补充题

1. 两个正态总体 $X \sim N(\mu_1, \sigma_1^2)$，$Y \sim N(\mu_2, \sigma_2^2)$ 且相互独立。$X_1, X_2, \cdots, X_{n_1}$ 为来自总体 X 的样本，$Y_1, Y_2, \cdots, Y_{n_2}$ 为来自总体 Y 的样本，未知 σ_1^2, σ_2^2，但当方差比 $\dfrac{\sigma_2^2}{\sigma_1^2} = c$ 为已知常数时，证明

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{(n_1-1)S_X^2 + (n_2-1)S_Y^2/c}{n_1+n_2-2}}\sqrt{\dfrac{1}{n_1} + \dfrac{c}{n_2}}} \sim t(n_1+n_2-2)。$$

证明：因

$$U = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim N(0,1)。$$

当方差比 $\dfrac{\sigma_2^2}{\sigma_1^2} = c$ 为已知常数时，记 $\sigma_2^2 = c\sigma_1^2 = c\sigma^2$，有

$$U = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{c}{n_2}}} \sim N(0,1)，$$

再将分母中的总体标准差 σ 替换为（综合）样本标准差 S_w，抽样分布就由标准正态分布变成 t 分布。

因 $X \sim N(\mu_1, \sigma^2)$，$Y \sim N(\mu_2, c\sigma^2)$，且相互独立，有

$$\frac{(n_1-1)S_X^2}{\sigma^2} \sim \chi^2(n_1-1)，\quad \frac{(n_2-1)S_Y^2}{c\sigma^2} \sim \chi^2(n_2-1)，$$

且相互独立，则

$$\chi^2 = \frac{(n_1-1)S_X^2}{\sigma^2} + \frac{(n_2-1)S_Y^2}{c\sigma^2} = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2/c}{\sigma^2} \sim \chi^2(n_1+n_2-2)。$$

且 $\bar{X}$、$\bar{Y}$、S_X^2、S_Y^2 相互独立，即 U 与 χ^2 相互独立，则由 t 分布的定义可知

$$T = \frac{U}{\sqrt{\chi^2/(n_1+n_2-2)}} = \frac{\dfrac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{c}{n_2}}}}{\sqrt{\dfrac{(n_1-1)S_X^2 + (n_2-1)S_Y^2/c}{\sigma^2}\Big/(n_1+n_2-2)}}$$

$$= \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{(n_1-1)S_X^2 + (n_2-1)S_Y^2/c}{n_1+n_2-2}} \cdot \sqrt{\dfrac{1}{n_1} + \dfrac{c}{n_2}}} \sim t(n_1+n_2-2)。$$

2. 设总体 $X \sim N(0, 1.2)$，$Y \sim N(1, 1)$，且相互独立。分别抽取容量为 8 与 10 的样本 $X_1, X_2, \cdots, X_8$ 与 $Y_1, Y_2, \cdots, Y_{10}$，二者样本均值与样本方差分别为 $\bar{X}$、S_X^2 与 $\bar{Y}$、S_Y^2。求概率 $P\{\bar{Y} > 0\}$，$P\{S_X^2 > 2.4\}$，

$P\{\bar{X} < \bar{Y}\}$， $P\{S_X^2 < 5.04 S_Y^2\}$。

解：因

$$U = \frac{\bar{Y} - \mu_2}{\sigma_2 \big/ \sqrt{n_2}} = \frac{\bar{Y} - 1}{1 \big/ \sqrt{10}} = \sqrt{10}(\bar{Y} - 1) \sim N(0,1),$$

则

$$P\{\bar{Y} > 0\} = P\{U = \sqrt{10}(\bar{Y} - 1) > -\sqrt{10}\} = 1 - \Phi(-\sqrt{10}) = \Phi(\sqrt{10}) = 0.9992 \text{。}$$

因

$$\chi^2 = \frac{(n_1 - 1)S_X^2}{\sigma_1^2} = \frac{7S_X^2}{1.2} \sim \chi^2(7),$$

则

$$P\{S_X^2 > 2.4\} = P\left\{\chi^2 = \frac{7S_X^2}{1.2} > 14\right\} = 1 - 0.9488 = 0.0512 \text{。}$$

因

$$U = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{X} - \bar{Y} + 1}{\sqrt{\frac{1.2}{8} + \frac{1}{10}}} = 2(\bar{X} - \bar{Y} + 1) \sim N(0,1),$$

则

$$P\{\bar{X} < \bar{Y}\} = P\{U = 2(\bar{X} - \bar{Y} + 1) < 2\} = \Phi(2) = 0.9772 \text{。}$$

因

$$F = \frac{S_X^2 \big/ \sigma_1^2}{S_Y^2 \big/ \sigma_2^2} = \frac{S_X^2}{1.2 S_Y^2} \sim F(7,9),$$

则

$$P\{S_X^2 < 5.04 S_Y^2\} = P\left\{F = \frac{S_X^2}{1.2 S_Y^2} < 4.2\right\} = 0.975 \text{。}$$