A Practical Guide to Implementing Regularized Stokeslets for Modelling

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In applications relevant to this tutorial, we are interested in viscous or low Reynolds number (Re) fluids where the inertial effect in the system is negligible. The governing equations for such a problem are

$$\begin{aligned}\mu\Delta\mathbf{u} &= \nabla p - \mathbf{F}, && (1)\\ \nabla\cdot\mathbf{u} &= 0, && (2)\end{aligned}$$

where $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{F}$ is the external force density acting on the fluid, μ is the fluid dynamic viscosity. These equations describe the motion of a steady-state, zero Reynolds number, incompressible fluid and are called the Stokes equations.

The solution to the Stokes equations corresponding to the force $\mathbf{F} = \mathbf{f}\phi_\epsilon(\mathbf{y}-\mathbf{x})$ with the blob function

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2+\epsilon^2)^{7/2}} \tag{3}$$

is

$$\mathbf{u}(\mathbf{y}) = \frac{\mathbf{f}(r^2+2\epsilon^2) + [\mathbf{f}\cdot(\mathbf{y}-\mathbf{x})](\mathbf{y}-\mathbf{x})}{8\pi\mu(r^2+\epsilon^2)^{3/2}}, \tag{4}$$

where $r = |\mathbf{y}-\mathbf{x}|$. In the velocity tensor format, this solution is

$$\mathbf{u}(\mathbf{y}) = \frac{1}{8\pi\mu}A(\mathbf{y},\mathbf{x})\mathbf{f} \tag{5}$$

with

$$A(\mathbf{y},\mathbf{x}) = \begin{bmatrix} a_{11}^\epsilon & a_{12}^\epsilon & a_{13}^\epsilon \\ a_{21}^\epsilon & a_{22}^\epsilon & a_{23}^\epsilon \\ a_{31}^\epsilon & a_{32}^\epsilon & a_{33}^\epsilon \end{bmatrix}. \tag{6}$$

The tensor $A(\mathbf{y},\mathbf{x})$ depends on the location of the force, $\mathbf{x} = (x_1,x_2,x_3)^T$, the location of the observation point, $\mathbf{y} = (y_1,y_2,y_3)^T$, and the parameter ϵ. In detail,

$$\begin{aligned} a_{11}^\epsilon &= \frac{(r^2+2\epsilon^2)+r_1^2}{r_\epsilon^3}; \quad a_{12}^\epsilon = \frac{r_1r_2}{r_\epsilon^3}; \quad a_{13}^\epsilon = \frac{r_1r_3}{r_\epsilon^3};\\ a_{21}^\epsilon &= \frac{r_2r_1}{r_\epsilon^3}; \quad a_{22}^\epsilon = \frac{(r^2+2\epsilon^2)+r_2^2}{r_\epsilon^3}; \quad a_{23}^\epsilon = \frac{r_2r_3}{r_\epsilon^3};\\ a_{31}^\epsilon &= \frac{r_3r_1}{r_\epsilon^3}; \quad a_{32}^\epsilon = \frac{r_3r_2}{r_\epsilon^3}; \quad a_{33}^\epsilon = \frac{(r^2+2\epsilon^2)+r_3^2}{r_\epsilon^3}. \end{aligned} \tag{7}$$

where

$$\begin{aligned} r_1 = y_1 - x_1, \quad r_2 = y_2 - x_2 \quad r_3 = y_3 - x_3, \\ r^2 = r_1^2 + r_2^2 + r_3^2 \quad \text{and} \quad r_\epsilon = \sqrt{r^2+\epsilon^2}. \end{aligned} \tag{8}$$

After computing the velocity of the tracer particle, $\mathbf{v}^0 = \mathbf{v}(\mathbf{y},0)$, we can update the position of the tracer particle. Since velocity $\mathbf{v}(\mathbf{y},t)$ is the derivative of position $\mathbf{y}$ with respect to time t,

$$\frac{d\mathbf{y}}{dt} = \mathbf{v}(\mathbf{y},t), \tag{9}$$

so we can integrate the velocity to get the updated position. The forward Euler method for (9) would yield

$$\mathbf{y}^{i+1} = \mathbf{y}^i + \Delta t\ \mathbf{v}^i, \tag{10}$$

where Δt is the time step for the time discretization.

Exercises

1. Let $\mathbf{f} = (2, 3, -1)^T$ be the force applied at the point $\mathbf{x} = (1, -1, 2)^T$. Let $\epsilon = 0.05$ and $\mu = 1$. We want to find the resulting velocity $\mathbf{u}(\mathbf{y})$ at the observation point $\mathbf{y} = (2, 0, 1)^T$.

   (a) Modify the RegStokes.m function to assemble the matrix in (6).

   (b) Use (5) to find the resulting velocity at the point $\mathbf{y} = (2, 0, 1)^T$ by running the MainForce.m function. You can check your answer by removing the semicolon at the end of the following line:

   ```
   u = 1/(8*pi*mu)*A*f;
   ```

2. Next we will adapt our code to update the position of the tracer particles using Forward Euler with time step $\Delta t = 0.01$.

   (a) Modify the ForwardEuler.m function so that y is updated using Equation (10).

   (b) Run the MainForce.m. Check your solution b removing the semicolons at the end of the following lines:

   ```
   %Update the position with Forward Euler
   y = ForwardEuler(y,u,delta_t);

   %Update the time
   t = i*delta_t;
   ```

   Because $N = 1$, this will only calculate a single time step and end at time $t = 0.01$. Change the value of N to end at time $t = 0.06$.

   (c) Update your code to include a new parameter, tFinal, representing the final time. Redefine N so that it depends on t, tFinal and delta_t.

3. In this exercise, we prescribe the velocity at this point, so $\mathbf{x}$ is the forcing location as well as the location of the observation point and find the force $\mathbf{f}$. Let $\mathbf{x} = (-3, 2, 1)^T$ and the prescribed velocity at $\mathbf{x}$ be $\mathbf{u}(\mathbf{x}) = (1, -.5, -1)^T$. Let $\epsilon = 0.05$ and $\mu = 1$.

   (a) In the MainVelocity.m function, modify the line

   ```
   A= RegStokes(   ,   ,epsilon);
   ```

   to write out the 3×3 tensor $A(\mathbf{x}, \mathbf{x})$ defined in (6).

   (b) Solve the system

   $$\mathbf{u}(\mathbf{x}) = \frac{1}{8\pi\mu} A(\mathbf{x}, \mathbf{x})\mathbf{f}$$

   to find $\mathbf{f}$ by solving the linear system. *Notice: the location of the force and the observation point are the same. This ensures the velocity tensor is a diagonal matrix and therefore invertible.*

   (c) Plug $A(\mathbf{x}, \mathbf{x})$ and $\mathbf{f}$ into Equation (5) to verify that your force results in the desired prescribed velocity.

   (d) ***BONUS*** Now use your ForwardEuler.m function so that that the position of forcing point is updated as it is carried along by its own velocity.

   (e) *** BONUS *** Let $\mathbf{y} = (1, -2, 3)^T$ be an observation point. Use the function you created in exercise 2 to find the velocity at that point $\mathbf{u}(\mathbf{y})$ resulting from the force you found in part (a) applied at $\mathbf{x}$.

Solutions

1.

$$\mathbf{u}(\mathbf{y}) \quad = \quad \frac{1}{8\pi(3+\epsilon^2)^{3/2}}(12+4\epsilon^2, 15+6\epsilon^2, -9-2\epsilon^2)^T \approx (0.0918, 0.1148, -0.0689)^T,$$

where $\mathbf{y} = (2,0,1)$, with $\mathbf{f} = (2,3,-1)$ at $\mathbf{x} = (1,-1,2)$.

2. (b) $\mathbf{u}(\mathbf{y}) \approx (0.0918, 0.1148, -0.0689)^T$ at $\mathbf{y} = (2,0,1)^T$, with $\mathbf{f}(\mathbf{x}) = (2,3,-1)^T$ at $\mathbf{x} = (1,-1,2)^T$. (d) Below are the approximations from Forward Euler.

t	$\mathbf{y}_1$
0	$(2,0,1)^T$
0.01	$(2.000918498864207,\ 0.001148314774860,\ 0.999311317046446)^T$
0.02	$(2.001836190618279,\ 0.002295716638611,\ 0.998623335402053)^T$
0.03	$(2.002753077333999,\ 0.003442207810047,\ 0.997936053142048)^T$
0.04	$(2.003669161074396,\ 0.004587790498958,\ 0.997249468350167)^T$
0.05	$(2.004584443893788,\ 0.005732466906183,\ 0.996563579118606)^T$

3. (a)

$$A(\mathbf{x},\mathbf{x}) = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

(b) $\mathbf{f}(\mathbf{x}) \approx (0.6283, -0.3142, -0.6283)^T$ at $\mathbf{x} = (-3,2,1)^T$.

(e) $\mathbf{u}(\mathbf{y}) \approx (0.0060, -0.0039, -0.0032)^T$ at $\mathbf{y} = (1,-2,3)^T$.