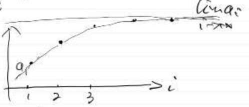
Infinite Sequence

$a_1, a_2, a_3, \ldots$

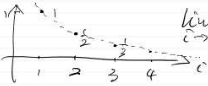
$S = \sum_{i=1}^{\infty} a_i$

Limit of a sequence

$\lim_{i \to +\infty} a_i$



e.g. $a_i = \frac{1}{i}$



$\lim_{i \to +\infty} \frac{1}{i} = 0$

e.g. $\lim_{i \to +\infty} \left(1 - \frac{1}{i}\right) = 1$

e.g. $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$

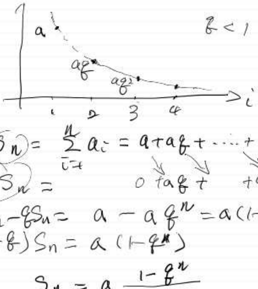
Limit of Inf Sequence

$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

$\sum_{i=1}^{+\infty} a_i = \lim_{n \to +\infty} S_n$

$\sum_{i=1}^{+\infty} a_i < \infty$, converge

e.g. $a_i = a q^{i-1}$



$S_n = \sum_{i=1}^{n} a_i = a + aq + \cdots + aq^{n-1}$

$qS_n = \quad 0 + aq + \quad + aq^{n-1} + aq^n$

$S_n - qS_n = a - aq^n = a(1 - q^n)$

$(1-q) S_n = a(1 - q^n)$

$S_n = a \frac{1 - q^n}{1 - q}$

$\lim_{n \to +\infty} S_n = a \frac{1 - 0}{1 - q} = \frac{a}{1-q} \quad q < 1$

e.g. $a = 1, \ q = \frac{1}{2}$

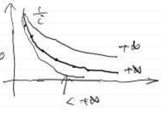
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$

$\sum_{i=1}^{+\infty} \left(\frac{1}{2}\right)^{i-1} = 2, \quad \sum_{i=1}^{+\infty} \left(\frac{1}{2}\right)^{i} = 1$

Some sequences:

$\sum_{i=1}^{+\infty} \frac{1}{i} = +\infty$

Harmonic sequence



$\sum_{i=1}^{+\infty} \frac{1}{i^{\alpha+1}} < +\infty$

for $\alpha > 0$.

$\sum_{i=1}^{\infty} \frac{1}{i^{1.1}} < \infty \qquad \sum_{i=1}^{+\infty} \frac{1}{i^{\frac{1}{2}}} = +\infty$

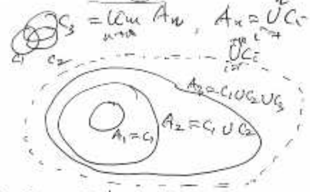
$\sum_{i=1}^{+\infty} \frac{1}{i^2} < \infty$

Infinite Union and Intersection of Sets

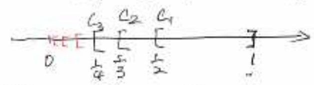
$$\bigcup_{i=1}^{n} C_i = \{x \mid \exists i,\ x \in C_i\}$$

$$\bigcup_{i=1}^{+\infty} C_i = \lim_{n\to+\infty} \bigcup_{i=1}^{n} C_i = \{x \mid \exists i,\ x \in C_i\}$$

$$= \lim_{n\to+\infty} A_n,\quad A_n = \bigcup_{i=1}^{n} C_i$$



e.g. $C_k = [\frac{1}{1+k}, 1]$, ex 1.2.7.



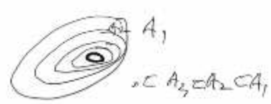
$$\bigcup_{i=1}^{+\infty} C_i = \lim_{n\to+\infty} \bigcup_{i=1}^{n} C_i$$
$$= \lim_{n\to+\infty} C_n$$
$$= \lim_{n\to+\infty} [\frac{1}{1+n}, 1]$$
$$= (0, 1] \quad (\text{not } [0,1])$$

Infinite Int.

$$\bigcap_{i=1}^{+\infty} C_i = \lim_{n\to+\infty} \bigcap_{i=1}^{n} C_i$$
$$= \{x \mid x \in C_i, \text{ for all } i = 1, 2, \ldots\}$$

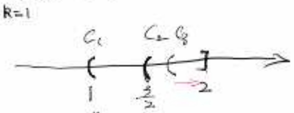
$A_1 = C_1$
$A_2 = C_1 \cap C_2$
$A_3 = C_1 \cap C_2 \cap C_3$

$$\bigcap_{i=1}^{+\infty} C_i = \lim_{n\to+\infty} A_n$$



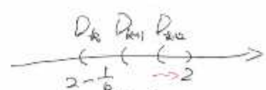
e.g. ex 1.2.9.

$$C_k = (2 - \frac{1}{k}, 2]$$
$$\bigcap_{k=1}^{+\infty} C_k = ?$$



$$A_n = \bigcap_{k=1}^{n} C_k = C_n$$
$$\bigcap_{k=1}^{+\infty} C_k = \lim_{n\to+\infty} A_n = \lim_{n\to+\infty} C_n = \{2\}$$

$$D_k = (2 - \frac{1}{k}, 2)$$



$$\bigcap_{k=1}^{+\infty} D_k = \lim_{k\to+\infty} D_k = \emptyset$$

Def. of the limit of non-decreasing set

$C_1 \subseteq C_2 \subseteq C_3 \subseteq C_4$

$$\lim_{n\to+\infty} C_n = \bigcup_{i=1}^{+\infty} C_i$$

Def of the limit of non-increasing set

$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$

$$\lim_{i\to+\infty} C_i = \bigcap_{i=1}^{+\infty} C_i$$