二次型 $f(x_1,x_2,\cdots,x_n)=\sum_{i=1}^{n}(x_i-\overline{x})^2=\sum_{i=1}^{n}x_i^2-n\overline{x}^2$，其中 $\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$. 用正交变换法化二次型 f 为标准形，并求出所用正交变换.

解：二次型 $f(x_1,x_2,\cdots,x_n)=\sum_{i=1}^{n}x_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2=\left(1-\frac{1}{n}\right)\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i<j}x_ix_j$ ，系数矩阵

$$A=\begin{pmatrix}1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n}\\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n}\end{pmatrix},$$

令行列式 $|\lambda E-A|=0$，可得

$$\begin{vmatrix}\lambda-1+\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ \frac{1}{n} & \lambda-1+\frac{1}{n} & \cdots & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{n} & \frac{1}{n} & \cdots & \lambda-1+\frac{1}{n}\end{vmatrix}=\lambda(\lambda-1)^{n-1}=0,$$

可得 A 的特征值为

$$\lambda_1=0,\quad \lambda_2=\lambda_3=\cdots=\lambda_n=1.$$

求解 $(0\cdot E-A)X=0$，得对应于 $\lambda_1=0$ 的特征向量 $X_1=(1,1,\cdots,1)^T$.

求解 $(1\cdot E-A)X=0$，得对应于 $\lambda_2=\lambda_3=\cdots=\lambda_n=1$ 的特征向量

$$X_2=\begin{pmatrix}1\\-1\\0\\\vdots\\0\end{pmatrix},\quad X_3=\begin{pmatrix}1\\0\\-1\\\vdots\\0\end{pmatrix},\quad \cdots,\quad X_n=\begin{pmatrix}1\\0\\0\\\vdots\\-1\end{pmatrix}.$$

正交化，得

$$\alpha_1=\begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix},\quad \alpha_2=\begin{pmatrix}1\\-1\\0\\\vdots\\0\end{pmatrix},\quad \alpha_3=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\\\vdots\\0\end{pmatrix},\quad \cdots,\quad \alpha_n=\begin{pmatrix}\frac{1}{n-1}\\\frac{1}{n-1}\\\frac{1}{n-1}\\\vdots\\-1\end{pmatrix},$$

再单位化，得

$$\alpha_1^* = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix}, \quad \alpha_2^* = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \alpha_3^* = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \alpha_n^* = \begin{pmatrix} \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n(n-1)}} \\ \vdots \\ -\frac{n-1}{\sqrt{n(n-1)}} \end{pmatrix},$$

得到正交矩阵

$$C = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & -\frac{2}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}} & 0 & 0 & \cdots & -\frac{n-1}{\sqrt{n(n-1)}} \end{pmatrix}.$$

作正交变换

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & -\frac{2}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}} & 0 & 0 & \cdots & -\frac{n-1}{\sqrt{n(n-1)}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix},$$

可将二次型 f 为标准形

$$f = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2 = y_2^2 + y_3^2 + \cdots + y_n^2 .$$