Derivation of $r(\pi^\tau, d^{p-2})$ on (?, ?)

Derivation using E(E(L(u, d(X))|u)):

$$d^{p-2} = \left(1 - \frac{p-2}{\|X\|^2}\right) X$$

$$R(u, d^{p-2}(X)) = E_X\left(\left\| u - \left(1 - \frac{p-2}{\|X\|^2}\right) X \right\|^2 \Big| u \right)$$

By previous class

$$= p - (p-2)^2 E\left(\frac{1}{\|X\|^2} \Big| u \right)$$

$$r(\pi^\tau, d^{p-2}(X)) = E_u\left[R(u, d^{p-2}(X)) \right]$$

$$= E_u\left(p - (p-2)^2 E_{X|u}\left(\frac{1}{\|X\|^2} \Big| u \right) \right)$$

$$= p - (p-2)^2 \cdot E_X\left(\frac{1}{\|X\|^2} \right)$$

$$= p - (p-2)^2 \cdot \frac{1}{(p-2)(\tau^2+1)}$$

$$= p - \frac{p-2}{\cancel{(p-2)}(\tau^2+1)}$$

$$= \frac{p\tau^2 + 2}{\tau^2+1}$$

Another derivation using E(E(L(u, d(X))|X)

$$d_i = \left(1 - \frac{p-2}{\|X\|^2}\right) X_i$$

$$u_i \mid X_i \sim N\left(\frac{\tau^2}{1+\tau^2} X_i,\ \frac{\tau^2}{1+\tau^2}\right)$$

$$X_i \sim N(0, 1+\tau_i^2)$$

$$E\left((u_i - d_i)^2 \mid X_i\right)$$

$$= \mathrm{Var}(u_i \mid X_i) + \left(E(u_i \mid X_i) - d_i\right)^2$$

$$= \frac{\tau^2}{1+\tau^2} + \left(\frac{\tau^2}{1+\tau^2} X_i - \left(1 - \frac{p-2}{\|X\|^2}\right) X_i\right)^2$$

$$= \frac{\tau^2}{1+\tau^2} + \left(\frac{\tau^2}{1+\tau^2} - 1 + \frac{p-2}{\|X\|^2}\right)^2 X_i^2$$

$$= \frac{\tau^2}{1+\tau^2} + \left(\frac{p-2}{\|X\|^2} - \frac{1}{1+\tau^2}\right)^2 X_i^2$$

$$\sum_{i=1}^{p} E\left((u_i - d_i)^2 \mid X_i\right)$$

$$= \frac{p\tau^2}{1+\tau^2} + \left(\frac{p-2}{\|X\|^2} - \frac{1}{1+\tau^2}\right)^2 \|X\|^2$$

$$= \frac{p\tau^2}{1+\tau^2} + \frac{(p-2)^2}{\|X\|^2} - \frac{2(p-2)}{1+\tau^2} + \frac{\|X\|^2}{(1+\tau^2)^2}$$

$$\cancel{E\left(\frac{\|X\|^2}{1+\tau}\right) =}$$

$$E\,\|X\|^2 = (1+\tau^2)\,p \text{, Since } X_i/\sqrt{1+\tau^2} \sim N(0,1)$$

$$E\left(1/\|X\|^2\right) = 1/\left[(p-2)(1+\tau^2)\right]$$

$$E\left(\sum_{i=1}^{p} E\left((u_i - d_i)^2 \mid X_i\right)\right)$$

$$= \frac{p\tau^2}{1+\tau^2} + (p-2)^2/\left[(p-2)(1+\tau^2)\right]$$

$$- \frac{2(p-2)}{1+\tau^2} + \left((1+\tau^2)\cdot p\right)/(1+\tau^2)^2$$

$$= \frac{p\tau^2}{1+\tau^2} + \frac{p-2}{1+\tau^2} - \frac{2(p-2)}{1+\tau^2} + \frac{p\cancel{(1+\tau^2)}}{1+\tau^2}.$$

$$= \frac{p\tau^2 + p - 2 - 2p + 4 + p}{1+\tau^2} = \frac{p\tau^2 + 2}{1+\tau^2}$$