补充资料：

拟合优度检验的 χ^2 统计量推导过程

当检验数据分成 r 类，需要检验在每一类数据中取值的概率是否为给定的 $p_i, i=1,2,\cdots,r$. 对容量为 n 的数据样本，设其中属于第 i 类的样品个数为 $n_i, i=1,2,\cdots,r$ ，（$n_1+n_2+\cdots+n_r=n$），证明，当 n 很大时

$$\chi^2=\sum_{i=1}^{r}\frac{(n_i-np_i)^2}{np_i}\dot{\sim}\chi^2(r-1)\text{。}$$

证明：因 $n_i\sim b(n,p_i)$ ，当 n 很大时，n_i 近似服从正态分布 $n_i\dot{\sim}N(np_i,np_i(1-p_i))$。又因 $i\neq j$ 时，

$$n_i+n_j\sim b(n,p_i+p_j)\text{，}$$

则

$$\mathrm{Var}(n_i+n_j)=n(p_i+p_j)(1-p_i-p_j)\text{，}$$

可得

$$\mathrm{Cov}(n_i,n_j)=\frac{\mathrm{Var}(n_i+n_j)-\mathrm{Var}(n_i)-\mathrm{Var}(n_j)}{2}\text{，}$$
$$=\frac{n(p_i+p_j)(1-p_i-p_j)-np_i(1-p_i)-np_j(1-p_j)}{2}=-np_ip_j\text{，}$$

设 $X_i=\dfrac{n_i-np_i}{\sqrt{np_i}}$，有 X_i 近似服从正态分布，且

$$E(X_i)=\frac{1}{\sqrt{np_i}}E(n_i-np_i)=0\text{，}\quad \mathrm{Var}(X_i)=\frac{1}{np_i}\mathrm{Var}(n_i)=\frac{1}{np_i}np_i(1-p_i)=1-p_i\text{，}$$

$$\mathrm{Cov}(X_i,X_j)=\frac{1}{\sqrt{np_i}\sqrt{np_j}}\mathrm{Cov}(n_i,n_j)=\frac{1}{\sqrt{np_i}\sqrt{np_j}}\cdot(-np_ip_j)=-\sqrt{p_ip_j}\text{，}$$

记 $\overrightarrow{X}=(X_1,X_2,\cdots,X_r)^T$，有 $\overrightarrow{X}$ 的均值向量和协方差矩阵分别为

$$E\overrightarrow{X}=0\text{，}$$

$$\mathrm{Cov}(\overrightarrow{X},\overrightarrow{X})=\begin{pmatrix}1-p_1 & -\sqrt{p_1p_2} & \cdots & -\sqrt{p_1p_r}\\ -\sqrt{p_2p_1} & 1-p_2 & \cdots & -\sqrt{p_2p_r}\\ \vdots & \vdots & \ddots & \vdots\\ -\sqrt{p_rp_1} & -\sqrt{p_rp_2} & \cdots & 1-p_r\end{pmatrix}\text{，}$$

因

$$D_r=|\lambda E_r-\mathrm{Cov}(\overrightarrow{X},\overrightarrow{X})|=\begin{vmatrix}\lambda-1+p_1 & \sqrt{p_1p_2} & \cdots & \sqrt{p_1p_r}\\ \sqrt{p_2p_1} & \lambda-1+p_2 & \cdots & \sqrt{p_2p_r}\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{p_rp_1} & \sqrt{p_rp_2} & \cdots & \lambda-1+p_r\end{vmatrix}$$

$$=\begin{vmatrix}\lambda-1+p_1 & \sqrt{p_1p_2} & \cdots & \sqrt{p_1p_r}\\ \sqrt{p_2p_1} & \lambda-1+p_2 & \cdots & \sqrt{p_2p_r}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda-1\end{vmatrix}+\begin{vmatrix}\lambda-1+p_1 & \sqrt{p_1p_2} & \cdots & \sqrt{p_1p_r}\\ \sqrt{p_2p_1} & \lambda-1+p_2 & \cdots & \sqrt{p_2p_r}\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{p_rp_1} & \sqrt{p_rp_2} & \cdots & p_r\end{vmatrix}$$

$$=(\lambda-1)D_{r-1}+\sqrt{p_r}\begin{vmatrix}\lambda-1+p_1 & \sqrt{p_1p_2} & \cdots & \sqrt{p_1p_r}\\ \sqrt{p_2p_1} & \lambda-1+p_2 & \cdots & \sqrt{p_2p_r}\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{p_1} & \sqrt{p_2} & \cdots & \sqrt{p_r}\end{vmatrix}$$

$$=(\lambda-1)D_{r-1}+\sqrt{p_r}\begin{vmatrix}\lambda-1 & 0 & \cdots & 0\\ 0 & \lambda-1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{p_1} & \sqrt{p_2} & \cdots & \sqrt{p_r}\end{vmatrix}=(\lambda-1)D_{r-1}+p_r(\lambda-1)^{r-1},$$

则由递推公式可得

$$D_r=(\lambda-1)D_{r-1}+p_r(\lambda-1)^{r-1}=(\lambda-1)[(\lambda-1)D_{r-2}+p_{r-1}(\lambda-1)^{r-2}]+p_r(\lambda-1)^{r-1}$$

$$=(\lambda-1)^2D_{r-2}+p_{r-1}(\lambda-1)^{r-1}+p_r(\lambda-1)^{r-1}$$

$$=(\lambda-1)^3D_{r-3}+p_{r-2}(\lambda-1)^{r-1}+p_{r-1}(\lambda-1)^{r-1}+p_r(\lambda-1)^{r-1}$$

$$=(\lambda-1)^r+p_1(\lambda-1)^{r-1}+\cdots+p_{r-1}(\lambda-1)^{r-1}+p_r(\lambda-1)^{r-1}$$

$$=(\lambda-1)^{r-1}(\lambda-1+p_1+\cdots+p_{r-1}+p_r)=\lambda(\lambda-1)^{r-1},$$

令

$$D_r=|\lambda E_r-\mathrm{Cov}(\vec{X},\vec{X})|=\lambda(\lambda-1)^{r-1}=0,$$

可得协方差矩阵 $\mathrm{Cov}(\vec{X},\vec{X})$ 的特征值为

$$\lambda_1=\lambda_2=\cdots=\lambda_{r-1}=1,\quad \lambda_r=0,$$

有 $\mathrm{Cov}(\vec{X},\vec{X})$ 正交相似于对角阵

$$\Lambda=\begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix},$$

即存在正交阵 C，使得

$$C^{-1}\mathrm{Cov}(\vec{X},\vec{X})C=C^T\mathrm{Cov}(\vec{X},\vec{X})C=\Lambda,$$

作正交变换

$$\vec{X}=C\vec{Y}=C(Y_1,Y_2,\cdots,Y_r)^T,$$

有 $\vec{Y}=C^T\vec{X}$，则 $\vec{Y}$ 的均值向量和协方差矩阵分别为

$$E\vec{Y}=C^T E\vec{X}=C^T\cdot 0=0,$$

$$\mathrm{Cov}(\vec{Y},\vec{Y})=\mathrm{Cov}(C^T\vec{X},C^T\vec{X})=C^T\,\mathrm{Cov}(\vec{X},\vec{X})C=\Lambda=\begin{pmatrix}1&0&\cdots&0&0\\0&1&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&0\\0&0&\cdots&0&0\end{pmatrix},$$

则 $Y_1,Y_2,\cdots,Y_{r-1},Y_r$ 相互独立，且 $Y_1,Y_2,\cdots,Y_{r-1}$ 都服从正态分布 $N(0,1)$，而 $EY_r=0$， $\mathrm{Cov}(Y_r)=0$，即 Y_r 几乎必然等于 0，因

$$\chi^2=\sum_{i=1}^{r}\frac{(n_i-np_i)^2}{np_i}=\sum_{i=1}^{r}X_i^2=\vec{X}^T\vec{X}=\vec{Y}^TC^TC\vec{Y}=\vec{Y}^T\vec{Y}=\sum_{i=1}^{r}Y_i^2=\sum_{i=1}^{r-1}Y_i^2,\quad a.s.。$$

故

$$\chi^2=\sum_{i=1}^{r}\frac{(n_i-np_i)^2}{np_i}\dot{\sim}\chi^2(r-1)。$$