A suite of code for dynamic modelling of slender offshore structures

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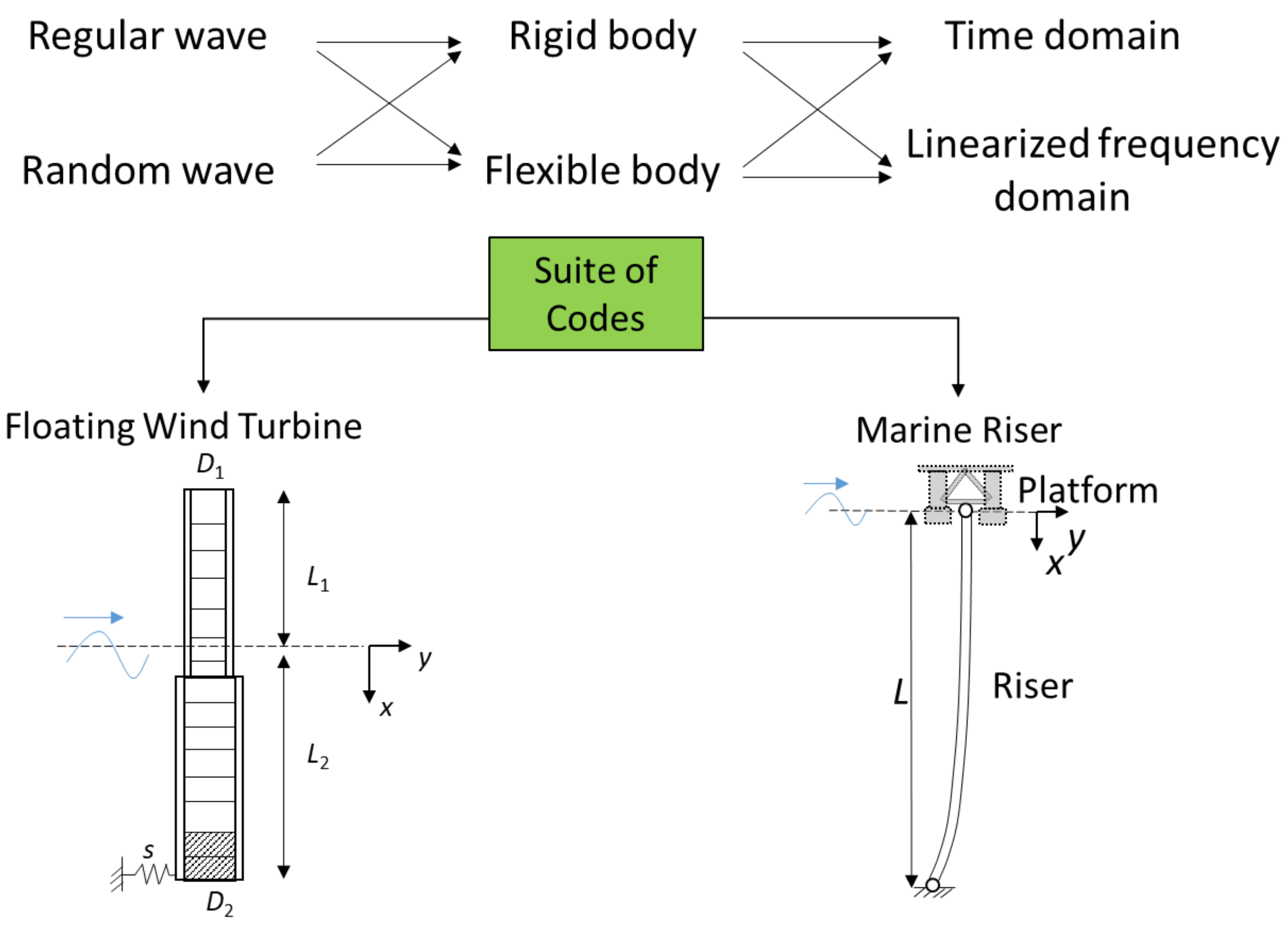


One Governing Equation – A Suite of Codes for Offshore Structural Dynamics

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{C}]\{\dot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} = \{\mathbf{Q}\}$$

Environment

Distributed nonlinear wave force

- Regular Harmonic Wave ✓
- Random Wave ✓

Analysis Method

- Linearized Frequency Domain ✓
- Non-linear Time Domain ✓

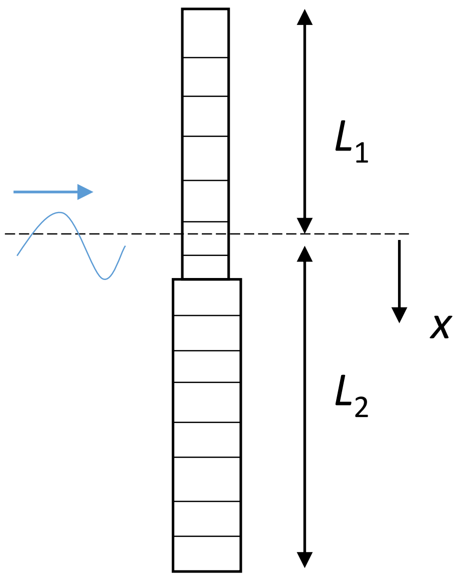
Structural Flexibility

Slender flexible MDOF structure

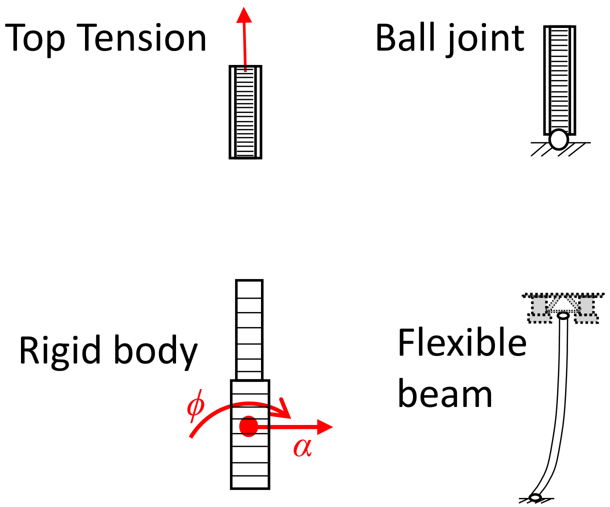
- Rigid Body ✓
- Flexible Structure ✓

Development Strategy

Structure modelled as strips to allow non-uniform structure design

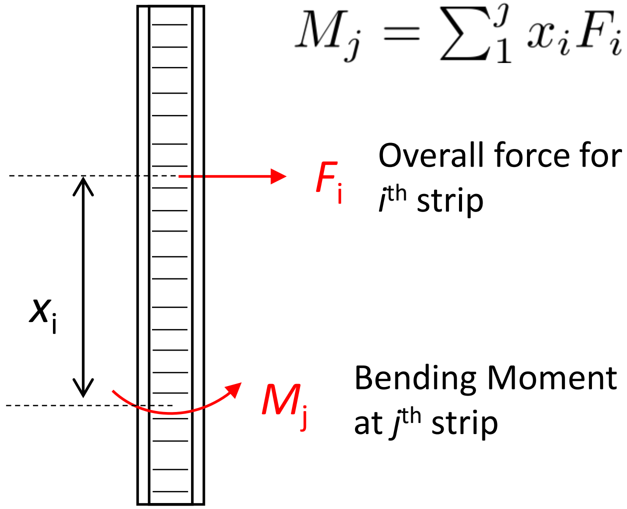


Shape function method allow versatile boundary conditions and flexible control of structural modes



Bending moment and stress can be recovered with 'dynamic force balance' and it is applicable to rigid bodies

$$M_j = \sum_1^j x_i F_i$$

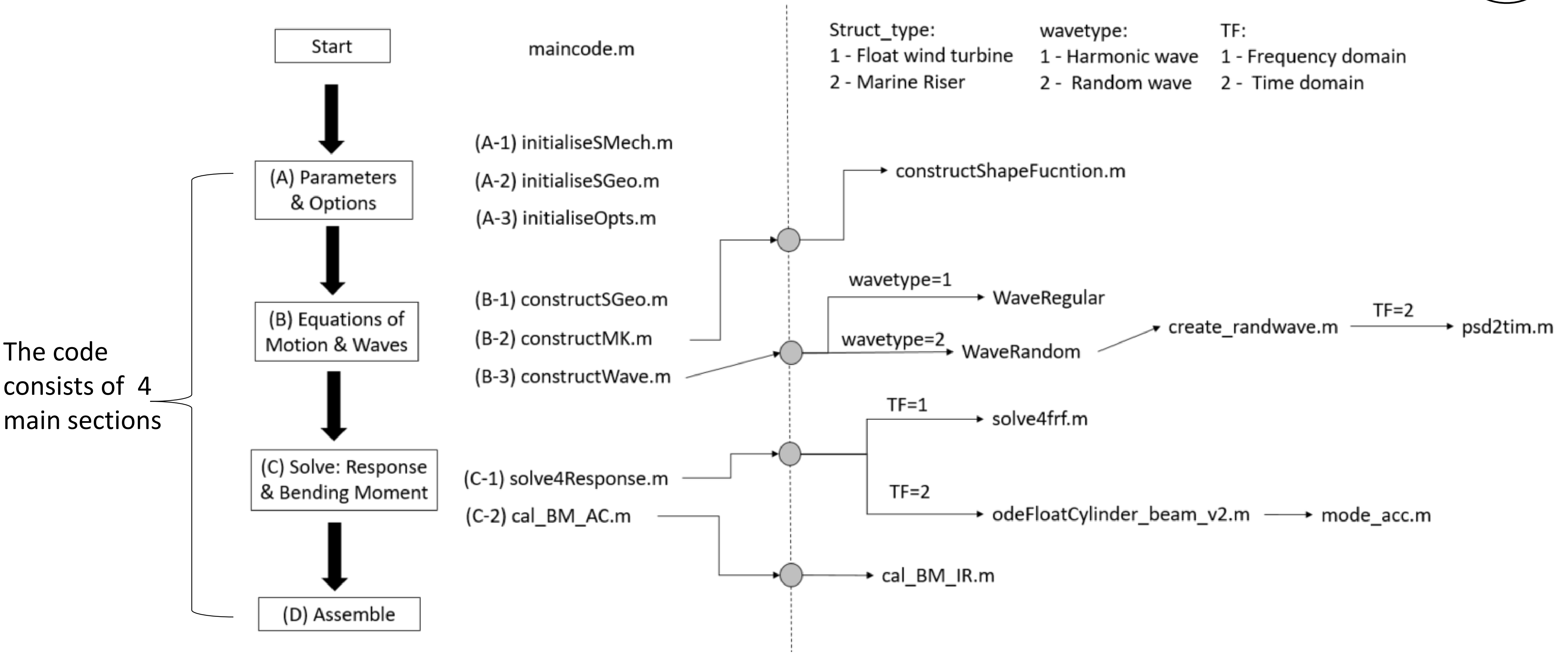


$$F_i = F_{\mathrm{wave}} - m\frac{\partial^2 v}{\partial t^2} - T_{\mathrm{eff}}\frac{\partial^2 v}{\partial x^2} + \frac{\mathrm{d}T_{\mathrm{eff}}}{\mathrm{d}x}\frac{\partial v}{\partial x}$$

$$T_{\mathrm{eff}}(x) = (\rho_{\mathrm{f}} - \rho_{\mathrm{s}})gA(x)x$$

Overview of the code structure

In the following slides, I will introduce the main theories used in the code



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Lagrange's Equation of Motion

Use shape functions to transform into generalised coordinates

$$\nu(x,t) = \sum_{m=1}^{n} \varphi_m(x) q_m(t)$$

Shape functions are key to the formulation: incorporates essential boundary conditions and flexibility of the structure

Use Lagrange's Equation to formulate equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right) + \frac{\partial D}{\partial \dot{q}_m} + \frac{\partial V}{\partial q_m} = Q_m$$

Damping is from fluid drag force and to be introduced at the section describing wave forces

Potential Energy	$V = \frac{1}{2}\int_{-L_1}^{L_2} E(x)I(x)\left(\frac{\partial^2\nu(x,t)}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_{-L_1}^{L_2} T_{\text{eff}}(x)\left(\frac{\partial\nu(x,t)}{\partial x}\right)^2 dx$
Kinetic Energy	$T = \frac{1}{2}\int_{-L_1}^{L_2} \rho(x)A(x)\left(\frac{\partial^2\nu(x,t)}{\partial t^2}\right)^2 dx$
Generalised Force	$Q_m = \int_{-L_1}^{L_2} F(x,t)\varphi_m(x)dx$

Equation of Motion for Linear Vibration

Lagrange's Equation
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_m}\right)+\frac{\partial D}{\partial \dot{q}_m}+\frac{\partial V}{\partial q_m}=Q_m$$

Assuming small motions for the structure, in which the generalised displacements from the equilibrium position are small so that the generalised coordinates and their time derivatives appear in the differential equations of motion at most to the first power, the scalar functions of kinetic energy (*T*), potential energy (*V*) and dissipation energy (*D*) can be formulated in the concise form as below

$$T=\frac{1}{2}\{\dot{\mathbf{q}}\}^{\mathsf{T}}\mathbf{M}\{\dot{\mathbf{q}}\}\,,\ V=\frac{1}{2}\{\mathbf{q}\}^{\mathsf{T}}\mathbf{K}\{\mathbf{q}\}\,,\ D=\frac{1}{2}\{\dot{\mathbf{q}}\}^{\mathsf{T}}\mathbf{C}\{\dot{\mathbf{q}}\}$$

Equation of Motion for Linear Vibration

$$[\mathbf{M}]\,\{\ddot{\mathbf{q}}\} + [\mathbf{C}]\,\{\dot{\mathbf{q}}\} + [\mathbf{K}]\,\{\mathbf{q}\} = \{\mathbf{Q}\}$$

Mass Matrix (**M**), Stiffness Matrix (**K**), Damping Matrix (**C**) and Generalised Forces (**Q**) in terms of generalised coordinates

$$\mathbf{M} = m_{\mathrm{eff}}\Delta x \mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi}$$

$$\mathbf{K} = EI\Delta x \mathbf{\Phi}''^{\mathsf{T}}\mathbf{\Phi}'' + \mathbf{\Phi}'^{\mathsf{T}}\mathbf{T}_{\mathrm{eff}}\mathbf{\Phi}'$$

$$\mathbf{C} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C_T}\mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}}\mathrm{diag}(\mathrm{c_D}(\mathbf{x}))\mathbf{\Phi}$$

$$\mathbf{Q} = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_m \end{bmatrix}^{\mathsf{T}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F}$$

$$\mathbf{q}(t) = \begin{Bmatrix} q_1 & q_2 & \ldots & q_m \end{Bmatrix}^{\mathsf{T}}$$

$$\mathbf{\Phi}(x) = \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & \cdots & \varphi_m(x) \end{bmatrix}$$

$$Q_m = \sum_{i=1}^{N} \varphi_m(x_i)F(x_i)$$

Link to the code
Section B: construct
(*constructMK.m*)

Undamped Modes

The equation of motion turns into a standard eigenvalue problem if we look for undamped free harmonic vibration (note that the structure itself is assumed to be linear and nonlinearity is only from wave interaction)

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\psi = \mathbf{0}$$

where $\boldsymbol{\psi}$ is the eigenvector representing normal modes of the free vibration.

Then the generalised coordinates can be expressed as

$$\mathbf{q} = \boldsymbol{\Psi}\mathbf{a}$$

where $\mathbf{a}$ is the modal coordinates and $\boldsymbol{\Psi} = \{\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2 \cdots\}$ is the eigenvector matrix with eigenvectors as its columns.

Link to the code
Section C: solve
(*solve4frf.m*)

Solution in Frequency Domain

In frequency domain, the frequency response function (FRF) is computed

$$\mathbf{a} = [-\omega^2\mathbf{I} + i\omega\mathbf{C}_b + \boldsymbol{\Omega}^2]^{-1}\mathbf{Q}_b$$

where **a** is the response in modal coordinates. $\boldsymbol{\Omega}^2$ is the diagonal matrix of natural frequencies, and $\mathbf{C}_b$ and $\mathbf{Q}_b$ are damping and force in modal coordinates:

$$\boldsymbol{\Psi}^\mathsf{T}\mathbf{M}\boldsymbol{\Psi} = \mathbf{I} \qquad \boldsymbol{\Psi}^\mathsf{T}\mathbf{K}\boldsymbol{\Psi} = \boldsymbol{\Omega}^2 \qquad \mathbf{C}_b = \boldsymbol{\Psi}^\mathsf{T}\mathbf{C}\boldsymbol{\Psi} \qquad \mathbf{Q}_b = \boldsymbol{\Psi}^\mathsf{T}\mathbf{Q}$$

and $\boldsymbol{\Psi}$ is the eigenvector matrix with eigenvectors as its columns. Direct inversion of the dynamic stiffness matrix is used because the damping matrix is in general not diagonal in modal coordinates. To maintain a small matrix size, mode acceleration is used:

$$\mathbf{q} = \mathbf{K}^{-1}\left[\mathbf{Q} - \mathbf{M}\boldsymbol{\Psi}(-\omega^2\mathbf{a}) - \mathbf{C}\boldsymbol{\Psi}(i\omega\mathbf{a})\right]$$

Using the mode acceleration method, only a small number of modes are kept for computation of FRF. As a result, the eigenvector matrix $\boldsymbol{\Psi}$ is the reduced eigenvector matrix consists of only first ζ modes. Note that the response consists of quasi-static contribution from all modes but the dynamic part from only the first ζ modes

Solution in Frequency Domain

Link to the code
Section C: solve
(*solve4frf.m*)

For frequency domain analysis, the system of equations need to be linear. Although we have assumed a linear wave theory, there is nonlinearity from Morison's equation because the drag force is dependent on the relatively velocity squared.

Assuming this can be linearized, then we are looking for a form like below with a linearization coefficient *γ*

$$\left|u_{\mathrm{r}}\right| u_{\mathrm{r}} = \left|u - \dot{\nu}\right| \left(u - \dot{\nu}\right) \approx \gamma \left(u - \dot{\nu}\right)$$

And for **harmonic** wave, this linearized coefficient is

$$\gamma(x,\omega) = \frac{8}{3\pi} \left|u_{\mathrm{r}}(x,\omega)\right|$$

As the linearization coefficient depends on the structural response, an iterative approach is used

And for **random** wave, this linearized coefficient is

$$\gamma(x) = \sqrt{\frac{8}{\pi}}\sigma_{\mathrm{u_r}}(x) = \sqrt{\frac{8}{\pi}} \sum_{\omega} \sqrt{2S_{\mathrm{u_r u_r}}(x,\omega)d\omega}$$

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Solution in Time Domain

Link to the code
Section C: solve
(*solve4XXXTimeResponse.m*)

To solve the response in time domain, the equation of motion needs to be rearranged into first order form like below in modal coordinates:

$$\dot{\mathbf{z}} = \mathbf{B} - \mathbf{A}\mathbf{z}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{\Omega}^2 & \mathbf{C}_b \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{Q}_b \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{bmatrix}$$

The first order differential equation can be solved using ODE functions (ode45 from Matlab is used here). After solving the ODE function, the response in generalised coordinates is recovered using mode acceleration

$$\mathbf{q} = \mathbf{K}^{-1}\left[\mathbf{Q} - \mathbf{M}\mathbf{\Psi}\ddot{\mathbf{a}} - \mathbf{C}\mathbf{\Psi}\dot{\mathbf{a}}\right]$$

Note that the acceleration in modal coordinate is not automatically obtained and it needs to be recovered separately

$$\ddot{\mathbf{a}} = \mathbf{Q}_b - \mathbf{C}_b\dot{\mathbf{a}} - \mathbf{\Omega}^2\mathbf{a}$$

Link to the code
Section C: solve
(*mode_acc.m*)

Link to the code
Section C: solve
(*solve4XXXTimeResp
onse.m*)

Solution in Time Domain

In frequency domain, the frequency response function (FRF) is computed with a unit wave amplitude. However, in time domain, the time history of the wave excitation is needed.

For a **harmonic wave** of frequency ω and amplitude a_w , the particle velocity and acceleration can be described as:

$$u(x,t) = i\omega a_{\mathrm{w}} e^{-k_{\mathrm{w}} x} e^{i\omega t} \qquad \dot{u}(x,t) = -\omega^2 a_{\mathrm{w}} e^{-k_{\mathrm{w}} x} e^{i\omega t}$$

where k_w is the wave number and $k_w = \omega^2/g$ based on linear wave dispersion relationship in deep water condition

Link to the code
Section B: construct
(*constructWave.m*)

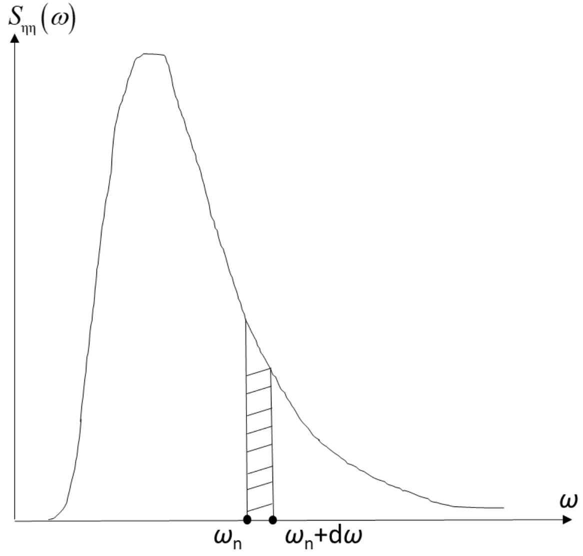
Solution in Time Domain

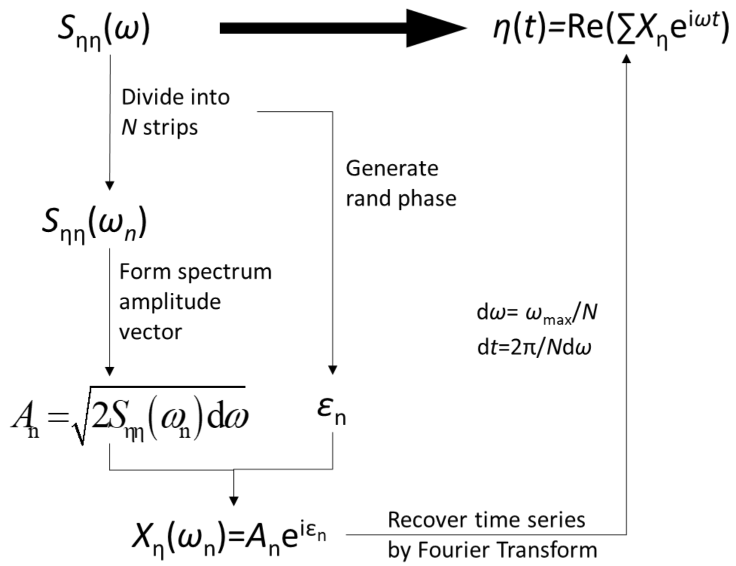
For a **random wave**, given a wave spectrum, the time history can be estimated and this needs to be statistically indistinguishable from the original signal. This process can be summarised in the flow chart shown below

- For wind waves, the surface elevation at a single position can be seen as an infinite sum of harmonic waves of random phase

- When successive ω_n are chosen to be equally spaced, with interval $\Delta\omega$, the wave amplitude for each component can be determined based on the knowledge that the area under the spectrum segment is equal to the variance of the wave component

- The time series can then be recovered by taking Fourier Transform of the frequency spectrum





Wave Force - Morison's Equation

The formulation of wave forces are imbedded in various codes in *Section (C) Solve*

The fluid forces are considered to be the sum of an inertia force and a drag force. The inertia force is due to fluid acceleration and the drag force is associated with relative velocity

$$F(x,t) = F_{\mathrm{I}}(x,t) + F_{\mathrm{DT}}(x,t)$$

The **inertia force** (the part that is proportional to structure acceleration is treated as an added mass and is included in the mass matrix)

$$F_{\mathrm{I}}(x,t) = \rho_{\mathrm{f}} A(x)\dot{u}(x,t) + C_a \rho_{\mathrm{f}} A(x)\dot{u}(x,t) = \rho_{\mathrm{f}} A(x) C_{\mathrm{m}} \dot{u}(x,t)$$

The **drag force** (where the second term for the drag force, c_D, can be treated as a damping coefficient and brought to the left hand side of the equation of motion)

$$\begin{aligned} F_{\mathrm{DT}}(x,t) &= \tfrac{1}{2}\rho_{\mathrm{f}} C_{\mathrm{d}} D(x) \left| u(x,t) - \dot{\nu}(x,t) \right| \left(u(x,t) - \dot{\nu}(x,t) \right) \\ &= \tfrac{1}{2}\rho_{\mathrm{f}} C_{\mathrm{d}} D(x) \left| u(x,t) - \dot{\nu}(x,t) \right| u(x,t) - \tfrac{1}{2}\rho_{\mathrm{f}} C_{\mathrm{d}} D(x) \left| u(x,t) - \dot{\nu}(x,t) \right| \dot{\nu}(x,t) \\ &= F_{\mathrm{D}}(x,t) - \tfrac{1}{2} c_{\mathrm{D}}(x,t) \end{aligned}$$

Link to the code
Section C: solve
(*cal_BM_AC.m*)

Bending Moment

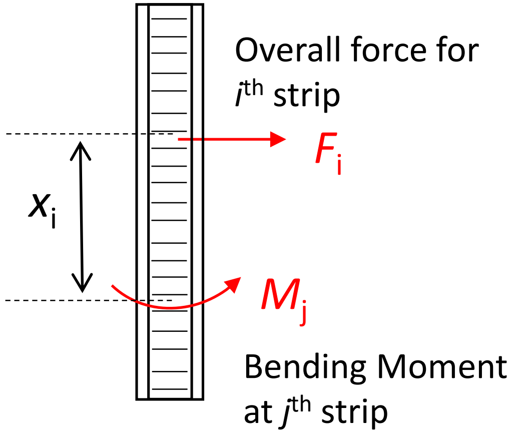
Once the structural response in generalised coordinate are computed, the spatial response can be recovered using the shape function specified

$$\nu(x,t) = \sum_{m=1}^{n} \varphi_m(x) q_m(t)$$

If the 2nd derivative of the shape function is nonzero, we can compute the bending moment directly

$$M(x,t) = EI\frac{\partial^2 \nu}{\partial x^2} = \sum_{m=1}^{n} \varphi''_m(x) q_m(t)$$

However, for rigid body structures, the 2nd derivative of the shape functions are zero. Therefore, we recover the forces at each strip along the structure and compute the bending moment. We assume the structure is in dynamic equilibrium with inertia forces included using D'Alembert principle. In the code, this is called 'Inertia Relief (IR)' because it bears similarity to inertia relief method used to compute static deflections for unconstrained structures.



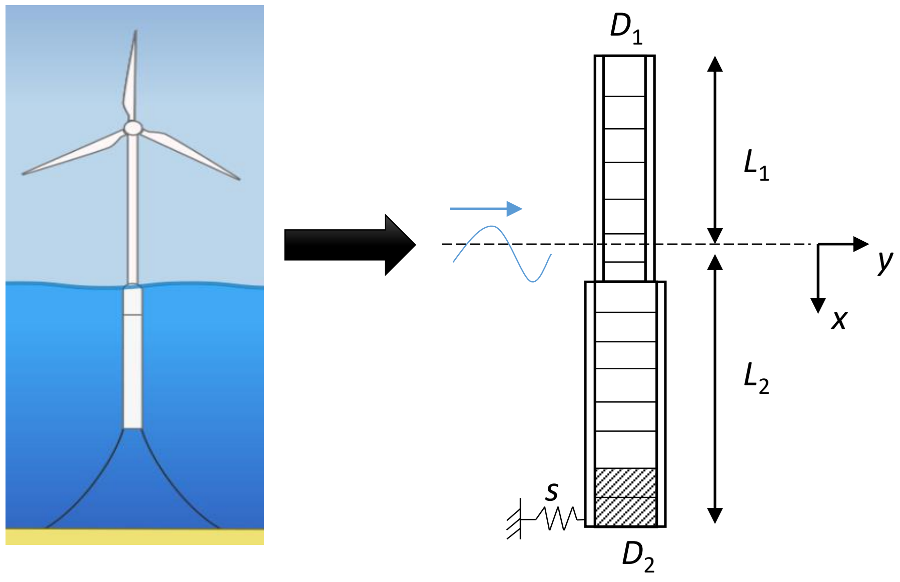
$$F_i = F_{\text{wave}} - m\frac{\partial^2 v}{\partial t^2} - T_{\text{eff}}\frac{\partial^2 v}{\partial x^2} + \frac{\mathrm{d}T_{\text{eff}}}{\mathrm{d}x}\frac{\partial v}{\partial x}$$

$$T_{\text{eff}}(x) = (m_a(x) - m_s(x))gx$$

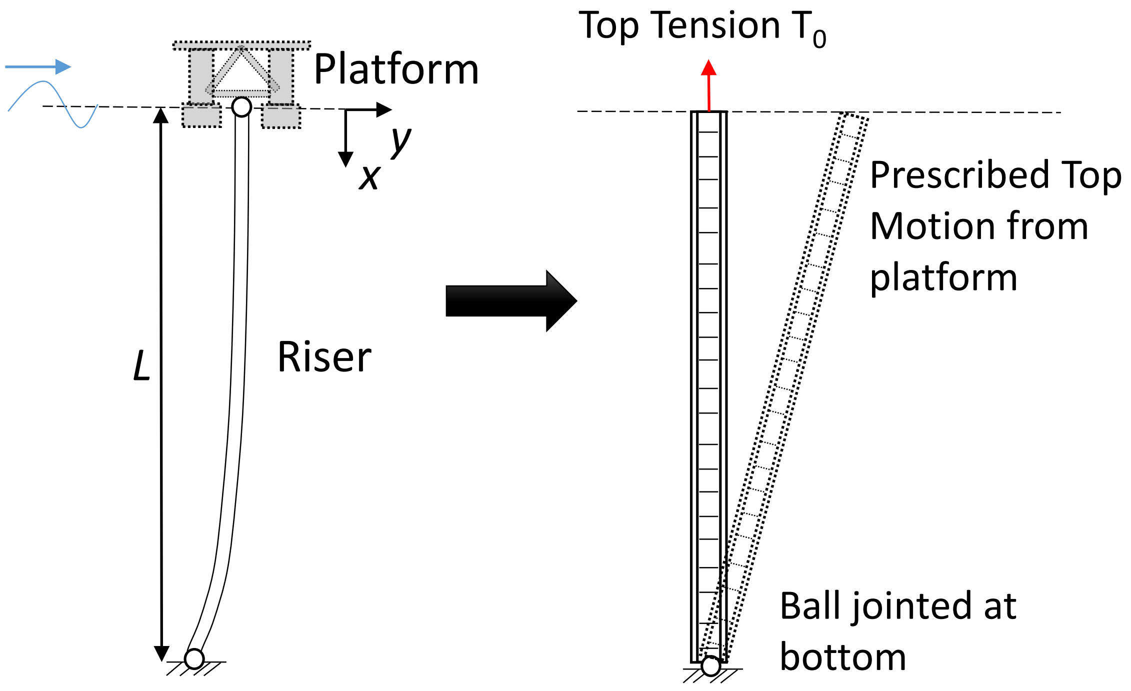
$$M_j = \sum_1^j x_i F_i$$

Demonstration Applications

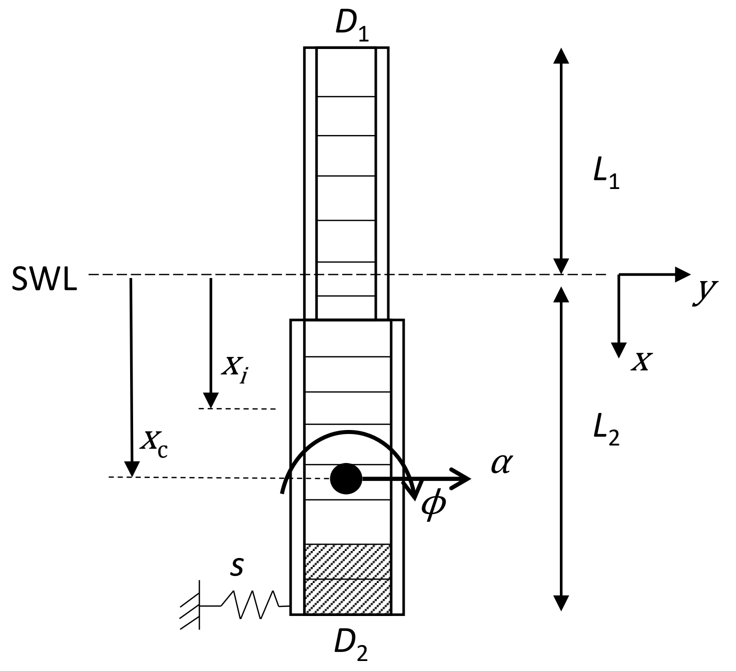
Floating Wind Turbine



Offshore Marine Riser



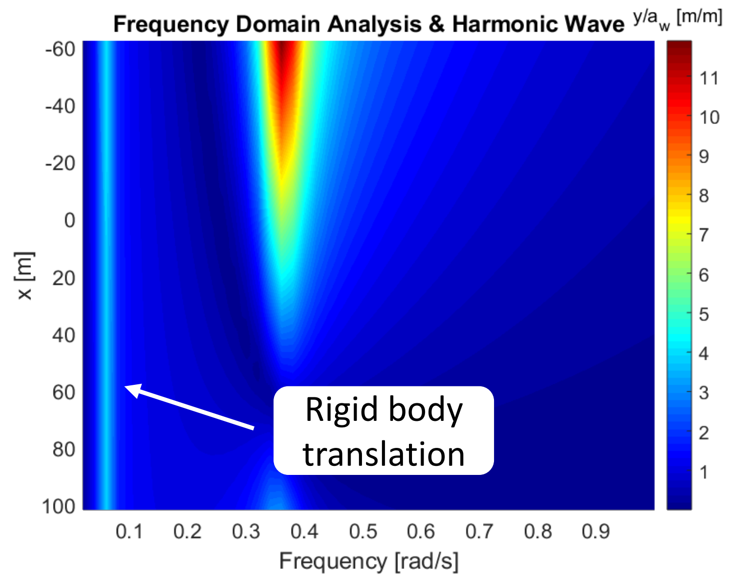
Preliminary results for a rigid body floating wind turbine (tower)



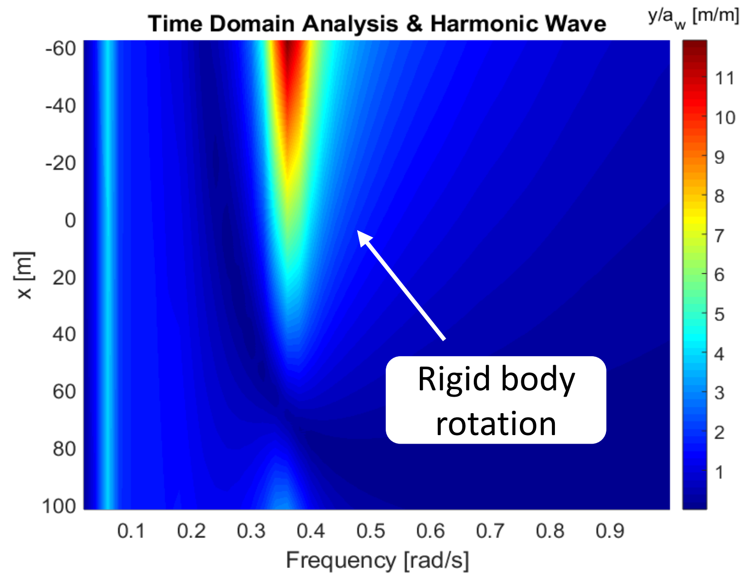
Harmonic Wave Excitation

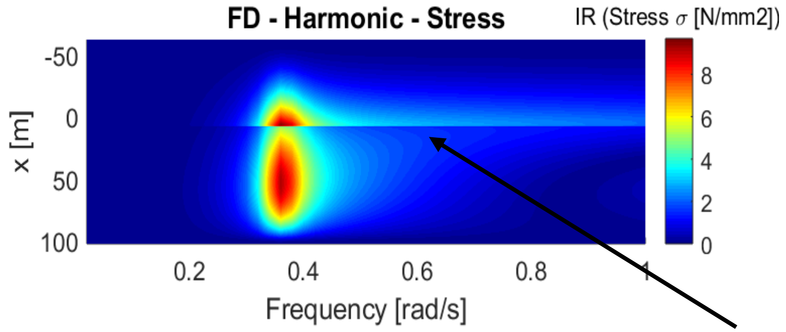
The results of displacement & bending stress from frequency and time domain analysis agree well

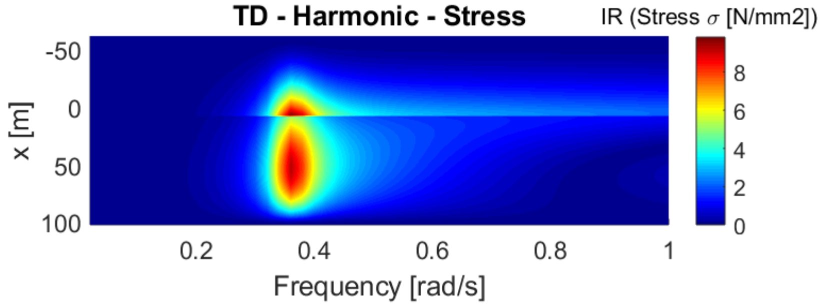
(Frequency domain analysis)



(Time domain analysis)







Jump of bending stress due to discontinuous change of diameter

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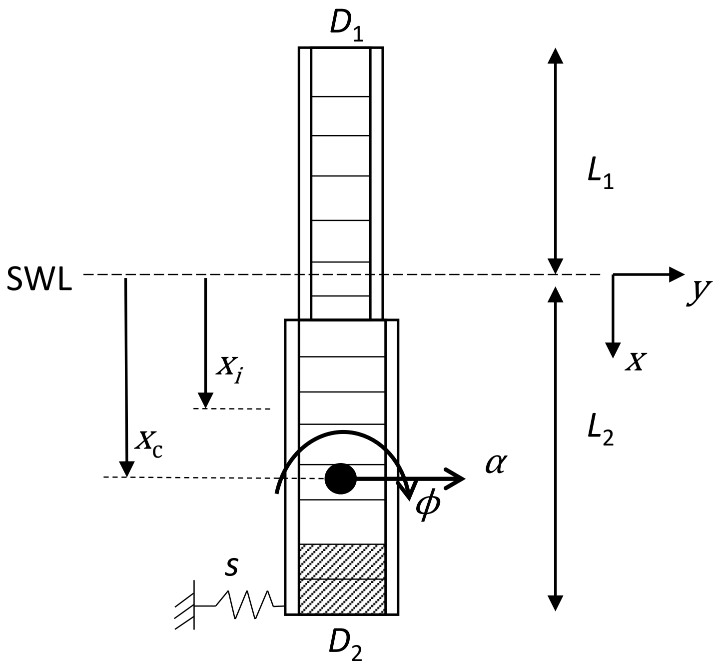
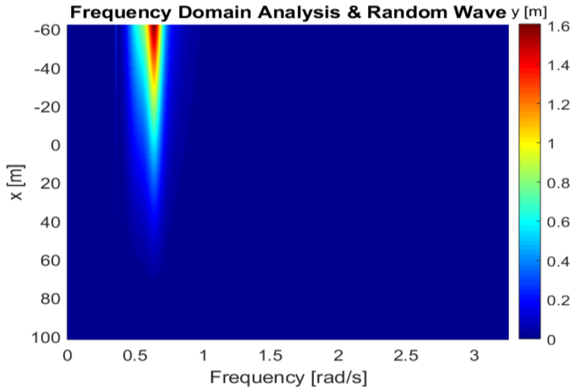
Preliminary results for a rigid body floating wind turbine (tower)

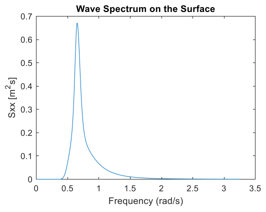


Random Wave Excitation (Typical Jonswap)

The results of displacement & bending stress from frequency and time domain analysis agree well

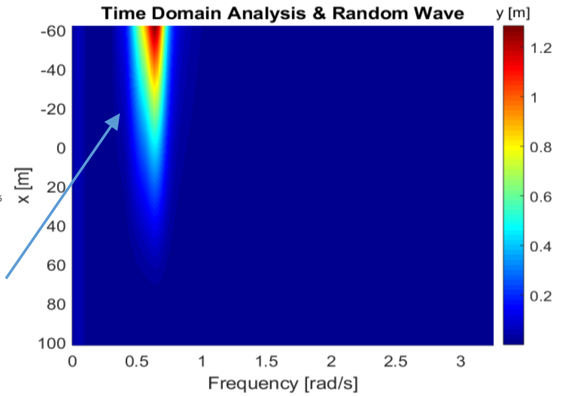
(Frequency domain analysis)

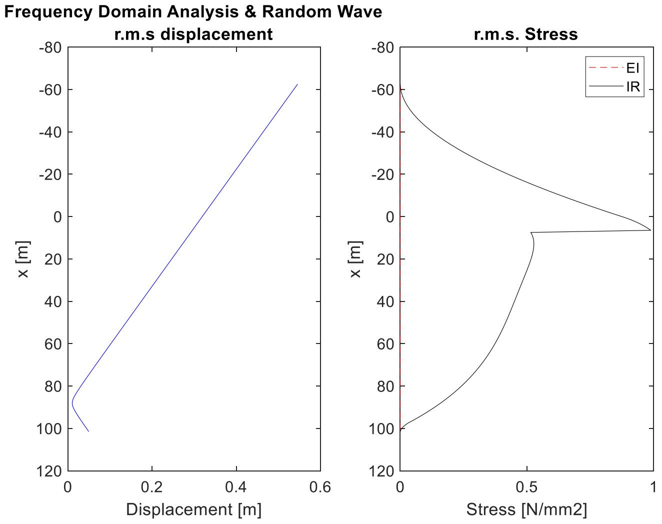


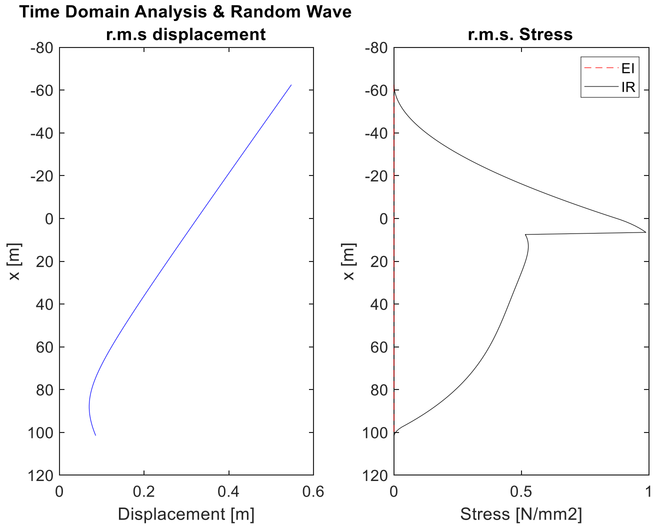


Dominated by peak wave frequency

(Time domain analysis)



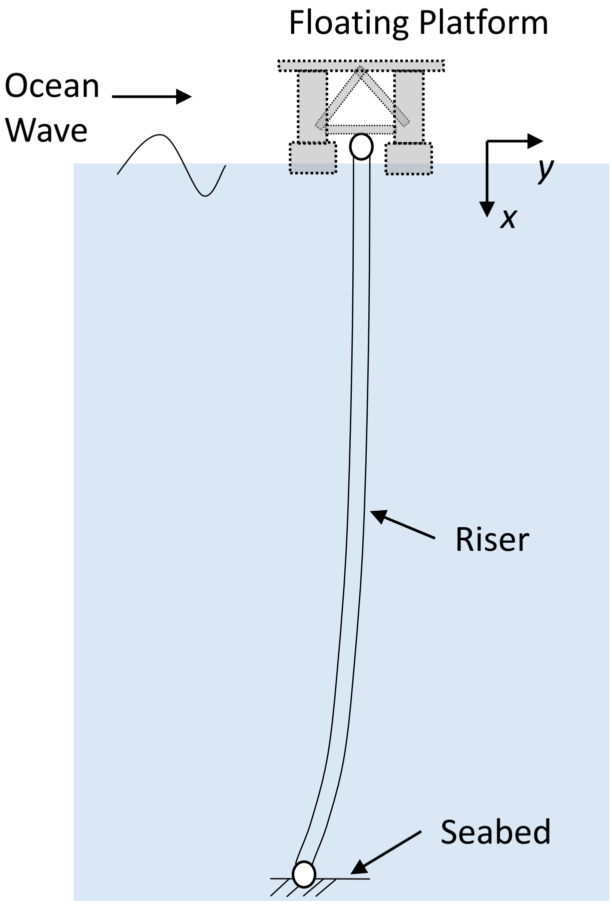




Preliminary results for a flexible marine riser

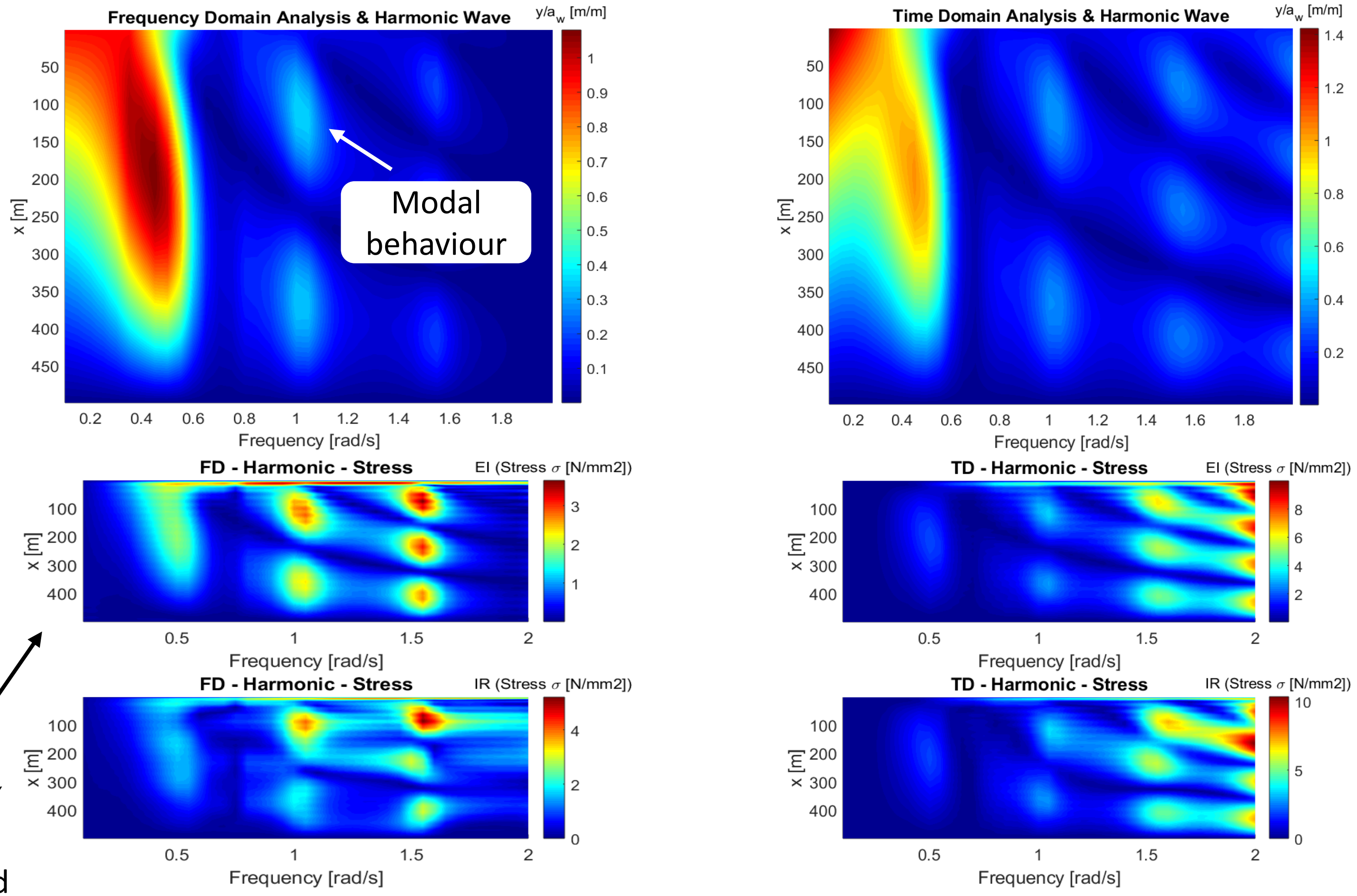
The results of displacement & bending stress from frequency and time domain analysis agree well

Harmonic Wave Excitation



(Frequency domain analysis)

(Time domain analysis)

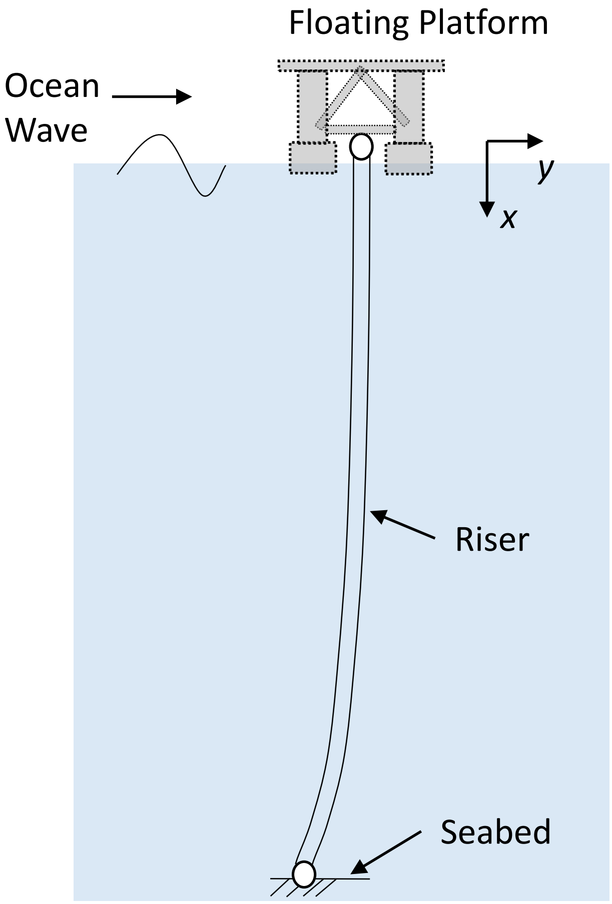


Both ‘EI’ and ‘IR’ can be used as 2nd derivative is available for flexible structure

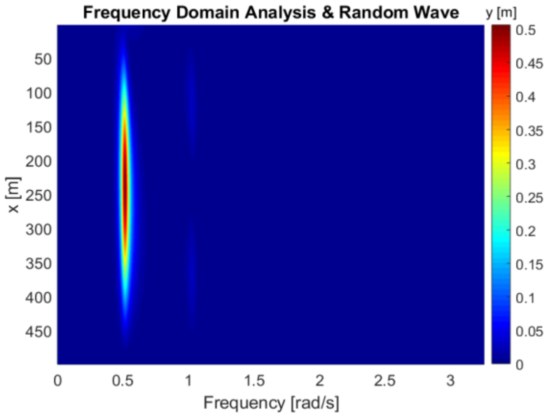
Preliminary results for a flexible marine riser

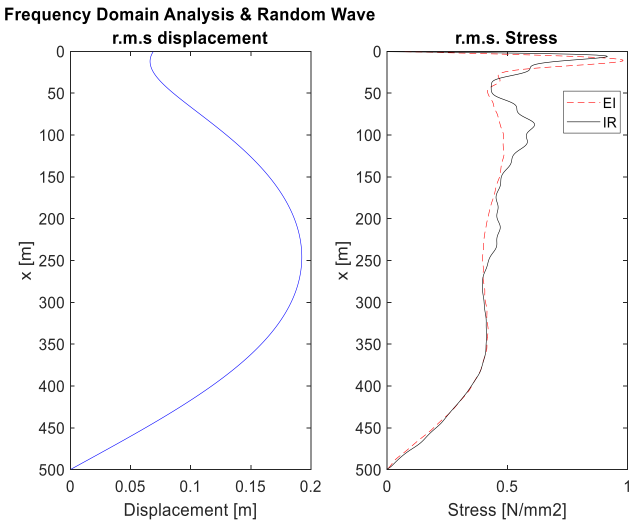
The results of displacement & bending stress from frequency and time domain analysis agree well

Random Wave Excitation (Typical Jonswap)

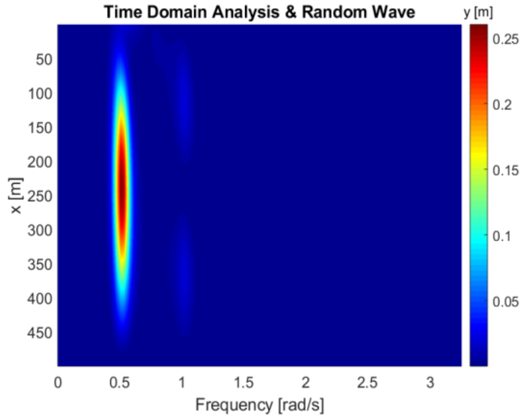


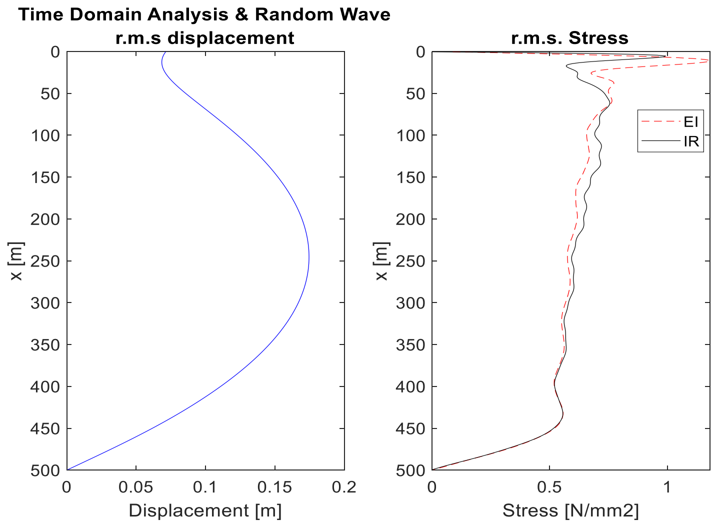
(Frequency domain analysis)





(Time domain analysis)





Acknowledgement

The work here has been funded by the Engineering and Physical Sciences Research Council through the award of a Programme Grant "Digital Twins for Improved Dynamic Design", Grant No. EP/R006768.

I would like to acknowledge additional contributors to this work:

- Prof. Robin Langley, University of Cambridge, UK
- Dr. Arnau Razquin, Universitat Politècnica de Catalunya (UPC), Spain