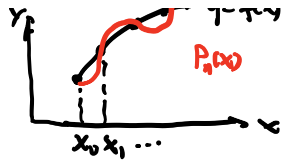
多项式插值



拉格朗日插值

$$f(x) \approx L_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x),$$

$$l_i(x) = \prod_{\substack{j=0\\ j\ne i}}^{n} \frac{x-x_j}{x_i-x_j}, \quad i=0,1,\cdots,n$$

拉格朗日基函数, $l_i(x_j)=\begin{cases}1, & j=i,\\ 0, & i\ne j.\end{cases}$

$$f(x_j)=L_n(x_j), \quad j=0,1,\cdots,n.$$

插值条件

$$f(x)-L_n(x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}\underbrace{(x-x_0)\cdots(x-x_n)}_{\omega_{n+1}(x)}}_{R_n(x)}$$

$R_n(x)$ 余项(误差)

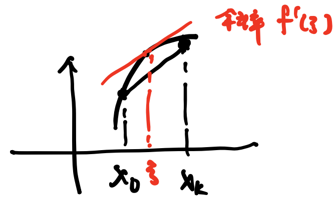
若 $f(x)=x^k$, $k\le n$, 则

$$f^{(n+1)}(x) = (x^k)^{(n+1)} = 0$$

$(x^n)'=nx^{n-1}$, $(x^n)''=n(n-1)x^{n-2}$

$$\cdots (x^n)^{(n)} = n(n-1)\cdots 1\, x^0 = n!$$

$$(x^n)^{(n+1)} = 0$$



拉格朗日中值定理

牛顿插值

差商:

$$f[x_0,x_k] = \frac{f(x_k)-f(x_0)}{x_k-x_0} = f'(\xi) \qquad \text{一阶差商}$$

$$f[x_0,x_1,x_k] = \frac{f[x_0,x_k]-f[x_0,x_1]}{x_k-x_1} = \frac{f''(\xi)}{2!} \qquad \text{二阶差商}$$

$$\vdots$$

$$f[x_0,x_1,x_2,\cdots,x_k] = \frac{f[x_0,x_1,x_2,\cdots,x_{k-2},x_k]-f[x_0,x_1,x_2,\cdots,x_{k-1}]}{x_k-x_{k-1}} = \frac{f^{(k)}(\xi)}{k!} \qquad k\text{阶差商}$$

$$f(x) = f(x_0) + f[x_0,x](x-x_0)$$

$$f[x_0,x] = f[x_0,x_1] + f[x_0,x_1,x](x-x_1)$$

$$f(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x](x-x_0)(x-x_1)$$

$$\vdots$$

$$f[x_0,x_1,\cdots,x_{n-1},x] = f[x_0,x_1,\cdots,x_n] + f[x_0,x_1,\cdots,x_n,x](x-x_n)$$

▲

$$\begin{aligned} f(x) = {} & \underbrace{f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots + f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})}_{P_n(x)} \\ & + \underbrace{f[x_0,x_1,\cdots,x_n,x]\,\omega_{n+1}(x)}_{R_n(x)} \end{aligned}$$

数学归纳法可证上面的公式：

假设(k)成立，$\underline{f[x_0,x_1,\cdots,x_n,x]} = \underline{f[x_0,x_1,\cdots,x_n,x_{n+1}]} + \underline{f[x_0,x_1,\cdots,x_{n+1},x](x-x_{n+1})}$

$$f(x)=f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$$
$$+f[x_0,x_1,\cdots,x_n,x_{n+1}](x-x_0)\cdots(x-x_n) + f[x_0,\cdots,x_{n+1},x]\,w_{n+2}(x)$$

$\hookrightarrow$ $f(x)=P_n(x)+R_n(x)$，其中：$R_n(x)=f[x_0,x_1,\cdots,x_n,x]\,\underset{\underset{(x-x_0)\cdots(x-x_n)}{\uparrow}}{w_{n+1}(x)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}w_{n+1}(x)$

$f(x_i)=P_n(x_i)+R_n(x_i)$，　　$R_n(x_i)=0$.

$i=0,1,\cdots,n$　　　　$f(x_i)=P_n(x_i)$ 插值条件

总结：$f(x)\approx P_n(x)=f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$

$\hookrightarrow$ $f(x_i)=P_n(x_i)$，$i=0,1,\cdots,n$ 插值条件

$f(x)-P_n(x):=R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}w_{n+1}(x)$，其中：$w_{n+1}(x)=(x-x_0)\cdots(x-x_n)$.

例：设函数值表

	x_0	x_1	x_2	x_3	x_4
x	-2	-1	0	1	3
$f(x)$	48	13	6	3	153

写出Newton插值多项式。

构造四次多项式 $P_4(x)$ 满足 $P_4(x_i)=f(x_i)$，即 $P_4(-2)=48$，$P_4(-1)=13$，$P_4(0)=6$，$P_4(1)=3$，$P_4(3)=153$.

$$P_4(x)=\underline{f(x_0)}+\underline{f[x_0,x_1]}(x-x_0)+\underline{f[x_0,x_1,x_2]}(x-x_0)(x-x_1)$$
$$+\underline{f[x_0,x_1,x_2,x_3]}(x-x_0)(x-x_1)(x-x_2)+\underline{f[x_0,x_1,x_2,x_3,x_4]}(x-x_0)(x-x_1)(x-x_2)\cdot(x-x_3)$$

	x_i	$f(x_i)$	一阶差商	二阶差商	三阶差商	四阶差商
x_0	-2	48				
x_1	-1	13	$\frac{13-48}{-1-(-2)}=-35$			
x_2	0	6	$\frac{6-13}{0-(-1)}=-7$	$\frac{-7-(-35)}{0-(-2)}=14$		
x_3	1	3	$\frac{3-6}{1-0}=-3$	$\frac{-3-(-7)}{1-(-1)}=2$	$\frac{2-14}{1-(-2)}=-4$	
x_4	3	153	$\frac{153-3}{3-1}=75$	$\frac{75-(-3)}{3-0}=26$	$\frac{26-2}{3-(-1)}=6$	$\frac{6-(-4)}{3-(-2)}=2$

埃尔米特插值（Hermite）

例：求 $p(x)$ 满足 $p(x_i)=f(x_i)$, $i=0,1,2$, $p'(x_1)=f'(x_1)$.

$p(x)$ 为三次多项式

设 $P_2(x)=f(x_0)+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)$

则 $f(x)-P_2(x)=R_2(x)=\dfrac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$

$R_2(x_i)=0$, 即 $f(x_i)=P_2(x_i)$, $i=0,1,2$.

$P_2'(x)=f[x_0,x_1]+f[x_0,x_1,x_2]\left((x-x_1)+(x-x_0)\right)$

$P_2'(x_1)=f[x_0,x_1]+f[x_0,x_1,x_2]\cdot(x_1-x_0)$

于是 $p(x)=P_2(x)+A(x-x_0)(x-x_1)(x-x_2)$

$p(x_i)=P_2(x_i)=f(x_i)$, $i=0,1,2$.

$$p'(x)=P_2'(x)+A\frac{d}{dx}[(x-x_0)(x-x_1)(x-x_2)]$$

$$f'(x_1)=p'(x_1)=f[x_0,x_1]+f[x_0,x_1,x_2](x_1-x_0)+A(x_1-x_0)(x_1-x_2)$$

$$A=\frac{f'(x_1)-\{f[x_0,x_1]+f[x_0,x_1,x_2](x_1-x_0)\}}{(x_1-x_0)(x_1-x_2)}$$

例：已知 $y=f(x)$ 满足如下数据表

x	-1	0	2
$f(x)$	0	-3	-3
$f'(x)$	-13	0	
$f''(x)$		0	

$p(x)=f(x_0)+f[x_0,x_1](x-x_0)$
$+f[x_0,x_1,x_2](x-x_0)(x-x_1)$
$+f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$
$+f[x_0,x_1,x_2,x_3,x_4](x-x_0)(x-x_1)(x-x_2)(x-x_3)$
$+f[x_0,x_1,x_2,x_3,x_4,x_5](x-x_0)(x-x_1)(x-x_2)(x-x_3)\cdot(x-x_4)$

求 5 次 Hermite 插值多项式。

	x_i	$f(x_i)$	一阶差商	二阶差商	三阶差商	四阶差商	五阶差商
x_0 ←	-1	0					
x_1 ←	-1	0	$f'(-1)=-13$				
x_2 ←	0	-3	$\frac{-3-0}{0-(-1)}=-3$	$\frac{-3-(-13)}{0-(-1)}=10$			
x_3 ←	0	-3	$f'(0)=0$	$\frac{0-(-3)}{0-(-1)}=3$	$\frac{3-10}{0-(-1)}=-7$		
x_4 ←	0	-3	$f'(0)=0$	$\frac{f''(0)}{2!}=0$	$\frac{0-3}{0-(-1)}=-3$	$\frac{-3-(-7)}{0-(-1)}=4$	

$x_5 \leftarrow \quad 2 \quad -3 \quad \frac{-3-(-3)}{2-0}=0 \quad \frac{0-0}{2-0}=0 \quad \frac{0-0}{2-0}=0 \quad \frac{0-(-3)}{2-(-1)}=1 \quad \frac{1-4}{2-(-1)}=-1$

$$f[\underbrace{x,\cdots,x}_{n+1\text{个}}]=\frac{f^{(n)}(x)}{n!}$$

$$P(x)=-13(x+1)+10(x+1)^2-7(x+1)^2x+4(x+1)^2x^2-(x+1)^2x^3$$

总结:

Newton

$$f(x)\approx f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$$
$$=P_n(x)$$

$$R_n(x)=f(x)-P_n(x)=f[x_0,x_1,\cdots,x_n,x](x-x_0)\cdots(x-x_n)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

$R_n(x_i)=0$, 即 $f(x_i)=P_n(x_i)$, $i=0,1,\cdots,n$

(1) $x_0<x_1<\cdots<x_n$

$f(x_k)=P_n(x_k)$, $f(x_{k+1})=P_n(x_{k+1})$ 为同一条件

(2) 部分结点相等的情况 如 $x_k=x_{k+1}$, $f'(x_k)=P_n'(x_k)$

Newton 插值公式也成立。