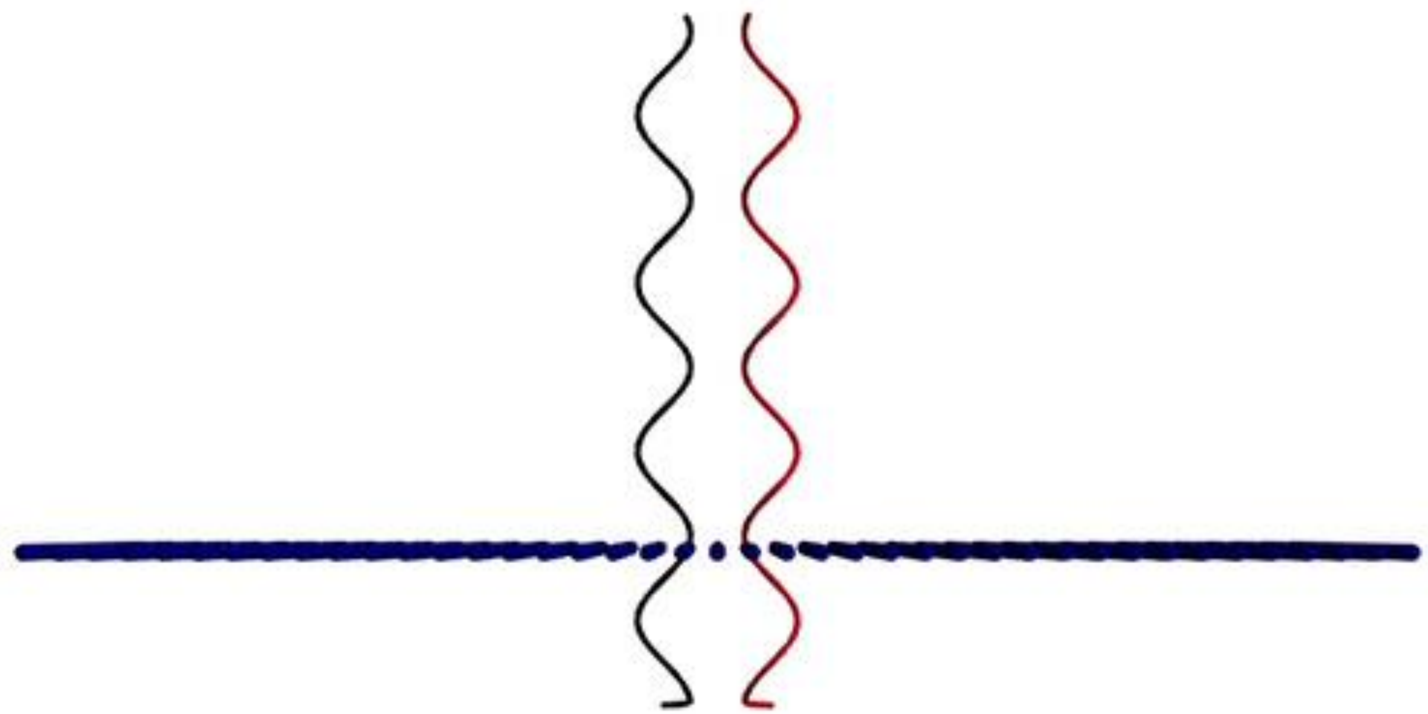
Scan this link to log in to MATLAB
https://www.mathworks.com/products/matlab.html

A Practical Guide to Implementing Regularized Stokeslets for Modelling

Scan this link to access the workshop materials
https://github.com/longhuazhao/MRS_IntroductoryGuide/tree/main

A Practical Guide to Implementing Regularized Stokeslets for Modelling



Amy Buchmann
University of San Diego

Eva Strawbridge
James Madison University

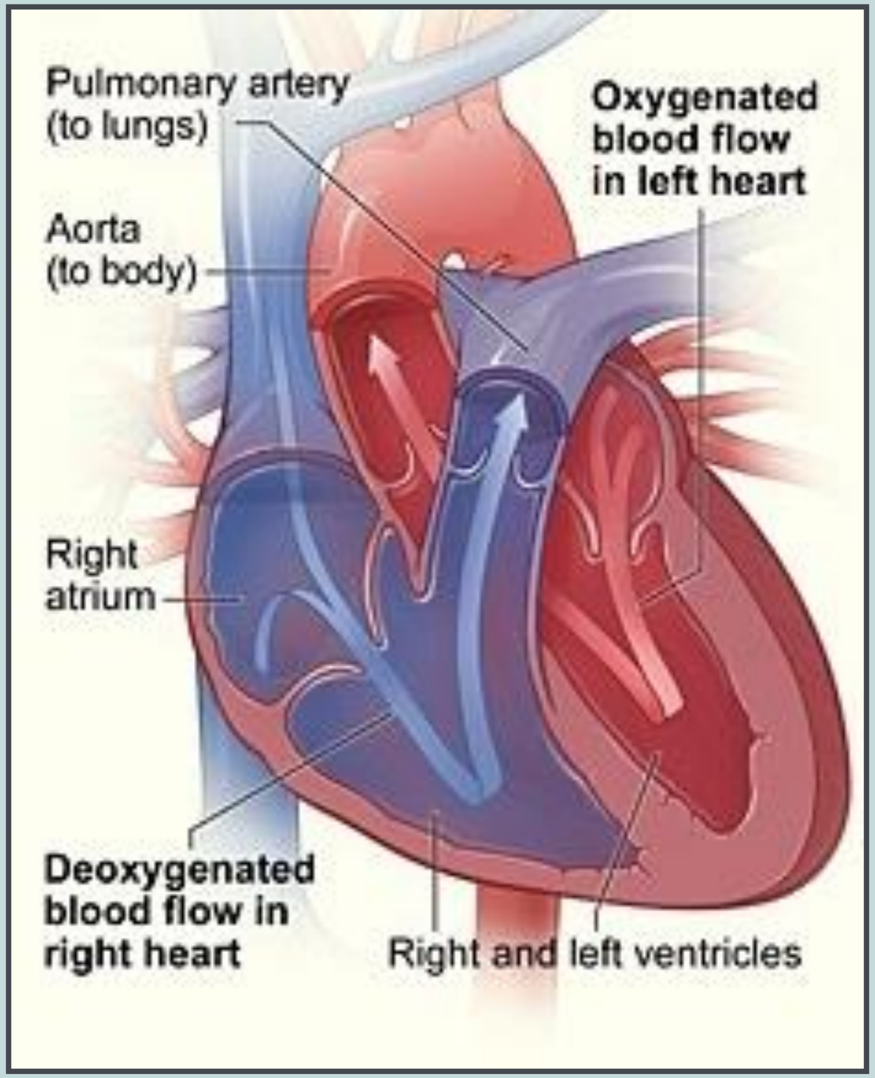
Longhua Zhao
Case Western Reserve University

Goals

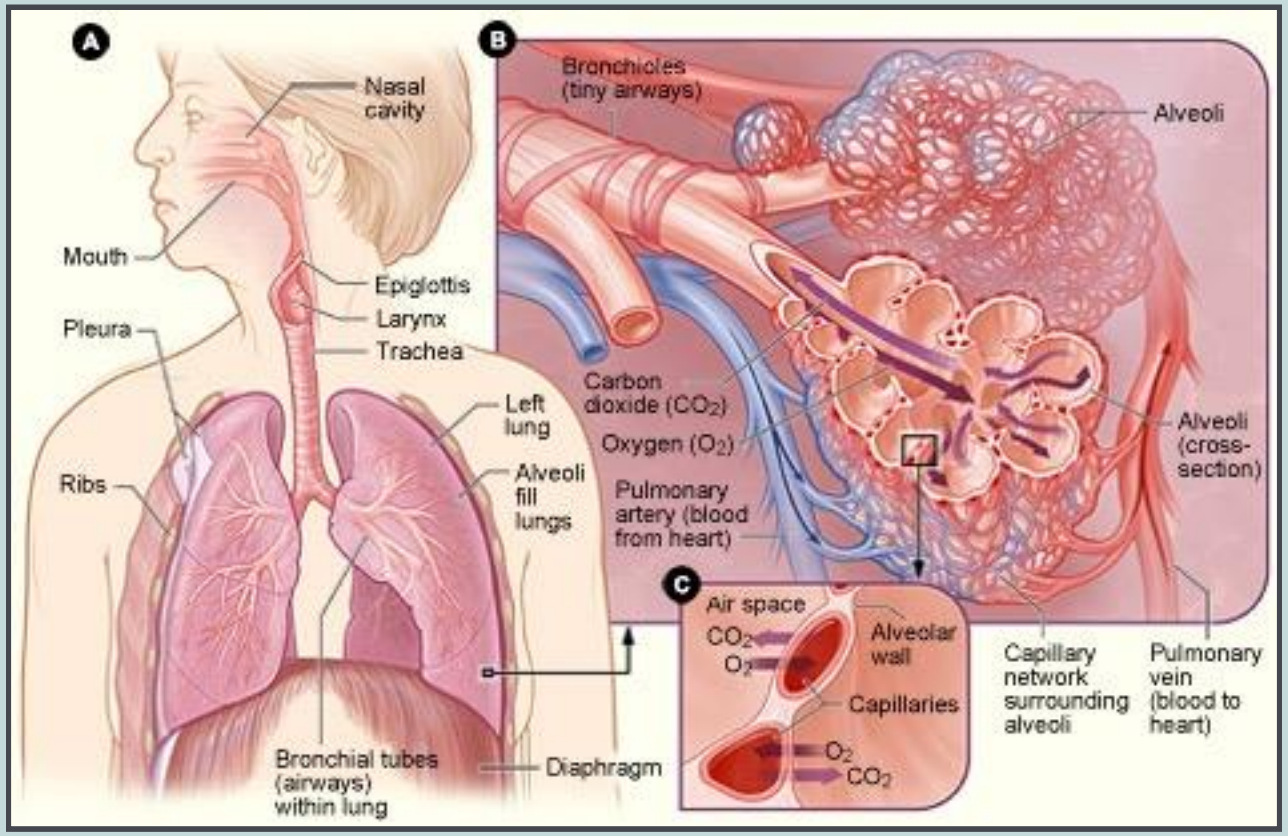
- To demonstrate how mathematics can be used to enhance our understanding of the world around us.
- In particular, concepts from undergraduate math courses can be used to study microscale biological fluid dynamics
  - Single Variable Calculus
  - Multivariable Calculus
  - Differential Equations
  - Linear Algebra

Biological Fluid Dynamics

Blood flow in the heart



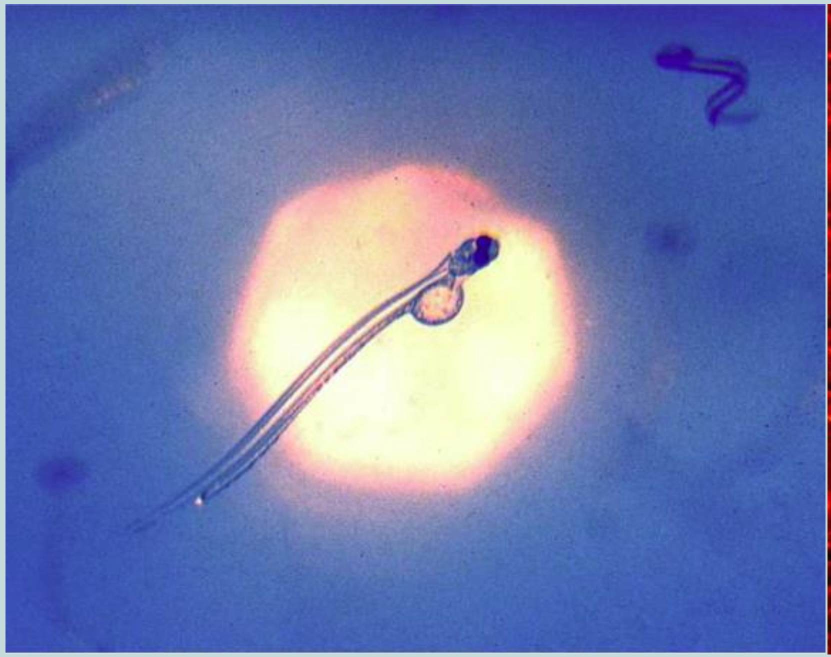
Airflow in the lungs

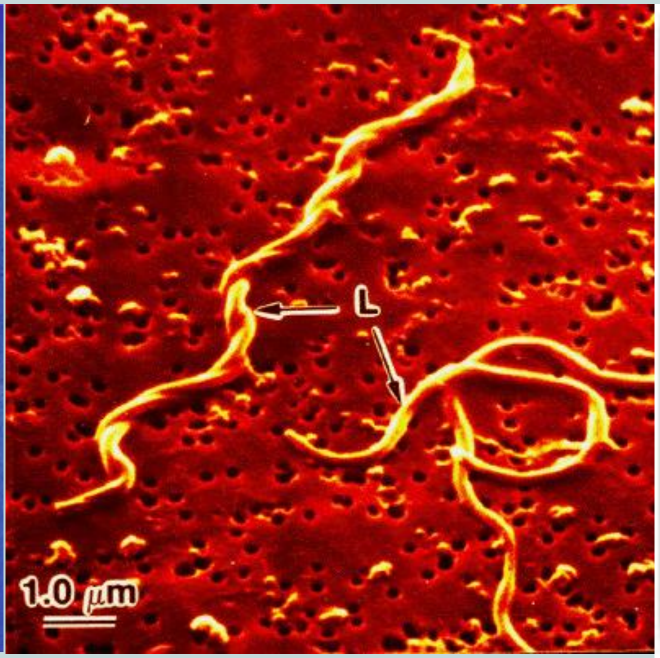


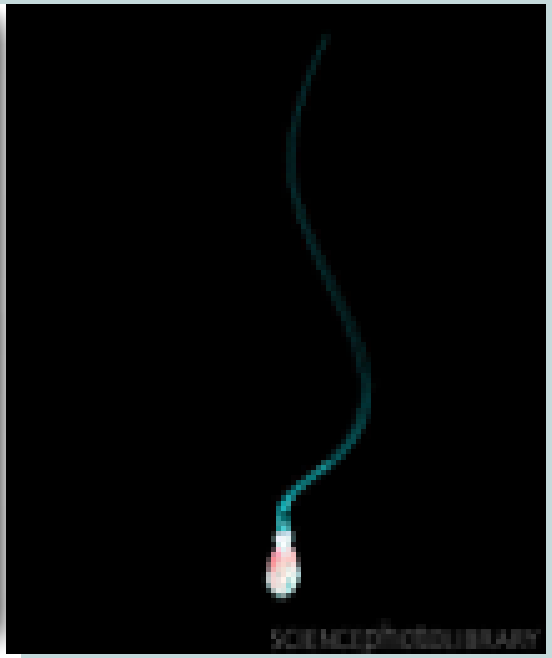
NIH

Microscale Biological Fluid Dynamics

- Bacteria (~ 25 μm)
  - How do helical flagella mix and pump fluid?
- Cilia (~ 10 μm)
  - How do cilia coordinate their movement to form a metachronal wave?

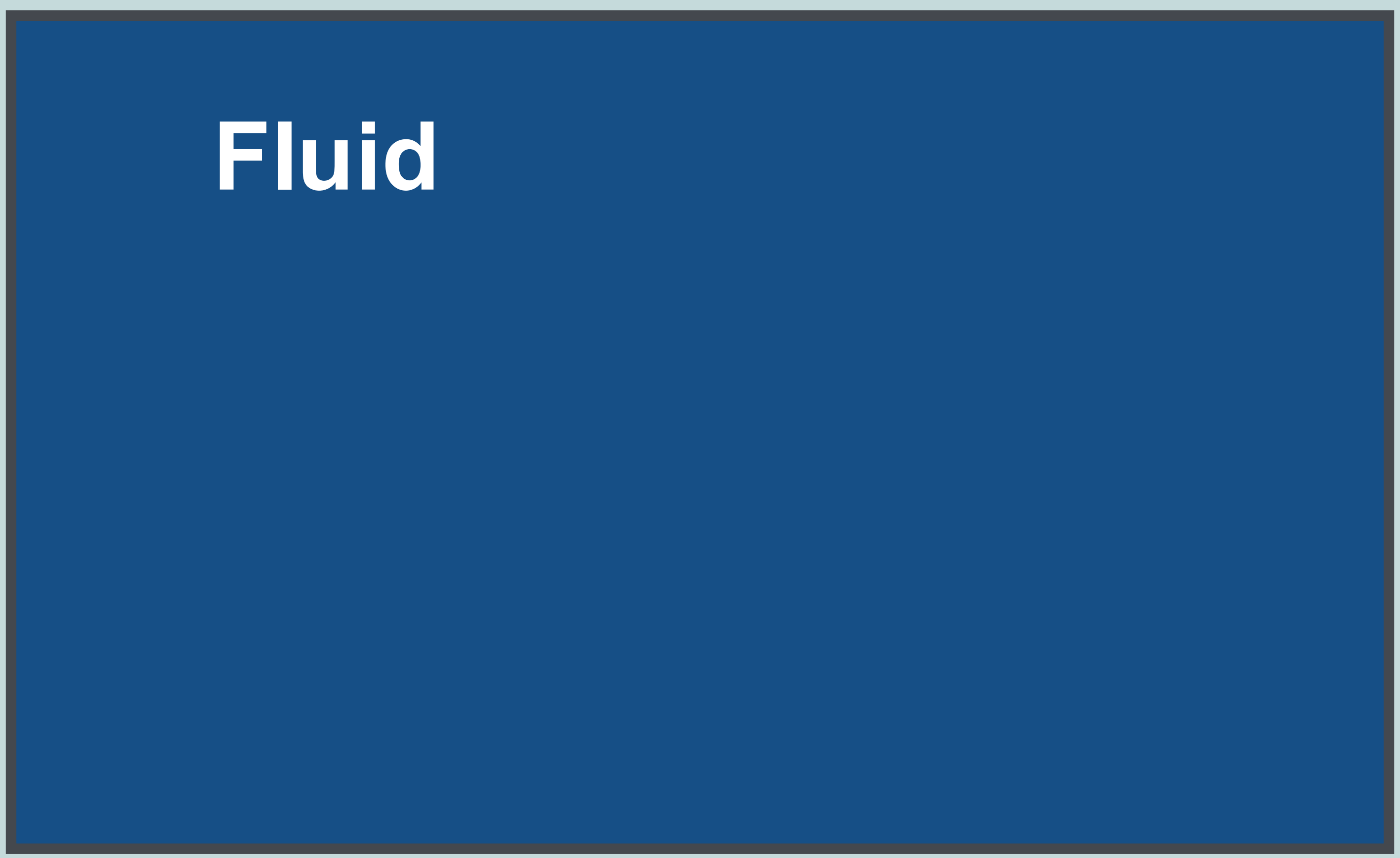






Fluid Structure Interactions

- We need a mathematical framework to describe



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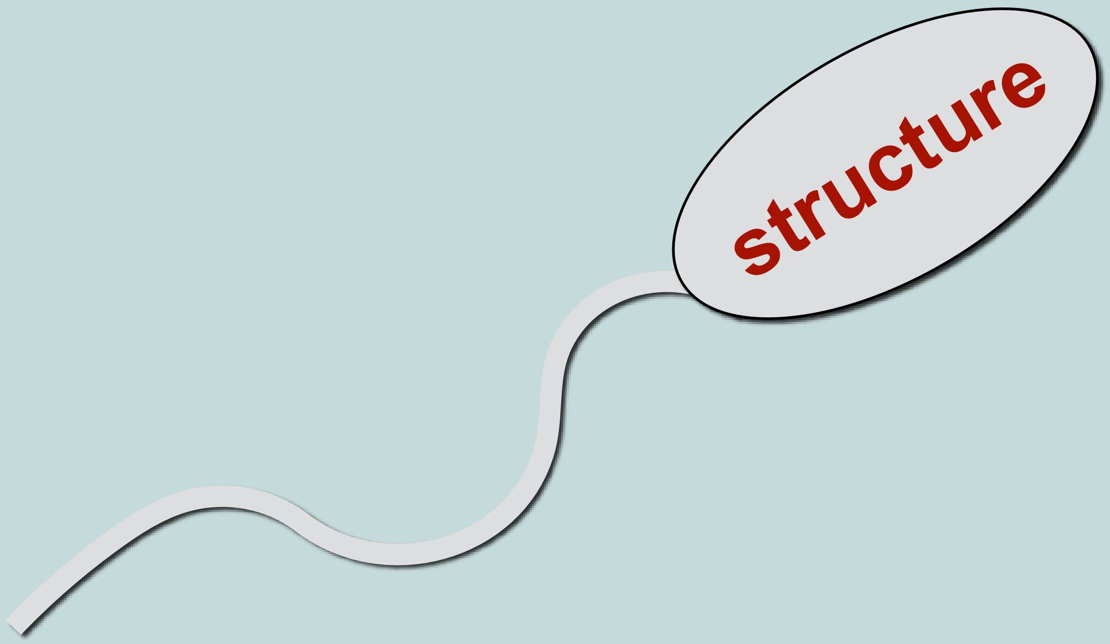

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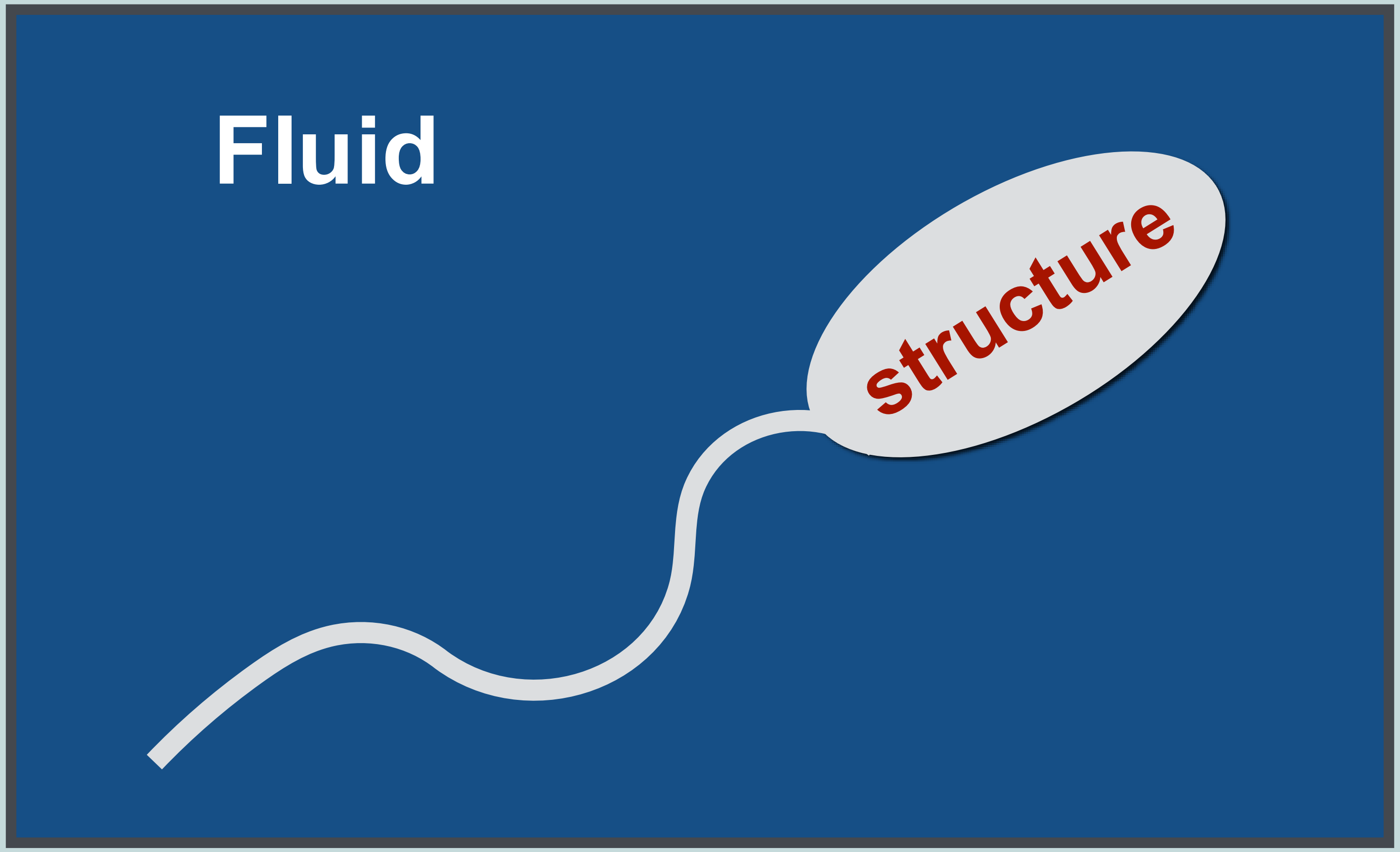

Calculus Concepts: Derivatives!

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- Derivatives: rates of change (single variable)

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$$

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$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

Calculus Concepts: Derivatives!

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- Partial Derivatives: rates of change (multivariable)

$$\frac{\partial}{\partial y}\left[2x^2y^3\right] = 6x^2y^2$$

Calculus Concepts: Derivatives!

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- Partial Derivatives: rates of change (multivariable)

$$\frac{\partial}{\partial y}\left[2x^2y^3\right] = 6x^2y^2$$

- Partial Differential Equations: equations with partial derivatives

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Fluid: Navier Stokes Equations

- The motion of liquids is well described by the Navier Stokes Equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}$$

"Force Balance"

$$\nabla\cdot\mathbf{u} = 0$$

"Conservation of Mass"

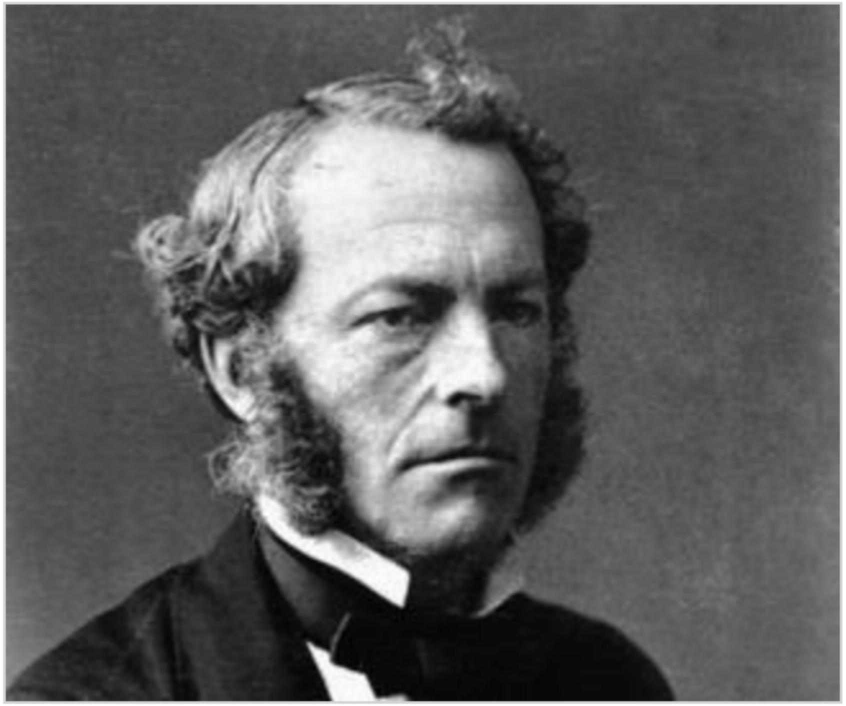
ρ : fluid density

$\mathbf{u}$: fluid velocity

p : fluid pressure

μ : fluid viscosity

Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Image: Sir George Gabriel Stokes (13 August 1819–1 February 1903). Public Domain

This problem is:	Unsolved

Rules:

Rules for the Millennium Prizes

Related Documents:

Official Problem Description

Related Links:

Lecture by Luis Cafarelli

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Open MATLABoverview.pdf

Fluid: Navier Stokes Equations

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$$\nabla\cdot\mathbf{u} = 0$$ "Conservation of Mass"

ρ : fluid density

$\mathbf{u}$: fluid velocity

p : fluid pressure

μ : fluid viscosity

Non-dimensionalization

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Left hand side

Reynolds Number

$$\boxed{\frac{\rho L U}{\mu}}\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+w^*\frac{\partial u^*}{\partial z^*}\right)$$

Right hand side

$$=\left(-\frac{\partial p^*}{\partial x^*}+\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{\partial^2 u^*}{\partial z^{*2}}\right)$$

Non-dimensionalization

$$
\begin{aligned}
\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) &= -\nabla p + \mu\nabla^2\mathbf{u} \\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}
$$

Introducing the non-dimensional variables

$$
\mathbf{x}^* = \frac{\mathbf{x}}{L} \qquad \mathbf{u}^* = \frac{\mathbf{u}}{U} \qquad p^* = \frac{pL}{\mu U} \qquad t^* = t\frac{U}{L}
$$

this becomes

$$
\begin{aligned}
Re\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^*\cdot\nabla\mathbf{u}^*\right) &= -\nabla p^* + \nabla^2\mathbf{u}^* \\
\nabla\cdot\mathbf{u}^* &= 0
\end{aligned}
$$

Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

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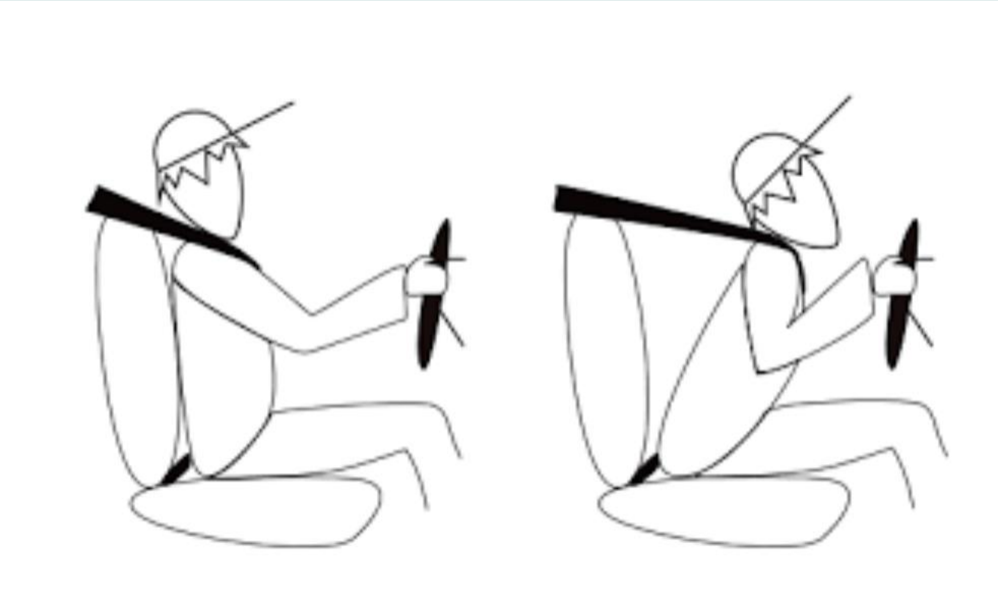
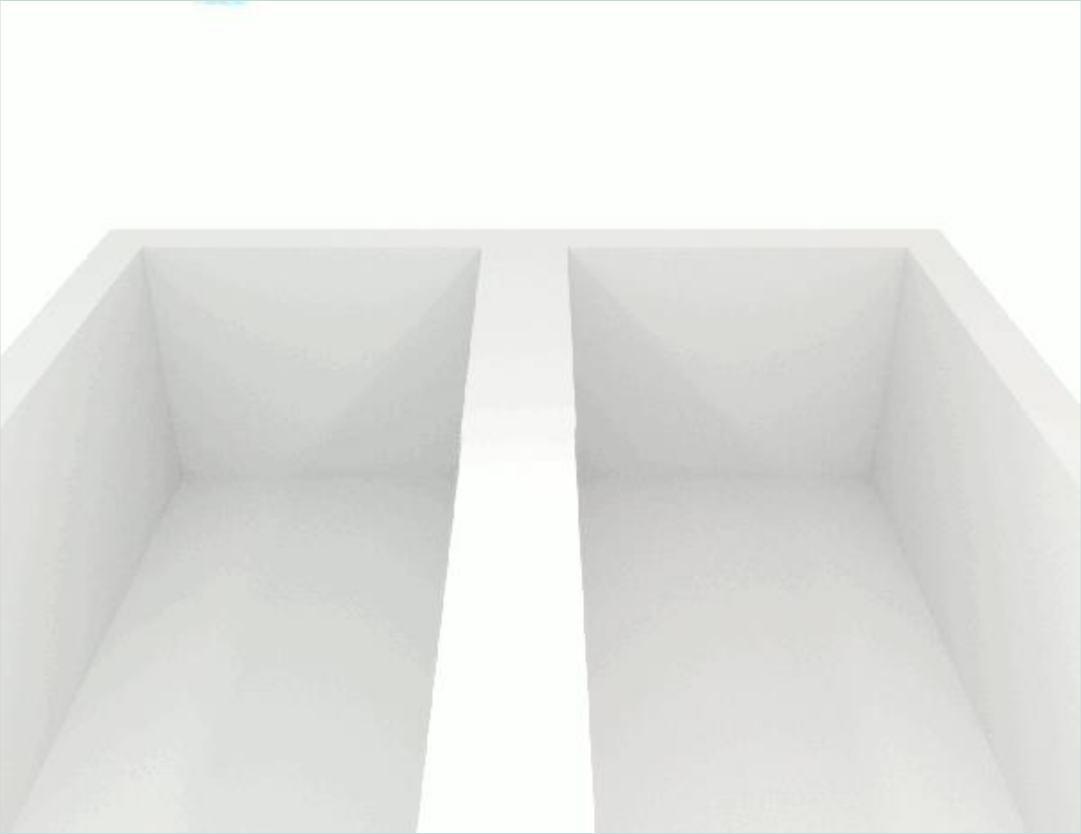


INERTIA: a property of matter by which it ***continues in its existing state*** of rest or uniform motion in a straight line, unless that state is changed by an external force.

Reynolds Number

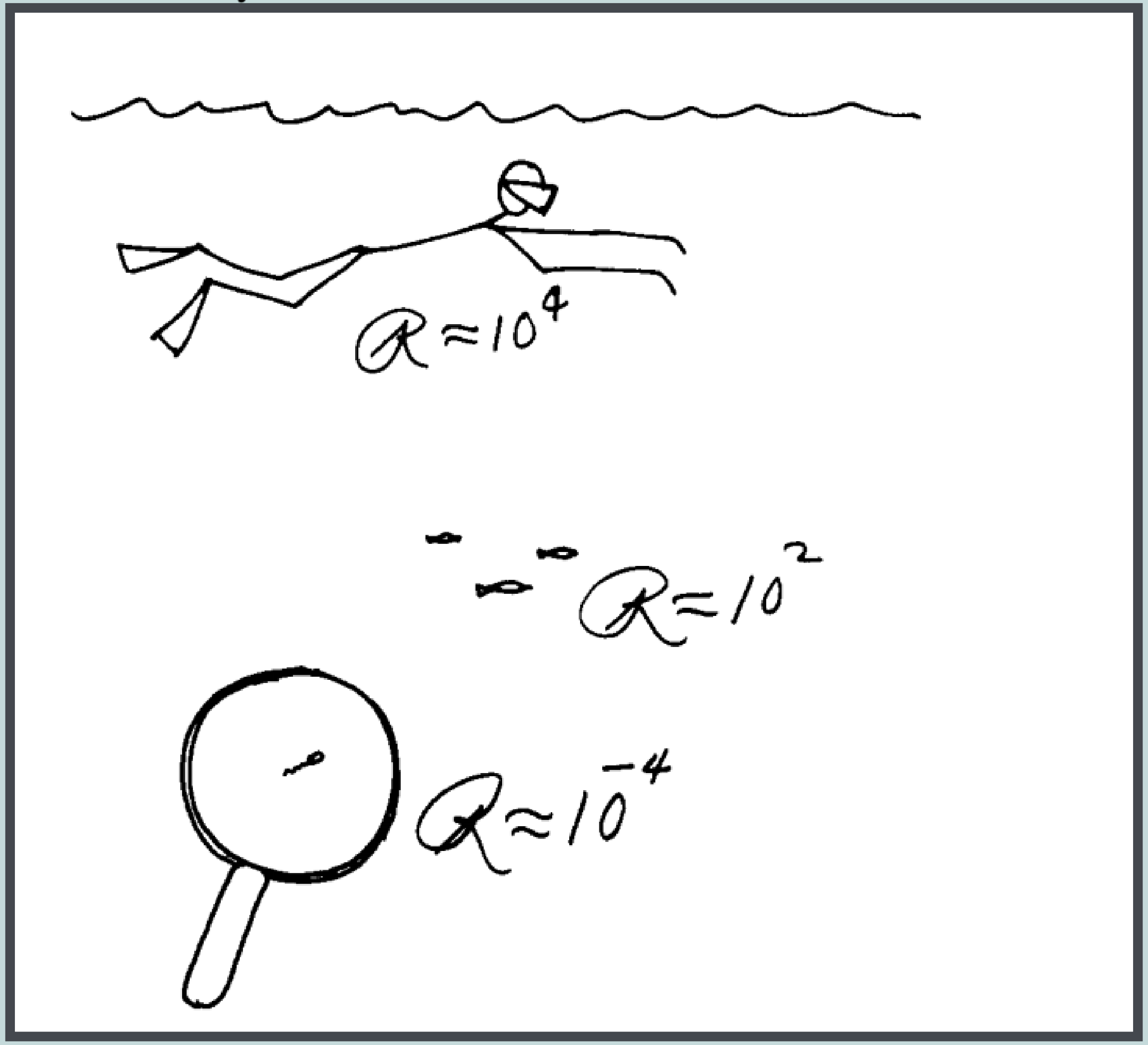
$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



VISCOSITY: The viscosity of a fluid is a measure of its **resistance to deformation** at a given rate.

Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



Purcell, Life at low Reynolds number 1976

Intermediate Reynolds Number

G.I. Taylor

Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

- High Reynolds Number - Inertial Forces Dominate
- Low Reynolds Number - Viscous Forces Dominate

Low Reynolds Number

G.I. Taylor

Non-dimensionalization

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right) \quad = \quad -\nabla p+\mu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} \quad = \quad 0$$

Introducing the non-dimensional variables

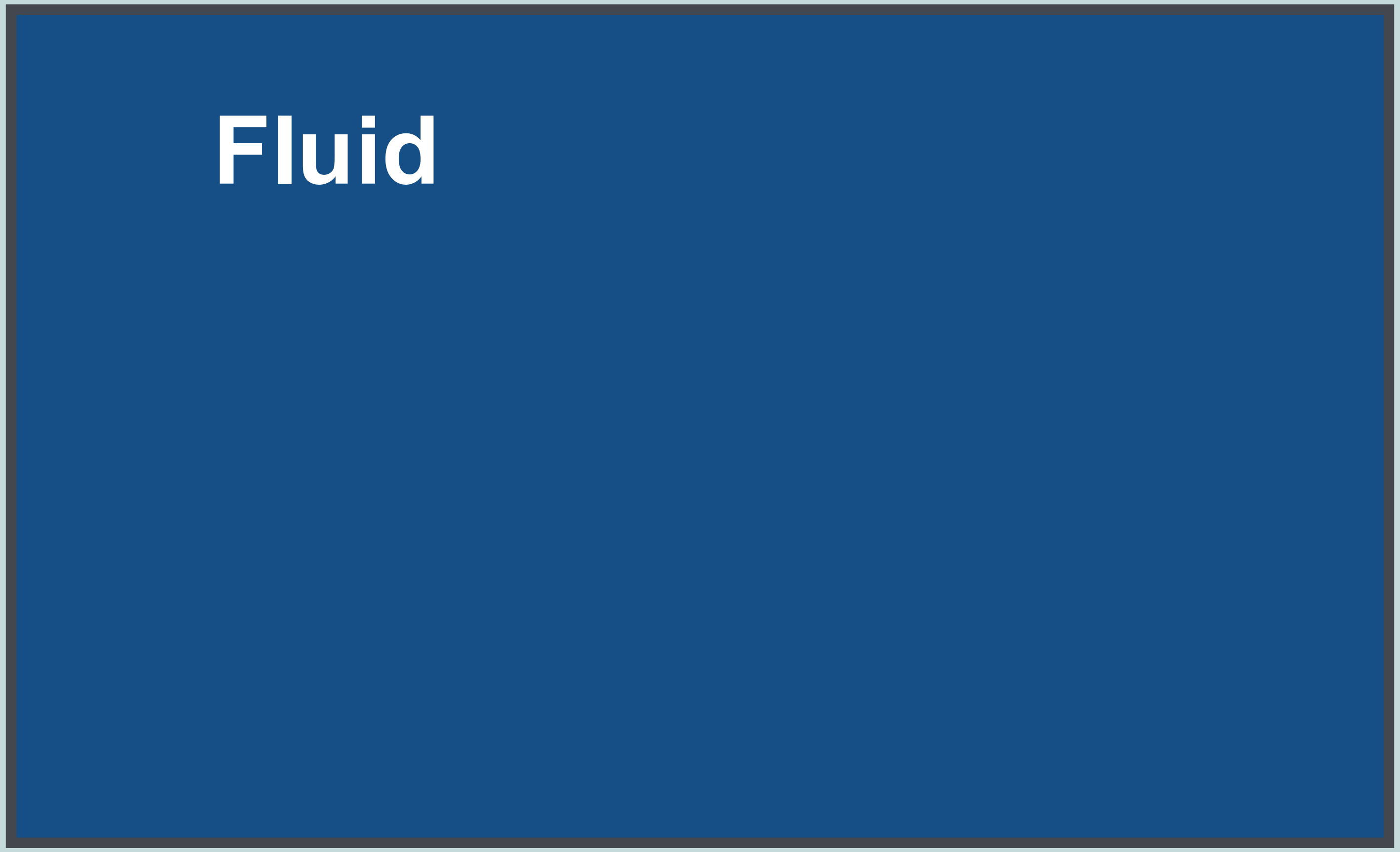
$$\mathbf{x}^*=\frac{\mathbf{x}}{L} \qquad \mathbf{u}^*=\frac{\mathbf{u}}{U} \qquad p^*=\frac{pL}{\mu U} \qquad t^*=t\frac{U}{L}$$

this becomes

$$Re\left(\frac{\partial \mathbf{u}^*}{\partial t^*}+\mathbf{u}^*\cdot\nabla\mathbf{u}^*\right) \quad = \quad -\nabla p^*+\nabla^2\mathbf{u}^*$$
$$\nabla\cdot\mathbf{u}^* \quad = \quad 0$$

Fluid Structure Interactions

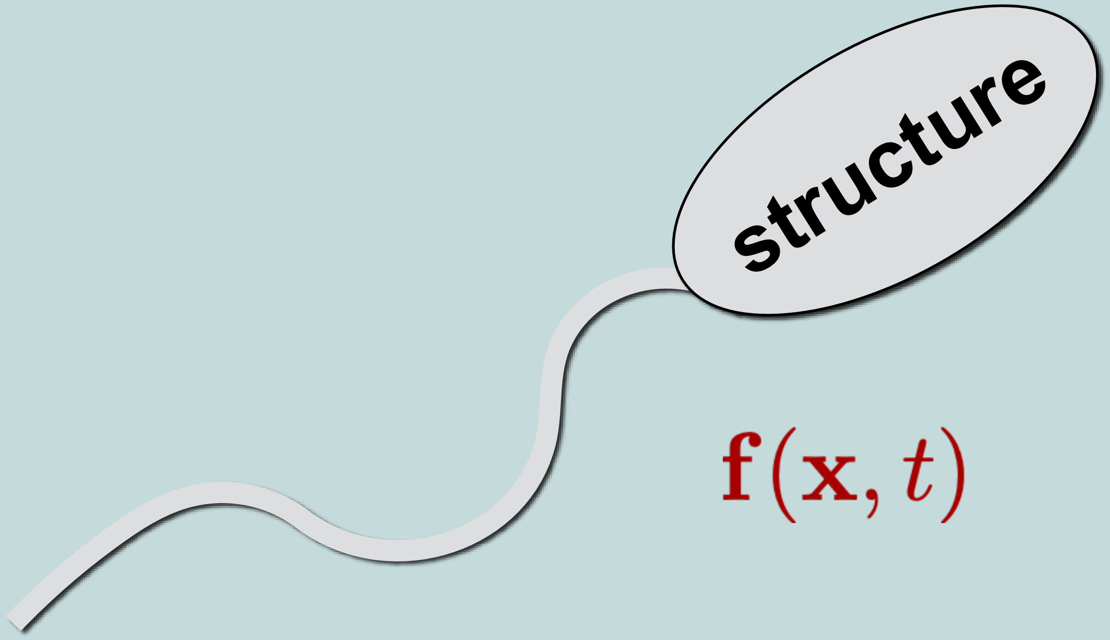
- We need a mathematical framework to describe



$$
\begin{aligned}
-\nabla p + \mu \nabla^2 \mathbf{u} &= \mathbf{0} \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
$$

Fluid Structure Interactions

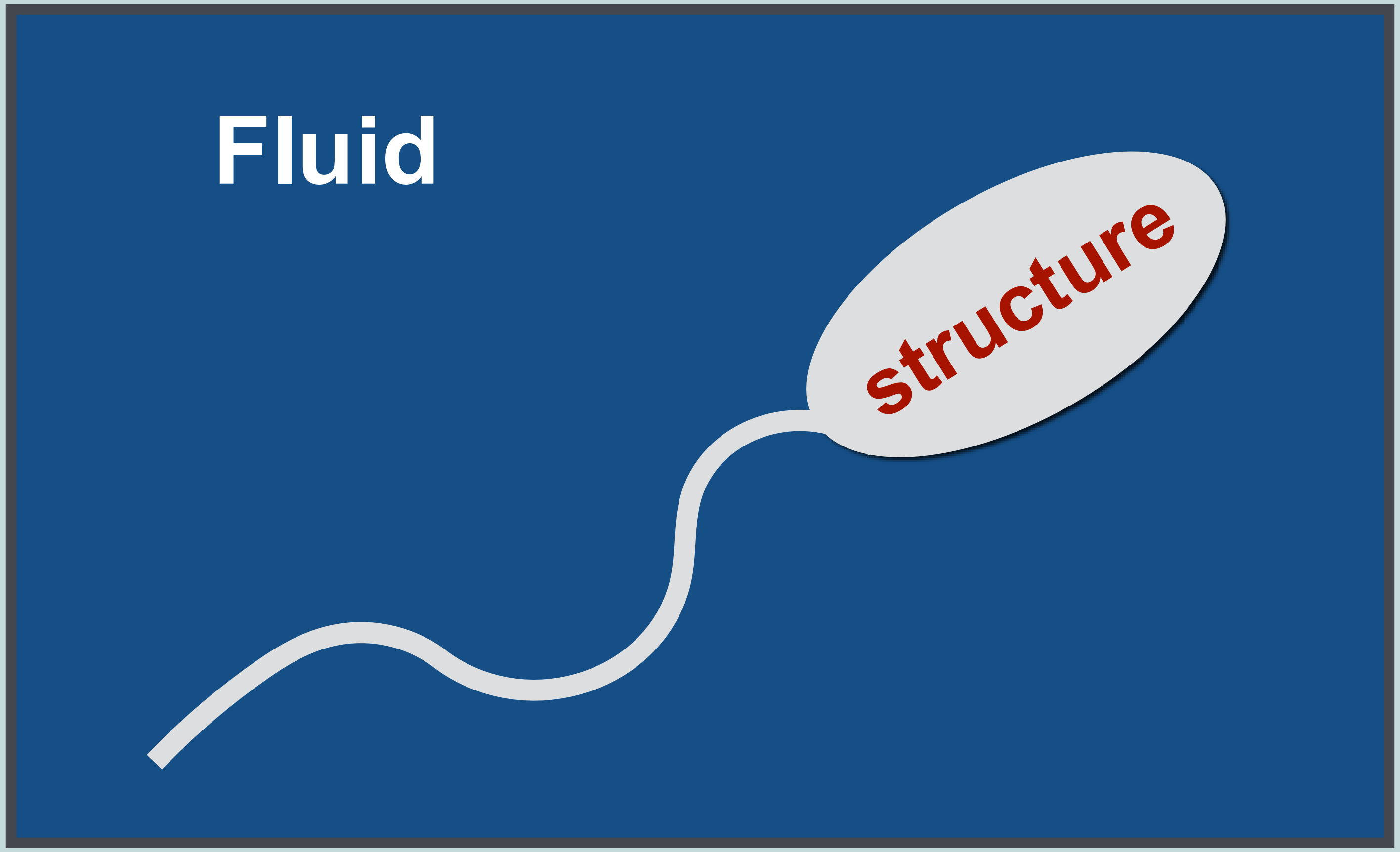
- We need a mathematical framework to describe



$\mathbf{f}(\mathbf{x}, t)$

Fluid Structure Interactions

- We need a mathematical framework to describe



$$\begin{aligned} -\nabla p + \mu \nabla^2 \mathbf{u} + \ \mathbf{f}(\mathbf{x}, t) &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

A force at a point?

- Dirac delta distribution

$$\int_{-\infty}^{\infty} f(y)\delta(y - x_0)dy = f(x_0)$$

- Fundamental solutions

$$\mu\nabla^2\mathbf{u} + \mathbf{f}_0\delta(\mathbf{x} - \mathbf{x}_0) = \nabla p$$
$$\nabla \cdot \mathbf{u} = 0$$

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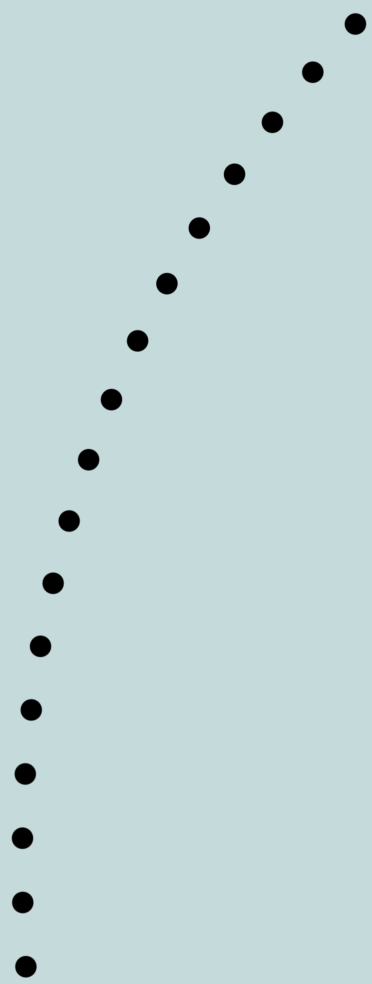
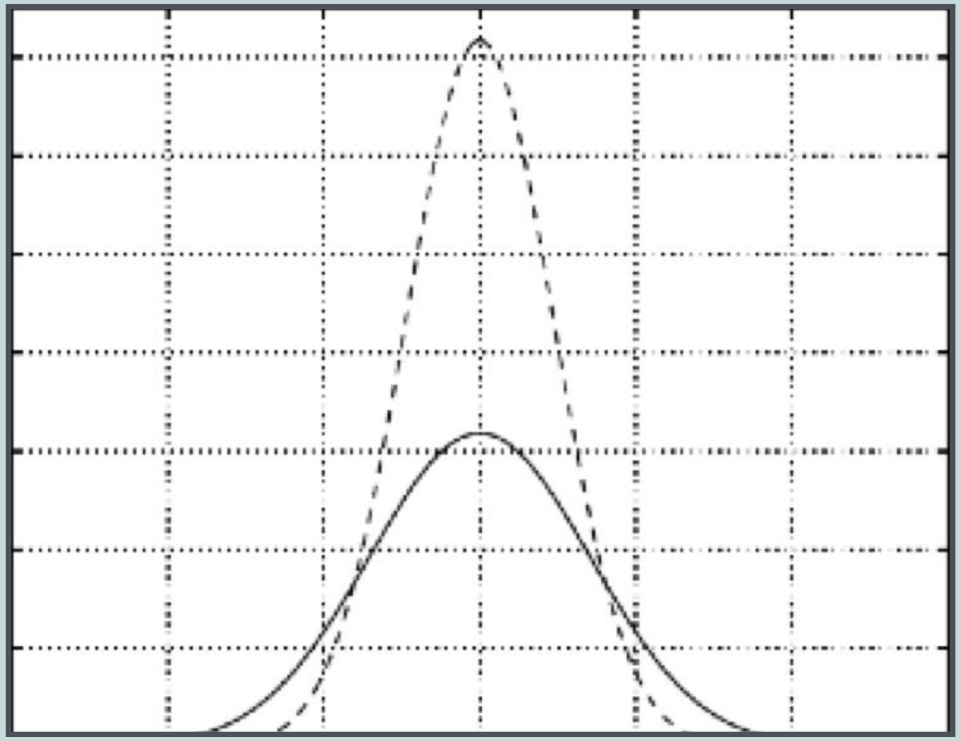


Method of regularized Stokeslets

Big Idea: Develop a regularized version of the fundamental solutions by designing smooth approximations of the Dirac delta distribution



- $\int \phi_\epsilon(\mathbf{x}) d\mathbf{x} = 1$
- $\phi_\epsilon(\mathbf{x}) = \phi_\epsilon(|\mathbf{x}|)$
- concentrated near $\mathbf{x} = \mathbf{0}$

$$\begin{aligned} \mu \nabla^2 \mathbf{u} + \mathbf{f_0} \phi_\epsilon(\hat{\mathbf{x}}) &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Method of regularized Stokeslets

$$\begin{aligned} \mu \Delta \mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

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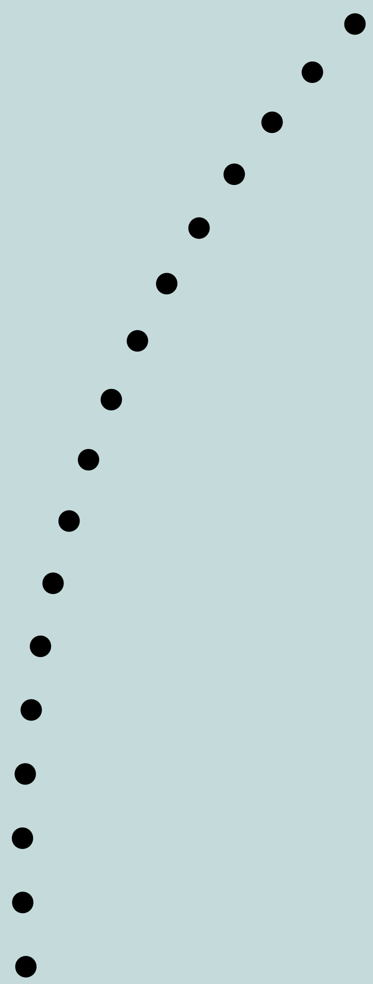
Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p\\ \nabla\cdot\mathbf{u} &= 0\end{aligned}$$



R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Method of regularized Stokeslets

$$
\begin{aligned}
\mu \Delta \mathbf{u} + \mathbf{F} &= \nabla p \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
$$

$$
\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)
$$

R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Method of regularized Stokeslets

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\begin{aligned}
\mu \Delta \mathbf{u} + \mathbf{F} &= \nabla p \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2 + \epsilon^2)^{7/2}}$$

R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u}+\mathbf{F} &= \nabla p\\ \nabla\cdot\mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x})=\mathbf{f}_k\phi_\epsilon(||\mathbf{x}-\mathbf{x}_k||)$$

$$\phi_\epsilon(r)=\frac{15\epsilon^4}{8\pi(r^2+\epsilon^2)^{7/2}}$$

$$\mathbf{u}(\mathbf{x})=\sum_{k=1}^{N}\left[\frac{\mathbf{f}_k(r_k^2+2\epsilon^2)+(\mathbf{f}_k\cdot(\mathbf{x}-\mathbf{x}_k))(\mathbf{x}-\mathbf{x}_k)}{8\pi\mu(r_k^2+\epsilon^2)^{3/2}}\right]$$

R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Method of regularized Stokeslets

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\begin{aligned}
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$$U = AF$$

R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Method of regularized Stokeslets

$$
\begin{aligned}
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\end{aligned}
$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$\boxed{U = AF}$$

We will code up an example of this today

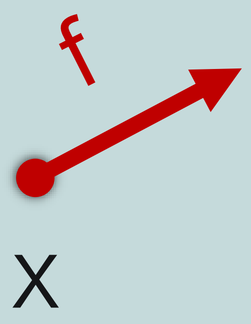
R Cortez. SIAM J. Sci. Comput., 23:1204, 2001.

Updating particle position with a prescribed force

x

Forcing point location

Updating particle position with a prescribed force

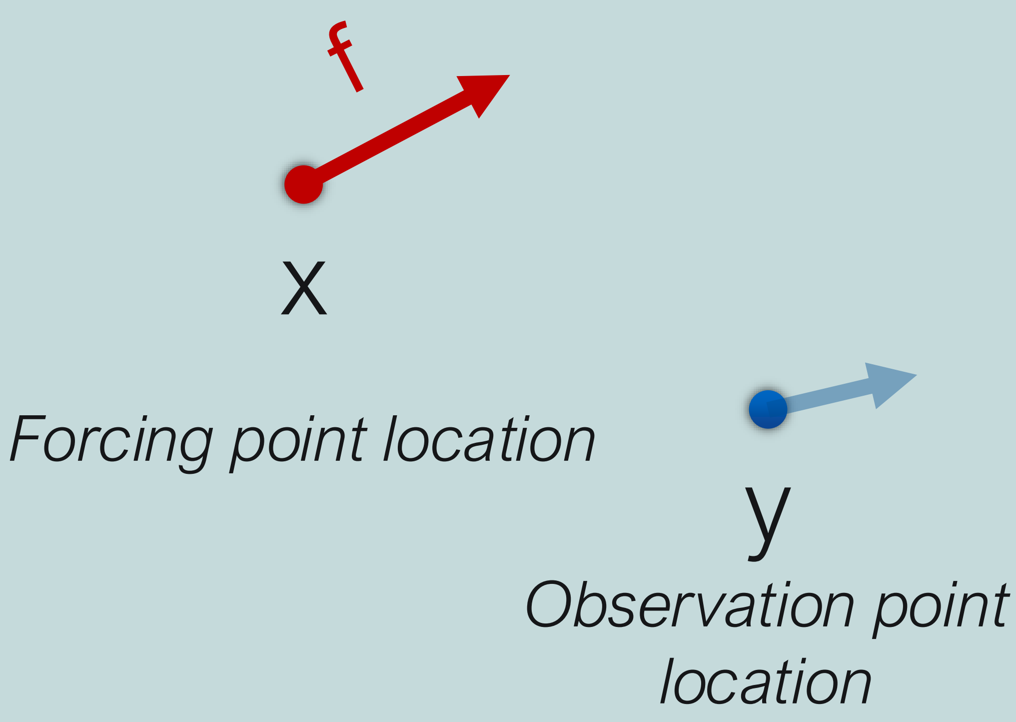


Forcing point location

1. Set the force **f** at **x**

e

Updating particle position with a prescribed force



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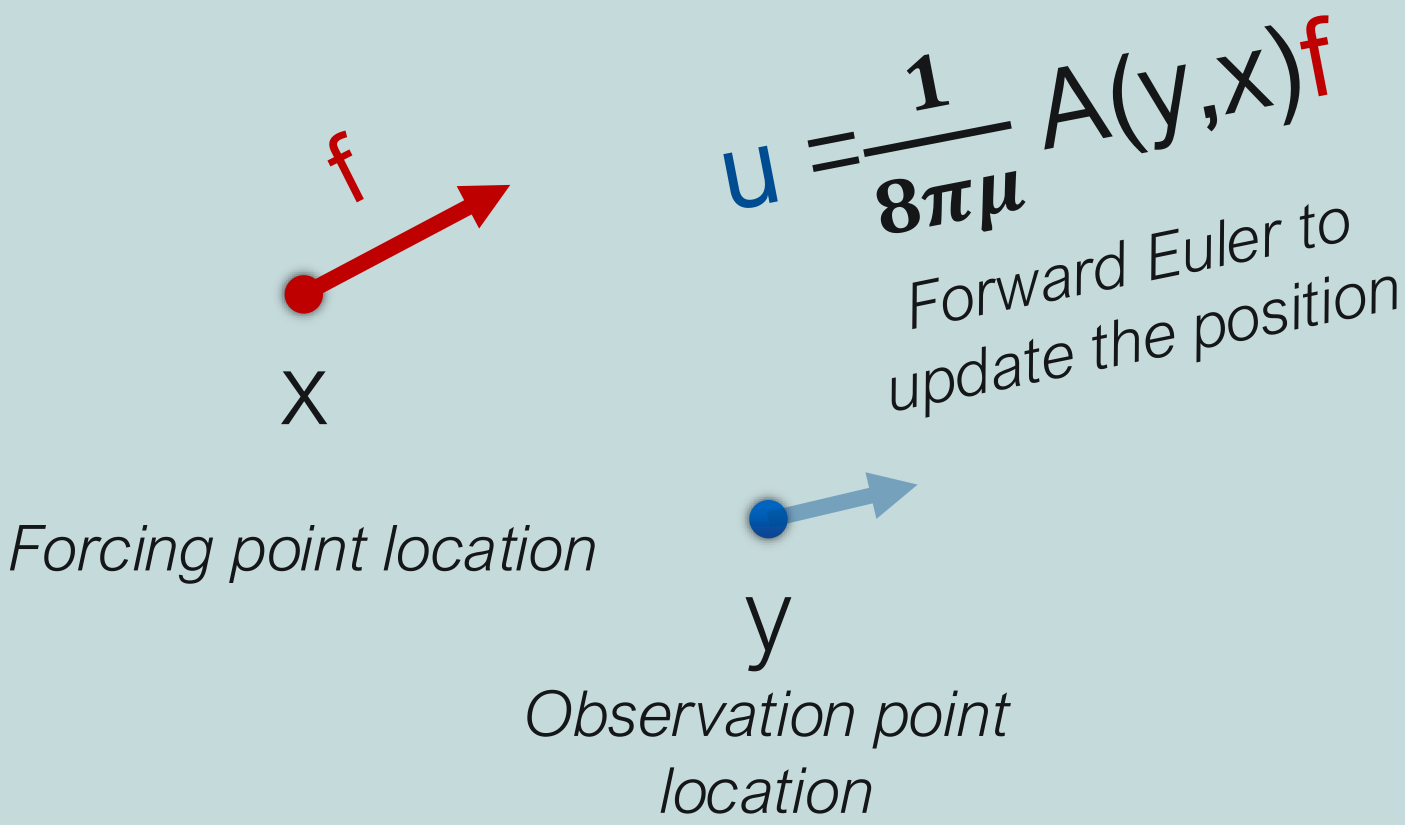
e

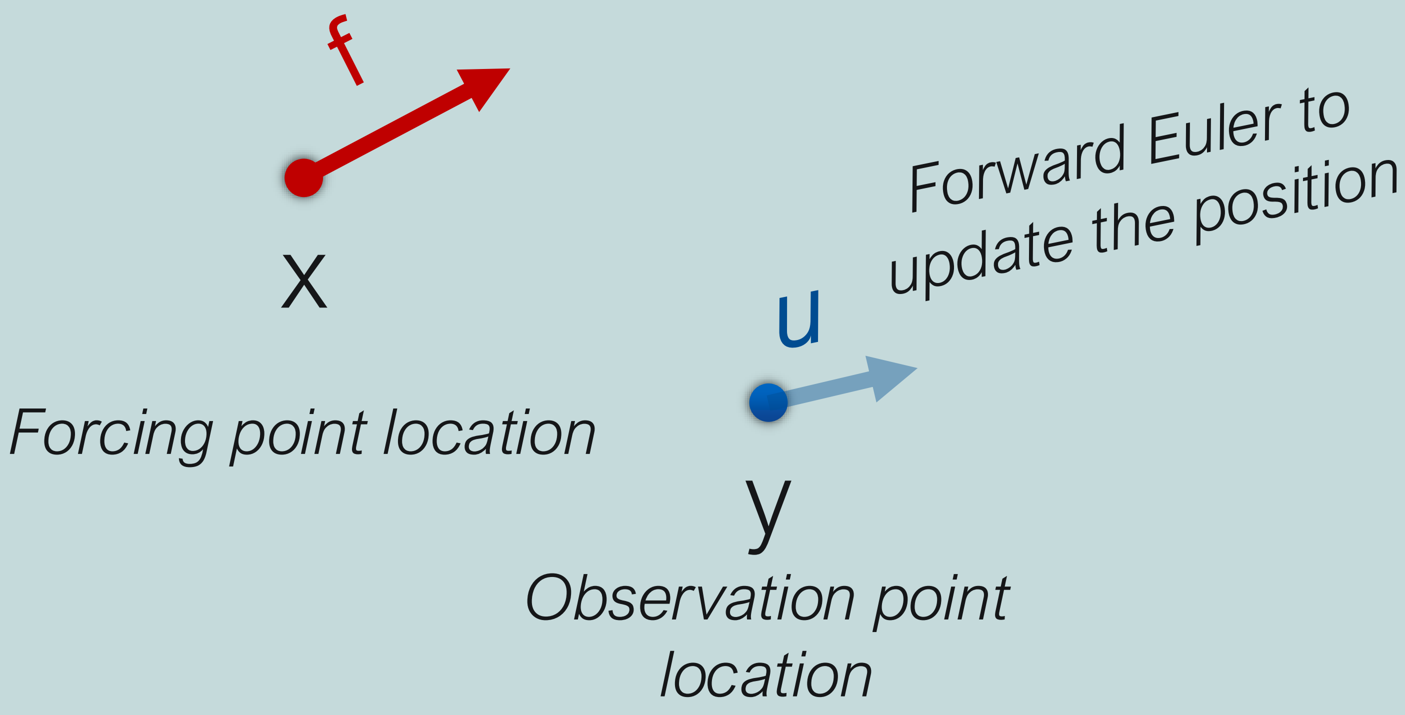
Updating particle position with a prescribed force



1. Set the force f at x
2. Compute $u = \frac{1}{8\pi\mu} A(y,x)\, f$

   to find the velocity u at y

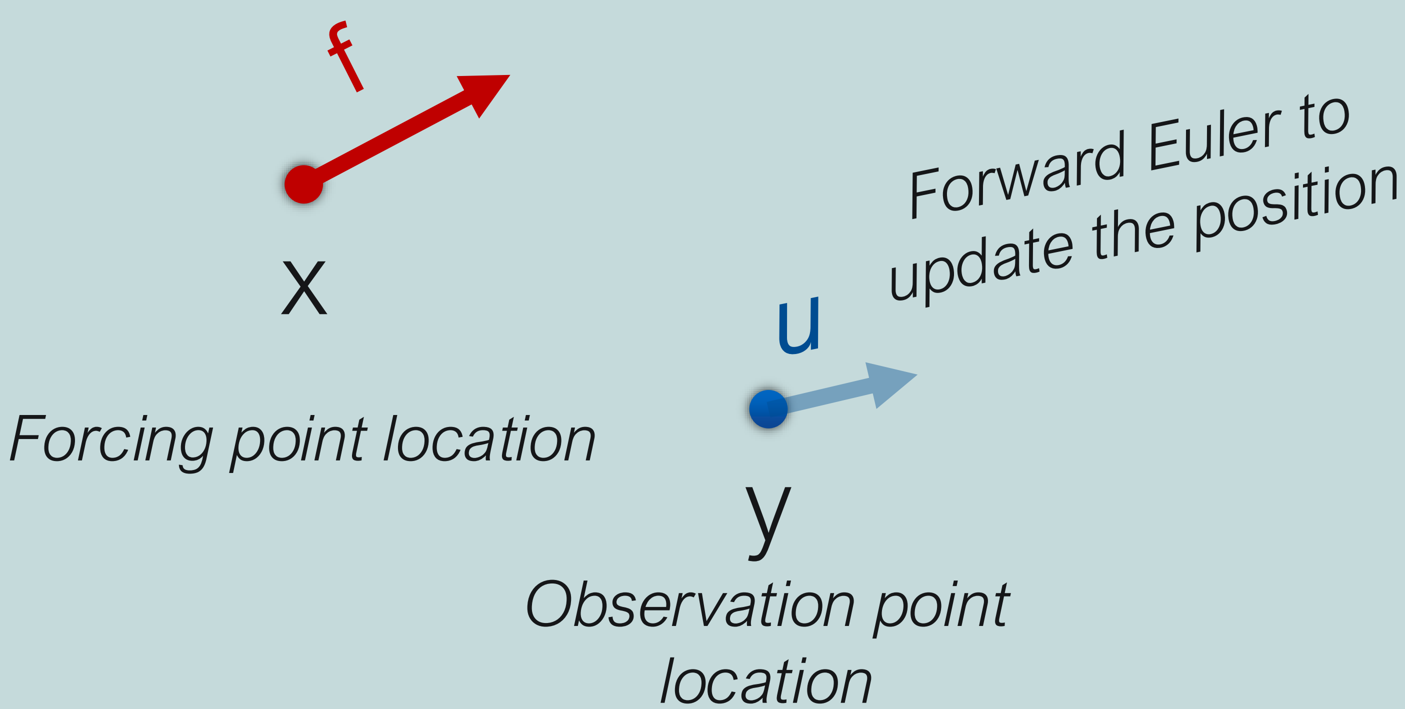
Updating particle position with a prescribed force



1. Set the force f at x
2. Compute u = $\frac{1}{8\pi\mu}$ A(y,x) f

   to find the velocity u at y
3. Use Forward Euler to update the position of y

Updating particle position with a prescribed force



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   to find the velocity u at y

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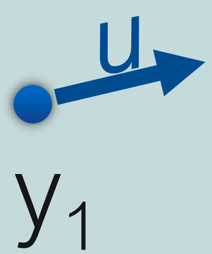
Open brief_MRS.pdf
Use MATLAB to do Exercises 1 and 2

https://github.com/longhuazhao/MRS_IntroductoryGuide/tree/main

Prescribing the Velocity

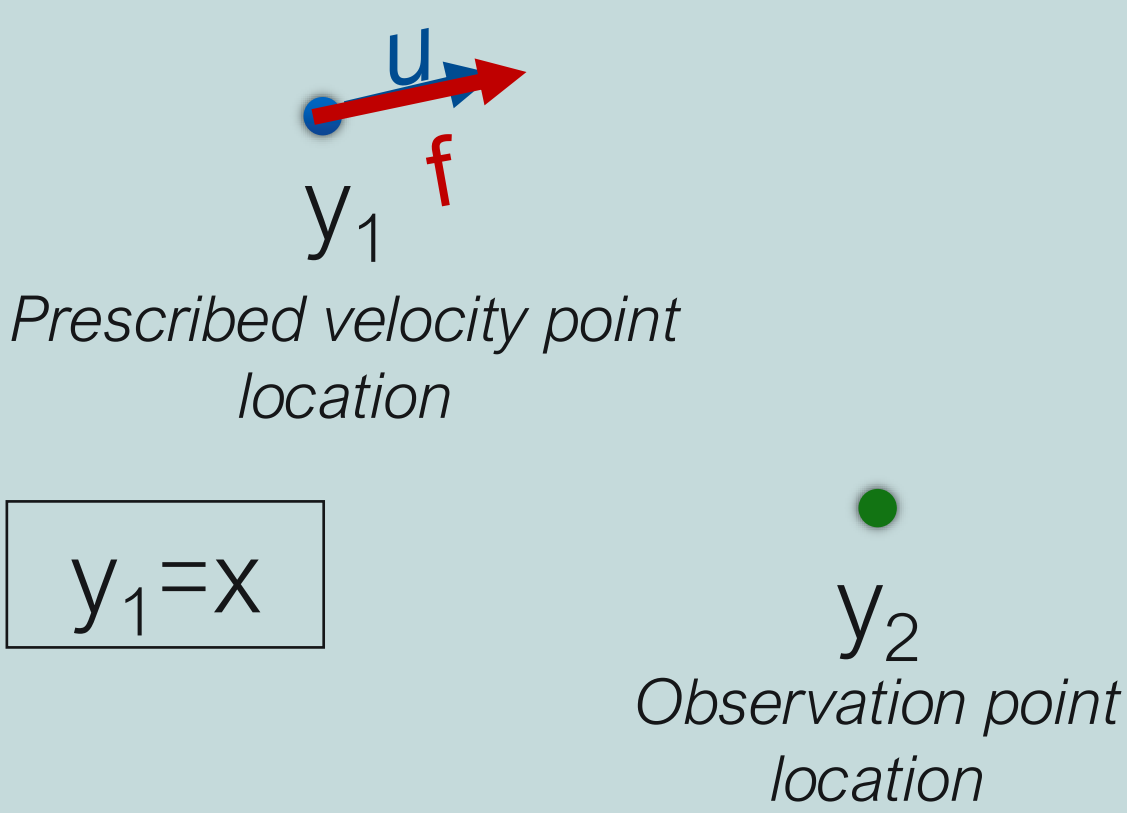


Prescribed velocity point location

Prescribing the Velocity

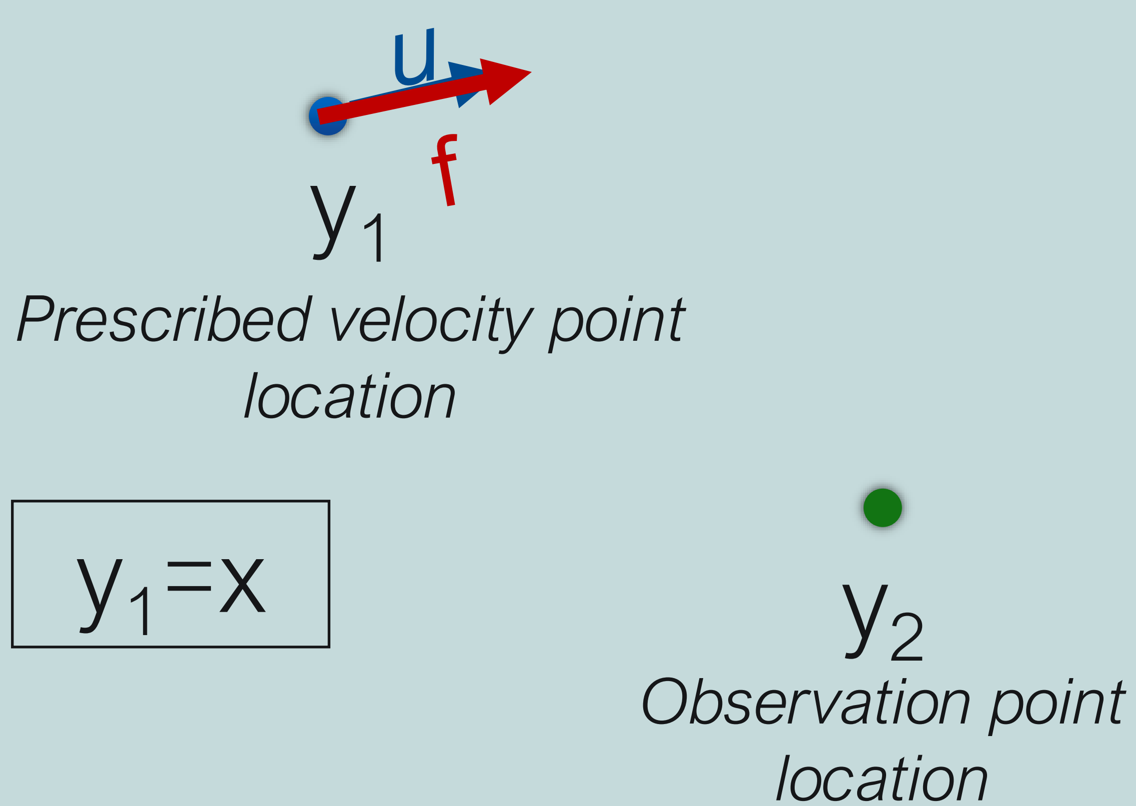
1. Prescribe the velocity $u(y_1)$

   at y_1



$\boxed{y_1=x}$

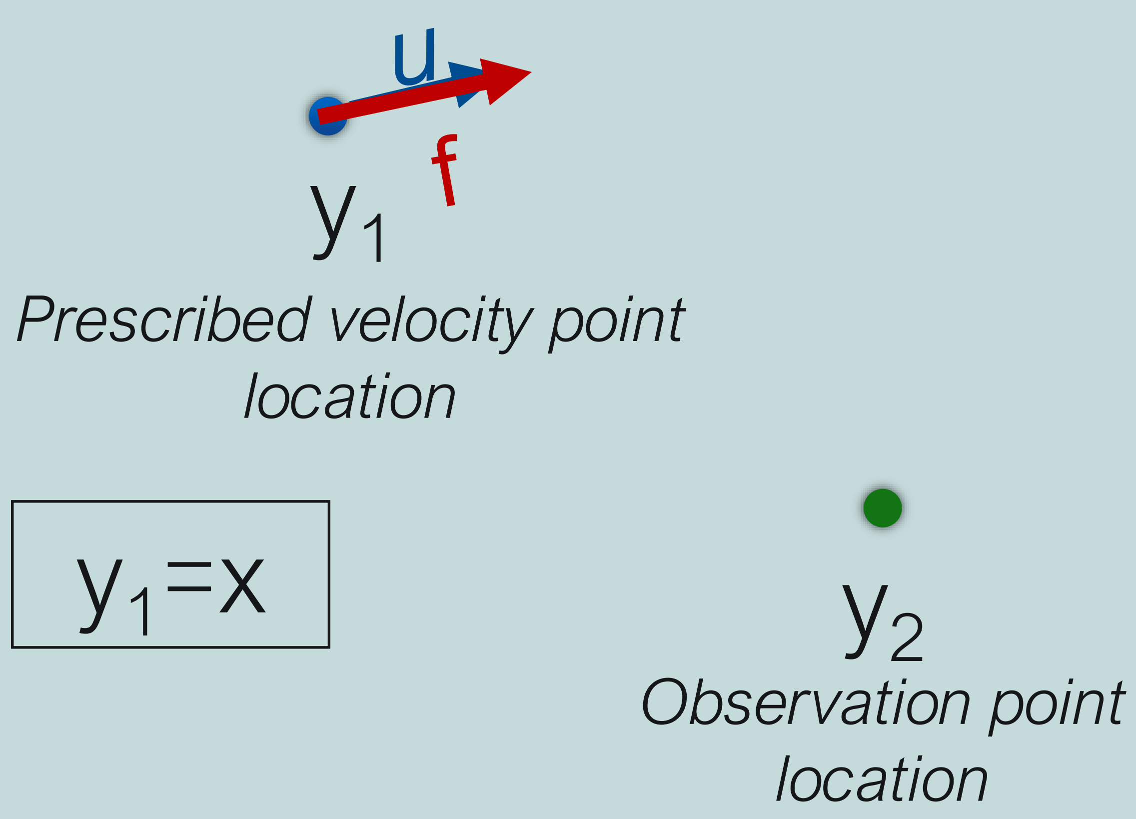
Prescribing the Velocity



1. Prescribe the velocity $u(y_1)$ at y_1
2. Solve $8\pi\mu A(x,x)f = u$ for f using $f = 8\pi\mu A(x,x)\backslash u$

Note that y_1=x because the observation point is the forcing point when prescribing the velocity

Prescribing the Velocity



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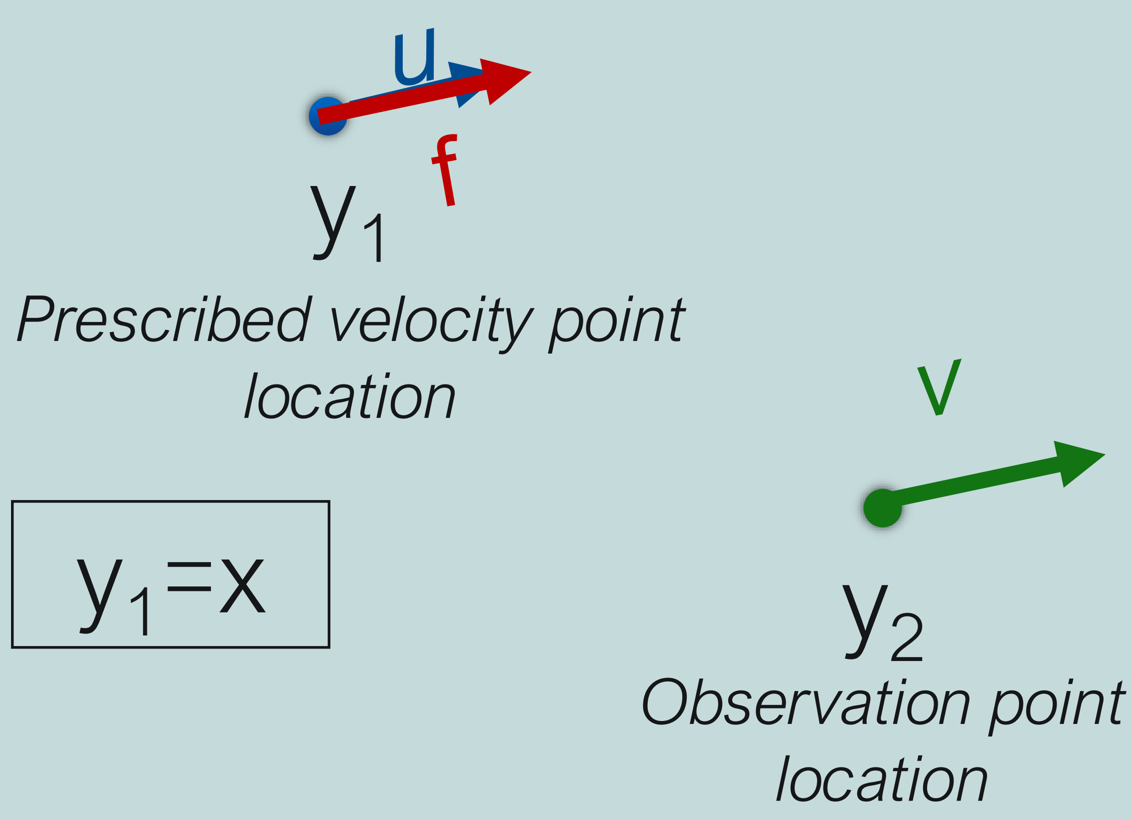
Prescribing the Velocity



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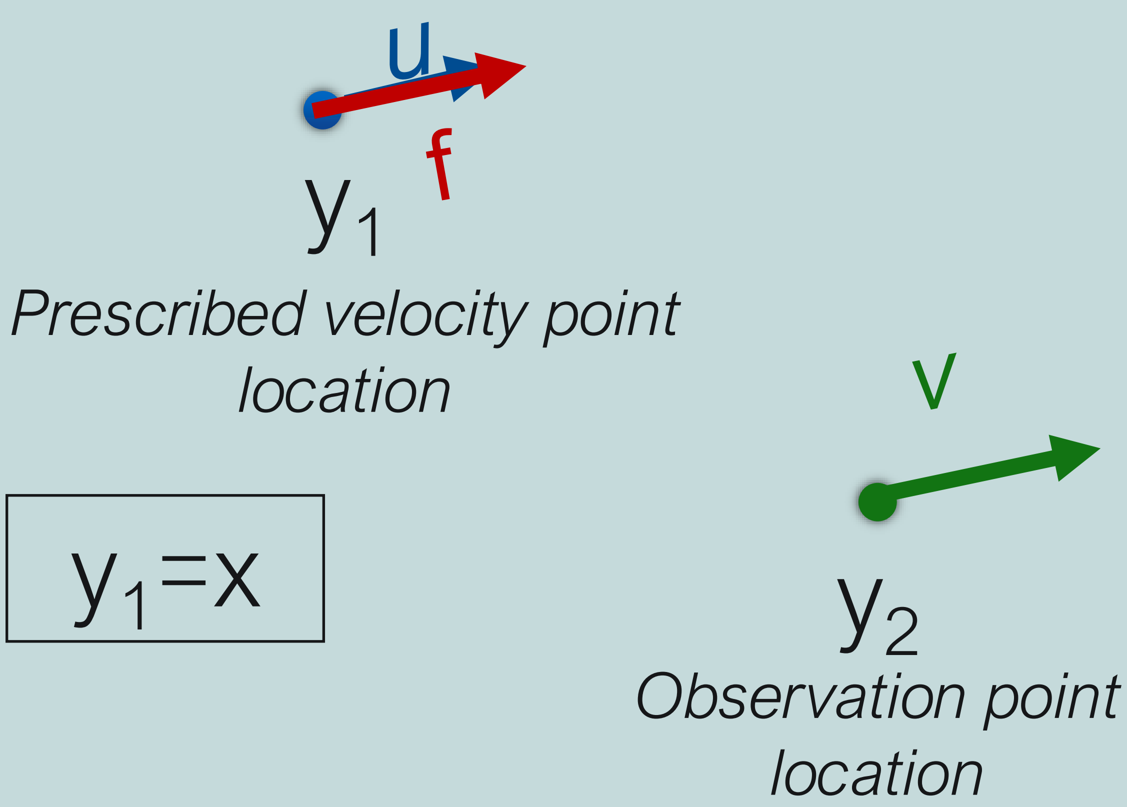
3. Compute $v = 8\pi\mu\, A(y,x)f$ to find the velocity v at y_2

Prescribing the Velocity



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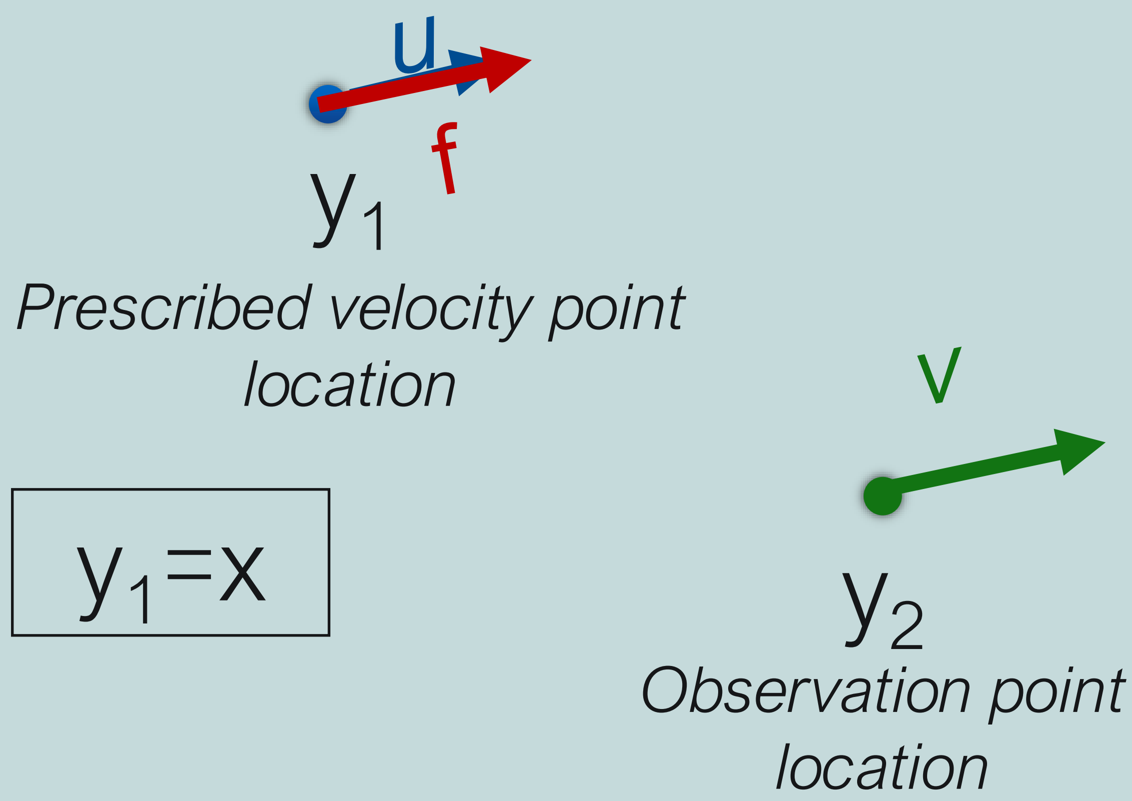
3. Compute $v = 8\pi\mu\, A(y,x)f$ to find the velocity v at y_2
4. Use Forward Euler to update the position of y_2

Prescribing the Velocity



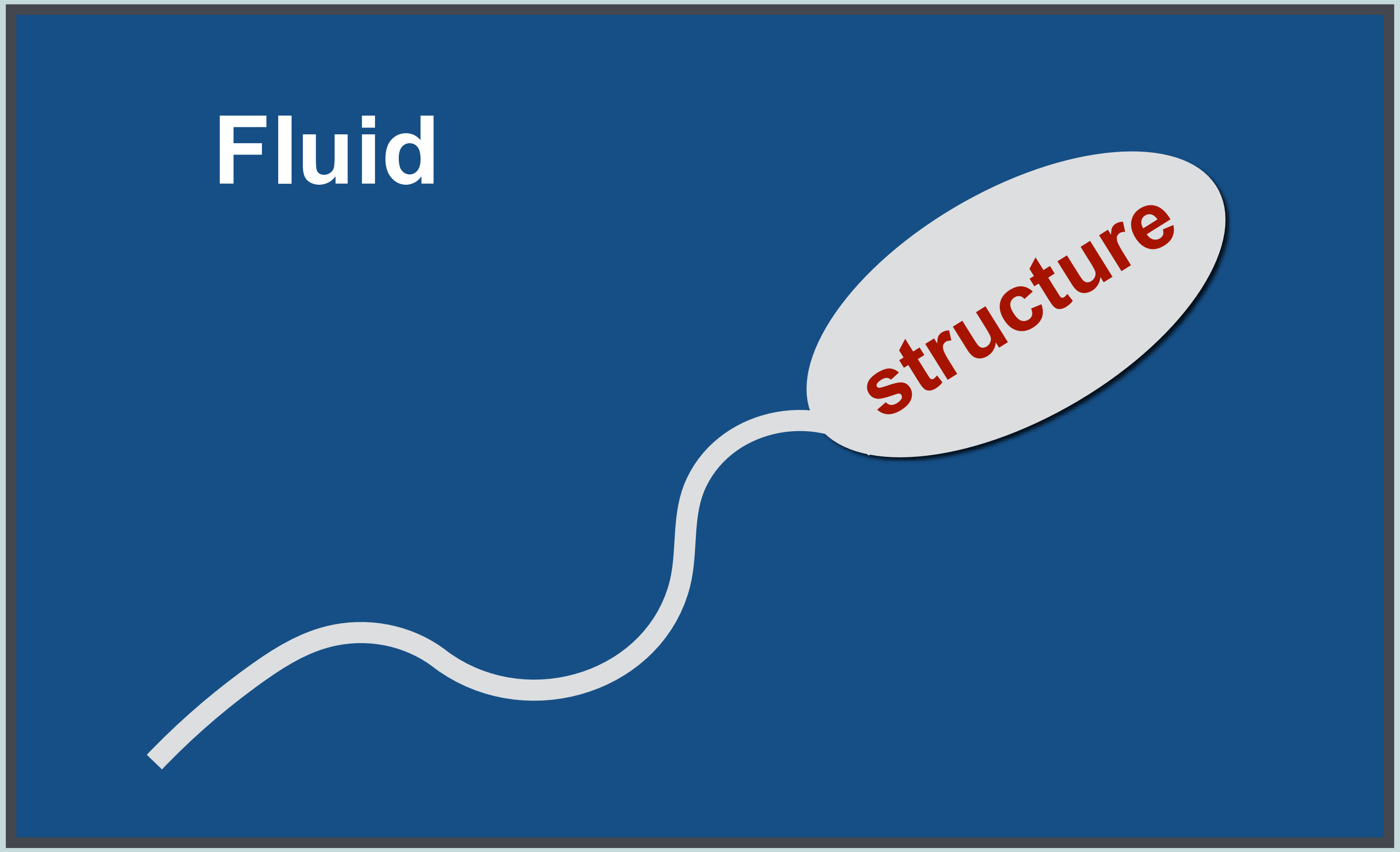
$y_1 = x$

1. Prescribe the velocity $u(y_1)$ at y_1
2. Solve $8\pi\mu A(x,x)f = u$ for f using $f = 8\pi\mu A(x,x)\backslash u$
3. Compute $v = 8\pi\mu\, A(y,x)f$ to find the velocity v at y_2
4. Use Forward Euler to update the position of y_2

Use MATLAB to do Exercise 3

Fluid Structure Interactions

- We need a mathematical framework to describe

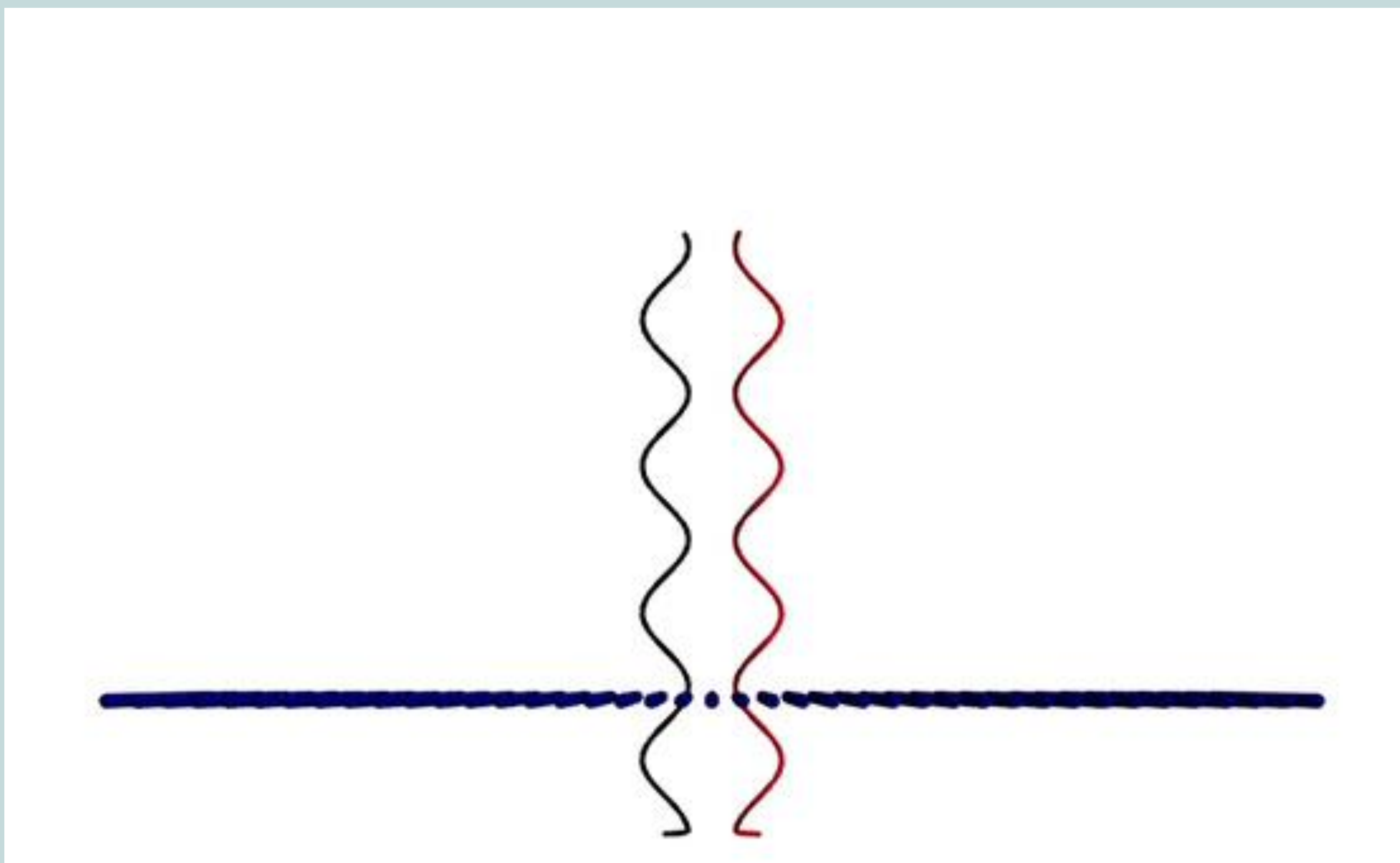


$$\begin{aligned} -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{x}, t) &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

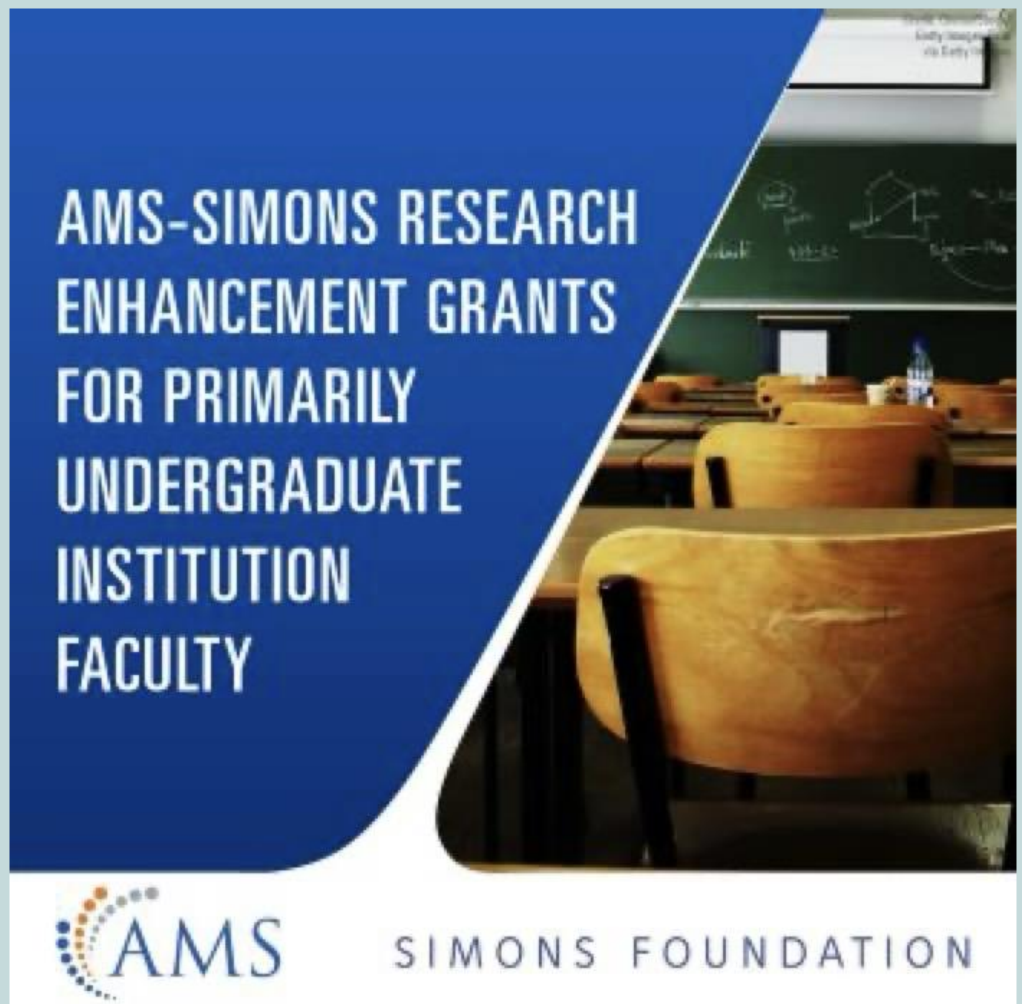
Coming soon…

AN INTRODUCTORY GUIDE TO THE METHOD OF REGULARIZED STOKESLETS: A PRACTICAL IMPLEMENTATION [*]

AMY BUCHMANN[†], MUHAN JI [‡], EVA STRAWBRIDGE [§], MCKENNA WITT [¶], AND LONGHUA ZHAO [‖]



Thank you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