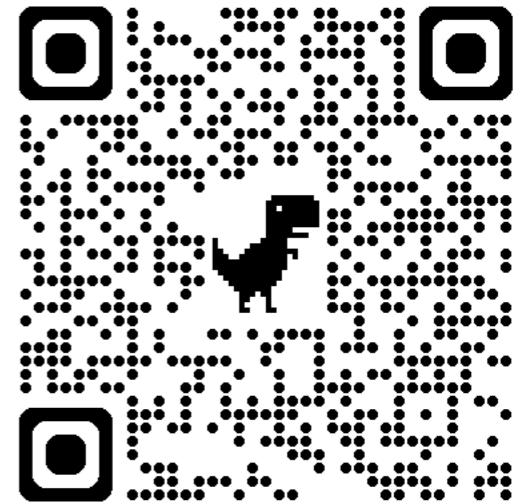


Scan this link to log in to MATLAB

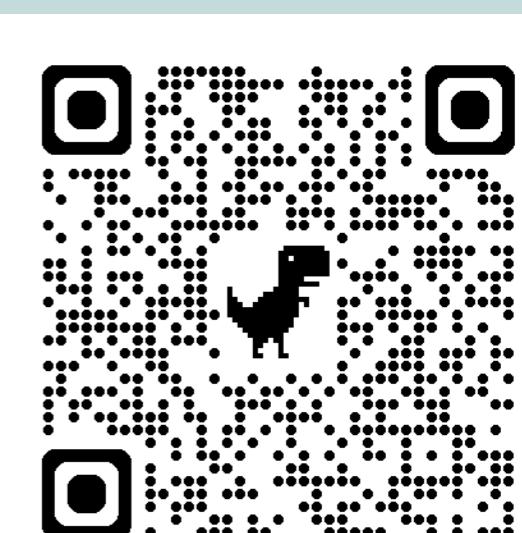
<https://www.mathworks.com/products/matlab.html>



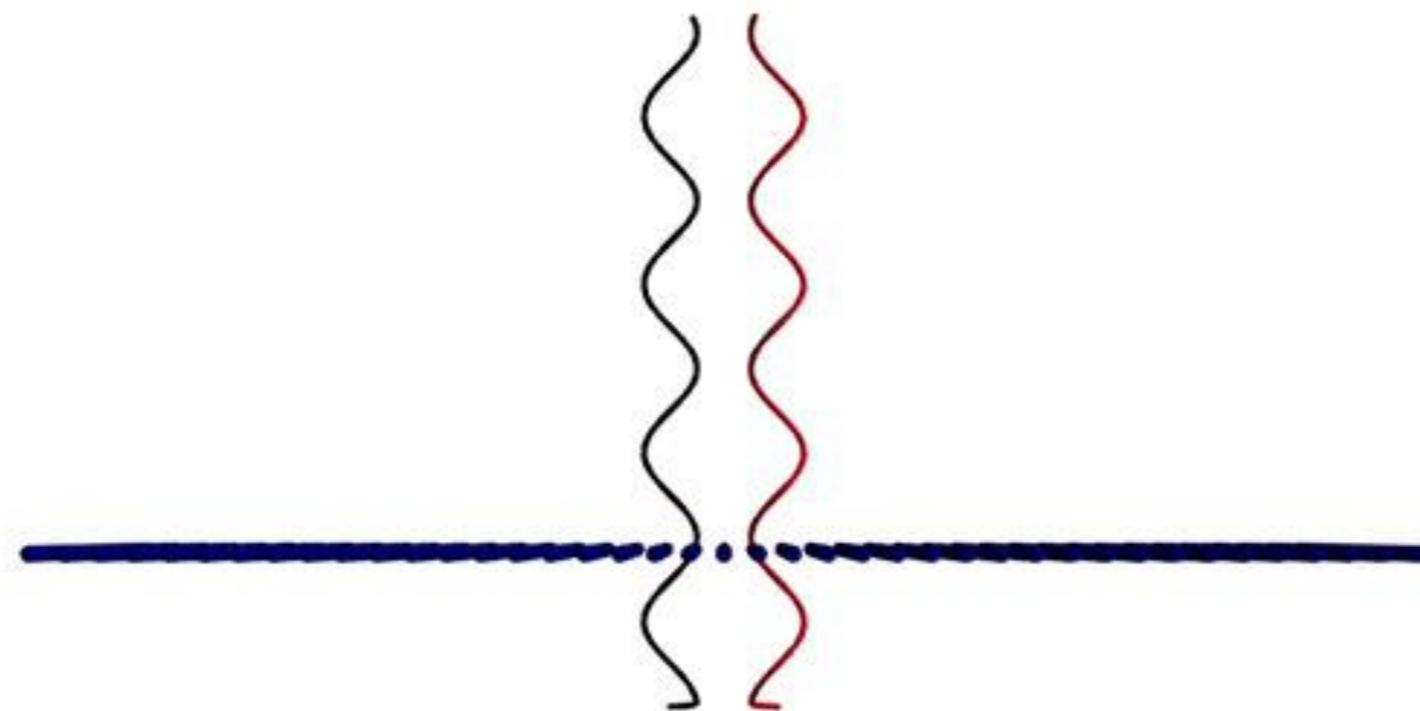
A Practical Guide to Implementing Regularized Stokeslets for Modelling

Scan this link to access the workshop
materials

https://github.com/longhuazhao/MRS_IntroductoryGuide/tree/main



A Practical Guide to Implementing Regularized Stokeslets for Modelling



Amy Buchmann
University of San
Diego

Eva Strawbridge
James Madison
University

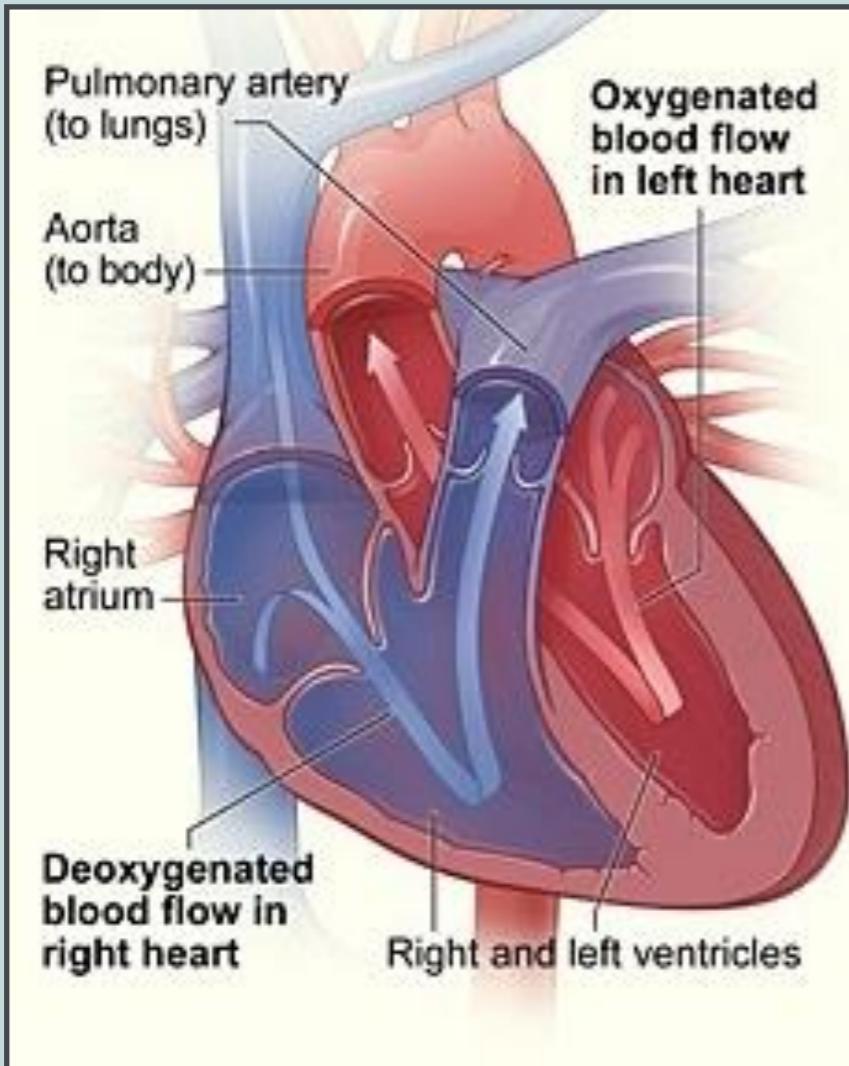
Longhua Zhao
Case Western
Reserve University

Goals

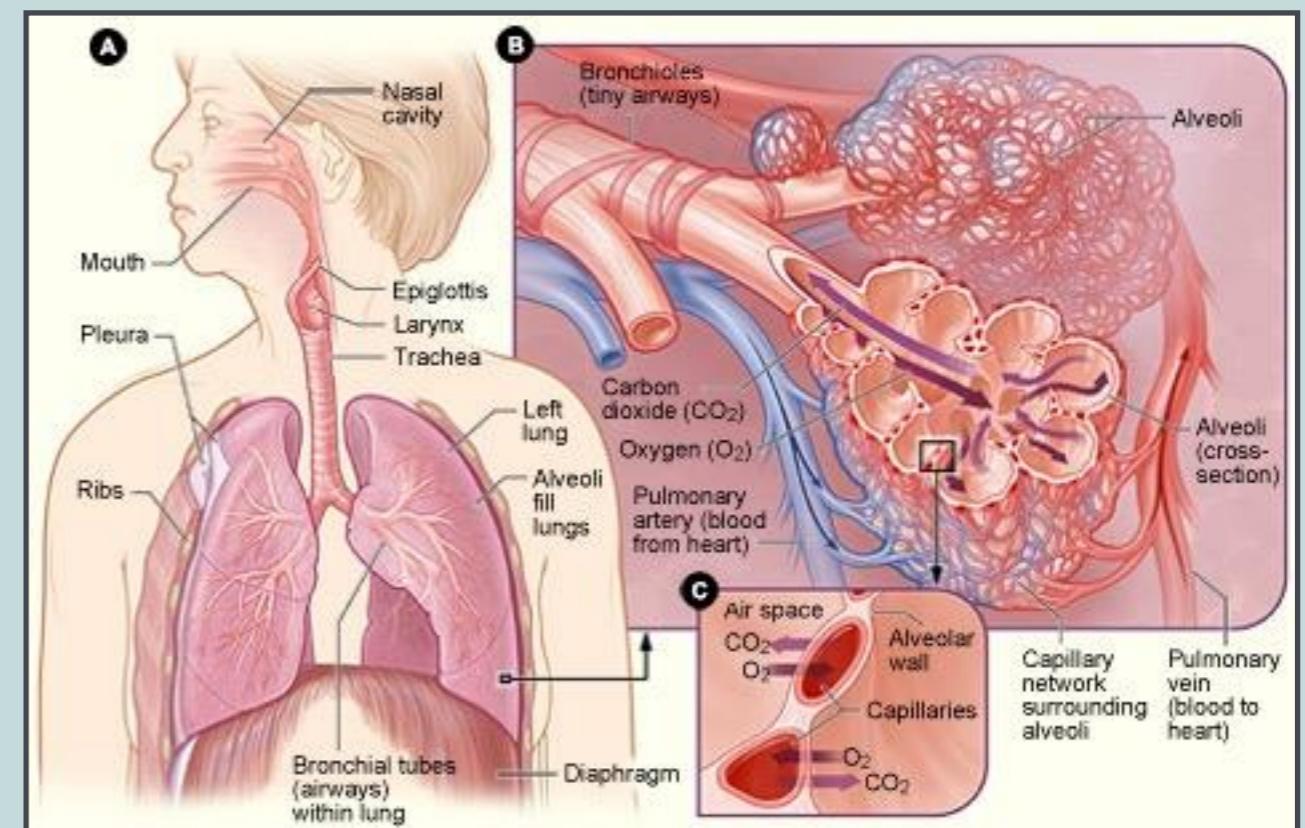
- ♦ To demonstrate how mathematics can be used to enhance our understanding of the world around us.
- ♦ In particular, concepts from undergraduate math courses can be used to study microscale biological fluid dynamics
 - Single Variable Calculus
 - Multivariable Calculus
 - Differential Equations
 - Linear Algebra

Biological Fluid Dynamics

Blood flow in the heart



Airflow in the lungs



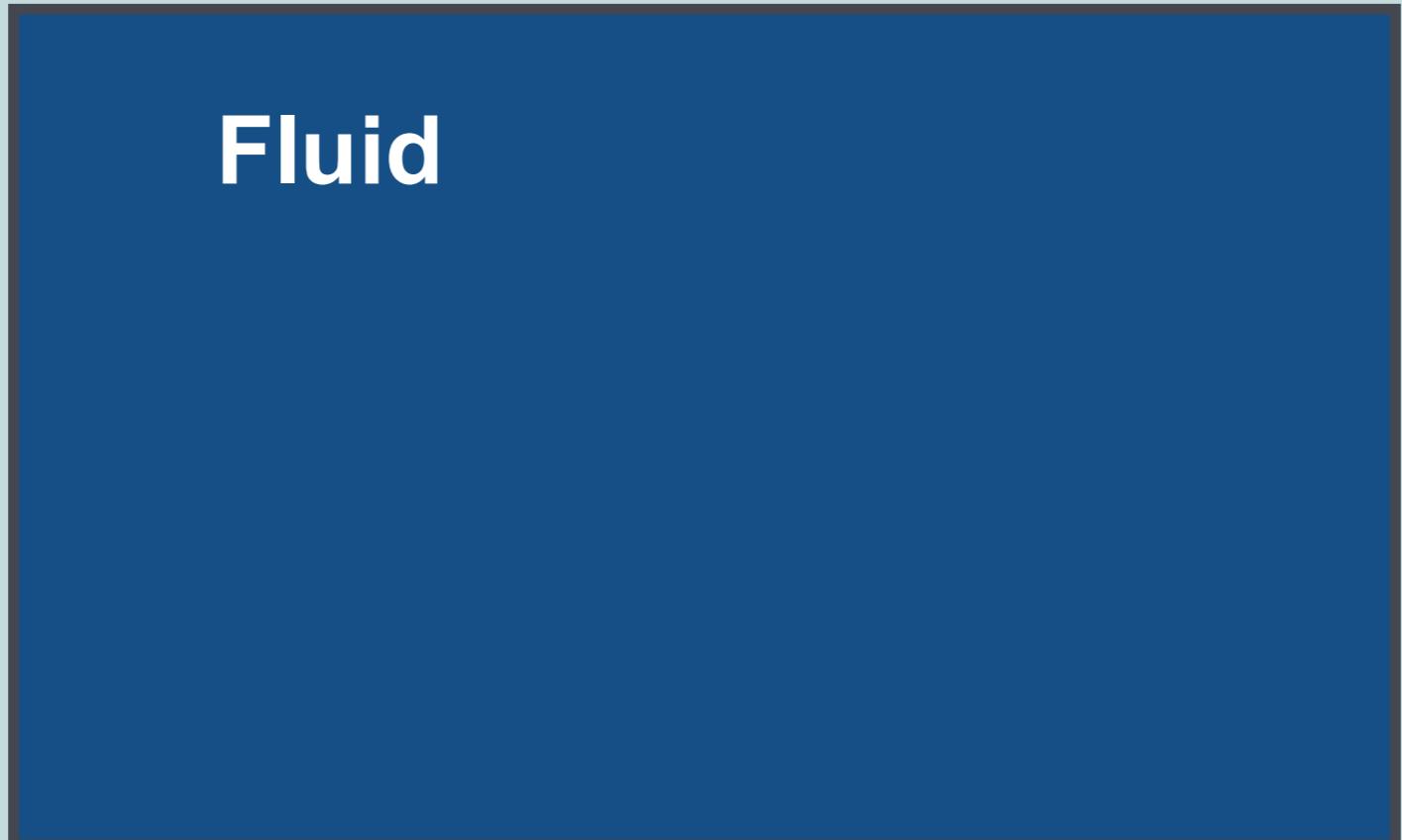
Microscale Biological Fluid Dynamics

- ♦ Bacteria ($\sim 25 \mu\text{m}$)
 - How do helical flagella mix and pump fluid?
- ♦ Cilia ($\sim 10 \mu\text{m}$)
 - How do cilia coordinate their movement to form a metachronal wave?



Fluid Structure Interactions

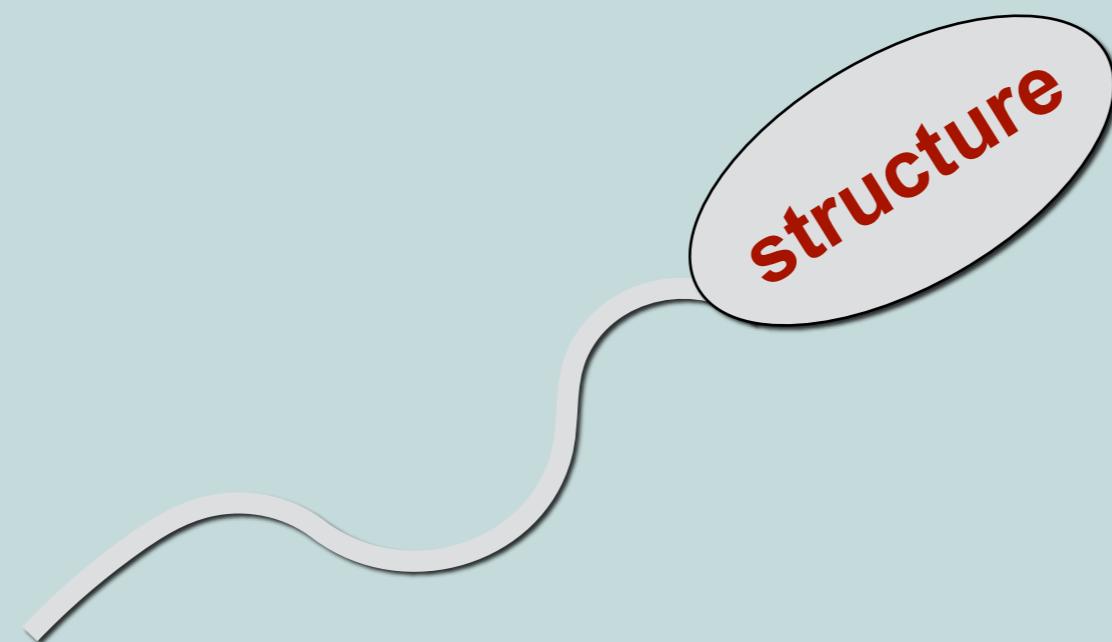
- ♦ We need a mathematical framework to describe



Fluid

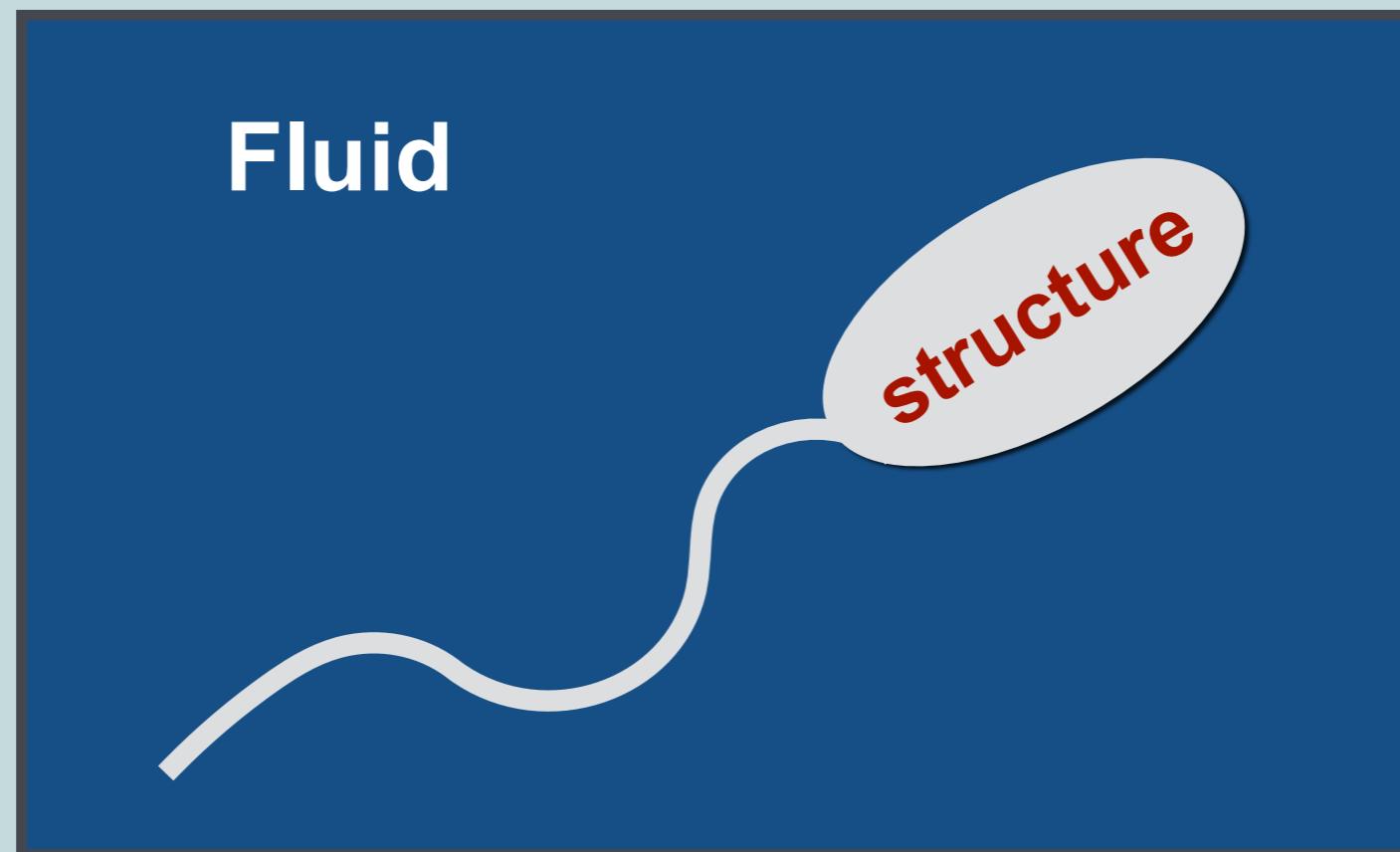
Fluid Structure Interactions

- ◆ We need a mathematical framework to describe



Fluid Structure Interactions

- ◆ We need a mathematical framework to describe



Calculus Concepts: Derivatives!

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- ◆ Derivatives: rates of change (single variable)

$$\frac{d}{dx} [\sin(x)] = \cos(x)$$

Calculus Concepts: Derivatives!

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$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

Calculus Concepts: Derivatives!

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$$\frac{\partial}{\partial y} [2x^2y^3] = 6x^2y^2$$

Calculus Concepts: Derivatives!

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- ◆ Partial Derivatives: rates of change (multivariable)

$$\frac{\partial}{\partial y} [2x^2y^3] = 6x^2y^2$$

- ◆ Partial Differential Equations: equations with partial derivatives

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Fluid: Navier Stokes Equations

- ◆ The motion of liquids is well described by the Navier Stokes Equations

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u}$$

“Force Balance”

$$\nabla \cdot \mathbf{u} = 0$$

“Conservation of Mass”

ρ : fluid density

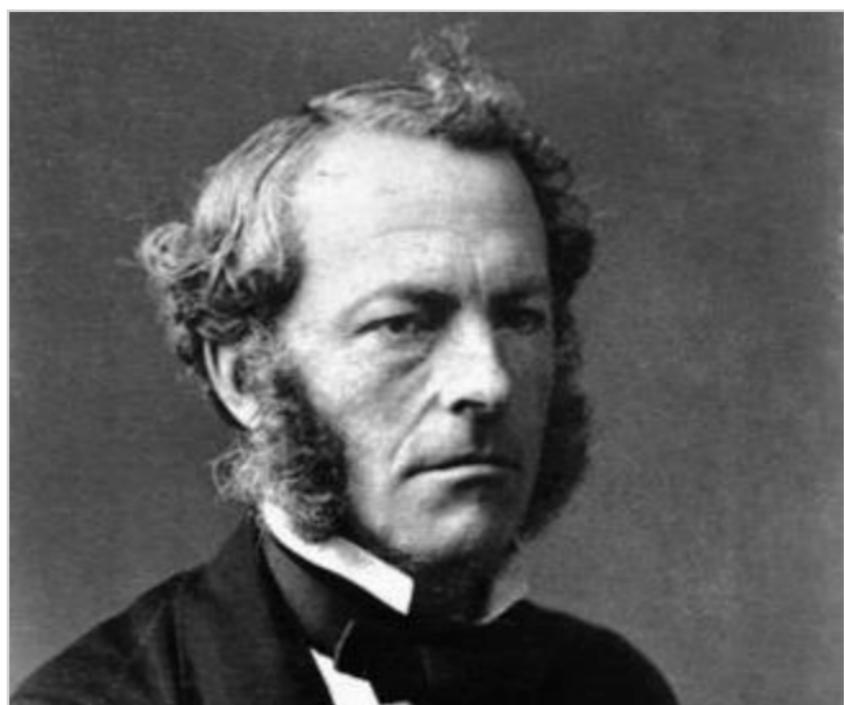
\mathbf{u} : fluid velocity

p : fluid pressure

μ : fluid viscosity

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Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Image: Sir George Gabriel Stokes (13 August 1819–1 February 1903). Public Domain

This problem is:

Unsolved

Rules:

[Rules for the Millennium Prizes](#)

Related Documents:

[Official Problem Description](#)

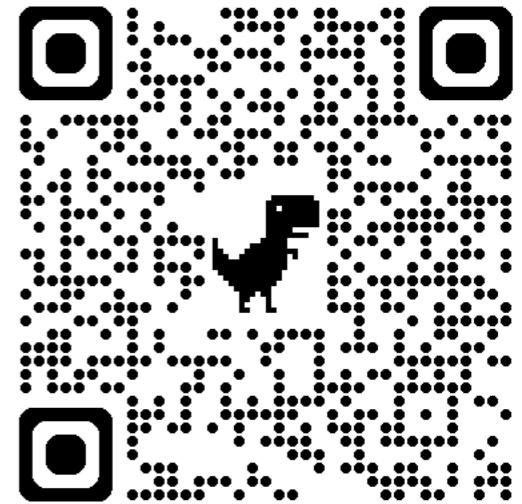
Related Links:

[Lecture by Luis Cafarelli](#)

MATLAB Overview

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Open MATLABOverview.pdf

Fluid: Navier Stokes Equations

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$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u}$$

“Force Balance”

$$\nabla \cdot \mathbf{u} = 0$$

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ρ : fluid density

\mathbf{u} : fluid velocity

p : fluid pressure

μ : fluid viscosity

Non-dimensionalization

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Left hand side

Reynolds
Number

$$\boxed{\frac{\rho L U}{\mu}} \left(\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} \right)$$

Right hand side

$$= \left(- \frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \right)$$

Non-dimensionalization

$$\begin{aligned}\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p + \mu \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

Introducing the non-dimensional variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{L} \quad \mathbf{u}^* = \frac{\mathbf{u}}{U} \quad p^* = \frac{pL}{\mu U} \quad t^* = t \frac{U}{L}$$

this becomes

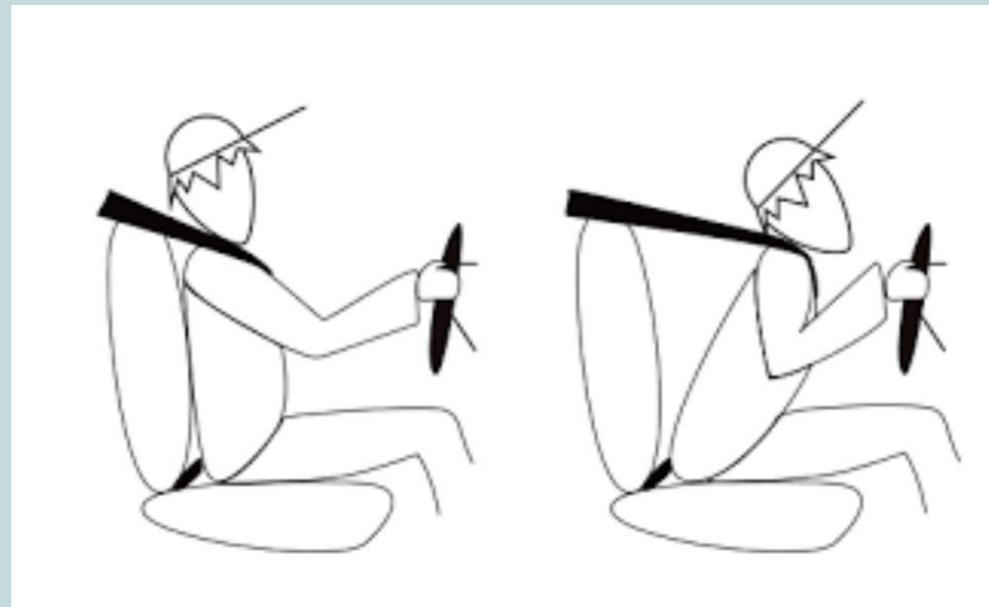
$$\begin{aligned}Re \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* \right) &= -\nabla p^* + \nabla^2 \mathbf{u}^* \\ \nabla \cdot \mathbf{u}^* &= 0\end{aligned}$$

Reynolds Number

$$Re = \frac{\rho UL}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

Reynolds Number

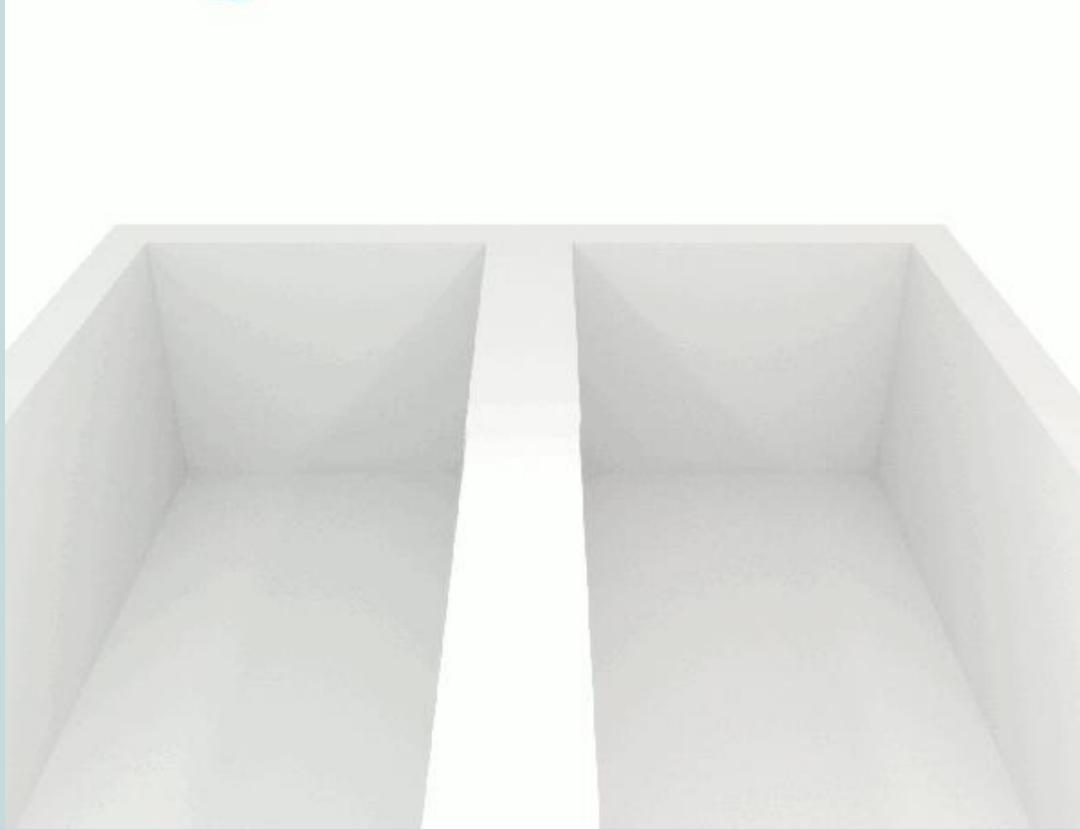
$$Re = \frac{\rho UL}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



INERTIA: a property of matter by which it ***continues in its existing state*** of rest or uniform motion in a straight line, unless that state is changed by an external force.

Reynolds Number

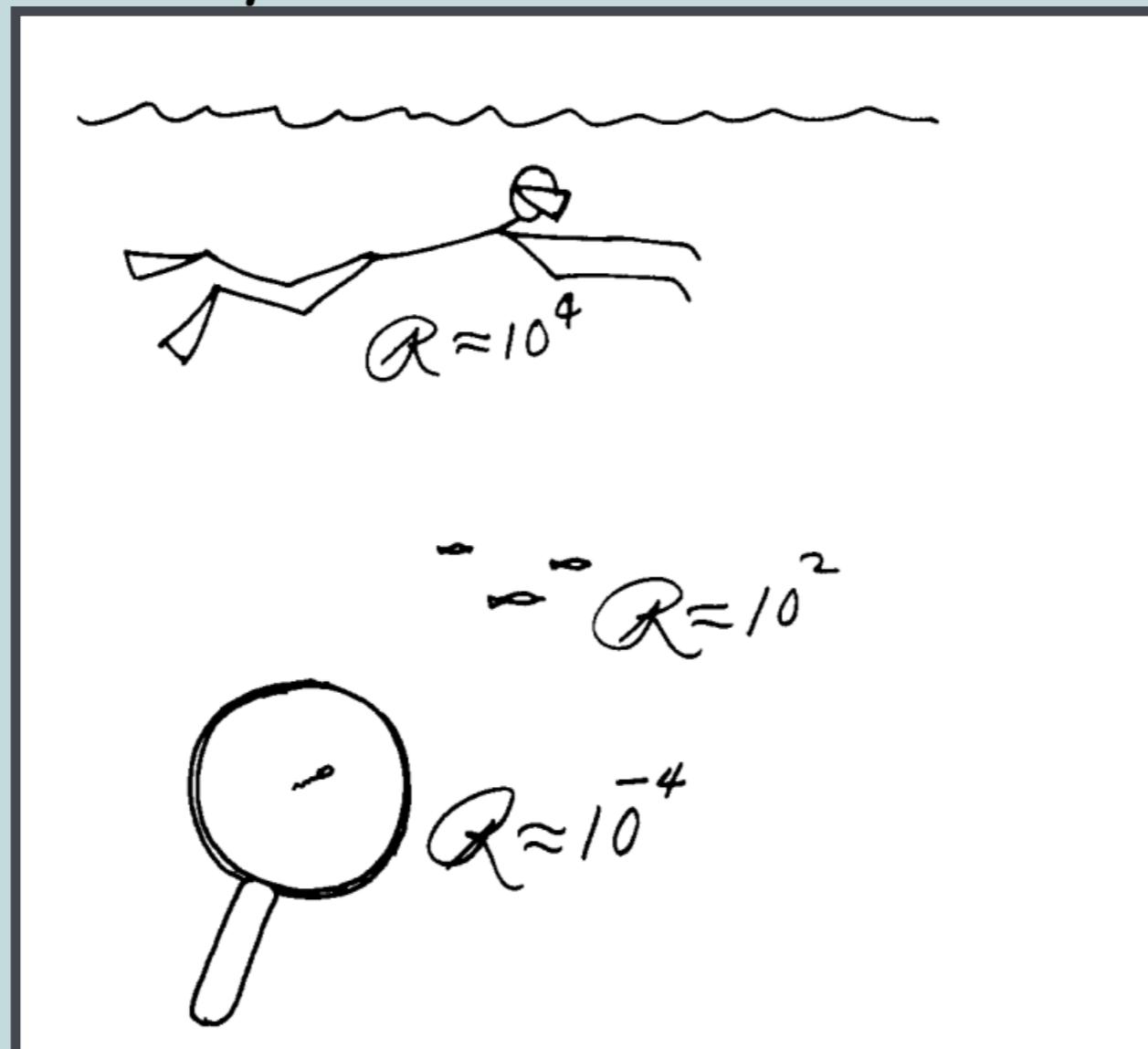
$$Re = \frac{\rho UL}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



VISCOSITY: The viscosity of a fluid is a measure of its **resistance to deformation** at a given rate.

Reynolds Number

$$Re = \frac{\rho UL}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



Intermediate Reynolds Number



G.I. Taylor

Reynolds Number

$$Re = \frac{\rho UL}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

- ♦ High Reynolds Number - Inertial Forces Dominate
- ♦ Low Reynolds Number - Viscous Forces Dominate

Low Reynolds Number



G.I. Taylor

Non-dimensionalization

$$\begin{aligned}\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p + \mu \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

Introducing the non-dimensional variables

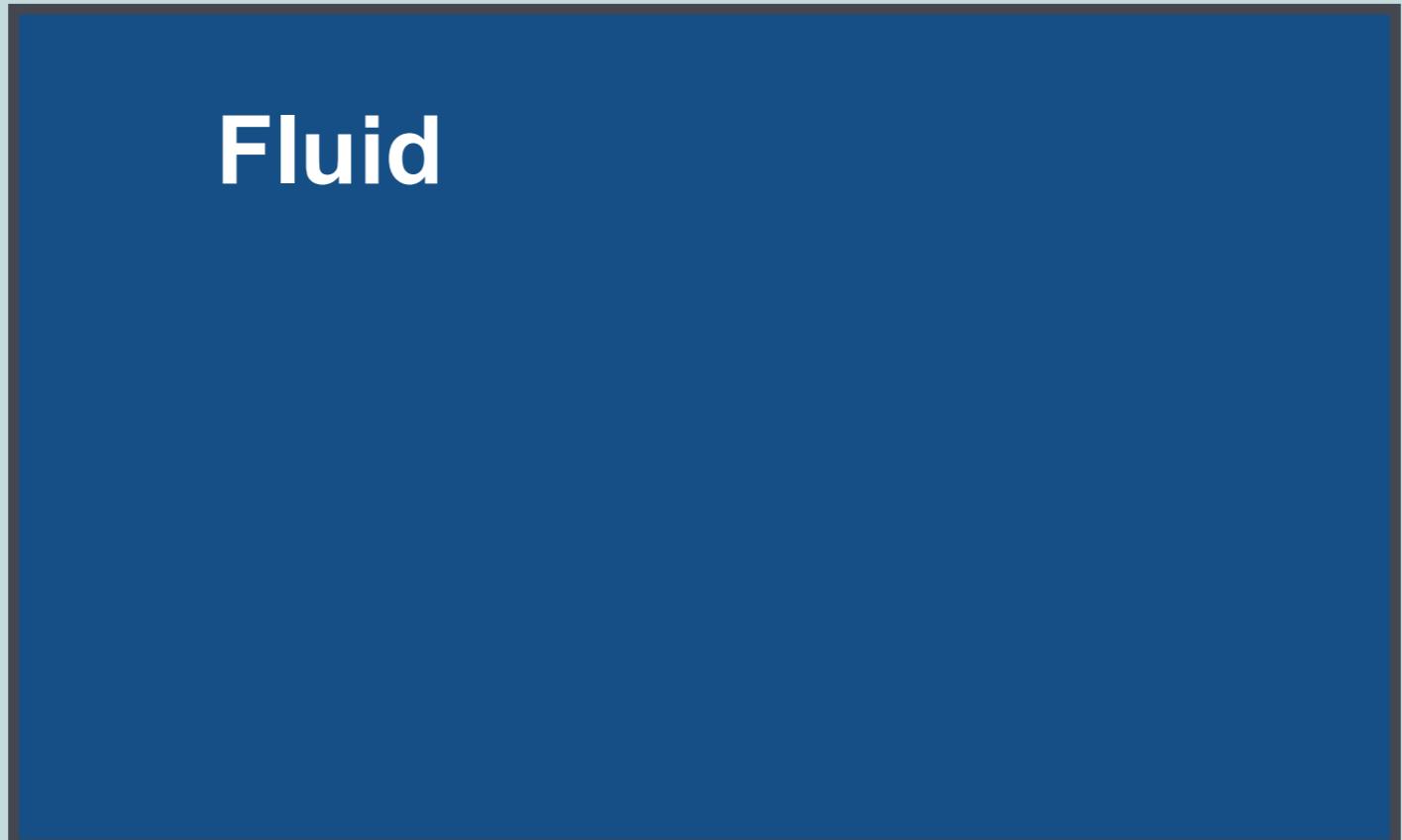
$$\mathbf{x}^* = \frac{\mathbf{x}}{L} \quad \mathbf{u}^* = \frac{\mathbf{u}}{U} \quad p^* = \frac{pL}{\mu U} \quad t^* = t \frac{U}{L}$$

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$$\begin{aligned}Re \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* \right) &= -\nabla p^* + \nabla^2 \mathbf{u}^* \\ \nabla \cdot \mathbf{u}^* &= 0\end{aligned}$$

Fluid Structure Interactions

- ♦ We need a mathematical framework to describe

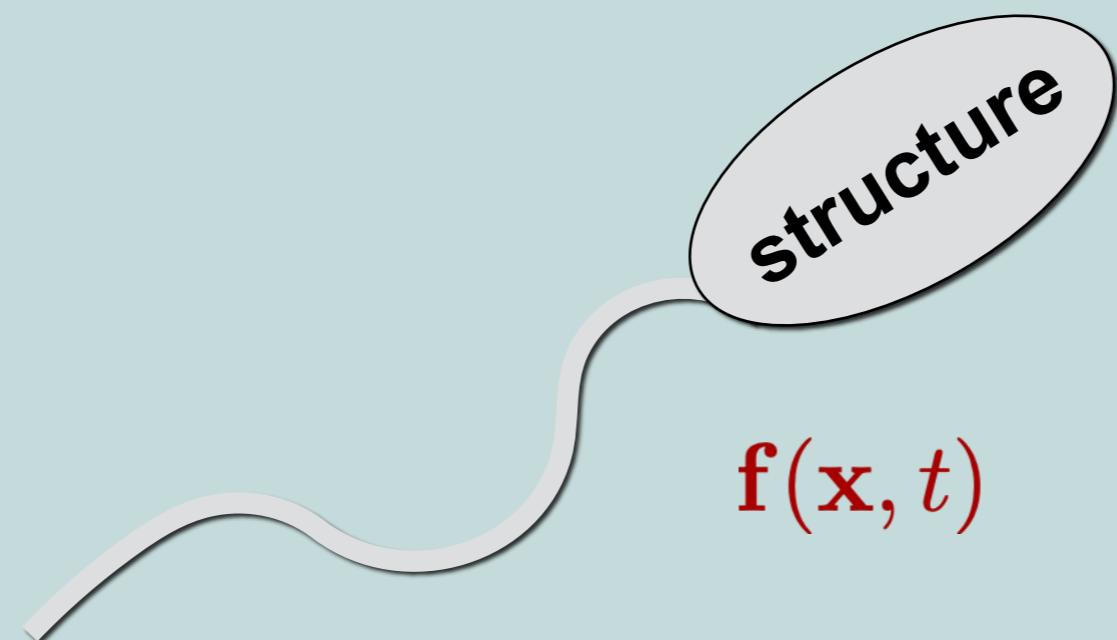


Fluid

$$\begin{aligned}-\nabla p + \mu \nabla^2 \mathbf{u} &= 0 \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

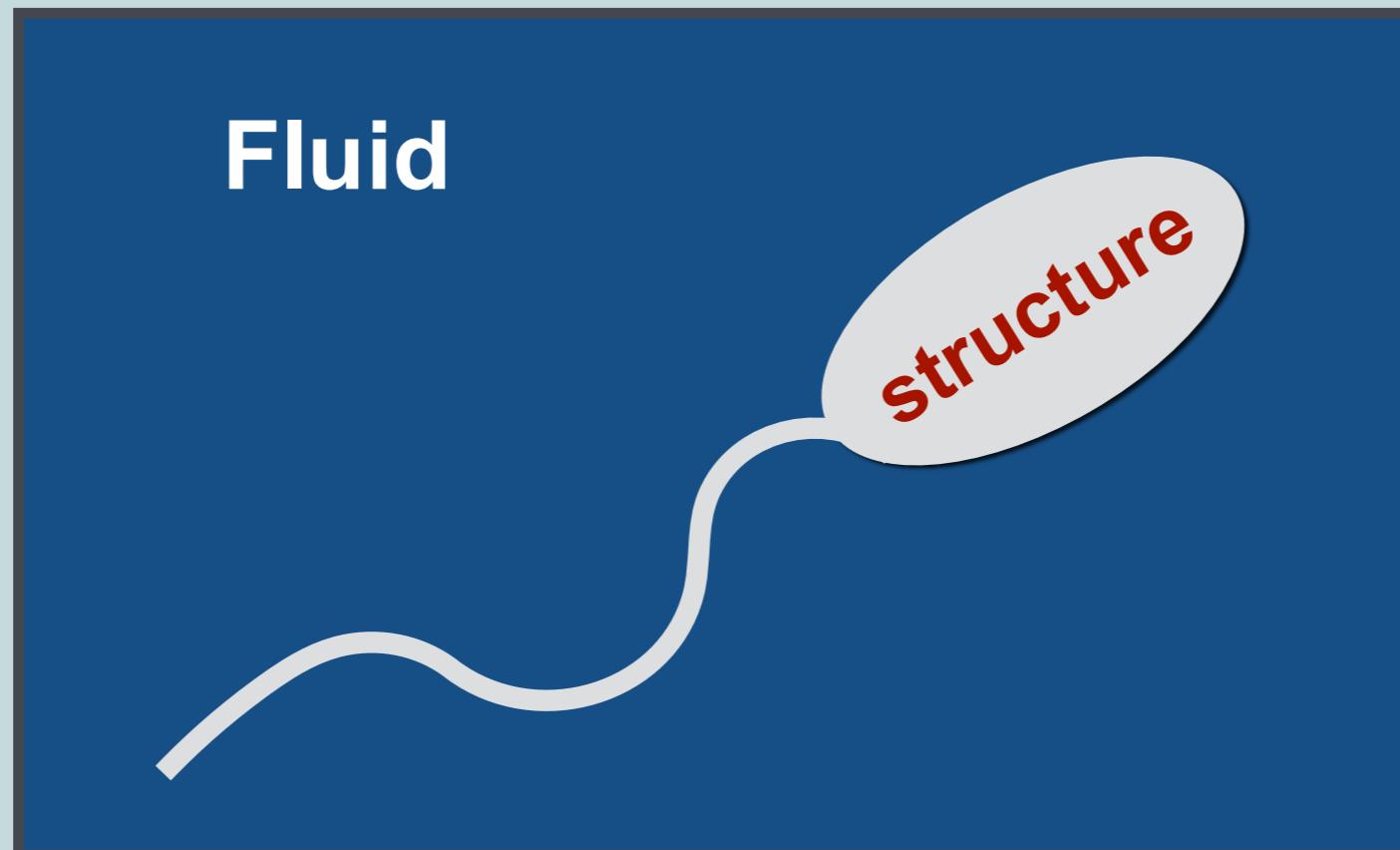
Fluid Structure Interactions

- ◆ We need a mathematical framework to describe



Fluid Structure Interactions

- ♦ We need a mathematical framework to describe



$$\begin{aligned}-\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{x}, t) &= 0 \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

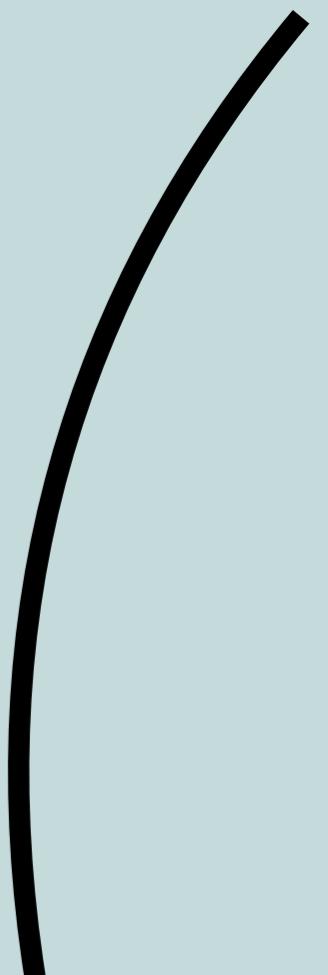
A force at a point?

- ◆ Dirac delta distribution

$$\int_{-\infty}^{\infty} f(y) \delta(y - x_0) dy = f(x_0)$$

- ◆ Fundamental solutions

$$\begin{aligned}\mu \nabla^2 \mathbf{u} + \mathbf{f}_0 \delta(\mathbf{x} - \mathbf{x}_0) &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$



A force at a point?

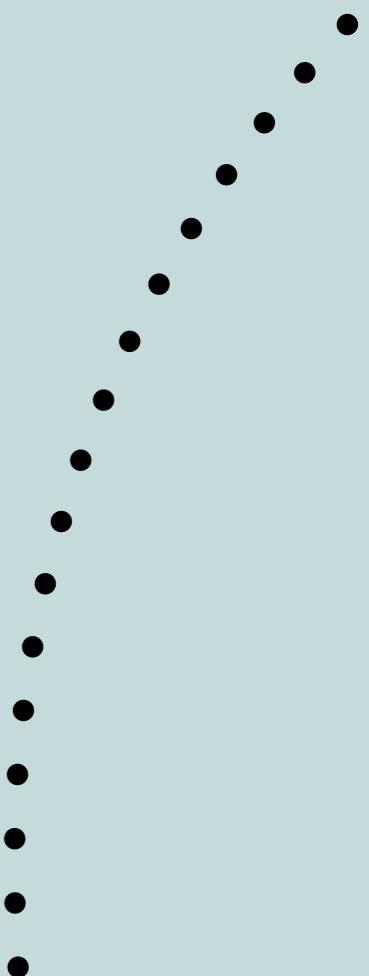
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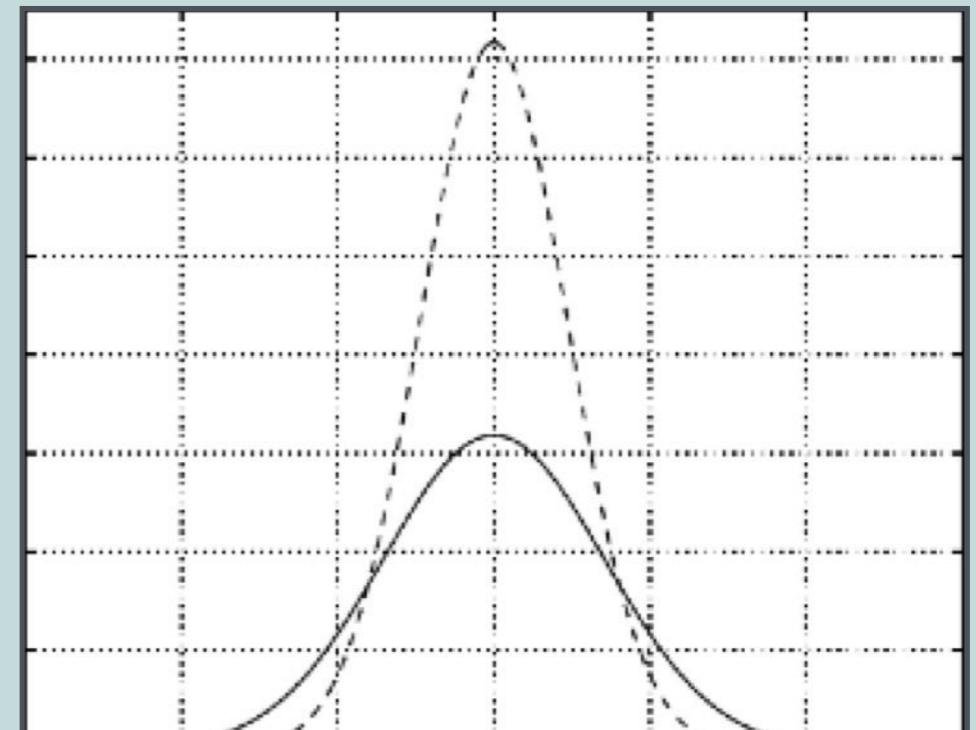
$$\nabla \cdot \mathbf{u} = 0$$



Method of regularized Stokeslets

Big Idea: Develop a regularized version of the fundamental solutions by designing smooth approximations of the Dirac delta distribution

$$\begin{aligned}\mu \nabla^2 \mathbf{u} + \mathbf{f}_0 \phi_\epsilon(\hat{\mathbf{x}}) &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

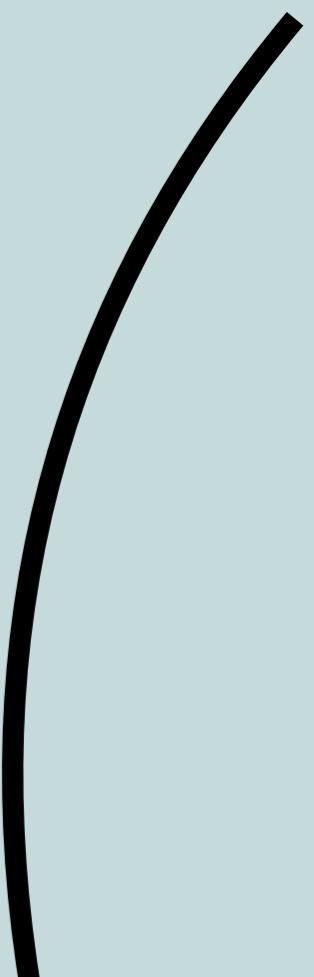


- $\int \phi_\epsilon(\mathbf{x}) d\mathbf{x} = 1$
- $\phi_\epsilon(\mathbf{x}) = \phi_\epsilon(|\mathbf{x}|)$
- concentrated near $\mathbf{x} = \mathbf{0}$

Method of regularized Stokeslets

$$\mu \Delta \mathbf{u} + \mathbf{F} = \nabla p$$

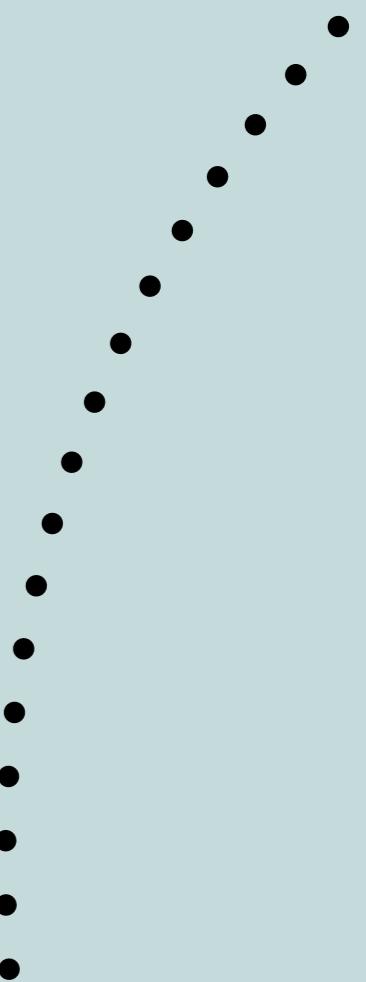
$$\nabla \cdot \mathbf{u} = 0$$



Method of regularized Stokeslets

$$\mu \Delta \mathbf{u} + \mathbf{F} = \nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$



Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$



Method of regularized Stokeslets

$$\begin{aligned}\mu \Delta \mathbf{u} + \mathbf{F} &= -\nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2 + \epsilon^2)^{7/2}}$$



Method of regularized Stokeslets

$$\begin{aligned}\mu \Delta \mathbf{u} + \mathbf{F} &= -\nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2 + \epsilon^2)^{7/2}}$$

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^N \left[\frac{\mathbf{f}_k(r_k^2 + 2\epsilon^2) + (\mathbf{f}_k \cdot (\mathbf{x} - \mathbf{x}_k))(\mathbf{x} - \mathbf{x}_k)}{8\pi\mu(r_k^2 + \epsilon^2)^{3/2}} \right]$$

Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= -\nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

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$$U = AF$$



Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$U = AF$$

We will code up
an example of
this today



Updating particle position with a prescribed force



X

Forcing point location

Updating particle position with a prescribed force

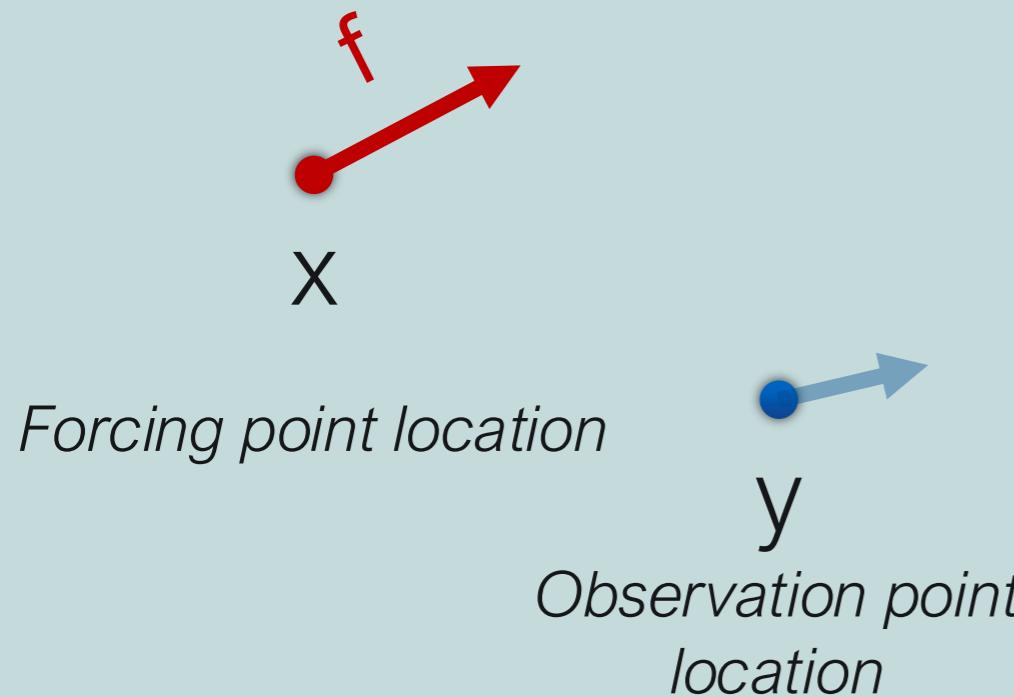
1. Set the force \mathbf{f} at \mathbf{x}



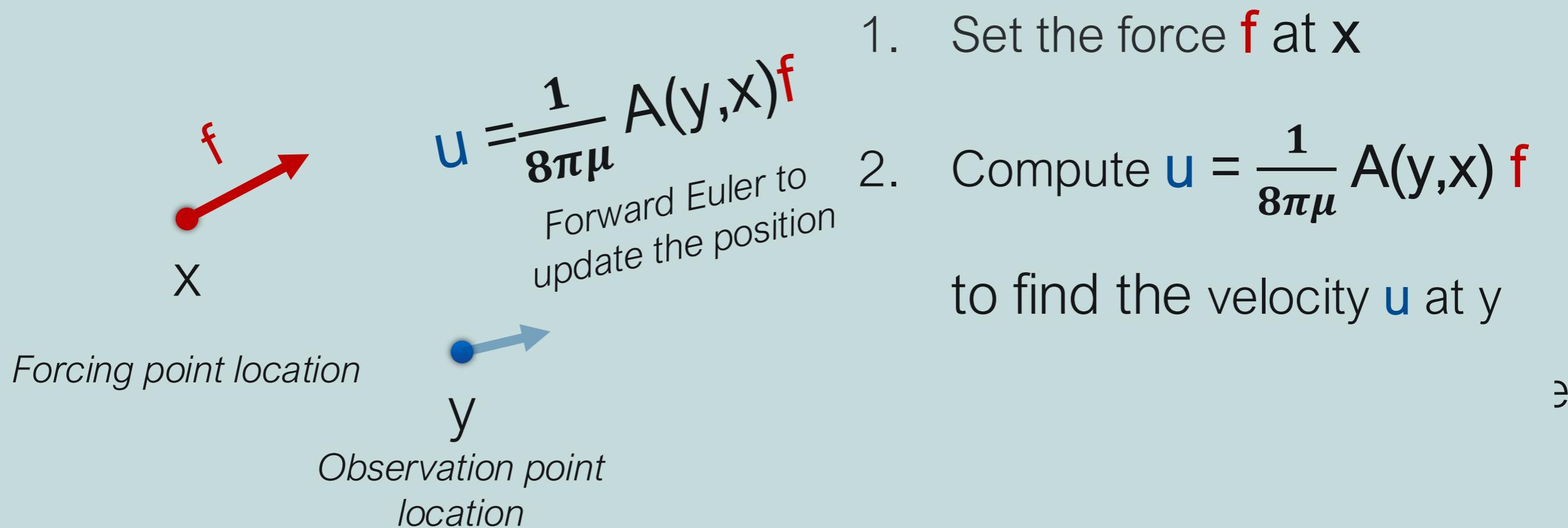
Forcing point location

Updating particle position with a prescribed force

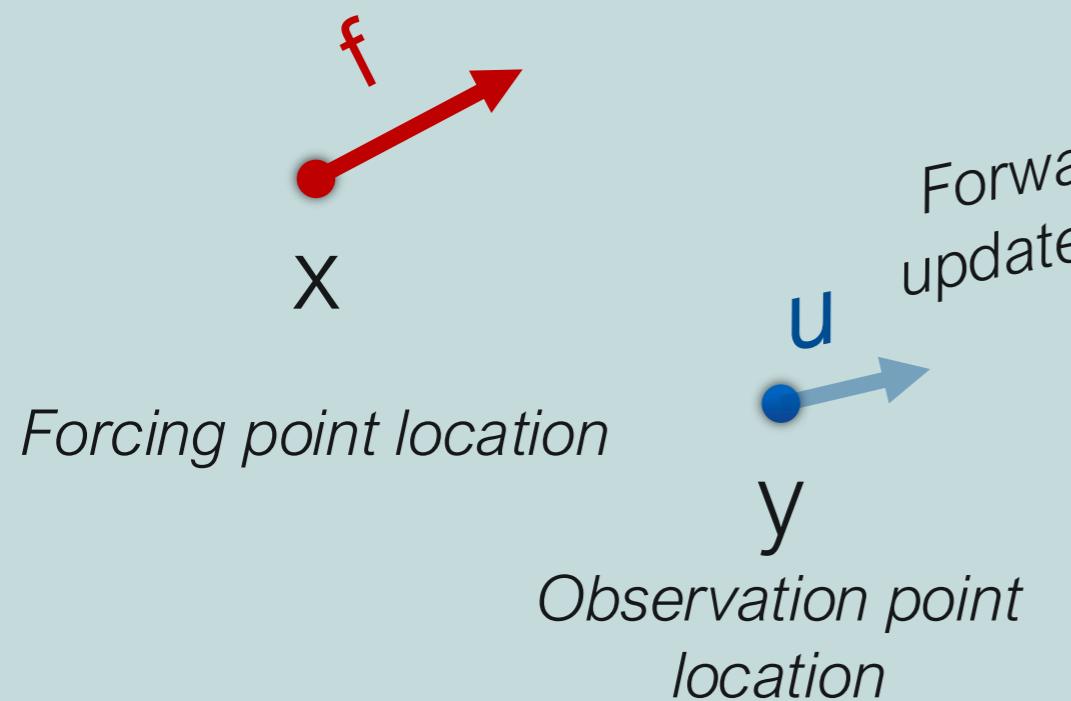
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Updating particle position with a prescribed force

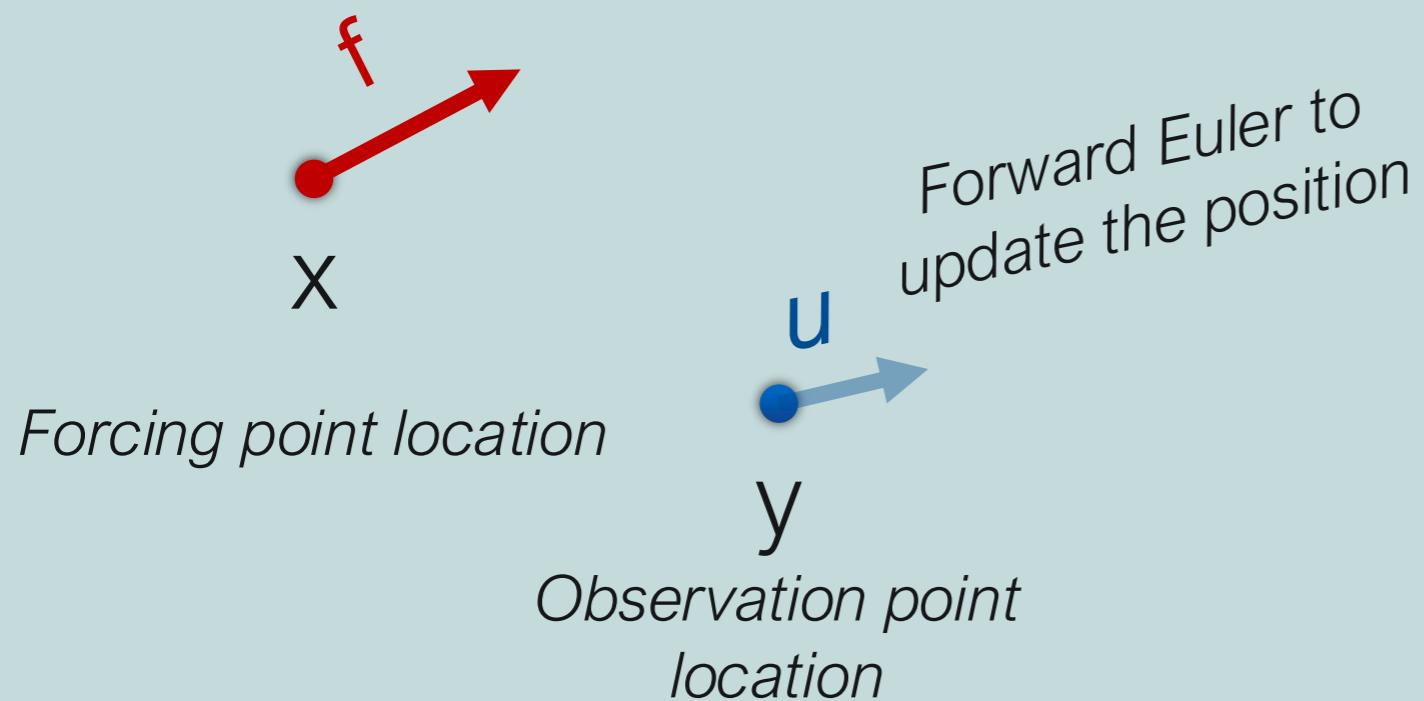


Updating particle position with a prescribed force

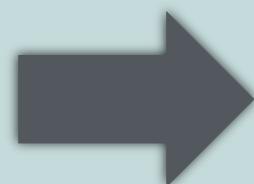


1. Set the force f at x
2. Compute $u = \frac{1}{8\pi\mu} A(y,x) f$ to find the velocity u at y
3. Use Forward Euler to update the position of y

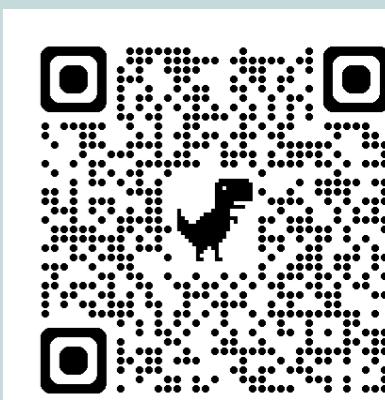
Updating particle position with a prescribed force



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Open brief_MRS.pdf
Use MATLAB to do Exercises 1 and 2



Prescribing the Velocity



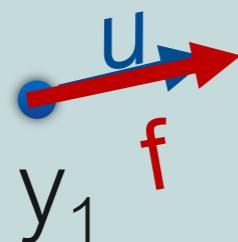
y_1

*Prescribed velocity point
location*

Prescribing the Velocity

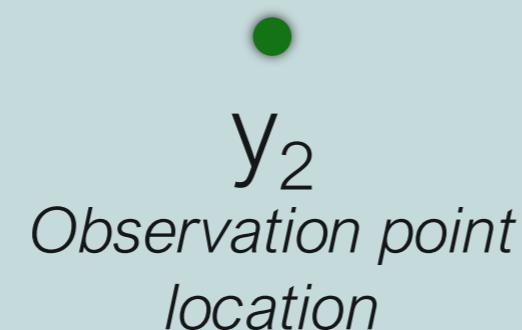
1. Prescribe the velocity $\mathbf{u}(y_1)$

at y_1



*Prescribed velocity point
location*

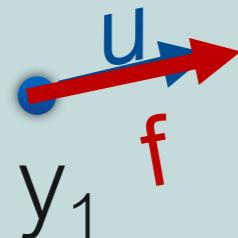
$y_1=x$



Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$

at y_1



Prescribed velocity point
location

2. Solve $\frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \mathbf{f} = \mathbf{u}$ for \mathbf{f}

using $\mathbf{f} = \frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \backslash \mathbf{u}$

$y_1 = \mathbf{x}$

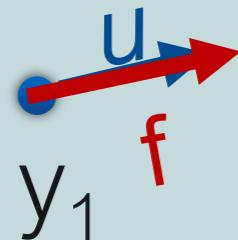
•
 y_2
Observation point
location

Note that $y_1 = \mathbf{x}$ because
the observation point is
the forcing point when
prescribing the velocity

Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$

at y_1



*Prescribed velocity point
location*

2. Solve $\frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \mathbf{f} = \mathbf{u}$ for \mathbf{f}

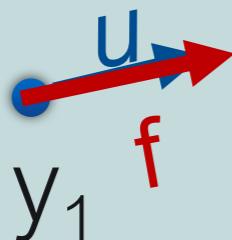
using $\mathbf{f} = \frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \backslash \mathbf{u}$

$y_1 = \mathbf{x}$

●
 y_2
*Observation point
location*

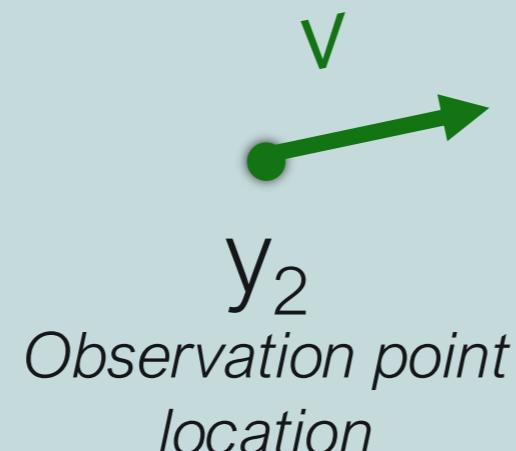
Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$ at y_1



Prescribed velocity point
location

$$y_1 = \mathbf{x}$$

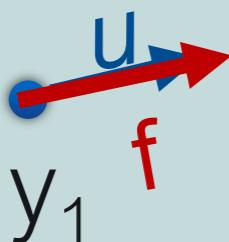


Observation point
location

2. Solve $\frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \mathbf{f} = \mathbf{u}$ for \mathbf{f} using $\mathbf{f} = \frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \setminus \mathbf{u}$
3. Compute $\mathbf{v} = \frac{1}{8\pi\mu} \mathbf{A}(\mathbf{y}, \mathbf{x}) \mathbf{f}$ to find the velocity \mathbf{v} at y_2

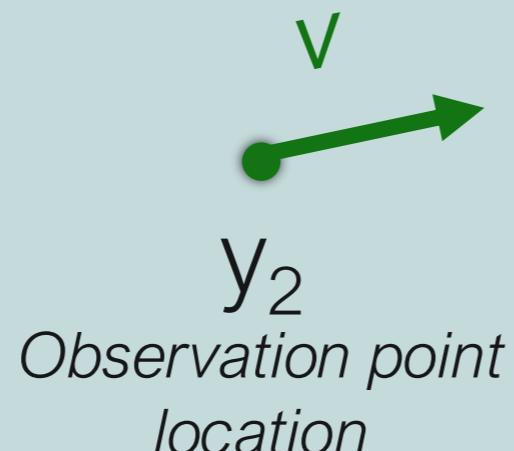
Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$ at y_1



Prescribed velocity point
location

$$y_1 = \mathbf{x}$$

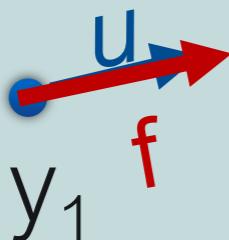


Observation point
location

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3. Compute $\mathbf{v} = \frac{1}{8\pi\mu} \mathbf{A}(\mathbf{y}, \mathbf{x}) \mathbf{f}$ to find the velocity \mathbf{v} at y_2
4. Use Forward Euler to update the position of y_2

Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$ at y_1



Prescribed velocity point
location

$$y_1 = x$$



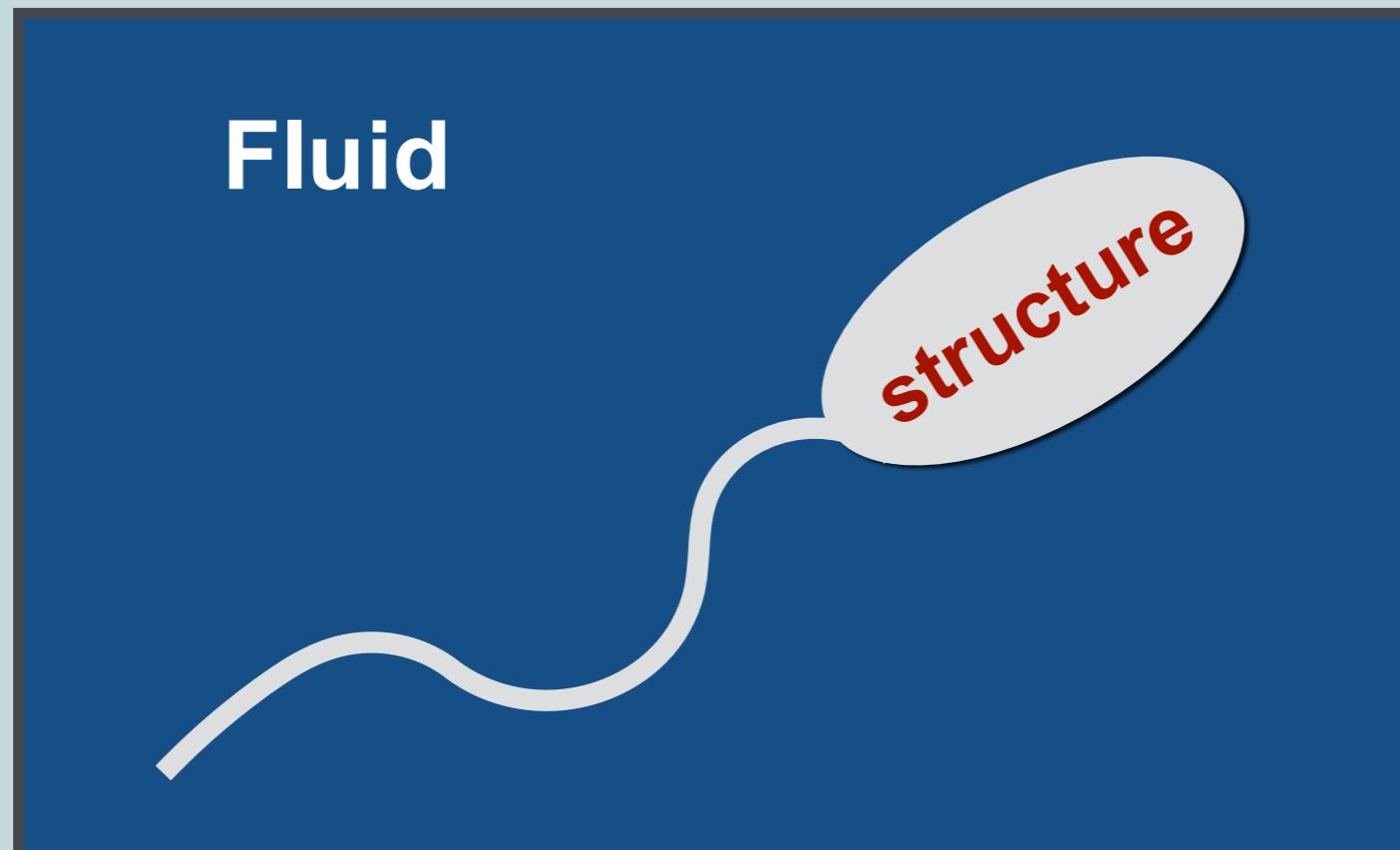
Observation point
location

Use MATLAB to do Exercise 3

2. Solve $\frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \mathbf{f} = \mathbf{u}$ for \mathbf{f} using $\mathbf{f} = \frac{1}{8\pi\mu} \mathbf{A}(\mathbf{x}, \mathbf{x}) \setminus \mathbf{u}$
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Fluid Structure Interactions

- ♦ We need a mathematical framework to describe

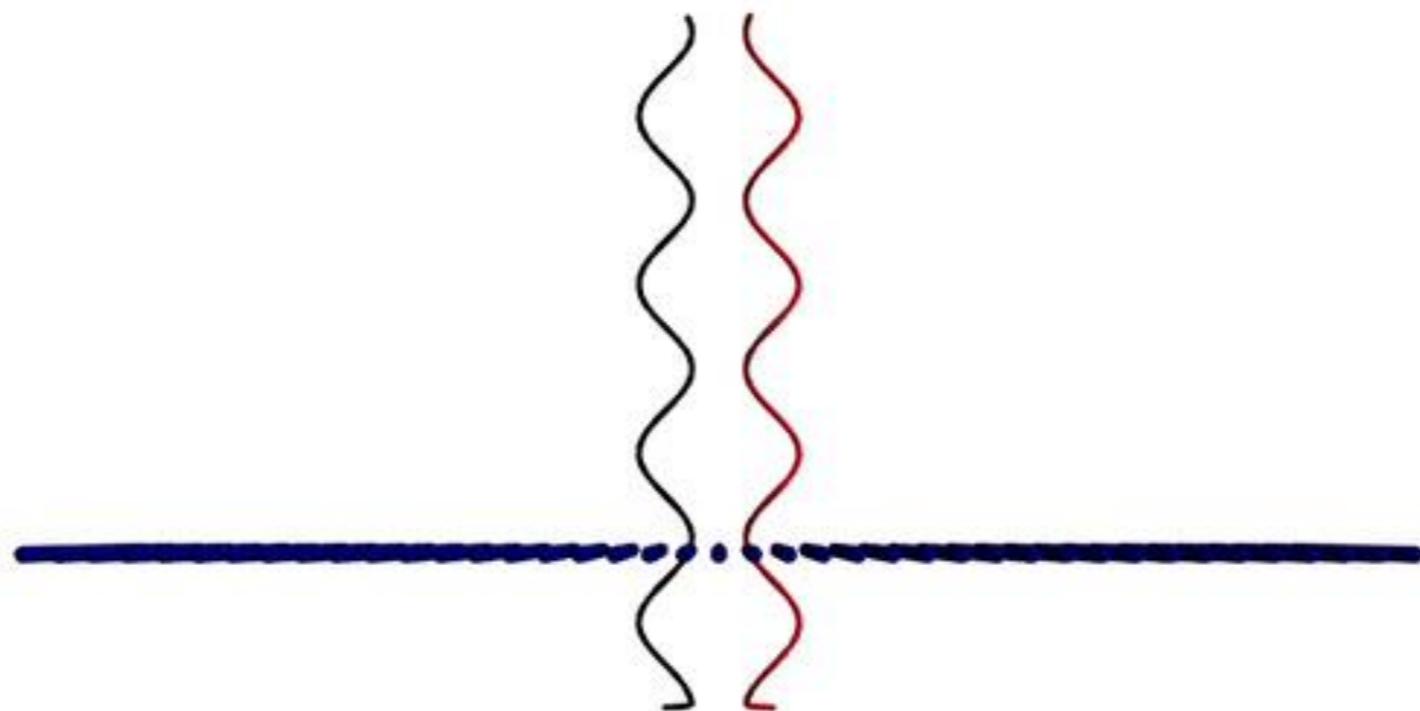


$$\begin{aligned}-\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{x}, t) &= 0 \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

Coming soon...

**AN INTRODUCTORY GUIDE TO THE METHOD OF
REGULARIZED STOKESLETS: A PRACTICAL IMPLEMENTATION ***

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LONGHUA ZHAO ^{||}



Thank you

- ♦ Please fill out the following survey



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