# Introduction to PDEs, Fall 2022

## **Homework 3** Solutions

**Name**:______________

1. Perform straightforward calculations to verify that

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = \frac{L}{2}\delta_{mn} = \begin{cases} \frac{L}{2}, \text{ if } m = n, \\ 0, \text{ if } m \neq n; \end{cases}$$

here $\delta$ is the so–called Kronecker delta function.

**Solution 1.** *(i). For the sine functions: if $m \neq n$, we can compute that*

$$\begin{aligned}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \int_0^L -\frac{\cos\frac{(m+n)\pi}{L} - \cos\frac{(m-n)\pi}{L}}{2}dx \\ &= (-\frac{1}{2})\Big[\frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\Big|_0^L - \frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\Big|_0^L\Big] = 0;\end{aligned}$$

*if $m = n$, we have that*

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \int_0^L \frac{1-\cos\frac{2m\pi x}{L}}{2}dx = \frac{L}{2} - \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\Big|_0^L = \frac{L}{2}.$$

*(ii). For the cosine function: if $m \neq n$, we can compute that*

$$\begin{aligned}\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx &= \int_0^L \frac{\cos\frac{(m+n)\pi}{L} + \cos\frac{(m-n)\pi}{L}}{2}dx \\ &= \frac{1}{2}\Big[\frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\Big|_0^L + \frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\Big|_0^L\Big] = 0;\end{aligned}$$

*if $m = n$, we have that*

$$\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \int_0^L \frac{1+\cos\frac{2m\pi x}{L}}{2}dx = \frac{L}{2} + \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\Big|_0^L = \frac{L}{2}.$$

*Therefore, we can conclude that*

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = \frac{L}{2}\delta_{mn} = \begin{cases} \frac{L}{2}, \textit{ if } m = n \\ 0, \textit{ if } m \neq n \end{cases}$$

*I would like to remark we will come across another important delta function, called Dirac-delta function later in this course. In general, when PDE people say delta function they mean the Dirac-delta, not the Kronecker-delta.*

2. We have shown that only a pair of the form $(X_n, \lambda_n) = \Big(\sin\frac{n\pi x}{L}, \big(\frac{n\pi}{L}\big)^2\Big)$, $n \in \mathbb{N}$, can satisfy the associated problem

$$\begin{cases} X'' + \lambda X = 0, x \in (0, L), \\ X(0) = X(L) = 0. \end{cases} \tag{0.1}$$

First of all, it is easy to see that $CX_n$ is also a solution of (0.1) for any $C \in \mathbb{R}$, however we conventionally choose $C = 1$ and write $X_n = \sin\frac{n\pi x}{L}$; or occasionally we choose its normalized version $X_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ since $\|X_n\|_{L^2(0,L)} = 1$. That being said, it is the shape of sin that matters, but not the magnitude, at least for (0.1).

Second of all, (0.1) is called an eigen-value problem, $(X_n, \lambda_n)$ an eigen-pair, an analogy to eigen-vectors and eigen-values in linear algebra. Let us recall the followings in linear algebra: consider a $n \times n$ matrix $A$, we call $(\mathbf{x}, \lambda)$ its eigen-pair, $\mathbf{x}$ (nonzero) the eigen-vector and $\lambda$ the eigen-value, if $A\mathbf{x} = \lambda \mathbf{x}$ holds ($\lambda = 0$ is allowed). I assume that you are aware that in linear algebra if $\mathbf{x}$ is an eigen-vector, so does $C\mathbf{x}$—I wish this also gives you another motivation why $C = 1$ is selected above. Now, for (0.1), one can formally treat $-\frac{d^2}{dx^2}$ as $A$, and then it writes $AX = \lambda X$, however the eigen-space $\{X_n\}$ (the space consists of all such eigen-functions) is infinite–dimensional, since $X_n$ for each $n \in \mathbb{N}$ is an element. This is a strong contrast to the linear algebra, when a $n \times n$ matrix has an eigen-space of at most $n$–dimension.

Moreover, it is well–known that if a $n \times n$ matrix $A$ is invertible, its eigen-vectors form a basis of $\mathbb{R}^n$ (go to review this if you are not aware). Then we shall see in the coming lectures that, similarly the eigen-functions of $-\frac{d}{dx^2}$ (or just solutions to the eigen-value problem (0.1)) $\{X_n\}_{n\in\mathbb{N}}$ form a basis of $L^2(0, L)$ with DBC, i.e., the square–integrable functions with Dirichlet boundary conditions. This is known as the Sturm–Liouville theory, one of the corner–stones in the studies of differential equations–more will be talked about later in class. Generally speaking, the studies of many PDE problem comes to the investigations of eigen-value problems, of course some of way more complicated that (0.1). However, we can study the cousins of (0.1):

Find eigen-paris $\{(X_k, \lambda_k)\}$ to the following eigen-value problems

$$\begin{cases} X'' + \lambda X = 0, x \in (0, L), \\ X'(0) = X'(L) = 0; \end{cases} \tag{0.2}$$

$$\begin{cases} X'' + \lambda X = 0, x \in (0, L), \\ X(0) = X'(L) = 0; \end{cases} \tag{0.3}$$

and

$$\begin{cases} X'' + \lambda X = 0, x \in (0, L), \\ X'(0) = X(L) = 0; \end{cases} \tag{0.4}$$

**Solution 2.** *I shall only work on (0.2) in detail while one can do the rest similarly. To find the eigen-pairs for (0.2), we divide our discussions of the (sign of) parameter $\lambda$ into the following three cases:*

*Case 1: $\lambda = -\mu^2 < 0, \mu \in \mathbb{R}$. Then we know that the solution of the ODE takes the form*

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x},\ c_1, c_2 \in \mathbb{R}$$

*In light of the boundary condition we have*

$$\begin{cases} X'(0) &= \mu c_1 - \mu c_2 = 0 \\ X'(L) &= \mu c_1 e^{\mu L} - \mu c_2 e^{-\mu L} = 0, \end{cases}$$

*which imply that $c_1 = c_2 = 0$ hence $X \equiv 0$. This is impossible since we look for nonzero eigen-functions and this case is ruled out.*

*Case 2: $\lambda = 0$. Then we can easily find that $X(x) = c_1 x + c_2$ and then $c_1 = 0$, $c_2 \in \mathbb{R}$ thanks to the BC.*

*Case 3: $\lambda = \mu^2 > 0, \mu \in \mathbb{R}$. In this case, the general solution takes the form*

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x, c_1, c_2 \in \mathbb{R}.$$

*Therefore, we can compute that*

$$\begin{cases} X'(0) &= \mu c_2 = 0 \\ X'(L) &= -\mu c_1 \sin \mu L + \mu c_2 \cos \mu L = 0, \end{cases}$$

*which imply that* $\sin \mu L = 0$. *Therefore we must have* $\mu = \frac{k\pi}{L}$, *for* $k = 1, 2, \cdots$ *and the eigen-function corresponding to* $\mu$ *is* $\cos \frac{k\pi x}{L}$.

*In light of both case 2 and case 3, we see that the eigen-pairs of (0.2) are*

$$\{(X_k, \lambda_k)\} = \Big\{\Big(\cos \frac{k\pi x}{L}, (\frac{k\pi}{L})^2\Big)\Big\}_{k=0}^{\infty}$$

***Remark:*** *It is necessary to mention that* $k$ *should start from* $0$ *here, which corresponds to a constant eigen-fucntion. This is a contrast from the case with DBC.*

*eigen-pairs of (0.3) are*

$$\{(X_k, \lambda_k)\} = \Big\{\Big(\sin \frac{(2k+1)\pi x}{2L}, (\frac{2k\pi + \pi}{2L})^2\Big)\Big\}_{k=0}^{\infty}$$

*eigen-pairs of (0.4) are*

$$\{(X_k, \lambda_k)\} = \Big\{\Big(\cos \frac{(2k+1)\pi x}{2L}, (\frac{2k\pi + \pi}{2L})^2\Big)\Big\}_{k=0}^{\infty}$$

3. Let us come back to the $L^2(0, L)$, the space of square integrable functions. In general, for $p \in (1, \infty)$, the $L^p$ space is defined to be

$$L^p(\Omega) := \Big\{f(x)\Big| \int_\Omega |f(x)|^p dx\Big\} < \infty;$$

some other conditions may be added/imposed such as $\Omega$, $f$ measurable, while I skip them in order not to bother you too much this time. Moreover, one is able to define *the length*, the so–called ***norm***, of any function $f \in L^p$ as

$$\|f\|_{L^p(\Omega)} := \Big(\int_\Omega |f|^p dx\Big)^{\frac{1}{p}}.$$

Find: (a) $\|f(x)\|_{L^2_{(0,1)}}$ for $f(x) = e^x$ and (b). $\|f(x)\|_{L^2_{(0,2)}}$ for $f(x) = x - 1$.

**Solution 3.** *(a). By the definition of* $L^p$*-norm, we find that*

$$\begin{aligned}\|f(x)\|_{L^2_{(0,1)}} &= \Big(\int_0^1 |f(x)|^2 dx\Big)^{\frac{1}{2}} = \Big(\int_0^1 (e^x)^2 dx\Big)^{\frac{1}{2}} \\ &= \sqrt{\frac{e^{2x}}{2}\Big|_0^1} = \sqrt{\frac{e^2 - 1}{2}};\end{aligned}$$

*(b). Similarly, we can find that*

$$\|f(x)\|_{L^2_{(0,2)}} = \Big(\int_0^2 |f(x)|^2 dx\Big)^{\frac{1}{2}} = \sqrt{\int_0^2 (x^2 - 2x + 1)dx} = \frac{\sqrt{6}}{3}.$$

4. We showed that $f(x) = \frac{1}{\sqrt{x}} \in L^1(0, 1)$ but not $L^2(0, 1)$. What would be your general theory/conditions about a function of the form $f(x) = x^\alpha \in L^p(0, 1)$, but not $L^q(0, 1)$. Assuming that $p, q \in (1, \infty)$ for simplicity.

**Solution 4.** *(i) It is easy to find that the anti–derivative of* $f(x)$ *is* $\int f(x)dx = 2\sqrt{x}$, *while that of* $f^2(x)$ *is* $\int f^2(x)dx = \ln x$, *whence it is obvious* $f \in L^1$, *but* $\notin L^2(0, 1)$;

*(ii) note that* $x^\alpha > 0$ *for all* $x > 0$. *In general, the antiderivative reads*

$$\int |f(x)|^p dx = \int x^{p\alpha} = \begin{cases} \ln x, & p\alpha = -1, \\ \frac{1}{p\alpha+1} x^{p\alpha+1} dx, & p\alpha \neq -1, \end{cases}$$

*then we readily see that* $x^{p\alpha}$ *is not integrable over* $(0, 1)$ *if* $p\alpha \leq -1$, *or equivalent* $x^\alpha \in L^p(0, 1)$ *if and only if* $p\alpha > -1$. *This implies that, for* $1 \leq p < q < \infty$, $x^\alpha \in L^p(0, 1)$, $\notin L^q(0, 1)$, *if* $-\frac{1}{p} < \alpha \leq -\frac{1}{q}$.

5. The orthogonality of functions is generalized from that of vectors with inner products of the latter being replaced by the inner product. One can also generalize the idea as follows: suppose that $w(x)$ is a nonnegative function on $[a, b]$. Let $f(x)$ and $g(x)$ be real-valued functions and their inner product on $[a, b]$ with respect to the weight $w$ is given by

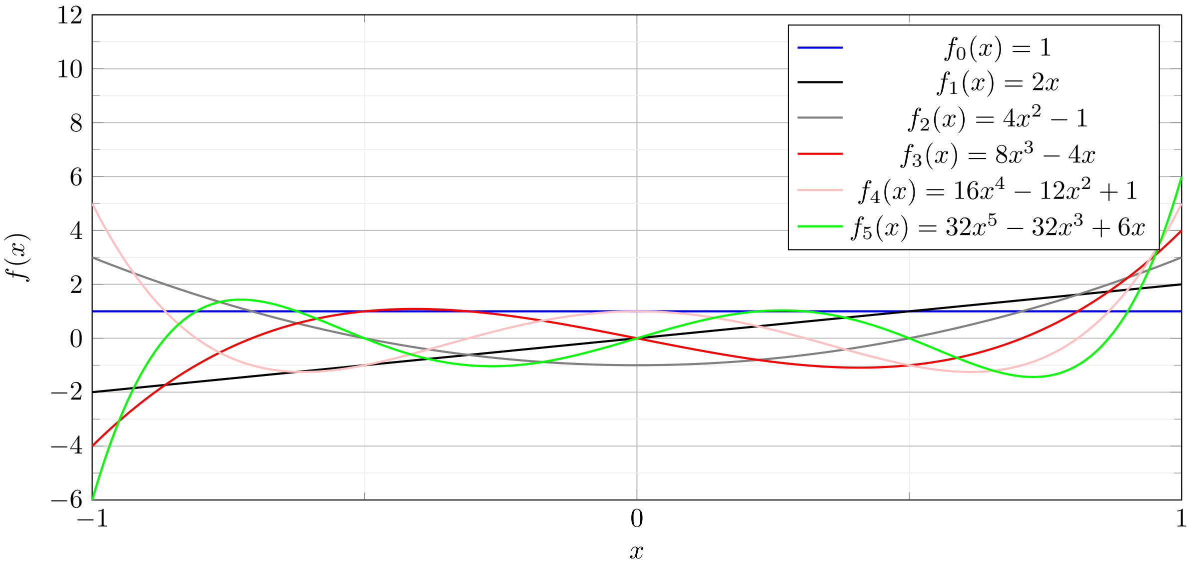
$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx.$$

Then we say $f$ and $g$ are orthogonal on $[a, b]$ with respect to the weight $w$ if $\langle f, g \rangle = 0$. Show that The functions

$$f_0(x) = 1, \quad f_1(x) = 2x, \quad f_2(x) = 4x^2 - 1, \quad f_3(x) = 8x^3 - 4x$$

are pairwise orthogonal on $[-1, 1]$ relative to the weight function $w(x) = \sqrt{1 - x^2}$. They are examples of **Chebyshev polynomials of the second kind**. Indeed, one can find that $f_4(x) = 16x^4 - 12x^2 + 1, \quad f_5(x) = 32x^5 - 32x^3 + 6x$ (you can but are not required to verify this). Plot all the functions $f_i(x)$, $i = 0, 1, , \ldots, 5$ on the same coordinate. Do you observe orthogonality? Justify or explain your observations.

*Proof.* The orthogonality can be verified by straightforward calculations, and I skip typing them here. One should be able to observe that a higher eigen-value has more zero roots, and its monotonicity changes more times.

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