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Chapter 4: Hypothesis Testing

=> Hypothesis testing (Model comparison)

=> Formulation : We want to know that

$$H_0 : \theta \in \Theta_0 \quad Vs \quad H_1 : \theta \in \Theta_1$$

(Null) (alternative)

=> Simple hypothesis

θ_0 has a single value

=> Composite hypothesis :

θ_0 has > 1 value

=> example :

$X_1, \cdots, X_n \mid \mu, \sigma^2 \sim N(\mu, \sigma^2)$

If σ^2 is known, $H_0 : \mu = \mu_0$ is simple

If σ^2 is un-known, $H_0 : \mu = \mu_0$ is composite

=> size of a test

$Pr(\text{"Reject } H_0\text{"}(X) \mid \theta) \leq \alpha$ for all $\theta \in \Theta_0$

then α is called the size of "Reject H_0"(X)

$$\alpha = \sup_{\theta} Pr(\text{"Reject } H_0 \mid \theta\text{"})$$

=> Critical Region based on $t(x)$

Rejected H_0: if $t(x) \in C_\alpha$

C_α is called Critical Region

=> Test function:

$$\phi(x) = \begin{cases} 1 & \text{(Rejected) if } t(x) \in C_\alpha \\ 0 & \text{O/W} \end{cases}$$

=> example:

$$X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$H_0: \mu = \mu_0$ VS H_1 $\mu > \mu_0$

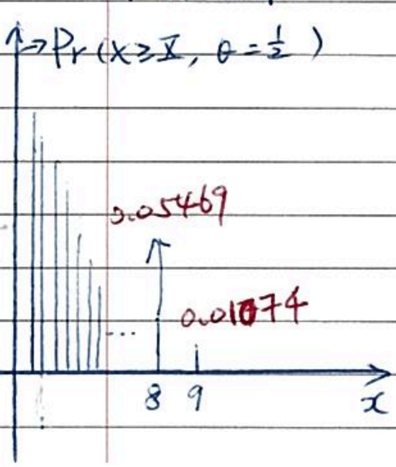
$$t(x) = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$$

test. function in case:

$$\phi(x) = \begin{cases} 1, & \text{if } t(x) > t_{n-1, \alpha} \\ 0 & \text{O/W} \end{cases}$$

=> Randomised test function:

Example. $X \sim Bin(n=10, \theta)$



$H_0: \theta = \frac{1}{2}$; VS. $H_1: \theta > \frac{1}{2}$

$t(x) = X$

Critical Region $X \geq K_\alpha$

Given $\alpha = 5\%$ we can't find

the integer K_α, s.t $P(X \geq K_\alpha \mid \theta = \frac{1}{2}) = 0.05$

Randomised test function:

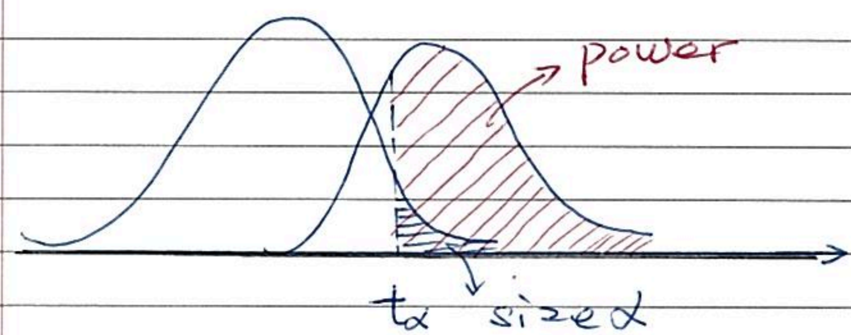
$$\phi(x) = \begin{cases} 1 & \text{if } x \geq 9 \\ \frac{67}{75} & \text{if } x = 8 \text{ (Reject } H_0 \text{ with } Pr. = \frac{67}{75}\text{)} \\ 0 & \text{if } x \leq 7 \end{cases}$$

$$E(\phi(x)) = P(\text{Reject } H_0)$$

$$= Pr(x \geq 9) \times 1 + Pr(x=8) \times Pr(\text{Reject } H_0, x=8)$$

$$= 0.01074 \times 1 + (0.05469 - 0.01074) \times \frac{67}{75} = 5\%$$

$\Rightarrow$ Power function:



$$W(\theta) = Pr(\text{Reject } H_0 \mid \theta) \quad \text{for } \theta \in H_1$$

$$= E(\phi(x) \mid \theta), \quad \text{for } \theta \in H_1$$

We have many way tests but we need to the best way

Ideally, a test function $\phi(x)$ is good if

(1) $E\{\phi_0(x) \mid \theta\} \leq \alpha$ for $\theta \in H_0$

(2) $E\{\phi_0(x) \mid \theta\} \geq E\{\phi(x) \mid \theta\}$

For each $\theta \in H_1$ and for all $\phi_{(x)}$ with size α

Within this framework, we have

(1) Simple H_0 $\leftrightarrow$ simple H_1

Neymon-Pearson formed optimal $\phi(x)$:

$$\phi(x) = \begin{cases} 1 & \text{if } \Lambda(x) = f(x \mid \theta \in H_1) / f(x \mid \theta \in H_0) > k_\alpha \\ r & \text{if } \Lambda(x) = k_\alpha \\ 0 & \text{if } \Lambda(x) < k_\alpha \end{cases}$$

(2) simple H_0 VS composite H_1

but $\Lambda(x)$ is monotone, we can have

optimal $\phi(x)$ for all $\theta \in H_1$

uniformly most powerful test

(3) Composite H_0 VS Composite H_1

we don't have optimal test

1.

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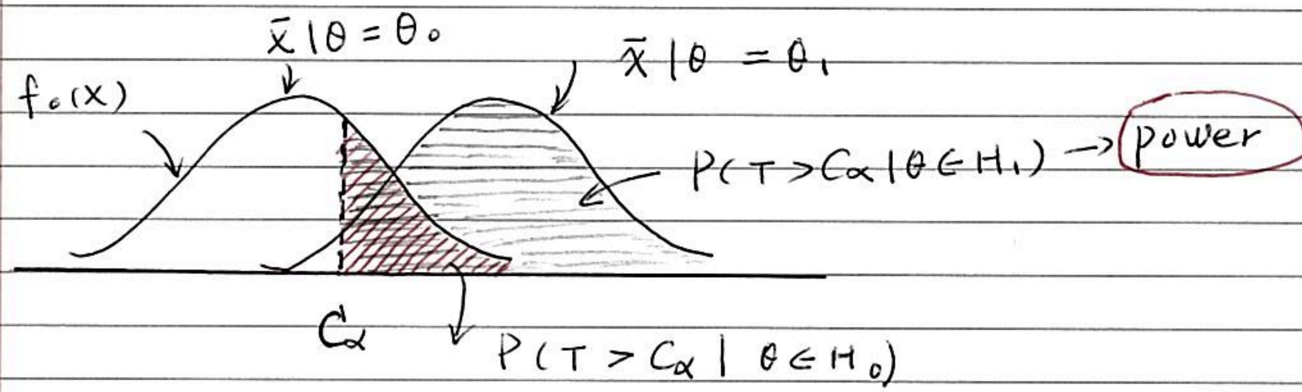
=> Test Function :

$$\phi(x) = \begin{cases} 1 & \text{if } x \in C_\alpha \\ \lambda(x) & \text{if } x = x_0 \\ 0 & \text{if } x \notin C_\alpha \cup \{x = x_0\} \end{cases}$$

=> size : $\sup_{\theta \in \Theta_0} E(\phi(x) \mid \theta)$

=> power : $W(\theta) = Pr(\phi(x) = 1 \mid \theta) = E(\phi(x) \mid \theta)$ for $\theta \in \Theta_1$



=> Neyman – Pearson theorem

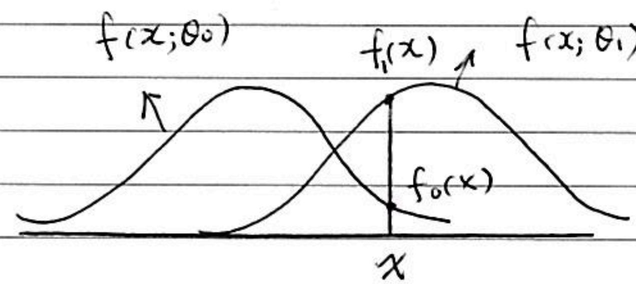
$H_0 : \theta = \theta_0$ Vs. $H_1 : \theta = \theta_1$

(LR) => Likelihood Ratio : Define the likelihood Ratio $\Lambda(x)$ by

$$\Lambda(x) = \frac{f_1(x)}{f_0(x)}$$

pdf of X : $f_1(x) = f(x; \theta_1)$

$f_0(x) = f(x; \theta_0)$



$\Rightarrow$ Likelihood Ratio Test (LRT)

The (Randomised) test with the test function $\phi_0(x)$ is said to be a likelihood ratio test if it is of the form

$$\phi_0(x) = \begin{cases} 1 & \text{if } \Lambda(x) = \frac{f_1(x)}{f_0(x)} > K \\ \gamma(x) & \text{if } \Lambda(x) = f_1(x)/f_0(x) = K. \\ 0 & \text{if } \Lambda(x) = f_1(x)/f_0(x) < K. \end{cases}$$

$\Rightarrow$ Theorem 4.1 (Neyman – Pearson)

(最优性)

(a) (optimality). For any K and $\gamma(x)$, the test ϕ_0 has maximum power among all tests whose sizes are no greater than the size of $\phi_0(x)$

proof:

Let ϕ be any test for which $E_{\theta_0}\phi(x) \le E_{\theta_0}\phi_0(x)$

Define $U(x) = \{\phi_0(x) - \phi(x)\}\{f_1(x) - K \cdot f_0(x)\}$,

When, $f_1(x) - K \cdot f_0(x) > 0$, we have $\phi_0(x) = 1$

So, $U(x) \geq 0$.

when $f_1(x) - K f_0(x) < 0$, we have $\phi_0(x) = 0$

So, $U(x) \geq 0$

When $f_1(x) - K f_0(x) = 0$, of course $U(x) = 0$

Thus $U(x) \geq 0$ for all x. Hence

$$0 \le \int U(x)\,dx$$

$$= \int \{\phi_0(x) - \phi(x)\}\{f_1(x) - k f_0(x)\}\,dx$$

$$= \int \phi_0(x)\cdot f_1(x)\,dx - \int \phi(x) f_1(x)\,dx$$

$$+ K\left\{\int \phi(x) f_0(x)\,dx - \int \phi_0(x) f_0(x)\,dx\right\}$$

$$= E_{\theta_1}\{\phi_0(x)\} - E_{\theta_1}\{\phi(x)\} + K\left\{E_{\theta_0}\{\phi(x)\} - E_{\theta_0}\{\phi_0(x)\}\right\}$$

$\because$ $E_{\theta_0}\{\phi(x)\} - E_{\theta_0}(\phi_0(x)) \le 0$, because of the assumption that the size of ϕ is no greater than the size of ϕ_0, therefore,

$$\int \phi_0(x) f_1(x)\,dx - \int \phi(x) f_1(x)\,dx \ge 0$$

which establishes that the power of ϕ cannot be greater than the power of $\phi_0(x)$.

(b) (Existence) Given $\alpha \in (0, 1)$, there exist constants K and γ_0 such that the LRT defined by this K and $\gamma(x) = \gamma_0$ for all x has size exactly α.

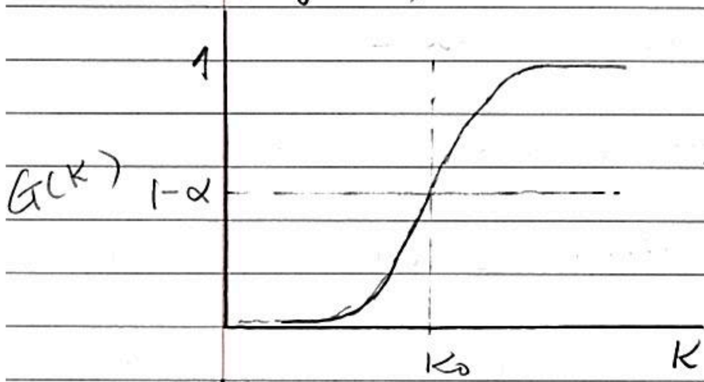
Proof: The probability distribution function

$G(K) = Pr_{\theta_0}\{\Lambda(x) \le K\}$ is non-Decreasing as K

increases; it is also right-continuous.

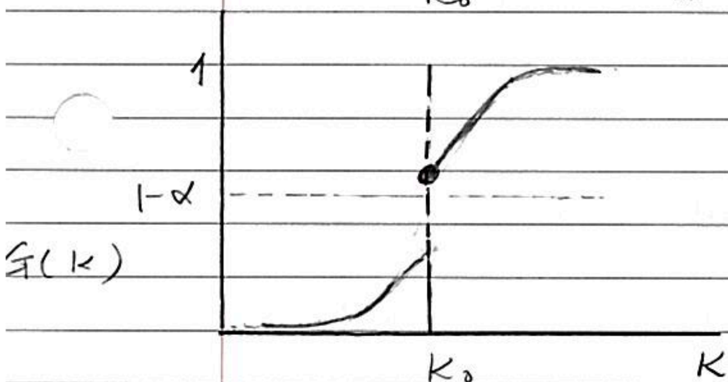
Try to find a value K_0 for which

$G(K_0) = 1-\alpha$, As can be seen from the below figure, There are two possibilities:



(1) Such K_0 exist or

(2) we cannot exactly solve the equation $G(K_0) = 1-\alpha$



but we can find a K_0 for which $G_-(K_0) = \Pr_{\theta_0}\{\Lambda(X) < K_0\}$

$\leqslant 1-\alpha \leqslant G(K_0)$

in (1), we are done (set $\gamma_0 = 0$)

in case (2), set $\gamma_0 = \dfrac{G(K_0)-(1-\alpha)}{G(K_0)-G_-(K_0)}$

Then it is an easy exercise to demonstrate that this test has size exactly α.

(C) (uniqueness). if the test ϕ has size α, and is of maximum power amongst all possible tests of size α, then ϕ is necessarily a likelihood ratio test, except possibly on a set of values of x which has probability 0 under both H_0 and H_1.

proof:

Let ϕ_0 be the LRT defined by the constant K and function $\gamma(x)$, and suppose ϕ is another test of the same size α and the same power as ϕ_0. Define $U(x)$ as in (a), then $U(x) \geq 0$ for all x, but, because ϕ and ϕ_0 have the same size and power, $\int U(x)\,dx = 0$. So the function $U(x)$ is non-negative and integrates to 0. Hence $U(x) = 0$ for all x, except possibly on a set, S say, of values of x, which has probability zero under both H_0 and H_1. This in turn means that, except on the set S $\phi(x) = \phi_0(x)$ or $f_1(x) = K f_0(x)$, so that $\phi(x)$ has the form of a LRT. This establishes the uniqueness result, and so completes the proof of the theorem.

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⇒ Neyman - pearson Lemma :

$$\phi_0(x) = \begin{cases} 1 & \text{if } \Lambda(x) = f_1(x)/f_0(x) > K \\ \gamma(x) & \text{if } \Lambda(x) = f_1(x)/f_0(x) = K \\ 0 & \text{if } \Lambda(x) = f_1(x)/f_0(x) < K \end{cases}$$

LRT

⇒ (b) Given $\alpha \in (0, 1)$, we can find K and $\gamma(x)$ such that: $E\{\phi(x) | \theta_0\} = \alpha$

(a)

⇒ $\phi_0(x)$ is specified with K and $\gamma(x)$ given in (b) so $E(\phi_0(x) | \theta_0) = \alpha$ For any test $\phi(x)$ with $E(\phi(x) | \theta_0) \le \alpha$

$E(\phi(x) | \theta_1) \le E(\phi_0(x) | \theta_1)$

⇒ proof :

suppose that : $U(x) = (\phi_0(x) - \phi(x))[f_1(x) - K f_0(x)]$

We see that $U(x) \geq 0$

$$\int U(x)\, dx = \int \{\phi_0(x) - \phi(x)\}[f_1(x) - K f_0(x)]\, dx$$

$$= E_{\theta_1}(\phi_0(x)|\theta_1) - K E_{\theta_0}(\phi_0(x)|\theta_0)$$

$$- E_{\theta_1}(\phi(x)|\theta_1) + K E_{\theta_0}(\phi(x)|\theta_0)$$

$$= \{E_{\theta_1}(\phi_0(x)) - E_{\theta_1}(\phi(x))\} + K\{E_{\theta_0}(\phi(x)) - E_{\theta_0}(\phi_0(x))\}$$

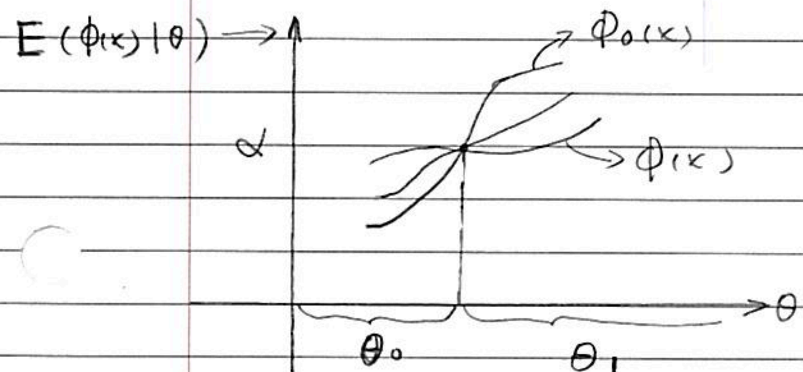
$$\Rightarrow E_{\theta_1}(\phi_0(x)) - E_{\theta_1}(\phi(x)) \geq 0$$

⇒ Definition of uniformly most powerful test of size α is a test $\phi_0(x)$ such that

(1) $E_\theta\{\phi_0(x)\} \le \alpha$ for all $\theta \in \Theta_0$

(2) given any other test $\phi(x)$ for which $E_\theta\{\phi(x)\} \le \alpha$ for all $\theta \in \Theta_0$, we have

$E_\theta\{\phi_0(x)\} \ge E_\theta\{\phi(x)\}$ for all $\theta \in \Theta_1$



⇒ Reviewer of Gamma (α, θ)

(1) θ is a scale

$$f(x|\theta) = \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{\theta}\right)^{\alpha} \cdot e^{-\frac{x}{\theta}} \quad \text{for } x>0$$

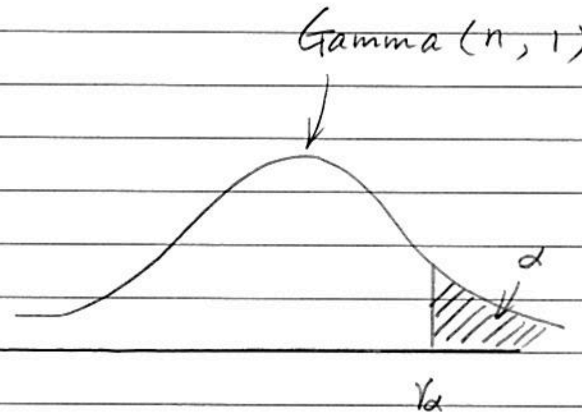
(2) Gamma $(\alpha=1, \theta) = \exp(\theta)$

$$f(x|\theta) = \frac{1}{\theta} \exp^{-\frac{x}{\theta}}$$

$x_1, \dots, x_n \overset{iid}{\sim} \exp(\theta)$

$\Sigma x_i \sim$ Gamma (n, θ)

(3) $\Sigma x_i / n \sim$ Gamma $(n, 1)$



Let γ_α be the upper α quantile of Gamma $(n, 1)$

$$E\left(\sum X_i / n\right) = \frac{1}{n} \sum E(X_i) = \frac{1}{n} \cdot n\theta = \theta$$

$\Rightarrow$ example 4.2.

$X_1, X_2, \cdots, X_n \overset{iid}{\sim} \exp(\theta)$

Let $X = (X_1, X_2, \cdots, X_n)$

$$f(x \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} \exp^{-x_i/\theta} = \frac{1}{\theta^n} \exp^{-\frac{\sum x_i}{\theta}}$$

We want to test

Test$_0$: $H_0 : \theta = \theta_0$ VS $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$

By N-P Lemma, the UMP test $\phi_0(x)$ is given

$$\phi_0(x) = \begin{cases} 1, & \text{if } f(x \mid \theta_1)/f(x \mid \theta_0) > K \\ 0 & \text{if } f(x \mid \theta_1)/f(x \mid \theta_0) < K \end{cases}$$

$$\frac{f(x \mid \theta_1)}{f(x \mid \theta_0)} = \frac{\theta_0^n}{\theta_1^n} e^{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \sum x_i}$$

$$\phi_0(x) = \begin{cases} 1 & \text{if } \sum x_i > t \\ 0 & \text{if } \sum x_i < t \end{cases}$$

what is t, we should determine t, given

α. $E(\phi_0(x) \mid \theta_0) = \alpha$

$$Pr\left(\sum X_i > t \mid \theta = \theta_0\right) = \alpha \Rightarrow Pr\left(\frac{\sum X_i}{\theta} > \frac{t}{\theta} \mid \theta = \theta_0\right) = \alpha$$

So, $\frac{t}{\theta_0} = r_{\alpha,n}$

the UMP is $\phi_0(x) = \begin{cases} 1 & \text{if } \Sigma x_i > \theta_0 r_{\alpha,n} \\ 0 & \text{if } \Sigma x_i \le \theta_0 r_{\alpha,n} \end{cases}$

By N-P Lemma

Note that:

(1) $E(\phi_0(x) \mid \theta = \theta_0) = \alpha$

(2) $E(\phi_0(x) \mid \theta = \theta_1) \geq E(\phi(x) \mid \theta = \theta_1)$ for all $\theta_1 > \theta_0$

and all $\phi(x)$ with $E(\phi(x) \mid \theta = \theta_0) \le \alpha$

(3) $\phi_0(x)$ doesn't depend on θ_1. $\Lambda(x)$ is monotone

with respect to Σx_i

(4) $E(\phi_0(x) \mid \theta) \le \alpha$ for all $\theta \le \theta_0$

$\Rightarrow$ proof: For $\theta \le \theta_0$ $E(\phi_0(x) \mid \theta)$

$= Pr\left(\sum_{i=1}^{n} x_i > \theta_0 \cdot r_{\alpha,n} \mid \theta\right) \times \underset{1}{\underset{\shortparallel}{\phi_0(x)}} + Pr(\Sigma x_i < \theta_0 \cdot r_{\alpha,n} \mid \theta) \cdot \underset{0}{\underset{\shortparallel}{\phi_0(x)}}$

$= Pr\left(\frac{\Sigma x_i}{\theta} > \frac{\theta_0}{\theta} \cdot r_{\alpha,n} \mid \theta\right)$ Let $Y = \frac{\Sigma x_i}{\theta}$

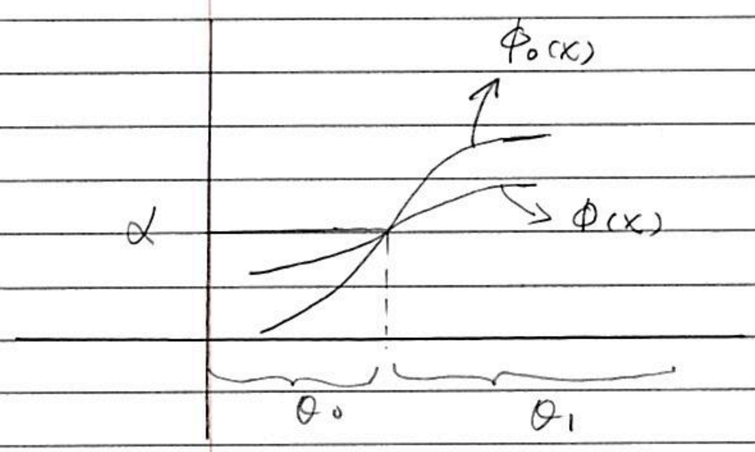
$= Pr\left(Y > \frac{\theta_0}{\theta} \cdot r_{\alpha,n} \mid \theta\right)$

where $Y \sim Gamma(n, 1)$

We can combining (1), (2), (3), (4) we see that

$\phi_0(x)$ is an UMP test with size α for

Test 1 $\theta \le \theta_0 \leftrightarrow \theta > \theta_0$


$\phi_0(x)$
$\phi(x)$
α
θ_0
θ_1

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⇒ Definition

A family of densities $\{ f(x|\theta) | \theta \in \Theta \}$ with scalar Q

This family has monotone likelihood ratio (MLR)

if $\Lambda(x) = f(x|\theta_1) / f(x|\theta_2)$ is an non-decreasing

function of a statistic $t(x)$ whenever $\theta_0 < \theta_1$

⇒ examples 1:

1) $x_1, \cdots x_n \overset{iid}{\sim} \exp(\lambda = \frac{1}{\theta})$; Let $X = (x_1, \cdots, x_n)$

$f(x|\theta) = \theta^{-n} \cdot e^{-\frac{1}{\theta} \Sigma x_i}$ For $\theta_0 < \theta_1$

$\Lambda(x) = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-(\frac{1}{\theta_1} - \frac{1}{\theta_0}) \Sigma x_i}$

Let $t(x) = \Sigma x_i$

$\Lambda(x)$ is ↗ with respect to $t(x)$

That is : $\Lambda(x) > K \iff \Sigma x_i > t_0$ (determined by θ_0)

⇒ example 2:

$x_1, \cdots x_n \overset{iid}{\sim} f(X|\theta) = C(\theta) \cdot h(x) \cdot e^{\theta \tau(x)}$

exponential family For $\theta_0 < \theta_1$

$$\Lambda(x) = \frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{C(\theta_1) \cdot h(x) e^{\theta_1 \tau(x)}}{C(\theta_0) h(x) e^{\theta_0 \tau(x)}}$$

$$= \frac{C(\theta_1)}{C(\theta_0)} \cdot e^{(\theta_1 - \theta_0) \tau(x)}$$

Let $t(x) = \tau(x)$, $\Lambda(x)$ ↗ function W.R.T $\tau(x)$

⇒ Theorem 4.2: suppose X has a distribution from a family with MLR with respect to $t(x)$; also the distribution of $t(x)$ is continuous

ⓐ The test:

$$\phi_0(x) = \begin{cases} 1 & \text{if } t(x) > t_0 \\ 0 & \text{if } t(x) \le t_0 \end{cases}$$

where t_0 is a value such that (s.t)

$$E(\phi_0(x) \mid \theta_0) = \alpha$$

[i.e, t_0 is determined by θ_0 and α]

$\phi_0(x)$ is the UMP test for

$$H_0: \theta \le \theta_0 \quad \text{Vs} \quad H_1: \theta > \theta_0$$

⇒ proof: suppose we want to test

$$H_0^*: \theta = \theta_0 \quad \text{Vs} \quad H_1^*: \theta = \theta_1 \quad (\theta_1 > \theta_0)$$

Let $\phi_0(x) = \begin{cases} 1 & \text{if } \Lambda(x) > K \\ 0 & \text{if } \Lambda(x) \le K \end{cases}$

Because $\Lambda(x)$ is monotone w.r.t $t(x)$

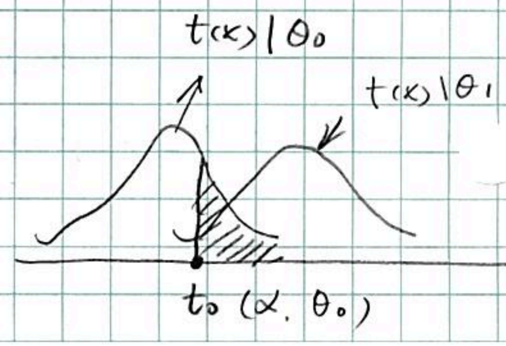
$$\phi_0(x) = \begin{cases} 1 & \text{if } t(x) > t_0 \\ 0 & \text{if } t(x) \le t_0 \end{cases}$$

We can determined t_0 by

$$E(\phi_0(x) \mid \theta_0) = \alpha$$

We note:

t_0 doesn't depend on θ_1



By N-P Lemma

$E\{\phi_0(x) \mid \theta_1\} \geq E\{\phi(x) \mid \theta_1\}$ for all ϕ with

$E(\phi(x) \mid \theta_0) = \alpha$

We note that: the above statement is true for

all $\theta_1 \in \Theta_1 = (\theta_1, +\infty)$

We remain to show that

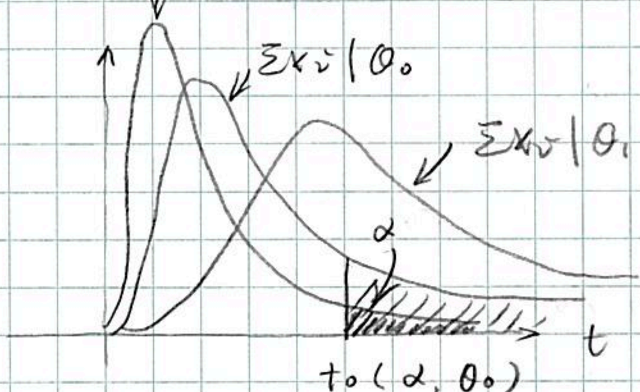
$E(\phi_0(x) \mid \theta) \leq \alpha$ for all $\theta \leq \theta_0$

$\Rightarrow$ Example: $X_1, \dots X_n \overset{iid}{\sim} \exp(\lambda = \frac{1}{\theta})$

$t(x) = \sum x_i$

scalar

$t(x) \mid \theta \sim \text{Gamma}(n, \beta = \theta)$, θ is a scala



$$\phi_0(x) = \begin{cases} 1 & \text{if } \Sigma x_i > t_0(\theta_0, \alpha) \\ 0 & \text{if } \Sigma x_i \leq t_0(\theta_0, \alpha) \end{cases}$$

By N-P Lemma: $E(\phi_0(x) \mid \theta_1) \geq E(\phi(x) \mid \theta_1)$

with $E(\phi(x) \mid \theta_0) = \alpha$

also for this example, we can see

$E\{\phi_0(x) \mid \theta_2\} < E(\phi_0(x) \mid \theta_0) = \alpha$

for all $\theta_2 \leq \theta_0$ [this statement is true, general

as long as $A(x)$ has MLR. So ϕ is UMP for $\theta \leq \theta_0$ vs

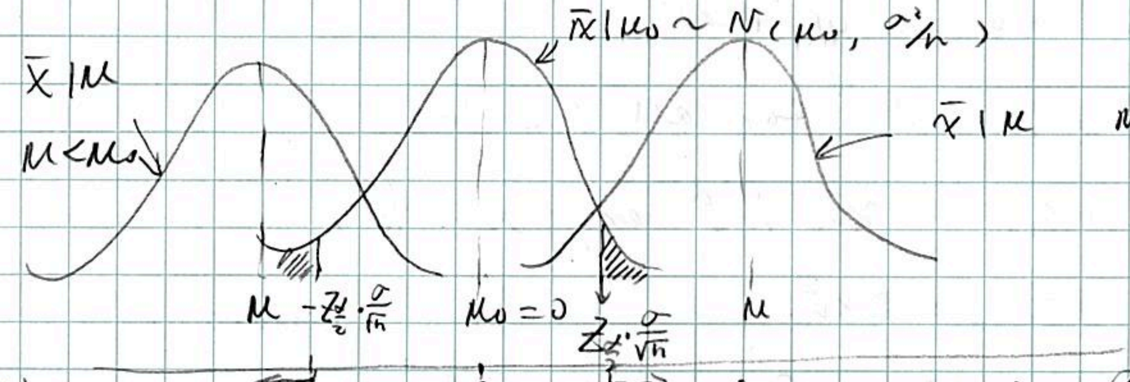
$\Rightarrow$ There is no UMP test for

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$$

Example: $X_1, \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 is known.

$\Lambda(x)$ is monotone W.R.T $t(x) = \sum x_i / n$

$H_0 : \mu = \mu_0 = 0 \Leftrightarrow H_1 : \mu \neq \mu_0 = 0$



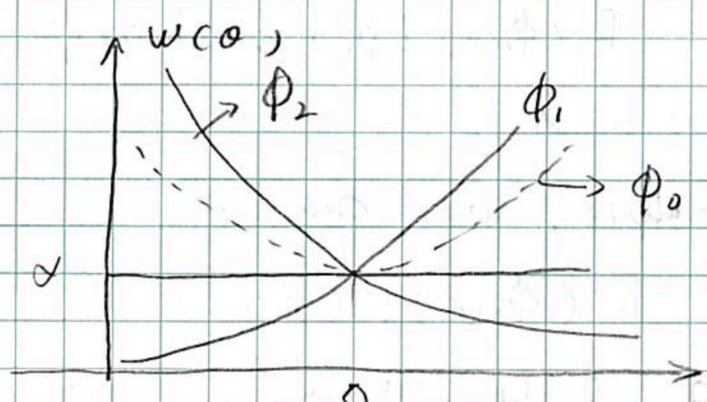
criteria region

$$\phi_0(x) = \begin{cases} 1 & \text{if } |\bar{x}| > Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \\ 0 & \text{o/w} \end{cases}$$

$$\phi_1(x) = \begin{cases} 1 & \text{if } |\bar{x}| > Z_{\alpha} \cdot \frac{\sigma}{\sqrt{n}} \\ 0 & \text{o/w} \end{cases}$$

$$\phi_2(x) = \begin{cases} 1 & \text{if } \bar{x} < Z_{\alpha} \cdot \frac{\sigma}{\sqrt{n}} \\ 0 & \text{o/w} \end{cases}$$

we can look power function:



when $\mu < 0$ ϕ_2 is Most power

$\mu > 0$ ϕ_1 is Most power

No ϕ_0 s.t $W(\mu)$ is Most power for $\mu \in \Theta_1$

$W(\mu) = E(\phi(x) | \mu)$

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⇒ Regular class

Likelihood function:

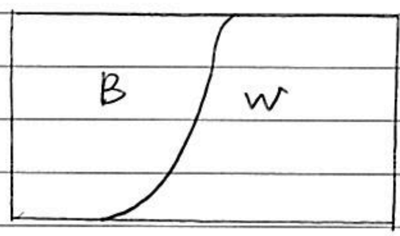
Mode : $x \mid \theta \sim f(x \mid \theta)$

$$\theta \to 1, \quad \theta \to 2, \quad \theta \to 3$$

$f(x \mid \theta)$

$L(\theta \mid x) = f(x \mid \theta)$ θ is variable and x is fixed

⇒ Example :



θ = propotion of black balls

$$x_i = \begin{cases} 1 & \text{if } i^{th} \text{ is black} \\ 0 & \text{o/w} \end{cases}$$

$x_1, \cdots, x_n \mid \theta \overset{iid}{\sim} \text{Bern}(\theta)$

$f(x_1, \cdots, x_n \mid \theta) = \theta^{n_1}(1-\theta)^{n-n_1}$

n_1 = # of black $= \sum_{i=1}^{n} x_i$

$L(\theta \mid x_1, \cdots x_n) = \theta^{n_1}(1-\theta)^{n-n_1}$

Data example $n_1 = 30 = \sum_{i=1}^{n} x_i$, $n = 100$

$L(\theta \mid x_1, \cdots, x_n) = \theta^{30} \cdot (1-\theta)^{70}$