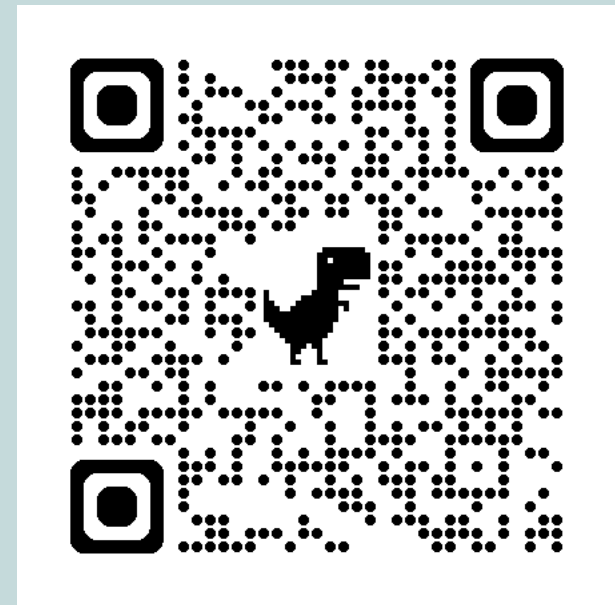


Scan this link to log in to MATLAB

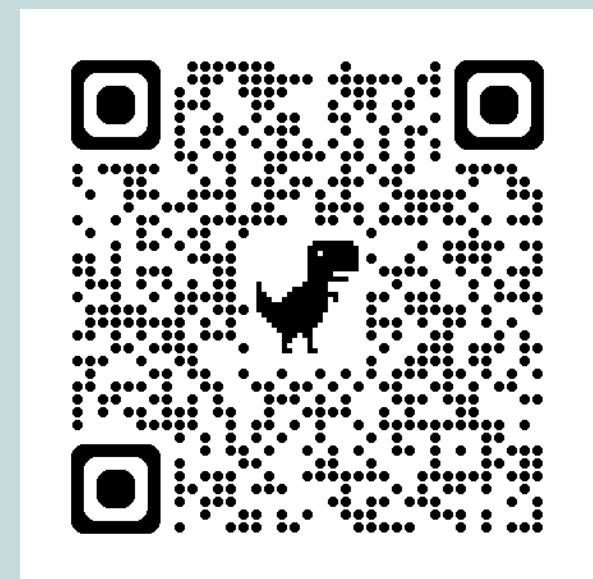
<https://www.mathworks.com/products/matlab.html>



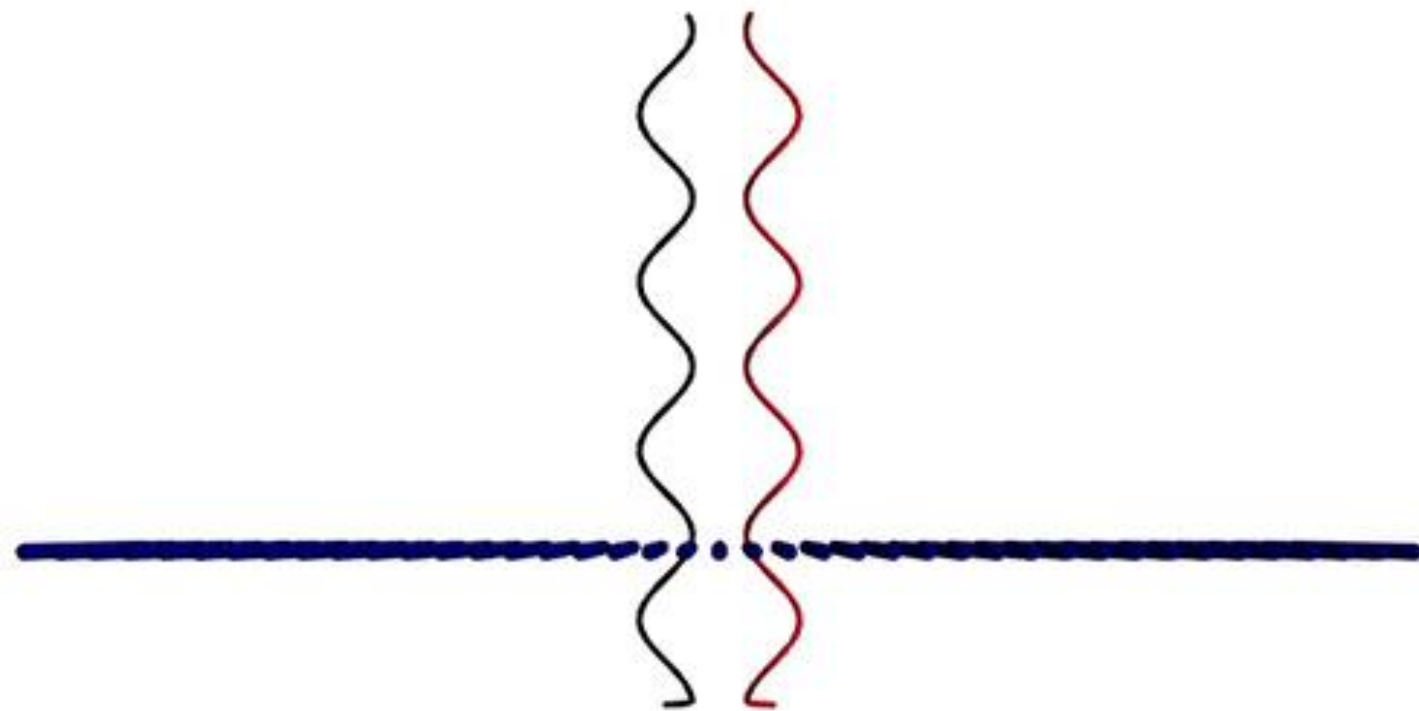
A Practical Guide to Implementing Regularized Stokeslets for Modelling

Scan this link to access the workshop
materials

https://github.com/longhuazhao/MRS_IntroductoryGuide/tree/main



A Practical Guide to Implementing Regularized Stokeslets for Modelling



Amy Buchmann
University of San
Diego

Eva Strawbridge
James Madison
University

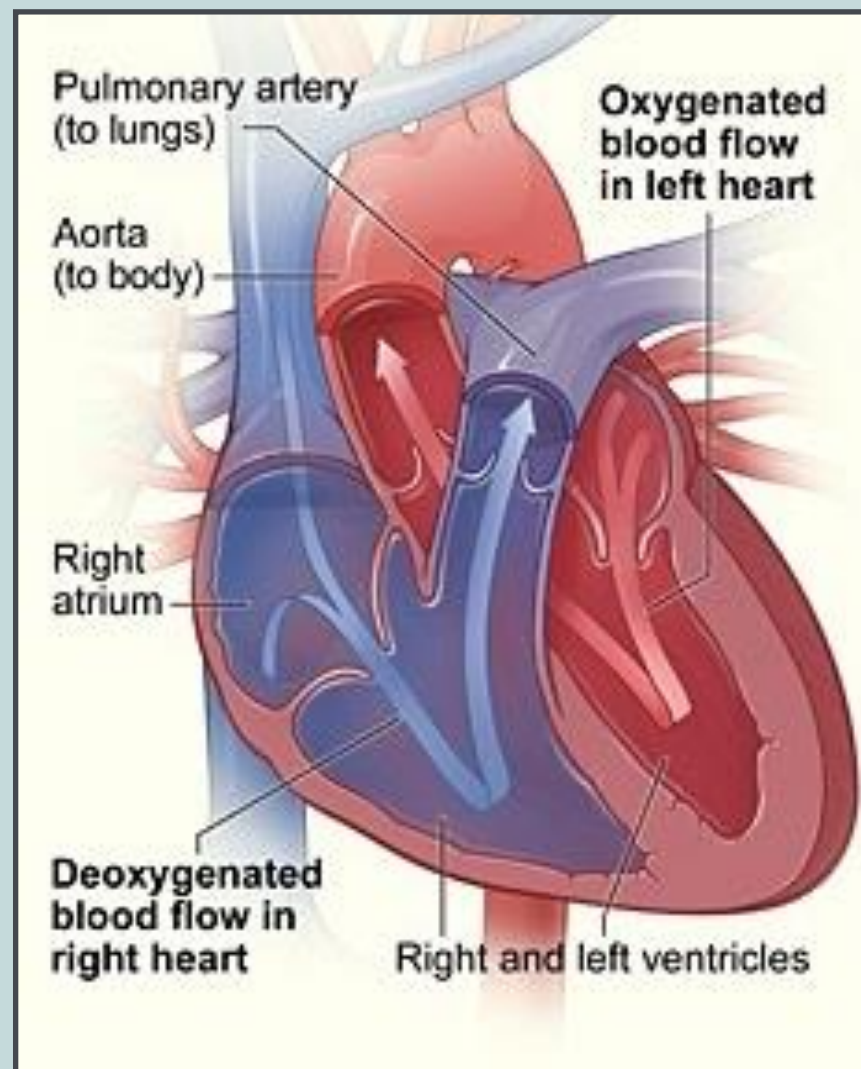
Longhua Zhao
Case Western
Reserve University

Goals

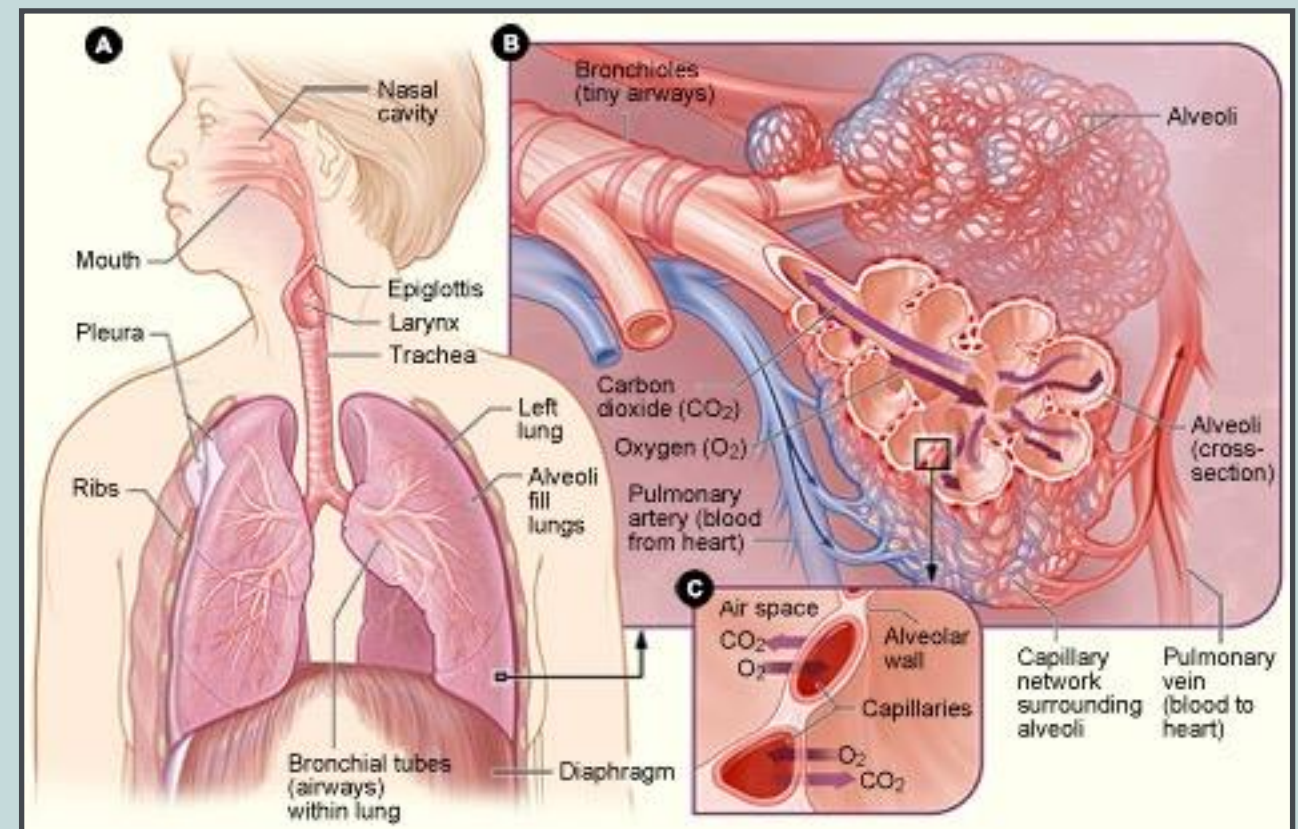
- ◆ To demonstrate how mathematics can be used to enhance our understanding of the world around us.
- ◆ In particular, concepts from undergraduate math courses can be used to study microscale biological fluid dynamics
 - Single Variable Calculus
 - Multivariable Calculus
 - Differential Equations
 - Linear Algebra

Biological Fluid Dynamics

Blood flow in the heart

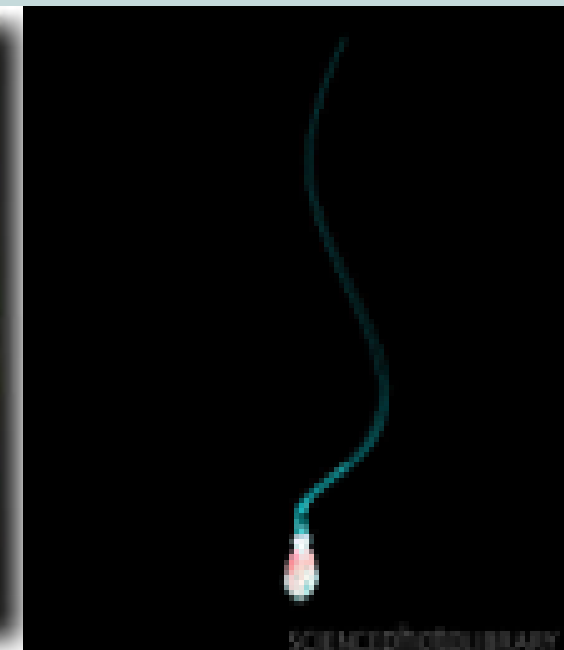
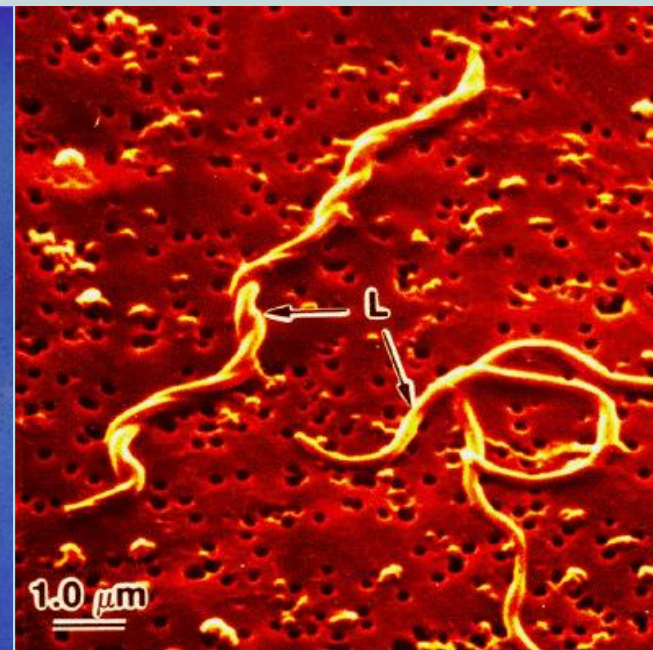
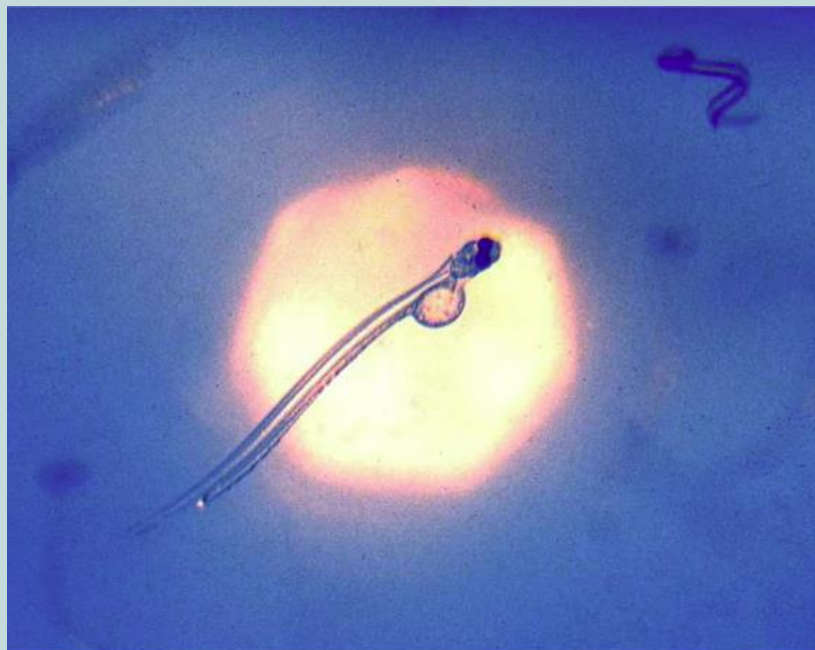


Airflow in the lungs



Microscale Biological Fluid Dynamics

- ♦ Bacteria ($\sim 25\ \mu\text{m}$)
 - How do helical flagella mix and pump fluid?
- ♦ Cilia ($\sim 10\ \mu\text{m}$)
 - How do cilia coordinate their movement to form a metachronal wave?



Fluid Structure Interactions

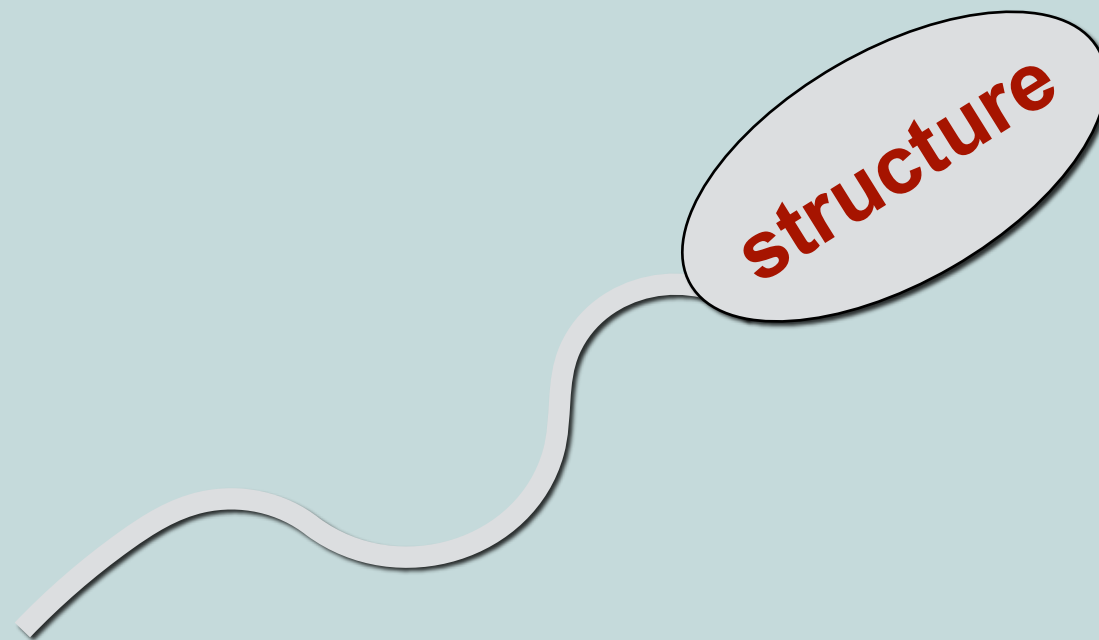
- ✦ We need a mathematical framework to describe



Fluid

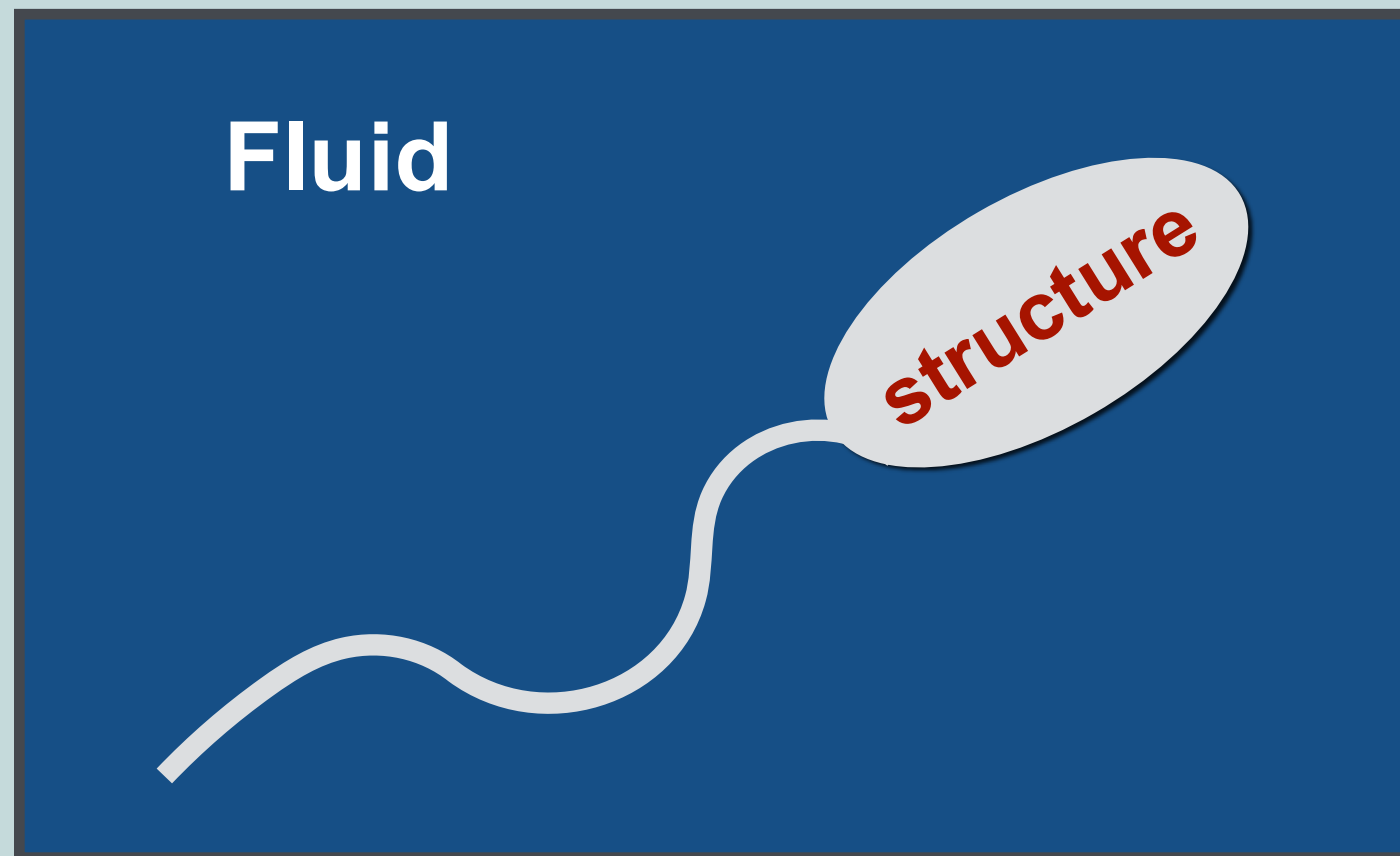
Fluid Structure Interactions

- ✦ We need a mathematical framework to describe



Fluid Structure Interactions

- ✦ We need a mathematical framework to describe



Calculus Concepts: Derivatives!

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- ♦ Derivatives: rates of change (single variable)

$$\frac{d}{dx} [\sin(x)] = \cos(x)$$

Calculus Concepts: Derivatives!

- ◆ Derivatives: rates of change (single variable)

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- ◆ Differential Equations: equations with derivatives

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

Calculus Concepts: Derivatives!

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- ◆ Partial Derivatives: rates of change (multivariable)

$$\frac{\partial}{\partial y} [2x^2y^3] = 6x^2y^2$$

Calculus Concepts: Derivatives!

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- ◆ Partial Derivatives: rates of change (multivariable)

$$\frac{\partial}{\partial y} [2x^2y^3] = 6x^2y^2$$

- ◆ Partial Differential Equations: equations with partial derivatives

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Fluid: Navier Stokes Equations

- ◆ The motion of liquids is well described by the Navier Stokes Equations

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u}$$

“Force Balance”

$$\nabla \cdot \mathbf{u} = 0$$

“Conservation of Mass”

ρ : fluid density

\mathbf{u} : fluid velocity

p : fluid pressure

μ : fluid viscosity

[ABOUT](#)[PROGRAMS](#)[MILLENNIUM PROBLEMS](#)[PEOPLE](#)[PUBLICATIONS](#)[EVENTS](#)[EUCLID](#)

Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.


Image: Sir George Gabriel Stokes (13 August 1819–1 February 1903). [Public Domain](#)

This problem is: Unsolved

Rules:

[Rules for the Millennium Prizes](#)

Related Documents:

 [Official Problem Description](#)

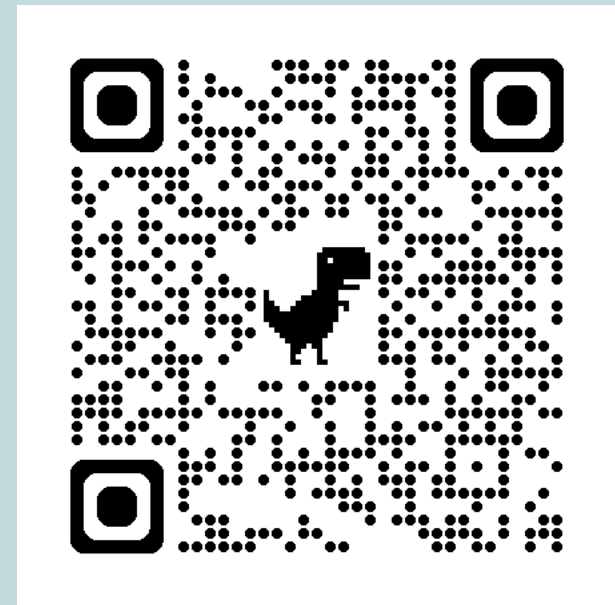
Related Links:

[Lecture by Luis Cafarelli](#)

MATLAB Overview

Scan this link to log in to MATLAB

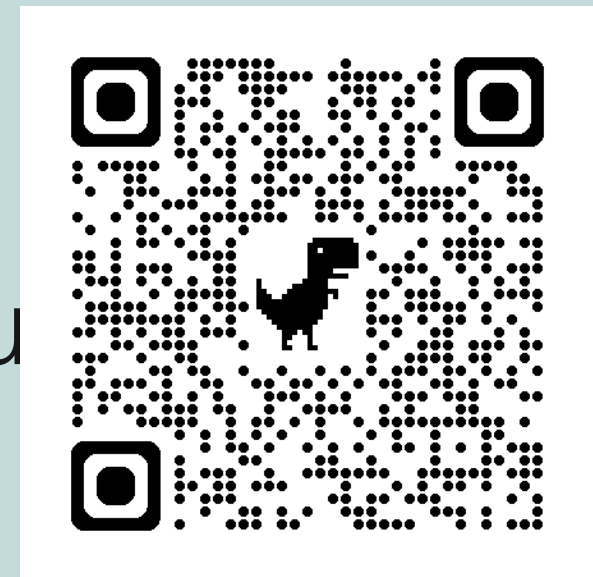
<https://www.mathworks.com/products/matlab.html>



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Open MATLABOverview.pdf

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ρ : fluid density

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Non-dimensionalization

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Left hand side

Reynolds
Number

$$\boxed{\frac{\rho L U}{\mu}} \left(\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} \right)$$

Right hand side

$$= \left(- \frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \right)$$

Non-dimensionalization

$$\begin{aligned}\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p + \mu \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

Introducing the non-dimensional variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{L} \quad \mathbf{u}^* = \frac{\mathbf{u}}{U} \quad p^* = \frac{pL}{\mu U} \quad t^* = t \frac{U}{L}$$

this becomes

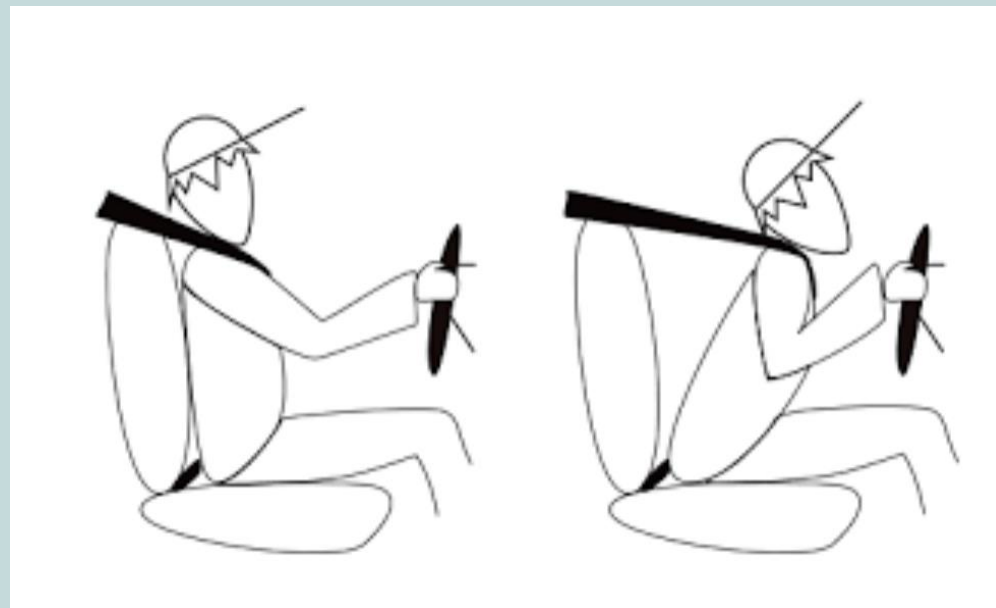
$$\begin{aligned}Re \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* \right) &= -\nabla p^* + \nabla^2 \mathbf{u}^* \\ \nabla \cdot \mathbf{u}^* &= 0\end{aligned}$$

Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

Reynolds Number

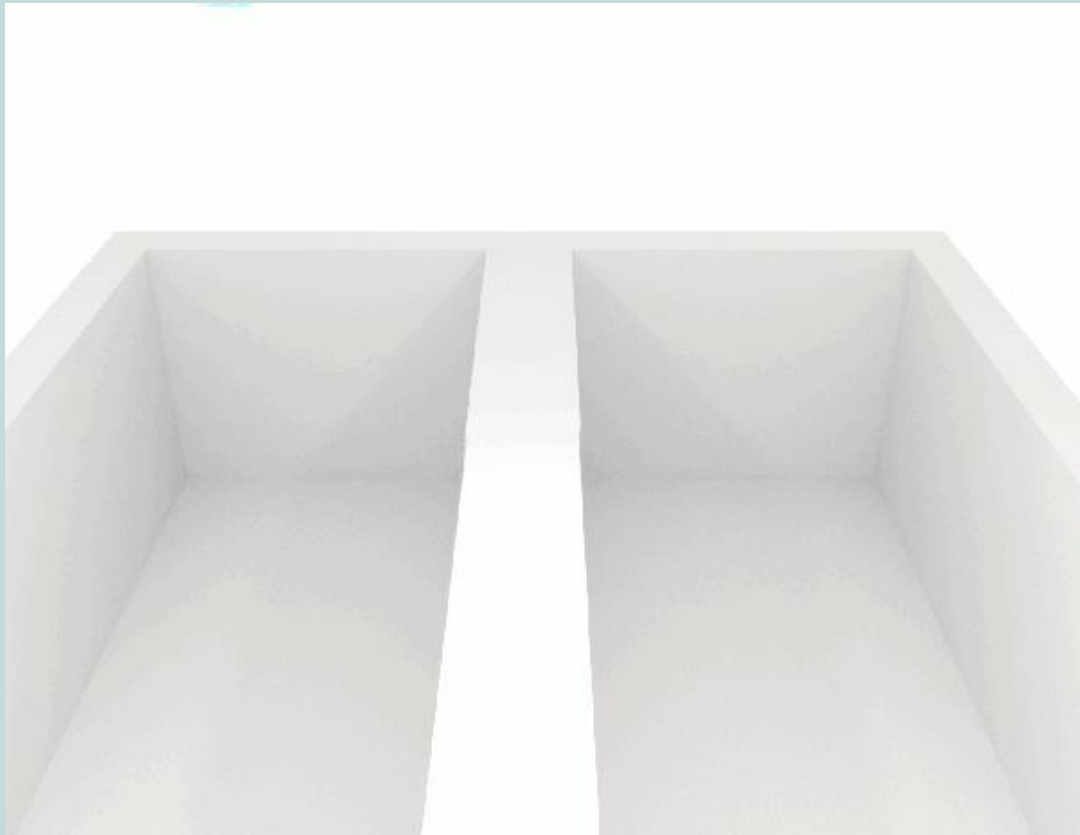
$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



INERTIA: a property of matter by which it ***continues in its existing state*** of rest or uniform motion in a straight line, unless that state is changed by an external force.

Reynolds Number

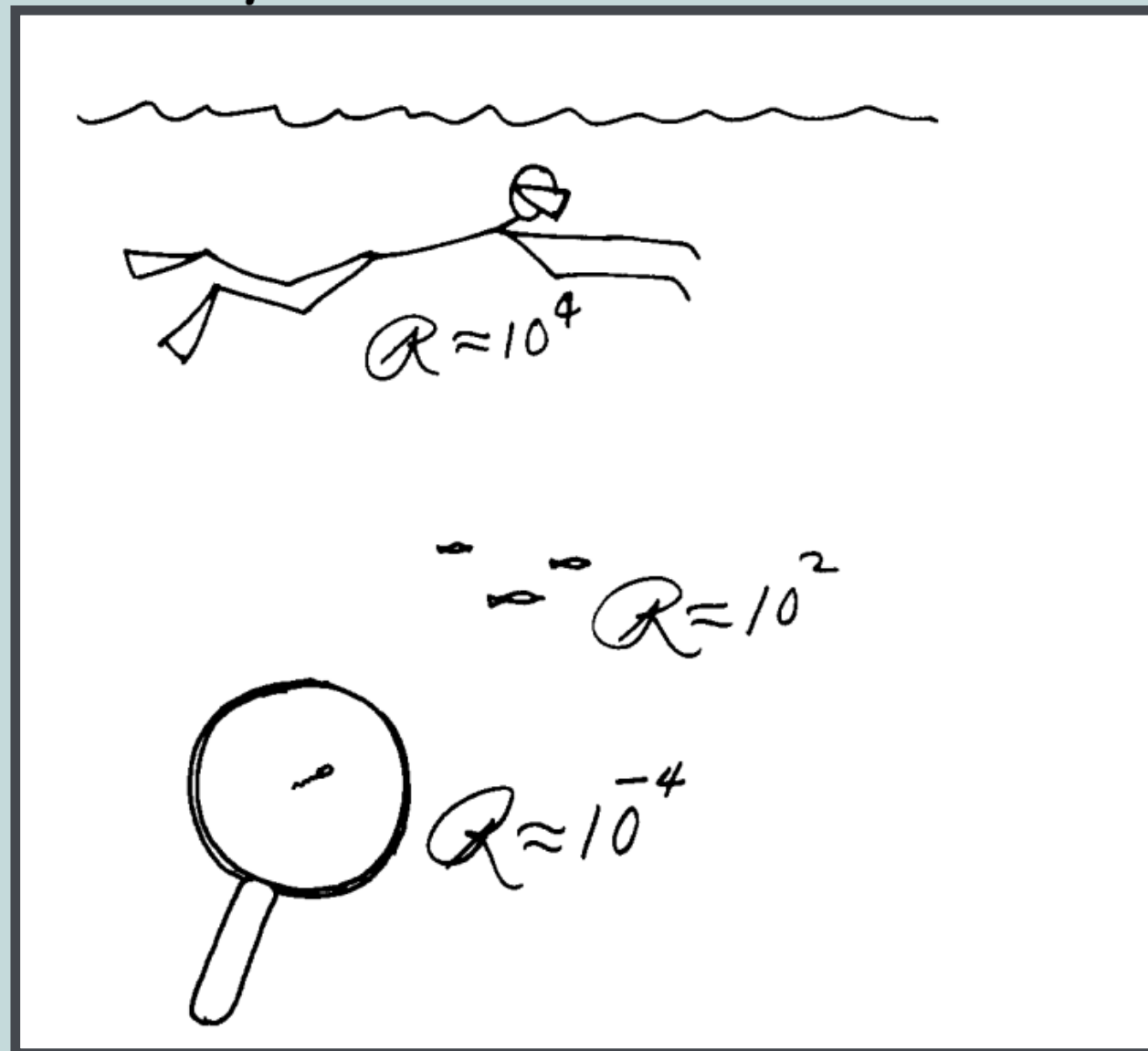
$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



VISCOSITY: The viscosity of a fluid is a measure of its **resistance to deformation** at a given rate.

Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$



Intermediate Reynolds Number



G.I. Taylor

Reynolds Number

$$Re = \frac{\rho U L}{\mu} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}}$$

- ♦ High Reynolds Number - Inertial Forces Dominate
- ♦ Low Reynolds Number - Viscous Forces Dominate

Low Reynolds Number



G.I. Taylor

Non-dimensionalization

$$\begin{aligned}\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p + \mu \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

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$$\mathbf{x}^* = \frac{\mathbf{x}}{L} \quad \mathbf{u}^* = \frac{\mathbf{u}}{U} \quad p^* = \frac{pL}{\mu U} \quad t^* = t \frac{U}{L}$$

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Fluid Structure Interactions

- ✦ We need a mathematical framework to describe

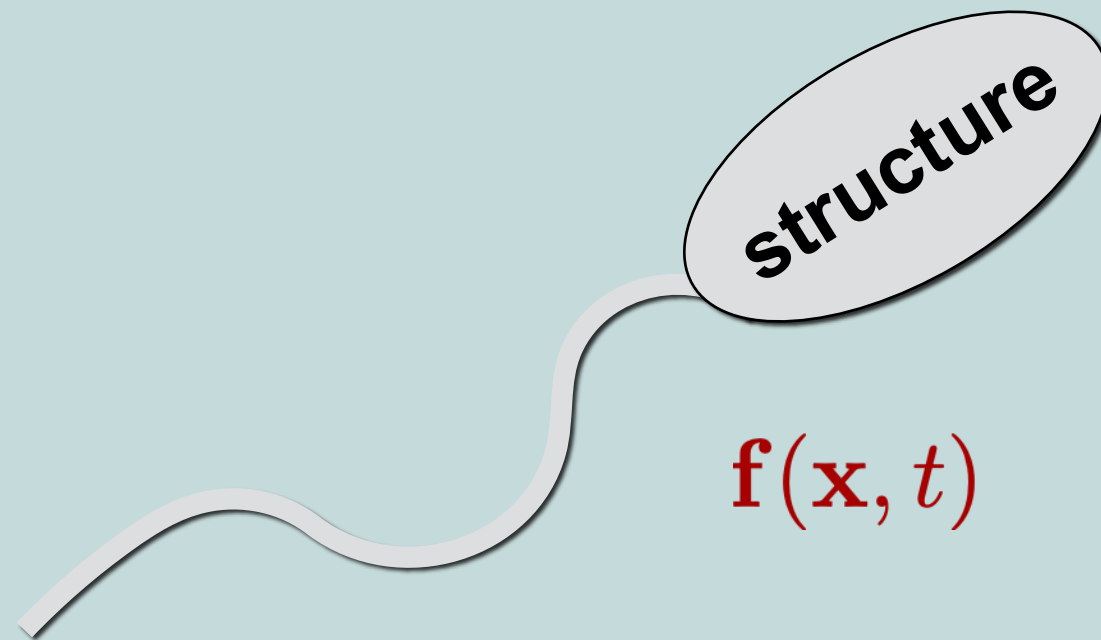


Fluid

$$\begin{aligned} -\nabla p + \mu \nabla^2 \mathbf{u} &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

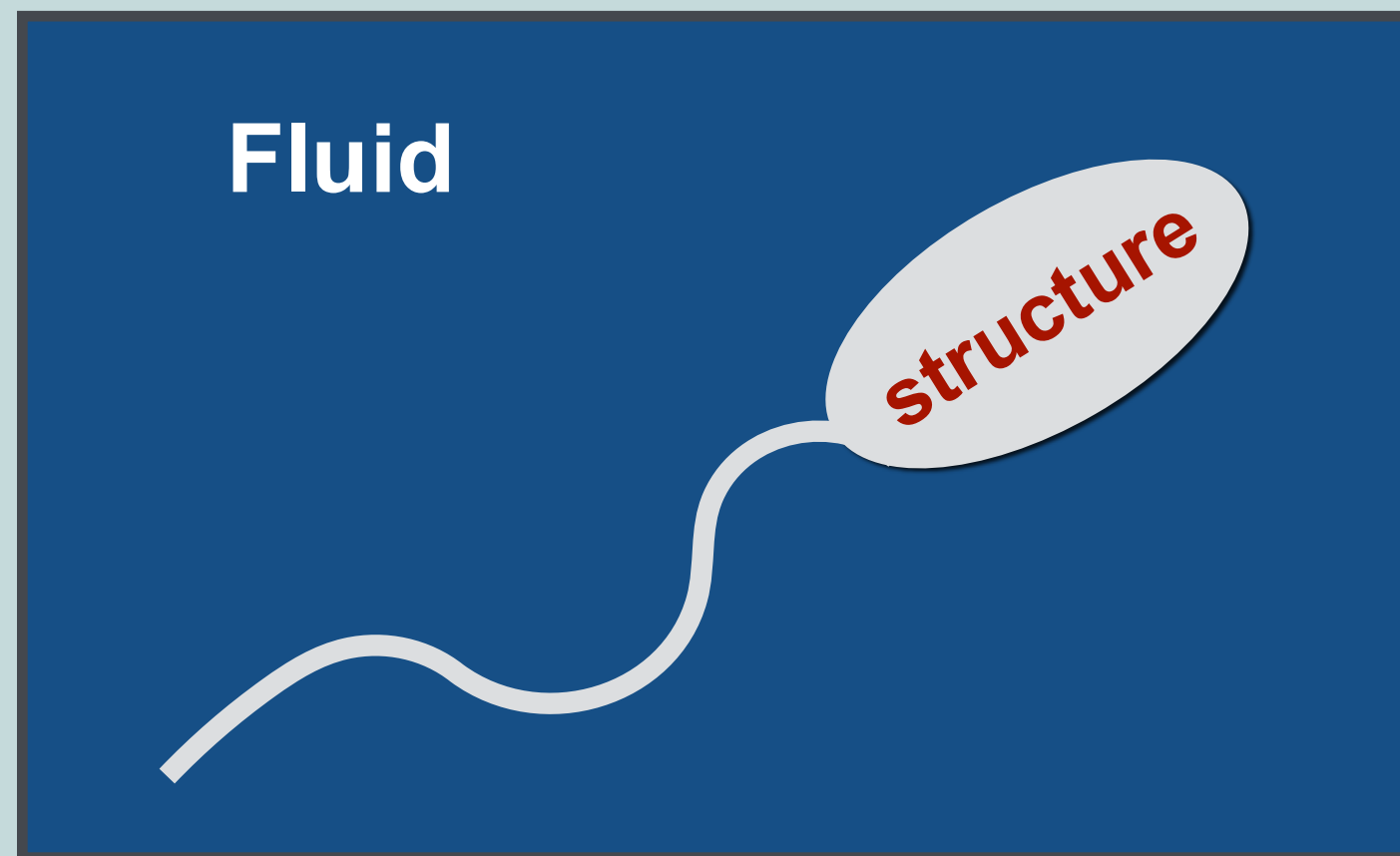
Fluid Structure Interactions

- ◆ We need a mathematical framework to describe



Fluid Structure Interactions

- ✦ We need a mathematical framework to describe



$$\begin{aligned} -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{x}, t) &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

A force at a point?

- ◆ Dirac delta distribution

$$\int_{-\infty}^{\infty} f(y) \delta(y - x_0) dy = f(x_0)$$

- ◆ Fundamental solutions

$$\begin{aligned} \mu \nabla^2 \mathbf{u} + \mathbf{f}_0 \delta(\mathbf{x} - \mathbf{x}_0) &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$



A force at a point?

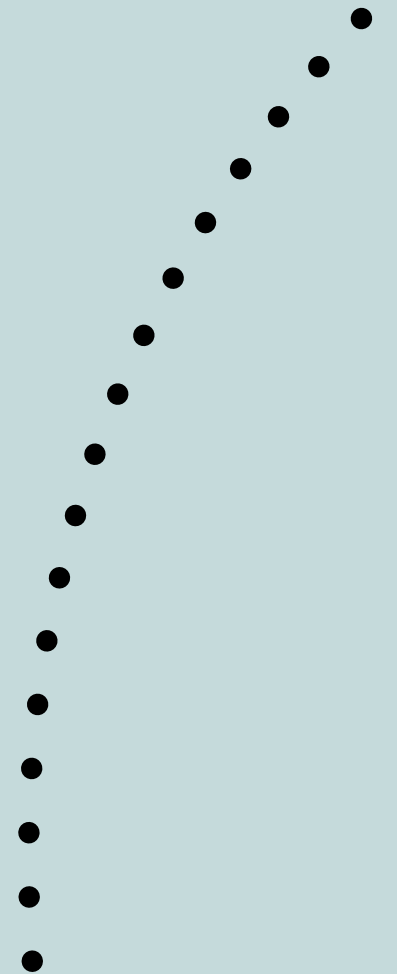
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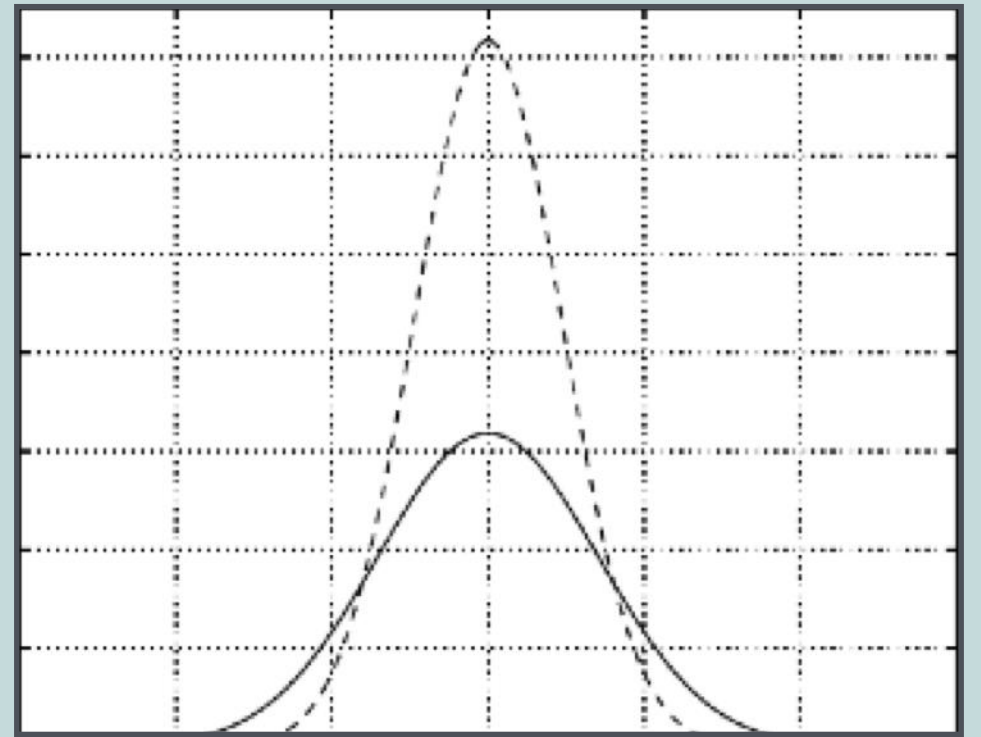
$$\mu \nabla^2 \mathbf{u} + \mathbf{f}_0 \delta(\mathbf{x} - \mathbf{x}_0) = \nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$



Method of regularized Stokeslets

Big Idea: Develop a regularized version of the fundamental solutions by designing smooth approximations of the Dirac delta distribution



$$\begin{aligned}\mu \nabla^2 \mathbf{u} + \mathbf{f}_0 \phi_\epsilon(\hat{\mathbf{x}}) &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

- $\int \phi_\epsilon(\mathbf{x}) d\mathbf{x} = 1$
- $\phi_\epsilon(\mathbf{x}) = \phi_\epsilon(|\mathbf{x}|)$
- concentrated near $\mathbf{x} = \mathbf{0}$

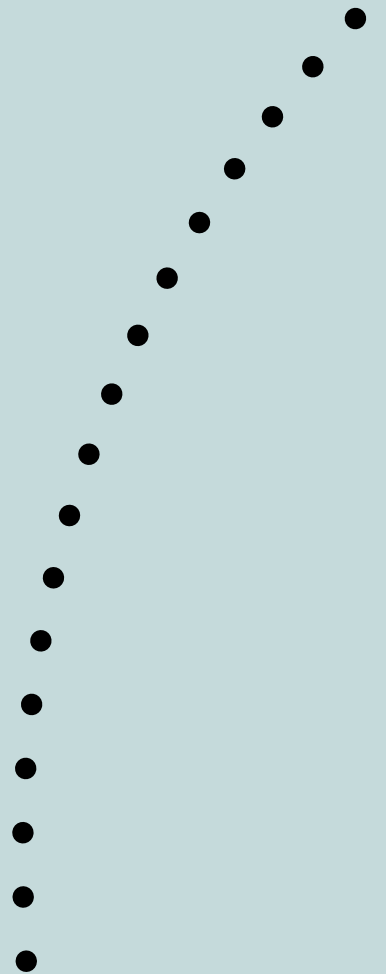
Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$



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$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$



Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2 + \epsilon^2)^{7/2}}$$



Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2 + \epsilon^2)^{7/2}}$$

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^N \left[\frac{\mathbf{f}_k(r_k^2 + 2\epsilon^2) + (\mathbf{f}_k \cdot (\mathbf{x} - \mathbf{x}_k))(\mathbf{x} - \mathbf{x}_k)}{8\pi\mu(r_k^2 + \epsilon^2)^{3/2}} \right]$$

Method of regularized Stokeslets

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$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$U = AF$$



Method of regularized Stokeslets

$$\begin{aligned}\mu\Delta\mathbf{u} + \mathbf{F} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

$$\mathbf{F}_k(\mathbf{x}) = \mathbf{f}_k \phi_\epsilon(||\mathbf{x} - \mathbf{x}_k||)$$

$$U = AF$$

We will code up
an example of
this today

Updating particle position with a prescribed force

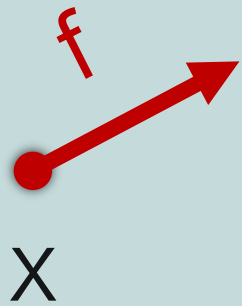


X

Forcing point location

Updating particle position with a prescribed force

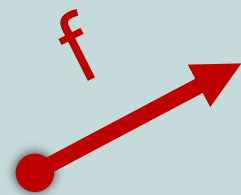
1. Set the force \mathbf{f} at \mathbf{x}



Forcing point location

Updating particle position with a prescribed force

1. Set the force \mathbf{f} at \mathbf{x}



\mathbf{x}

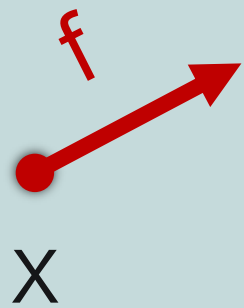
Forcing point location



\mathbf{y}

*Observation point
location*

Updating particle position with a prescribed force



Forcing point location

$$u = \frac{1}{8\pi\mu} A(y, x) f$$

Forward Euler to
update the position

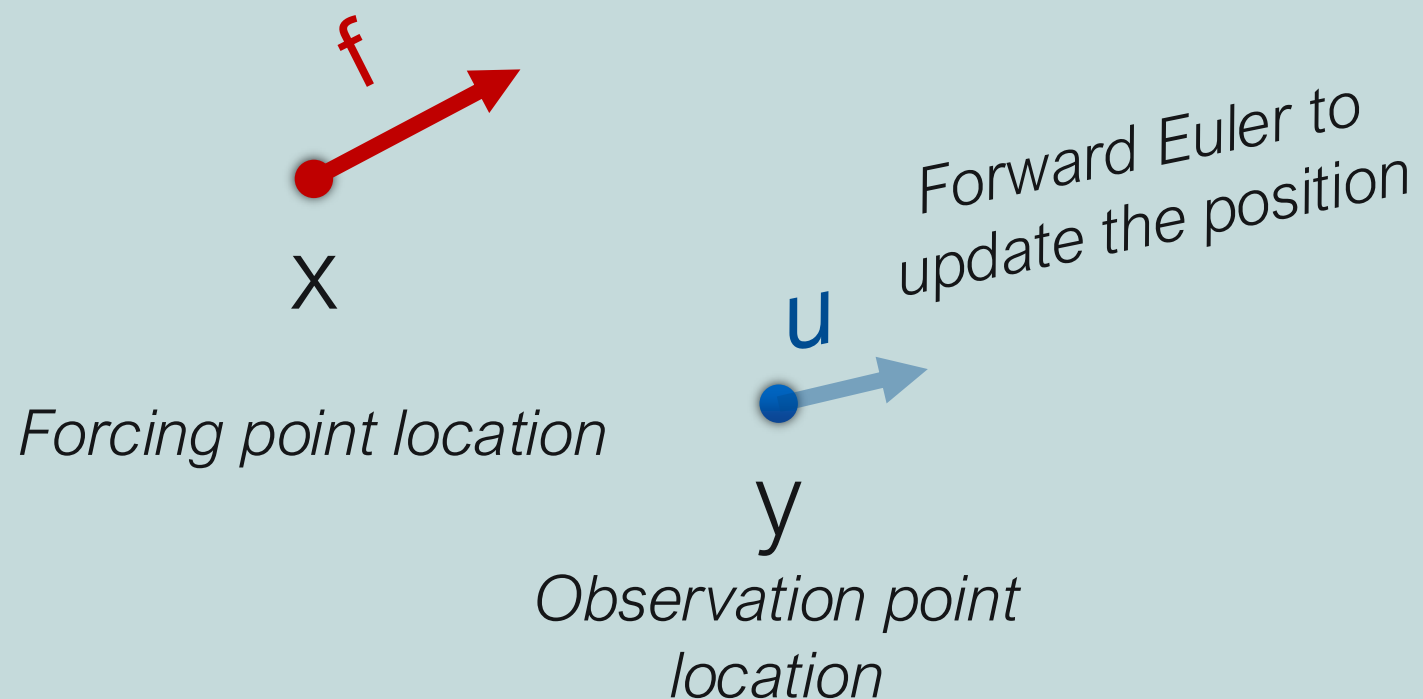


Observation point
location

1. Set the force f at x

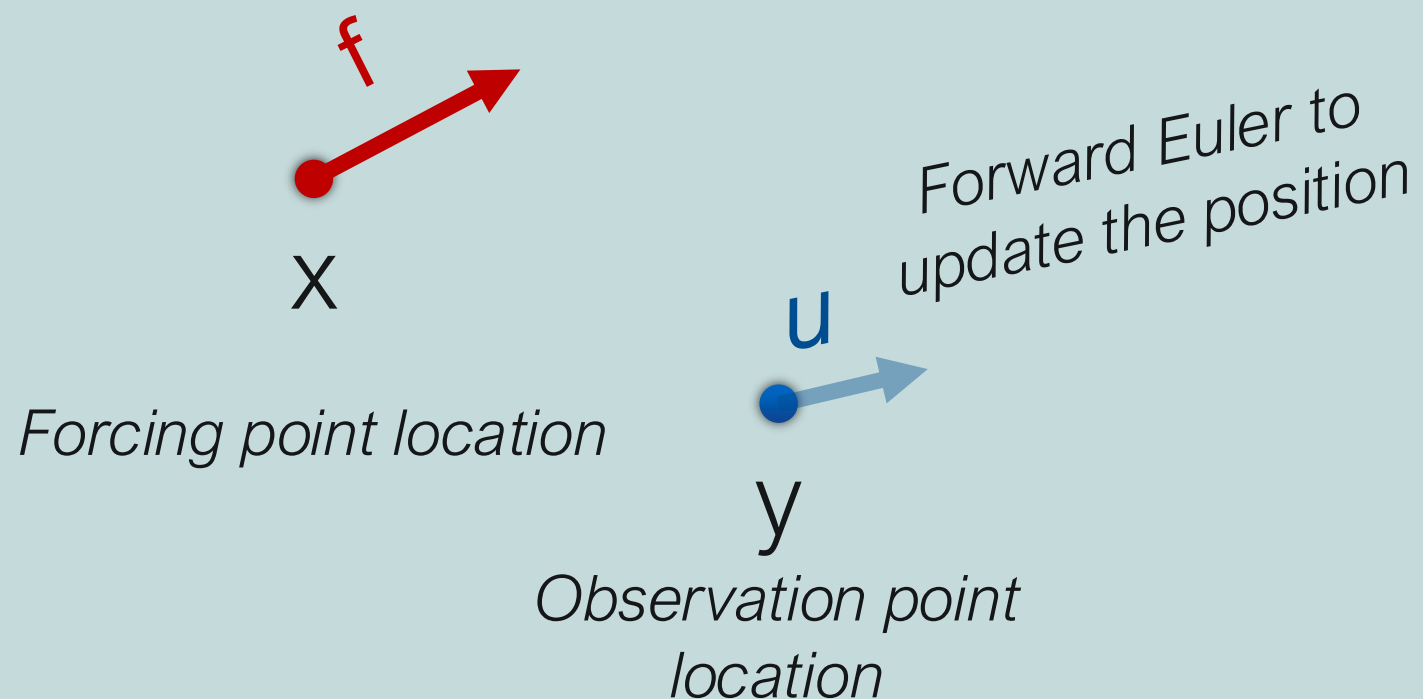
2. Compute $u = \frac{1}{8\pi\mu} A(y, x) f$
to find the velocity u at y

Updating particle position with a prescribed force



1. Set the force **f** at **x**
2. Compute $\mathbf{u} = \frac{1}{8\pi\mu} A(\mathbf{y}, \mathbf{x}) \mathbf{f}$ to find the velocity **u** at **y**
3. Use Forward Euler to update the position of **y**

Updating particle position with a prescribed force



1. Set the force \mathbf{f} at \mathbf{x}
2. Compute $\mathbf{u} = \frac{1}{8\pi\mu} A(\mathbf{y}, \mathbf{x}) \mathbf{f}$ to find the velocity \mathbf{u} at \mathbf{y}
3. Use Forward Euler to update the position of \mathbf{y}



Open brief_MRS.pdf
Use MATLAB to do Exercises 1 and 2



Prescribing the Velocity

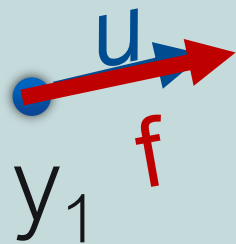


y_1

*Prescribed velocity point
location*

Prescribing the Velocity

1. Prescribe the velocity $u(y_1)$
at y_1



*Prescribed velocity point
location*

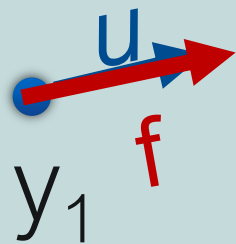
$$y_1 = x$$



y_2
*Observation point
location*

Prescribing the Velocity

1. Prescribe the velocity $u(y_1)$ at y_1



Prescribed velocity point location

2. Solve $\frac{1}{8\pi\mu}A(x,x)f = u$ for f

using $f = \frac{1}{8\pi\mu}A(x,x)u$

$$y_1 = x$$

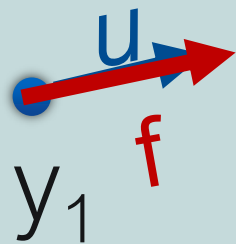


y_2
Observation point location

Note that $y_1 = x$ because the observation point is the forcing point when prescribing the velocity

Prescribing the Velocity

1. Prescribe the velocity $u(y_1)$ at y_1



Prescribed velocity point location

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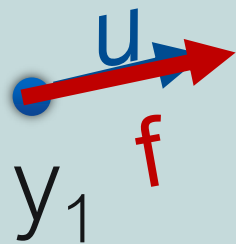
$$y_1 = x$$



y_2
Observation point location

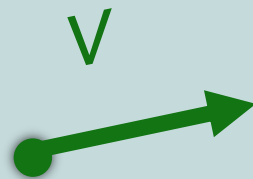
Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$ at y_1



Prescribed velocity point location

$$y_1 = x$$

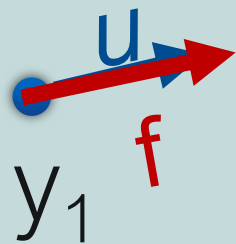


y_2
Observation point location

2. Solve $\frac{1}{8\pi\mu}A(x,x)\mathbf{f} = \mathbf{u}$ for \mathbf{f}
using $\mathbf{f} = \frac{1}{8\pi\mu}A(x,x)\backslash\mathbf{u}$
3. Compute $\mathbf{v} = \frac{1}{8\pi\mu}A(y,x)\mathbf{f}$ to find the velocity \mathbf{v} at y_2

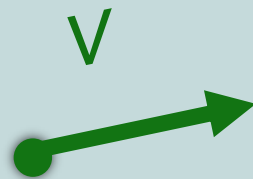
Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$ at y_1



Prescribed velocity point location

$$y_1 = x$$

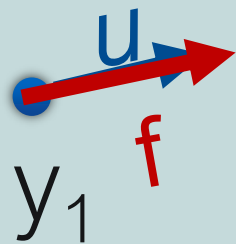


y_2
Observation point location

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4. Use Forward Euler to update the position of y_2

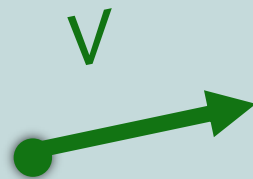
Prescribing the Velocity

1. Prescribe the velocity $\mathbf{u}(y_1)$ at y_1



Prescribed velocity point location

$$y_1 = x$$



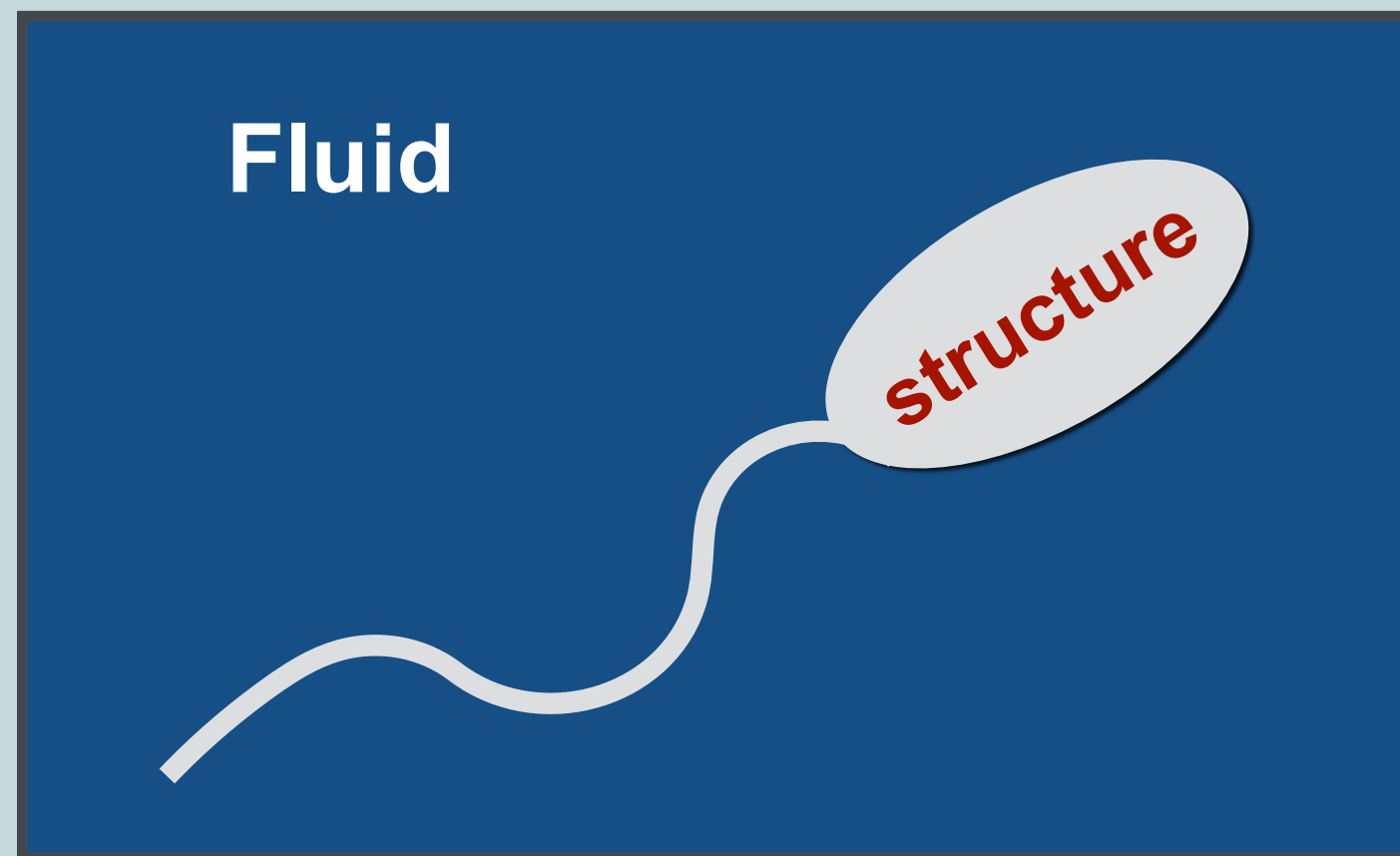
Observation point location

Use MATLAB to do Exercise 3

2. Solve $\frac{1}{8\pi\mu}A(x,x)\mathbf{f} = \mathbf{u}$ for \mathbf{f} using $\mathbf{f} = \frac{1}{8\pi\mu}A(x,x) \backslash \mathbf{u}$
3. Compute $\mathbf{v} = \frac{1}{8\pi\mu}A(y,x)\mathbf{f}$ to find the velocity \mathbf{v} at y_2
4. Use Forward Euler to update the position of y_2

Fluid Structure Interactions

- ✦ We need a mathematical framework to describe

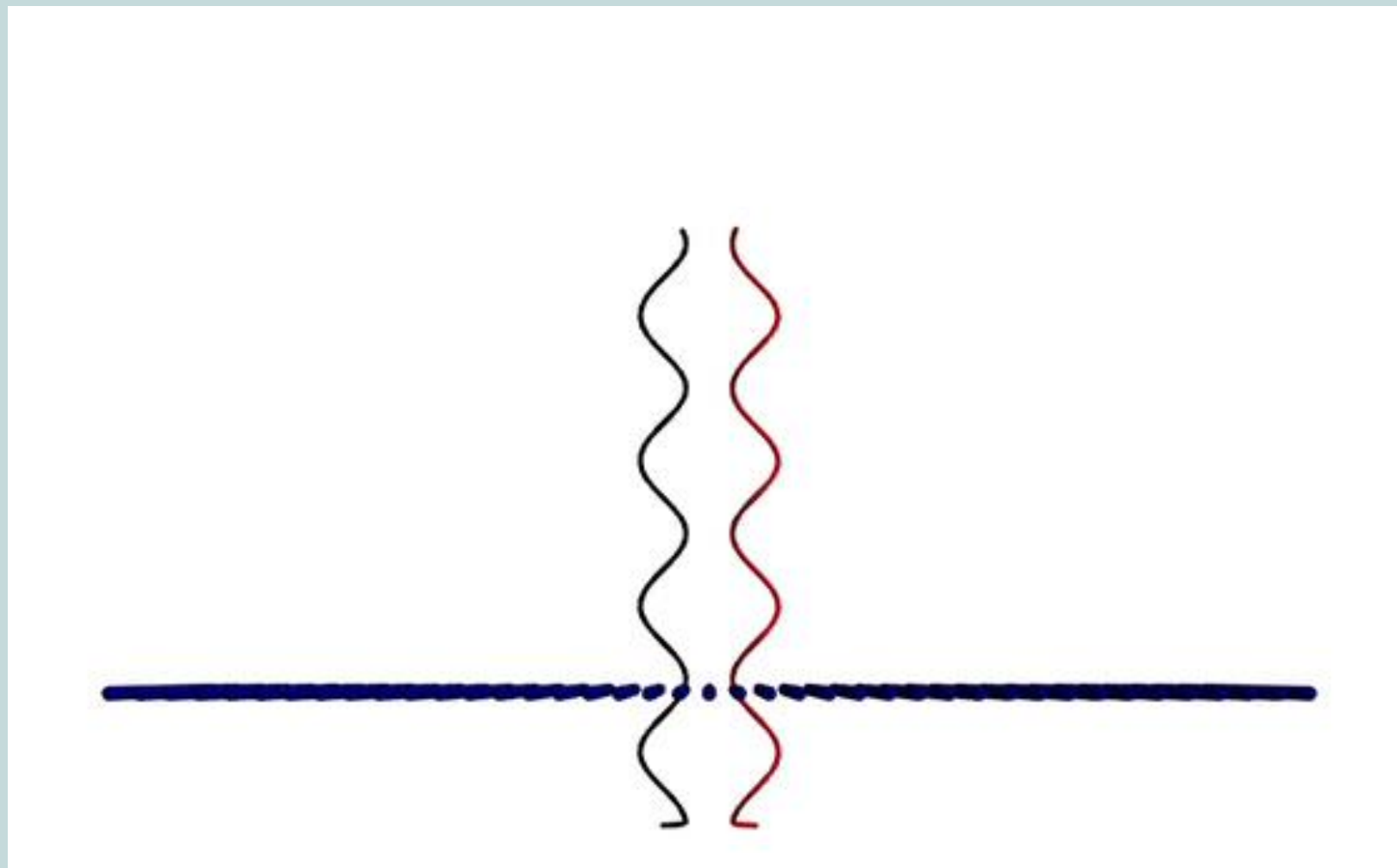


$$\begin{aligned} -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{x}, t) &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

Coming soon...

**AN INTRODUCTORY GUIDE TO THE METHOD OF
REGULARIZED STOKESLETS: A PRACTICAL IMPLEMENTATION ***

AMY BUCHMANN[†], MUHAN JI [‡], EVA STRAWBRIDGE [§], MCKENNA WITT [¶], AND
LONGHUA ZHAO ^{||}



Thank you

- ◆ Please fill out the following survey



References

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