

A Practical Guide to Implementing Regularized Stokeslets for Modelling

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In applications relevant to this tutorial, we are interested in viscous or low Reynolds number (Re) fluids where the inertial effect in the system is negligible. The governing equations for such a problem are

$$\mu \Delta \mathbf{u} = \nabla p - \mathbf{F}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

where \mathbf{u} is the velocity, p is the pressure, \mathbf{F} is the external force density acting on the fluid, μ is the fluid dynamic viscosity. These equations describe the motion of a steady-state, zero Reynolds number, incompressible fluid and are called the Stokes equations.

The solution to the Stokes equations corresponding to the force $\mathbf{F} = \mathbf{f}\phi_\epsilon(\mathbf{y} - \mathbf{x})$ with the blob function

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2 + \epsilon^2)^{7/2}} \quad (3)$$

is

$$\mathbf{u}(\mathbf{y}) = \frac{\mathbf{f}(r^2 + 2\epsilon^2) + [\mathbf{f} \cdot (\mathbf{y} - \mathbf{x})](\mathbf{y} - \mathbf{x})}{8\pi\mu(r^2 + \epsilon^2)^{3/2}}, \quad (4)$$

where $r = |\mathbf{y} - \mathbf{x}|$. In the velocity tensor format, this solution is

$$\mathbf{u}(\mathbf{y}) = \frac{1}{8\pi\mu} A(\mathbf{y}, \mathbf{x}) \mathbf{f} \quad (5)$$

with

$$A(\mathbf{y}, \mathbf{x}) = \begin{bmatrix} a_{11}^\epsilon & a_{12}^\epsilon & a_{13}^\epsilon \\ a_{21}^\epsilon & a_{22}^\epsilon & a_{23}^\epsilon \\ a_{31}^\epsilon & a_{32}^\epsilon & a_{33}^\epsilon \end{bmatrix}. \quad (6)$$

The tensor $A(\mathbf{y}, \mathbf{x})$ depends on the location of the force, $\mathbf{x} = (x_1, x_2, x_3)^T$, the location of the observation point, $\mathbf{y} = (y_1, y_2, y_3)^T$, and the parameter ϵ . In detail,

$$\begin{aligned} a_{11}^\epsilon &= \frac{(r^2 + 2\epsilon^2) + r_1^2}{r_\epsilon^3}; & a_{12}^\epsilon &= \frac{r_1 r_2}{r_\epsilon^3}; & a_{13}^\epsilon &= \frac{r_1 r_3}{r_\epsilon^3}; \\ a_{21}^\epsilon &= \frac{r_2 r_1}{r_\epsilon^3}; & a_{22}^\epsilon &= \frac{(r^2 + 2\epsilon^2) + r_2^2}{r_\epsilon^3}; & a_{23}^\epsilon &= \frac{r_2 r_3}{r_\epsilon^3}; \\ a_{31}^\epsilon &= \frac{r_3 r_1}{r_\epsilon^3}; & a_{32}^\epsilon &= \frac{r_3 r_2}{r_\epsilon^3}; & a_{33}^\epsilon &= \frac{(r^2 + 2\epsilon^2) + r_3^2}{r_\epsilon^3}. \end{aligned} \quad (7)$$

And

$$r_1 = y_1 - x_1, \quad r_2 = y_2 - x_2, \quad r_3 = y_3 - x_3, \quad (8)$$

$$r^2 = r_1^2 + r_2^2 + r_3^2 \quad \text{and} \quad r_\epsilon = \sqrt{r^2 + \epsilon^2}.$$

After computing the velocity of a tracer particle, $\mathbf{v}^0 = \mathbf{v}(\mathbf{y}, 0)$, we can update the position of the tracer particle. Since velocity $\mathbf{v}(\mathbf{y}, t)$ is the derivative of position \mathbf{y} with respect to time t ,

$$\frac{d\mathbf{y}}{dt} = \mathbf{v}(\mathbf{y}, t), \quad (9)$$

so we can integrate the velocity to get the updated position. The forward Euler method for (9) would yield

$$\mathbf{y}^{i+1} = \mathbf{y}^i + \Delta t \mathbf{v}^i, \quad (10)$$

where Δt is the time step for the time discretization.

Exercises

1. Let $\mathbf{f} = (2, 3, -1)^T$ be the force applied at the point $\mathbf{x} = (1, -1, 2)^T$, $\epsilon = 0.05$, and $\mu = 1$. We want to find the resulting velocity $\mathbf{u}(\mathbf{y})$ at the observation point $\mathbf{y} = (2, 0, 1)^T$.

- (a) Modify the RegStokes.m function to assemble the matrix in (6).
- (b) Use (5) to find the resulting velocity at $\mathbf{y} = (2, 0, 1)^T$ by running the MainForce.m function. You can check your answer by removing the semicolon at the end of the following line:

```
u = 1/(8*pi*mu)*A*f;
```

2. Next we will adapt our code to update the position of the tracer particles using Forward Euler with time step $\Delta t = 0.01$.

- (a) Modify the ForwardEuler.m function so that \mathbf{y} is updated using Equation (10).
- (b) Run MainForce.m. Check your solution by removing the semicolons at the end of the following lines:

```
%Update the position with Forward Euler
y = ForwardEuler(y,u,delta_t);
```

```
%Update the time
t = i*delta_t;
```

Because $N = 1$, this will only calculate a single time step and end at time $t = 0.01$. Change the value of N to end at time $t = 0.06$.

- (c) Update your code to include a new parameter, $tFinal$, representing the final time. Redefine N so that it depends on t , $tFinal$ and Δt .
- 3. In this exercise, we prescribe the velocity at this point, so \mathbf{x} is the forcing location as well as the location of the observation point and find the force \mathbf{f} . Let $\mathbf{x} = (-3, 2, 1)^T$ and the prescribed velocity at \mathbf{x} be $\mathbf{u}(\mathbf{x}) = (1, -.5, -1)^T$. Let $\epsilon = 0.05$ and $\mu = 1$.

- (a) In the MainVelocity.m function, modify the line

```
A= RegStokes( , ,epsilon);
```

to write out the 3×3 tensor $A(\mathbf{x}, \mathbf{x})$ defined in (6).

- (b) Solve the system

$$\mathbf{u}(\mathbf{x}) = \frac{1}{8\pi\mu} A(\mathbf{x}, \mathbf{x}) \mathbf{f}$$

to find \mathbf{f} by solving the linear system. *Notice: the location of the force and the observation point are the same.*

- (c) Plug $A(\mathbf{x}, \mathbf{x})$ and \mathbf{f} into Equation (5) to verify that your force results in the desired prescribed velocity.
- (d) *** BONUS *** Let $\mathbf{y} = (1, -2, 3)^T$ be a new observation point. Find the velocity at this point $\mathbf{u}(\mathbf{y})$ resulting from the force you found in part (a) applied at \mathbf{x} .
- (e) ***TAKEHOME*** Now use your ForwardEuler.m function so that the position of forcing point is also updated as it is carried along by its own velocity.

Solutions

1.

$$\mathbf{u}(\mathbf{y}) = \frac{1}{8\pi(3+\epsilon^2)^{3/2}}(12+4\epsilon^2, 15+6\epsilon^2, -9-2\epsilon^2)^T \approx (0.0918, 0.1148, -0.0689)^T,$$

where $\mathbf{y} = (2, 0, 1)$, with $\mathbf{f} = (2, 3, -1)$ at $\mathbf{x} = (1, -1, 2)$.

2. (b) Below are the approximations from Forward Euler.

t	\mathbf{y}_1
0	$(2, 0, 1)^T$
0.01	$(2.000918498864207, 0.001148314774860, 0.999311317046446)^T$
0.02	$(2.001836190618279, 0.002295716638611, 0.998623335402053)^T$
0.03	$(2.002753077333999, 0.003442207810047, 0.997936053142048)^T$
0.04	$(2.003669161074396, 0.004587790498958, 0.997249468350167)^T$
0.05	$(2.004584443893788, 0.005732466906183, 0.996563579118606)^T$
0.06	$(2.005498927837842, 0.006876239223654, 0.995878383547968)^T$

3. (a)

$$A(\mathbf{x}, \mathbf{x}) = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

(b) $\mathbf{f}(\mathbf{x}) \approx (0.6283, -0.3142, -0.6283)^T$ at $\mathbf{x} = (-3, 2, 1)^T$.

(d) $\mathbf{u}(\mathbf{y}) \approx (0.0060, -0.0039, -0.0032)^T$ at $\mathbf{y} = (1, -2, 3)^T$.