§2.4 基变换与坐标变换

一、线性组合的形式记法

二、基变换

三、坐标变换

一、线性组合的形式记法

1、设V为数域 P上的 n 维线性空间，$\alpha_1, \alpha_2, \cdots, \alpha_n$ 为 V中的一组向量，$\beta \in V$ ，若

$$\beta = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$$

则可记作

$$\beta = (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

2、设V为数域 P 上 n 维线性空间， $\alpha_1,\alpha_2,\cdots,\alpha_n$ **；**

$\beta_1,\beta_2,\cdots,\beta_n$ **为V中的两组向量，若**

$$\begin{cases} \beta_1 = a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{n1}\alpha_n \\ \beta_2 = a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{n2}\alpha_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_n = a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{nn}\alpha_n \end{cases}$$

则可记作

$$(\beta_1,\beta_2,\cdots,\beta_n) = (\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

注：在形式书写法下有下列运算规律

1) $\alpha_1,\alpha_2,\cdots,\alpha_n\in V, a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n\in P$

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}+(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}a_1+b_1\\a_2+b_2\\\vdots\\a_n+b_n\end{pmatrix}$$

若$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关，则

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}\Leftrightarrow\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}$$

2） $\alpha_1,\alpha_2,\cdots,\alpha_n$; $\beta_1,\beta_2,\cdots,\beta_n$ 为V中的两组向量，

矩阵 $A,B\in P^{n\times n}$，则

$$((\alpha_1,\alpha_2,\cdots,\alpha_n)A)B=(\alpha_1,\alpha_2,\cdots,\alpha_n)(AB);$$

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)A+(\alpha_1,\alpha_2,\cdots,\alpha_n)B$$
$$=(\alpha_1,\alpha_2,\cdots,\alpha_n)(A+B);$$

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)A+(\beta_1,\beta_2,\cdots,\beta_n)A$$
$$=(\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n)A;$$

若 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 线性无关，则

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)A=(\alpha_1,\alpha_2,\cdots,\alpha_n)B\Leftrightarrow A=B$$

二、基变换

1、定义

设V为数域P上n维线性空间，$\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$；

$\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n'$ 为V中的两组基，若

$$\begin{cases} \varepsilon_1' = a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{n1}\varepsilon_n \\ \varepsilon_2' = a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + \cdots + a_{n2}\varepsilon_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varepsilon_n' = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{nn}\varepsilon_n \end{cases} \quad ①$$

即，

$$(\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n')=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad ②$$

则称矩阵 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

为由基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 到基 $\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n'$ 的**过渡矩阵**;

称 ① 或 ② 为由基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 到基 $\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n'$

的**基变换公式**.

2、性质

1）过渡矩阵都是可逆矩阵；反过来，任一可逆矩阵都可看成是两组基之间的过渡矩阵.

证明：若 $\alpha_1,\alpha_2,\cdots,\alpha_n$； $\beta_1,\beta_2,\cdots,\beta_n$ 为V的两组基，

且由基 $\alpha_1,\alpha_2,\cdots,\alpha_n$到$\beta_1,\beta_2,\cdots,\beta_n$ 的过渡矩阵为A，

即 $(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A$ ③

又由基 $\beta_1,\beta_2,\cdots,\beta_n$到$\alpha_1,\alpha_2,\cdots,\alpha_n$ 也有一个过渡矩阵，

设为B，即 $(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\beta_1,\beta_2,\cdots,\beta_n)B$ ④

比较③ 、④两个等式，有

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)AB$$

$\because$ $\alpha_1,\alpha_2,\cdots,\alpha_n$ 是V的一组基,

$\therefore$ $AB=E$. 即，A是可逆矩阵, 且$A^{-1}=B$.

反过来，设 $A=(a_{ij})_{n\times n}$ 为P上任一可逆矩阵，

任取V的一组基 $\alpha_1,\alpha_2,\cdots,\alpha_n$,

令$\beta_j=\sum_{i=1}^{n}a_{ij}\alpha_i=(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)\begin{pmatrix}a_{1j}\\a_{2j}\\\vdots\\a_{nj}\end{pmatrix},j=1,2,\cdots,n$

于是有， $(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A$

由A可逆，有 $(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\beta_1,\beta_2,\cdots,\beta_n)A^{-1}$

即，$\alpha_1,\alpha_2,\cdots,\alpha_n$也可由 $\beta_1,\beta_2,\cdots,\beta_n$ 线性表出.

$\therefore$ $\alpha_1,\alpha_2,\cdots,\alpha_n$与$\beta_1,\beta_2,\cdots,\beta_n$等价.

故 $\beta_1,\beta_2,\cdots,\beta_n$ 线性无关，从而也为V的一组基.

并且A就是$\alpha_1,\alpha_2,\cdots,\alpha_n$到$\beta_1,\beta_2,\cdots,\beta_n$ 的过渡矩阵.

2）若由基$\alpha_1,\alpha_2,\cdots,\alpha_n$到基$\beta_1,\beta_2,\cdots,\beta_n$的过渡矩阵为A,

则由基$\beta_1,\beta_2,\cdots,\beta_n$到基$\alpha_1,\alpha_2,\cdots,\alpha_n$的过渡矩阵为$A^{-1}$.

3）若由基$\alpha_1,\alpha_2,\cdots,\alpha_n$到基$\beta_1,\beta_2,\cdots,\beta_n$的过渡矩阵为A, 由基 $\beta_1,\beta_2,\cdots,\beta_n$到基$\gamma_1,\gamma_2,\cdots,\gamma_n$的过渡矩阵为B，则由基 $\alpha_1,\alpha_2,\cdots,\alpha_n$到基$\gamma_1,\gamma_2,\cdots,\gamma_n$的过渡矩阵为AB.

事实上, 若 $(\beta_1,\beta_2,\cdots,\beta_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A$

$$(\gamma_1,\gamma_2,\cdots,\gamma_n)=(\beta_1,\beta_2,\cdots,\beta_n)B$$

则有，$(\gamma_1,\gamma_2,\cdots,\gamma_n)=((\alpha_1,\alpha_2,\cdots,\alpha_n)A)B$

$$=(\alpha_1,\alpha_2,\cdots,\alpha_n)AB$$

三、坐标变换

1、定义：V为数域P上n维线性空间 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$；

$\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n'$ 为V中的两组基，且

$$(\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n')=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad ⑤$$

设 $\xi\in V$且ξ在基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 与基 $\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n'$

下的坐标分别为$(x_1,x_2,\cdots,x_n)$与 $(x_1',x_2',\cdots,x_n')$，

即，$\xi=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$与 $\xi=(\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n')\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{pmatrix}$

因为,$(\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n')=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\cdots&\cdots&\cdots&\cdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix}$,

从而，

$$(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\cdots&\cdots&\cdots&\cdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix}\begin{pmatrix}x_1'\\x_2'\\\vdots\\x_n'\end{pmatrix}$$

则 $$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} \qquad ⑥$$

或 $$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad ⑦$$

称⑥或⑦为向量ξ在基变换⑤下的**坐标变换公式**.

例1 在P^n中，求由基$\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 到基$\eta_1,\eta_2,\cdots,\eta_n$ 的过渡矩阵及由基$\eta_1,\eta_2,\cdots,\eta_n$ 到基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 的过渡矩阵．其中

$$\varepsilon_1=(1,0,\cdots,0),\varepsilon_2=(0,1,\cdots,0),\cdots,\varepsilon_n=(0,\cdots,0,1)$$

$$\eta_1=(1,1,\cdots,1),\eta_2=(0,1,\cdots,1),\cdots,\eta_n=(0,\cdots,0,1)$$

并求向量$\alpha=(a_1,a_2,\cdots,a_n)$在基$\eta_1,\eta_2,\cdots,\eta_n$下的坐标.

解：$\because$ $\begin{cases}\eta_1=\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n\\ \eta_2=\quad\ \ \varepsilon_2+\cdots+\varepsilon_n\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \eta_n=\qquad\qquad\quad\ \varepsilon_n\end{cases}$

$$\therefore\ (\eta_1,\eta_2,\cdots,\eta_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix}1&0&\cdots&0\\1&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&1\end{pmatrix}$$

从而，$$(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=(\eta_1,\eta_2,\cdots,\eta_n)\begin{pmatrix}1&0&\cdots&0\\1&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&1\end{pmatrix}^{-1}$$

$$=(\eta_1,\eta_2,\cdots,\eta_n)\begin{pmatrix}1&0&0&\cdots&0\\-1&1&0&\cdots&0\\0&-1&1&\cdots&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&1\end{pmatrix}$$

故，由基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 到基 $\eta_1,\eta_2,\cdots,\eta_n$ 的过渡矩阵为

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

由基 $\eta_1,\eta_2,\cdots,\eta_n$ 到基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 的过渡矩阵为

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$\alpha=(a_1,a_2,\cdots,a_n)$在基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$下的坐标就是

$(a_1,a_2,\cdots,a_n)$

设α在基 $\eta_1,\eta_2,\cdots,\eta_n$下的坐标为$(x_1,x_2,\cdots,x_n)$，则

$$\alpha=\sum_{i=1}^{n}a_i\varepsilon_i=\begin{pmatrix}\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{pmatrix}$$

$$=\begin{pmatrix}\eta_1 & \eta_2 & \cdots & \eta_n\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ -1 & 1 & 0 & \cdots & 0\\ 0 & -1 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 - a_1 \\ \vdots \\ a_n - a_{n-1} \end{pmatrix},$$

所以α在基$\eta_1,\eta_2,\cdots,\eta_n$下的坐标为

$$(a_1, a_2 - a_1, \cdots, a_n - a_{n-1}).$$

例2 在$P[x]_3$中，设

$$\eta_1 = x^2 - x + 1, \eta_2 = x - 1, \eta_3 = 1,$$
$$\alpha_1 = x^2 + x + 3, \alpha_2 = x - 5, \alpha_3 = x^2 + 2x - 1,$$

（1）求由基η_1,η_2,η_3到基 $\alpha_1,\alpha_2,\alpha_3$ 的过渡矩阵；

（2）设向量 β 在基 $\alpha_1,\alpha_2,\alpha_3$ 下的坐标为 $(1,-1,2)$，

求向量 β 在基η_1,η_2,η_3下的坐标.

解：(1) 设 $\varepsilon_1 = x^2, \varepsilon_2 = x, \varepsilon_3 = 1,$

则有

$$(\eta_1, \eta_2, \eta_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3)\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix},$$

$$(\alpha_1, \alpha_2, \alpha_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3)\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & -5 & -1 \end{pmatrix},$$

设 $(\alpha_1, \alpha_2, \alpha_3) = (\eta_1, \eta_2, \eta_3)A,$ 则有

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3)\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & -5 & -1 \end{pmatrix} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}A,$$

从而，$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & -5 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} A,$$

解得，$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & -5 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & -4 & 1 \end{pmatrix}$$

(2) $$\beta = \alpha_1 - \alpha_2 + 2\alpha_3$$

$$= (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = (\eta_1, \eta_2, \eta_3) A \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = (\eta_1, \eta_2, \eta_3) \begin{pmatrix} 3 \\ 7 \\ 10 \end{pmatrix}$$

求向量 β 在基 η_1, η_2, η_3 下的坐标为 $(3, 7, 10)$.