 2: Evolution of the compounded heat kernel.



Formation of δ as $t \to 0^+$

Smoothing effect of diffusion rate

Figure 3: Formation of δ and effect of diffusion rate.

Solution 6. *There are some additional plots I added in Figure 2 and Figure 3 to examine some interesting and also important phenomena such as the convergence of the heat kernel to a δ-function as $t \to 0^+$, the smoothing effect of diffusion rate D, etc. We will provide rigorous proof later in this class.*

7. Consider the following problem

$$\begin{cases} u_t = Du_{xx} - \alpha u_x - ru, & x \in (-\infty, 0), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x) \geq, \neq 0, & x \in (\infty, 0), \\ u(-\infty, t) = e^{-rt}K > 0, u(0,t) = 0, & t \in \mathbb{R}^+, \end{cases} \tag{0.10}$$

where D, α, r and K are positive constants.

Let's visit it truncated problem

$$\begin{cases} u_t = Du_{xx} - \alpha u_x - ru, & x \in (-L, 0), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x) \geq, \neq 0, & x \in (\infty, 0), \\ u(-L, t) = e^{-rt}K > 0, u(0,t) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{0.11}$$

(i) Solve (0.11) in terms of infinite series. Hint: its boundary condition is inhomogeneous;

(ii) Send L to infinity and then find the limiting solution in terms of an integral.

Solution 7. *(i) We first recognize that the boundary condition is not homogeneous. Therefore, we introduce an auxiliary function w $v(x,t) := u(x,t) + w(x,t)$ such that*

$$\begin{aligned} v(-L,t) =& u(-L,t) + w(-L,t) = e^{-rt}K + w(-L,t) = 0, \\ v(0,t) =& u(0,t) + w(0,t) = 0. \end{aligned}$$

One simple and natural choice is $w(x,t) = e^{-rt}\frac{Kx}{L}$, *and from this we have that*

$$u(x,t) = v(x,t) - e^{-rt}\frac{Kx}{L},$$

which then implies

$$\begin{cases} v_t = Dv_{xx} - \alpha v_x - rv + \overbrace{\alpha e^{-rt}\frac{K}{L}}^{:=\Phi(t)}, & x \in (-L,0), t \in \mathbb{R}^+, \\ v(x,0) = \overbrace{\phi(x) + \frac{Kx}{L}}^{:=\varphi(x)}, & x \in (-L,0), \\ v(-L,t) = 0, v(0,t) = 0, & t \in \mathbb{R}^+. \end{cases}$$

Let us write the eigen-expansions

$$v(x,t) = \sum_{k=1}^{\infty} C_k(t) \sin\frac{k\pi x}{L}.$$

and substitute it into the v*-equation to collect that*

$$\begin{aligned} \sum_{k=1}^{\infty} C_k'(t)\sin\frac{k\pi x}{L} = & -D\sum_{k=1}^{\infty}(\frac{k\pi}{L})^2 C_k(t)\sin\frac{k\pi x}{L} - \alpha\sum_{k=1}^{\infty}\frac{k\pi}{L}C_k(t)\cos\frac{k\pi x}{L} \\ & - r\sum_{k=1}^{\infty} C_k(t)\sin\frac{k\pi x}{L} + \Phi(t). \end{aligned}$$

In what follows, we can multiply this identity by $\sin\frac{k\pi x}{L}$ *and then integrate over* $(-L,0)$ *to collect the ODEs for coefficients* $C_k(t)$ *such as*

$$C_k'(t) = -\Big[D(\frac{k\pi}{L})^2 + r\Big]C_k(t) + \frac{2}{L}\Phi(t)\int_{-L}^{0}\sin\frac{k\pi x}{L}dx. \tag{0.12}$$

Though all seem routine, we raise a flag here that cos *terms do not disappear since it is not orthogonal to the* sin *in* $L^2((-L,0))$ *in* (0.12) *and this is a very common mistake therein. Indeed, one has that*

$$\int_0^L \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} \Lambda_{mn} := \frac{nL\left(1-(-1)^{m+n}\right)}{\pi(m^2-n^2)}, & m \neq n, \\ 0, & m = n, \end{cases}$$

Note that they are indeed orthogonal in $L^2((-L,L))$, *a whole period, but not the half period* $(-L,0)$ *or* $(0,L)$. *Then we find that*

$$C_k'(t)\frac{L}{2} = -\Big[D(\frac{k\pi}{L})^2 + r\Big]C_k(t)\frac{L}{2} - \alpha\sum_{m\in\mathbb{N}^+, m\neq k}^{\infty}\frac{m\pi}{L}\Lambda_{mk}C_m(t) + \Phi(t)\int_{-L}^{0}\sin\frac{k\pi x}{L}dx.$$

These ODEs are strongly coupled (e.g., you can solve each of them separately but all together). That being said, we are required to solve

$$\begin{pmatrix} C_1'(t) \\ C_2'(t) \\ \vdots \\ C_k'(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \\ \vdots \\ C_k(t) \\ \vdots \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_k(t) \\ \vdots \end{pmatrix}, \tag{0.13}$$

where you are asked fill the coefficient matrix $A = (a_{i,j})$ *and source vector* f_i *yourself. Note that the counterpart for* (0.13) *is a pure diagonal matrix. Yes,* (0.13) *is still a linear system and one* *can solve it by transforming the ODE*[†] *into diagonally separated problems. However, I do not see how you can do this easily. One way is to apply MATLAB ODE solver and I stop the discussion here.*

We recognize that this issue arises due to the presence of αv_x, and this is called the advection. Instead of moving randomly, the advection is directed and makes the problem realistic and mathematically tricky. Therefore, we are motivated to get rid of the advection term first before moving forward. There are at least two ways we can deal with this issue. I shall only sketch them here, and leave the calculations to advanced and motivated students.

The first one is to introduce yet another new function such that the advection disappear. To see this, let us denote $h(x,t) = e^{\theta x} v(x,t)$ for some constant θ, and then you shall see that the advection term disappears in the h-equation for properly chosen θ (hint: $\theta = -\frac{\alpha}{2D}$). Then you can play with the rest as before without running into (0.13).

The second way is to consider the operator $\mathcal{L} := \frac{d^2}{dx^2} - \alpha \frac{d}{dx}$, and study the eigenvalue problem

$$\begin{cases} \mathcal{L}X(x) = \lambda X(x), & x \in (-L, 0), \\ X(-L) = X(0) = 0. \end{cases}$$

You do it exactly the same as in the case when $\alpha = 0$, i.e., discuss the sign of λ and then solve the ODEs. The solutions are a family of eigenpairs $\{X_k(x), \lambda_k\}$, and one can then write $u(x,t) = \sum C_k(t) X_k(x)$[‡] *and do the rest calculations.*

[†] well, this is where your ODE knowledge plays

[‡] C_k is a generic notation and is not the same as above