A Practical Guide to Implementing Regularized Stokeslets for Modelling

Amy Buchmann, Eva Strawbridge, and Longhua Zhao

In applications relevant to this tutorial, we are interested in viscous or low Reynolds number (Re) fluids where the inertial effect in the system is negligible. The governing equations for such a problem are

$$\begin{aligned} \mu\Delta\mathbf{u} &= \nabla p - \mathbf{F}, & (1)\\ \nabla\cdot\mathbf{u} &= 0, & (2)\end{aligned}$$

where $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{F}$ is the external force density acting on the fluid, μ is the fluid dynamic viscosity respectively. These equations describe the motion of a steady-state, zero Reynolds number, incompressible fluid and are called the Stokes equations.

The solution to the Stokes equations corresponding to the force $\mathbf{F} = \mathbf{f}\phi_\epsilon(\mathbf{y}-\mathbf{x})$ with the blob function

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2+\epsilon^2)^{7/2}} \tag{3}$$

is

$$\mathbf{u}(\mathbf{y}) = \frac{\mathbf{f}(r^2+2\epsilon^2) + [\mathbf{f}\cdot(\mathbf{y}-\mathbf{x})](\mathbf{y}-\mathbf{x})}{8\pi\mu(r^2+\epsilon^2)^{3/2}}, \tag{4}$$

where $r = |\mathbf{y}-\mathbf{x}|$. In the velocity tensor format, this solution is

$$\mathbf{u}(\mathbf{y}) = \frac{1}{8\pi\mu}A(\mathbf{y},\mathbf{x})\mathbf{f} \tag{5}$$

with

$$A(\mathbf{y},\mathbf{x}) = \begin{bmatrix} a_{11}^\epsilon & a_{12}^\epsilon & a_{13}^\epsilon \\ a_{21}^\epsilon & a_{22}^\epsilon & a_{23}^\epsilon \\ a_{31}^\epsilon & a_{32}^\epsilon & a_{33}^\epsilon \end{bmatrix}. \tag{6}$$

The tensor $A(\mathbf{y},\mathbf{x})$ depends on the location of the force, $\mathbf{x} = (x_1, x_2, x_3)^T$, the location of the observation point, $\mathbf{y} = (y_1, y_2, y_3)^T$, and the parameter ϵ. In detail,

$$\begin{aligned} a_{11}^\epsilon &= \frac{(r^2+2\epsilon^2)+r_1^2}{r_\epsilon^3}; \quad a_{12}^\epsilon = \frac{r_1 r_2}{r_\epsilon^3}; \quad a_{13}^\epsilon = \frac{r_1 r_3}{r_\epsilon^3}; \\ a_{21}^\epsilon &= \frac{r_2 r_1}{r_\epsilon^3}; \quad a_{22}^\epsilon = \frac{(r^2+2\epsilon^2)+r_2^2}{r_\epsilon^3}; \quad a_{23}^\epsilon = \frac{r_2 r_3}{r_\epsilon^3}; \\ a_{31}^\epsilon &= \frac{r_3 r_1}{r_\epsilon^3}; \quad a_{32}^\epsilon = \frac{r_3 r_2}{r_\epsilon^3}; \quad a_{33}^\epsilon = \frac{(r^2+2\epsilon^2)+r_3^2}{r_\epsilon^3}. \end{aligned} \tag{7}$$

where

$$\begin{gathered} r_1 = y_1 - x_1, \quad r_2 = y_2 - x_2 \quad r_3 = y_3 - x_3, \\ r^2 = r_1^2 + r_2^2 + r_3^2 \quad \text{and} \quad r_\epsilon = \sqrt{r^2+\epsilon^2}. \end{gathered} \tag{8}$$

After computing the velocity of the tracer particle, $\mathbf{v}^0 = \mathbf{v}(\mathbf{y}, 0)$, we can update the position of the tracer particle. Since velocity $\mathbf{v}(\mathbf{y}, t)$ is the derivative of position $\mathbf{y}$ with respect to time t,

$$\frac{d\mathbf{y}}{dt} = \mathbf{v}(\mathbf{y}, t), \tag{9}$$

so we can integrate the velocity to get the updated position. The forward Euler method for (9) would yield

$$\mathbf{y}^{i+1} = \mathbf{y}^i + \Delta t\ \mathbf{v}^i, \tag{10}$$

where Δt is the time step for the time discretization.

Exercises

1. Let $\mathbf{f} = (2, 3, -1)^T$ be the force applied at the point $\mathbf{x} = (1, -1, 2)^T$. Let $\epsilon = 0.05$. We want to find the resulting velocity $\mathbf{u}(\mathbf{y})$ at the observation point $\mathbf{y} = (2, 0, 1)^T$.
   (a) Assemble the matrix in (6) and use (5) to find the resulting velocity at the point $\mathbf{y} = (2, 0, 1)^T$.
   (b) Alternatively, you can use (4) to find the resulting velocity at the point $\mathbf{y} = (2, 0, 1)^T$.

2. Write a function that takes in a forcing location, $\mathbf{x}$, a corresponding force, $\mathbf{f}$, an observation point, $\mathbf{y}$, values for ϵ and μ, and calculates each element a_{ij}^{ϵ} of the velocity tensor in (7) to construct $A(\mathbf{y}, \mathbf{x})$, and then uses (5) to find the resulting velocity at the observation point, $\mathbf{u}(\mathbf{y})$.
   (a) Use your function to reproduce the result from Exercise 1.
   (b) Visualize the vector $\mathbf{f}$ at the point $\mathbf{x}$ and the vector $\mathbf{u}(\mathbf{y})$ at the point $\mathbf{y}$. Use the MATLAB command *view([-1 1 1])* to specify the line of sight.
   (c) The vectors will be hard to visualize due to the difference in their magnitudes. Normalize each vector and redo (b) to visualize the normalized vectors.
   (d) Adapt your code from part (a) to update the position of the tracer particles using Forward Euler (10) and $\Delta t = 0.01$ to update the position of the particle starting at $t = 0$ and ending at $t = 0.05$. *Note: This time step is large for Forward Euler. For simulations, we recommend using* $\Delta t = 0.001$ *or smaller.*

3. Let $\mathbf{x} = (-3, 2, 1)^T$ be a forcing location with force $\mathbf{f}$. In this exercise, we will prescribe the velocity at this point, so $\mathbf{x}$ is the forcing location as well as the location of the observation point, hence, $\mathbf{x} = \mathbf{y}$. Let the prescribed velocity at $\mathbf{y}$ be $\mathbf{u}(\mathbf{y}) = (1, -.5, -1)^T$. Let $\epsilon = 0.05$.
   (a) Write out the 3×3 tensor $A(\mathbf{y}, \mathbf{x})$ defined in (6).
   (b) Solve the system
   $$\mathbf{u}(\mathbf{y}) = \frac{1}{8\pi\mu} A(\mathbf{y}, \mathbf{x})\mathbf{f}$$
   to find $\mathbf{f}$ by calculating A^{-1}. *Notice: the location of the force and the observation point are the same. This ensures the velocity tensor is a diagonal matrix and therefore invertible.*
   (c) Plug $A(\mathbf{y}, \mathbf{x})$ and $\mathbf{f}$ into Equation (5) to verify that your force results in the desired prescribed velocity.

4. Write a function that takes in an observation point $\mathbf{y}$, the prescribed velocity at that point, $\mathbf{u}(\mathbf{y})$, and values for ϵ and μ, and calculates the force $\mathbf{f}$ that would result in the prescribed velocity when the force is applied at $\mathbf{y}$. *Note: It may be helpful to start by copying your code from Exercise 2 and modifying it accordingly.*
   (a) Use your function to reproduce the results of Exercise 3, and calculate $\mathbf{f}$.
   (b) Use the code you wrote for Exercise 2 to verify that the force you calculated in part (a) results in the desired prescribed velocity.
   (c) Let $\mathbf{y}_2 = (1, -2, 3)^T$ be a second observation point. Use the function you created in exercise 2 to find the velocity at that point $\mathbf{u}(\mathbf{y}_2)$ resulting from the force you found in part (a) applied at $\mathbf{x}$.

Solutions

1.

$$\mathbf{u}(\mathbf{y}) \quad = \quad \frac{1}{8\pi(3+\epsilon^2)^{3/2}}(12+4\epsilon^2, 15+6\epsilon^2, -9-2\epsilon^2)^T \approx (0.0918, 0.1148, -0.0689)^T,$$

where $\mathbf{y} = (2, 0, 1)$, with $\mathbf{f} = (2, 3, -1)$ at $\mathbf{x} = (1, -1, 2)$.

2. (a) $\mathbf{u}(\mathbf{y}) \approx (0.0918, 0.1148, -0.0689)^T$ at $\mathbf{y} = (2, 0, 1)^T$, with $\mathbf{f}(\mathbf{x}) = (2, 3, -1)^T$ at $\mathbf{x} = (1, -1, 2)^T$. (d) Below are the approximations from Forward Euler.

t	$\mathbf{y}_1$
0	$(2, 0, 1)^T$
0.01	$(2.000918498864207,\ 0.001148314774860,\ 0.999311317046446)^T$
0.02	$(2.001836190618279,\ 0.002295716638611,\ 0.998623335402053)^T$
0.03	$(2.002753077333999,\ 0.003442207810047,\ 0.997936053142048)^T$
0.04	$(2.003669161074396,\ 0.004587790498958,\ 0.997249468350167)^T$
0.05	$(2.004584443893788,\ 0.005732466906183,\ 0.996563579118606)^T$

3. (a)

$$A(\mathbf{y}, \mathbf{x}) = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

(b)

$$A^{-1} = \begin{bmatrix} \frac{1}{40} & 0 & 0 \\ 0 & \frac{1}{40} & 0 \\ 0 & 0 & \frac{1}{40} \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} \frac{\pi}{5} \\ -\frac{\pi}{10} \\ -\frac{\pi}{5} \end{bmatrix}$$

4 (a) $\mathbf{f}(\mathbf{x}) \approx (0.6283, -0.3142, -0.6283)^T$ at $\mathbf{x} = (-3, 2, 1)^T$.
(c) $\mathbf{u}(\mathbf{y_2}) \approx (0.0060, -0.0039, -0.0032)^T$ at $\mathbf{y_2} = (1, -2, 3)^T$.