Title

§9 拓扑线性空间

Date: M T W T F S S

Notes Review

1. 两个例子：拓扑线性空间

拓扑空间 线性空间 但加法数乘不连续

2. $X.\ \tau \subseteq P(X)$

非必要线性　X的一些子集构成的集合

(1) $\phi.\ X \in \tau$

(2) $A.B \in \tau \Rightarrow A \cap B \in \tau$

(3) $A_\alpha \in \tau \quad \alpha \in I \Rightarrow \bigcup\limits_{\alpha \in I} A_\alpha \in I$

注：若X为线性空间，τ为X上的一个拓扑（≠拓扑线性空间）

且有 若 $x_\alpha \xrightarrow{\tau} x \quad y_\beta \xrightarrow{\tau} y \quad a_{\alpha'} \xrightarrow{d_R} a \Rightarrow a_{\alpha'} x_\alpha + b_{\beta'} y_\beta \xrightarrow{\tau} ax+by$

则称 (X,τ) 为拓扑线性空间

2. $a_{\alpha'} \xrightarrow{d_R} a$: $\forall \varepsilon>0 \ \exists \alpha_0$ s.t. 对 $\forall \alpha > \alpha_0 \ |a_\alpha - a| < \varepsilon$

$(a_n)_{\underline{n\in N}}$ 可数. 指标集　$(a_\alpha)_{[0,1]}$ 不可数

eg $a_\alpha = 1+\alpha \quad a_{0.1} = 1+0.1 \quad a_{0.01} = 1+0.01 \quad a_\alpha \xrightarrow{d_R} 1$

3. 数乘 $x_\alpha \xrightarrow{\tau} x \quad a \in R \quad a_\alpha \to a \quad x \in X \Rightarrow a x_\alpha \xrightarrow{\tau} ax$ 且 $a_\alpha x \to ax$

加法 $x_\alpha \xrightarrow{\tau} x \quad \forall y \in X \Rightarrow x_\alpha + y \xrightarrow{\tau} x+y$

eg normed vector space $(X, \|\cdot\|)$ $\quad x_\alpha \xrightarrow{\|\cdot\|} x \quad a \in R \Rightarrow a x_\alpha \xrightarrow{\|\cdot\|} ax$

Proof: $\|a x_\alpha - ax\| = \|a(x_\alpha - x\| = |a| \|x_\alpha - x\| = 0$

$a_\alpha - a \quad \forall x \in X \quad \|a_\alpha x - ax\| = \|(a_\alpha - a)x\| = |a_\alpha - a| \|x\| \to 0$

$x_\alpha + y \xrightarrow{\|\cdot\|} x+y$

$\|x_\alpha + y - (x+y)\| = \|x_\alpha - x\| \longrightarrow 0$

内积空间 → 赋范线性空间 → 拓扑线性空间
↓
度量空间 $\not\longrightarrow$ ↑

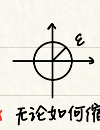
4. (X,τ) $\quad A \in \tau \Leftrightarrow A$ 为开集

A^c 为闭集 若 $A \in \tau$

A为平衡集 若 $x \in A \Rightarrow ax \in A \quad \forall |a| \le 1$

A为吸收集 $\forall x \in X \ \exists \lambda \in R$ s.t. $\begin{cases} \forall \lambda' \\ |\lambda'| \le \lambda \end{cases}$ 有 $\lambda' x \in A$

eg1 $A = D(0,\varepsilon)$

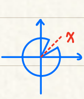


$\lambda = \frac{\varepsilon}{r_x}$ $\quad |\lambda' x| \le |\lambda x| = |\frac{\varepsilon}{r_x} \cdot x| = \frac{\varepsilon}{r_x} \cdot r_x = \varepsilon \Rightarrow \lambda x \in A$

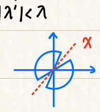
$\Uparrow$

$\forall |\lambda'| < \lambda$

eg2



无论如何缩小都不在A中



不是吸收集

Content 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

5. 若A是吸收集. $A\subseteq B$ 则B也是吸收集

6. 邻域：$A\subseteq X$. $\exists\, x\in U\in\tau$ s.t. $U\subseteq A$ 则A是X的邻域

A.B 均为X的邻域 $A\cap B$ 也是X的邻域

证明：$\exists\, U_A.\ U_B\in\tau$ s.t. $\begin{cases} x\in U_A\subseteq A \\ x\in U_B\subseteq B \end{cases} \Rightarrow x\in U_A\cap U_B\subseteq A\cap B$

A_α 是邻域 $\bigcup_{\alpha\in I} A_\alpha$ 也是X的邻域

证明：$x\in U_\alpha\subseteq A_\alpha$

$x\in U_\alpha\in\tau \quad \subseteq \bigcup_{\alpha\in I} A_\alpha$

$\Rightarrow \bigcup_{\alpha\in I} A_\alpha$ 也X的一个邻域

7. $(X.\tau)$ 为拓扑线性空间

$\tau=\{x+U \mid x\in X$ U为包含 $\overset{X}{\underset{\Downarrow}{}}0$ (原点)的开集

$N(0)=\{A \mid 0\in U\subseteq A\}$

$N(x)=\{x+A \mid 0\in U\subseteq A\}=x+N(0)$

$X_\alpha\xrightarrow{\tau}X \quad \forall\, x\in U\in\tau \quad \exists\,\alpha_0 \quad s.t.\ \alpha\geq\alpha_0$

$X_\alpha\in U \quad X_\alpha-X\xrightarrow{\tau}0 \quad \forall\, 0\in U'\in I \quad \exists\,\alpha_0$ s.t. $\alpha\geq\alpha_0$ $x_\alpha-x\in U'\Rightarrow x_\alpha\in x+U$

"+" 连续性和开集 $(x+U)$ 的结构有密切关系

X线性. τ 拓扑 判断开集的结构来 $(X.\tau)$ 是否为拓扑线性空间

8. $(X.\tau)$ N为包含0的邻域 $(X.\tau)$ 是拓扑线性空间

$\Leftrightarrow$ (1) $\forall A\in N\Rightarrow 0\in A$

(2) $\forall A\in N \quad \exists B\in N$ s.t. $B+B\subseteq A$

(3) $\forall A\in N \quad 0\neq\lambda\in R$ 有 $\lambda A\in N$

(4) $\forall A\in N$ A为吸收集

(5) $\forall A\in N$ A为平衡集

Proof: (1) 略

(2) (X,τ) "+": $X\times X\to X \quad f_+: \underline{X\times X}\to X$ $\quad (x,y)\to x+y$

$(X\times X,\ \tau\times\tau)$ 乘积拓扑

乘积拓扑 $(X,\tau_1),(X,\tau_2)$

$(X\times Y,\tau_1\times\tau_2)$: $\tau_1\times\tau_2=\{U\times V \mid U\in\tau_1\ V\in\tau_2\}$

设 $\vec{V}=f_+^{-1}(U)\Rightarrow$ V一定是X×X中的开集

$U\times V=\{(x,y) \mid x\in U,\ y\in V\}$

$\vec{V}=U_1\times U_2$ 令 $V=U_1\cap U_2\in\tau$ $(V\times V\subseteq\vec{V})$

$\Rightarrow V+V=f_+(V\times V)\subseteq U\subseteq A$ 令 $V=B\in N$

$A\in N\Rightarrow\exists\, 0\in\overset{\tau}{\underset{\Downarrow}{}}U\subseteq A$

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Review

(3) $A \in N \Rightarrow \lambda A \in N \quad \lambda \neq 0$

"·" $R \times X \to X$ $\quad (X, \tau)$

$(\lambda, x) \mapsto \lambda x$

$R \times X = \{(\lambda, x) \mid \lambda \in R,\ x \in X\}$

$d \times \tau = \{U \times V \mid U$ 为 R 中开集 $V \in \tau\}$

$0 \in \lambda \in \tau \subseteq \lambda A \iff \lambda A \in N$

$f: (X, \tau_1) \to (X, \tau_2)$ 连续 $\quad \exists\, U \in \tau$ s.t. $f(U) \subseteq V \subseteq \tau_2 \quad f((-\lambda, \lambda) \times V) \subseteq U \in \tau$

$\Rightarrow \lambda V \subseteq U \in \tau \quad \lambda V = \bigcup_{V \subseteq U} \lambda V \in \tau$

$X_\alpha \xrightarrow{\tau} 0 \Rightarrow \lambda X_\alpha \xrightarrow{\tau} 0$

(4) $\exists\, 0 \in U \subseteq A$ 要证 U 为吸收集

$\alpha \in [0, \lambda\alpha]$

$\lambda\alpha \xrightarrow{d_R} 0 \quad \forall x \in X$

$\Rightarrow \lambda\alpha X \xrightarrow{\tau} 0$

U 为吸收集: $\forall x \in X\ \exists \lambda \in R^+$ s.t. $\forall |\lambda'| \le \lambda$ 有 $\lambda x \in U$

$0 \in U \in \tau \quad \underline{\exists\, \alpha_0}$. s.t. $\alpha \geq 0$ 有 $\lambda\alpha X \in U$

$\lambda = \lambda\alpha_0$

$\forall |\lambda'| < \lambda = \lambda\alpha_0 \Rightarrow |\lambda'| x = \lambda\alpha' x \in U$

1)-5) $\Leftarrow \quad x_\alpha \xrightarrow{\tau} x \Rightarrow x_\alpha + y \xrightarrow{\tau} x + y$

$\forall\, x + y \in U\ \exists \alpha_0$ s.t. $\alpha \geq \alpha_0 \quad x_\alpha + y \in U$

$\forall x \in -y + U\ \exists \alpha_0$ s.t. $\alpha \geq \alpha_0 \quad x_\alpha \in -y + U \Rightarrow x_\alpha + y \in$

$X_\alpha \xrightarrow{\tau} X \quad \lambda \in R \quad \lambda X_\alpha \in \lambda X$

$\forall x \in U \quad \exists \alpha_0$ s.t. $\lambda x \in \lambda U \quad \exists U_0$ s.t. $\alpha \geq \alpha_0 \quad \lambda X_{\alpha_0} \in \lambda U$

$\lambda\alpha \xrightarrow{d_R} 0 \quad x \in X$

$\Rightarrow \lambda\alpha x \xrightarrow{\tau} 0 \quad$ 由(4)吸收集 $\forall 0 \in U\ \exists \alpha_0\ (\lambda\alpha_0 = \lambda) \quad |\lambda'| < \lambda \quad \lambda\alpha X = \lambda' x \in U \Rightarrow \lambda\alpha x \xrightarrow{\tau} 0$