

# **Statistical Inference**

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# Preface

This is a concise course about statistical inference.

## Key Features

- Use simulation and graphs to illustrate the concepts in probability theory and statistical inference
- Rigorous derivation of the key theorems in statistical inference

## Audience

This course requires a strong command of multivariate calculus, alongside a rigorous foundation in intermediate probability theory including asymptotic theory for probability. Students should also possess prior exposure to applied statistical methods and familiar with basic statistical concepts such as p-value and confidence interval.

# 1 Introduction to Statistical Inference

## 1.1 Population Model (Data Model)

We begin with observations (units)  $X_1, X_2, \dots, X_n$ . These may be vectors. We regard these observations as a realization of random variables.

**Definition 1.1** (Population Distribution). We assume that  $X_1, X_2, \dots, X_n \sim f(x)$ . The function  $f(x)$  is called the **population distribution**.

### Assumptions and Scope

For simplicity, we often assume the data are Independent and Identically Distributed (i.i.d.). The assumption of identical distribution can be relaxed to regression settings in which the distributions of  $x_i$ 's are independent but dependent on covariate  $x_i$ .

In **Parametric Statistics**, we assume  $f(x)$  is of a known analytic form but involves unknown parameters.

**Example 1.1** (Parametric Model: Normal). Consider the Normal distribution:

$$f(x; \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (1.1)$$

Here, the parameter space is  $\Theta = \{(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma \in [0, +\infty)\}$ . The goal is to learn aspects of the unknown  $\theta$  from observations  $X_1, \dots, X_n$ .

**Example 1.2** (Parametric Model: Bernoulli). Consider a sequence of binary outcomes (e.g., Success/Failure) where each  $X_i \in \{0, 1\}$ . We assume  $X_i \sim \text{Bernoulli}(\theta)$ . The probability mass function is:

$$f(x; \theta) = \theta^x (1 - \theta)^{1-x} \quad (1.2)$$

Here, the parameter space is  $\Theta = [0, 1]$ , where  $\theta$  represents the probability of success.

## 1.2 Probabilistic Model vs. Statistical Inference

There is a fundamental distinction between probability and statistics regarding the parameter  $\theta$ . We can visualize this using a “shooting target” analogy:

- **$\theta$  (The Center):** The true, unknown bullseye location.
- **$x$  (The Shots):** The observed holes on the target board.
- **Probability (Deductive):** The center  $\theta$  is **known**. We predict where the shots  $x$  will land.
- **Statistics (Inductive):** The shots  $x$  are **observed** on the board. The center  $\theta$  is unknown. We hypothesize different potential centers to see which one best explains the shots.

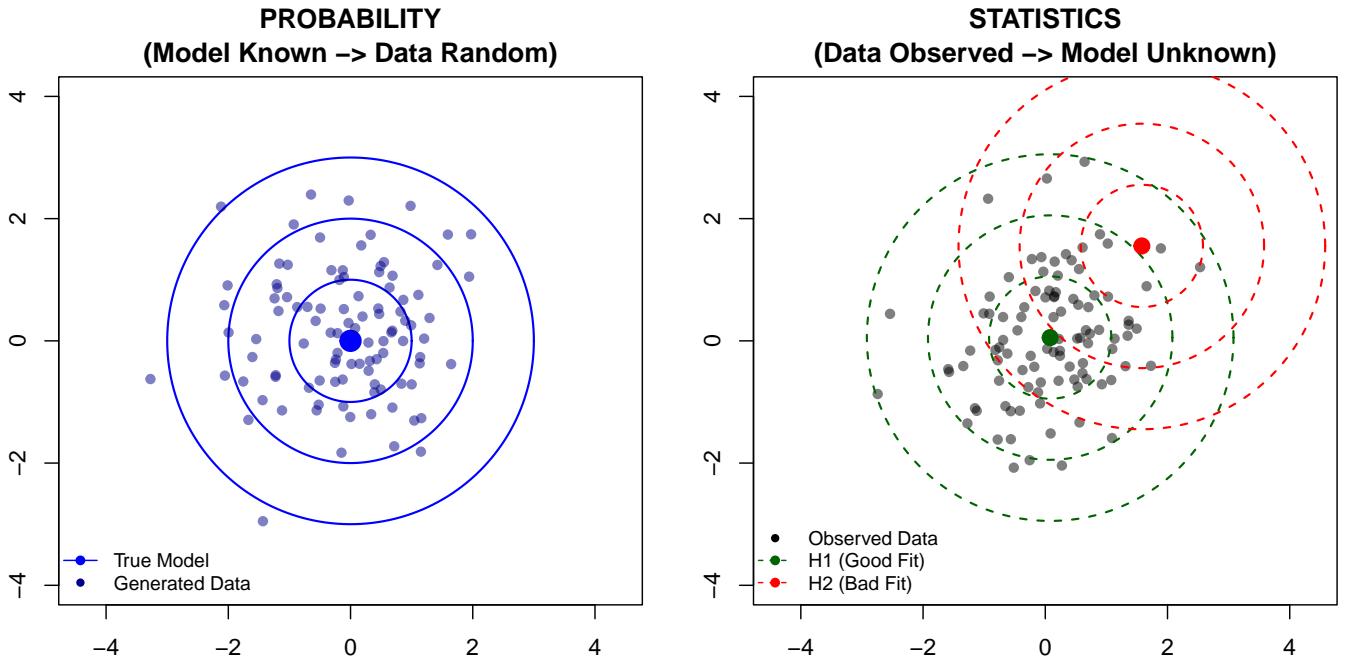


Figure 1.1: Probability vs Statistics. Left: Probability—The model is fixed (Blue center/contours), generating random data. Right: Statistics—Data is fixed (Black points); we test two hypothesized models: H1 (Green) centered at the sample mean (Good Fit) and H2 (Red) shifted by (1.5, 1.5) (Bad Fit).

### 1.3 A Motivating Example: The Lady Tasting Tea

To illustrate the concepts of statistical inference, we consider the famous experiment described by R.A. Fisher.

A lady claims she can distinguish whether milk was poured into the cup before or after the tea. To test this claim, we prepare  $n$  cups of tea.

- **Random Variable:** Let  $X_i = 1$  if she identifies the cup correctly, and 0 otherwise.
- **Parameter:** Let  $\theta$  be the probability that she correctly identifies a cup.
- **The Data:** Suppose we observe that she identifies 70% of cups correctly ( $\bar{x} = 0.7$ ), which is a summary of the observed vector of  $x_i$ , for example,

$$x = (0, 1, 1, 0, 1, 1, 0, 1, 1, 1) \quad (1.3)$$

### 1.3.1 Small Sample (n=10)

We observe **7 out of 10** correct ( $k = 7$ ).

$$\bar{x} = 0.7 \quad (1.4)$$

### 1.3.2 Large Sample (n=40)

We observe **28 out of 40** correct ( $k = 28$ ).

$$\bar{x} = 0.7 \quad (1.5)$$

## 1.4 Questions to Answer in Statistical Inference

Using this example, we identify the four main types of statistical inference.

### Point Estimation

We want to use a single number to capture the parameter:  $\hat{\theta} = \theta(X_1, \dots, X_n)$ .

- *Tea Example:* Our best guess for her success rate is  $\hat{\theta} = 0.7$ .

### Hypothesis Testing

We want to test a theory about the parameter:  $H_0$  vs  $H_1$ .

- *Tea Example:* Is she just guessing? We test  $H_0 : \theta = 0.5$  vs  $H_1 : \theta > 0.5$ .

### Model Assessment

We want to test a theory about the parameter:  $H_0$  vs  $H_1$ .

- *Example:* Can we use a reduced model? What level of complexity of  $f(x; \theta)$  is necessary?

### Interval Estimation

We want to construct an interval likely to contain the parameter:  $\theta \in (L, U)$ .

- *Tea Example:* We might say her true skill  $\theta$  is likely between 0.45 and 0.95.

### Prediction

We want to predict a new observation  $Y_{n+1}$  given previous data.

- *Tea Example:* If we give her an  $(n + 1)$ -th cup, what is the probability she identifies it correctly?

## 1.5 The Likelihood Function

The bridge between probability and statistics is the Likelihood Function.

**Definition 1.2** (Likelihood Function). Let  $f(x_1, \dots, x_n; \theta)$  be the joint probability density (or mass) function of the data given the parameter  $\theta$ . When we view this function as a function of  $\theta$  for fixed observed data  $x_1, \dots, x_n$ , we call it the **likelihood function**, denoted  $L(\theta)$ .

$$L(\theta) = f(x_1, \dots, x_n; \theta) \quad (1.6)$$

### Example: Lady Tasting Tea

For our Tea Tasting data, the likelihood is proportional to the Binomial probability:

$$L(\theta) = \binom{n}{k} \theta^k (1 - \theta)^{n-k} \quad (1.7)$$

#### 1.5.1 n=10 (k=7)

Here,  $L(\theta) = \binom{10}{7} \theta^7 (1 - \theta)^3$ .

$\theta$	Calculation $\binom{10}{7} \theta^7 (1 - \theta)^3$	$L(\theta)$
0.0	$120 \times 0^7 \times 1^3$	0.0000
0.2	$120 \times 0.2^7 \times 0.8^3$	0.0008
0.4	$120 \times 0.4^7 \times 0.6^3$	0.0425
0.6	$120 \times 0.6^7 \times 0.4^3$	0.2150
0.7	$120 \times 0.7^7 \times 0.3^3$	<b>0.2668 (Max)</b>
0.8	$120 \times 0.8^7 \times 0.2^3$	0.2013
1.0	$120 \times 1^7 \times 0^3$	0.0000

#### 1.5.2 n=40 (k=28)

Here,  $L(\theta) = \binom{40}{28} \theta^{28} (1 - \theta)^{12}$ . Notice how the likelihood becomes **narrower** (more peaked) with more data, even though the peak remains at 0.7.

$\theta$	Calculation $\binom{40}{28} \theta^{28} (1 - \theta)^{12}$	$L(\theta)$
0.0	$5.5868535 \times 10^9 \times 0^{28} \times 1^{12}$	0.0000
0.2	$5.5868535 \times 10^9 \times 0.2^{28} \times 0.8^{12}$	0.0000
0.4	$5.5868535 \times 10^9 \times 0.4^{28} \times 0.6^{12}$	0.0001
0.6	$5.5868535 \times 10^9 \times 0.6^{28} \times 0.4^{12}$	0.0576

$\theta$	Calculation $\binom{40}{28}\theta^{28}(1-\theta)^{12}$	$L(\theta)$
0.7	$5.5868535 \times 10^9 \times 0.7^{28} \times 0.3^{12}$	<b>0.1366 (Max)</b>
0.8	$5.5868535 \times 10^9 \times 0.8^{28} \times 0.2^{12}$	0.0443
1.0	$5.5868535 \times 10^9 \times 1^{28} \times 0^{12}$	0.0000

## Questions

- Is an estimator like  $\bar{x}$ , which is called Maximum Likelihood Estimator (MLE), a good estimator in general?
- What do you discover from actually observing the two likelihood functions of different sample size  $n$ ?
- Is the likelihood function central to all inference problems?
- What are the essential ‘parameters’ of the likelihood function?

There are two primary frameworks for “How” to perform these inferences.

## 1.6 Frequentist Inference

- **Concept:**  $\theta$  is unknown but fixed; Data  $X$  is random.
- **Sampling Distribution:** We analyze how  $\hat{\theta}$  behaves under hypothetical repeated sampling.

### Example: Frequentist Test of Lady Tasting Tea

We test  $H_0 : \theta = 0.5$  (Guessing) vs  $H_1 : \theta > 0.5$  (Skill). We analyze the behavior of  $\bar{X}$  assuming  $H_0$  is true. The rejection region (one-sided) is shaded red.

#### 1.6.1 n=10 (k=7)

We calculate the P-value: Probability of observing  $\geq 7$  correct out of 10, assuming  $\theta = 0.5$ .

#### 1.6.2 n=40 (k=28)

We calculate the P-value: Probability of observing  $\geq 28$  correct out of 40. With a larger sample size, the same proportion (0.7) provides **stronger evidence** against the null.

Likelihood  $L(\theta)$  for  $n = 10$ ,  $k = 7$

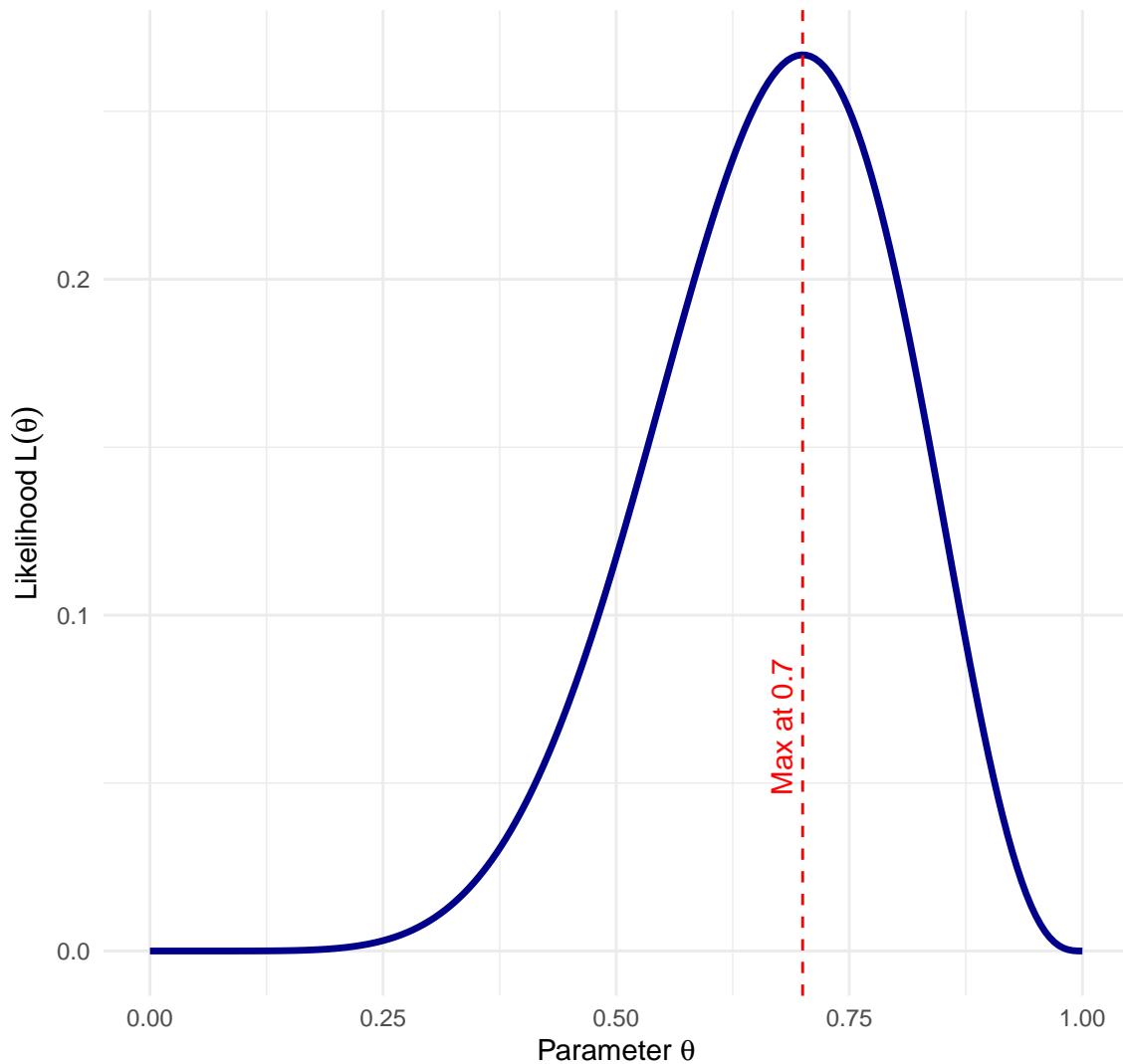


Figure 1.2: Likelihood Function ( $n= 10$  )

Likelihood  $L(\theta)$  for  $n = 40$ ,  $k = 28$

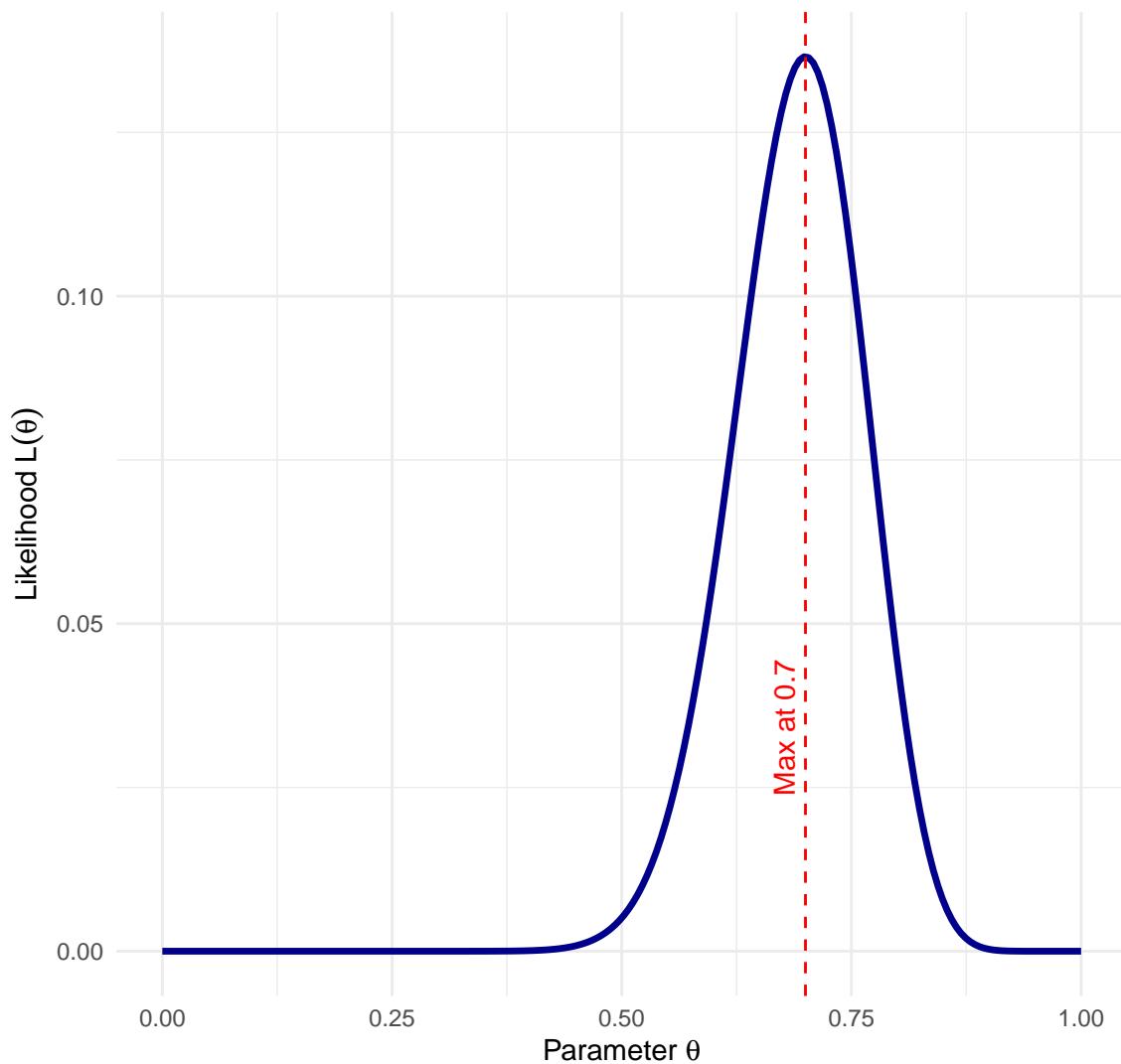


Figure 1.3: Likelihood Function (n= 40 )

### Sampling Distribution ( $n = 10$ )

Testing  $H_0: \theta = 0.5$  vs  $H_1: \theta > 0.5$

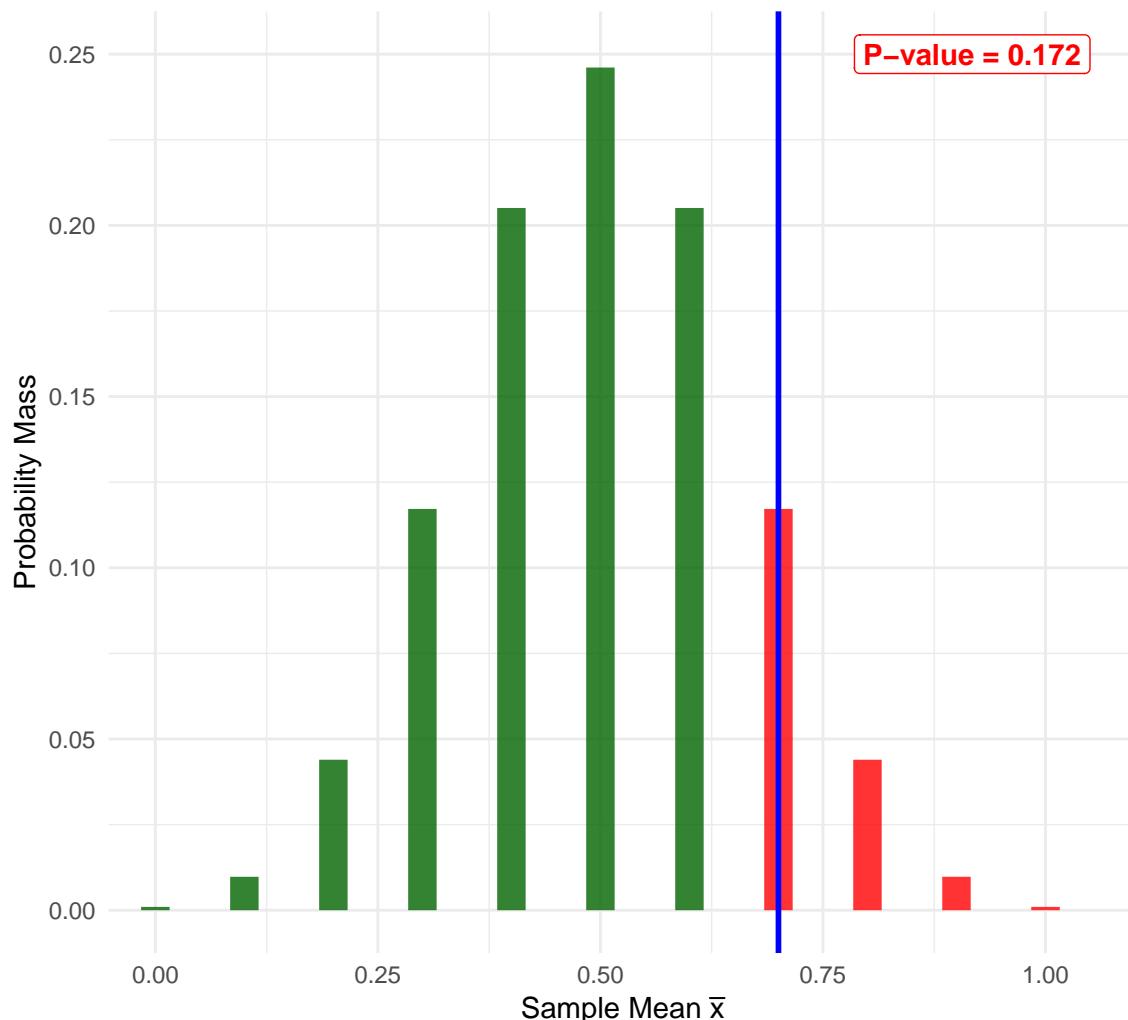


Figure 1.4: Sampling Distribution ( $n= 10$  )

Sampling Distribution ( $n = 40$ )

Testing  $H_0: \theta = 0.5$  vs  $H_1: \theta > 0.5$

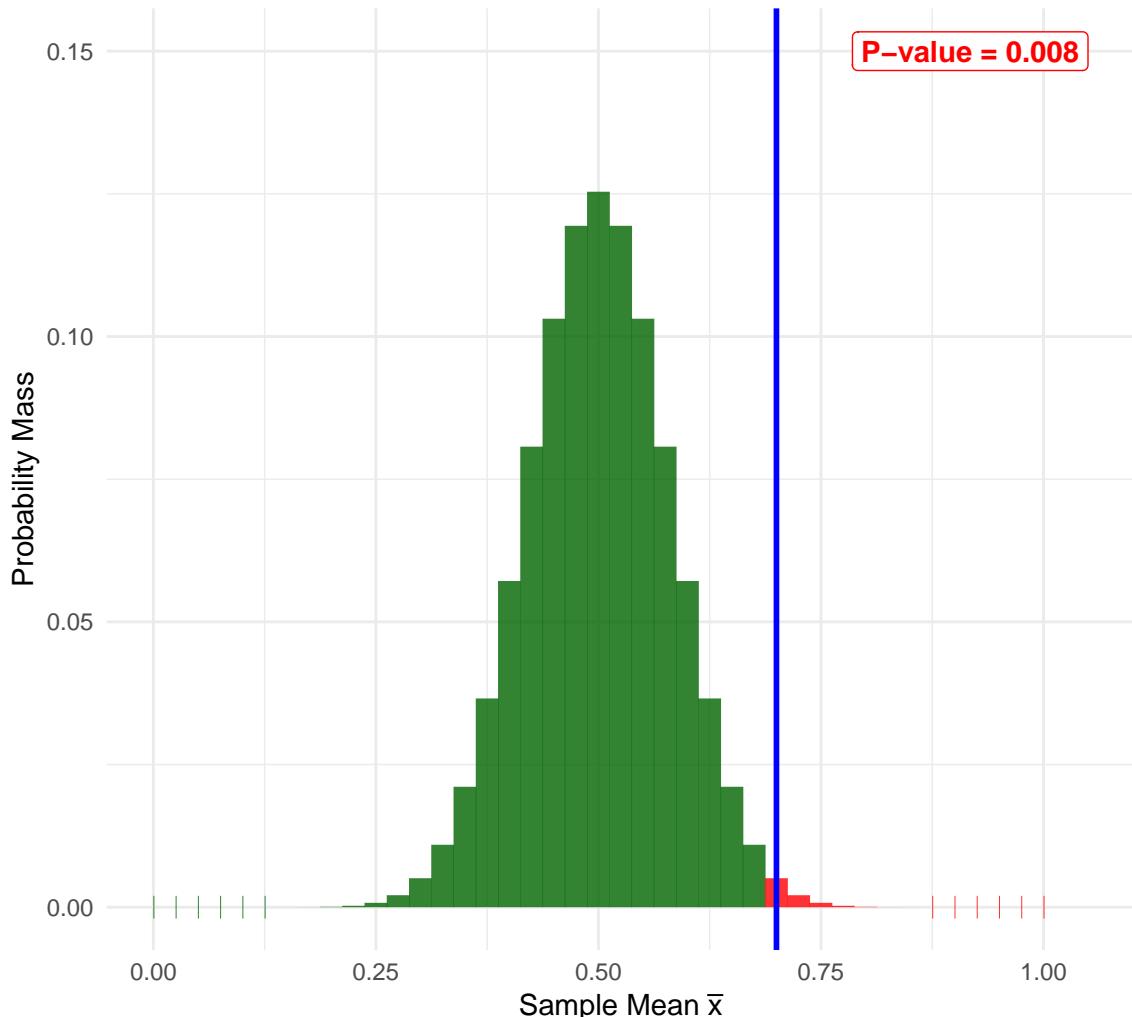


Figure 1.5: Sampling Distribution ( $n= 40$  )

### 1.6.3 Questions to Answer

In this course, we will answer several challenging questions related to general parametric models in the Frequentist framework.

- **MLE:** Can we use the Maximum Likelihood Estimator (MLE)  $\hat{\theta}$  for general models even no closed-form solution exists? Is MLE a good method?
- **Sampling Distributions:** What is the distribution of  $\hat{\theta}_{\text{MLE}}$ ? What's its mean and standard deviation?
- **Confidence Intervals:** How to construct CI with  $\hat{\theta}$ ?
- **Hypothesis Testing:** How do we derive powerful tests from the likelihood function? How to assess goodness-of-fit of parametric models with their likelihood information?

## 1.7 Bayesian Inference

- **Concept:**  $\theta$  is regarded as a random variable.
- **Posterior:** Posterior  $\propto$  Likelihood  $\times$  Prior.

### Example: Bayesian Analysis of the Lady Tasting Tea

Prior: Beta(1, 1) (Uniform).

#### 1.7.1 n=10 (k=7)

Posterior: Beta( $1 + 7, 1 + 3$ ) = Beta(8, 4)

#### 1.7.2 n=40 (k=28)

Posterior: Beta( $1 + 28, 1 + 12$ ) = Beta(29, 13).

### 1.7.3 Questions to Answer

We will also tackle the specific technical challenges involved in Bayesian analysis.

- **Posterior Derivation:** How do we derive the posterior distribution  $f(\theta|x)$  for various likelihoods and priors?
- **Comparing with Other methods:** Are Bayesian methods good or not for general inference?
- **Computation:** When the posterior cannot be derived analytically, how do we use computational techniques like Markov Chain Monte Carlo (MCMC) to sample from it?
- **Summarization:** How do we construct Credible Intervals (e.g., Highest Posterior Density regions) from posterior samples?
- **Prediction:** How do we solve the integral required to compute the posterior predictive distribution for future data?
- **Prior:** How to choose our prior? What's its effect on our inference?

## Bayesian Update (n = 10)

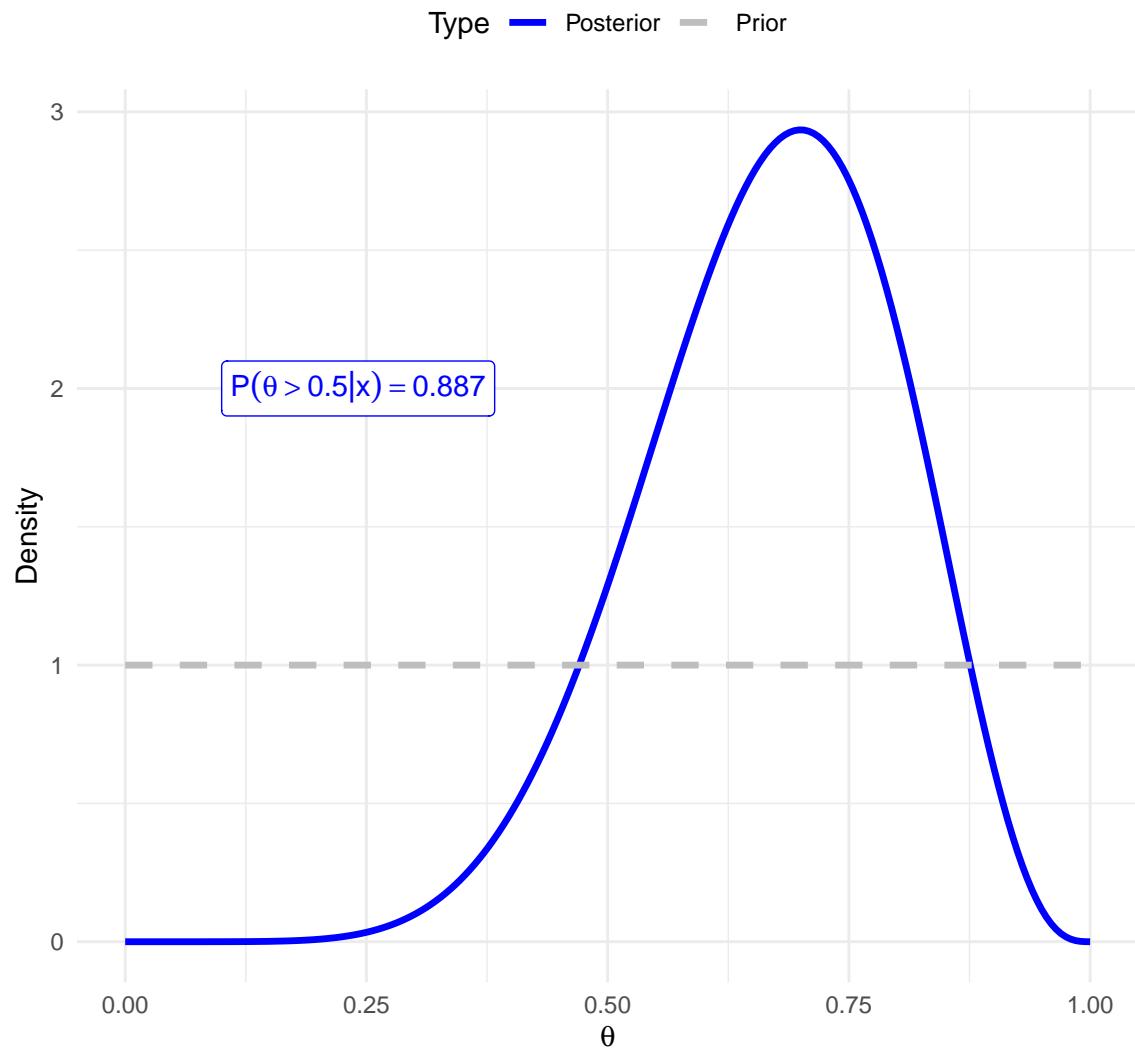


Figure 1.6: Bayesian Update (n= 10 )

## Bayesian Update (n = 40)

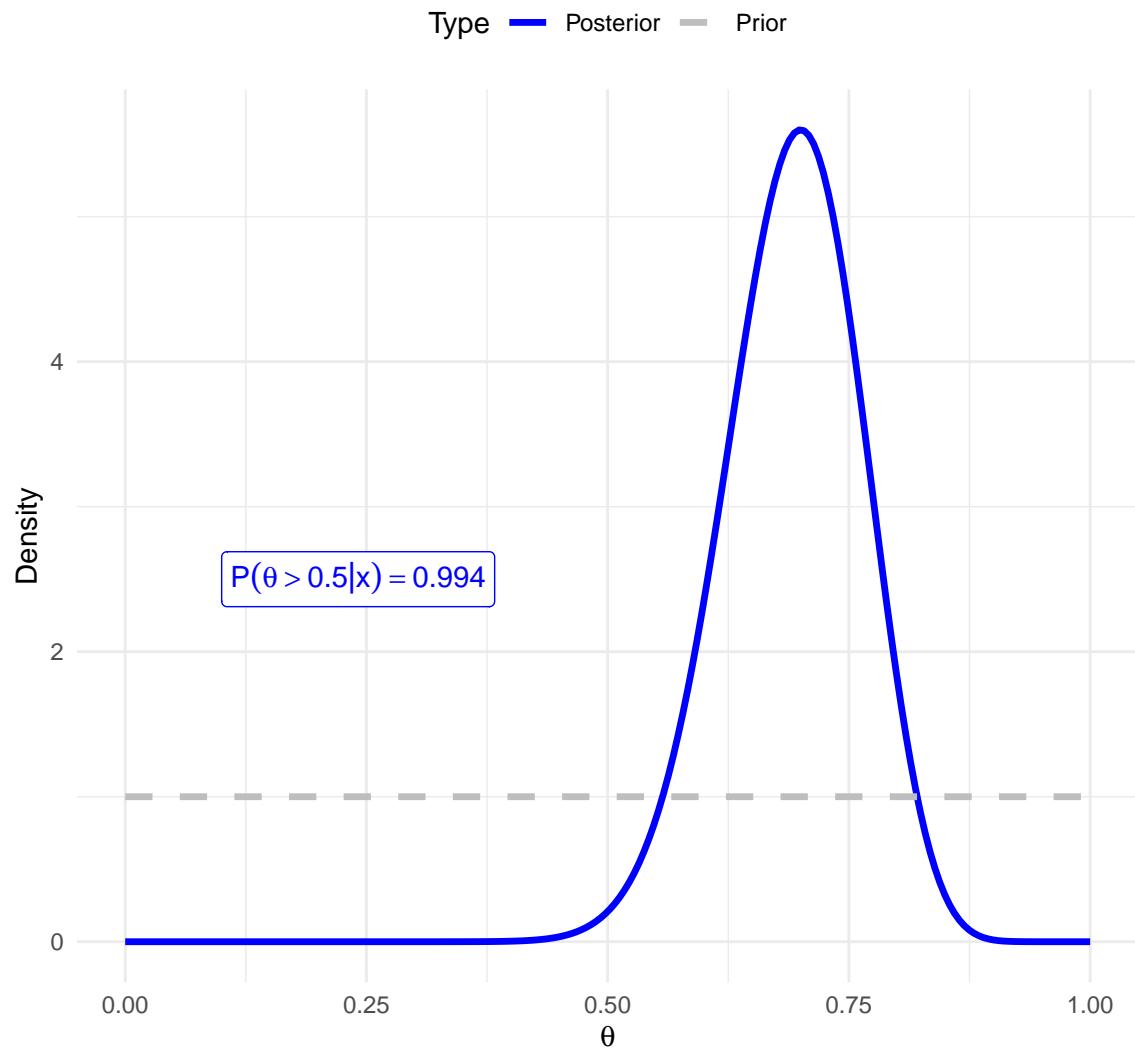


Figure 1.7: Bayesian Update (n= 40 )

- **Model Comparison and Assessment:** How to assess a Bayesian model?

# 2 Decision Theory

## 2.1 Formulation of Decision Theory

In decision theory, we formalize the process of making decisions under uncertainty using the following components:

1. **Parameter Space ( $\Theta$ ):** The set of all possible states of nature or values that the parameter can take.  $\theta \in \Theta$  (e.g., mean, variance).
2. **Sample Space ( $\mathcal{X}$ ):** The space where the data  $X$  lies. Example:  $X = (X_1, X_2, \dots, X_n)$  where  $X_i \in \mathbb{R}$ . So  $\mathcal{X} \in \mathbb{R}^n$ .
3. **Family of Probability Distributions:**  $\{P_\theta(x) : \theta \in \Theta\}$ . This describes how likely we are to see the data  $X$  given a specific parameter  $\theta$ .
  - If  $X$  is continuous:  $P_\theta(x) = f(x, \theta)$  (Probability Density Function).
  - If  $X$  is discrete:  $P_\theta(x) = f(x, \theta)$  (Probability Mass Function).
4. **Action Space ( $\mathcal{A}$ ):** The set of all actions or decisions available to the experimenter.
5. **Loss Function:**  $L : \Theta \times \mathcal{A} \rightarrow \mathbb{R}$ .  $L(\theta, a)$  specifies the loss incurred if the true parameter is  $\theta$  and we take action  $a$ . Generally,  $L(\theta, a) \geq 0$ .

## 2.2 Decision Rules and Risk Functions

### 2.2.1 Decision Rule

A decision rule is a function  $d : \mathcal{X} \rightarrow \mathcal{A}$ . It dictates the action  $d(x)$  we take when we observe data  $x$ .

### 2.2.2 Risk Function

The risk function is the expected loss for a given decision rule  $d$  as a function of the parameter  $\theta$ .

$$R(\theta, d) = E_\theta[L(\theta, d(X))] \quad (2.1)$$

## 2.3 Examples of Decision Problems

### 2.3.1 Example 1: Hypothesis Testing

We want to test  $H_0$  vs  $H_1$ .

- **Action Space:**  $\mathcal{A} = \{0, 1\}$  (0=“Accept  $H_0$ ”, 1=“Reject  $H_0$ ”).
- **Loss Function (0-1 Loss):** 0 if correct, 1 if wrong.
- **Risk Function:**
  - If  $\theta \in H_0$ :  $R(\theta, d) = P(\text{Type I Error})$ .
  - If  $\theta \in H_1$ :  $R(\theta, d) = P(\text{Type II Error})$ .

### 2.3.2 Example 2: Point Estimation

We want to estimate a parameter  $\theta$ .

- **Action Space:**  $\mathcal{A} = \Theta$ .
- **Loss Function (Squared Error):**  $L(\theta, a) = (\theta - a)^2$ .
- **Risk Function (MSE):**  $R(\theta, d) = \text{Var}(\bar{x}) + \text{Bias}^2$ .

### 2.3.3 Example 3: Interval Estimation

We want to estimate a range for the parameter.

- **Action Space:**  $\mathcal{A} = \{(l, u) : l \in \mathbb{R}, u \in \mathbb{R}, l \leq u\}$ .

## 2.4 The Duchess and the Emerald Necklace

**Scenario:** You are the Duchess of Omnium. You have two necklaces: a priceless **Real** one and a valueless **Imitation**. They are indistinguishable to you. One is in the **Left Drawer (Box 1)**, the other is in the **Right Drawer (Box 2)**.

**The Data (Great Aunt):** You consult your Great Aunt. She inspects the Left Drawer first, then the Right.

- If the **Real** necklace is in the **Left** ( $\theta = 1$ ): She identifies it correctly. (Infallible).
- If the **Real** necklace is in the **Right** ( $\theta = 2$ ): She sees the fake first, gets confused, and guesses randomly (50/50).

### 2.4.1 Formulation

1. **Parameter Space:**  $\Theta = \{1, 2\}$  (1=Real Left, 2=Real Right).
2. **Action Space:**  $\mathcal{A} = \{1, 2\}$  (1=Wear Left, 2=Wear Right).
3. **Loss Function:** 0 if correct, 1 if wrong.

### 2.4.2 Risk Calculation for Deterministic Rules

We consider four deterministic rules  $d(X)$ . We calculate the risk ( $R_1$  for  $\theta = 1$  and  $R_2$  for  $\theta = 2$ ) for each.

#### Rule $d_1$ (Always Left)

State	Component	$X = 1$	$X = 2$	Risk (Sum)
$\theta = 1$	Loss $L(1, d)$	0	0	$R_1 = 0$
	Prob $P(X   \theta = 1)$	1	0	
$\theta = 2$	Loss $L(2, d)$	1	1	$R_2 = 1$
	Prob $P(X   \theta = 2)$	0.5	0.5	

#### Rule $d_2$ (Always Right)

State	Component	$X = 1$	$X = 2$	Risk (Sum)
$\theta = 1$	Loss $L(1, d)$	1	1	$R_1 = 1$
	Prob $P(X   \theta = 1)$	1	0	
$\theta = 2$	Loss $L(2, d)$	0	0	$R_2 = 0$
	Prob $P(X   \theta = 2)$	0.5	0.5	

#### Rule $d_3$ (Follow Aunt)

State	Component	$X = 1$	$X = 2$	Risk (Sum)
$\theta = 1$	Loss $L(1, d)$	0	1	$R_1 = 0$
	Prob $P(X   \theta = 1)$	1	0	
$\theta = 2$	Loss $L(2, d)$	1	0	$R_2 = 0.5$
	Prob $P(X   \theta = 2)$	0.5	0.5	

#### Rule $d_4$ (Do Opposite)

State	Component	$X = 1$	$X = 2$	Risk (Sum)
$\theta = 1$	Loss $L(1, d)$	1	0	$R_1 = 1$
	Prob $P(X   \theta = 1)$	1	0	
$\theta = 2$	Loss $L(2, d)$	0	1	$R_2 = 0.5$
	Prob $P(X   \theta = 2)$	0.5	0.5	

## 2.5 Principles for Choosing a Decision Rule

Since no single rule minimizes risk for all  $\theta$ , we rely on several principles to order and select decision rules.

### 2.5.1 Admissibility

A decision rule  $d$  is **admissible** if it is not “dominated” by any other rule.

- **Domination:** A rule  $d$  dominates  $d'$  if  $R(\theta, d) \leq R(\theta, d')$  for all  $\theta$ , with strict inequality for at least one  $\theta$ .
- **Inadmissibility:** If a rule is dominated, it is inadmissible and can be discarded (we can do better or equal in every possible state).

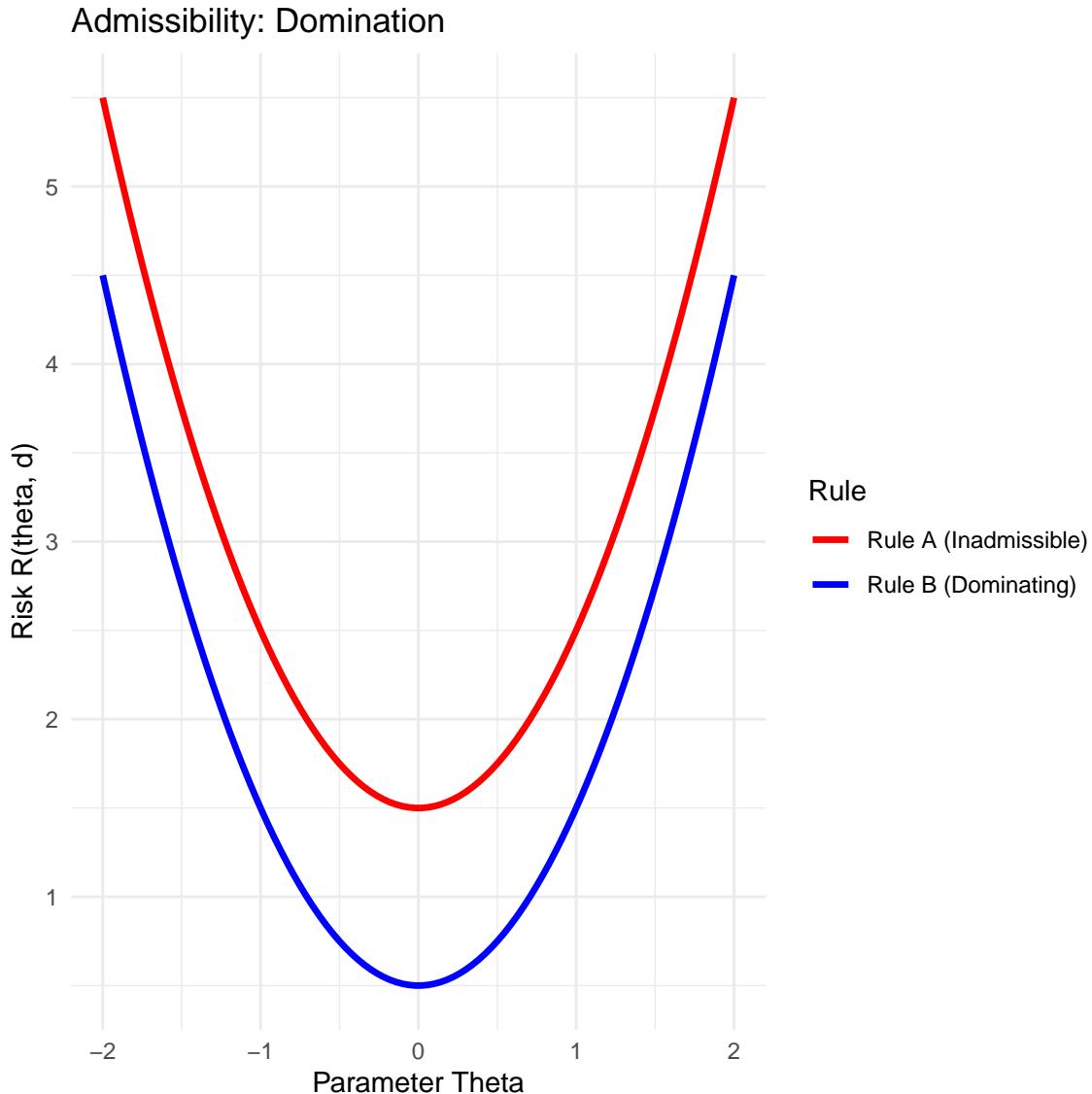


Figure 2.1: Illustration of Domination: Rule A (Red) is inadmissible because Rule B (Blue) has lower risk for all values of theta.

### 2.5.2 Minimax Principle

The Minimax principle is a conservative approach that guards against the worst-case scenario. It selects the rule that minimizes the maximum risk.

$$\min_d \left[ \sup_{\theta} R(\theta, d) \right] \quad (2.2)$$

In the plot below, while Rule B has lower risk in the center, it has a very high maximum risk. Rule A is “flatter” and has a lower maximum value, making it the **Minimax** choice.

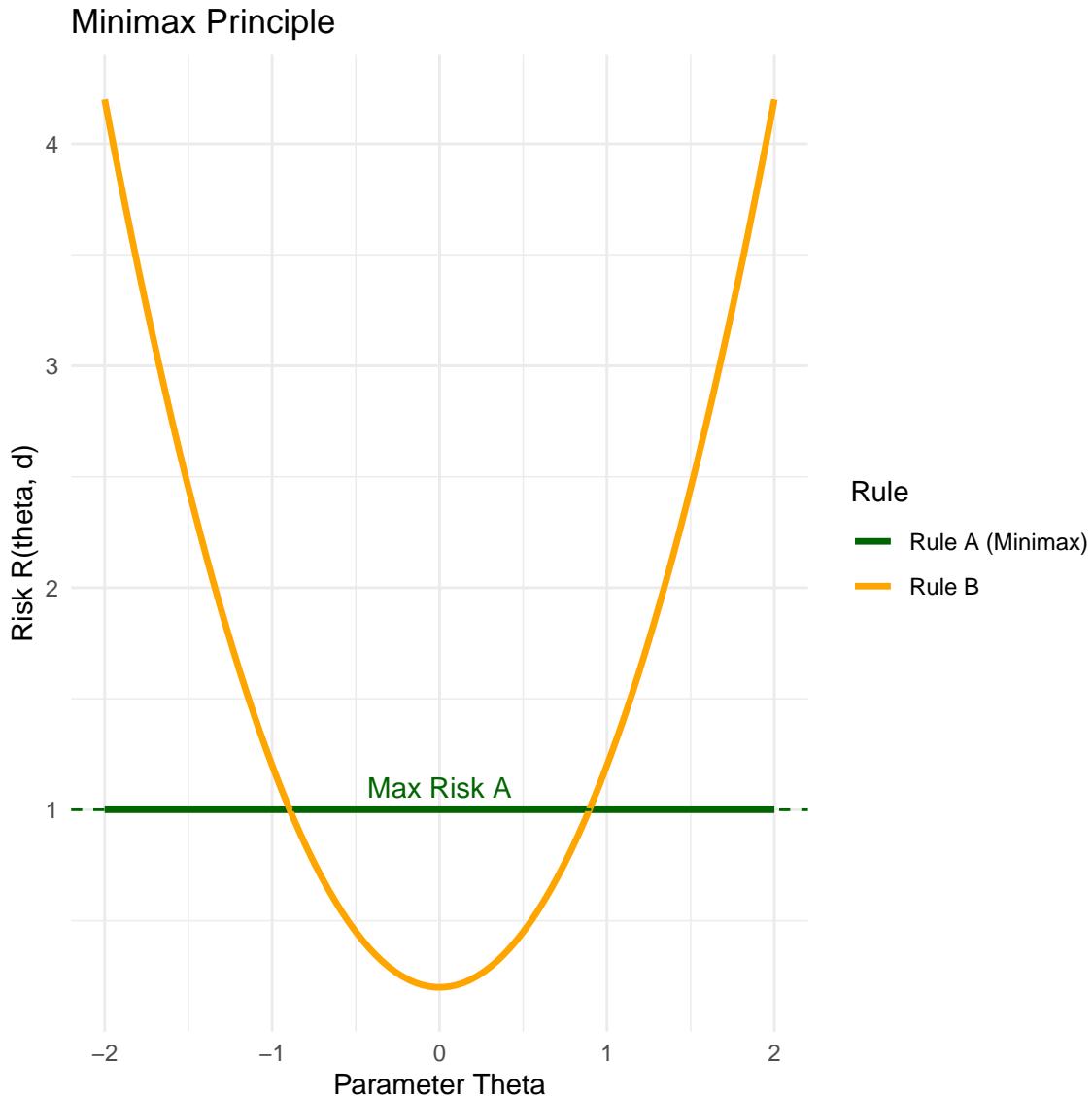


Figure 2.2: Illustration of Minimax: Rule A has a lower peak risk than Rule B, making Rule A the Minimax choice.

### 2.5.3 Bayes Decision Rules

The Bayes principle incorporates prior knowledge. If we assign a probability distribution (prior)  $\pi(\theta)$  to the parameter, we can calculate the **Bayes Risk**, which is the weighted average of the risk function. We choose the rule that minimizes this average.

$$r(\pi, d) = E_{\pi}[R(\theta, d)] = \int_{\Theta} R(\theta, d)\pi(\theta)d\theta \quad (2.3)$$

## 2.6 Risk Set for Finite Parameter Space

For finite parameter spaces (e.g.,  $\Theta = \{1, 2\}$ ), we can visualize the problem in 2D space where the axes are  $R_1 = R(\theta_1)$  and  $R_2 = R(\theta_2)$ .

### 2.6.1 The Risk Set ( $S$ )

The set of all possible risk vectors is called the Risk Set  $S$ .

- **Deterministic Rules:** These are the vertices of the set.
- **Randomized Rules:** By choosing rule  $d_i$  with probability  $p$  and  $d_j$  with probability  $1 - p$ , we can achieve any risk on the line segment connecting them.
- **Convexity:** The Risk Set is the **convex hull** of the deterministic rules.

### 2.6.2 Visualizing Admissibility

The admissible rules lie on the **lower-left boundary** of the set. Any point to the “north-east” of another point is dominated (inadmissible).

### 2.6.3 Visualizing Minimax

The Minimax rule is found by intersecting the Risk Set with the line  $y = x$  ( $R_1 = R_2$ ).

- We look for the point in  $S$  that touches the  $45^\circ$  line at the lowest value.
- If the set is entirely below the line, we minimize  $R_2$ . If entirely above, we minimize  $R_1$ .

### 2.6.4 Visualizing Bayes Rules

A Bayes rule minimizes  $\pi_1 R_1 + \pi_2 R_2 = k$ . This equation represents a line with slope  $m = -\pi_1/\pi_2$ .

- To find the Bayes rule, we find the **tangent line** to the Risk Set  $S$  with slope  $-\pi_1/\pi_2$ .

## Geometric Concepts in Decision Theory

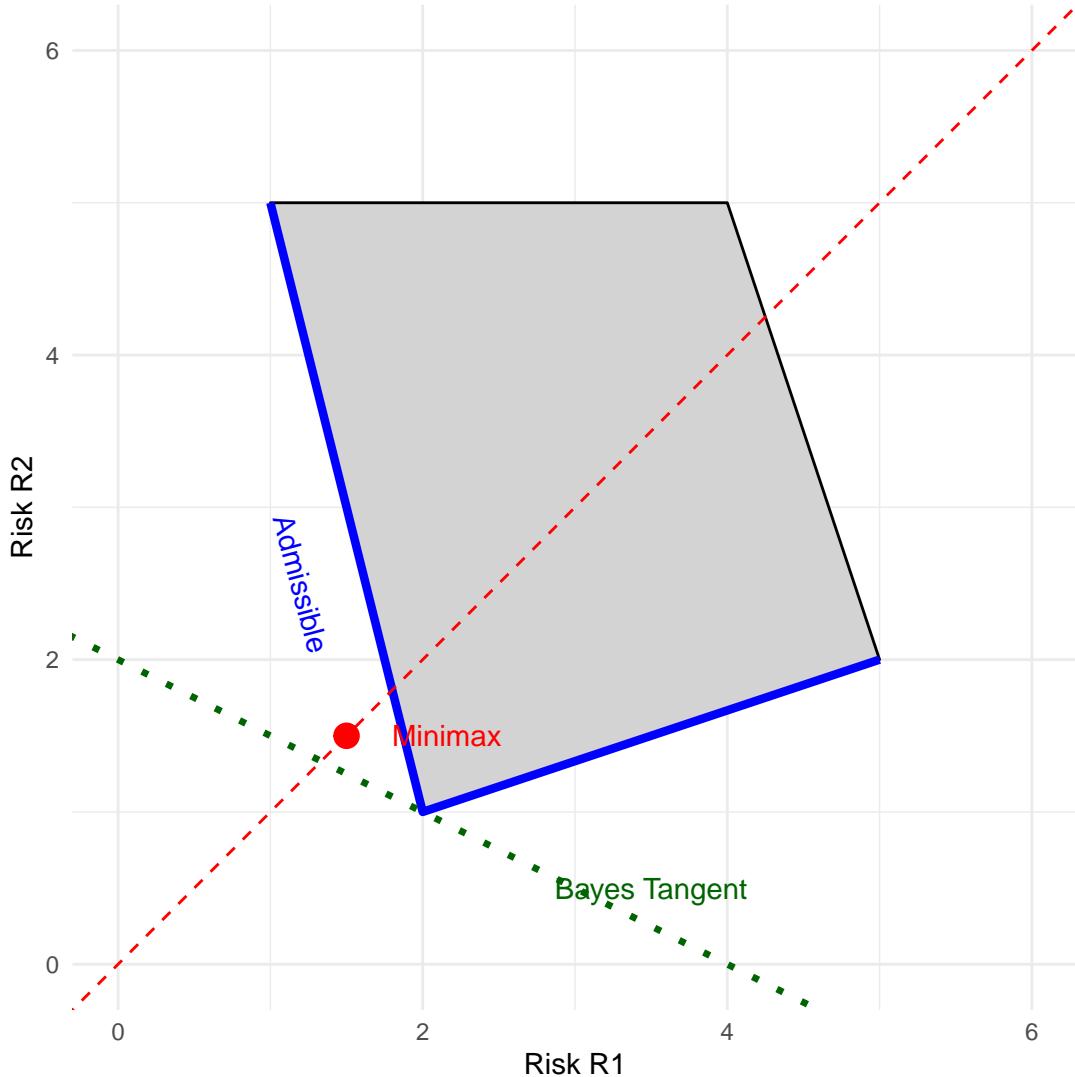


Figure 2.3: Geometric Interpretation: The gray polygon is the Risk Set S. The blue boundary represents admissible rules. The red point is the Minimax rule. The green line represents a Bayes rule for a specific prior.

## 2.7 Revisiting the Necklace Example: Geometric Solution

We now apply the geometric interpretation to the Necklace problem using the risks calculated in Section 2.4.

- $d_1: (0, 1)$
- $d_2: (1, 0)$
- $d_3: (0, 0.5)$
- $d_4: (1, 0.5)$

### 2.7.1 Analysis

#### 1. Admissibility:

- $d_4$  has risk  $(1, 0.5)$ .  $d_3$  has risk  $(0, 0.5)$ . Since  $0 < 1$ ,  $d_3$  strictly dominates  $d_4$ . Thus  $d_4$  is **inadmissible**.
- The efficient frontier connects  $d_3$  and  $d_2$ .

#### 2. Minimax Solution:

The Minimax rule lies on the segment connecting  $d_3(0, 0.5)$  and  $d_2(1, 0)$ .

- Let the randomized rule be  $\delta^* = pd_3 + (1 - p)d_2$ .
- $R(\delta^*) = p \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} + (1 - p) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-p \\ 0.5p \end{pmatrix}$ .
- Set  $R_1 = R_2$ :  $1 - p = 0.5p \Rightarrow 1 = 1.5p \Rightarrow p = 2/3$ .
- **Result:** The Minimax rule is to choose  $d_3$  with probability  $2/3$  and  $d_2$  with probability  $1/3$ .

## 2.8 Theorems Relating Minimax and Bayes Rules

In practice, finding a Minimax rule directly is mathematically difficult. A standard strategy is to “guess” a Least Favorable Prior  $\pi$ —defined as the prior distribution that maximizes the minimum Bayes risk (i.e., the prior against which it is hardest to defend)—find the corresponding Bayes rule, and then check if it satisfies specific conditions to confirm it is Minimax.

### 2.8.1 Constant Risk Bayes Rule Is Minimax (Proof by Contradiction)

**Theorem 2.1** (Constant Risk Bayes Rule Is Minimax). *Let  $\delta^\pi$  be a Bayes estimator with respect to a prior  $\pi$ . If the risk function of  $\delta^\pi$  is constant on the parameter space  $\Theta$ , such that  $R(\theta, \delta^\pi) = c$  for all  $\theta \in \Theta$ , then  $\delta^\pi$  is a minimax estimator.*

*Proof.* Assume, for the sake of contradiction, that  $\delta^\pi$  is **not** a minimax estimator.

By definition, if  $\delta^\pi$  is not minimax, there must exist some other estimator  $\delta'$  that has a strictly smaller maximum risk. That is:

$$\sup_{\theta \in \Theta} R(\theta, \delta') < \sup_{\theta \in \Theta} R(\theta, \delta^\pi) \quad (2.4)$$

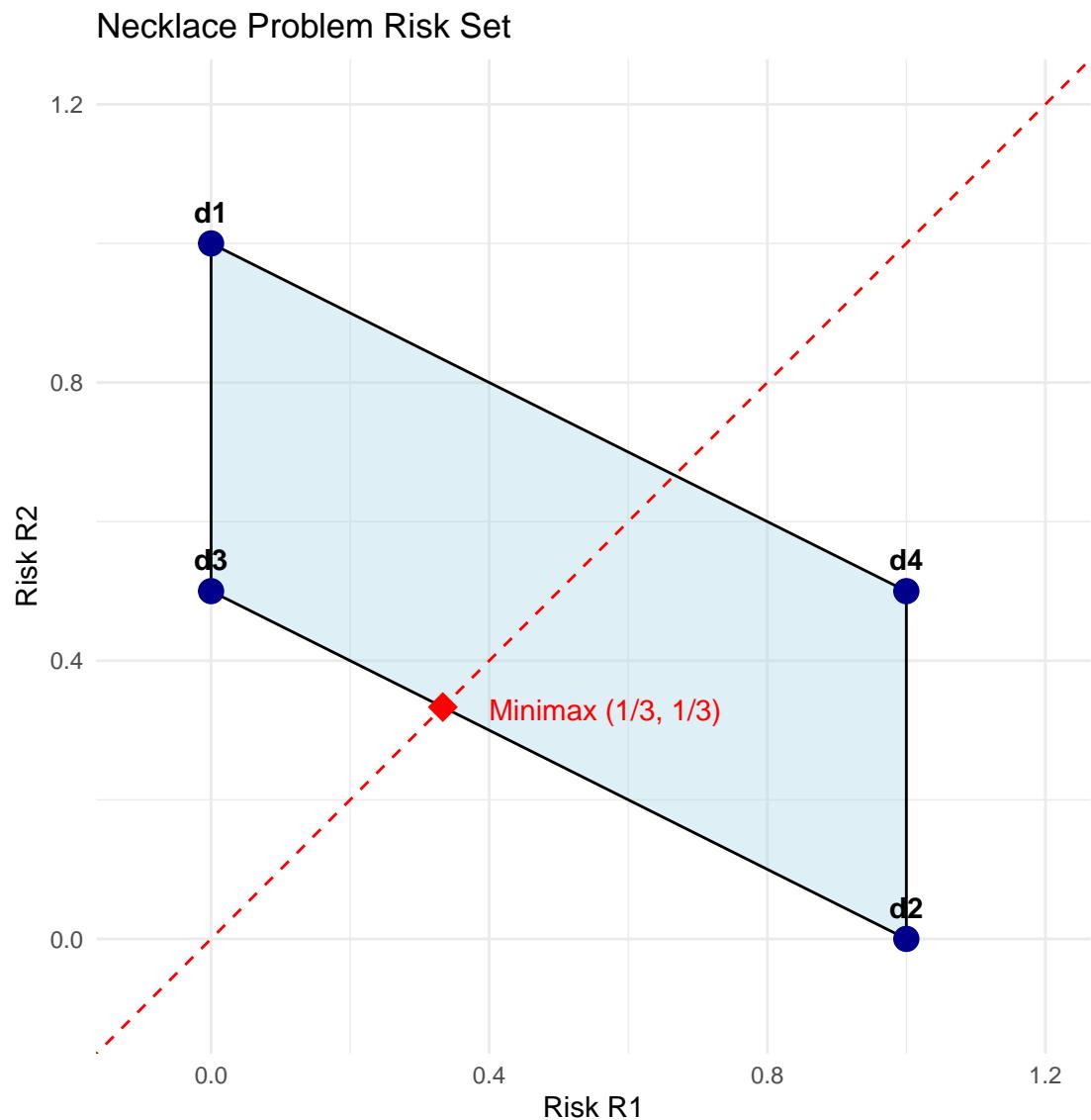


Figure 2.4: Necklace Problem Solution. The Minimax rule (red diamond) is the specific randomized combination of  $d_3$  and  $d_2$  that equalizes the risk.

Since we are given that  $R(\theta, \delta^\pi) = c$  for all  $\theta \in \Theta$ , its supremum is simply  $c$ . Therefore, our assumption implies:

$$\sup_{\theta \in \Theta} R(\theta, \delta') < c \quad (2.5)$$

Now, consider the Bayes risk of  $\delta'$  with respect to the prior  $\pi$ . The Bayes risk is the weighted average of the risk function:

$$r(\pi, \delta') = \int_{\Theta} R(\theta, \delta') \pi(\theta) d\theta \quad (2.6)$$

Since  $R(\theta, \delta') \leq \sup_{\theta \in \Theta} R(\theta, \delta')$  for all  $\theta$ , and we assumed this supremum is strictly less than  $c$ , it follows that:

$$r(\pi, \delta') \leq \sup_{\theta \in \Theta} R(\theta, \delta') < c \quad (2.7)$$

However, we know that  $c$  is the Bayes risk of  $\delta^\pi$ :

$$r(\pi, \delta^\pi) = \int_{\Theta} c \pi(\theta) d\theta = c \quad (2.8)$$

Substituting this into our inequality, we get:

$$r(\pi, \delta') < r(\pi, \delta^\pi) \quad (2.9)$$

This result contradicts the fact that  $\delta^\pi$  is a **Bayes estimator**. By definition, a Bayes estimator must minimize the Bayes risk, meaning  $r(\pi, \delta^\pi) \leq r(\pi, \delta)$  for any estimator  $\delta$ .

Because our assumption that  $\delta^\pi$  is not minimax leads to a contradiction of the Bayes optimality of  $\delta^\pi$ , the assumption must be false. Thus,  $\delta^\pi$  must be minimax.  $\square$

The plot below visualizes this logic. If an estimator  $\delta'$  (Blue) were to be “better” in a minimax sense than  $\delta^\pi$  (Red), its entire curve would have to stay below the maximum value  $c$ . However, if it stays below  $c$  everywhere, its average (Bayes risk) would necessarily be lower than  $c$ , which is impossible if  $\delta^\pi$  is the Bayes estimator.

```
# Define Parameter Space Theta
theta <- seq(0, 1, length.out = 200)

# 1. Constant Risk Bayes Estimator (risk = C)
c_val <- 0.6
risk_bayes <- rep(c_val, length(theta))

# 2. An estimator that would contradict Bayes optimality
# (Always below the constant risk line)
risk_contradiction <- 0.5 + 0.05 * cos(2 * pi * theta)

# Plotting
plot(theta, risk_bayes, type = 'l', lwd = 3, col = "red",
      ylim = c(0, 1), ylab = "Risk R(theta, d)", xlab = expression(theta),
      main = "Proof by Contradiction Geometry")
```

```

# Add the "Better" Estimator (which is impossible)
lines(theta, risk_contradiction, col = "blue", lwd = 2, lty = 2)

# Shaded area showing the "Impossible" Bayes Risk improvement
polygon(c(theta, rev(theta)), c(risk_contradiction, rev(risk_bayes)),
         col = rgb(0, 0, 1, 0.1), border = NA)

# Add Legend
legend("topright",
       legend = c("Constant Risk Bayes (c)", "Hypothetical 'Better' Est."),
       col = c("red", "blue"), lwd = 2, lty = c(1, 2))

```

## Proof by Contradiction Geometry

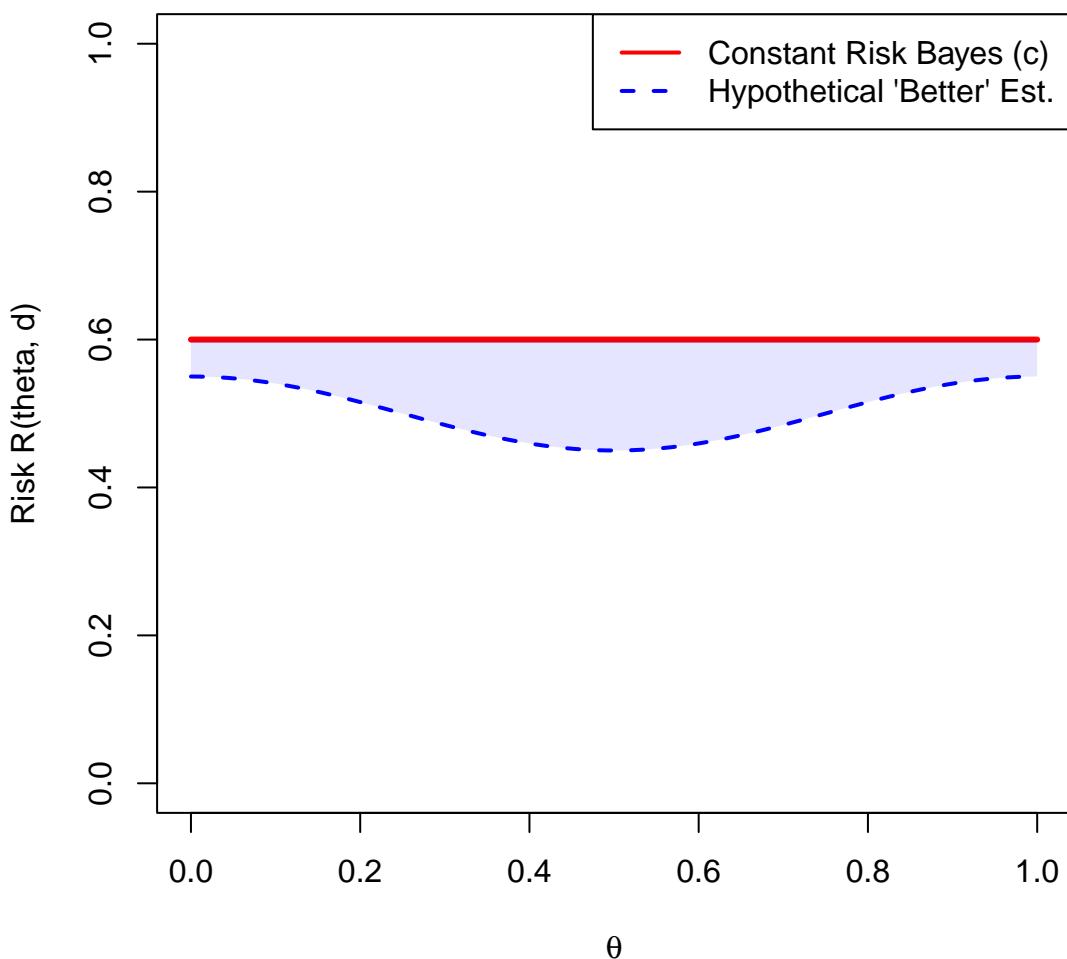


Figure 2.5: Visualizing the Contradiction: If the blue curve's maximum were below the red line, its average risk would be lower than the Bayes risk of the red estimator.

## 2.8.2 Minimaxity via Limiting Bayes Risks

Sometimes the Minimax rule corresponds to an “improper” prior (a prior that does not integrate to 1, like a uniform distribution on the real line). We approach these via a limiting sequence.

**Theorem 2.2** (Minimaxity of Limit-Attaining Rules). *Let  $\{\delta_n\}$  be a sequence of Bayes rules with respect to priors  $\{\pi_n\}$ . Let  $r(\pi_n, \delta_n)$  be the associated Bayes risks. If there exists a rule  $\delta_0$  such that:*

$$\sup_{\theta} R(\theta, \delta_0) \leq \lim_{n \rightarrow \infty} r(\pi_n, \delta_n) \quad (2.10)$$

*Then  $\delta_0$  is Minimax.*

*Proof.*

1. **Define Limit:** Let  $V = \lim_{n \rightarrow \infty} r(\pi_n, \delta_n)$ . We are given that  $\sup_{\theta} R(\theta, \delta_0) \leq V$ .

2. **Contradiction Setup:** Suppose  $\delta_0$  is *not* Minimax. Then there exists a rule  $\delta^*$  such that:

$$\sup_{\theta} R(\theta, \delta^*) < \sup_{\theta} R(\theta, \delta_0) \leq V \quad (2.11)$$

Let  $\sup_{\theta} R(\theta, \delta^*) = V - \epsilon$  for some  $\epsilon > 0$ .

3. **Bounded Risk of  $\delta^*$ :** The Bayes risk of  $\delta^*$  is bounded by its maximum risk:

$$r(\pi_n, \delta^*) = \int R(\theta, \delta^*) \pi_n(\theta) d\theta \leq V - \epsilon \quad (2.12)$$

Therefore,  $\lim_{n \rightarrow \infty} r(\pi_n, \delta^*) \leq V - \epsilon$ .

4. **Optimality of  $\delta_n$ :** Since  $\delta_n$  is the Bayes rule for  $\pi_n$ , it minimizes Bayes risk. This creates the inequality pair shown in the figure (Orange  $\leq$  Blue):

$$r(\pi_n, \delta_n) \leq r(\pi_n, \delta^*) \quad (2.13)$$

5. **The Contradiction:** Combining the inequalities, we get:

$$\lim_{n \rightarrow \infty} r(\pi_n, \delta_n) \leq \lim_{n \rightarrow \infty} r(\pi_n, \delta^*) \leq V - \epsilon \quad (2.14)$$

This implies  $V \leq V - \epsilon$ , which is impossible. Thus  $\delta_0$  must be Minimax. ■

□

## 2.8.3 Procedure: Verifying Minimaxity

The theorem above provides a practical recipe for identifying Minimax rules, particularly in unbounded parameter spaces (where a standard Least Favorable Prior often does not exist). The procedure is often used “backwards”—we guess a rule and then construct a sequence to prove it is Minimax.

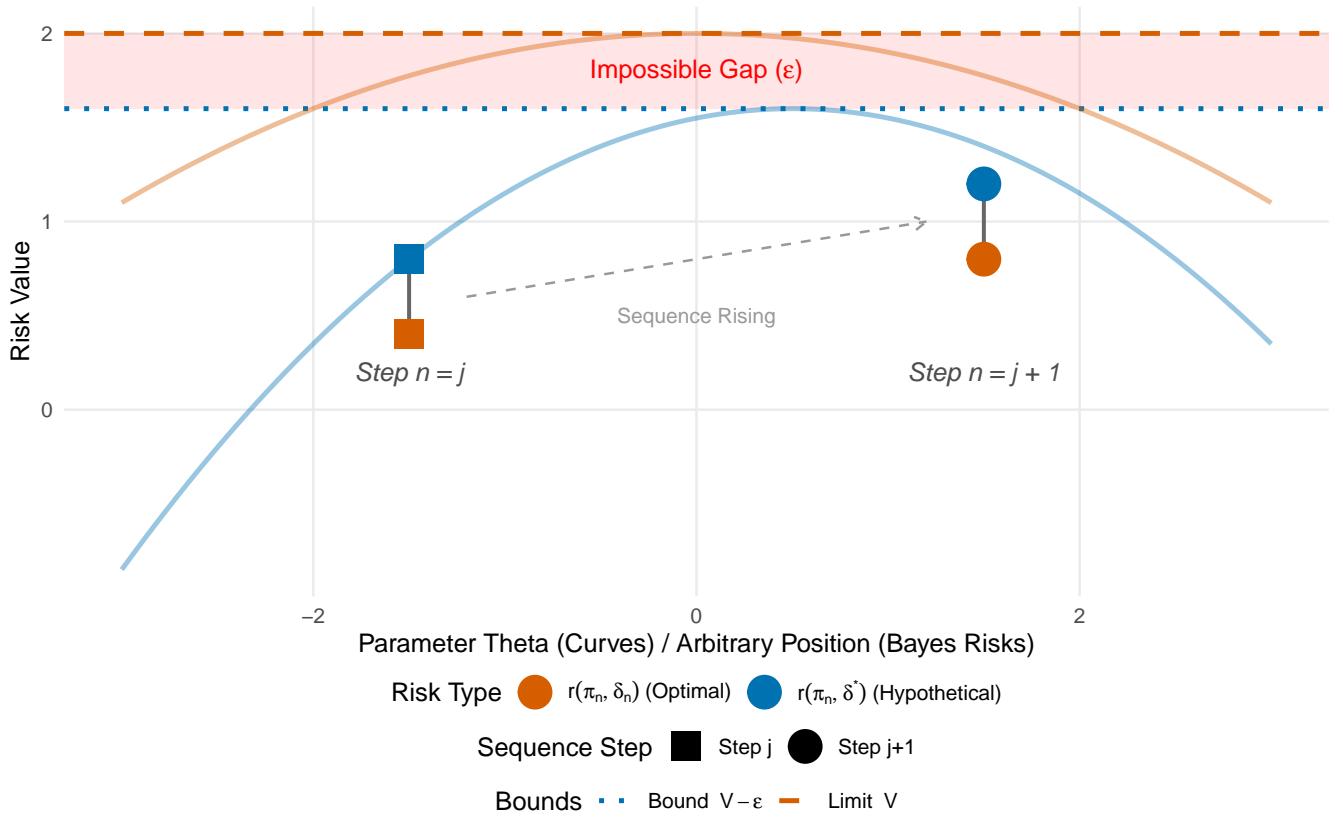


Figure 2.6: Visual Proof: We examine the Bayes risks at two steps,  $n = j$  (squares) and  $n = j + 1$  (circles). In both steps, the optimal risk  $r(\pi_n, \delta_n)$  (orange) must be lower than the hypothetical risk  $r(\pi_n, \delta^*)$  (blue). Even as the sequence rises ( $j+1$  is higher than  $j$ ), the blue points are capped by the bound  $V - \epsilon$ . This ‘traps’ the orange points, making it impossible for them to ever reach the Limit  $V$ .

- Propose a Candidate Rule ( $\delta_0$ ):** Identify a rule that intuitively seems robust. Typically, we look for an **Equalizer Rule**, which is a rule with constant risk ( $R(\theta, \delta_0) = C$  for all  $\theta$ ). If the risk is constant, then  $\sup_{\theta} R(\theta, \delta_0) = C$ .
- Construct a Sequence of Priors ( $\pi_n$ ):** Choose a sequence of priors that becomes increasingly “diffuse” or “flat” as  $n \rightarrow \infty$  (e.g., Uniform on  $[-n, n]$  or Normal with variance  $n$ ). These approximate the “improper” prior corresponding to the candidate rule.
- Compute Bayes Risks ( $r_n$ ):** Calculate the Bayes risk  $r(\pi_n, \delta_n)$  for each prior in the sequence. Note that you do not necessarily need the formula for the Bayes rule  $\delta_n$  itself, only its associated risk.
- Verify the Condition:** Check if the limit of the Bayes risks approaches the maximum risk of your candidate:

$$\lim_{n \rightarrow \infty} r(\pi_n, \delta_n) = \sup_{\theta} R(\theta, \delta_0) \quad (2.15)$$

If this holds,  $\delta_0$  is Minimax.

**Example 2.1** (The Normal Mean). Consider a single observation  $X \sim N(\theta, 1)$  with squared error loss  $L(\theta, \delta) = (\theta - \delta)^2$ . We suspect the sample mean (in this case, just  $X$  itself) is the Minimax estimator.

#### Step 1: Candidate Rule

Let  $\delta_0(X) = X$ . The risk is the variance of the estimator:

$$R(\theta, \delta_0) = E[(\theta - X)^2] = \text{Var}(X) = 1 \quad (2.16)$$

Since the risk is constant (1) for all  $\theta$ ,  $\sup_{\theta} R(\theta, \delta_0) = 1$ .

#### Step 2: Sequence of Priors

We choose a sequence of Normal priors  $\pi_n \sim N(0, n)$ . As  $n$  increases, the variance increases, making the prior flatter over the real line.

#### Step 3: Bayes Risks

For a Normal prior  $\theta \sim N(0, \tau^2)$  and data  $X \sim N(\theta, \sigma^2)$ , the Bayes risk is known to be:

$$r(\pi, \delta_{\pi}) = \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} \quad (2.17)$$

Substituting our values ( $\sigma^2 = 1, \tau^2 = n$ ):

$$r(\pi_n, \delta_n) = \frac{1 \cdot n}{1 + n} = \frac{n}{n + 1} \quad (2.18)$$

#### Step 4: Verification

We take the limit of the sequence of Bayes risks:

$$\lim_{n \rightarrow \infty} r(\pi_n, \delta_n) = \lim_{n \rightarrow \infty} \frac{n}{n + 1} = 1 \quad (2.19)$$

Comparing this to our candidate:

$$\sup_{\theta} R(\theta, \delta_0) = 1 \leq 1 \quad (2.20)$$

The condition holds. Therefore,  $\delta_0(X) = X$  is the Minimax estimator for  $\theta$ .

## 2.8.4 Bayes Rule as a Working Horse to Find a Minimax Rule

### 2.8.4.1 The Minimax Theorem (Saddle Point)

This theorem connects the search for a Minimax rule to the search for a Least Favorable Prior. It justifies the strategy of “finding the worst prior and solving it.”

**Theorem 2.3** (The Minimax Theorem). *Let  $\mathcal{D}$  be the set of all decision rules and  $\Pi$  be the set of all prior distributions. Let  $r(\pi, \delta)$  denote the Bayes risk.*

*The Minimax value equals the Maximin Bayes value:*

$$\inf_{\delta \in \mathcal{D}} \sup_{\pi \in \Pi} r(\pi, \delta) = \sup_{\pi \in \Pi} \inf_{\delta \in \mathcal{D}} r(\pi, \delta) \quad (2.21)$$

Furthermore, a pair  $(\delta_0, \pi_0)$  is a **Saddle Point** if for all  $\delta \in \mathcal{D}$  and  $\pi \in \Pi$ :

$$r(\pi_0, \delta) \geq r(\pi_0, \delta_0) \geq r(\pi, \delta_0) \quad (2.22)$$

If such a saddle point exists, then:

1.  $\delta_0$  is a **Minimax rule**.
2.  $\pi_0$  is a **Least Favorable Prior**.

*Proof.* **Goal:** We wish to show that if  $(\delta_0, \pi_0)$  is a saddle point, then  $\sup_{\theta} R(\theta, \delta_0) \leq \sup_{\theta} R(\theta, \delta)$  for any other rule  $\delta$ .

**1. Interpret the Saddle Point Inequalities:** The condition is given as two simultaneous inequalities:

$$\begin{aligned} (A) \quad & r(\pi_0, \delta_0) \leq r(\pi_0, \delta) \quad \text{for all } \delta \\ (B) \quad & r(\pi, \delta_0) \leq r(\pi_0, \delta_0) \quad \text{for all } \pi \end{aligned} \quad (2.23)$$

**2. Analyze Inequality (A):** Since  $r(\pi_0, \delta_0) \leq r(\pi_0, \delta)$  for all  $\delta$ ,  $\delta_0$  minimizes the Bayes risk with respect to  $\pi_0$ .

- Therefore,  $\delta_0$  is the **Bayes rule** for  $\pi_0$ .

**3. Analyze Inequality (B):** Since  $r(\pi, \delta_0) \leq r(\pi_0, \delta_0)$  for all  $\pi$ , the prior  $\pi_0$  maximizes the average risk of  $\delta_0$ .

- Since the supremum over all priors includes point-mass priors (which yield the risk at a single  $\theta$ ), maximizing over  $\pi$  is equivalent to maximizing over  $\theta$ :

$$\sup_{\pi} r(\pi, \delta_0) = \sup_{\theta} R(\theta, \delta_0) \quad (2.24)$$

- Therefore, Inequality (B) implies:

$$\sup_{\theta} R(\theta, \delta_0) = r(\pi_0, \delta_0) \quad (2.25)$$

**4. Combine to Prove Minimality:** Let  $\delta^*$  be any arbitrary decision rule. We compute its worst-case risk:

$$\begin{aligned}
 \sup_{\theta} R(\theta, \delta^*) &= \sup_{\pi} r(\pi, \delta^*) && (\text{Max risk} = \text{Max average risk}) \\
 &\geq r(\pi_0, \delta^*) && (\text{Supremum} \geq \text{specific value}) \\
 &\geq r(\pi_0, \delta_0) && (\text{From Inequality A: } \delta_0 \text{ is Bayes for } \pi_0) \\
 &= \sup_{\theta} R(\theta, \delta_0) && (\text{From Step 3})
 \end{aligned} \tag{2.26}$$

**5. Conclusion:** We have shown that for any  $\delta^*$ :

$$\sup_{\theta} R(\theta, \delta^*) \geq \sup_{\theta} R(\theta, \delta_0) \tag{2.27}$$

Thus,  $\delta_0$  minimizes the maximum risk.  $\delta_0$  is Minimax. ■ □

#### 2.8.4.2 Alternating Optimization on the Risk Surface

The Minimax solution can be found computationally by iteratively optimizing one variable while holding the other fixed.

1. **Fix Prior  $\pi$ , Minimize Risk:** We search the valley bottom for the current  $\pi$ .
2. **Fix Rule  $\delta$ , Maximize Risk:** We search the hill top for the current  $\delta$ .

This creates a “zigzag” path on the surface that converges to the saddle point.

## 2.9 Admissibility of Bayes Rules

Bayes rules are generally good candidates for admissibility. If a rule is Bayes, it is likely efficient, provided the prior doesn’t ignore parts of the parameter space.

**Theorem 2.4** (Admissibility of Bayes Rules (Finite Support)). *If the parameter space  $\Theta$  is finite (or countable) and the prior  $\pi$  assigns positive probability to every  $\theta \in \Theta$  (i.e.,  $\pi(\theta) > 0$  for all  $\theta$ ), then any Bayes rule  $\delta_\pi$  is admissible.*

*Proof.*

1. **Contradiction Setup:** Suppose  $\delta_\pi$  is inadmissible. Then there exists a rule  $\delta'$  that dominates it. By definition of domination:

- $R(\theta, \delta') \leq R(\theta, \delta_\pi)$  for all  $\theta$ .
- $R(\theta_k, \delta') < R(\theta_k, \delta_\pi)$  for at least one  $\theta_k$ .

2. **Bayes Risk Difference:** Consider the difference in Bayes risk:

$$r(\pi, \delta_\pi) - r(\pi, \delta') = \sum_{\theta \in \Theta} \pi(\theta) [R(\theta, \delta_\pi) - R(\theta, \delta')] \tag{2.28}$$

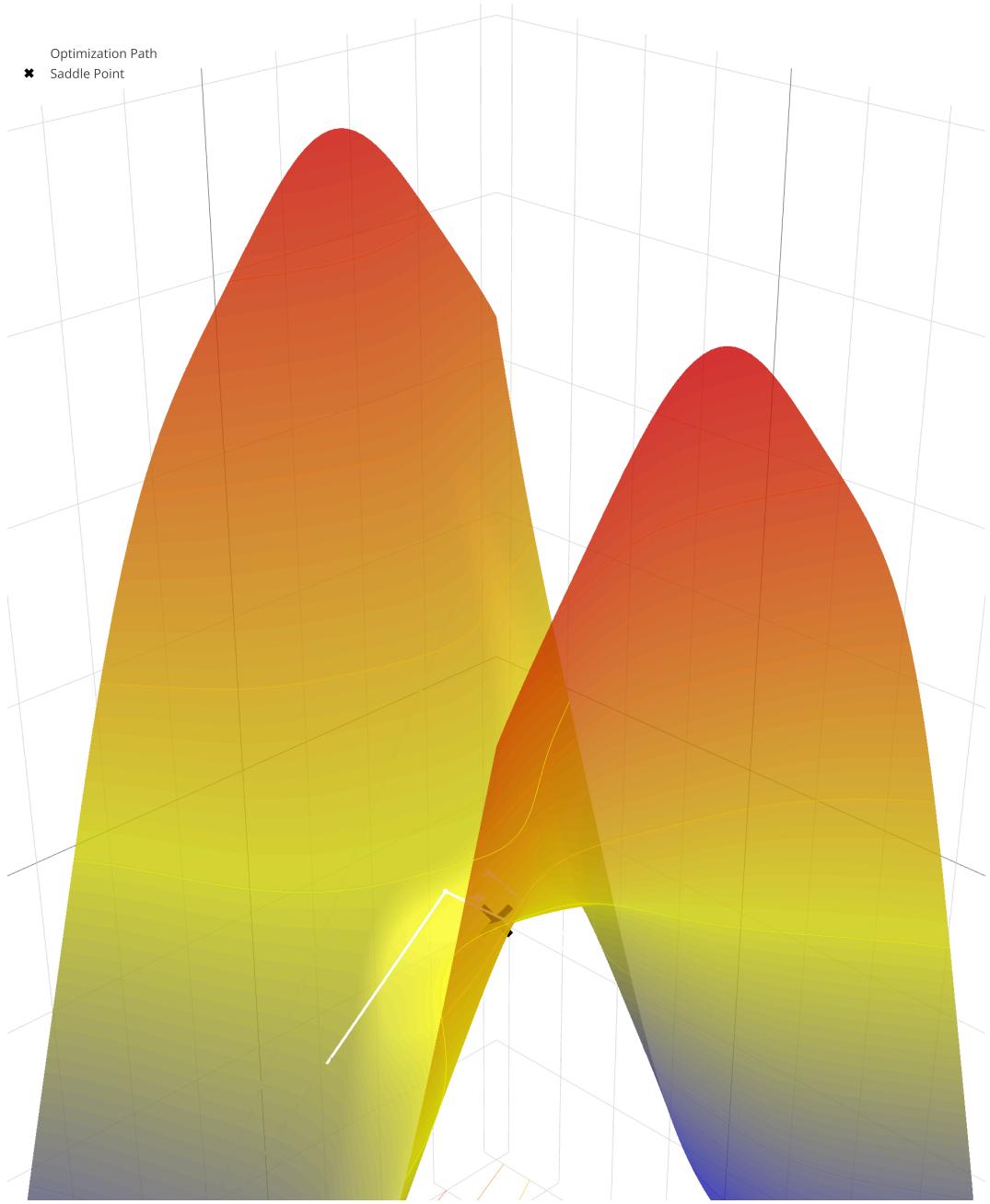


Figure 2.7: The ‘Wiggle Mountain’ of Risk. The surface represents Bayes Risk  $r(\pi, \delta)$ . The white zigzag line shows the iterative algorithm: starting from an arbitrary prior, we alternate between finding the best  $\delta$  (moving along the valley) and the worst  $\pi$  (climbing the hill). This path spirals inward, converging to the red Saddle Point (Minimax solution) in the center.

### 3. Strict Positivity:

- Since  $\delta'$  dominates  $\delta_\pi$ , each term  $[R(\theta, \delta_\pi) - R(\theta, \delta')]$  is non-negative ( $\geq 0$ ).
- At  $\theta_k$ , the term is strictly positive ( $> 0$ ).
- We assumed the prior has full support, so  $\pi(\theta) > 0$  for all  $\theta$ .

4. **Summation:** A sum of non-negative terms where at least one term is strictly positive must be strictly positive.

$$r(\pi, \delta_\pi) - r(\pi, \delta') > 0 \implies r(\pi, \delta') < r(\pi, \delta_\pi) \quad (2.29)$$

5. **Conclusion:** This contradicts the definition that  $\delta_\pi$  is a Bayes rule (which must minimize Bayes risk). Therefore,  $\delta_\pi$  is admissible. ■

□

### 2.9.1 Admissibility of Unique Bayes Rules

If the Bayes rule is unique, we can drop the requirement that the parameter space be discrete or finite.

**Theorem 2.5** (Admissibility of Unique Bayes Rules). *Let  $\delta_\pi$  be a Bayes rule with respect to  $\pi$ . If  $\delta_\pi$  is the unique Bayes rule (up to risk equivalence), then  $\delta_\pi$  is admissible.*

*Proof.*

1. **Contradiction Setup:** Suppose  $\delta_\pi$  is inadmissible. Then there exists a rule  $\delta'$  such that:  $R(\theta, \delta') \leq R(\theta, \delta_\pi)$  for all  $\theta$ , with strict inequality for some set of  $\theta$ .

2. **Bayes Risk Inequality:** Taking the expectation with respect to  $\pi$ :

$$r(\pi, \delta') = \int R(\theta, \delta') \pi(\theta) d\theta \leq \int R(\theta, \delta_\pi) \pi(\theta) d\theta = r(\pi, \delta_\pi) \quad (2.30)$$

3. **Minimality:** Since  $\delta_\pi$  is Bayes, it minimizes the risk, so  $r(\pi, \delta_\pi) \leq r(\pi, \delta')$ . Combining these gives  $r(\pi, \delta') = r(\pi, \delta_\pi)$ .

4. **Uniqueness:** This implies that  $\delta'$  is also a Bayes rule. However, we assumed that  $\delta_\pi$  is the **unique** Bayes rule. Therefore,  $\delta'$  must be equal to  $\delta_\pi$  (in terms of risk functions).

5. **Conclusion:** If  $\delta'$  and  $\delta_\pi$  have identical risk functions, then  $\delta'$  cannot strictly dominate  $\delta_\pi$ . This contradicts the assumption of inadmissibility. Thus,  $\delta_\pi$  is admissible. ■

□

# 3 Bayesian Inference

## 3.1 Posterior Distributions

The foundation of Bayesian inference relies on the relationship between the prior distribution, the likelihood of the data, and the posterior distribution. This relationship is governed by Bayes' Theorem (or Law).

**Definition 3.1** (Posterior Distribution). Suppose we have a parameter  $\theta$  with a prior distribution denoted by  $\pi(\theta)$ . If we observe data  $x$  drawn from a distribution with probability density function (pdf)  $f(x; \theta)$ , then the **posterior density** of  $\theta$  given the data  $x$  is defined as:

$$\pi(\theta|x) = \frac{\pi(\theta)f(x; \theta)}{m(x)} \quad (3.1)$$

where  $m(x)$  is the **marginal distribution** (or marginal likelihood) of the data, calculated as:

$$m(x) = \int_{\Theta} \pi(\theta)f(x; \theta)d\theta \quad (3.2)$$

In this context,  $m(x)$  acts as a normalizing constant. Since it depends only on the data  $x$  and not on the parameter  $\theta$ , it ensures that the posterior density integrates to 1 but does not influence the **shape** of the posterior distribution. Thus, we often state the proportional relationship:

$$\pi(\theta|x) \propto \pi(\theta)f(x; \theta) \quad (3.3)$$

### 3.1.1 Discrete Posterior Calculation

**Example 3.1** (Discrete Posterior Calculation). Consider the following table where we calculate the posterior probabilities for a discrete parameter space.

Let the parameter  $\theta$  take values  $\{1, 2, 3\}$  with prior probabilities  $\pi(\theta)$ . Let the data  $x$  take values  $\{0, 1, 2, \dots\}$ . Given:

- Prior  $\pi(\theta)$ :  $\pi(1) = 1/3, \pi(2) = 1/3, \pi(3) = 1/3$ .
- Likelihood  $\pi(x|\theta)$ :
  - If  $\theta = 1, x \sim \text{Uniform on } \{0, 1\}$  (Prob = 1/2).
  - If  $\theta = 2, x \sim \text{Uniform on } \{0, 1, 2\}$  (Prob = 1/3).
  - If  $\theta = 3, x \sim \text{Uniform on } \{0, 1, 2, 3\}$  (Prob = 1/4).

Suppose we observe  $x = 2$ . The calculation of the posterior probabilities is summarized in the table below:

	$\theta = 1$	$\theta = 2$	$\theta = 3$	Sum
<b>Prior</b> $\pi(\theta)$	1/3	1/3	1/3	1
<b>Likelihood</b> $\pi(x = 2 \theta)$	0	1/3	1/4	-
<b>Product</b> $\pi(\theta)\pi(x \theta)$	0	1/9	1/12	7/36
<b>Posterior</b> $\pi(\theta x)$	0	4/7	3/7	1

The marginal sum (evidence) is calculated as  $0 + 1/9 + 1/12 = 4/36 + 3/36 = 7/36$ . The posterior values are obtained by dividing the product row by this sum.

### 3.1.2 Binomial-beta Conjugacy

**Example 3.2** (Binomial-beta Conjugacy). Consider an experiment where  $x|\theta \sim \text{Bin}(n, \theta)$ . The likelihood function is:

$$f(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (3.4)$$

Suppose we choose a Beta distribution as the prior for  $\theta$ , such that  $\theta \sim \text{Beta}(a, b)$ . The prior density is:

$$\pi(\theta) = \frac{\theta^{a-1} (1-\theta)^{b-1}}{B(a, b)} \quad (3.5)$$

where  $B(a, b)$  is the Beta function defined as  $\int_0^1 \theta^{a-1} (1-\theta)^{b-1} d\theta$ .

To find the posterior, we multiply the prior and the likelihood:

$$\pi(\theta|x) \propto \theta^{a-1} (1-\theta)^{b-1} \cdot \theta^x (1-\theta)^{n-x} \quad (3.6)$$

Combining terms with the same base:

$$\pi(\theta|x) \propto \theta^{a+x-1} (1-\theta)^{b+n-x-1} \quad (3.7)$$

We can recognize this kernel as a Beta distribution. Therefore, we conclude that the posterior distribution is:

$$\theta|x \sim \text{Beta}(a+x, b+n-x) \quad (3.8)$$

#### Properties of the Posterior:

- The posterior mean is:

$$E^{\theta|X}[\theta] = \frac{a+X}{a+b+n} \quad (3.9)$$

As  $n \rightarrow \infty$ , this approximates the maximum likelihood estimate  $\frac{X}{n}$ .

- The posterior variance is:

$$\text{Var}^{\theta|X}(\theta) = \frac{(a+X)(n+b-X)}{(a+b+n)^2(a+b+n+1)} \quad (3.10)$$

For large  $n$ , this approximates  $\frac{X(n-X)}{n^3} = \frac{\hat{p}(1-\hat{p})}{n}$ .

#### Numerical Illustration:

Suppose we are estimating a probability  $\theta$ .

- Prior:**  $\theta \sim \text{Beta}(2, 2)$  (Mean = 0.5).
- Data:** 10 trials, 8 successes ( $n = 10, x = 8$ ).
- Posterior:**  $\theta|x \sim \text{Beta}(2 + 8, 2 + 2) = \text{Beta}(10, 4)$  (Mean  $\approx 0.71$ ).

The plot below shows the prior (dashed) and posterior (solid) densities.

### 3.1.3 Normal-normal Conjugacy (known Variance)

**Example 3.3** (Normal-normal Conjugacy (known Variance)). Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed (i.i.d.) variables such that  $X_i \sim N(\mu, \sigma^2)$ , where  $\sigma^2$  is known.

We assign a Normal prior to the mean  $\mu$ :  $\mu \sim N(\mu_0, \sigma_0^2)$ .

To find the posterior  $\pi(\mu|x_1, \dots, x_n)$ , let  $x = (x_1, \dots, x_n)$ . The posterior is proportional to:

$$\pi(\mu|x) \propto \pi(\mu) \cdot f(x|\mu) \quad (3.11)$$

$$\propto \exp\left\{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2}\right\} \cdot \exp\left\{-\sum_{i=1}^n \frac{(x_i - \mu)^2}{2\sigma^2}\right\} \quad (3.12)$$

#### Posterior Precision:

It is often more convenient to work with **precision** (the inverse of variance). Let:

- $\tau_0 = 1/\sigma_0^2$  (Prior precision)
- $\tau = 1/\sigma^2$  (Data precision)
- $\tau_1 = 1/\sigma_1^2$  (Posterior precision)

The relationship is additive:

$$\tau_1 = \tau_0 + n\tau \quad (3.13)$$

$$\text{Posterior Precision} = \text{Prior Precision} + \text{Precision of Data} \quad (3.14)$$

The posterior mean  $\mu_1$  is a weighted average of the prior mean and the sample mean:

$$\mu_1 = \frac{\mu_0\tau_0 + n\bar{x}\tau}{\tau_0 + n\tau} \quad (3.15)$$

## Beta Prior vs Posterior

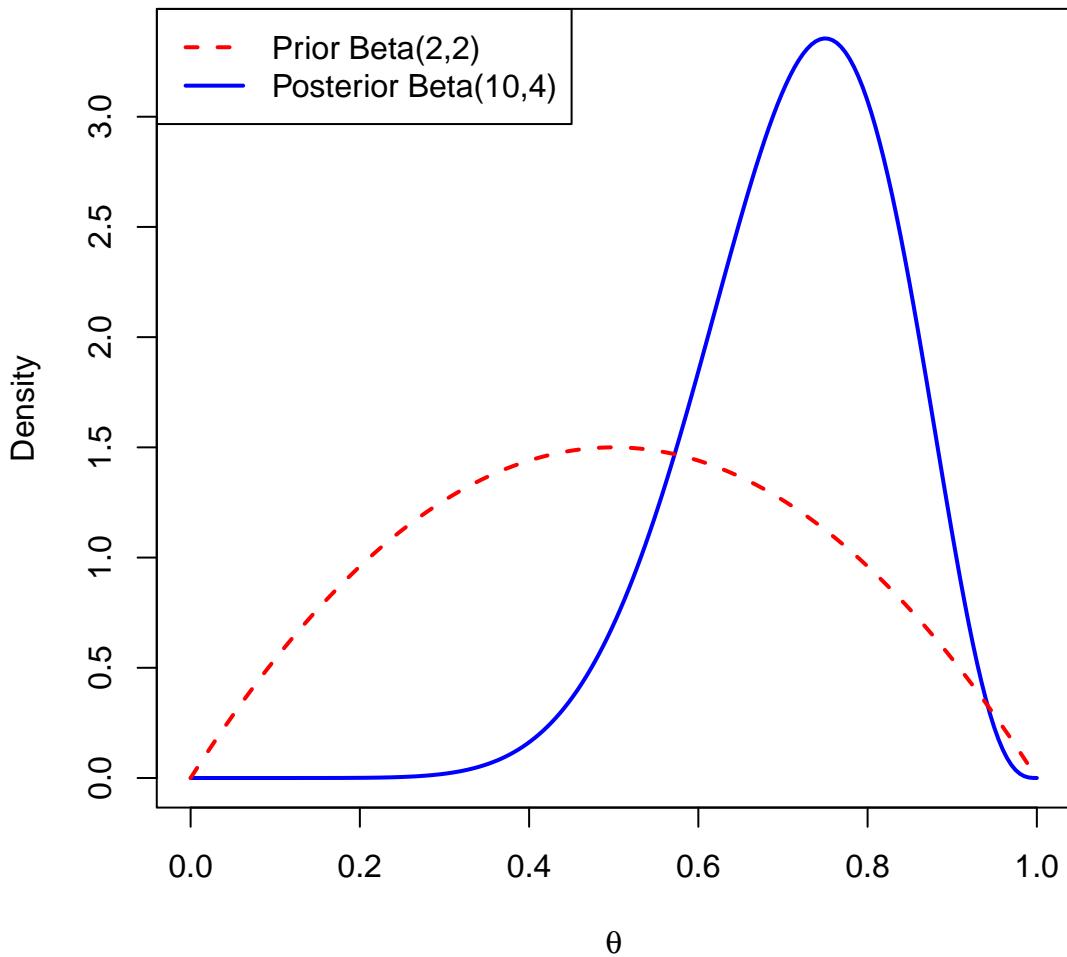


Figure 3.1: Prior vs Posterior for Beta-Binomial Example

So, the posterior distribution is:

$$\mu|x_1, \dots, x_n \sim N\left(\frac{\mu_0\tau_0 + n\bar{x}\tau}{\tau_0 + n\tau}, \frac{1}{\tau_0 + n\tau}\right) \quad (3.16)$$

### Numerical Illustration:

Suppose we estimate a mean height  $\mu$ .

- **Known Variance:**  $\sigma^2 = 100$  ( $\tau = 0.01$ ).
- **Prior:**  $\mu \sim N(175, 25)$  (Precision  $\tau_0 = 0.04$ ).
- **Data:**  $n = 10, \bar{x} = 180$ . (Total data precision  $n\tau = 0.1$ ).
- **Posterior:**
  - Precision  $\tau_1 = 0.04 + 0.1 = 0.14$ .
  - Variance  $\sigma_1^2 \approx 7.14$ .
  - Mean  $\mu_1 = \frac{175(0.04) + 180(0.1)}{0.14} \approx 178.6$ .

Figure 3.2 illustrates the prior (dashed) and posterior (solid) normal densities.

```
mu_vals <- seq(150, 200, length.out = 200)

# Prior: N(175, 25) -> SD = 5
prior_norm <- dnorm(mu_vals, mean = 175, sd = 5)

# Posterior: N(178.6, 7.14) -> SD = Sqrt(7.14) Approx 2.67
posterior_norm <- dnorm(mu_vals, mean = 178.6, sd = sqrt(7.14))

plot(mu_vals, posterior_norm, type = 'l', lwd = 2, col = "blue",
      xlab = expression(mu), ylab = "Density",
      main = "Normal Prior vs Posterior",
      ylim = c(0, max(c(prior_norm, posterior_norm))))
lines(mu_vals, prior_norm, col = "red", lty = 2, lwd = 2)
legend("topleft", legend = c("Prior N(175, 25)", "Posterior N(178.6, 7.14)"),
       col = c("red", "blue"), lty = c(2, 1), lwd = 2)
```

## Normal Prior vs Posterior

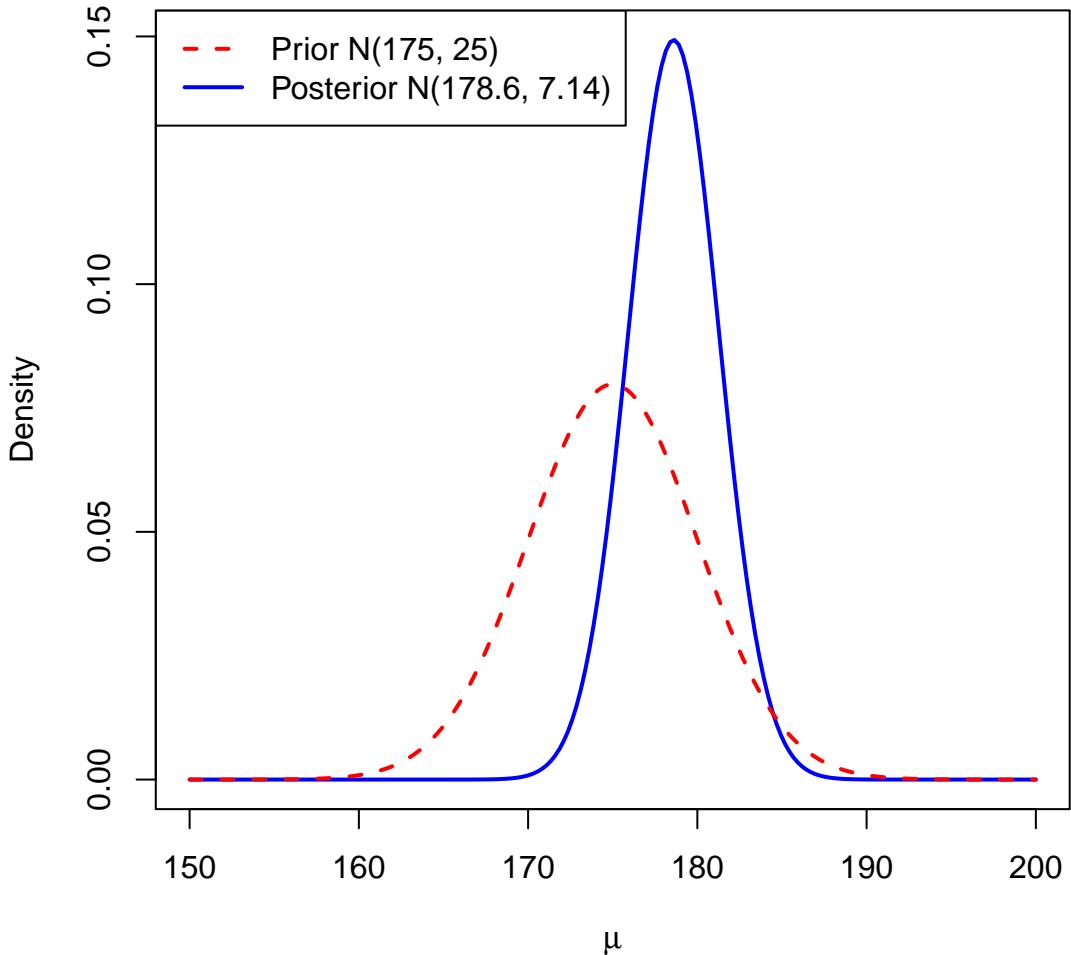


Figure 3.2: Prior vs Posterior for Normal-Normal Example

### 3.1.4 Normal with Unknown Mean and Variance

**Example 3.4** (Normal with Unknown Mean and Variance). Consider  $X_1, \dots, X_n \sim N(\mu, 1/\tau)$ , where both  $\mu$  and the precision  $\tau$  are unknown.

We use a **Normal-Gamma** conjugate prior with parameters  $\mu_0, \tau_0, \alpha_0, w_0$ :

- $\tau \sim \text{Gamma}(\alpha_0/2, \alpha_0 w_0/2)$

$$\pi(\tau) \propto \tau^{\alpha_0/2-1} \exp\left\{-\frac{\alpha_0 w_0}{2}\tau\right\} \quad (3.17)$$

- $\mu|\tau \sim N(\mu_0, 1/(\tau_0\tau))$

$$\pi(\mu|\tau) \propto \tau^{1/2} \exp\left\{-\frac{\tau_0\tau}{2}(\mu - \mu_0)^2\right\} \quad (3.18)$$

The joint prior is:

$$\pi(\mu, \tau) \propto \tau^{(\alpha_0+1)/2-1} \exp \left\{ -\frac{\tau}{2} (\alpha_0 w_0 + \tau_0(\mu - \mu_0)^2) \right\} \quad (3.19)$$

### The Likelihood:

To derive the Maximum Likelihood Estimators (MLEs), we work with the log-likelihood function  $l(\mu, \tau) = \log L(\mu, \tau)$ :

$$\begin{aligned} l(\mu, \tau) &= \log \left( \tau^{n/2} \exp \left\{ -\frac{\tau}{2} [S_{xx} + n(\bar{x} - \mu)^2] \right\} \right) \\ &= \frac{n}{2} \log \tau - \frac{\tau}{2} [S_{xx} + n(\bar{x} - \mu)^2] + \text{const} \end{aligned} \quad (3.20)$$

### MLE for $\mu$

Differentiating  $l(\mu, \tau)$  with respect to  $\mu$  and setting to zero:

$$\frac{\partial l}{\partial \mu} = n\tau(\bar{x} - \mu) = 0 \implies \hat{\mu}_{\text{MLE}} = \bar{x} \quad (3.21)$$

### MLE for $\sigma^2$

Differentiating  $l(\mu, \tau)$  with respect to  $\tau$ , setting to zero, and substituting  $\mu = \bar{x}$ :

$$\frac{\partial l}{\partial \tau} = \frac{n}{2\tau} - \frac{S_{xx}}{2} = 0 \implies \hat{\tau}_{\text{MLE}} = \frac{n}{S_{xx}} \quad (3.22)$$

Using the invariance property ( $\sigma^2 = 1/\tau$ ):

$$\hat{\sigma}_{\text{MLE}}^2 = \frac{S_{xx}}{n} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \quad (3.23)$$

### Derivation of the Posterior:

Multiplying the prior by the likelihood gives the joint posterior density. We organize the terms to separate the marginal distribution of  $\tau$  from the conditional distribution of  $\mu$ :

$$\begin{aligned} \pi(\mu, \tau | x) &\propto \underbrace{\tau^{(\alpha_0+n)/2-1} \exp \left\{ -\frac{\tau}{2} \left[ \alpha_0 w_0 + S_{xx} + \frac{n\tau_0}{n+\tau_0}(\bar{x} - \mu_0)^2 \right] \right\}}_{\text{Marginal of } \tau} \\ &\times \underbrace{\tau^{1/2} \exp \left\{ -\frac{(n+\tau_0)\tau}{2} \left( \mu - \frac{\tau_0\mu_0 + n\bar{x}}{n+\tau_0} \right)^2 \right\}}_{\text{Conditional of } \mu | \tau} \end{aligned} \quad (3.24)$$

### Results:

- **Conditional Posterior of  $\mu | \tau, x$ :**

$$\mu | \tau, x \sim N(\mu', 1/(\tau' \tau)) \quad (3.25)$$

$$E^{\mu | \tau, X}[\mu] = \frac{\tau_0 \mu_0 + n \bar{x}}{\tau_0 + n} \quad (3.26)$$

where

$$\tau' = \tau_0 + n \quad (3.27)$$

$$\mu' = \frac{\tau_0 \mu_0 + n \bar{x}}{\tau_0 + n} \quad (3.28)$$

- **Marginal Posterior of  $\tau|x$ :** The marginal posterior is  $\tau|x \sim \text{Gamma}(\alpha', \beta')$  with:

$$\alpha' = \frac{\alpha_0 + n}{2}, \quad \beta' = \frac{\alpha_0 w_0 + n\hat{\sigma}_{\text{MLE}}^2 + \frac{n\tau_0}{n+\tau_0}(\bar{x} - \mu_0)^2}{2} \quad (3.29)$$

Using the approximation  $E^{\sigma^2|X}[\sigma^2] \approx 1/E^{\tau|X}[\tau] = \beta'/\alpha'$ , the posterior expectation of the variance is a weighted average of the prior variance, the data variance, and the discrepancy between the prior and data means:

$$E^{\sigma^2|X}[\sigma^2] \approx \frac{\alpha_0 w_0 + n\hat{\sigma}_{\text{MLE}}^2 + \frac{1}{1/n+1/\tau_0}(\bar{x} - \mu_0)^2}{\alpha_0 + n} \quad (3.30)$$

- **Conditional Posterior of  $\tau|\mu, x$ :** If  $\mu$  is considered known, the posterior for  $\tau$  combines the prior  $\alpha_0, w_0$  with the deviations from  $\mu$ . Note that the prior term  $\pi(\mu|\tau)$  contributes an extra factor of  $\tau^{1/2}$  to the shape.

$$\tau|\mu, x \sim \text{Gamma}(\alpha'', \beta'') \quad (3.31)$$

Where:

$$\alpha'' = \frac{\alpha_0 + n + 1}{2}, \quad \beta'' = \frac{\alpha_0 w_0 + \sum_{i=1}^n (x_i - \mu)^2 + \tau_0(\mu - \mu_0)^2}{2} \quad (3.32)$$

The approximate expectation of the variance is:

$$E^{\sigma^2|\mu, X}[\sigma^2] \approx \frac{\alpha_0 w_0 + \sum_{i=1}^n (x_i - \mu)^2 + \tau_0(\mu - \mu_0)^2}{\alpha_0 + n + 1} \quad (3.33)$$

## 3.2 Finding Bayes Rules via Minimizing Posterior Expected Loss

The general form of Bayes rule is derived by minimizing risk.

**Definition 3.2** (Risk Function and Bayes Risk). **Setup and Notation:**

- $\theta \in \Theta$ : parameter of interest (unknown state of nature)
- $x \in X$ : observed data
- $\pi(\theta)$ : prior probability distribution over the parameter space
- $f(x; \theta)$ : likelihood or sampling distribution of the data given the parameter
- $d : X \rightarrow A$ : decision rule mapping observed data to an action/decision
- $\mathcal{L}(\theta, a)$ : loss function measuring the loss incurred when the true parameter is  $\theta$  and action  $a$  is taken

**Definition:**

- **Risk Function:** For a given decision rule  $d$  and parameter value  $\theta$ ,

$$R(\theta, d) = \int_X \mathcal{L}(\theta, d(x)) f(x; \theta) dx = E^{X|\theta}[\mathcal{L}(\theta, d(X))] \quad (3.34)$$

is the expected loss with respect to the sampling distribution when the true parameter is  $\theta$ .

- **Bayes Risk:** For a decision rule  $d$  and prior distribution  $\pi$ ,

$$r(\pi, d) = \int_{\Theta} R(\theta, d)\pi(\theta)d\theta = E^{\theta}[R(\theta, d)] \quad (3.35)$$

is the expected risk averaging over both the parameter uncertainty (prior) and the data variability (likelihood).

- **Posterior Bayes Loss:** The minimum possible expected loss given observed data  $x$  is denoted as  $\rho^{\text{Bayes}}(\pi, x)$ . It represents the expected posterior loss of the Bayes rule:

$$\rho^{\text{Bayes}}(\pi, x) = \inf_d E^{\theta|x}[\mathcal{L}(\theta, d)] \quad (3.36)$$

**Theorem 3.1** (Minimization of Bayes Risk). *Minimizing the Bayes risk  $r(\pi, d)$  is equivalent to minimizing the posterior expected loss for each observed  $x$ . That is, the Bayes rule  $d(x)$  is defined as*

$$d^{\text{Bayes}}(x) = \arg \min_a E^{\theta|x}[\mathcal{L}(\theta, a)] \quad (3.37)$$

The value of the minimum expected posterior loss is  $\rho^{\text{Bayes}}(\pi, x)$ .

*Proof.* We start by writing the Bayes risk essentially as a double integral over the parameters and the data. Substituting the definition of the risk function  $R(\theta, d)$ :

$$\begin{aligned} r(\pi, d) &= \int_{\Theta} R(\theta, d)\pi(\theta)d\theta \\ &= \int_{\Theta} \left[ \int_X \mathcal{L}(\theta, d(x))f(x|\theta)dx \right] \pi(\theta)d\theta \end{aligned} \quad (3.38)$$

Assuming the conditions for Fubini's Theorem are met, we switch the order of integration:

$$r(\pi, d) = \int_X \left[ \int_{\Theta} \mathcal{L}(\theta, d(x))f(x|\theta)\pi(\theta)d\theta \right] dx \quad (3.39)$$

Recall that the joint density can be factored as  $f(x, \theta) = f(x|\theta)\pi(\theta) = \pi(\theta|x)m(x)$ , where  $m(x)$  is the marginal density of the data. Substituting this into the inner integral:

$$\begin{aligned} r(\pi, d) &= \int_X \left[ \int_{\Theta} \mathcal{L}(\theta, d(x))\pi(\theta|x)m(x)d\theta \right] dx \\ &= \int_X m(x) \left[ \int_{\Theta} \mathcal{L}(\theta, d(x))\pi(\theta|x)d\theta \right] dx \end{aligned} \quad (3.40)$$

Since the marginal density  $m(x)$  is non-negative, minimizing the total integral  $r(\pi, d)$  with respect to the decision rule  $d(\cdot)$  is equivalent to minimizing the term inside the brackets for every  $x$  (specifically where  $m(x) > 0$ ). The term inside the brackets is the **Posterior Expected Loss**:

$$\int_{\Theta} \mathcal{L}(\theta, d(x)) \pi(\theta|x) d\theta = E^{\theta|X}[\mathcal{L}(\theta, d(X))] \quad (3.41)$$

□

### ! Important

Therefore, to minimize the Bayes risk, one effectively minimizes the posterior expected loss for each  $x$ . This relationship relies on the key identity for the total expectation of the loss:

$$r(\pi, d) = E^X [E^{\theta|X}(\mathcal{L}(\theta, d(X)))] = E^{\theta} [E^{X|\theta}(\mathcal{L}(\theta, d(X)))] \quad (3.42)$$

In the first expression, the outer expectation  $E^X$  is taken with respect to the **marginal density of the data**,  $m(x)$ , defined as:

$$m(x) = \int_{\Theta} f(x|\theta) \pi(\theta) d\theta \quad (3.43)$$

In the second expression, the outer expectation  $E^{\theta}$  is taken with respect to the **prior density**  $\pi(\theta)$ .

The following diagram summarizes the general workflow for deriving a Bayes estimator:

## 3.3 Special Bayes Rules

### 3.3.1 Squared Error Loss (point Estimate)

$$\mathcal{L}(\theta, a) = (\theta - a)^2 \quad (3.44)$$

To find the optimal estimator  $d(x)$ , we minimize the posterior expected loss  $E^{\theta|X}[(\theta - d(X))^2]$ . Taking the derivative with respect to  $d$  and setting it to 0:

$$-2E^{\theta|X}(\theta - d) = 0 \implies d(X) = E^{\theta|X}[\theta] \quad (3.45)$$

**Result:** The Bayes rule under squared error loss is the **posterior mean**.

### 3.3.2 Scale-Invariant Squared Error Loss

Consider the loss function that penalizes relative errors rather than absolute errors. This is particularly useful when the magnitude of the parameter  $\theta$  varies significantly, and an error of 1.0 is “worse” when  $\theta = 1$  than when  $\theta = 1000$ .

$$\mathcal{L}(\theta, d) = \left( \frac{d - \theta}{\theta} \right)^2 = \left( \frac{d}{\theta} - 1 \right)^2 \quad (3.46)$$

To find the Bayes rule, we minimize the posterior expected loss  $E^{\theta|X}[\mathcal{L}(\theta, d)]$ :

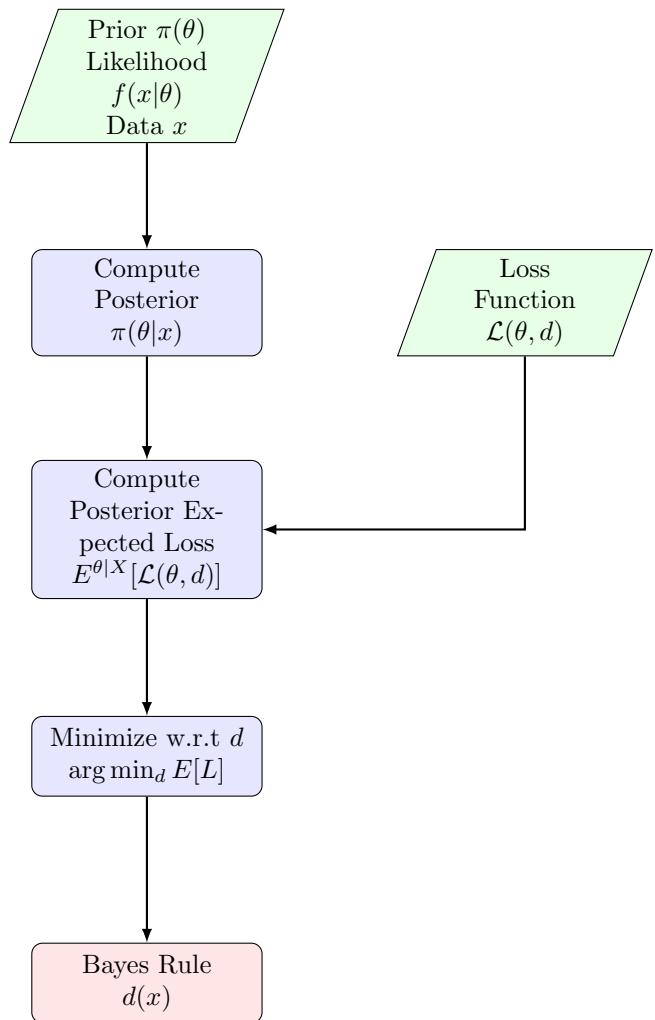


Figure 3.3: Workflow for Finding the Bayes Rule

$$Q(d) = E^{\theta|X} \left[ \frac{d^2}{\theta^2} - \frac{2d}{\theta} + 1 \right] = d^2 E^{\theta|X}[\theta^{-2}] - 2d E^{\theta|X}[\theta^{-1}] + 1 \quad (3.47)$$

Differentiating with respect to  $d$  and setting to zero:

$$\frac{\partial Q}{\partial d} = 2d E^{\theta|X}[\theta^{-2}] - 2E^{\theta|X}[\theta^{-1}] = 0 \quad (3.48)$$

Solving for  $d$ :

$$d(X) = \frac{E^{\theta|X}[\theta^{-1}]}{E^{\theta|X}[\theta^{-2}]} \quad (3.49)$$

**Result:** The Bayes rule under scale-invariant squared error loss is the ratio of the posterior mean of  $\theta^{-1}$  to the posterior mean of  $\theta^{-2}$ .

### 3.3.3 Absolute Error Loss

$$\mathcal{L}(\theta, d) = |\theta - d| \quad (3.50)$$

To find the Bayes rule, we minimize the posterior expected loss:

$$\psi(d) = E^{\theta|X}[|\theta - d|] = \int_{-\infty}^{\infty} |\theta - d| dF(\theta|x) \quad (3.51)$$

where  $F(\theta|x)$  is the cumulative distribution function (CDF) of the posterior. Splitting the integral at the decision point  $d$ :

$$\psi(d) = \int_{-\infty}^d (d - \theta) dF(\theta|x) + \int_d^{\infty} (\theta - d) dF(\theta|x) \quad (3.52)$$

We find the minimum by analyzing the rate of change of  $\psi(d)$  with respect to  $d$ . Differentiating (or taking the subgradient for non-differentiable points):

$$\frac{\partial}{\partial d} \psi(d) = \int_{-\infty}^d 1 dF(\theta|x) - \int_d^{\infty} 1 dF(\theta|x) = P(\theta \leq d|x) - P(\theta > d|x) \quad (3.53)$$

Setting this derivative to zero implies we seek a point where the probability mass to the left equals the probability mass to the right:

$$P(\theta \leq d|x) = P(\theta > d|x) \quad (3.54)$$

Since the total probability is 1, this condition simplifies to finding  $d$  such that the cumulative probability is 1/2.

### General Case (Discrete or Mixed Distributions)

In cases where the posterior distribution is discrete or has jump discontinuities (e.g., the CDF jumps from 0.4 to 0.6 at a specific value), an exact solution to  $F(d) = 0.5$  may not exist. To generalize, the Bayes rule is defined as any **median**  $m$  of the posterior distribution.

A median is formally defined as any value  $m$  that satisfies the following two conditions simultaneously:

- $P(\theta \leq m|x) \geq \frac{1}{2}$
- $P(\theta \geq m|x) \geq \frac{1}{2}$

**Result:** The Bayes rule under absolute error loss is the **posterior median**.

#### 3.3.4 Weighted Absolute Error Loss (min-normalization)

$$\mathcal{L}(\theta, d) = \frac{|\theta - d|}{\min(\theta, 1 - \theta)} \quad (3.55)$$

This loss function penalizes errors extremely heavily when the true parameter  $\theta$  is near the boundaries (0 or 1). Because the denominator approaches zero at the boundaries, the “cost” of an error becomes infinite, forcing the estimator to be very cautious (conservative) if the posterior has significant mass near 0 or 1.

To find the Bayes rule, we minimize the posterior expected loss. Let  $\pi(\theta|x)$  denote the posterior density.

$$\psi(d) = E^{\theta|x} \left[ \frac{|\theta - d|}{\min(\theta, 1 - \theta)} \right] = \int \frac{|\theta - d|}{\min(\theta, 1 - \theta)} \pi(\theta|x) d\theta \quad (3.56)$$

Let  $w(\theta) = \frac{1}{\min(\theta, 1 - \theta)}$ . We can view this integral as an expectation with respect to a **weighted posterior density**  $\pi^*(\theta|x)$ :

$$\pi^*(\theta|x) \propto w(\theta)\pi(\theta|x) = \frac{\pi(\theta|x)}{\min(\theta, 1 - \theta)} \quad (3.57)$$

**Result:** The Bayes rule is the **median** of the weighted posterior distribution  $\pi^*(\theta|x)$ .

**Algorithm 3.1** (Importance Sampling for Weighted Median). **Goal:** Estimate the median of  $\pi^*(\theta|x) \propto w(\theta)\pi(\theta|x)$  using samples from  $\pi(\theta|x)$ .

1. **Sample:** Generate  $M$  independent draws  $\theta_1, \dots, \theta_M$  from the standard posterior  $\pi(\theta|x)$ .

2. **Weight:** For each  $i = 1, \dots, M$ , compute the importance weight:

$$W_i = w(\theta_i) = \frac{1}{\min(\theta_i, 1 - \theta_i)} \quad (3.58)$$

3. **Sort:** Reorder the samples such that  $\theta_{(1)} \leq \theta_{(2)} \leq \dots \leq \theta_{(M)}$ . Permute the weights  $W_{(1)}, \dots, W_{(M)}$  to match this ordering.

4. **Accumulate:** Compute the cumulative weights:

$$S_k = \sum_{j=1}^k W_{(j)} \quad \text{for } k = 1, \dots, M \quad (3.59)$$

5. **Select:** Find the smallest index  $k^*$  such that the cumulative weight exceeds half the total weight:

$$k^* = \min\{k : S_k \geq 0.5 \times S_M\} \quad (3.60)$$

6. **Output:** Return the estimator  $\hat{\delta} = \theta_{(k^*)}$ .

### Numerical Example: Beta(2, 10)

We compare the “exact” weighted median (found by numerical integration) with the Monte Carlo estimate for a skewed distribution.

```
# 1. Setup
set.seed(2025)
M <- 10
alpha <- 2
beta <- 10

theta_samples <- rbeta(M, alpha, beta)
w <- function(theta) { 1 / pmin(theta, 1 - theta) }

# 2. Process
weights <- w(theta_samples)
ord <- order(theta_samples)
sorted_theta <- theta_samples[ord]
sorted_weights <- weights[ord]
cum_weights <- cumsum(sorted_weights)
total_weight <- sum(sorted_weights)
threshold <- 0.5 * total_weight

# Find k
k_idx <- which(cum_weights >= threshold)[1]

# 3. Create Data Frame
selection_table <- data.frame(
  idx = 1:M,
  theta = sorted_theta,
  weight = sorted_weights,
  cum_weight = cum_weights,
  check = ifelse(cum_weights >= threshold, "$\\geq$ Threshold", "$<$ Threshold"),
  sel = ifelse(1:M == k_idx, "$\\leftarrow k$ (Median)", ""))
)
```

```

# 4. Set LaTeX Column Names
# Note: We use double backslashes \\ for LaTeX commands inside R strings
colnames(selection_table) <- c(
  "$i$",
  "$\\theta_{(i)}$",
  "$w_{(i)}$",
  "$\\sum_{j=1}^i w_{(j)}$",
  "Condition",
  "Selection"
)
# Print Context
cat("Total Weight ($\\sum w_i$):", total_weight, "\n")

```

Total Weight ( $\sum w_i$ ): 48.91008

```
cat("Threshold ($0.5 \\times \\sum w_i$):", threshold, "\n\n")
```

Threshold ( $0.5 \times \sum w_i$ ): 24.45504

```

# 5. Render Table with escape = FALSE
# escape = FALSE is crucial; otherwise, it prints the dollar signs literally
knitr::kable(selection_table,
             digits = 4,
             align = "c",
             escape = FALSE)

```

$i$	$\theta_{(i)}$	$w_{(i)}$	$\sum_{j=1}^i w_{(j)}$	Condition	Selection
1	0.1256	7.9601	7.9601	< Threshold	
2	0.1462	6.8412	14.8013	< Threshold	
3	0.1563	6.3965	21.1978	< Threshold	
4	0.1714	5.8329	27.0307	$\geq$ Threshold	
5	0.2221	4.5024	31.5330	$\geq$ Threshold	
6	0.2265	4.4144	35.9475	$\geq$ Threshold	
7	0.2676	3.7376	39.6850	$\geq$ Threshold	
8	0.2840	3.5214	43.2064	$\geq$ Threshold	
9	0.2990	3.3445	46.5509	$\geq$ Threshold	
10	0.4239	2.3592	48.9101	$\geq$ Threshold	$\leftarrow k$ (Median)

Actual Weighted Median with 1000 draws of  $\theta$

```

# 1. Setup Parameters
set.seed(123)
M <- 1000
alpha <- 2
beta <- 10

# 2. Generate Samples and Weights
theta_samples <- rbeta(M, alpha, beta)
# Weight function: w(theta) = 1 / min(theta, 1-theta)
w <- function(theta) { 1 / pmin(theta, 1 - theta) }
weights <- w(theta_samples)

# 3. Sort and Calculate Cumulative Weights
ord <- order(theta_samples)
sorted_theta <- theta_samples[ord]
sorted_weights <- weights[ord]

cum_weights <- cumsum(sorted_weights)
total_weight <- sum(sorted_weights)
threshold <- 0.5 * total_weight

# 4. Find the Weighted Median Index k
k_idx <- which(cum_weights >= threshold)[1]
mc_weighted_median <- sorted_theta[k_idx]

# 5. Compare with Theoretical Value (calculated previously)
# Re-calculating theoretical for completeness of this chunk
weighted_dens_unnorm <- function(theta) { w(theta) * dbeta(theta, alpha, beta) }
C <- integrate(weighted_dens_unnorm, 0, 1)$value
weighted_cdf <- function(q) { integrate(weighted_dens_unnorm, 0, q)$value / C }
theo_median <- uniroot(function(x) weighted_cdf(x) - 0.5, c(0.001, 0.999))$root

# 6. Display Results
results <- data.frame(
  "Method" = c("Theoretical (Integration)", "Monte Carlo (M=1000)", "Standard Median (Unweighted)"),
  "Value" = c(theo_median, mc_weighted_median, qbeta(0.5, alpha, beta))
)

knitr::kable(results, digits = 4, align = "l", caption = "Weighted Median Estimation")

```

Table 3.3: Weighted Median Estimation

Method	Value
Theoretical (Integration)	0.0670
Monte Carlo (M=1000)	0.0692

Method	Value
Standard Median (Unweighted)	0.1480

### 3.3.5 Hypothesis Testing (0-1 Loss)

Consider the hypothesis test  $H_0 : \theta \in \Theta_0$  versus  $H_1 : \theta \in \Theta_1$ . We define the decision space as  $\mathcal{A} = \{0, 1\}$ , where  $a = 0$  means accepting  $H_0$  and  $a = 1$  means rejecting  $H_0$  (accepting  $H_1$ ).

#### Case 1: 0-1 Loss

The standard 0-1 loss function assigns a penalty of 1 for an incorrect decision and 0 for a correct one:

Table 3.4: Standard 0-1 Loss Function

State of Nature ( $\theta$ )	Action $a = 0$ (Accept $H_0$ )	Action $a = 1$ (Reject $H_0$ )
$\theta \in \Theta_0$ ( $H_0$ True)	0 (Correct)	1 (Type I Error)
$\theta \in \Theta_1$ ( $H_1$ True)	1 (Type II Error)	0 (Correct)

To find the Bayes rule, we minimize the **posterior expected loss** for a given  $x$ , denoted as  $E^{\theta|X}[\mathcal{L}(\theta, a)]$ .

- **Expected Loss for choosing  $a = 0$  (Accept  $H_0$ ):**

$$E^{\theta|X}[\mathcal{L}(\theta, 0)] = 0 \cdot P(\theta \in \Theta_0|x) + 1 \cdot P(\theta \in \Theta_1|x) = P(\theta \in \Theta_1|x) \quad (3.61)$$

- **Expected Loss for choosing  $a = 1$  (Reject  $H_0$ ):**

$$E^{\theta|X}[\mathcal{L}(\theta, 1)] = 1 \cdot P(\theta \in \Theta_0|x) + 0 \cdot P(\theta \in \Theta_1|x) = P(\theta \in \Theta_0|x) \quad (3.62)$$

The Bayes rule selects the action with the smaller expected loss. Thus, we choose  $a = 1$  if:

$$P(\theta \in \Theta_0|x) \leq P(\theta \in \Theta_1|x) \quad (3.63)$$

This confirms that under 0-1 loss, the Bayes rule simply selects the hypothesis with the higher posterior probability. The optimal Bayes decision rule  $d(x)$  is given by:

$$d(x) = \begin{cases} 1 & \text{if } P(\Theta_0|x) \leq \frac{1}{2} \quad (\text{Reject } H_0) \\ 0 & \text{if } P(\Theta_0|x) > \frac{1}{2} \quad (\text{Accept } H_0) \end{cases} \quad (3.64)$$

#### Case 2: General Loss (Asymmetric Costs)

In many practical applications, the cost of errors is not symmetric. For example, a Type I error (false rejection) might be more costly than a Type II error. Let  $c_1$  be the cost of a Type I error and  $c_2$  be the cost of a Type II error. Usually, we normalize one cost to 1.

Table 3.5: Loss Function with Type I Error Cost  $c$

State of Nature ( $\theta$ )	Action $a = 0$ (Accept $H_0$ )	Action $a = 1$ (Reject $H_0$ )
$\theta \in \Theta_0$ ( $H_0$ True)	0	$c$ (Type I Error)
$\theta \in \Theta_1$ ( $H_1$ True)	1 (Type II Error)	0

We again calculate the posterior expected loss:

- **Expected Loss for  $a = 0$ :**

$$E^{\theta|X}[\mathcal{L}(\theta, 0)] = 0 \cdot P(\Theta_0|x) + 1 \cdot P(\Theta_1|x) = P(\Theta_1|x) \quad (3.65)$$

- **Expected Loss for  $a = 1$ :**

$$E^{\theta|X}[\mathcal{L}(\theta, 1)] = c \cdot P(\Theta_0|x) + 0 \cdot P(\Theta_1|x) = cP(\Theta_0|x) \quad (3.66)$$

We reject  $H_0$  ( $a = 1$ ) if the expected loss of doing so is lower:

$$cP(\Theta_0|x) \leq P(\Theta_1|x) \quad (3.67)$$

Since  $P(\Theta_1|x) = 1 - P(\Theta_0|x)$ , we can rewrite this condition as:

$$cP(\Theta_0|x) \leq 1 - P(\Theta_0|x) \implies (1 + c)P(\Theta_0|x) \leq 1 \quad (3.68)$$

$$P(\Theta_0|x) \leq \frac{1}{1 + c} \quad (3.69)$$

**Result:** With asymmetric costs, we accept  $H_1$  only if the posterior probability of the null hypothesis is sufficiently small (below the threshold  $\frac{1}{1+c}$ ). If the cost of false rejection  $c$  is high, we require stronger evidence against  $H_0$ . The optimal Bayes decision rule  $d(x)$  is given by:

$$d(x) = \begin{cases} 1 & \text{if } P(\Theta_0|x) \leq \frac{1}{1+c} \quad (\text{Reject } H_0) \\ 0 & \text{if } P(\Theta_0|x) > \frac{1}{1+c} \quad (\text{Accept } H_0) \end{cases} \quad (3.70)$$

### 3.3.6 Classification Prediction

In classification problems, the parameter of interest is a discrete class label  $y$  taking values in a set of categories  $\{1, 2, \dots, K\}$ . The goal is to predict the true class label based on observed features  $x$ .

We typically employ the **0-1 loss function**, which assigns a penalty of 1 for a misclassification and 0 for a correct prediction:

$$\mathcal{L}(y, \hat{y}) = \begin{cases} 0 & \text{if } \hat{y} = y \text{ (Correct Classification)} \\ 1 & \text{if } \hat{y} \neq y \text{ (Misclassification)} \end{cases} \quad (3.71)$$

To find the optimal classification rule (the Bayes Classifier), we minimize the posterior expected loss, which is equivalent to minimizing the probability of misclassification.

$$E^{Y|X}[\mathcal{L}(y, \hat{y})] = \sum_y \mathcal{L}(y, \hat{y}) P(y|x) \quad (3.72)$$

Since the loss is 1 only when the predicted class  $\hat{y}$  differs from the true class  $y$ , this sum simplifies to:

$$E^{Y|X}[\mathcal{L}(y, \hat{y})] = \sum_{y \neq \hat{y}} 1 \cdot P(y|x) = P(y \neq \hat{y}|x) = 1 - P(y = \hat{y}|x) \quad (3.73)$$

Minimizing the misclassification rate  $1 - P(y = \hat{y}|x)$  is mathematically equivalent to maximizing the probability of being correct,  $P(y = \hat{y}|x)$ .

### Result:

The Bayes rule for classification is to predict the class with the highest posterior **predictive** probability. In the context of machine learning and pattern recognition, this decision rule is known as the **Bayes Optimal Classifier**.

$$\hat{y}_{\text{Bayes}}(x) = \arg \max_{k \in \{1, \dots, K\}} P(y = k|x) \quad (3.74)$$

### 3.3.7 Interval Estimation as a Decision Problem

We can motivate the choice of a Credible Interval by defining a specific loss function for interval estimation. We define the **action space**  $\mathcal{A}$  as the set of all intervals of fixed radius  $\delta > 0$  centered at  $d$ , i.e.,  $\mathcal{A} = \{[d - \delta, d + \delta] \mid d \in \mathbb{R}\}$ .

The loss function is defined as:

$$\mathcal{L}(\theta, d) = \begin{cases} 0 & \text{if } |\theta - d| \leq \delta \quad (\theta \in [d - \delta, d + \delta]) \\ 1 & \text{if } |\theta - d| > \delta \quad (\theta \notin [d - \delta, d + \delta]) \end{cases} \quad (3.75)$$

### Derivation of the Bayes Rule

We minimize the **Expected Posterior Loss**, which is simply the probability that  $\theta$  falls outside the interval:

$$E^{\theta|X}[\mathcal{L}(\theta, d)] = 1 \cdot P(|\theta - d| > \delta|x) = 1 - P(d - \delta \leq \theta \leq d + \delta|x) \quad (3.76)$$

Minimizing this loss is equivalent to maximizing the posterior probability mass contained within the interval. Thus, the Bayes estimator  $d$  is:

$$d_{\text{Bayes}} = \arg \max_d \int_{d-\delta}^{d+\delta} \pi(\theta|x) d\theta \quad (3.77)$$

To find the optimal  $d$ , we differentiate the integral with respect to  $d$  and set it to zero:

$$\frac{\partial}{\partial d} \left( \int_{d-\delta}^{d+\delta} \pi(\theta|x) d\theta \right) = \pi(d + \delta|x) - \pi(d - \delta|x) = 0 \quad (3.78)$$

This yields the condition  $\pi(d + \delta|x) = \pi(d - \delta|x)$ .

The optimal  $d$  centers the interval such that the posterior density heights at the two endpoints are equal. This is the defining characteristic of a **Highest Posterior Density (HPD)** interval.

### 90% Credible Intervals (Skewed Posterior)

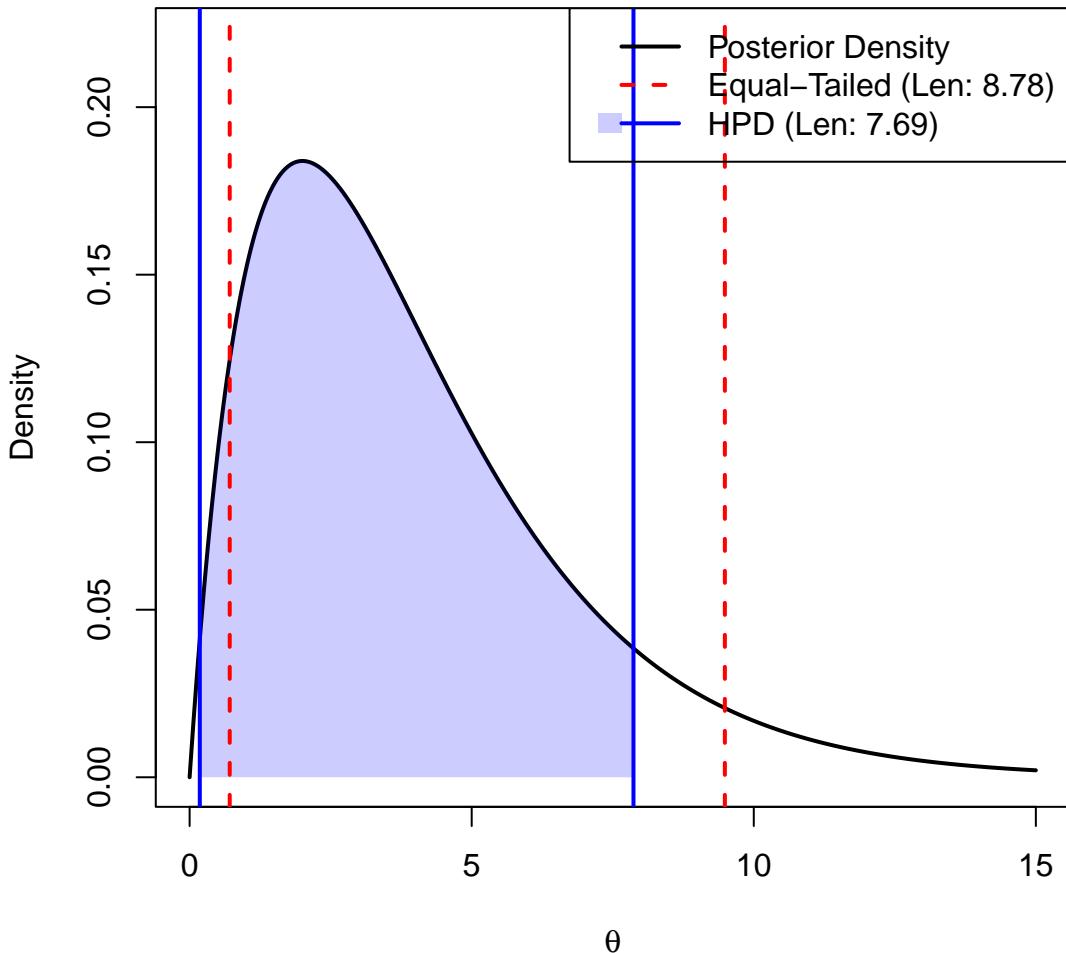


Figure 3.4: Comparison of HPD and Equal-Tailed Intervals for a Skewed Distribution

#### Comparison with Equal-Tailed Intervals:

- **Equal-Tailed Interval:** We simply cut off  $\alpha/2$  probability from each tail of the distribution. This is easy to compute but may not be the shortest interval if the distribution is skewed.
- **HPD Interval:** This is the shortest possible interval for the given coverage. For unimodal distributions, the probability density at the two endpoints of the HPD interval is identical.

The plot below illustrates a skewed posterior distribution (Gamma). Notice how the **HPD Interval (Blue)** is shifted toward the mode (the peak) to capture the highest density values, resulting in a shorter interval length compared to the **Equal-Tailed Interval (Red)**.

## 3.4 Finding Minimax Rules with Bayes Rules

Theorem 2.1 states that if a Bayes estimator  $\delta^\pi$  (derived from a prior  $\pi$ ) yields a constant risk  $R(\theta, \delta^\pi) = c$  across the entire parameter space  $\Theta$ , then that estimator is necessarily minimax.

This result is a cornerstone of decision theory because it provides a sufficient condition for minimaxity. While the minimax criterion focuses on the “worst-case scenario” by minimizing the maximum possible risk, the Bayes criterion focuses on the “average-case scenario” relative to a prior. When the risk is constant, these two perspectives align: the average risk equals the maximum risk, and no other estimator can achieve a lower maximum without also having a lower Bayes risk, which would contradict the optimality of the Bayes rule.

### 3.4.1 Binomial Minimax Estimator

**Example 3.5.** Let  $X \sim \text{Bin}(n, \theta)$  and  $\theta \sim \text{Beta}(a, b)$ . The squared error loss is  $\mathcal{L}(\theta, d) = (\theta - d)^2$ . The Bayes estimator is the posterior mean:

$$d(X) = \frac{a + X}{a + b + n} \quad (3.79)$$

We calculate the risk  $R(\theta, d)$ :

$$R(\theta, d) = E^{X|\theta} \left[ \left( \theta - \frac{a + X}{a + b + n} \right)^2 \right] \quad (3.80)$$

Let  $c = a + b + n$ .

$$R(\theta, d) = \frac{1}{c^2} E^{X|\theta} [(c\theta - a - X)^2] \quad (3.81)$$

Using the bias-variance decomposition and knowing  $E^{X|\theta}[X] = n\theta$  and  $E^{X|\theta}[X^2] = (n\theta)^2 + n\theta(1 - \theta)$ , we expand the risk function. To make the risk constant (independent of  $\theta$ ), we set the coefficients of  $\theta$  and  $\theta^2$  to zero. Solving the resulting system of equations yields:

$$a = b = \frac{\sqrt{n}}{2} \quad (3.82)$$

Thus, the minimax estimator is:

$$d(X) = \frac{X + \sqrt{n}/2}{n + \sqrt{n}} \quad (3.83)$$

This differs from the standard MLE  $\hat{\theta} = X/n$  and the uniform prior Bayes estimator ( $a = b = 1$ ).

According to Theorem 2.2, let  $\{\delta_n\}$  be a sequence of Bayes rules with respect to priors  $\{\pi_n\}$ , and let  $r(\pi_n, \delta_n)$  be the associated Bayes risks. If there exists a rule  $\delta_0$  such that

$$\sup_{\theta} R(\theta, \delta_0) \leq \lim_{n \rightarrow \infty} r(\pi_n, \delta_n) \quad (3.84)$$

then  $\delta_0$  is a minimax estimator.

We can rewrite the Minimax estimator  $d(X)$  as a linear combination of the sample proportion (MLE)  $\hat{\theta} = X/n$  and the prior mean  $p_0 = 1/2$ :

$$d(X) = \underbrace{\left( \frac{n}{n + \sqrt{n}} \right)}_w \underbrace{\left( \frac{X}{n} \right)}_{\hat{p}} + \underbrace{\left( \frac{\sqrt{n}}{n + \sqrt{n}} \right)}_{1-w} \underbrace{\left( \frac{1}{2} \right)}_{p_0} \quad (3.85)$$

$$d(X) = w\hat{p} + (1-w)p_0 \quad (3.86)$$

### Interpretation:

- $p_0 = 0.5$ : The estimator shrinks the data toward a neutral prior mean of 0.5 (representing maximum uncertainty).
- $w = \frac{n}{n + \sqrt{n}}$ : The weight assigned to the data. As the sample size  $n$  increases,  $w \rightarrow 1$ , and the minimax estimator converges to the MLE.

**Example 3.6 (Exponential Minimax Estimation).** Let  $X_1, \dots, X_n$  be a sample from an  $\text{Exp}(\theta)$  distribution with mean  $\theta$ . We consider the **Scale-Invariant Loss Function**:

$$\mathcal{L}(\theta, d) = \left( \frac{d}{\theta} - 1 \right)^2 \quad (3.87)$$

### Likelihood and MLE

The probability density function for a single observation is  $f(x_i | \theta) = \frac{1}{\theta} e^{-x_i/\theta}$ . The likelihood function for the sample is:

$$L(\theta | x) = \theta^{-n} e^{-\frac{1}{\theta} \sum_{i=1}^n x_i} \quad (3.88)$$

The Maximum Likelihood Estimator is standard:  $\hat{\theta}_{\text{MLE}} = \bar{X}$ .

### Minimax Estimation Setup

We propose the estimator  $d_0(X) = \frac{\sum X_i}{n+1}$ . To show this is a minimax estimator, we consider a sequence of priors  $\pi_k$  and examine the limit of their Bayes risks.

### Prior Density

We assume the prior  $\pi_k(\theta)$  follows an **Inverse-Gamma** distribution with shape  $\alpha_k$  and scale  $\beta_k$ . The density is given by:

$$\pi_k(\theta) = \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)} \theta^{-\alpha_k-1} e^{-\beta_k/\theta}, \quad \theta > 0 \quad (3.89)$$

### Posterior Analysis

Let  $T = \sum X_i$ . The posterior density is proportional to:

$$\pi(\theta | x) \propto (\theta^{-n} e^{-T/\theta}) \cdot (\theta^{-\alpha_k-1} e^{-\beta_k/\theta}) \propto \theta^{-(n+\alpha_k)-1} e^{-(T+\beta_k)/\theta} \quad (3.90)$$

This is an Inverse-Gamma distribution with parameters  $\alpha^* = n + \alpha_k$  and  $\beta^* = T + \beta_k$ .

### Calculation of the Bayes Estimator

Using the result derived in the **Scale-Invariant Squared Error Loss** section, the Bayes estimator is:

$$d_{\pi_k}(X) = \frac{E^{\theta|X}[\theta^{-1}]}{E^{\theta|X}[\theta^{-2}]} \quad (3.91)$$

For an  $\text{Inverse-Gamma}(\alpha^*, \beta^*)$  variable, the required moments are:

- $E^{\theta|X}[\theta^{-1}] = \frac{\alpha^*}{\beta^*}$
- $E^{\theta|X}[\theta^{-2}] = \frac{\alpha^*(\alpha^*+1)}{(\beta^*)^2}$

Substituting these into the estimator formula:

$$d_{\pi_k}(X) = \frac{\frac{\alpha^*}{\beta^*}}{\frac{\alpha^*(\alpha^*+1)}{(\beta^*)^2}} = \frac{\beta^*}{\alpha^* + 1} = \frac{T + \beta_k}{n + \alpha_k + 1} \quad (3.92)$$

### Bayes Risk Limit

The Bayes risk  $r(\pi_k, d_{\pi_k})$  is the expected value of the minimum posterior loss. Substituting  $d_{\pi_k}$  back into the loss equation:

$$r(\pi_k, d_{\pi_k}) = 1 - \frac{(E^{\theta|X}[\theta^{-1}])^2}{E^{\theta|X}[\theta^{-2}]} = 1 - \frac{n + \alpha_k}{n + \alpha_k + 1} = \frac{1}{n + \alpha_k + 1} \quad (3.93)$$

Taking the limit as the prior parameters approach zero ( $\alpha_k \rightarrow 0$ ):

$$\lim_{k \rightarrow \infty} r(\pi_k, d_{\pi_k}) = \frac{1}{n + 1} \quad (3.94)$$

### Minimax Verification

We compute the frequentist risk of our candidate estimator  $d_0(X) = \frac{T}{n+1}$ . Let  $Y = T/\theta \sim \text{Gamma}(n, 1)$ . Note that  $E^{X|\theta}[Y] = n$  and  $\text{Var}^{X|\theta}(Y) = n$ .

$$\begin{aligned} R(\theta, d_0) &= E^{X|\theta} \left[ \left( \frac{d_0}{\theta} - 1 \right)^2 \right] = E^{X|\theta} \left[ \left( \frac{Y}{n+1} - 1 \right)^2 \right] \\ &= \text{Var}^{X|\theta} \left( \frac{Y}{n+1} \right) + \left( E^{X|\theta} \left[ \frac{Y}{n+1} \right] - 1 \right)^2 \\ &= \frac{n}{(n+1)^2} + \left( \frac{n}{n+1} - 1 \right)^2 \\ &= \frac{1}{n+1} \end{aligned} \quad (3.95)$$

Since  $R(\theta, d_0) = \lim_{k \rightarrow \infty} r(\pi_k, d_{\pi_k}) = \frac{1}{n+1}$  for all  $\theta$ ,  $d_0$  is a **minimax estimator**.

## 3.5 Stein's Paradox and the James-Stein Estimator

### 3.5.1 The Problem of Estimating Normal Mean

In high-dimensional estimation ( $p \geq 3$ ), the Maximum Likelihood Estimator (MLE) is inadmissible under squared error loss. The **James-Stein Estimator** dominates the MLE, meaning it achieves lower risk for all values of  $\theta$ .

Consider the setting:

- Data:  $X \sim N_p(\theta, I)$

- Prior:  $\theta \sim N_p(0, \sigma^2 I)$
- James-Stein Estimator:

$$d^{JS}(X) = \left(1 - \frac{p-2}{\|X\|^2}\right) X \quad (3.96)$$

The James-Stein estimator improves upon the MLE by shrinking the individual observations toward a common mean (usually zero). The magnitude of this shrinkage depends on the total sum of squares of the observations.

- When the variance of  $\theta$  is large,  $\|X\|^2$  tends to be large, resulting in less shrinkage.
- When the variance of  $\theta$  is small,  $\|X\|^2$  is smaller, leading to a larger shrinkage factor.

The following R code simulates these two cases and displays them side-by-side with a shared y-axis for direct comparison.

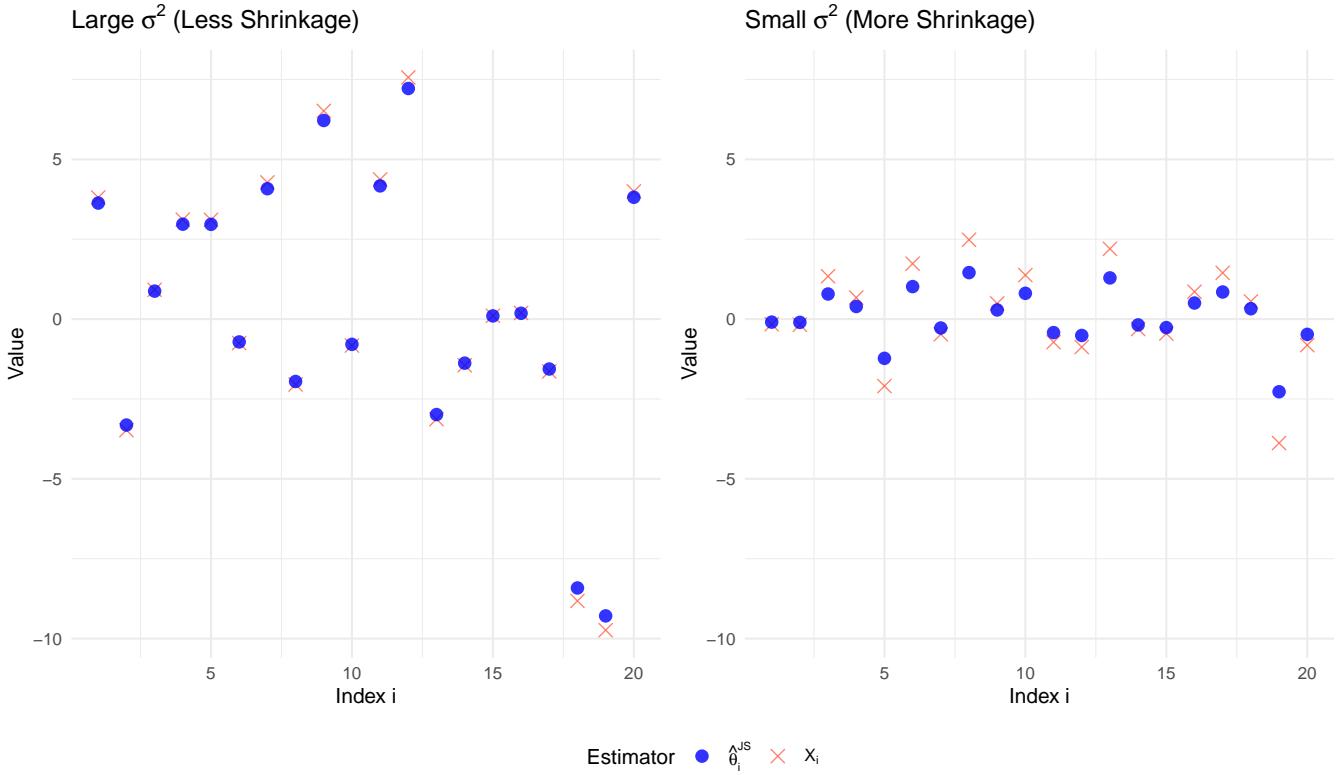


Figure 3.5: Visualization of JS Estimator

### 3.5.2 The Maximum Likelihood Estimator

Since the observations have the covariance matrix  $I$  (the identity matrix), the individual components  $X_1, \dots, X_p$  are independent, with  $X_i \sim N(\theta_i, 1)$ .

The joint likelihood function is the product of the individual probability density functions:

$$L(\theta; x) = \prod_{i=1}^p \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(X_i - \theta_i)^2}{2}\right) \quad (3.97)$$

To find the estimator, we maximize the log-likelihood function  $\ell(\theta)$ :

$$\begin{aligned} \ell(\theta) &= \ln \left( \prod_{i=1}^p \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(X_i - \theta_i)^2}{2}\right) \right) \\ &= \sum_{i=1}^p \left[ \ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{(X_i - \theta_i)^2}{2} \right] \end{aligned} \quad (3.98)$$

Maximizing this sum is equivalent to minimizing the sum of squared errors  $\sum(X_i - \theta_i)^2$ . We can solve for each component  $\theta_i$  separately. Differentiating with respect to  $\theta_i$ :

$$\frac{\partial \ell}{\partial \theta_i} = (X_i - \theta_i) \quad (3.99)$$

Setting the derivative to zero gives the critical point:

$$X_i - \hat{\theta}_i = 0 \implies \hat{\theta}_{i,\text{MLE}} = X_i \quad (3.100)$$

Since this holds for every component  $i = 1, \dots, p$ , the Maximum Likelihood Estimator for the entire vector is simply the observation vector itself:

$$d^{\text{MLE}}(X) = X \quad (3.101)$$

### 3.5.3 A Bayes Rule

We first derive the Bayes rule with respect to a specific conjugate prior. Instead of using matrix notation, we can look at the problem component-wise, as the observations are independent.

Consider the model where we observe  $p$  independent components:

$$X_i | \theta_i \sim N(\theta_i, 1), \quad \text{for } i = 1, \dots, p \quad (3.102)$$

We place independent centered normal priors on each unknown parameter  $\theta_i$ :

$$\theta_i \sim N(0, \sigma^2), \quad \text{for } i = 1, \dots, p \quad (3.103)$$

Since the components are independent, we can derive the Bayes rule for a single scalar component  $X_i$  estimating  $\theta_i$ . The total risk will simply be the sum of the component risks.

For a single component, the posterior distribution of  $\theta_i$  given  $X_i$  is Normal, with parameters determined by the standard conjugate formulas:

- **Posterior Precision (inverse variance):** The posterior precision is the sum of the prior precision and the data precision.

$$\frac{1}{v_{\text{post}}} = \frac{1}{\sigma^2} + \frac{1}{1} = \frac{1 + \sigma^2}{\sigma^2} \quad (3.104)$$

Therefore, the posterior variance is:

$$v_{\text{post}} = \frac{\sigma^2}{1 + \sigma^2} \quad (3.105)$$

- **Posterior Mean (Bayes Estimator):** The posterior mean is the precision-weighted average of the prior mean (0) and the data mean ( $X_i$ ).

$$\begin{aligned} E^{\theta_i | X_i}[\theta_i] &= v_{\text{post}} \left( \frac{0}{\sigma^2} + \frac{X_i}{1} \right) \\ &= \frac{\sigma^2}{1 + \sigma^2} X_i \\ &= \left( 1 - \frac{1}{1 + \sigma^2} \right) X_i \end{aligned} \quad (3.106)$$

Since this holds for all  $i$ , the Bayes rule for the vector  $\theta$  is applying this shrinkage factor to each component:

$$d^{\text{Bayes}}(X) = \left( 1 - \frac{1}{1 + \sigma^2} \right) X \quad (3.107)$$

### 3.5.3.1 Bayes Risk of the Bayes Rule

To compute the Bayes risk, we sum the risks of the individual components. For squared error loss, the posterior expected loss for one component is simply the posterior variance derived above:

$$E^{\theta_i | X_i}[(\theta_i - d^{\text{Bayes}}(X_i))^2] = v_{\text{post}} = \frac{\sigma^2}{1 + \sigma^2} \quad (3.108)$$

The total Bayes risk is the sum of these variances over  $p$  components:

$$r(\pi, d^{\text{Bayes}}) = \sum_{i=1}^p \frac{\sigma^2}{1 + \sigma^2} = \frac{p\sigma^2}{1 + \sigma^2} \quad (3.109)$$

### 3.5.3.2 Minimality of the MLE

The James-Stein result is particularly striking when compared to the performance of the standard estimator.

**Theorem 3.2** (Minimality of the Maximum Likelihood Estimator). *Let  $X \sim N_p(\theta, I)$ . Under the squared error loss function  $\mathcal{L}(\theta, d) = \|\theta - d\|^2$ , the standard Maximum Likelihood Estimator  $d^0(X) = X$  is a **minimax rule**. That is, it minimizes the maximum possible risk over the parameter space:*

$$\sup_{\theta \in \mathbb{R}^p} R(\theta, d^0) = \inf_d \sup_{\theta \in \mathbb{R}^p} R(\theta, d) = p \quad (3.110)$$

*Proof.*

Click to view proof by least favorable prior

The risk of the MLE is  $R(\theta, d^0) = p$  for all  $\theta$ . Since it is a constant risk estimator, its maximum risk is simply  $p$ . To prove it is minimax, we show that  $p$  is the limit of the Bayes risks for a sequence of conjugate priors  $\theta_i \sim N(0, \sigma^2)$ . As derived above, the Bayes risk for the optimal Bayes estimator  $d^{\text{Bayes}}$  is:

$$r(\pi_{\sigma^2}, d^{\text{Bayes}}) = \frac{p\sigma^2}{1 + \sigma^2} \quad (3.111)$$

As  $\sigma^2 \rightarrow \infty$  (the prior becomes “flat”), the Bayes risk approaches  $p$ :

$$\lim_{\sigma^2 \rightarrow \infty} \frac{p\sigma^2}{1 + \sigma^2} = p \quad (3.112)$$

By the property that the maximum risk of an estimator is always at least the Bayes risk of any prior, and specifically greater than or equal to the limit of Bayes risks for a sequence of priors, we establish that no estimator can have a maximum risk lower than  $p$ . Since  $d^0$  achieves this maximum risk, it is minimax.  $\square$

### 3.5.4 Stein's Lemma

#### i Notation: The Divergence Operator

The symbol  $\nabla \cdot g(X)$  (read as “divergence of  $g$ ”) is simply a shorthand notation for the sum of the partial derivatives:

$$\nabla \cdot g(X) \equiv \sum_{i=1}^p \frac{\partial g_i(X)}{\partial X_i} \quad (3.113)$$

It represents the total “outward flow” of the vector field  $g$  from a local point.

**Lemma 3.1** (Stein's Lemma). *Let  $X \sim N_p(\theta, I)$  be a multivariate normal random vector, and let  $g : \mathbb{R}^p \rightarrow \mathbb{R}^p$  be a continuously differentiable function such that  $E^{X|\theta}[|\partial g_i / \partial X_i|] < \infty$ . Then:*

$$E^{X|\theta} [(X - \theta)^T g(X)] = E^{X|\theta} [\nabla \cdot g(X)] = E^{X|\theta} \left[ \sum_{i=1}^p \frac{\partial g_i(X)}{\partial X_i} \right] \quad (3.114)$$

The term  $\nabla \cdot g(X)$  represents the **divergence** of the vector field  $g$ , which intuitively measures the local rate of expansion or outward flux of the function  $g$  at the point  $X$ ; in this statistical context, it quantifies the aggregate sensitivity of the function components to changes in the data.

*Proof.*

It suffices to show the result for a single component in 1 dimension, as the multivariate case follows by summation due to independence. Let  $X_i \sim N(\theta_i, 1)$  and let  $\phi(t)$  be the standard normal density function. The joint density is  $f(x) = \prod \phi(x_j - \theta_j)$ .

Consider the expectation of the  $i$ -th term:

$$E^{X|\theta}[(X_i - \theta_i)g_i(X)] = \int_{\mathbb{R}^p} (x_i - \theta_i)g_i(x) \left( \prod_{j=1}^p \phi(x_j - \theta_j) \right) dx \quad (3.115)$$

Focusing on the integral with respect to  $x_i$ :

$$\int_{-\infty}^{\infty} (x_i - \theta_i)\phi(x_i - \theta_i)g_i(x)dx_i \quad (3.116)$$

Recall that  $\phi'(z) = -z\phi(z)$ . Therefore,  $(x_i - \theta_i)\phi(x_i - \theta_i) = -\frac{\partial}{\partial x_i}\phi(x_i - \theta_i)$ . We use integration by parts with:

$$u = g_i(x) \quad \text{and} \quad dv = -\frac{\partial}{\partial x_i}\phi(x_i - \theta_i)dx_i \quad (3.117)$$

Thus:

$$\int_{-\infty}^{\infty} g_i(x)(x_i - \theta_i)\phi(x_i - \theta_i)dx_i = [-g_i(x)\phi(x_i - \theta_i)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial g_i(x)}{\partial x_i}\phi(x_i - \theta_i)dx_i \quad (3.118)$$

Assuming  $g(x)$  does not grow exponentially fast, the boundary term vanishes. The remaining integral is the expectation of the partial derivative. Summing over all  $i = 1 \dots p$  gives the divergence  $\nabla \cdot g(X)$ . □

In high-dimensional statistics, Stein's Lemma is often expressed using the inner product of the random vector and the function vector field, which highlights the alignment between the data and the transformation.

**Corollary 3.1** (Stein's Lemma (Vector Form)). *Let  $X \sim N_p(\theta, I)$  and  $g : \mathbb{R}^p \rightarrow \mathbb{R}^p$  be a weakly differentiable function. Then:*

$$E^{X|\theta}[X^T g(X)] = \theta^T E^{X|\theta}[g(X)] + E^{X|\theta}[\nabla \cdot g(X)] \quad (3.119)$$

💡 Remark: Connection to Non-Central  $\chi^2$  Moments

This identity provides an elegant way to derive the mean of a non-central chi-square distribution without performing complex integration.

Consider the case where  $g(X) = X$ . Here,  $\nabla \cdot X = p$ . Plugging this into the vector form:

$$E^{X|\theta}[X^T X] = \theta^T E^{X|\theta}[X] + E^{X|\theta}[p] \quad (3.120)$$

Since  $E^{X|\theta}[X] = \theta$ , we immediately obtain:

$$E^{X|\theta}[\|X\|^2] = \|\theta\|^2 + p \quad (3.121)$$

This is precisely the mean of a  $\chi_p^2(\lambda)$  distribution with non-centrality parameter  $\lambda = \|\theta\|^2$ . Essentially, Stein's Lemma decomposes the second moment into the **signal component** ( $\|\theta\|^2$ ) and the **geometric noise component** ( $p$ ).

**Lemma 3.2** (Stein's Lemma for Radial Fields). *Let  $X \sim N_p(\theta, I)$  and consider a radial vector field of the form  $g(X) = c(\|X\|^2)X$ , where  $c : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable scalar function. Then:*

$$E^{X|\theta} [(X - \theta)^T g(X)] = E^{X|\theta} [p \cdot c(\|X\|^2) + 2\|X\|^2 \cdot c'(\|X\|^2)] \quad (3.122)$$

where  $c'(z) = \frac{d}{dz}c(z)$ .

*Proof.*

We apply the general Stein's Lemma by calculating the divergence of the radial field  $g(X) = c(\|X\|^2)X$ . Using the product rule for divergence:

$$\nabla \cdot (c(\|X\|^2)X) = c(\|X\|^2)(\nabla \cdot X) + X^T(\nabla c(\|X\|^2)) \quad (3.123)$$

**Step 1: The geometric spread.** The divergence of the identity map  $X$  in  $p$  dimensions is simply the sum of the partial derivatives of each component with respect to itself:

$$\nabla \cdot X = \sum_{i=1}^p \frac{\partial X_i}{\partial X_i} = p \quad (3.124)$$

**Step 2: The radial stretch.** To find  $\nabla c(\|X\|^2)$ , we use the chain rule. Let  $h(X) = \|X\|^2 = \sum X_i^2$ . Then  $\nabla h(X) = 2X$ .

$$\nabla c(\|X\|^2) = c'(\|X\|^2)\nabla(\|X\|^2) = 2c'(\|X\|^2)X \quad (3.125)$$

Substituting this back into the divergence formula:

$$\begin{aligned} \nabla \cdot g(X) &= p \cdot c(\|X\|^2) + X^T(2c'(\|X\|^2)X) \\ &= p \cdot c(\|X\|^2) + 2c'(\|X\|^2)\|X\|^2 \end{aligned} \quad (3.126)$$

Taking the expectation of both sides completes the proof.

### Connection to the $\chi^2$ Distribution

This version of the lemma is particularly useful because when  $\theta = 0$ , the quantity  $\|X\|^2$  follows a central  $\chi_p^2$  distribution.

- **Verifying the Mean:** If we set  $c(\|X\|^2) = 1$ , then  $g(X) = X$ . The lemma gives  $E^{X|\theta=0}[\|X\|^2] = p + 2\|X\|^2(0) = p$ , which is the expected value of a  $\chi_p^2$  variable.
- **The James-Stein Weight:** If we set  $c(\|X\|^2) = \frac{1}{\|X\|^2}$ , then  $c'(z) = -\frac{1}{z^2}$ .

$$\nabla \cdot g(X) = \frac{p}{\|X\|^2} + 2\|X\|^2 \left( -\frac{1}{\|X\|^4} \right) = \frac{p-2}{\|X\|^2} \quad (3.127)$$

This explains why the  $p - 2$  constant appears in the James-Stein estimator—it is the net result of  $p$  dimensions of “spreading” minus 2 dimensions of “radial thinning.”

□

**Example 3.7** (An Example for Verifying Stein's Lemma). Let  $X \sim N(\theta, 1)$  be a univariate normal random variable with unit variance. Let  $g(x) = x^2$ . Stein's Lemma states that:

$$E^{X|\theta}[(X - \theta)g(X)] = E^{X|\theta}[g'(X)] \quad (3.128)$$

**Step 1: Calculate the Right-Hand Side (RHS)** First, we find the derivative of  $g(x)$ :

$$g'(x) = \frac{d}{dx}(x^2) = 2x \quad (3.129)$$

Now, compute the expectation of the derivative:

$$\text{RHS} = E^{X|\theta}[g'(X)] = E^{X|\theta}[2X] = 2E^{X|\theta}[X] = 2\theta \quad (3.130)$$

**Step 2: Calculate the Left-Hand Side (LHS)** We evaluate the expectation of the cross-product term. Substitute  $X = Z + \theta$ , where  $Z \sim N(0, 1)$  is a standard normal variable. Then  $X - \theta = Z$ .

$$\begin{aligned} \text{LHS} &= E^{X|\theta}[(X - \theta)X^2] \\ &= E^{X|\theta}[Z(Z + \theta)^2] \quad \text{where } Z \sim N(0, 1) \\ &= E^{X|\theta}[Z(Z^2 + 2\theta Z + \theta^2)] \\ &= E^{X|\theta}[Z^3] + 2\theta E^{X|\theta}[Z^2] + \theta^2 E^{X|\theta}[Z] \end{aligned} \quad (3.131)$$

We use the known moments of the standard normal distribution  $Z$ :

- $E[Z] = 0$  (mean)
- $E[Z^2] = 1$  (variance)
- $E[Z^3] = 0$  (skewness of symmetric distribution)

Substituting these values back:

$$\text{LHS} = 0 + 2\theta(1) + \theta^2(0) = 2\theta \quad (3.132)$$

**Conclusion** We observe that:

$$\text{LHS} = 2\theta \quad \text{and} \quad \text{RHS} = 2\theta \quad (3.133)$$

Thus, Stein's Lemma holds for this specific case.

**Example 3.8** (A Radial Field Example Verifying Stein's Lemma). Let  $X \sim N_p(\theta, I)$  and  $g(X) = \|X\|^2 X$ . We verify the Radial Field Lemma:

$$E^{X|\theta}[(X - \theta)^T g(X)] = E^{X|\theta}[p \cdot c(\|X\|^2) + 2\|X\|^2 \cdot c'(\|X\|^2)] \quad (3.134)$$

Here,  $c(z) = z$ , which implies  $c'(z) = 1$ .

**RHS (Divergence):** Using the radial formula:

$$\begin{aligned} \nabla \cdot g(X) &= p(\|X\|^2) + 2\|X\|^2(1) \\ &= (p + 2)\|X\|^2 \end{aligned} \quad (3.135)$$

The expectation is  $(p + 2)E^{X|\theta}[\|X\|^2]$ . Since  $\|X\|^2$  is a non-central  $\chi_p^2$  with non-centrality parameter  $\|\theta\|^2$ , we know  $E^{X|\theta}[\|X\|^2] = p + \|\theta\|^2$ . Thus, RHS =  $(p + 2)(p + \|\theta\|^2)$ .

**LHS (Alignment):**

$$\begin{aligned} E^{X|\theta} [(X - \theta)^T (\|X\|^2 X)] &= E^{X|\theta} [\|X\|^2 (X^T X - \theta^T X)] \\ &= E^{X|\theta} [\|X\|^4 - \theta^T X \|X\|^2] \end{aligned} \quad (3.136)$$

To simplify, let  $X = \theta + Z$  where  $Z \sim N_p(0, I)$ . Recall the moments of the non-central chi-square distribution or expand the terms:  $E[\|X\|^4] = p(p+2) + 2(p+2)\|\theta\|^2 + \|\theta\|^4$ . For the cross term  $E[\theta^T X \|X\|^2]$ , we find it equals  $\|\theta\|^4 + (p+2)\|\theta\|^2$ .

Subtracting these:

$$\begin{aligned} \text{LHS} &= [p(p+2) + 2(p+2)\|\theta\|^2 + \|\theta\|^4] - [\|\theta\|^4 + (p+2)\|\theta\|^2] \\ &= p(p+2) + (p+2)\|\theta\|^2 \\ &= (p+2)(p + \|\theta\|^2) \end{aligned} \quad (3.137)$$

### Conclusion

The results match exactly. The alignment of the cubic radial field with the noise is perfectly predicted by the sum of its geometric expansion ( $p\|X\|^2$ ) and its radial stretch ( $2\|X\|^2$ ).

### 3.5.5 Inadmissibility of the MLE in High Dimensions (Stein's Phenomenon)

**Theorem 3.3.** Let  $X \sim N_p(\theta, I)$  be a  $p$ -dimensional random vector with  $p \geq 3$ . Under the squared error loss function  $\mathcal{L}(\theta, d) = \|\theta - d\|^2$ , the standard Maximum Likelihood Estimator  $d^0(X) = X$  is **inadmissible**.

*Proof of Inadmissibility.* To show that  $d^0(X) = X$  is inadmissible, we compare its risk to that of the James-Stein estimator  $d^{JS}(X)$ .

Let  $g(X) = c(\|X\|^2)X$  where  $c(\|X\|^2) = \frac{p-2}{\|X\|^2}$ . We can write the James-Stein estimator as  $d^{JS}(X) = X - g(X)$ .

The risk is the expected squared error loss:

$$\begin{aligned} R(\theta, d^{JS}) &= E^{X|\theta} [\|(X - \theta) - g(X)\|^2] \\ &= E^{X|\theta} [\|X - \theta\|^2] - 2E^{X|\theta} [(X - \theta)^T g(X)] + E^{X|\theta} [\|g(X)\|^2] \end{aligned} \quad (3.138)$$

The first term is the risk of the MLE, which is  $p$ .

For the second term, we apply **Stein's Lemma for Radial Fields** (Lemma 3.2). We first compute the scalar function and its derivative:

$$c(z) = \frac{p-2}{z} \implies c'(z) = -\frac{p-2}{z^2} \quad (3.139)$$

Substituting these into the radial divergence formula from Lemma 3.2:

$$\begin{aligned}
\nabla \cdot g(X) &= p \cdot c(\|X\|^2) + 2\|X\|^2 \cdot c'(\|X\|^2) \\
&= p \left( \frac{p-2}{\|X\|^2} \right) + 2\|X\|^2 \left( -\frac{p-2}{\|X\|^4} \right) \\
&= \frac{p(p-2)}{\|X\|^2} - \frac{2(p-2)}{\|X\|^2} \\
&= \frac{(p-2)^2}{\|X\|^2}
\end{aligned} \tag{3.140}$$

Applying the lemma to the cross-term:

$$2E^{X|\theta} [(X - \theta)^T g(X)] = 2E^{X|\theta} [\nabla \cdot g(X)] = 2(p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \tag{3.141}$$

The third term in the risk expansion is the squared magnitude of the shrinkage:

$$\|g(X)\|^2 = \left\| \frac{p-2}{\|X\|^2} X \right\|^2 = \frac{(p-2)^2}{\|X\|^4} \|X\|^2 = \frac{(p-2)^2}{\|X\|^2} \tag{3.142}$$

Substituting these results back into the risk equation:

$$\begin{aligned}
R(\theta, d^{JS}) &= p - 2(p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] + (p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \\
&= p - (p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right]
\end{aligned} \tag{3.143}$$

Since  $p \geq 3$ , the constant  $(p-2)^2$  is strictly positive. Because  $1/\|X\|^2 > 0$  with probability 1, the risk of the James-Stein estimator is strictly less than  $p$  for all  $\theta \in \mathbb{R}^p$ .

Thus,  $d^{JS}$  dominates the MLE, proving the MLE is inadmissible.  $\square$

### 3.5.6 How much JS Estimator Improves over MLE

The exact risk function of the James-Stein estimator  $d^{JS}(X)$  under squared error loss for  $X \sim N_p(\theta, I)$  is given by:

$$R(\theta, d^{JS}) = p - (p-2)^2 E \left[ \frac{1}{\chi_p^2(\|\theta\|^2/2)} \right] \tag{3.144}$$

where  $\chi_p^2(\|\theta\|^2/2)$  is a non-central chi-square random variable with  $p$  degrees of freedom and non-centrality parameter  $\lambda = \|\theta\|^2/2$ .

Using the approximation  $E[1/\chi_p^2(\lambda)] \approx 1/E[\chi_p^2(\lambda)] = 1/(p + \|\theta\|^2)$ , we can see the approximate behavior of the risk:

$$R(\theta, d^{JS}) \approx p - \frac{(p-2)^2}{p + \|\theta\|^2} \quad (3.145)$$

- **Aggressive Shrinkage near the Origin:** When  $\|\theta\|^2$  is small, the denominator  $p + \|\theta\|^2$  is small, making the subtracted term large. This results in a risk substantially lower than the MLE risk of  $p$ .
- **Diminishing Improvement with Large Signal:** As  $\|\theta\|^2$  becomes large, the term  $\frac{(p-2)^2}{p + \|\theta\|^2}$  approaches zero. Consequently, the risk of the James-Stein estimator approaches  $p$ , and the improvement over the MLE becomes negligible.

**Risk Ratio near the Origin:** As the true parameter vector shrinks to zero ( $\|\theta\| \rightarrow 0$ ), the ratio of the risks converges to a constant fraction. Using the exact expectation  $E^{X|\theta=0}[1/\|X\|^2] = 1/(p-2)$ :

$$\lim_{\|\theta\| \rightarrow 0} \frac{R(\theta, d^{JS})}{R(\theta, d^{\text{MLE}})} = \frac{p - (p-2)^2 \left( \frac{1}{p-2} \right)}{p} = \frac{p - (p-2)}{p} = \frac{2}{p} \quad (3.146)$$

For a dimension like  $p = 10$ , the James-Stein estimator incurs only 20% of the risk of the MLE near the origin.

**i** Is  $d^{JS}$  Minimax?

Yes. Since the MLE is minimax with constant risk  $p$ , the minimax risk value for this problem is  $p$ . Because  $R(\theta, d^{JS}) < p$  for all  $\theta$  and  $\lim_{\|\theta\| \rightarrow \infty} R(\theta, d^{JS}) = p$ , the maximum risk of the James-Stein estimator is exactly  $p$ . Therefore,  $d^{JS}$  achieves the minimax risk level and is a minimax estimator.

### 3.5.7 Using Normalized Loss (Optional)

We consider the **Normalized Squared Error Loss** function, which penalizes errors relative to the magnitude of the true parameter vector:

$$\mathcal{L}(\theta, d) = \frac{\|d - \theta\|^2}{\|\theta\|^2}, \quad \theta \neq 0 \quad (3.147)$$

#### 1. Risk of the MLE ( $d^{\text{MLE}} = X$ )

The risk of the Maximum Likelihood Estimator is straightforward because the standard Mean Squared Error (MSE) of  $X$  is constant ( $p$ ):

$$R(\theta, d^{\text{MLE}}) = E^{X|\theta} \left[ \frac{\|X - \theta\|^2}{\|\theta\|^2} \right] = \frac{1}{\|\theta\|^2} E^{X|\theta} [\|X - \theta\|^2] \quad (3.148)$$

Since  $X \sim N_p(\theta, I)$ , we have  $E^{X|\theta}[\|X - \theta\|^2] = p$ .

$$R(\theta, d^{\text{MLE}}) = \frac{p}{\|\theta\|^2} \quad (3.149)$$

#### 2. Risk of the James-Stein Estimator ( $d^{JS}$ )

### Risk Comparison of Estimators ( $p = 17$ )

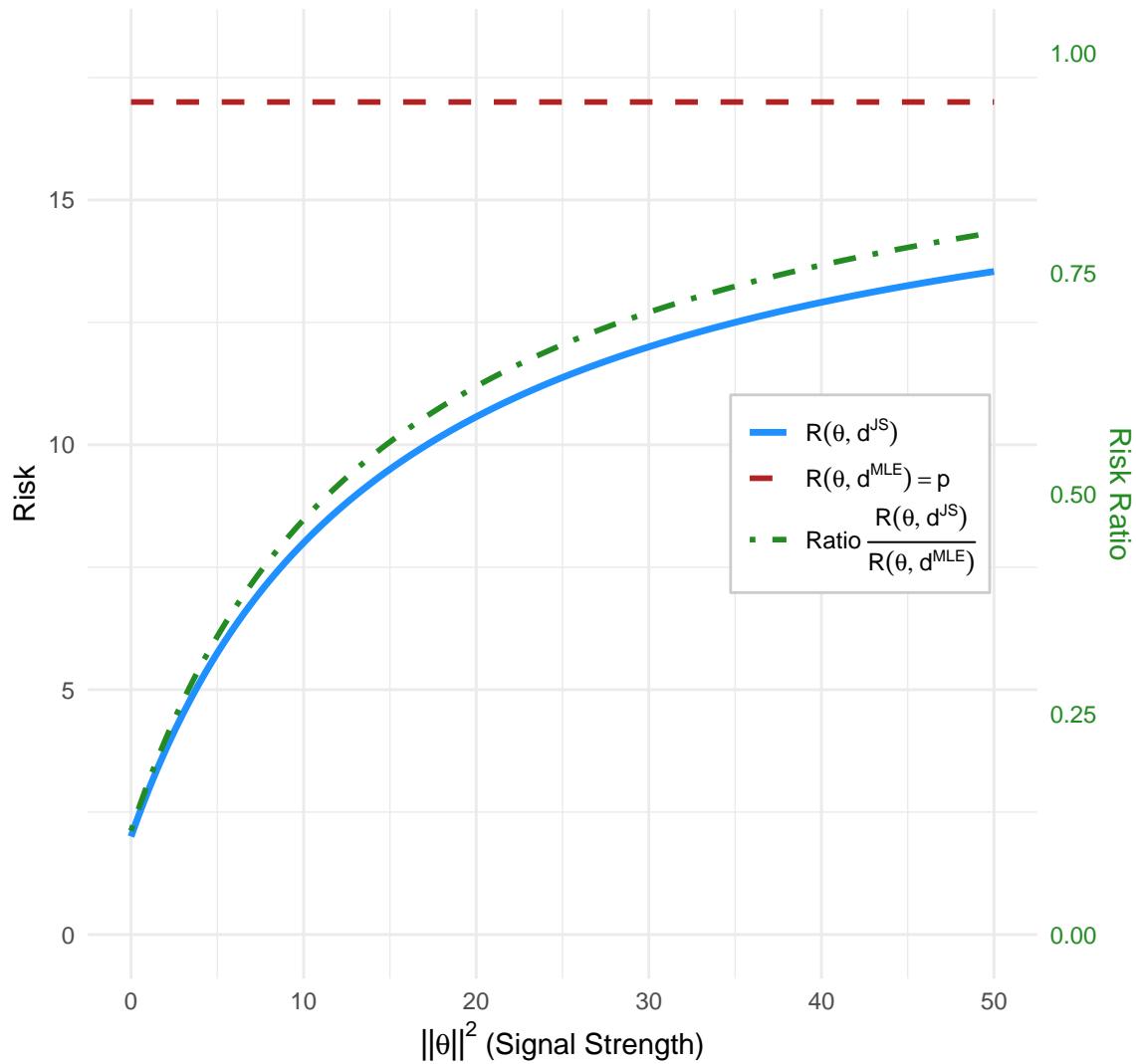


Figure 3.6: Risk comparison of James-Stein vs MLE ( $p=10$ ). Note: Ratio uses the right axis.

For the James-Stein estimator  $d^{JS} = \left(1 - \frac{p-2}{\|X\|^2}\right) X$ , we utilize the known result for its standard MSE risk:

$$E^{X|\theta}[\|d^{JS} - \theta\|^2] = p - (p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \quad (3.150)$$

The risk under the normalized loss is simply this term scaled by  $1/\|\theta\|^2$ :

$$R(\theta, d^{JS}) = \frac{1}{\|\theta\|^2} \left( p - (p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \right) \quad (3.151)$$

### 3. Comparison and Dominance

We compare the risks by taking the difference:

$$R(\theta, d^{\text{MLE}}) - R(\theta, d^{JS}) = \frac{1}{\|\theta\|^2} (p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \quad (3.152)$$

Since  $(p-2)^2 > 0$  (for  $p \geq 3$ ) and the expectation of a positive random variable is positive, this difference is strictly positive for all  $\theta \neq 0$ .

$$R(\theta, d^{JS}) < R(\theta, d^{\text{MLE}}) \quad (3.153)$$

- **Global Dominance:** The James-Stein estimator dominates the MLE under this loss function as well, achieving lower risk everywhere in the parameter space.
- **Behavior near  $\theta \approx 0$ :** As  $\|\theta\| \rightarrow 0$ , both risks diverge to infinity. We analyze their relative performance by examining the ratio of the risks:

$$\frac{R(\theta, d^{JS})}{R(\theta, d^{\text{MLE}})} = \frac{p - (p-2)^2 E^{X|\theta}[1/\|X\|^2]}{p} = 1 - \frac{(p-2)^2}{p} E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \quad (3.154)$$

At  $\theta = 0$ ,  $\|X\|^2 \sim \chi_p^2$ , and the expectation is  $E^{X|\theta=0}[1/\|X\|^2] = 1/(p-2)$ . Substituting this into the ratio:

$$\lim_{\|\theta\| \rightarrow 0} \frac{R(\theta, d^{JS})}{R(\theta, d^{\text{MLE}})} = 1 - \frac{(p-2)^2}{p(p-2)} = 1 - \frac{p-2}{p} = \frac{2}{p} \quad (3.155)$$

Thus, near the origin, the James-Stein estimator reduces the risk by a factor of  $p/2$ . For large dimensions (e.g.,  $p = 10$ ), the JS estimator has only 20% of the risk of the MLE.

#### 3.5.8 Bayes Risk of James-stein Estimator (Optional)

We can derive the Bayes Risk  $r(\pi, d^{JS})$  of this estimator using two equivalent methods: minimizing the expected frequentist risk, or minimizing the expected posterior loss.

**Theorem 3.4** (Bayes Risk of James-stein Estimator). *For  $p \geq 3$ , the Bayes risk of the James-Stein estimator  $d^{JS}$  with respect to the prior  $\theta \sim N(0, \sigma^2 I)$  is:*

$$r(\pi, d^{JS}) = \frac{p\sigma^2 + 2}{\sigma^2 + 1} \quad (3.156)$$

*Proof.*

Method 1: Integration over the Prior (Frequentist Risk approach)

The Bayes risk is defined as  $r(\pi, d) = E^\pi[R(\theta, d)]$ .

First, recall the frequentist risk of the James-Stein estimator for a fixed  $\theta$ . Using Stein's Lemma, the risk is given by:

$$R(\theta, d^{JS}) = p - (p-2)^2 E^{X|\theta} \left[ \frac{1}{\|X\|^2} \right] \quad (3.157)$$

To find the Bayes risk, we take the expectation of this risk with respect to the prior  $\pi(\theta)$ :

$$r(\pi, d^{JS}) = \int R(\theta, d^{JS}) \pi(\theta) d\theta = p - (p-2)^2 E^\pi \left[ E^{X|\theta} \left( \frac{1}{\|X\|^2} \right) \right] \quad (3.158)$$

By the law of iterated expectations,  $E^\pi[E^{X|\theta}(\cdot)]$  is equivalent to the expectation with respect to the marginal distribution of  $X$ , denoted as  $m(x)$ . Under the conjugate prior, the marginal distribution is  $X \sim N(0, (1+\sigma^2)I)$ . Consequently, the quantity  $\frac{\|X\|^2}{1+\sigma^2}$  follows a Chi-squared distribution with  $p$  degrees of freedom ( $\chi_p^2$ ). The expectation of the inverse chi-square is:

$$E^X \left[ \frac{1}{\|X\|^2} \right] = \frac{1}{1+\sigma^2} E \left[ \frac{1}{\chi_p^2} \right] = \frac{1}{1+\sigma^2} \cdot \frac{1}{p-2} \quad (3.159)$$

Substituting this back into the risk equation:

$$\begin{aligned} r(\pi, d^{JS}) &= p - (p-2)^2 \cdot \frac{1}{(p-2)(1+\sigma^2)} \\ &= p - \frac{p-2}{1+\sigma^2} \\ &= \frac{p(1+\sigma^2) - (p-2)}{1+\sigma^2} \\ &= \frac{p\sigma^2 + p - p + 2}{1+\sigma^2} = \frac{p\sigma^2 + 2}{\sigma^2 + 1} \end{aligned} \quad (3.160)$$

□

*Proof.*

Method 2: Integration over the Marginal (Posterior Loss approach)

Alternatively, we can compute the Bayes risk by first finding the posterior expected loss for a given  $x$ , and then averaging over the marginal distribution of  $x$ :

$$r(\pi, d) = E^X [E^{\theta|X} [\mathcal{L}(\theta, d(X))]] \quad (3.161)$$

### Step 1: Posterior Expected Loss

The posterior distribution of  $\theta$  given  $x$  is:

$$\theta|x \sim N\left(\frac{\sigma^2}{1+\sigma^2}x, \frac{\sigma^2}{1+\sigma^2}I\right) \quad (3.162)$$

The expected squared error loss can be decomposed into the variance (trace) and the squared bias:

$$E^{\theta|X}[||\theta - d^{JS}(X)||^2] = \text{tr}(\text{Var}^{\theta|X}(\theta)) + ||E^{\theta|X}[\theta] - d^{JS}(X)||^2 \quad (3.163)$$

- **Trace term:**

$$\text{tr}\left(\frac{\sigma^2}{1+\sigma^2}I_p\right) = \frac{p\sigma^2}{1+\sigma^2} \quad (3.164)$$

- **Squared Bias term:** Let  $B = \frac{1}{1+\sigma^2}$ . Then  $E^{\theta|X}[\theta] = (1-B)X$ . The estimator is  $d^{JS}(X) = (1 - \frac{p-2}{||X||^2})X$ . The difference is:

$$E^{\theta|X}[\theta] - d^{JS}(X) = \left((1-B) - \left(1 - \frac{p-2}{||X||^2}\right)\right)X = \left(\frac{p-2}{||X||^2} - B\right)X \quad (3.165)$$

Squaring the norm gives:

$$\left(\frac{p-2}{||X||^2} - B\right)^2 ||X||^2 = \frac{(p-2)^2}{||X||^2} - 2B(p-2) + B^2||X||^2 \quad (3.166)$$

### Step 2: Expectation with respect to Marginal $X$

We now take the expectation  $E^X[\cdot]$  of the posterior loss. Recall  $X \sim N(0, (1+\sigma^2)I)$ , so  $E^X[||X||^2] = p(1+\sigma^2)$  and  $E^X[1/||X||^2] = \frac{1}{(p-2)(1+\sigma^2)}$ .

- **Expectation of Trace term:** Constant, remains  $\frac{p\sigma^2}{1+\sigma^2}$ .
- **Expectation of Bias term:**

$$\begin{aligned} E^X\left[\frac{(p-2)^2}{||X||^2} - \frac{2(p-2)}{1+\sigma^2} + \frac{||X||^2}{(1+\sigma^2)^2}\right] &= (p-2)^2 \frac{1}{(p-2)(1+\sigma^2)} - \frac{2(p-2)}{1+\sigma^2} + \frac{p(1+\sigma^2)}{(1+\sigma^2)^2} \\ &= \frac{p-2}{1+\sigma^2} - \frac{2p-4}{1+\sigma^2} + \frac{p}{1+\sigma^2} \\ &= \frac{p-2-2p+4+p}{1+\sigma^2} \\ &= \frac{2}{1+\sigma^2} \end{aligned} \quad (3.167)$$

### Step 3: Combine Terms

$$r(\pi, d^{JS}) = \underbrace{\frac{p\sigma^2}{1+\sigma^2}}_{\text{Variance Part}} + \underbrace{\frac{2}{1+\sigma^2}}_{\text{Bias Part}} = \frac{p\sigma^2 + 2}{\sigma^2 + 1} \quad (3.168)$$

Both methods yield the same result.

□

### 3.5.9 Practical Application: One-way ANOVA and “Borrowing Strength”

**Example 3.9.** Consider a One-Way ANOVA setting where we wish to estimate the means of  $p$  different independent groups (e.g., the true batting averages of  $p = 10$  baseball players, or the efficacy of  $p = 5$  different hospital treatments).

- **Model:** Let  $X_i \sim N(\theta_i, \sigma^2)$  be the observed sample mean for group  $i$ , for  $i = 1, \dots, p$ .
- **Goal:** Estimate the vector of true means  $\theta = (\theta_1, \dots, \theta_p)$  simultaneously. The loss is the sum of squared errors:  $L(\theta, \hat{\theta}) = \sum(\theta_i - \hat{\theta}_i)^2$ .

**The MLE Approach (Total Separation):** The standard estimator is  $\hat{\theta}_i^{\text{MLE}} = X_i$ . This estimates each group entirely independently, using only data from that specific group. If a specific player has a lucky streak, their estimate is very high; if they are unlucky, it is very low.

**The James-Stein Approach (Shrinkage / Pooling):** In this context, the James-Stein estimator (specifically the variation shrinking toward the grand mean  $\bar{X}$ ) is:

$$\hat{\theta}_i^{JS} = \bar{X} + \left(1 - \frac{(p-3)\sigma^2}{\sum(X_i - \bar{X})^2}\right)(X_i - \bar{X}) \quad (3.169)$$

**Why is this better?** Even though the groups might be physically independent (e.g., distinct hospitals), the James-Stein estimator “borrows strength” from the ensemble.

- **Noise Reduction:** Extreme observations  $X_i$  are likely to contain more positive noise than signal. Shrinking them toward the global average  $\bar{X}$  reduces this variance.
- **Stein’s Paradox:** While  $\hat{\theta}_i^{JS}$  introduces bias (estimates are pulled toward the center), the reduction in variance is so significant that the **Total Risk** (sum of squared errors over all groups) is strictly lower than that of the MLE, provided  $p \geq 3$ .

Thus, estimating the groups *together* yields a more accurate global picture than estimating them *separately*, even if the groups are independent.

### 3.5.10 Why Is This Paradoxical?

The result that  $d^{JS}$  dominates  $d^0$  is called **Stein’s Paradox** because it defies intuition in several ways:

- **Independence Irrelevance:** The result holds even if the components  $X_i$  are completely unrelated (e.g.,  $X_1$  is the price of tea in China,  $X_2$  is the temperature in Saskatoon, and  $X_3$  is the weight of a local cat). It seems absurd that combining unrelated data improves the estimate of each, but the combined risk is indeed lower.
- **No “Free Lunch”:** The James-Stein estimator does not improve every individual component  $\theta_i$  simultaneously for every realization. Instead, it minimizes the **total** risk  $\sum E(\hat{\theta}_i - \theta_i)^2$ . It sacrifices accuracy on outliers (by biasing them) to gain significant stability on the bulk of the data.
- **Destruction of Symmetry:** The MLE is invariant under translation and rotation. The James-Stein estimator breaks this symmetry by shrinking toward an arbitrary point (usually the origin or the grand mean), yet it yields a better objective performance.

### 3.5.11 What We Learned

- **Bias-Variance Tradeoff:** This is the most famous example where introducing **bias** (shrinkage) leads to a massive reduction in **variance**, thereby reducing the overall Mean Squared Error (MSE). Unbiasedness is not always a virtue in estimation.
- **Inadmissibility in High Dimensions:** Intuitions formed in 1D or 2D (where MLE is admissible) fail in higher dimensions ( $p \geq 3$ ). The volume of space grows so fast that “standard” diffuse priors or MLEs become inefficient.
- **Hierarchical Modeling:** Stein’s result provides the theoretical foundation for **Hierarchical Bayesian Models**. When we assume parameters come from a common distribution (e.g.,  $\theta_i \sim N(\mu, \tau^2)$ ), we naturally derive shrinkage estimators that “borrow strength” across groups, formalized as Empirical Bayes or fully Bayesian methods.

## 3.6 Empirical Bayes Rules

The James-Stein estimator provides a natural entry point into the concept of **Empirical Bayes (EB)**. While the Stein estimator was originally derived using frequentist risk arguments, it can be intuitively understood as a Bayesian estimator where the parameters of the prior distribution are estimated from the data itself.

### 3.6.1 The General Empirical Bayes Framework

In a standard Bayesian analysis, the hyperparameters of the prior are fixed based on subjective belief or external information. In contrast, Empirical Bayes uses the observed data to “learn” the prior.

The workflow typically follows these steps:

1. **Hierarchical Model:** We assume the data  $X$  comes from a distribution  $f(x|\theta)$ , and the parameter  $\theta$  comes from a prior  $\pi(\theta|\eta)$  controlled by hyperparameters  $\eta$ .
2. **Marginal Likelihood (Evidence):** We integrate out the parameter  $\theta$  to obtain the marginal distribution of the data given the hyperparameters:

$$m(x|\eta) = \int f(x|\theta)\pi(\theta|\eta)d\theta \quad (3.170)$$

3. **Estimation of Hyperparameters:** Instead of fixing  $\eta$ , we estimate it by maximizing the marginal likelihood (Type-II Maximum Likelihood) or using method-of-moments:

$$\hat{\eta} = \arg \max_{\eta} m(x|\eta) \quad (3.171)$$

4. **Posterior Inference:** We proceed with standard Bayesian inference, but we substitute the estimated estimate  $\hat{\eta}$  into the posterior:

$$\pi(\theta|x, \hat{\eta}) \propto f(x|\theta)\pi(\theta|\hat{\eta}) \quad (3.172)$$

**Discussion:**

- “**Borrowing Strength**”: EB allows us to pool information across independent groups to estimate the common structure (the prior) governing them.
- **The Critique:** A purist Bayesian might object that using the data twice (once to estimate the prior, once to estimate  $\theta$ ) underestimates the uncertainty. A fully Bayesian Hierarchical model would instead place a “hyperprior” on  $\eta$  and integrate it out.

### 3.6.2 Deriving James-Stein as Empirical Bayes

The James-Stein estimator can be viewed as an **Empirical Bayes** procedure, where the hyperparameters of the prior are estimated directly from the data rather than being specified *a priori*.

**Model:**

- **Likelihood:**  $X_i | \mu_i \sim N(\mu_i, 1)$  for  $i = 1, \dots, p$ .
- **Prior:**  $\mu_i \sim N(0, \sigma^2)$ , where  $\sigma^2$  is an unknown hyperparameter.

#### Step 1: The Ideal Bayes Estimator

If  $\sigma^2$  were known, the posterior distribution of  $\mu_i$  would be Normal. The optimal estimator under squared error loss is the posterior mean:

$$E^{\mu_i | X_i, \sigma^2}[\mu_i] = \frac{\sigma^2}{1 + \sigma^2} X_i = \left(1 - \frac{1}{1 + \sigma^2}\right) X_i \quad (3.173)$$

We define the shrinkage factor  $B = \frac{1}{1 + \sigma^2}$ .

#### Step 2: Marginal Estimation

The marginal distribution of the data (integrating out  $\mu_i$ ) is:

$$X_i \sim N(0, 1 + \sigma^2) \quad (3.174)$$

Consequently, the sum of squares  $S = \|X\|^2 = \sum X_i^2$  follows a scaled Chi-squared distribution:

$$S \sim (1 + \sigma^2)\chi_p^2 \quad (3.175)$$

#### Step 3: Estimating the Shrinkage Factor

We need an estimator for  $B = \frac{1}{1 + \sigma^2}$ . From the properties of the inverse Chi-square distribution, we know  $E^X[1/\chi_p^2] = \frac{1}{p-2}$  for  $p > 2$ . Therefore:

$$E^X\left[\frac{p-2}{S}\right] = \frac{p-2}{1 + \sigma^2} E^X\left[\frac{1}{\chi_p^2}\right] = \frac{p-2}{1 + \sigma^2} \cdot \frac{1}{p-2} = \frac{1}{1 + \sigma^2} = B \quad (3.176)$$

Thus,  $\hat{B} = \frac{p-2}{\|X\|^2}$  is an unbiased estimator of the optimal shrinkage factor  $B$ .

#### Step 4: The Empirical Bayes Rule

Plugging  $\hat{B}$  into the ideal Bayes estimator recovers the James-Stein rule:

$$\delta^{EB}(X) = \left(1 - \hat{B}\right) X = \left(1 - \frac{p-2}{\|X\|^2}\right) X \quad (3.177)$$

**Remarks:**

- (1) **Adaptive Shrinkage:** The James-Stein estimator automatically adjusts the amount of shrinkage based on the observed total magnitude  $\|X\|^2$ . If the data suggests the true means are spread far from zero,  $\|X\|^2$  will be large,  $\hat{B}$  will be small, and we shrink less.
- (2) **Unbiasedness of B:** Interestingly, while  $\hat{B}$  is an unbiased estimator of the shrinkage factor, the resulting James-Stein estimator itself is biased toward the origin. This is a classic example of sacrificing unbiasedness to minimize total risk.

## 3.7 Hierarchical Modeling via MCMC

In complex Bayesian settings where the posterior distribution cannot be derived analytically, we utilize hierarchical structures to represent levels of uncertainty and Markov Chain Monte Carlo (MCMC) to approximate the resulting distributions.

### 3.7.1 Hierarchical Model Structure

A hierarchical model decomposes a complex joint distribution into a series of conditional levels. The general mathematical form is:

$$\begin{aligned} \text{Level 1 (Data Likelihood): } & X_i | \mu_i, \sigma^2 \sim f(x_i | \mu_i, \sigma^2) \\ \text{Level 2 (Parameters): } & \mu_i | \theta, \tau^2 \sim \pi(\mu_i | \theta, \tau^2) \\ \text{Level 3 (Hyperparameters): } & \theta, \tau^2 \sim \pi(\theta, \tau^2) \end{aligned} \quad (3.178)$$

The goal is to compute the joint posterior distribution of all unobserved parameters given the data  $X = \{X_1, \dots, X_n\}$ :

$$p(\mu, \theta, \tau^2 | X) \propto \left[ \prod_{i=1}^n f(x_i | \mu_i, \sigma^2) \pi(\mu_i | \theta, \tau^2) \right] \pi(\theta, \tau^2) \quad (3.179)$$

### 3.7.2 Graphical Model Representation (tree Structure)

The following tree diagram illustrates the conditional dependencies. Note that the parameters  $\mu_i$  are conditionally independent given the hyperparameter  $\theta$ , which facilitates “borrowing strength” across groups.

### 3.7.3 MCMC Estimation

In hierarchical models, the joint posterior distribution  $p(\mu, \theta | X)$  often lacks a closed-form analytical solution due to the integration required for the normalizing constant. We use **Markov Chain Monte Carlo (MCMC)** to draw sequence of samples  $\{\mu^{(t)}, \theta^{(t)}\}$  that converge to the target posterior distribution.

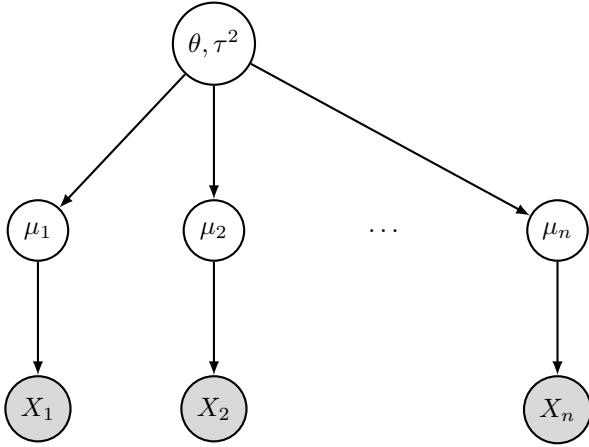


Figure 3.7: Hierarchical Tree Structure

### 3.7.3.1 Gibbs Sampling Algorithm

**Algorithm 3.2.** *Gibbs sampling is an algorithm for sampling from a multivariate distribution by sequentially sampling from the full conditional distributions. To sample from a target distribution  $p(\theta_1, \theta_2, \dots, \theta_k)$ , the algorithm iterates through each variable, updating it conditioned on the current values of all other variables:*

$$\begin{aligned}
 \theta_1^{(t+1)} &\sim p(\theta_1 | \theta_2^{(t)}, \theta_3^{(t)}, \dots, \theta_k^{(t)}) \\
 \theta_2^{(t+1)} &\sim p(\theta_2 | \theta_1^{(t+1)}, \theta_3^{(t)}, \dots, \theta_k^{(t)}) \\
 &\vdots \\
 \theta_k^{(t+1)} &\sim p(\theta_k | \theta_1^{(t+1)}, \theta_2^{(t+1)}, \dots, \theta_{k-1}^{(t+1)})
 \end{aligned} \tag{3.180}$$

#### Example 3.10 (Gibbs Sampling for Groups of Normal Data). The Model

To apply the general Gibbs sampling framework  $\theta_1, \theta_2, \dots, \theta_k$  to our specific hierarchical model, we identify the variables as follows:

- **Data Observations ( $X_i$ ):** These are the known, measured values at the lowest level of the hierarchy (e.g., test scores of students in school  $i$ ). In the Gibbs sampler, these remain fixed and condition the updates of the parameters.
- **Group-Level Parameters ( $\theta_1 = \mu_i$ ):** These represent the latent means for each specific group or cluster. In the update step,  $\mu_i$  acts as the first block of variables. It is updated by “compromising” between the local data  $X_i$  and the global characteristic  $\theta$ .
- **Global Hyperparameter ( $\theta_2 = \theta$ ):** This represents the common mean across all groups. It acts as the second block in the sampler. Its update depends on the current state of all  $\mu_i$  values, effectively “pooling” information from all groups to estimate the overall population center.

#### Gibbs Update in Hierarchical Models

In the hierarchical tree structure provided earlier, let our parameter vector be  $(\mu_i, \theta)$ . The “orthogonality” of the updates becomes clear when we derive the full conditionals for a Gaussian case:

- **Case  $\theta_1 = \mu_i$ :** Sample  $\mu_i^{(t+1)}$  from  $p(\mu_i | X_i, \theta^{(t)})$ . This is a normal distribution with:

$$\mu_i^{(t+1)} \sim N\left(\frac{\tau^2 X_i + \sigma^2 \theta^{(t)}}{\sigma^2 + \tau^2}, \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}\right) \quad (3.181)$$

- **Case  $\theta_2 = \theta$ :** Sample  $\theta^{(t+1)}$  from  $p(\theta | \mu^{(t+1)})$ . Assuming a flat prior  $\pi(\theta) \propto 1$ :

$$\theta^{(t+1)} \sim N\left(\frac{1}{n} \sum_{i=1}^n \mu_i^{(t+1)}, \frac{\tau^2}{n}\right) \quad (3.182)$$

**Visual Characteristic:** Gibbs sampling moves along the coordinate axes because it updates one parameter at a time while holding others constant.

### 3.7.3.2 Metropolis-hastings (MH) Sampling

When the full conditional distributions are not easy to sample from, we use the Metropolis-Hastings algorithm. At each step  $t$ :

- **Propose:** Draw a candidate state  $\theta^*$  from a proposal distribution  $q(\theta^* | \theta^{(t)})$ .
- **Accept/Reject:** Calculate the acceptance probability:

$$\alpha = \min\left(1, \frac{p(\theta^* | X) q(\theta^{(t)} | \theta^*)}{p(\theta^{(t)} | X) q(\theta^* | \theta^{(t)})}\right) \quad (3.183)$$

- Set  $\theta^{(t+1)} = \theta^*$  with probability  $\alpha$ ; otherwise, set  $\theta^{(t+1)} = \theta^{(t)}$ .

**Visual Characteristic:** MH sampling moves in arbitrary directions and can “stay put” if a proposal is rejected, exploring the space via a random walk.

## 3.8 Case Study: 1998 Major League Baseball Home Run Race

In 1998, the baseball world was captivated by Mark McGwire and Sammy Sosa as they chased Roger Maris’ 1961 record of 61 home runs in a single season. While McGwire and Sosa finished with 70 and 66 home runs respectively, we consider whether such performance could have been predicted using pre-season exhibition data.

For a set of  $i = 1, \dots, 17$  players (including McGwire and Sosa), we observe their batting records in pre-season exhibition matches. Our goal is to estimate each player’s home run “strike rate” for the competitive season.

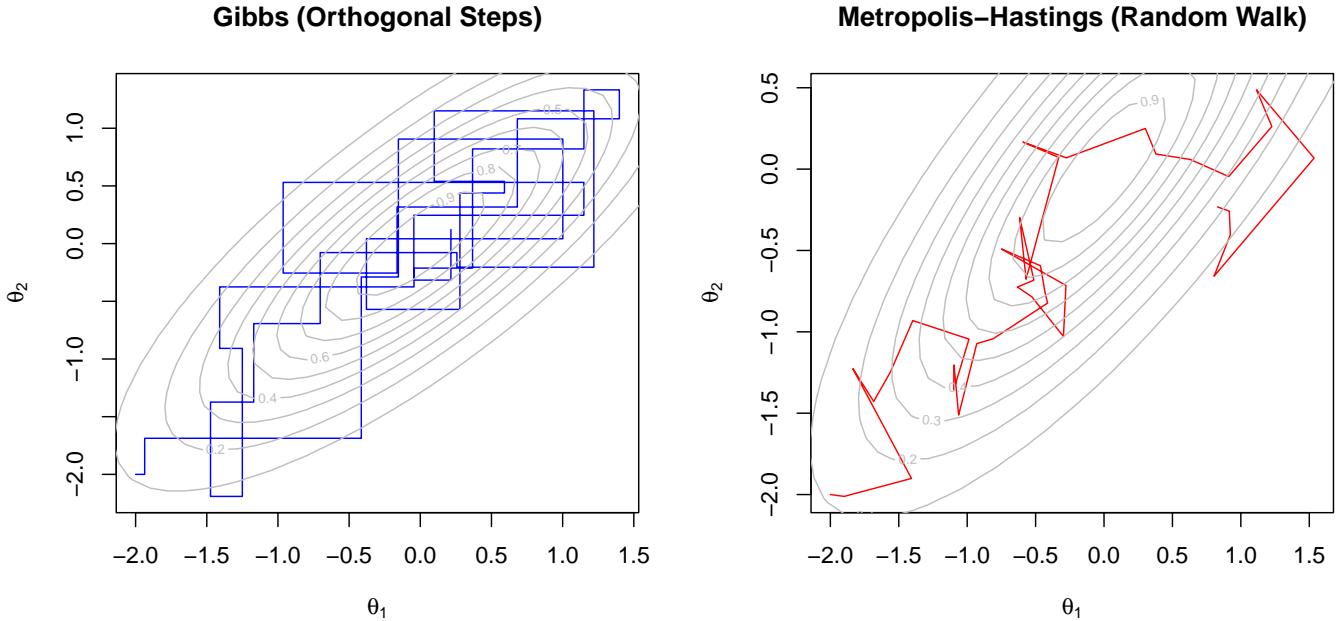


Figure 3.8: Comparison of Sampling Paths

### 3.8.1 Transforming Data

We utilize the pre-season home runs ( $y_i$ ) and at-bats ( $n_i$ ) for 17 players. The data is transformed using a variance-stabilizing transformation to approximate a normal distribution with known variance  $\sigma^2 = 1$ .

$$x_i = \sqrt{n_i} \arcsin \left( 2 \frac{y_i}{n_i} - 1 \right) \quad (3.184)$$

The goal is to estimate the latent parameter  $\mu_i$  for each player and compare it to the “true” regular season performance.

### 3.8.2 True Season Parameter ( $\mu_i$ or $p_i^{season}$ )

To validate our estimates, we define the “true” parameter value  $\mu_i$  using the player’s performance over the full competitive season. Let  $Y_i$  be the total home runs and  $N_i$  be the total at-bats in the regular season. The true transformed rate is calculated as:

$$\mu_i^{season} = \sqrt{n_i} \arcsin \left( 2 \frac{Y_i}{N_i} - 1 \right) \quad (3.185)$$

Note that while we use the season-long probability ( $Y_i/N_i$ ), we scale it by the pre-season sample size ( $\sqrt{n_i}$ ). This ensures that  $\mu_i^{season}$  is on the same scale as our observations  $x_i$ , allowing for direct comparison of the estimation error.

Table 3.6: 1998 MLB Statistics: Raw Counts, Probabilities, and Transformed Data

Player	$y_i$	$n_i$	$p_i^{\text{pre}}$	$x_i$	$Y_i$	$N_i$	$p_i^{\text{seas}}$	$\mu_i$
1	7	58	0.121	-6.559	70	509	0.138	-6.176
2	9	59	0.153	-5.901	66	643	0.103	-7.055
3	4	74	0.054	-9.476	56	633	0.088	-8.317
4	7	84	0.083	-9.029	46	645	0.071	-9.441
5	3	69	0.043	-9.558	45	606	0.074	-8.463
6	6	63	0.095	-7.488	44	555	0.079	-7.937
7	2	60	0.033	-9.323	43	619	0.069	-8.035
8	10	54	0.185	-5.005	40	609	0.066	-7.734
9	2	53	0.038	-8.589	37	552	0.067	-7.622
10	2	60	0.033	-9.323	34	540	0.063	-8.238
11	4	66	0.061	-8.720	32	561	0.057	-8.843
12	3	66	0.045	-9.270	30	440	0.068	-8.469
13	2	72	0.028	-10.487	29	585	0.050	-9.518
14	5	64	0.078	-8.034	28	531	0.053	-8.859
15	3	42	0.071	-6.673	23	454	0.051	-7.237
16	2	38	0.053	-6.829	21	504	0.042	-7.149
17	6	58	0.103	-6.975	15	244	0.061	-8.146

In this analysis, we model the home run strike rates of 17 Major League Baseball players using pre-season exhibition data from 1998. We apply five statistical methods ranging from simple independent estimation to advanced Bayesian decision theory.

### 3.8.3 Methods for Estimating $\mu_i$ (transformed Scale)

#### 3.8.3.1 Method 1: Simple Estimation (MLE)

The Maximum Likelihood Estimator (MLE) assumes each player's performance is independent. It relies solely on the observed pre-season data.

$$\hat{\mu}_i^{\text{MLE}} = X_i \quad (3.186)$$

```
# Simple Estimate Is Just the Data Itself
mu_mle <- baseball_data$x

# MSE Calculation (transformed Scale)
mse_mle <- mean((mu_mle - baseball_data$true_mu)^2)
```

### 3.8.3.2 Method 2: Empirical Bayes (james-stein)

The James-Stein estimator introduces a global mean  $\bar{X}$  and shrinks individual estimates toward it. This assumes the players come from a common population distribution.

$$\hat{\mu}_i^{JS} = \bar{X} + \left(1 - \frac{k-3}{\sum(X_i - \bar{X})^2}\right)(X_i - \bar{X}) \quad (3.187)$$

where  $k = 17$  is the number of players.

```
theta_hat <- mean(baseball_data$x)
S <- sum((baseball_data$x - theta_hat)^2)
shrinkage_factor <- 1 - (14 / S)

mu_js <- theta_hat + shrinkage_factor * (baseball_data$x - theta_hat)

# MSE Calculation (transformed Scale)
mse_js <- mean((mu_js - baseball_data$true_mu)^2)
```

### 3.8.3.3 Method 3: Fully Bayesian MCMC (brms)

We use a hierarchical Bayesian model where parameters are treated as random variables. We implement this using `brms`.

$$\begin{aligned} X_i &\sim N(\mu_i, 1) \\ \mu_i &\sim N(\theta, \tau^2) \\ \theta &\sim N(0, 10) \\ \tau &\sim \text{Cauchy}(0, 2) \end{aligned} \quad (3.188)$$

```
baseball_data$sei <- rep(1, length(baseball_data$x))
# Fit Random Intercept Model: X | Se(1) ~ 1 + (1|player)
fit_brms <- brm(
  formula = x | se(sei, sigma = TRUE) ~ 1 + (1 | Player),
  data = baseball_data,
  prior = c(
    prior(normal(0, 10), class = "Intercept"),
    prior(cauchy(0, 2), class = "sd")
  ),
  chains = 2, iter = 4000, warmup = 1000, seed = 123,
  refresh = 0
)

# Extract Point Estimates (posterior Means)
post_means <- fitted(fit_brms)[, "Estimate"]
```

```

mu_brms <- post_means

# MSE Calculation (transformed Scale)
mse_brms <- mean((mu_brms - baseball_data$true_mu)^2)

```

### 3.8.4 Comparison of Estimates of $\mu_i$

#### Full Comparison of Estimates (Transformed Scale)

The following table presents the transformed data ( $x_i$ ) and the true season parameter ( $\mu_i$ ) alongside the estimates from the three methods. The rows are sorted by  $x_i$  to visualize how the shrinkage methods (James-Stein and Bayesian) pull the estimates away from the extremes and toward the population mean compared to the raw MLE.

Table 3.7: Comparison of Estimates (Sorted by Pre-season  $x_i$ )

Player	$x_i$ (MLE)	$\hat{\mu}_{JS}$	$\hat{\mu}_{Bayes}$	$\mu_{true}$
13	-10.487	-9.589	-8.746	-9.518
5	-9.558	-9.006	-8.478	-8.463
3	-9.476	-8.954	-8.470	-8.317
7	-9.323	-8.858	-8.412	-8.035
10	-9.323	-8.858	-8.415	-8.238
12	-9.270	-8.825	-8.412	-8.469
4	-9.029	-8.673	-8.331	-9.441
11	-8.720	-8.479	-8.260	-8.843
9	-8.589	-8.397	-8.206	-7.622
14	-8.034	-8.048	-8.054	-8.859
6	-7.488	-7.705	-7.897	-7.937
17	-6.975	-7.384	-7.754	-8.146
16	-6.829	-7.292	-7.714	-7.149
15	-6.673	-7.194	-7.663	-7.237
1	-6.559	-7.122	-7.628	-6.176
2	-5.901	-6.709	-7.441	-7.055
8	-5.005	-6.146	-7.186	-7.734

#### Plots of Squared Errors (Sorted by $x_i$ )

This plot displays the Squared Error for each player. The x-axis represents the players sorted from lowest pre-season performance to highest.

```

# Calculate Squared Errors Using the SORTED Dataframe
err_mle <- (df_sorted$x_i - df_sorted$mu_true)^2
err_js   <- (df_sorted$mu_js - df_sorted$mu_true)^2
err_brms <- (df_sorted$mu_bayes - df_sorted$mu_true)^2

# Determine Y-axis Range

```

```

y_max <- max(c(err_mle, err_js, err_brms))

# Plot MLE Errors (baseline)
plot(1:17, err_mle, type = "b", pch = 1, col = "black", lty = 2,
      xlab = "Player Index (Sorted by Pre-season Performance)",
      ylab = expression(Squared~Error~~(hat(mu) - mu[true])^2),
      main = "Estimation Error Comparison (Sorted)",
      ylim = c(0, y_max))

# Add James-stein Errors
lines(1:17, err_js, type = "b", pch = 19, col = "blue")

# Add Bayesian (brms) Errors
lines(1:17, err_brms, type = "b", pch = 17, col = "red")

# Add Grid and Legend
grid()
legend("topleft",
       title = "Mean Squared Error",
       legend = c(paste0("MLE: ", round(mse_mle, 3)),
                  paste0("JS: ", round(mse_js, 3)),
                  paste0("Bayes: ", round(mse_brms, 3))),
       col = c("black", "blue", "red"),
       pch = c(1, 19, 17),
       lty = c(2, 1, 1))

```

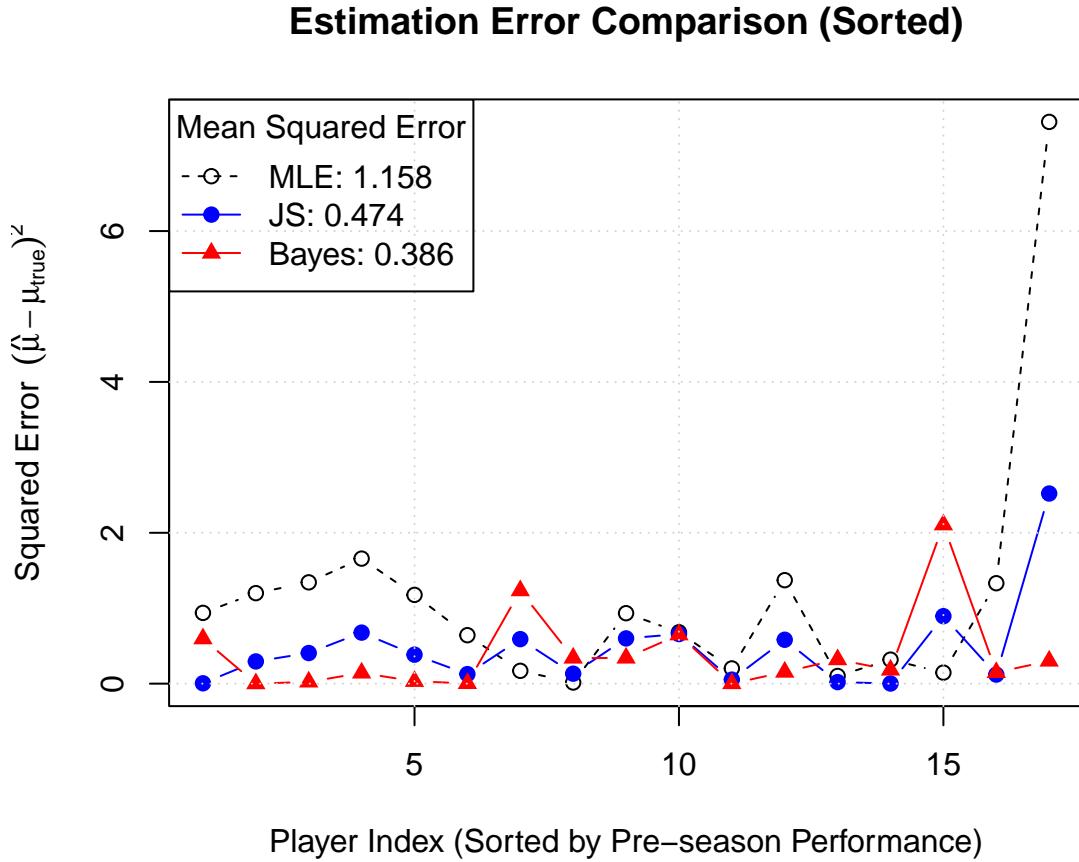


Figure 3.9: Squared Error by Sorted Player Index (Transformed Scale)

### 3.8.5 Methods for Estimating $p_i$ Directly

#### 3.8.5.1 Method 1-3: Converting $\hat{\mu}_i$ Back to $p_i$

The first three methods (MLE, James-Stein, and Normal-Normal Bayes) estimated the parameter  $\mu_i$  on the transformed scale. To obtain the probability estimates  $\hat{p}_i$ , we apply the inverse of the variance-stabilizing transformation:

$$\hat{p}_i = \frac{1}{2} \left( \sin \left( \frac{\hat{\mu}_i}{\sqrt{n_i}} \right) + 1 \right) \quad (3.189)$$

where  $\hat{\mu}_i$  corresponds to the estimate derived from Method 1, 2, or 3, and  $n_i$  is the number of pre-season at-bats for player  $i$ .

#### 3.8.5.2 Method 4: Hierarchical Logistic Regression (logit-normal)

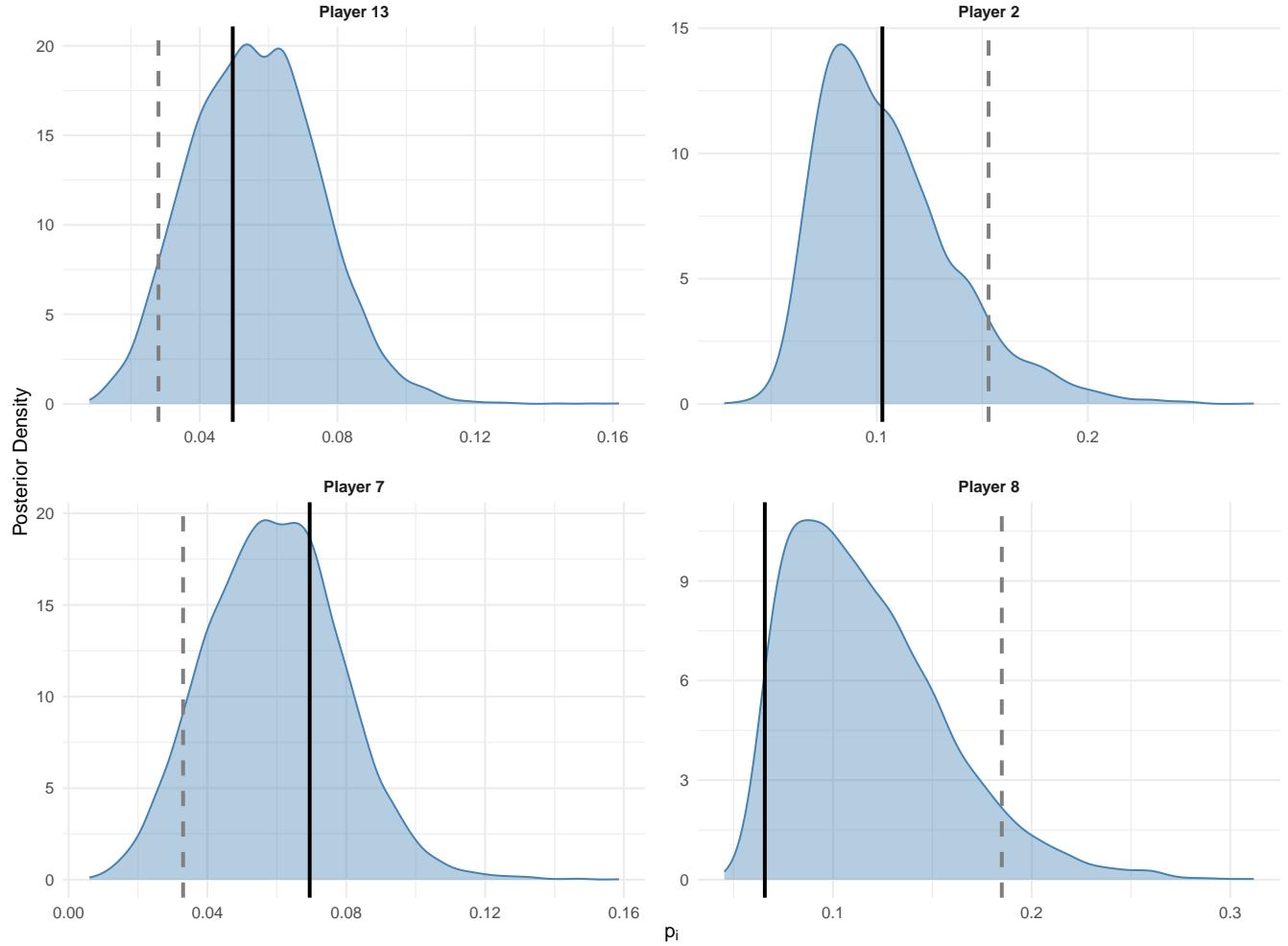
In this fourth method, we model the probability  $p_i$  directly using a hierarchical structure on the log-odds scale, rather than transforming the data.

We assume the count  $y_i$  follows a Binomial distribution. The log-odds (logit) of the success rate  $p_i$  are drawn from a common Normal distribution with unknown mean  $\mu_0$  and standard deviation  $\tau_0$ .

$$\begin{aligned}
 y_i | p_i &\sim \text{Binomial}(n_i, p_i) \\
 \text{logit}(p_i) &\sim N(\mu_0, \tau_0^2) \\
 \mu_0 &\sim N(0, 10) \\
 \tau_0 &\sim \text{Cauchy}(0, 2)
 \end{aligned} \tag{3.190}$$

We implement this in `brms` using the `binomial` family with a logit link. The individual point estimate  $\hat{p}_i$  is the **posterior mean** of  $p_i$ . Note that because the inverse-logit function is non-linear, the posterior mean of  $p_i$  is not simply the inverse-logit of the posterior mean of the random effect; `brms` handles this integration automatically via the `fitted()` function.

Posterior Distributions of HR Probabilities for Extreme Players  
Dashed Grey: Pre-season (Observed) | Solid Black: Remainder of Season (Actual)



### 3.8.5.3 Method 5: Optimal Bayes Estimator w.r.t. Relative Absolute Error

While the posterior mean (Method 4) minimizes the Mean Squared Error (MSE), it is not necessarily optimal for the **Relative Standardized Error** metric we defined earlier:

$$L(p, \hat{p}) = \frac{|p - \hat{p}|}{\min(p, 1-p)} \quad (3.191)$$

This is a form of weighted absolute error loss, where the weight is  $w(p) = \frac{1}{\min(p, 1-p)}$ . Theoretical derivation shows that the estimator minimizing the expected posterior loss for this function is the **Weighted Posterior Median**.

We compute this by extracting the full posterior samples from the Logit-Normal model (Method 4) and calculating the weighted median for each player.

```
# 1. Extract Posterior Samples (n_samples X 17 Players)
# Posterior_epred Gives Samples of the Expected Count (N * P)
post_counts <- posterior_epred(fit_logit)

# Convert to Probability Scale by Dividing by Trials
p_samples <- sweep(post_counts, 2, baseball_data$Pre_AtBats, "/")

# 2. Extract Posterior Means (Method 4)
# This provides the missing p_hat_logit variable
p_hat_logit <- colMeans(p_samples)

# 3. Define Function for Weighted Median
# Finds the Value 'q' Such That Sum(weights Where X <= q) >= 0.5 * Total_weight
get_weighted_median <- function(samples) {
  # Calculate weights based on the loss function denominator
  # Avoid division by exact zero (unlikely but safer)
  denom <- pmin(samples, 1 - samples)
  denom[denom < 1e-6] <- 1e-6
  weights <- 1 / denom

  # Normalize weights
  weights_norm <- weights / sum(weights)

  # Sort samples and weights
  ord <- order(samples)
  samp_sorted <- samples[ord]
  w_sorted <- weights_norm[ord]

  # Find cutoff
  cum_w <- cumsum(w_sorted)
  idx <- which(cum_w >= 0.5)[1]

  return(samp_sorted[idx])
}
```

```

}

# 4. Apply to All Players (Method 5)
p_hat_optimal <- apply(p_samples, 2, get_weighted_median)

```

### 3.8.5.4 Comparison of All Five Estimates of $p_i$

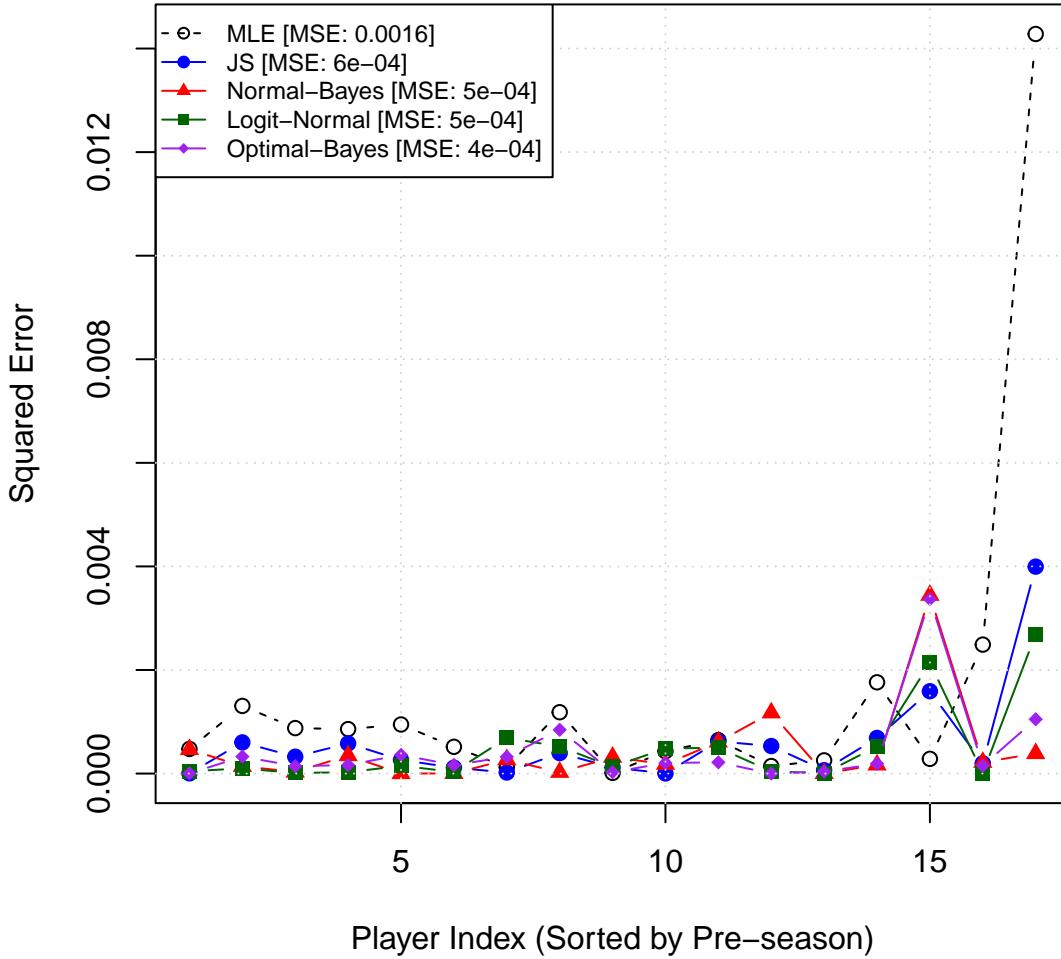
We now compare all five methods: MLE, James-Stein (transformed), Bayes Normal-Normal (transformed), Hierarchical Logit-Normal (Posterior Mean), and Optimal Bayes (Weighted Median).

Table 3.8: Comparison of Estimated Probabilities ( $p_i$ ) across Five Methods

Player	Season Avg ( $p_i$ )	MLE	James-Stein	Normal-Bayes	Logit-Normal	Optimal-Bayes
13	0.050	0.028	0.048	0.071	0.056	0.048
7	0.069	0.033	0.045	0.058	0.060	0.051
10	0.063	0.033	0.045	0.058	0.060	0.051
9	0.067	0.038	0.043	0.048	0.062	0.054
5	0.074	0.043	0.058	0.074	0.062	0.055
12	0.068	0.045	0.058	0.070	0.063	0.055
16	0.042	0.053	0.037	0.025	0.068	0.060
3	0.088	0.054	0.069	0.083	0.066	0.059
11	0.057	0.061	0.068	0.075	0.068	0.062
15	0.051	0.071	0.052	0.037	0.073	0.065
14	0.053	0.078	0.078	0.077	0.075	0.067
4	0.071	0.083	0.094	0.106	0.078	0.071
6	0.079	0.095	0.087	0.081	0.082	0.073
17	0.061	0.103	0.088	0.074	0.084	0.075
1	0.138	0.121	0.098	0.079	0.091	0.079
2	0.103	0.153	0.117	0.088	0.105	0.090
8	0.066	0.185	0.129	0.085	0.118	0.098

#### 1. MSE Comparison

## Squared Error by Method



## 2. Comparison of Relative Absolute Error

We also evaluate the methods using the relative error metric that penalizes deviations based on the rarity of the event:

$$\text{Metric}_i = \frac{|p_i^{\text{true}} - \hat{p}_i|}{\min(p_i^{\text{true}}, 1 - p_i^{\text{true}})} \quad (3.192)$$

## 3.9 Bayesian Predictive Distributions

A key feature of Bayesian analysis is the ability to make inference about future observations, rather than just the model parameters. The **posterior predictive distribution** describes the probability of observing a new data point  $y^*$  given the observed data  $y$ .

**Definition 3.3** (Posterior Predictive Distribution). Let  $f(y^*|\theta)$  be the sampling distribution of a future observation  $y^*$  given parameter  $\theta$ , and let  $\pi(\theta|y)$  be the posterior distribution of  $\theta$  given observed data  $y$ . The posterior

## Assessment of Estimation Methods

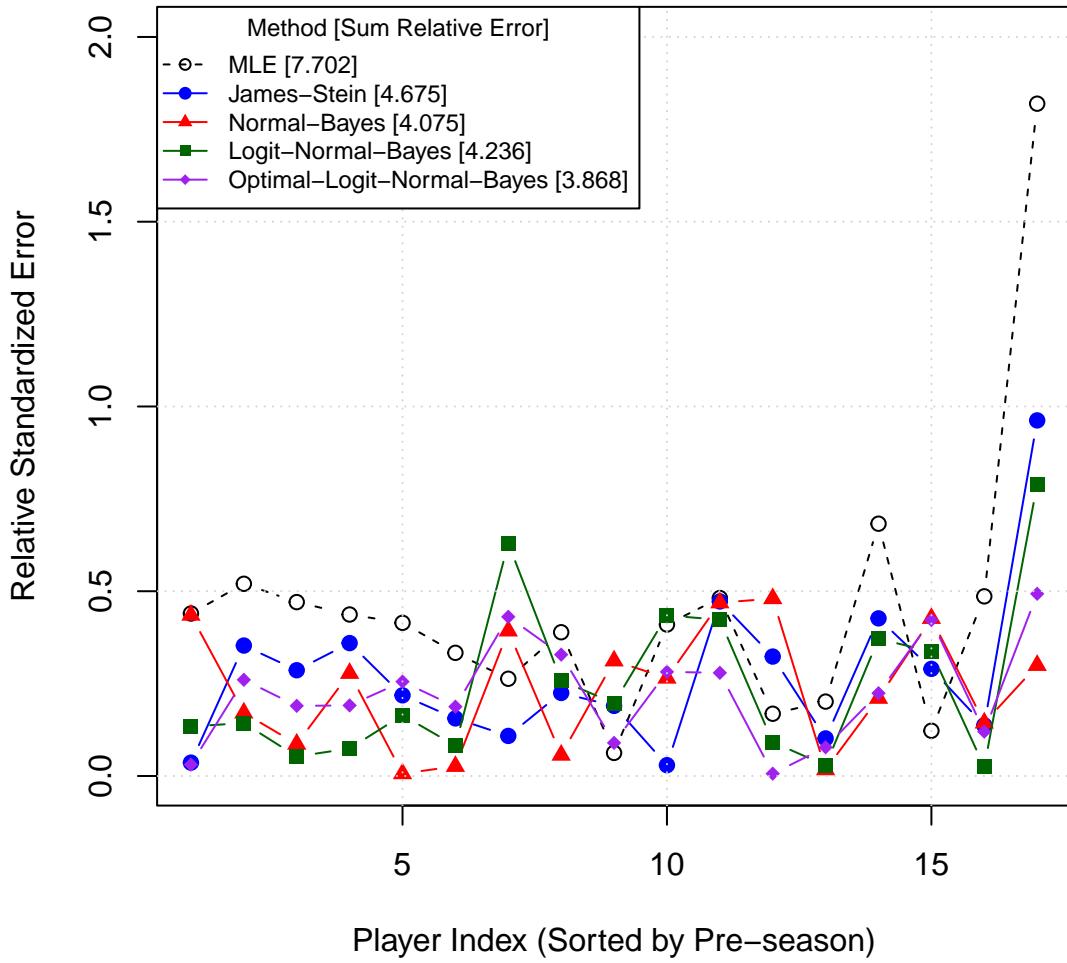


Figure 3.10: Relative Error Assessment: Five Methods

predictive density is obtained by marginalizing over the parameter  $\theta$ :

$$f(y^*|y) = \int_{\Theta} f(y^*|\theta)\pi(\theta|y) d\theta \quad (3.193)$$

This distribution incorporates two distinct sources of uncertainty:

- **Sampling Uncertainty (Aleatoric):** The inherent variability of the data generation process, represented by the variance in  $f(y^*|\theta)$ .
- **Parameter Uncertainty (Epistemic):** The uncertainty regarding the true value of  $\theta$ , represented by the variance in the posterior  $\pi(\theta|y)$ .

As sample size  $n \rightarrow \infty$ , the parameter uncertainty vanishes (the posterior approaches a point mass), and the predictive distribution converges to the true data-generating distribution.

**Example 3.11** (Normal-normal Predictive Distribution). Consider a case where the data  $y_1, \dots, y_n$  are independent and normally distributed with unknown mean  $\mu$  and known variance  $\sigma^2$ :

$$Y_i|\mu \sim N(\mu, \sigma^2) \quad (3.194)$$

Assume a conjugate prior for the mean:  $\mu \sim N(\mu_0, \sigma_0^2)$ . The posterior distribution is  $\mu|y \sim N(\mu_n, \sigma_n^2)$ , where  $\mu_n$  and  $\sigma_n^2$  are the updated posterior hyperparameters.

The predictive distribution for a new observation  $y^*$  is derived as:

$$\begin{aligned} f(y^*|y) &= \int_{-\infty}^{\infty} f(y^*|\mu)\pi(\mu|y) d\mu \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y^*-\mu)^2}{2\sigma^2}} \times \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(\mu-\mu_n)^2}{2\sigma_n^2}} d\mu \end{aligned} \quad (3.195)$$

This convolution of two Gaussians results in a new Gaussian distribution:

$$y^*|y \sim N(\mu_n, \sigma^2 + \sigma_n^2) \quad (3.196)$$

Here, the total predictive variance is the sum of the data variance ( $\sigma^2$ ) and the posterior uncertainty about the mean ( $\sigma_n^2$ ).

# 4 Hypothesis Testing

## 4.1 General Terminologies

### 4.1.1 Hypothesis Testing

We formulate the problem of hypothesis testing as deciding between two competing claims about a parameter  $\theta$ :

$$H_0 : \theta \in \Theta_0 \quad (\text{Null Hypothesis}) \quad (4.1)$$

$$H_1 : \theta \in \Theta_1 \quad (\text{Alternative Hypothesis}) \quad (4.2)$$

**Definition 4.1** (Simple and Composite Hypotheses). A hypothesis is called **simple** if it specifies a single value for the parameter (e.g.,  $\Theta_0$  contains only one point). It is called **composite** if it specifies more than one value.

**Example 4.1** (Normal Mean Test). Let  $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ .

- If  $\sigma^2$  is known,  $H_0 : \mu = \mu_0$  is a simple hypothesis.
- If  $\sigma^2$  is unknown,  $H_0 : \mu = \mu_0$  is a composite hypothesis (since  $\sigma^2$  can vary).

### 4.1.2 Test Functions and Size

A test is defined by a **critical region**  $C_\alpha$  such that we reject  $H_0$  if the data  $x \in C_\alpha$ . Equivalently, we can define a **test function**  $\phi(x)$  representing the probability of rejecting  $H_0$  given data  $x$ .

- A non-randomized test is given as follows:

$$\phi(x) = I(x \in C_\alpha) = \begin{cases} 1 & \text{if } x \in C_\alpha \text{ (Reject } H_0) \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

- A randomized test,  $\phi(x)$  can take values in  $[0, 1]$ , which can be expressed typically as follows:

$$\phi(x) = \begin{cases} 1 & \text{if } x \in C_1 \\ \gamma & \text{if } x \in C_* \\ 0 & \text{otherwise} \end{cases} \quad (4.4)$$

where:

- $C_1$  is the region where we strictly reject  $H_0$ .
- $C_*$  is the boundary region (often where  $T(x) = k$ ) where we reject  $H_0$  with probability  $\gamma$ .
- More generally,  $\phi(x)$  is just a function of  $x$  with values in  $[0, 1]$ , which represents the probability that we will reject  $H_0$ .

**Example 4.2** (Randomized Test for Binomial). Let  $X \sim \text{Bin}(n = 10, \theta)$ . Consider testing  $H_0 : \theta = 1/2$  vs  $H_1 : \theta > 1/2$  with target size  $\alpha = 0.05$ .

Suppose we choose a critical region  $X \geq k$ .

- If  $k = 9$ ,  $P(X \geq 9 | \theta = 0.5) \approx 0.0107$ .
- If  $k = 8$ ,  $P(X \geq 8 | \theta = 0.5) \approx 0.0547$ .

Since we cannot achieve exactly 0.05 with a non-randomized test (the survival function jumps over 0.05), we must use a randomized test function.

The randomized test is defined as:

$$\phi(x) = \begin{cases} 1 & \text{if } x \in C_1 \text{ (i.e., } x \geq 9\text{)} \\ \gamma & \text{if } x \in C_* \text{ (i.e., } x = 8\text{)} \\ 0 & \text{otherwise} \end{cases} \quad (4.5)$$

From the figure, we see that  $\alpha = 0.05$  lies between  $P(X \geq 9)$  and  $P(X \geq 8)$ . We always reject the “tail” where probabilities are strictly less than  $\alpha$  (here  $x \geq 9$ ). At the boundary  $x = 8$ , we cannot reject with probability 1 (which would give total size 0.0547), nor with probability 0 (which would give total size 0.0107).

We choose  $\gamma$  to bridge this gap:

$$\begin{aligned} \alpha &= P(X \geq 9) + \gamma \cdot P(X = 8) \\ 0.05 &= 0.01074 + \gamma \cdot (P(X \geq 8) - P(X \geq 9)) \\ 0.05 &= 0.01074 + \gamma \cdot (0.05469 - 0.01074) \end{aligned} \quad (4.6)$$

Solving for  $\gamma$ :

$$\gamma = \frac{0.05 - 0.01074}{0.04395} \approx \frac{39}{44} \approx 0.89 \quad (4.7)$$

## 4.2 Power and Size Function

**Definition 4.2** (Size of a Test). The **size** of a test  $\phi(x)$  is the maximum probability of rejecting the null hypothesis when it is true:

$$\text{Size}(\phi) = \sup_{\theta \in \Theta_0} W_\phi(\theta) = \sup_{\theta \in \Theta_0} E_\theta[\phi(X)] \quad (4.8)$$

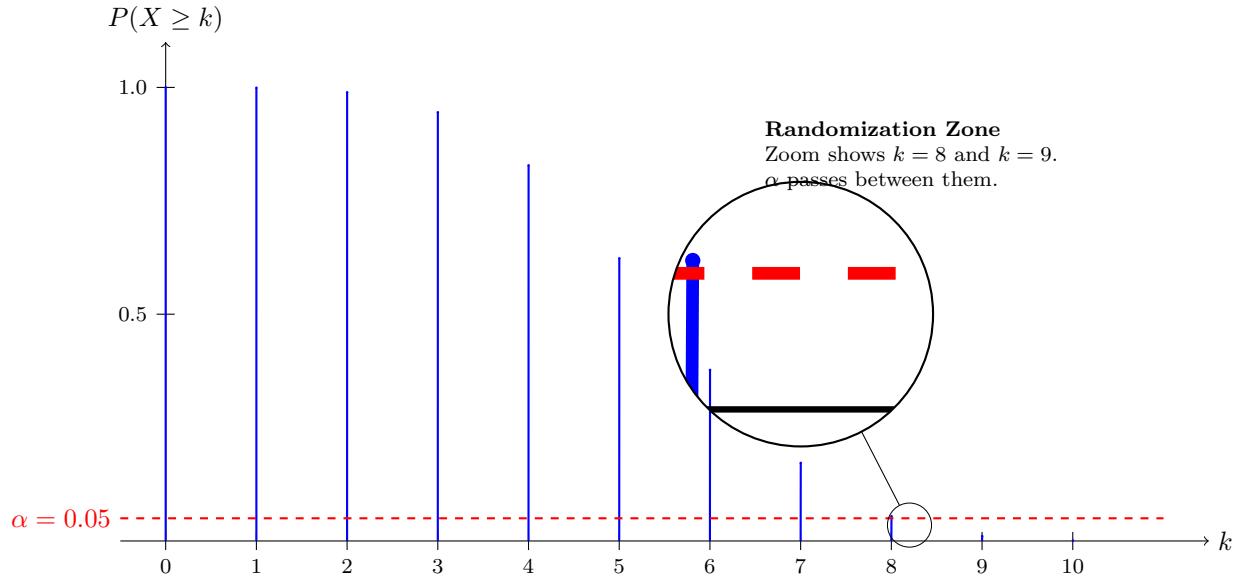


Figure 4.1: Survival Function  $P(X \geq k)$  with randomization detail

#### 4.2.1 Power Definitions

We distinguish between the power function varying over parameters and the power metric of a specific test.

**1. Power Function ( $W_\phi(\theta)$ )** The probability of rejecting  $H_0$  as a function of the parameter  $\theta$ :

$$W_\phi(\theta) = E_\theta[\phi(X)] \quad (4.9)$$

**2. Power of the Test (Power( $\phi$ ))** In the context of a specific alternative hypothesis (e.g.,  $H_1 : \theta = \theta_1$ ), we define the power as a scalar functional of  $\phi$ :

$$\text{Power}(\phi) = E_{\theta_1}[\phi(X)] \quad (4.10)$$

Ideally, we want:

- $W_\phi(\theta) \leq \text{Size}(\phi)$  for all  $\theta \in \Theta_0$  (Control Type I error).
- Power( $\phi$ ) to be as large as possible (Maximize sensitivity to  $H_1$ ).

### 4.3 The Neyman-Pearson Lemma

Consider testing a simple null against a simple alternative:  $H_0 : \theta = \theta_0$  vs  $H_1 : \theta = \theta_1$ .

We define the **Likelihood Ratio**  $\Lambda(x)$  as:

$$\Lambda(x) = \frac{f_1(x)}{f_0(x)} = \frac{f(x; \theta_1)}{f(x; \theta_0)} \quad (4.11)$$

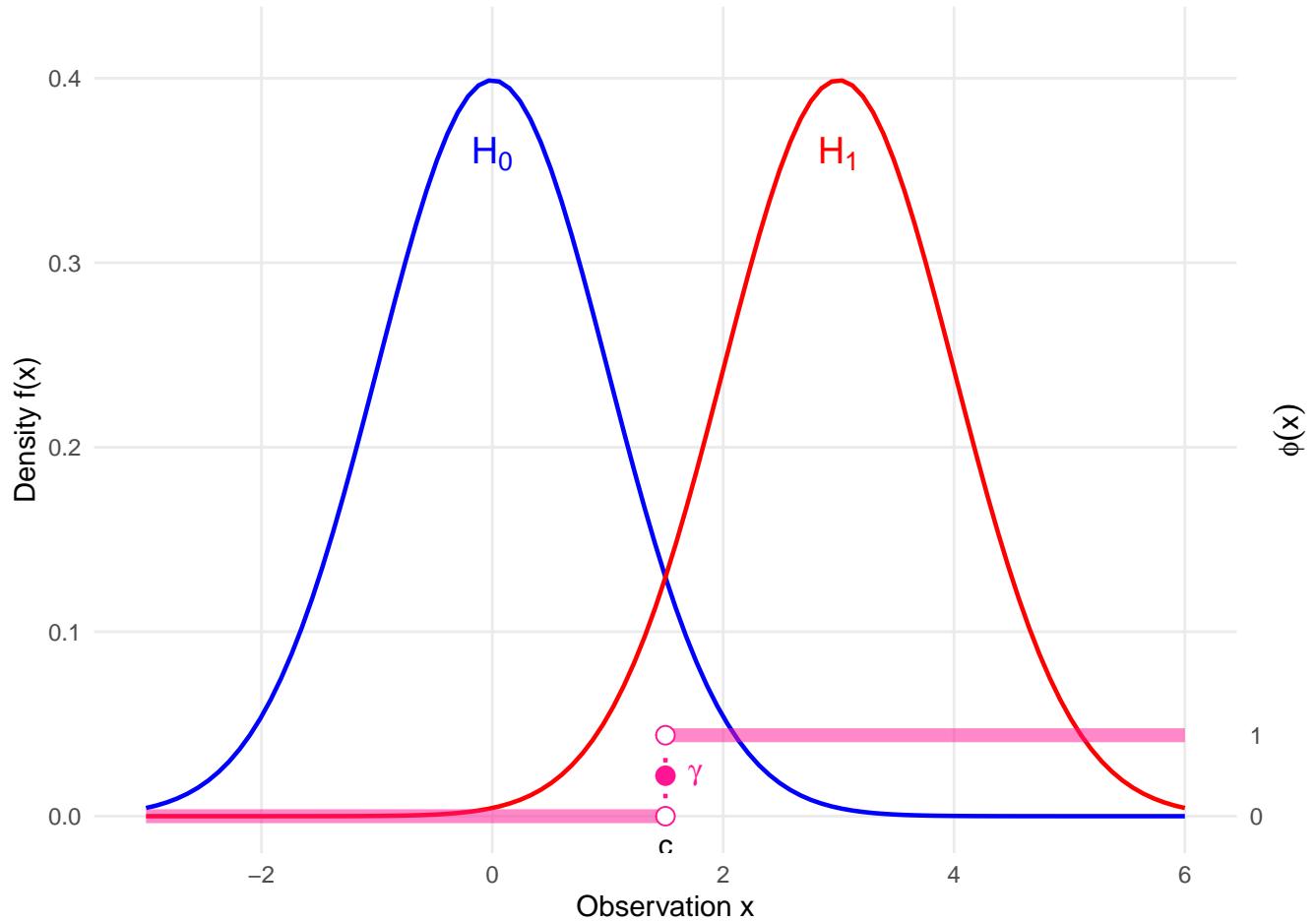


Figure 4.2: Illustration of a Test Function  $\phi(x)$  (Pink) relative to Size ( $H_0$ , Blue) and Power ( $H_1$ , Red).

**Definition 4.3** (Likelihood Ratio Test (LRT)). A test  $\phi(x)$  is a Likelihood Ratio Test if it has the form:

$$\phi(x) = \begin{cases} 1 & \text{if } \Lambda(x) > k \\ \gamma(x) & \text{if } \Lambda(x) = k \\ 0 & \text{if } \Lambda(x) < k \end{cases} \quad (4.12)$$

where  $k \geq 0$  is a constant and  $0 \leq \gamma(x) \leq 1$ .

### 4.3.1 Neyman-Pearson Lemma

**Theorem 4.1** (Neyman-Pearson Lemma).

- a) **Optimality:** For any  $k$  and  $\gamma(x)$ , the LRT  $\phi_0(x)$  defined above has maximum power among all tests whose size is less than or equal to the size of  $\phi_0(x)$ .
- b) **Existence:** Given  $\alpha \in (0, 1)$ , there exist constants  $k$  and  $\gamma_0$  such that the LRT defined by this  $k$  and  $\gamma(x) = \gamma_0$  has size exactly  $\alpha$ .
- c) **Uniqueness:** If a test  $\phi$  has size  $\alpha$  and is of maximum power among all tests of size  $\alpha$ , then  $\phi$  is necessarily an LRT, except possibly on a set of measure zero under  $H_0$  and  $H_1$ .

### 4.3.2 A Derivation with The Lagrange Multiplier Approach

To make the optimality of the Likelihood Ratio Test (LRT) intuitive, we can frame the search for the best test function  $\phi(x)$  as a constrained optimization problem.

We want to maximize the power of the test:

$$\text{Power}(\phi) = \int \phi(x) f_1(x) dx \quad (4.13)$$

subject to the constraint on the size of the test  $\alpha$ :

$$\text{Size}(\phi) = \int \phi(x) f_0(x) dx = \alpha \quad (4.14)$$

Using the method of Lagrange multipliers, we define the objective function  $L$  with a multiplier  $k$ :

$$L(\phi, k) = \int \phi(x) f_1(x) dx - k \left( \int \phi(x) f_0(x) dx - \alpha \right) \quad (4.15)$$

Rearranging the terms inside the integral, we get:

$$L(\phi, k) = \int \phi(x)[f_1(x) - kf_0(x)]dx + k\alpha \quad (4.16)$$

To maximize  $L$  with respect to  $\phi(x)$ , we look at the integrand. Since  $0 \leq \phi(x) \leq 1$ , we should choose  $\phi(x)$  to be as large as possible whenever its coefficient is positive, and as small as possible whenever its coefficient is negative:

- If  $f_1(x) - kf_0(x) > 0$ , set  $\phi(x) = 1$ .
- If  $f_1(x) - kf_0(x) < 0$ , set  $\phi(x) = 0$ .
- If  $f_1(x) - kf_0(x) = 0$ , the value of  $\phi(x)$  does not affect the integral (this is where  $\gamma$  comes in).

This decision rule is equivalent to:

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{f_1(x)}{f_0(x)} > k \\ 0 & \text{if } \frac{f_1(x)}{f_0(x)} < k \end{cases} \quad (4.17)$$

This is precisely the form of the Likelihood Ratio Test. The “shadow price” or Lagrange multiplier  $k$  represents the critical threshold that balances the gain in power against the cost of increasing the Type I error.

### 4.3.3 Proof of NP Lemma

*Proof. Proof of (a) Optimality:* Let  $\phi_{LRT}$  be the LRT with size  $\alpha$ , and  $\phi$  be any other test with size  $\leq \alpha$ . Define the function  $U(x)$  as the difference in test functions weighted by the linear combination of densities:

$$U(x) = (\phi_{LRT}(x) - \phi(x))(f_1(x) - kf_0(x)) \quad (4.18)$$

We analyze the sign of  $U(x)$  by looking at the sign of its two factors in three cases:

- If  $f_1(x) - kf_0(x) > 0$  (implies  $\Lambda(x) > k$ ). Since  $\phi_{LRT}(x) = 1$  and  $\phi(x) \leq 1$ , we have:

$$\begin{aligned} \phi_{LRT}(x) - \phi(x) &\geq 0 \\ U(x) = (\phi_{LRT}(x) - \phi(x))(f_1(x) - kf_0(x)) &\geq 0 \end{aligned} \quad (4.19)$$

- If  $f_1(x) - kf_0(x) < 0$  (implies  $\Lambda(x) < k$ ). Since  $\phi_{LRT}(x) = 0$  and  $\phi(x) \geq 0$ , we have:

$$\begin{aligned} \phi_{LRT}(x) - \phi(x) &\leq 0 \\ U(x) = (\phi_{LRT}(x) - \phi(x))(f_1(x) - kf_0(x)) &\geq 0 \end{aligned} \quad (4.20)$$

- If  $f_1(x) - kf_0(x) = 0$ . The product is zero regardless of the test functions.

$$U(x) = 0 \quad (4.21)$$

Combining these cases, we conclude that the product is non-negative for all  $x$ :

$$U(x) = (\phi_{LRT}(x) - \phi(x))(f_1(x) - kf_0(x)) \geq 0 \quad (4.22)$$

Therefore, integrating  $U(x)$  over the entire domain:

$$\int U(x)dx = \int (\phi_{LRT} - \phi)(f_1 - kf_0)dx \geq 0 \quad (4.23)$$

Expanding the integral:

$$\int \phi_{LRT}f_1 - \int \phi f_1 - k \left( \int \phi_{LRT}f_0 - \int \phi f_0 \right) \geq 0 \quad (4.24)$$

Converting to expectations:

$$E_{\theta_1}[\phi_{LRT}] - E_{\theta_1}[\phi] - k(E_{\theta_0}[\phi_{LRT}] - E_{\theta_0}[\phi]) \geq 0 \quad (4.25)$$

Since  $E_{\theta_0}[\phi_{LRT}] = \alpha$  and  $E_{\theta_0}[\phi] \leq \alpha$ , the difference  $(E_{\theta_0}[\phi_{LRT}] - E_{\theta_0}[\phi]) \geq 0$ . Given that  $k \geq 0$ :

$$E_{\theta_1}[\phi_{LRT}] - E_{\theta_1}[\phi] \geq 0 \implies \text{Power}(\phi_{LRT}) \geq \text{Power}(\phi) \quad (4.26)$$

**Proof of (b) Existence:** Let  $G(k) = P_{\theta_0}(\Lambda(X) \leq k)$ .  $G(k)$  is the cumulative distribution function of the random variable  $\Lambda(X)$ , so it is non-decreasing. We seek  $k_0$  such that  $1 - G(k_0) \approx \alpha$ . Because of discrete jumps, we might not hit  $\alpha$  exactly. We choose  $k_0$  such that:

$$P_{\theta_0}(\Lambda(X) > k_0) \leq \alpha \leq P_{\theta_0}(\Lambda(X) \geq k_0) \quad (4.27)$$

Set  $\gamma_0 = \frac{\alpha - P_{\theta_0}(\Lambda(X) > k_0)}{P_{\theta_0}(\Lambda(X) = k_0)}$ . □

## 4.4 Uniformly Most Powerful (UMP) Tests

When the alternative hypothesis is composite ( $H_1 : \theta \in \Theta_1$ ), we seek a test that is “best” for *all*  $\theta \in \Theta_1$ .

**Definition 4.4** (Uniformly Most Powerful Test). A test  $\phi_0(x)$  of size  $\alpha$  is **Uniformly Most Powerful (UMP)** if:

1.  $E_\theta[\phi_0(X)] \leq \alpha$  for all  $\theta \in \Theta_0$ .
2. For any other test  $\phi(x)$  satisfying (1),  $E_\theta[\phi_0(X)] \geq E_\theta[\phi(X)]$  for all  $\theta \in \Theta_1$ .

## 4.5 Monotone Likelihood Ratio (MLR)

**Definition 4.5** (Monotone Likelihood Ratio). A family of densities  $\{f(x; \theta)\}$  has a **Monotone Likelihood Ratio (MLR)** with respect to a statistic  $T(x)$  if for any  $\theta_1 > \theta_0$ , the ratio:

$$\frac{f(x; \theta_1)}{f(x; \theta_0)} \quad (4.28)$$

is a non-decreasing function of  $T(x)$ .

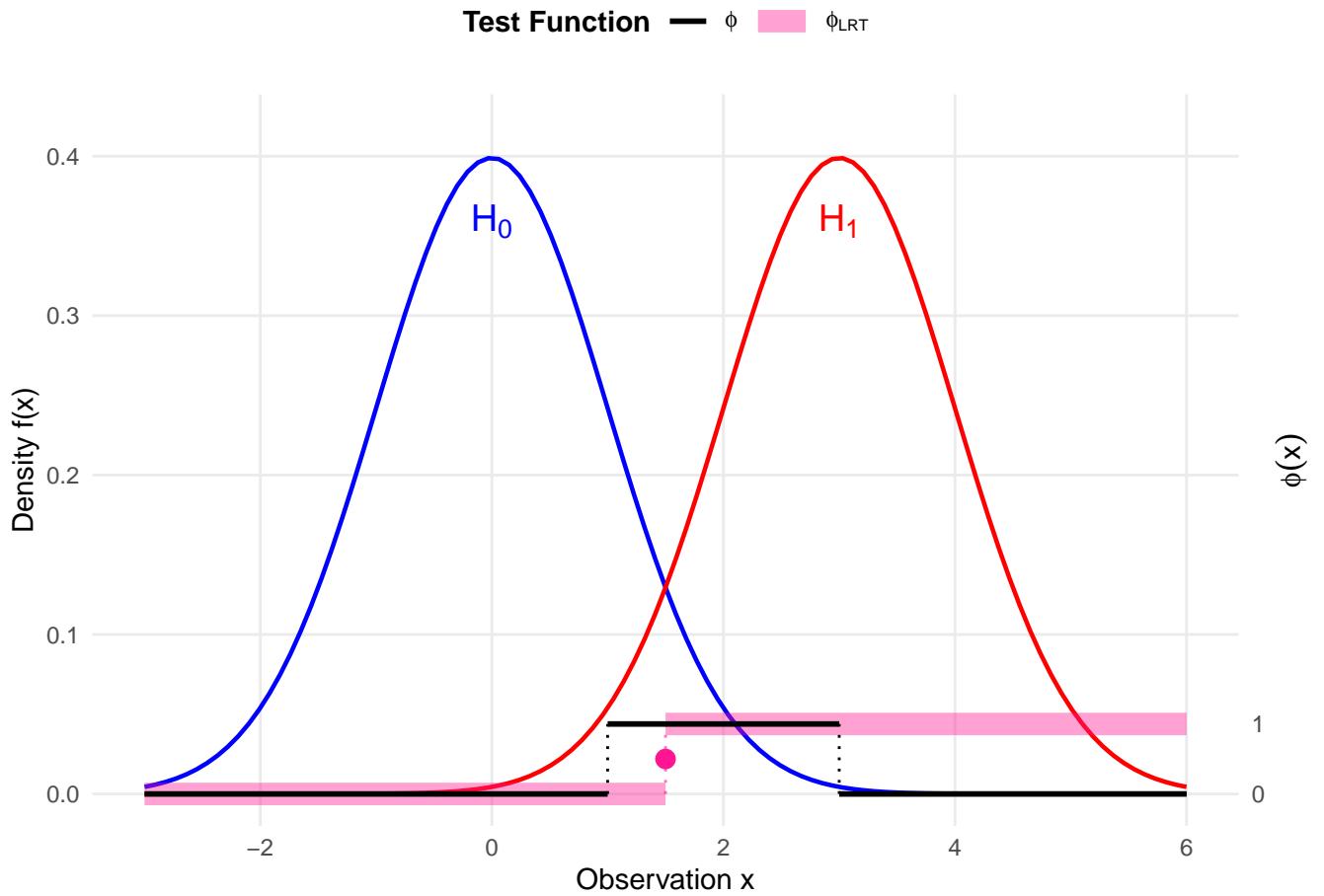


Figure 4.3: Visualizing the NP Lemma: The thick, transparent pink line is  $\phi_{LRT}$ . The thin solid black line is  $\phi$ . Overlap is visible as a black line inside pink.

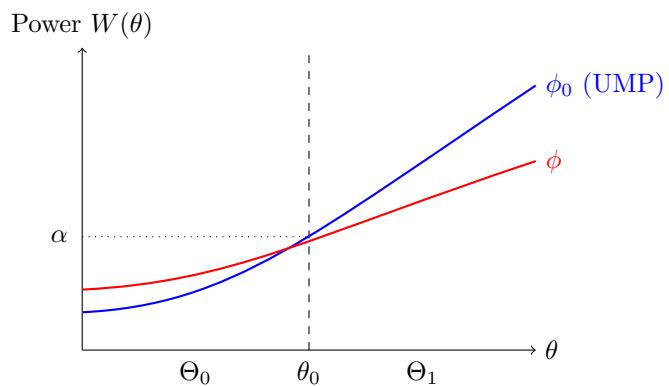


Figure 4.4: Power functions of UMP test vs. another test

Common examples include the one-parameter Exponential Family:  $f(x; \theta) = h(x)c(\theta) \exp\{w(\theta)T(x)\}$ . If  $w(\theta)$  is increasing, the family has MLR w.r.t  $T(x)$ .

## Karlin-Rubin Theorem

**Theorem 4.2** (Karlin-Rubin Theorem). *Suppose  $X$  has a distribution from a family with MLR with respect to  $T(X)$ , and the distribution of  $T(X)$  is continuous. Consider testing  $H_0 : \theta \leq \theta_0$  vs  $H_1 : \theta > \theta_0$ . The test:*

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > t_0 \\ 0 & \text{if } T(x) \leq t_0 \end{cases} \quad (4.29)$$

where  $t_0$  is determined by  $P_{\theta_0}(T(X) > t_0) = \alpha$ , is the UMP size  $\alpha$  test.

*Proof of Theorem 4.2 (UMP for MLR Families).* **The Test:** Define the test  $\phi(x)$  as:

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > t_0 \\ 0 & \text{if } T(x) \leq t_0 \end{cases} \quad (4.30)$$

where  $t_0$  is determined such that the power at the boundary is  $\alpha$ , i.e.,  $W_{LR}(\theta_0) = \alpha$ .

**(1) Application of Neyman-Pearson Lemma** Because the family has a Monotone Likelihood Ratio (MLR) in  $T(x)$ , for any specific alternative  $\theta_1 > \theta_0$ , the likelihood ratio  $\Lambda(x)$  is an increasing function of  $T(x)$ . Therefore, the rejection region  $T(x) > t_0$  corresponds to  $\Lambda(x) > k$ . By the Neyman-Pearson Lemma,  $\phi(x)$  is the Most Powerful (MP) test for testing  $H'_0 : \theta = \theta_0$  vs  $H'_1 : \theta = \theta_1$ .

**(2) Monotonicity of the Power Function** We claim that  $W_{LR}(\theta)$  is a non-decreasing function of  $\theta$ .

*Proof:* Let  $\theta_2 < \theta_1$ . Consider testing  $\theta = \theta_2$  vs  $\theta = \theta_1$ . Let  $\beta = W_{LR}(\theta_2)$ . Define a constant dummy test  $\phi^*(x) = \beta$  for all  $x$ . The power of this test is constant:  $W_{\phi^*}(\theta) = \beta$ . Since  $\phi(x)$  corresponds to the likelihood ratio test form (reject for large  $T$ ) for  $\theta_2$  vs  $\theta_1$ , it is the MP test of size  $\beta$ . By the optimality of the NP Lemma, the power of  $\phi$  at  $\theta_1$  must be at least the power of the competitor  $\phi^*$ :

$$W_{LR}(\theta_1) \geq W_{\phi^*}(\theta_1) = \beta = W_{LR}(\theta_2) \quad (4.31)$$

Thus,  $W_{LR}(\theta)$  is non-decreasing.

**(3) Size Control** Since  $W_{LR}(\theta)$  is non-decreasing and we set  $W_{LR}(\theta_0) = \alpha$ :

$$W_{LR}(\theta) \leq W_{LR}(\theta_0) = \alpha \quad \text{for all } \theta \leq \theta_0 \quad (4.32)$$

This proves the test satisfies the size constraint for the composite null  $H_0 : \theta \leq \theta_0$ .

**(4) UMP Property** Let  $\phi'(x)$  be any other test with size  $\leq \alpha$  for  $H_0 : \theta \leq \theta_0$ . This implies  $W_{\phi'}(\theta_0) \leq \alpha$ . For any specific  $\theta_1 > \theta_0$ , we treat the problem as testing  $\theta_0$  vs  $\theta_1$ . Since  $\phi(x)$  is the MP test for  $\theta_0$  vs  $\theta_1$  (from Step 1), and  $\phi'$  is a valid competitor (size  $\leq \alpha$  at  $\theta_0$ ), we have:

$$W_{LR}(\theta_1) \geq W_{\phi'}(\theta_1) \quad (4.33)$$

Since this holds for all  $\theta_1 > \theta_0$ ,  $\phi(x)$  is the Uniformly Most Powerful (UMP) test.  $\square$

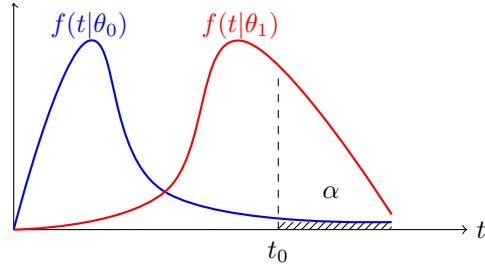


Figure 4.5: Distribution of statistic  $T$  under  $H_0$  and  $H_1$  with MLR

**Example 4.3** (UMP Test for Exponential/Gamma). Let  $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Exp}(\theta)$  with pdf  $f(x) = \frac{1}{\theta} e^{-x/\theta}$ . Test  $H_0 : \theta = \theta_0$  vs  $H_1 : \theta > \theta_0$ . The sum  $T = \sum X_i$  is a sufficient statistic, and  $T \sim \text{Gamma}(n, \theta)$ . The Likelihood Ratio for  $\theta_1 > \theta_0$  is:

$$\frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^{-n} e^{-\sum x_i / \theta_1}}{\theta_0^{-n} e^{-\sum x_i / \theta_0}} = \left( \frac{\theta_0}{\theta_1} \right)^n \exp \left\{ \left( \frac{1}{\theta_0} - \frac{1}{\theta_1} \right) \sum x_i \right\} \quad (4.34)$$

Since  $\theta_1 > \theta_0$ , the term  $(\frac{1}{\theta_0} - \frac{1}{\theta_1})$  is positive. Thus,  $\Lambda(x)$  is an increasing function of  $\sum x_i$ . Rejecting for large  $\Lambda(x)$  is equivalent to rejecting for  $\sum x_i > C$ .

This test form does not depend on the specific  $\theta_1$ , so it is UMP for all  $\theta > \theta_0$ .

#### 4.5.1 Note on Two-Sided Hypotheses

For testing  $H_0 : \theta = \theta_0$  vs  $H_1 : \theta \neq \theta_0$  (e.g., in a Normal distribution), a UMP test generally **does not exist**. This is because the “best” rejection region for  $\theta > \theta_0$  (right tail) is completely different from the “best” region for  $\theta < \theta_0$  (left tail).