

Chapter : 7.2. Condition Inference

\Rightarrow Example : δ indicates 2 meas tools

$\delta=1 \rightarrow$ the tool is more precise (e.g. using $n_1=90$ sample)

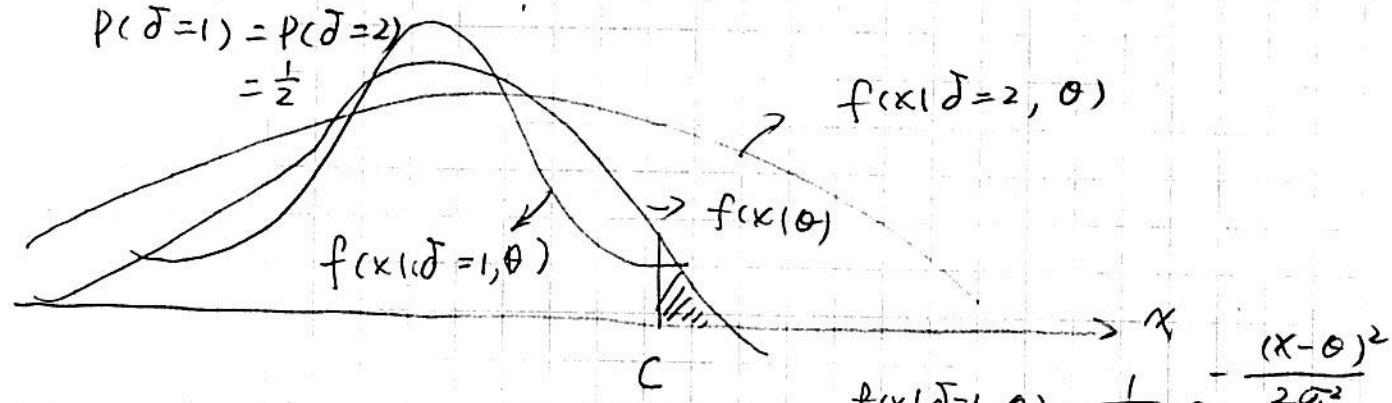
$\delta=2 \rightarrow$ the tool is less precise (e.g. using $n_2=10$ sample)

Model:

$$X | \delta=1 \sim N(\theta, \sigma_1^2), \sigma_1 = 1$$

$$X | \delta=2 \sim N(\theta, \sigma_2^2), \sigma_2 = 3$$

$$P(\delta=1) = P(\delta=2)$$



$$H_0: \theta = 0 \quad \text{vs} \quad H_1: \theta \neq 0$$

Test statistic: X

$$f(x|\delta=1, \theta) = \frac{1}{\sqrt{2\sigma_1^2}} e^{-\frac{(x-\theta)^2}{2\sigma_1^2}}$$

$$f(x|\delta=2, \theta) = \frac{1}{\sqrt{2\sigma_2^2}} e^{-\frac{(x-\theta)^2}{2\sigma_2^2}}$$

$$\text{Rejection Region: } \{x > C\}$$

$$f(x|\theta) = \frac{1}{2} f(x|\delta=1, \theta) + \frac{1}{2} f(x|\delta=2, \theta)$$

We will determine C by $\Pr(X > C | \theta = 0) = 0.05$

\Rightarrow Procedure 1: using $f(x|\theta=0)$

We will solve:

$$\Pr(X > C | \theta = 0) = 0.05$$

$$\frac{1}{2} P(N(0, \sigma_1^2) > C) + \frac{1}{2} P(N(0, \sigma_2^2) > C) = 0.05$$

\Rightarrow Procedure 2: observed; X and δ

Using $f(x|\delta, \theta)$ to determine C

If $\delta=1$, $X|0 \sim N(0, \sigma_1^2)$

$$P(X > C_1 | \theta=0) = 0.05$$

$$\Rightarrow C_1 = 0 + Z_{0.05} \cdot \sigma_1 = 1.645$$

which distribution
we used.

If $\delta=2$ $X|0=0 \sim N(0, \sigma_2^2)$

$$P(X > C_2 | \theta=0, \delta=2) = 0.05$$

$$\Rightarrow C_2 = 0 + 3 \times Z_{0.05} = 3 \times 1.645 = 4.92$$

Rejection Region: $X > 0 + Z_{0.05} \cdot \sigma_{\delta}$

What's the problems for procedure 1:

1) ignore δ

2) The justification for C is based on $P(X > C) = \alpha$
without knowing δ .

The α is probability averaging all data sets.

3) C is determined before we observe any data
value of X and δ

\Rightarrow Definition: $\theta = (\varphi, \lambda)$

φ is of our interest

λ is nuisance parameter

$T = (S, C)$ is MSS for (φ, λ)

If (a) the distribution of C doesn't depend on φ .
but on λ

(b) the distribution of $S|C$ depends on φ , but
not on λ .

Then, we say

C is an ancillary statistic for φ , and S is
conditionally sufficient for φ given C ,

\Rightarrow example: θ ($\lambda = \phi$) the distribution of δ doesn't
depend on θ .

The distribution $X|\delta$ depend on θ ,

$\delta \rightarrow$ called ancillary (not useless) statistic for θ

\Rightarrow Conditional principle:

our inference should be conditional on all
ancillary statistic.

\Rightarrow example 2:

$$X_1, \dots, X_n | \theta \sim \text{unif}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$$

$$f(x|\theta) = I(X_{(1)} - \frac{1}{2} < \theta < X_{(n)} + \frac{1}{2})$$

so, the $S = (X_{(1)}, X_{(n)})$ is MSS.

$$T = \frac{X_{(1)} + X_{(n)}}{2} \leftarrow \text{Not sufficient}$$

$$W = X_{(n)} - X_{(1)} \leftarrow \text{ancillary}$$

$f_T(t)$ depend on θ ; $f_W(w)$ does not depend on θ

(T, W) is MSS.

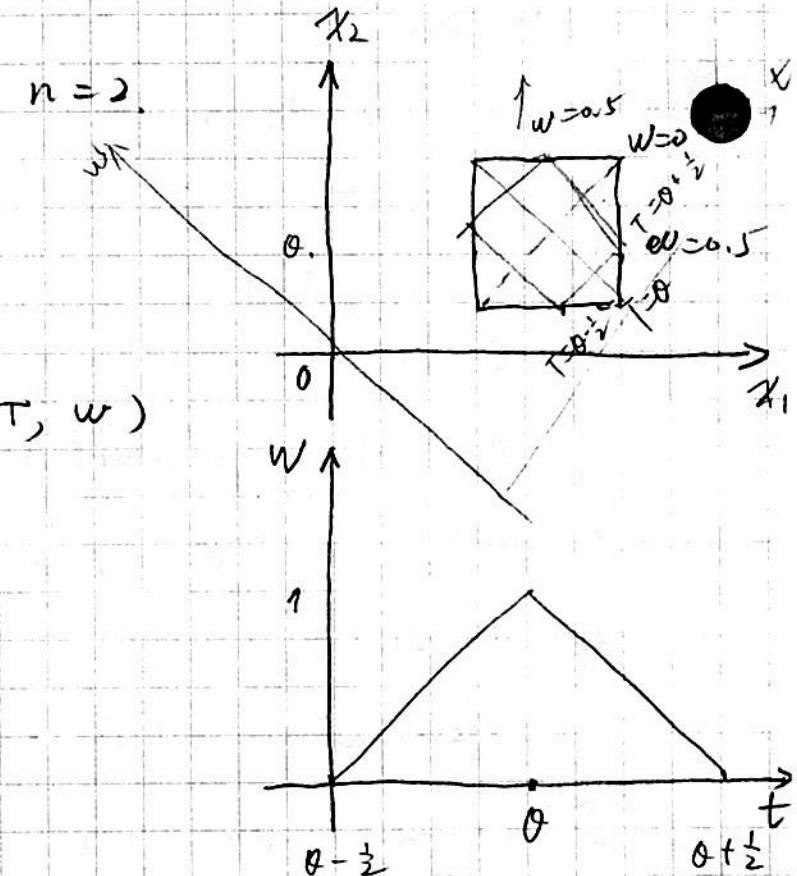
\Rightarrow look at a special case. $n=2$.

$$\begin{cases} T = \frac{x_1 + x_2}{2} \\ W = |x_1 - x_2| \end{cases}$$

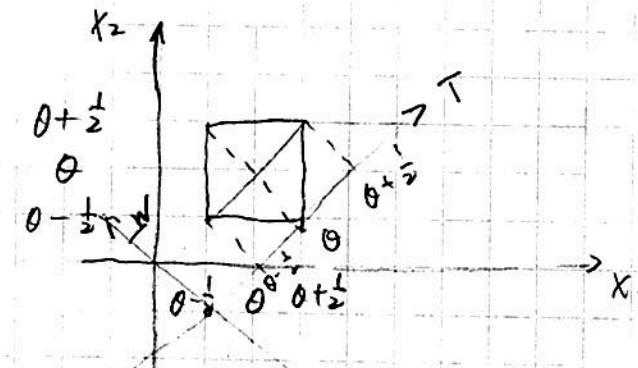
The joint distribution of (T, W)

$$(T, W) \sim \text{Unif}(A)$$

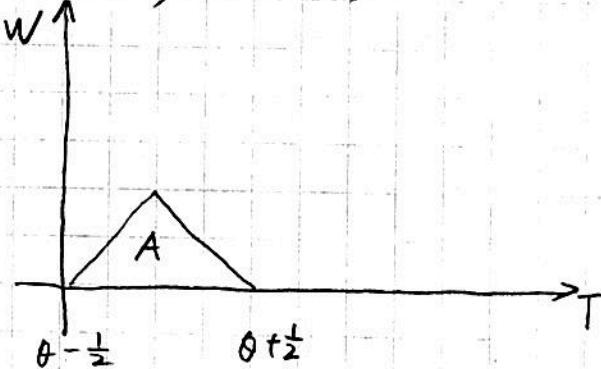
A is the triangle.



\Rightarrow example: $X_1, \dots, X_2 \stackrel{iid}{\sim} \text{unif}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$



$$T = \frac{X_{(1)} + X_{(2)}}{2} = \frac{X_1 + X_2}{2}$$



$$(T, W) \sim \text{unif}(A)$$

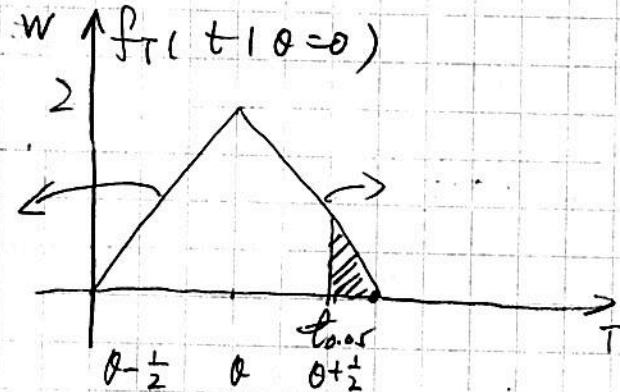
$$W = X_{(2)} - X_{(1)} = |X_{(1)} - X_{(2)}|$$

\Rightarrow hypothesis test:

$$H_0: \theta \leq 0 \text{ against } H_1: \theta > 0$$

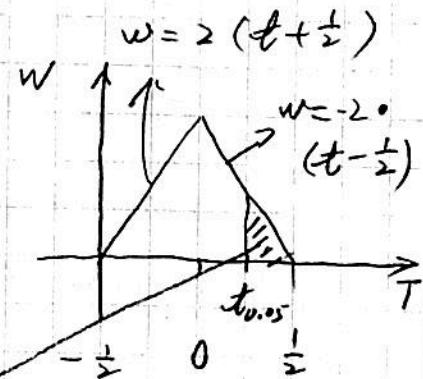
procedure 1: (unconditional)

using marginal distribution of T



Rejection Region: $\{T > t\}$

$$P(T > t_{0.05}) = 0.05 \quad (\alpha = 0.05)$$



where, $t_{0.05} = 0.342$

$$\begin{aligned} & (\frac{1}{2} - t_{0.05}) \times [-2(t_{0.05} - \frac{1}{2})] \times \frac{1}{2} \\ & = 0.025 \end{aligned}$$

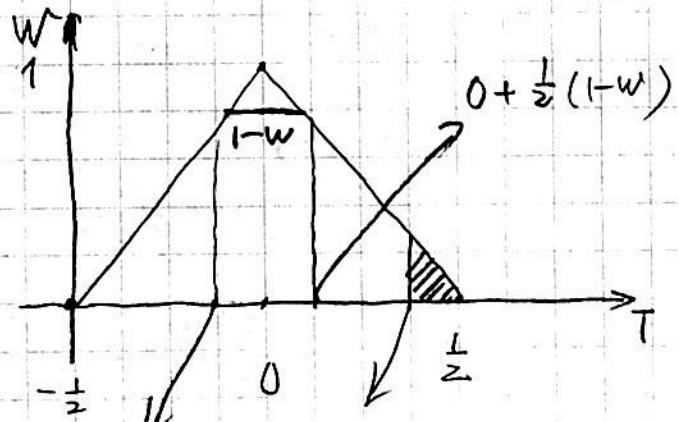
procedure 2: (conditional inference)

using the distribution $f(t|w, \theta)$

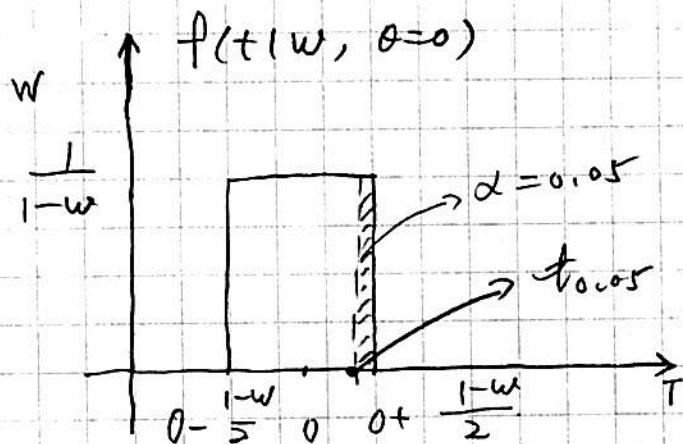
$$T|w, \theta=0 \sim \text{Unif}\left(\frac{w}{2} - \frac{1}{2}, \frac{1}{2} - \frac{w}{2}\right)$$

$$w = 2(t + \frac{1}{2}) \Rightarrow t = \frac{w}{2} - \frac{1}{2}$$

$$w = -2(t - \frac{1}{2}) \Rightarrow t = \frac{1}{2} - \frac{w}{2}$$



$$t_{0.05} = 0.342$$



$$\left(\frac{1-w}{2} - t_{0.05}^*\right) = 0.05 \times (1-w)$$

$$\frac{1-w}{2} - 0.05(1-w) = t_{0.05}^*$$

$$t_{0.05}^* = 0.45(1-w)$$

we should reject when:

$$\text{Rejection region is: } t > 0.45(1-w) = t_{0.05}^*$$

Looking at specific data set: $\begin{cases} t = 0.2 \\ w = 0.7 \end{cases}$

$$\begin{cases} t = 0.2 \\ w = 0.7 \end{cases}$$

By the procedure 1

No reject

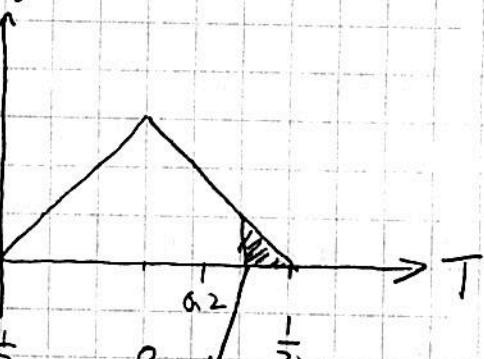
\Rightarrow Procedure 2:

$$T | w = 0.7, \theta = 0 \sim \text{Unif}\left(0 - \frac{0.3}{2}, 0 + \frac{0.3}{2}\right) \quad t_{0.05} = 0.342$$

$$= \text{Unif}(-0.15, 0.15)$$

By procedure 2. We will reject H_0 .

$$t^*_{0.05} = 0.45 \times 0.3 = 0.135$$



The procedure 2. is better than procedure 1.

\Rightarrow the procedure 3 (Bayesian)

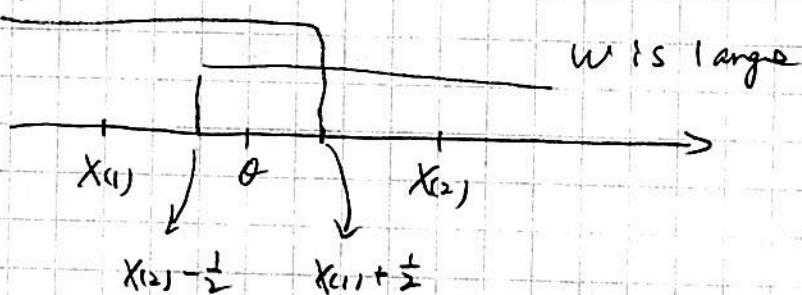
$\Pr(H_0 | x) < c$, we will Reject H_0 (in our case $c = \frac{1}{2}$)

We will compute $\Pr(\theta \leq 0 | x_1, x_2) = ?$

$$f(x_1, x_2 | \theta) = I(x_{(2)} - \frac{1}{2} < \theta < x_{(1)} + \frac{1}{2})$$

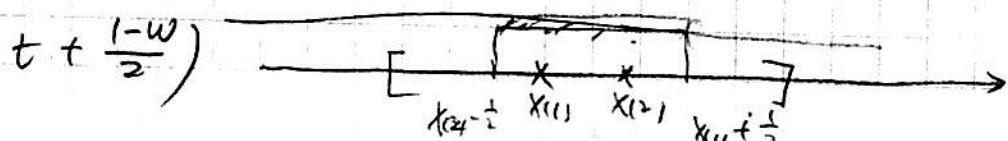
$$\theta \sim I(-M < \theta < M)$$

$$\theta | x_1, x_2 \sim \text{Unif}\left(x_{(2)} - \frac{1}{2}, x_{(1)} + \frac{1}{2}\right)$$



another way.

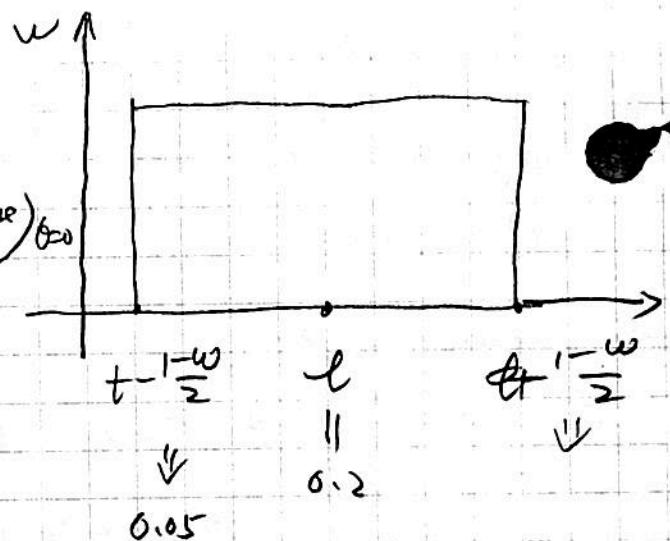
$$\theta | x_1, x_2 \sim \text{Unif}\left(t - \frac{1-w}{2}, t + \frac{1-w}{2}\right) \quad w \text{ is large}$$



Given $\begin{cases} \ell = 0.2 \\ -w = 0.7 \end{cases}$

$\Pr(\ell < 0 | \ell_1, \ell_2) = 0$ (it is impossible)

We reject H_0 .



Review:

For test 2:

Concept:

size

skill

Chapter 4: ✓ size, ✓ power, ✓ UMP
✓ N-P lemma
✓ MLR.

Find UMP

in MLR family

Chapter 5: ✓ exp. family.

✓ natural statistic.
✓ natural parameter

verify
exp. family

Chapter 6, ✓ MLE.
✓ complete sufficient.
✓ R-B theorem

Find UMVUE
with R-B theorem

$E(d, \kappa) | T$

① Fisher information
 $I(\theta)$

② CRLB

③ Asymptotic

distribution of Once

Chapter 8: likelihood function

Including threshold by Wilks' theorem

④ Likelihood Ratio test