

chapter: 7.2. Condition Inference

=> Example: δ indicator 2 means tools

$\delta=1 \rightarrow$ the tool is more precise (e.g. using $n_1=90$ sample)

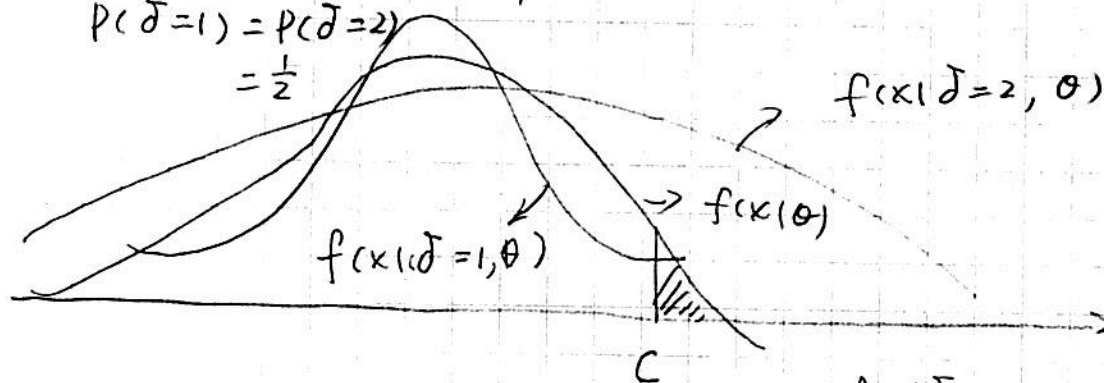
$\delta=2 \rightarrow$ the tool is less precise (e.g. using $n_2=10$ sample)

Model:

$$X | \delta=1 \sim N(0, \sigma_1^2), \sigma_1=1$$

$$X | \delta=2 \sim N(0, \sigma_2^2), \sigma_2=3$$

$$P(\delta=1) = P(\delta=2) = \frac{1}{2}$$



$$H_0: \theta=0 \quad \text{vs} \quad H_1: \theta \neq 0$$

Test statistic: X

Rejection Region: $\{x > c\}$

$$f(x|\delta=1, \theta) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\theta)^2}{2\sigma_1^2}}$$

$$f(x|\delta=2, \theta) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\theta)^2}{2\sigma_2^2}}$$

$$f(x|\theta) = \frac{1}{2} f(x|\delta=1, \theta) + \frac{1}{2} f(x|\delta=2, \theta)$$

We will determine c by $\Pr(X > c | \theta=0) = 0.05$

=> Procedure 1: using $f(x|\theta=0)$

We will solve:

$$\Pr(X > c | \theta=0) = 0.05$$

$$\frac{1}{2} \Pr(N(0, \sigma_1^2) > c) + \frac{1}{2} \Pr(N(0, \sigma_2^2) > c) = 0.05$$

=> procedure 2: observed; x and δ

using $f(x|\delta, \theta)$ to determine C

If $\delta=1$, $X|\theta=0 \sim N(0, \sigma_1^2)$

$$P(X > C_1 | \theta=0) = 0.05$$

$$\Rightarrow C_1 = 0 + Z_{\alpha} \times 1 = 1.645$$

which distribution
we used.

If $\delta=2$ $X|\theta=0 \sim N(0, \sigma_2^2)$

$$P(X > C_2 | \theta=0, \delta=2) = 0.05$$

$$\Rightarrow C_2 = 0 + 3 \times Z_{\alpha} = 3 \times 1.645 = 4.92$$

Rejection Region: $X > 0 + Z_{\alpha} \cdot \sigma_{\delta}$

what's the problems for procedure 1:

1) ignore δ

2) The justification for C is based on $P(X > C) = \alpha$
without knowing δ .

The α is probability averaging all data sets.

3) C is determined before we observe any data
value of X and δ

\Rightarrow Definition: $\theta = (\varphi, \lambda)$

φ is of our interest

λ is nuisance parameter

$\Gamma = (S, C)$ is MSS for (φ, λ)

If (a) the distribution of C doesn't depend on φ but on λ

(b) the distribution of $S|C$ depends on φ but not on λ .

Then, we say

C is an ancillary statistic for φ and S is conditionally sufficient for φ given C ,

\Rightarrow example: θ ($\lambda = \phi$) the distribution of J doesn't depend on θ .

The distribution $X|J$ depend on θ ,

$J \rightarrow$ called ancillary (not useless) statistic for θ

\Rightarrow Conditional principle:

our inference should be conditional on all ancillary statistic.

\Rightarrow example 2:

$$X_1, \dots, X_n | \theta \sim \text{unif}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$$

$$f(x|\theta) = I(X_{(1)} - \frac{1}{2} < \theta < X_{(n)} + \frac{1}{2})$$

So, the $S = (X_{(1)}, X_{(n)})$ is MSS.

$$T = \frac{X_{(1)} + X_{(n)}}{2} \leftarrow \text{Not sufficient}$$

$$W = X_{(n)} - X_{(1)} \leftarrow \text{ancillary}$$

$f_T(t)$ depend on θ ; $f_W(w)$ does not depend on θ

(T, W) is MSS.

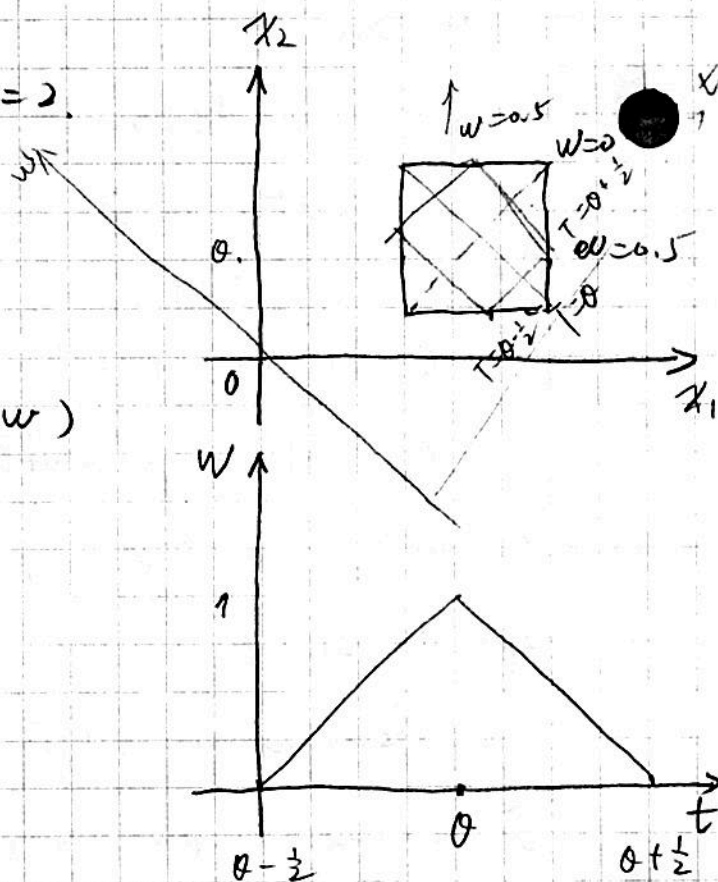
\Rightarrow look at a special case $n=2$.

$$\begin{cases} T = \frac{x_1 + x_2}{2} \\ W = |x_1 - x_2| \end{cases}$$

the joint distribution of (T, W)

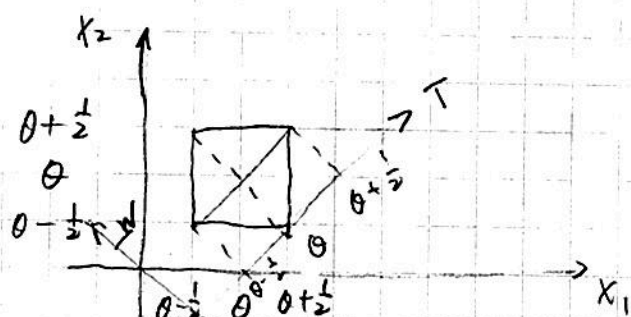
$(T, W) \sim \text{Unif}(A)$

A is the triangle.

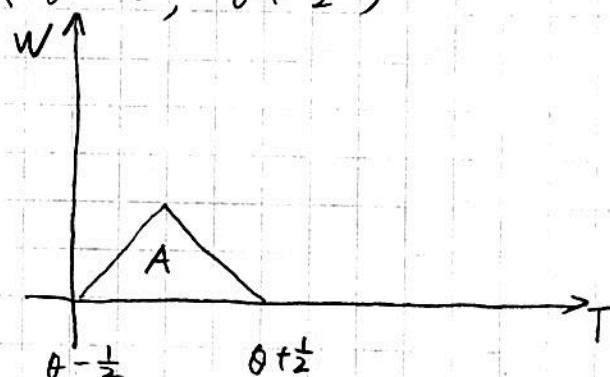


<< Mar 6, 2015 >> STAT 846:

\Rightarrow example: $X_1, \dots, X_n \stackrel{iid}{\sim} \text{unif}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$



$$T = \frac{X_{(1)} + X_{(n)}}{2} = \frac{X_1 + X_2}{2}$$



$$(T, W) \sim \text{unif}(A)$$

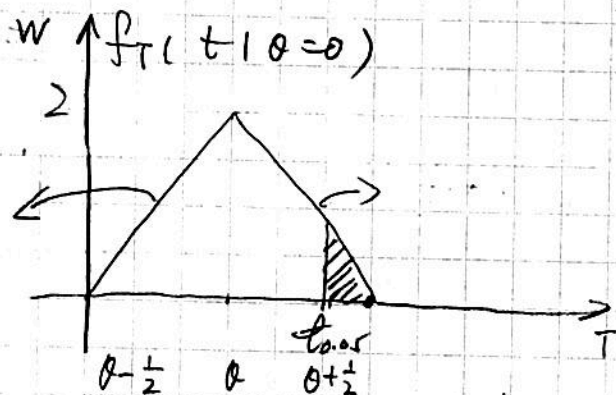
$$W = X_{(2)} - X_{(1)} = |X_{(1)} - X_{(2)}|$$

\Rightarrow hypothesis test:

$$H_0: \theta \leq 0 \text{ against } H_1: \theta > 0$$

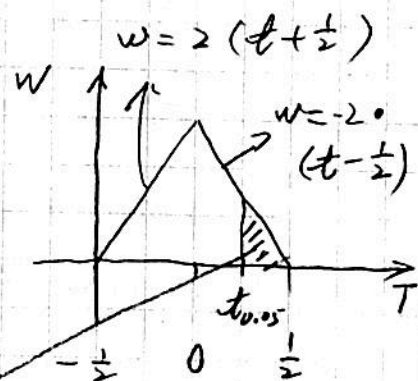
procedure 1: (unconditional)

using marginal distribution of T



Rejection Region: $\{T > t\}$

$$P(T > t_{0.05}) = 0.05 \quad (\alpha = 0.05)$$



where, $t_{0.05} = 0.342$

$$\left(\frac{1}{2} - t_{0.05}\right) \times \left[-2\left(t_{0.05} - \frac{1}{2}\right)\right] \times \frac{1}{2} = 0.025$$

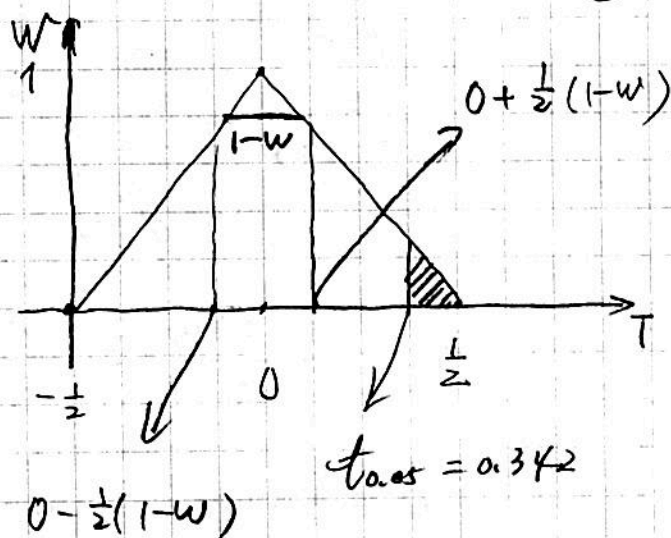
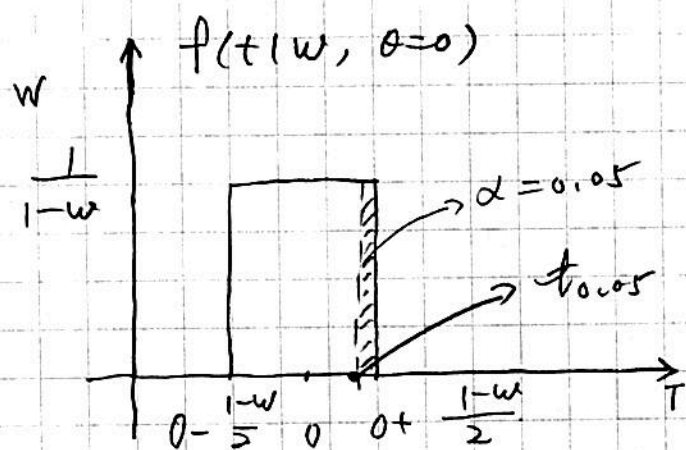
procedure 2: (conditional inference)

using the distribution $f(t|w, \theta)$

$$T|w, \theta=0 \sim \text{unif}\left(\frac{w}{2} - \frac{1}{2}, \frac{1}{2} - \frac{w}{2}\right)$$

$$w = 2\left(t + \frac{1}{2}\right) \Rightarrow t = \frac{w}{2} - \frac{1}{2}$$

$$w = -2\left(t - \frac{1}{2}\right) \Rightarrow t = \frac{1}{2} - \frac{w}{2}$$



$$\left(\frac{1-w}{2} - t_{0.05}^*\right) = 0.05 \times (1-w)$$

$$\frac{1-w}{2} - 0.05(1-w) = t_{0.05}^*$$

$$t_{0.05}^* = 0.45(1-w)$$

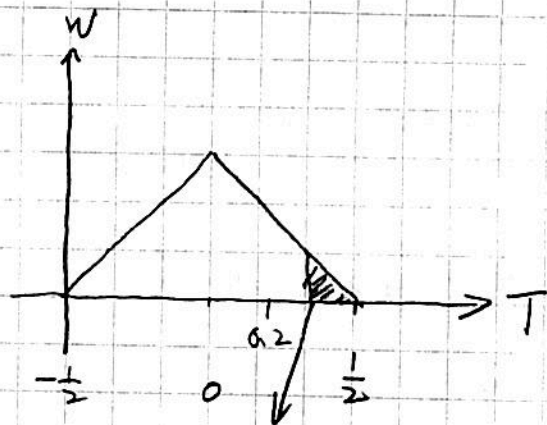
we should reject when:

Rejection Region is: $t > 0.45(1-w) = t_{0.05}^*$

Looking at specific data set: $\begin{cases} t=0.2 \\ w=0.7 \end{cases}$

$$\begin{cases} t = 0.2 \\ w = 0.7 \end{cases}$$

By the procedure 1
No reject

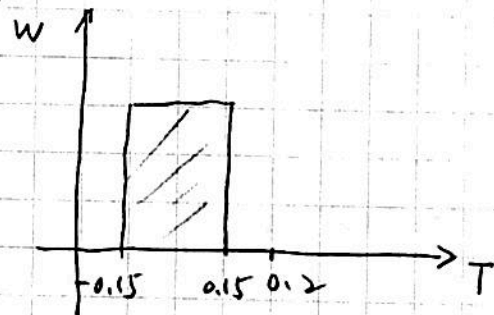


⇒ Procedure 2:

$$T|W=0.7, \theta=0 \sim \text{Unif}\left(0 - \frac{0.3}{2}, 0 + \frac{0.3}{2}\right) = \text{Unif}(-0.15, 0.15)$$

$$t_{0.05} = 0.342$$

By procedure 2, we will reject H_0 .



$$t_{0.05}^* = 0.45 \times 0.3 = 0.135$$

the procedure 2 is better than procedure 1.

⇒ the procedure 3 (Bayesian)

$\Pr(H_0 | X) < \alpha$, we will reject H_0 (in our case $\alpha = \frac{1}{2}$)

We will compute $\Pr(\theta \leq 0 | X_1, X_2) = ?$

$$f(X_1, X_2 | \theta) = I\left(X_{(2)} - \frac{1}{2} < \theta < X_{(1)} + \frac{1}{2}\right)$$

$$\theta \sim I(-M < \theta < M)$$

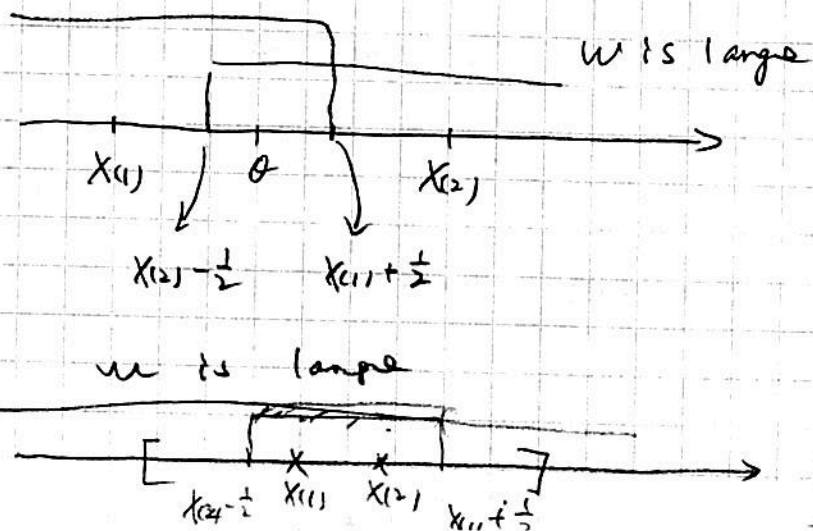
$$\theta | X_1, X_2 \sim \text{Unif}\left(X_{(2)} - \frac{1}{2}, X_{(1)} + \frac{1}{2}\right)$$

$$\text{range}(\theta | X_1, X_2) = 1 - w$$

another way

$$\theta | X_1, X_2 \sim \text{Unif}\left(t - \frac{1-w}{2}, t + \frac{1-w}{2}\right)$$

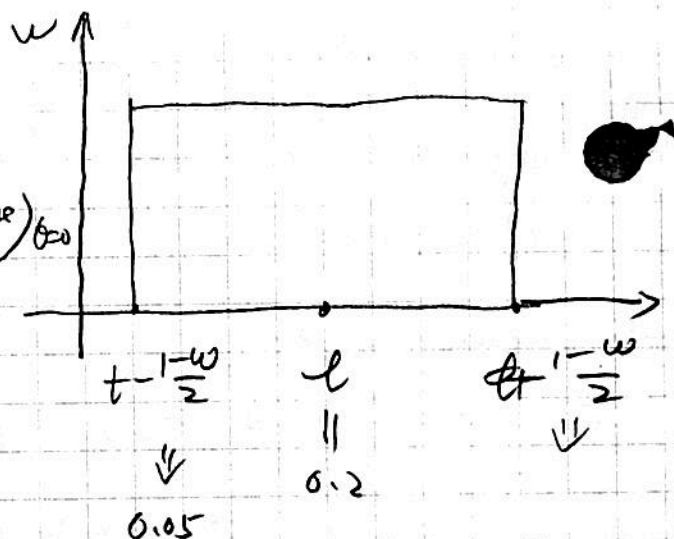
$$t + \frac{1-w}{2}$$



Given $\begin{cases} \ell = 0.2 \\ -w = 0.7 \end{cases}$

$$Pr(\theta < 0 | x_1, x_2) = 0 \text{ (it is impossible)}_{\theta=0}$$

We reject H_0 .



Review:

For test 2:

Concept:

Stat

Skill

Chapter 4: \checkmark size, \checkmark power, \checkmark UMP, \checkmark NP lemma, \checkmark MLR.

Find UMP

in MLR family

Chapter 5: \checkmark exp. family.

\checkmark natural statistic,
 \checkmark natural parameter

verify
exp family

Chapter 6: \checkmark M.S.S.

\checkmark complete sufficient S.
 \checkmark R-B theorem

Find UMVUE
with R-B theorem

Chapter 8: likelihood function

$E(d_i(x) | T)$

① Fisher information
 $I(\theta)$

② CRLB

③ Asymptotic

distribution of Once

including threshold by Wilk's theorem

④ Likelihood Ratio test