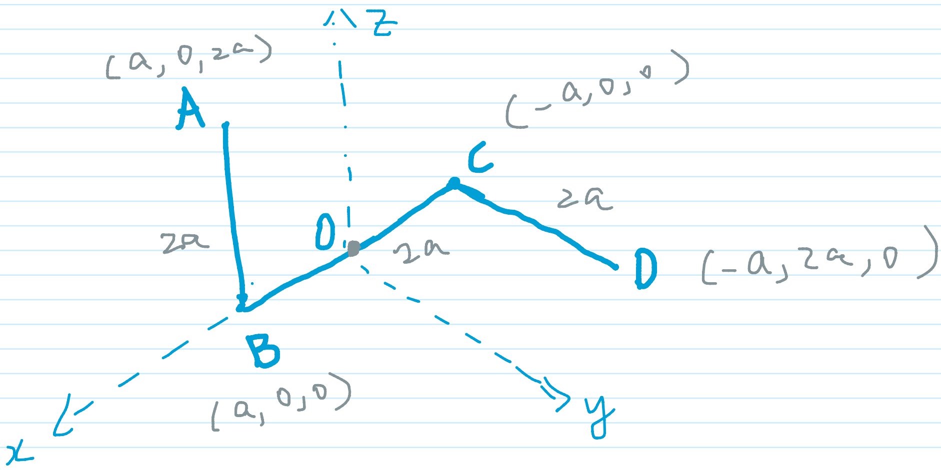
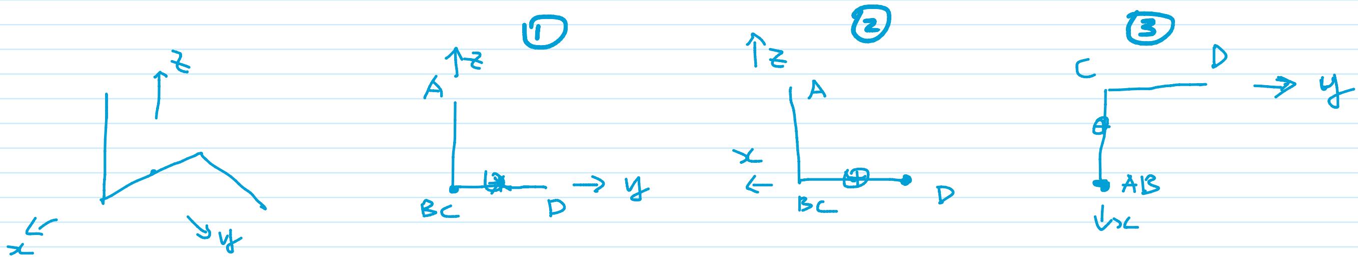
Inertia matrix for a paper clip

A uniform wire ABCB of length 6a, mass 3m, is bent at right angles at points B and C, as shown in the figure on the side (bend a paper clip!)

Find moments of inertia with respect to axes Oxyz and the mass centre G



The inertia matrix is symmetric, so 3 by 3 matrix has 6 unique entries. Although the algebra is straightforward, it is easy to make mistakes, especially for the products of inertia. Important to draw the three views to clearly define the products of inertia



	①	②	③
AB:	$I_{yz} = 0$	$I_{xz} \neq 0$	$I_{xy} = 0$
BC:	$I_{yz} = 0$	$I_{xz} = 0$	$I_{xy} = 0$
CD:	$= 0$	$= 0$	$\neq 0$

Now refer back to the 3D view for calculations. Take AB as an example:

$$I_{xx} = \frac{4}{3} m a^2$$

parallel axis

$$I_{yy} = \frac{1}{3} m a^2 + 2 m a^2$$

$$I_{zz} = 0 + m a^2$$

$$I_{xz} = \int_0^{2a} a z \, dm = \frac{m}{2a} \int_0^{2a} a z \, dz = \frac{m}{2} \frac{1}{2} z^2 \Big|_0^{2a} = m a^2$$

Once the calculations are done for the three individual rods, the inertia matrix is obtained by assembly

$$\begin{bmatrix} 8/3 & 1 & -1 \end{bmatrix}$$

assembly

$$I_O = \begin{bmatrix} 8/3 & 1 & -1 \\ & 11/3 & 0 \\ \text{sym} & & 11/3 \end{bmatrix} ma^2$$

For inertia matrix with respect to the centre of mass, G, the parallel axis theorem can be used (in reverse):

$$I_O = I_G + m \begin{bmatrix} Y^2 + Z^2 & -XY & -XZ \\ & Z^2 + X^2 & -YZ \\ & & X^2 + Y^2 \end{bmatrix}$$

(X, Y, Z) distance between O and G