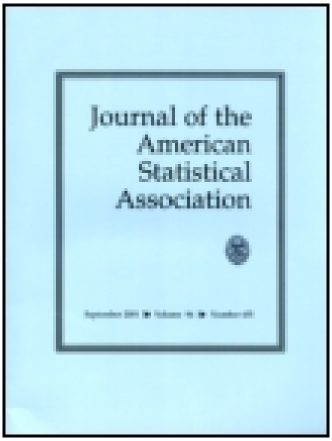

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The Power of Student's t-test

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THE POWER OF STUDENT'S t-TEST

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Tables are given which correspond to the power of various tests which use the Student t-distribution. Noncentrality parameters are given to 5 decimal places for tests which have Type I error equal to 0.050, 0.025, 0.010 and 0.005. Type II errors are covered as follows: 0.01, 0.05, 0.10 (0.10) 0.90. One-sided and two-sided tests are discussed.

1. INTRODUCTION

NEYMAN and Tokarska [3] published a table in 1936 which has been the basis of various tables and charts for several different statistical tests which have been widely used. The table given in this paper is a recomputation of the table given by Neyman and Tokarska using a modern high-speed computer with some extension in degrees of freedom and decimal places of accuracy. These tables are most easily described in terms of tests of hypotheses.

We assume that a sample, $x_1, x_2, \cdots, x_n$, of n observations taken at random from a normal distribution with mean μ and standard deviation σ is at hand. We wish to test the hypothesis that $\mu=\mu_0$ against the alternative hypothesis that $\mu>\mu_0$. The procedure is to reject the hypothesis H: $\mu=\mu_0$ if $(\bar{x}-\mu_0)\sqrt{n}/s>t_\alpha$ and to accept H if otherwise where $\bar{x}$ and s are the sample mean and standard deviation and where t_α is the α-th percentage point of the Student t-distribution with $f=n-1$ degrees of freedom, i.e.,

$$\Pr\{\text{Student } t > t_\alpha\} = \alpha.$$

The hypothesis $\mu=\mu_0$ is rejected with probability α when it is true.

If the mean of the normal distribution is μ_1 instead of μ_0, then the power of the above test is equal to

$$\Pr\{\text{noncentral } t > t_\alpha \mid \delta = (\mu_1 - \mu_0)\sqrt{n}/\sigma\} = 1 - \beta$$

where δ is the noncentrality parameter for the noncentral t-distribution with $f=n-1$ degrees of freedom.

Table I gives values of t_α for $\alpha=0.05$, 0.025, 0.01, and 0.005 for $f=1$ (1) 30 (5) 100 (10) 200, ∞. These values of t_α (rounded to 5 decimal places) were computed by the method given by Owen [4, p. 108]. They were then checked by numerical integration using a method given by Romberg [7] and [8], and also by three of the formulas given by Amos [1]. All discrepancies between the various computations were resolved so that we are confident that all of the figures given in Table I are correct to the number of decimal places given.

Tables II through V give values of δ corresponding to Type II Errors of .01, .05, .10 (.10).90 for the values of α and f covered in Table I. Again these values were computed by the method given by Owen [4, p. 108]. They were

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then checked by numerical integration using a method given by Romberg [7] and [8] and also by two of the formulas given by Amos [1] for the noncentral t-distribution in terms of hypergeometric functions. All discrepancies were resolved so that we are confident that all of the figures given in Tables II through V are correct to the number of decimal places given.

Other tables of percentage points of the non-central t-distribution have been prepared, in particular the ones by Resnikoff and Lieberman [6] and by Owen [5]. It is possible to obtain values of β for 10 values of δ from the tables in Resnikoff and Lieberman [6], but interpolation will have to be used. If, in addition, preassigned values of β are desired then the interpolation problem becomes difficult, and it is impossible to get accuracy to five decimal places in this process. Owen [5, p. 23] gives a table along the lines of the tables given here but not for as many values of α, β and degrees of freedom. Also different rounding processes were used in Owen [5] from the ones used in this paper, so that there are some last figure discrepancies of the tabulated values given here which overlap with Owen [5].

2. USES OF THE TABLES

The need for tables with the accuracy given here was first brought to the author's attention when a simulation study was underway which was to determine the robustness of the test of hypothesis mentioned in Section I when the underlying distribution was one of several non-normal distributions. It became clear that the simulation was quite sensitive to the critical values used and to the deltas. Hence the need for more accuracy than given in Neyman and Tokarska's table was demonstrated.

In applying these tables to the test of hypothesis given in Section I, the procedure is straightforward when one knows α, β, and $n=f+1$. Then δ may be read from the table. In case α, β, and δ are given and n has to be found, the graphs given by Croarkin [2] should be used to get a preliminary estimate of n. Note that in using these graphs δ as defined above must be divided by the square root of n.

Suppose we are given two random samples of the same size from two normal populations, $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$, where the x's and y's are independent with mean values μ_x and μ_y and common unknown variances $\sigma_x^2=\sigma_y^2=\sigma^2$. We wish to test the null hypothesis H: $\mu_x=\mu_y$ against the alternative hypothesis H_1 that $\mu_x>\mu_y$. The procedure is to reject the hypothesis H if

$$\sqrt{n}\,(\bar{x}-\bar{y})/\sqrt{s_x^2+s_y^2} > t_\alpha$$

where t_α is a critical value of the Student t-distribution based on $2(n-1)$ degrees of freedom.

In this case take $\delta=\sqrt{n}(\mu_x-\mu_y)/\sigma$ and then the probability of rejecting the hypothesis when $\mu_x>\mu_y$ is equal to

$$\Pr\{\text{noncentral } t > t_\alpha \mid \delta\} = 1-\beta$$

where the noncentral t-distribution has $2(n-1)$ degrees of freedom. If α, β, and n are given then $f=2(n-1)$ and δ may be read from the tables. If α, β, and δ are given and n is to be found one should refer to the Croarkin [2] graphs

TABLE I. CRITICAL VALUES OF THE STUDENT t-DISTRIBUTION

f	α			
	.050	.010	.025	.005
1	6.31375	31.82052	12.70620	63.65674
2	2.91999	6.96456	4.30265	9.92484
3	2.35336	4.54070	3.18245	5.84091
4	2.13185	3.74695	2.77645	4.60409
5	2.01505	3.36493	2.57058	4.03214
6	1.94318	3.14267	2.44691	3.70743
7	1.89458	2.99795	2.36462	3.49948
8	1.85955	2.89646	2.30600	3.35539
9	1.83311	2.82144	2.26216	3.24984
10	1.81246	2.76377	2.22814	3.16927
11	1.79588	2.71808	2.20099	3.10581
12	1.78229	2.68100	2.17881	3.05454
13	1.77093	2.65031	2.16037	3.01228
14	1.76131	2.62449	2.14479	2.97684
15	1.75305	2.60248	2.13145	2.94671
16	1.74588	2.58349	2.11991	2.92078
17	1.73961	2.56693	2.10982	2.89823
18	1.73406	2.55238	2.10092	2.87844
19	1.72913	2.53948	2.09302	2.86093
20	1.72472	2.52798	2.08596	2.84534
21	1.72074	2.51765	2.07961	2.83136
22	1.71714	2.50832	2.07387	2.81876
23	1.71387	2.49987	2.06866	2.80734
24	1.71088	2.49216	2.06390	2.79694
25	1.70814	2.48511	2.05954	2.78744
26	1.70562	2.47863	2.05553	2.77871
27	1.70329	2.47266	2.05183	2.77068
28	1.70113	2.46714	2.04841	2.76326
29	1.69913	2.46202	2.04523	2.75639
30	1.69726	2.45726	2.04227	2.75000
35	1.68957	2.43772	2.03011	2.72381
40	1.68385	2.42326	2.02108	2.70446
45	1.67943	2.41212	2.01410	2.68959
50	1.67591	2.40327	2.00856	2.67779
55	1.67303	2.39608	2.00404	2.66822
60	1.67065	2.39012	2.00030	2.66028
65	1.66864	2.38510	1.99714	2.65360
70	1.66691	2.38081	1.99444	2.64790
75	1.66543	2.37710	1.99210	2.64298
80	1.66412	2.37387	1.99006	2.63869
85	1.66298	2.37102	1.98827	2.63491
90	1.66196	2.36850	1.98667	2.63157

TABLE I. (*Continued*)

f	α			
	.050	.010	.025	.005
95	1.66105	2.36624	1.98525	2.62858
100	1.66023	2.36422	1.98397	2.62589
110	1.65882	2.36073	1.98177	2.62126
120	1.65765	2.35782	1.97993	2.61742
130	1.65666	2.35537	1.97838	2.61418
140	1.65581	2.35328	1.97705	2.61140
150	1.65508	2.35146	1.97591	2.60900
160	1.65443	2.34988	1.97490	2.60691
170	1.65387	2.34848	1.97402	2.60506
180	1.65336	2.34724	1.97323	2.60342
190	1.65291	2.34613	1.97253	2.60195
200	1.65251	2.34514	1.97190	2.60063
∞	1.64485	2.32635	1.95996	2.57583

for a preliminary value of n and then compute more exact values from the tables given here, if necessary.

If the tables presented here were set up to give the coordinates for the Croarkin graphs, the computations indicated below would be needed.

f	n $f+1$	δ	Δ_1 $\delta/\sqrt{}$	$(f+2)/2$	Δ_2 $2\delta/\sqrt{n+1}$
1	2	16.46586	11.64312	1.5	19.01314
2	3	6.88234	3.97352	2.0	6.88234
3	4	5.46629	2.73314	2.5	4.88920

where the entries given are for $\alpha=0.05$ and $\beta=0.01$.

For the test given in Section I, one would plot $n=f+1$ against Δ_1 to get Croarkin's graphs. For the test of equality of two means given above in this section, one would plot $(f+2)/2$, which is the size sample for each of the two populations, against Δ_2.

3. TWO-SIDED TESTS

For tests where the alternative hypothesis specifies that $\mu\neq\mu_0$ in the test of Section 1 or $\mu_x\neq\mu_y$ in the test of Section II, but the direction can be either up or down, i.e. $\mu>\mu_0$ or $\mu<\mu_0$ and $\mu_x>\mu_y$ or $\mu_x<\mu_y$ we have what is known as a two-sided test. In this case the rejection procedures for the test of Section I specify that H is rejected if either $(\bar{x}-\mu_0)\sqrt{n}/s>t_{\alpha/2}$ or $(\bar{x}-\mu_0)\sqrt{n}/s<-t_{\alpha/2}$ and the power of the test is

$$\Pr\{\text{noncentral } t > t_{\alpha/2}\} + \Pr\{\text{noncentral } t < -t_{\alpha/2}\} = 1-\beta$$

where $\delta=(\mu_1-\mu_0)\sqrt{n}/\sigma$.

TABLE II. VALUES OF THE NON-CENTRALITY PARAMETER FOR A ONE-SIDED TEST WITH $\alpha = .050$

f	β = Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
1	16.46586	12.52898	10.51465	8.19226	6.62535	5.38002	4.31164	3.35208	2.46068	1.59502	.64436
2	6.88234	5.51589	4.80923	3.97900	3.40067	2.92180	2.48801	2.06814	1.63457	1.14725	.50477
3	5.46629	4.45636	3.92820	3.29945	2.85399	2.47892	2.13312	1.79200	1.43225	1.01798	.45513
4	4.95463	4.06728	3.59995	3.03975	2.63987	2.30095	1.98652	1.67436	1.34292	.95833	.43076
5	4.69864	3.86994	3.43174	2.90458	2.52690	2.20582	1.90709	1.60971	1.29302	.92434	.41647
6	4.54674	3.75160	3.33014	2.82210	2.45738	2.14682	1.85745	1.56896	1.26128	.90247	.40712
7	4.44669	3.67302	3.26231	2.76664	2.41037	2.10672	1.82354	1.54097	1.23935	.88725	.40055
8	4.37599	3.61713	3.21389	2.72685	2.37650	2.07771	1.79892	1.52058	1.22330	.87606	.39568
9	4.32344	3.57538	3.17761	2.69691	2.35094	2.05577	1.78024	1.50507	1.21106	.86748	.39192
10	4.28290	3.54304	3.14944	2.67358	2.33098	2.03859	1.76559	1.49288	1.20141	.86071	.38895
11	4.25067	3.51725	3.12692	2.65490	2.31496	2.02478	1.75380	1.48304	1.19361	.85523	.38653
12	4.22447	3.49622	3.10854	2.63961	2.30183	2.01345	1.74411	1.47495	1.18719	.85070	.38454
13	4.20273	3.47872	3.09322	2.62685	2.29086	2.00397	1.73598	1.46816	1.18180	.84689	.38285
14	4.18443	3.46396	3.08029	2.61606	2.28157	1.99593	1.72910	1.46240	1.17721	.84365	.38141
15	4.16880	3.45133	3.06920	2.60680	2.27359	1.98903	1.72317	1.45744	1.17326	.84086	.38017
16	4.15529	3.44041	3.05961	2.59877	2.26667	1.98303	1.71803	1.45312	1.16982	.83842	.37909
17	4.14352	3.43087	3.05123	2.59176	2.26062	1.97779	1.71352	1.44935	1.16681	.83629	.37814
18	4.13315	3.42245	3.04382	2.58556	2.25526	1.97314	1.70953	1.44600	1.16414	.83439	.37729
19	4.12395	3.41499	3.03726	2.58005	2.25050	1.96901	1.70598	1.44302	1.16176	.83270	.37654
20	4.11575	3.40833	3.03139	2.57512	2.24625	1.96531	1.70280	1.44035	1.15963	.83119	.37587
21	4.10838	3.40232	3.02610	2.57068	2.24241	1.96198	1.69993	1.43794	1.15770	.82982	.37525
22	4.10172	3.39690	3.02132	2.56667	2.23893	1.95896	1.69733	1.43575	1.15596	.82858	.37470
23	4.09568	3.39198	3.01698	2.56303	2.23578	1.95622	1.69497	1.43377	1.15437	.82745	.37419
24	4.09018	3.38749	3.01302	2.55969	2.23289	1.95371	1.69281	1.43196	1.15292	.82642	.37373
25	4.08514	3.38338	3.00940	2.55664	2.23025	1.95141	1.69083	1.43029	1.15159	.82547	.37331
26	4.08051	3.37960	3.00606	2.55384	2.22782	1.94930	1.68901	1.42876	1.15036	.82460	.37292
27	4.07624	3.37611	3.00298	2.55124	2.22557	1.94734	1.68732	1.42734	1.14923	.82379	.37256
28	4.07229	3.37288	3.00013	2.54884	2.22349	1.94553	1.68576	1.42602	1.14817	.82304	.37222
29	4.06863	3.36989	2.99749	2.54661	2.22156	1.94385	1.68431	1.42481	1.14720	.82235	.37191
30	4.06522	3.36710	2.99502	2.54454	2.21976	1.94228	1.68296	1.42367	1.14628	.82169	.37162

TABLE II. (*Continued*)

f	β=Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
35	4.05122	3.35564	2.98488	2.53599	2.21234	1.93582	1.67739	1.41897	1.14252	.81901	.37041
40	4.04085	3.34713	2.97736	2.52964	2.20683	1.93102	1.67324	1.41548	1.13972	.81701	.36951
45	4.03286	3.34057	2.97155	2.52473	2.20257	1.92730	1.67003	1.41277	1.13755	.81546	.36882
50	4.02652	3.33536	2.96694	2.52083	2.19917	1.92434	1.66748	1.41062	1.13582	.81423	.36827
55	4.02135	3.33111	2.96316	2.51764	2.19640	1.92192	1.66538	1.40885	1.13440	.81321	.36780
60	4.01707	3.32759	2.96005	2.51500	2.19410	1.91992	1.66365	1.40739	1.13323	.81238	.36743
65	4.01347	3.32462	2.95741	2.51277	2.19217	1.91823	1.66219	1.40616	1.13224	.81167	.36711
70	4.01038	3.32207	2.95515	2.51086	2.19050	1.91677	1.66093	1.40509	1.13138	.81105	.36683
75	4.00772	3.31989	2.95322	2.50922	2.18907	1.91553	1.65985	1.40419	1.13065	.81054	.36660
80	4.00539	3.31796	2.95151	2.50777	2.18781	1.91442	1.65890	1.40338	1.13000	.81007	.36639
85	4.00335	3.31628	2.95002	2.50651	2.18671	1.91347	1.65807	1.40268	1.12944	.80967	.36621
90	4.00153	3.31478	2.94868	2.50538	2.18573	1.91261	1.65733	1.40205	1.12894	.80931	.36605
95	3.99991	3.31344	2.94750	2.50437	2.18485	1.91184	1.65666	1.40149	1.12849	.80898	.36590
100	3.99845	3.31224	2.94643	2.50347	2.18406	1.91115	1.65607	1.40098	1.12808	.80869	.36577
110	3.99594	3.31017	2.94459	2.50191	2.18271	1.90996	1.65504	1.40012	1.12738	.80819	.36554
120	3.99386	3.30845	2.94306	2.50061	2.18158	1.90898	1.65419	1.39940	1.12680	.80778	.36536
130	3.99210	3.30699	2.94177	2.49952	2.18062	1.90814	1.65347	1.39879	1.12631	.80743	.36520
140	3.99059	3.30574	2.94066	2.49858	2.17981	1.90743	1.65285	1.39826	1.12589	.80713	.36506
150	3.98929	3.30467	2.93971	2.49777	2.17910	1.90682	1.65232	1.39782	1.12553	.80687	.36495
160	3.98814	3.30372	2.93886	2.49705	2.17848	1.90627	1.65184	1.39741	1.12521	.80664	.36484
170	3.98714	3.30290	2.93813	2.49643	2.17794	1.90580	1.65143	1.39707	1.12493	.80644	.36476
180	3.98624	3.30215	2.93747	2.49587	2.17745	1.90537	1.65106	1.39675	1.12468	.80626	.36467
190	3.98544	3.30149	2.93688	2.49537	2.17701	1.90499	1.65073	1.39648	1.12445	.80610	.36460
200	3.98473	3.30091	2.93636	2.49493	2.17663	1.90465	1.65044	1.39623	1.12426	.80596	.36454
∞	3.97120	3.28971	2.92641	2.48647	2.16925	1.89820	1.64485	1.39151	1.12045	.80323	.36330

Two-Sided $\alpha = 0.100$

max	.00001	.00001	.00002	.00007	.00020	.00053	.00131	.00330	.00917	.03377	*
min	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00012	.00250	.02776	*

* Use $\delta=0$ for all entries with $\alpha=0.10$, $\beta=0.90$.

TABLE III. VALUES OF THE NON-CENTRALITY PARAMETER FOR A ONE-SIDED TEST WITH $\alpha = .010$

f	β=Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
1	82.00469	62.39786	52.36594	40.79977	32.99613	26.79405	21.47321	16.69493	12.26715	8.06562	4.00058
2	15.21723	12.25885	10.73648	8.95904	7.73217	6.72733	5.82985	4.97785	4.12233	3.20151	2.07513
3	9.33749	7.70736	6.86300	5.86825	5.17260	4.59438	4.06868	3.55829	3.03020	2.43674	1.65823
4	7.52027	6.28443	5.64011	4.87537	4.33551	3.88251	3.46642	3.05769	2.62881	2.13817	1.47801
5	6.68320	5.62334	5.06784	4.40494	3.93403	3.53662	3.16945	2.80653	2.42305	1.98063	1.37849
6	6.21267	5.24891	4.74177	4.13432	3.70106	3.33411	2.99393	2.65649	2.29856	1.88373	1.31573
7	5.91456	5.01007	4.53276	3.95957	3.54964	3.20167	2.87839	2.55702	2.21533	1.81827	1.27265
8	5.71003	4.84524	4.38794	3.83778	3.44360	3.10849	2.79671	2.48634	2.15585	1.77114	1.24131
9	5.56152	4.72496	4.28190	3.74819	3.36529	3.03944	2.73597	2.43359	2.11126	1.73562	1.21750
10	5.44903	4.63345	4.20100	3.67958	3.30515	2.98625	2.68906	2.39272	2.07661	1.70791	1.19880
11	5.36100	4.56157	4.13732	3.62540	3.25753	2.94405	2.65175	2.36014	2.04891	1.68568	1.18375
12	5.29030	4.50366	4.08590	3.58155	3.21891	2.90975	2.62138	2.33357	2.02628	1.66748	1.17136
13	5.23231	4.45602	4.04354	3.54533	3.18696	2.88134	2.59618	2.31149	2.00743	1.65228	1.16100
14	5.18390	4.41616	4.00803	3.51492	3.16010	2.85741	2.57492	2.29284	1.99149	1.63941	1.15219
15	5.14291	4.38233	3.97786	3.48903	3.13720	2.83700	2.55678	2.27690	1.97785	1.62838	1.14463
16	5.10776	4.35326	3.95191	3.46674	3.11745	2.81938	2.54110	2.26311	1.96604	1.61882	1.13806
17	5.07727	4.32800	3.92934	3.44732	3.10024	2.80401	2.52740	2.25106	1.95571	1.61043	1.13230
18	5.05060	4.30587	3.90955	3.43027	3.08512	2.79049	2.51535	2.24045	1.94660	1.60305	1.12721
19	5.02706	4.28632	3.89204	3.41517	3.07171	2.77850	2.50466	2.23103	1.93851	1.59647	1.12267
20	5.00615	4.26892	3.87646	3.40173	3.05977	2.76781	2.49512	2.22262	1.93128	1.59060	1.11862
21	4.98744	4.25333	3.86248	3.38966	3.04904	2.75820	2.48654	2.21505	1.92477	1.58530	1.11496
22	4.97059	4.23929	3.84988	3.37877	3.03935	2.74952	2.47878	2.20820	1.91888	1.58050	1.11164
23	4.95537	4.22659	3.83849	3.36891	3.03058	2.74166	2.47176	2.20200	1.91354	1.57616	1.10863
24	4.94153	4.21502	3.82810	3.35992	3.02257	2.73448	2.46534	2.19633	1.90866	1.57218	1.10587
25	4.92890	4.20446	3.81861	3.35171	3.01525	2.72792	2.45947	2.19114	1.90419	1.56854	1.10335
26	4.91732	4.19477	3.80991	3.34416	3.00853	2.72188	2.45407	2.18637	1.90007	1.56518	1.10102
27	4.90667	4.18586	3.80189	3.33721	3.00233	2.71632	2.44909	2.18196	1.89628	1.56208	1.09887
28	4.89684	4.17762	3.79448	3.33079	2.99661	2.71118	2.44449	2.17789	1.89277	1.55922	1.09688
29	4.88774	4.17000	3.78762	3.32484	2.99130	2.70641	2.44022	2.17411	1.88951	1.55655	1.09503
30	4.87930	4.16291	3.78125	3.31930	2.98636	2.70197	2.43624	2.17060	1.88647	1.55407	1.09331

TABLE III. (*Continued*)

f	β = Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
35	4.84478	4.13392	3.75513	3.29661	2.96609	2.68375	2.41991	2.15614	1.87399	1.54387	1.08620
40	4.81939	4.11254	3.73585	3.27984	2.95110	2.67026	2.40782	2.14541	1.86472	1.53629	1.08092
45	4.79992	4.09612	3.72104	3.26693	2.93955	2.65987	2.39848	2.13714	1.85757	1.53043	1.07683
50	4.78451	4.08311	3.70929	3.25669	2.93038	2.65160	2.39106	2.13055	1.85187	1.52575	1.07357
55	4.77203	4.07256	3.69975	3.24837	2.92293	2.64489	2.38503	2.12520	1.84723	1.52195	1.07091
60	4.76170	4.06383	3.69186	3.24148	2.91676	2.63932	2.38003	2.12076	1.84339	1.51880	1.06870
65	4.75302	4.05648	3.68521	3.23568	2.91156	2.63463	2.37582	2.11701	1.84014	1.51614	1.06684
70	4.74562	4.05022	3.67954	3.23072	2.90712	2.63063	2.37221	2.11381	1.83737	1.51386	1.06525
75	4.73923	4.04480	3.67464	3.22644	2.90327	2.62716	2.36909	2.11104	1.83497	1.51189	1.06386
80	4.73367	4.04009	3.67037	3.22271	2.89993	2.62414	2.36638	2.10863	1.83288	1.51017	1.06266
85	4.72877	4.03593	3.66661	3.21942	2.89698	2.62148	2.36398	2.10650	1.83103	1.50865	1.06160
90	4.72444	4.03226	3.66329	3.21651	2.89437	2.61912	2.36186	2.10461	1.82939	1.50731	1.06066
95	4.72057	4.02897	3.66031	3.21390	2.89203	2.61701	2.35996	2.10292	1.82793	1.50610	1.05981
100	4.71711	4.02603	3.65764	3.21157	2.88994	2.61512	2.35826	2.10141	1.82662	1.50503	1.05906
110	4.71113	4.02096	3.65304	3.20755	2.88632	2.61186	2.35533	2.09880	1.82435	1.50317	1.05775
120	4.70616	4.01673	3.64922	3.20420	2.88331	2.60914	2.35288	2.09662	1.82246	1.50161	1.05666
130	4.70197	4.01317	3.64599	3.20137	2.88078	2.60685	2.35081	2.09478	1.82087	1.50030	1.05574
140	4.69840	4.01014	3.64324	3.19896	2.87861	2.60489	2.34905	2.09322	1.81951	1.49918	1.05496
150	4.69530	4.00750	3.64085	3.19687	2.87673	2.60319	2.34752	2.09185	1.81832	1.49821	1.05427
160	4.69261	4.00521	3.63877	3.19505	2.87509	2.60171	2.34619	2.09067	1.81730	1.49736	1.05368
170	4.69022	4.00318	3.63693	3.19343	2.87365	2.60040	2.34501	2.08962	1.81639	1.49661	1.05315
180	4.68811	4.00139	3.63530	3.19201	2.87236	2.59924	2.34397	2.08869	1.81558	1.49595	1.05269
190	4.68622	3.99978	3.63384	3.19073	2.87121	2.59820	2.34303	2.08786	1.81485	1.49535	1.05227
200	4.68454	3.99834	3.63254	3.18959	2.87019	2.59728	2.34220	2.08712	1.81421	1.49483	1.05190
∞	4.65270	3.97120	3.60790	3.16797	2.85075	2.57969	2.32635	2.07300	1.80195	1.48473	1.04480

Differences for Two-Sided $\alpha = 0.02$

max	.00000	.00000	.00000	.00001	.00001	.00001	.00001	.00002	.00006	.00026	.00241
min	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00001

TABLE IV. VALUES OF THE NON-CENTRALITY PARAMETER FOR A ONE-SIDED TEST WITH $\alpha = .025$

f	β = Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
1	32.83021	24.98070	20.96447	16.33400	13.20985	10.72688	8.59670	6.68374	4.91110	3.22886	1.57683
2	9.66503	7.77166	6.79561	5.65349	4.86258	4.21231	3.62853	3.07027	2.50321	1.88045	1.08726
3	6.88500	5.65407	5.01380	4.25590	3.72267	3.27667	2.86831	2.46853	2.05055	1.57425	.93625
4	5.94343	4.92798	4.39587	3.76107	3.31027	2.92993	2.57861	2.23142	1.86458	1.44134	.86513
5	5.48897	4.57370	4.09168	3.51390	3.10147	2.75194	2.42769	2.10583	1.76415	1.36778	.82430
6	5.22572	4.36657	3.91262	3.36689	2.97617	2.64421	2.33552	2.02839	1.70153	1.32127	.79796
7	5.05531	4.23144	3.79519	3.26976	2.89286	2.57215	2.27352	1.97597	1.65884	1.28928	.77960
8	4.93646	4.13661	3.71243	3.20092	2.83355	2.52064	2.22900	1.93816	1.62790	1.26596	.76610
9	4.84906	4.06650	3.65106	3.14964	2.78922	2.48201	2.19551	1.90963	1.60446	1.24821	.75577
10	4.78216	4.01261	3.60375	3.10997	2.75483	2.45197	2.16941	1.88734	1.58610	1.23426	.74760
11	4.72937	3.96992	3.56620	3.07840	2.72739	2.42796	2.14851	1.86945	1.57133	1.22301	.74099
12	4.68665	3.93527	3.53566	3.05266	2.70499	2.40832	2.13138	1.85476	1.55918	1.21374	.73552
13	4.65141	3.90662	3.51037	3.03130	3.68636	2.39198	2.11711	1.84251	1.54904	1.20598	.73093
14	4.62184	3.88252	3.48907	3.01328	2.67063	2.37815	2.10502	1.83213	1.54043	1.19938	.72702
15	4.59668	3.86196	3.47088	2.99787	2.65717	2.36631	2.09466	1.82321	1.53302	1.19370	.72365
16	4.57502	3.84424	3.45518	2.98455	2.64552	2.35606	2.08568	1.81549	1.52660	1.18877	.72073
17	4.55617	3.82880	3.44149	2.97292	2.63534	2.34709	2.07782	1.80871	1.52097	1.18445	.71815
18	4.53962	3.81521	3.42944	2.96267	2.62636	2.33917	2.07088	1.80273	1.51599	1.18061	.71586
19	4.52498	3.80318	3.41876	2.95358	2.61839	2.33214	2.06471	1.79741	1.51156	1.17720	.71383
20	4.51194	3.79245	3.40922	2.94547	2.61127	2.32585	2.05919	1.79265	1.50759	1.17415	.71201
21	4.50024	3.78282	3.40066	2.93817	2.60487	2.32020	2.05423	1.78836	1.50402	1.17139	.71036
22	4.48970	3.77413	3.39293	2.93158	2.59908	2.31509	2.04974	1.78448	1.50078	1.16890	.70887
23	4.48015	3.76625	3.38592	2.92560	2.59383	2.31044	2.04566	1.78095	1.49784	1.16663	.70751
24	4.47145	3.75907	3.37953	2.92014	2.58903	2.30620	2.04193	1.77773	1.49515	1.16456	.70627
25	4.46349	3.75250	3.37367	2.91514	2.58463	2.30231	2.03851	1.77478	1.49269	1.16265	.70513
26	4.45619	3.74646	3.36830	2.91054	2.58059	2.29874	2.03536	1.77205	1.49041	1.16090	.70408
27	4.44947	3.74090	3.36334	2.90630	2.57686	2.29544	2.03246	1.76954	1.48831	1.15927	.70310
28	4.44326	3.73576	3.35876	2.90239	2.57341	2.29239	2.02978	1.76722	1.48638	1.15777	.70221
29	4.43750	3.73099	3.35450	2.89875	2.57021	2.28955	2.02728	1.76506	1.48457	1.15638	.70137
30	4.43215	3.72655	3.35055	2.89536	2.56723	2.28691	2.02495	1.76304	1.48288	1.15507	.70058

TABLE IV. (*Continued*)

f	β=Type II Error .01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
35	4.41023	3.70837	3.33431	2.88145	2.55498	2.27606	2.01539	1.75476	1.47596	1.14972	.69736
40	4.39403	3.69490	3.32228	2.87114	2.54588	2.26799	2.00828	1.74860	1.47080	1.14572	.69496
45	4.38157	3.68453	3.31300	2.86317	2.53885	2.26175	2.00278	1.74383	1.46680	1.14262	.69309
50	4.37170	3.67631	3.30565	2.85685	2.53327	2.25680	1.99841	1.74004	1.46363	1.14016	.69161
55	4.36367	3.66961	3.29965	2.85170	2.52872	2.25276	1.99484	1.73694	1.46103	1.13814	.69039
60	4.35703	3.66407	3.29469	2.84744	2.52495	2.24942	1.99189	1.73438	1.45888	1.13648	.68939
65	4.35144	3.65940	3.29051	2.84384	2.52177	2.24659	1.98940	1.73221	1.45706	1.13507	.68854
70	4.34667	3.65542	3.28694	2.84076	2.51905	2.24418	1.98726	1.73036	1.45551	1.13386	.68781
75	4.34254	3.65197	3.28385	2.83810	2.51670	2.24208	1.98541	1.72875	1.45416	1.13281	.68717
80	4.33895	3.64896	3.28115	2.83578	2.51465	2.24026	1.98380	1.72735	1.45298	1.13190	.68662
85	4.33579	3.64632	3.27878	2.83374	2.51284	2.23866	1.98239	1.72612	1.45195	1.13110	.68614
90	4.33298	3.64397	3.27667	2.83192	2.51123	2.23722	1.98112	1.72502	1.45103	1.13037	.68570
95	4.33048	3.64187	3.27480	2.83030	2.50980	2.23595	1.98000	1.72405	1.45021	1.12974	.68532
100	4.32823	3.63999	3.27311	2.82885	2.50852	2.23481	1.97898	1.72316	1.44947	1.12916	.68497
110	4.32436	3.63675	3.27020	2.82635	2.50630	2.23284	1.97724	1.72165	1.44820	1.12818	.68437
120	4.32114	3.63405	3.26778	2.82426	2.50445	2.23119	1.97579	1.72039	1.44714	1.12735	.68387
130	4.31842	3.63178	3.26573	2.82249	2.50289	2.22980	1.97456	1.71932	1.44624	1.12665	.68345
140	4.31609	3.62982	3.26398	2.82098	2.50155	2.22861	1.97351	1.71840	1.44547	1.12605	.68308
150	4.31409	3.62815	3.26248	2.81969	2.50041	2.22759	1.97261	1.71762	1.44481	1.12554	.68278
160	4.31233	3.62667	3.26115	2.81854	2.49939	2.22669	1.97180	1.71692	1.44423	1.12509	.68250
170	4.31079	3.62538	3.25999	2.81754	2.49850	2.22590	1.97111	1.71632	1.44372	1.12469	.68226
180	4.30941	3.62422	3.25895	2.81664	2.49771	2.22519	1.97048	1.71577	1.44326	1.12433	.68204
190	4.30819	3.62319	3.25803	2.81585	2.49700	2.22457	1.96993	1.71529	1.44286	1.12402	.68185
200	4.30709	3.62227	3.25720	2.81513	2.49637	2.22400	1.96943	1.71486	1.44249	1.12374	.68168
∞	4.28631	3.60482	3.24152	2.80159	2.48436	2.21331	1.95996	1.70662	1.43556	1.11834	.67841
	Two-Sided α=0.050										
max	.00000	.00000	.00001	.00001	.00002	.00004	.00012	.00033	.00105	.00414	.03414
min	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00017	.02605

TABLE V. VALUES OF THE NON-CENTRALITY PARAMETER FOR A ONE-SIDED TEST WITH $\alpha = .005$

f	β = Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
1	163.98913	124.78031	104.71894	81.58946	65.98411	53.58147	42.94112	33.38575	24.53127	16.12924	8.00018
2	21.49002	17.32248	15.17911	12.67842	10.95409	9.54355	8.28570	7.09430	5.90226	4.62778	3.09501
3	11.75934	9.72680	8.67593	7.44048	6.57888	5.86481	5.21782	4.59236	3.94892	3.23193	2.30486
4	8.96737	7.52086	6.76874	5.87856	5.25227	4.72850	4.24910	3.78006	3.29027	2.73345	1.99115
5	7.72734	6.53333	5.90934	5.16682	4.64105	4.19864	3.79109	3.38953	2.96673	2.48110	1.82419
6	7.04582	5.98692	5.43127	4.76741	4.29521	3.89624	3.52724	3.16209	2.77578	2.32950	1.72104
7	6.62077	5.64407	5.12993	4.51387	4.07428	3.70186	3.35649	3.01380	2.65018	2.22862	1.65118
8	6.33258	5.41036	4.92374	4.33939	3.92149	3.56678	3.23725	2.90969	2.56143	2.15678	1.60082
9	6.12526	5.24143	4.77420	4.21226	3.80974	3.46762	3.14938	2.83265	2.49546	2.10306	1.56283
10	5.96939	5.11389	4.66100	4.11564	3.72453	3.39178	3.08199	2.77338	2.44452	2.06138	1.53317
11	5.84819	5.01434	4.57243	4.03981	3.65749	3.33197	3.02871	2.72640	2.40402	2.02815	1.50938
12	5.75134	4.93454	4.50128	3.97873	3.60336	3.28359	2.98552	2.68824	2.37106	2.00101	1.48988
13	5.67226	4.86919	4.44293	3.92850	3.55878	3.24367	2.94983	2.65665	2.34372	1.97845	1.47362
14	5.60649	4.81471	4.39419	3.88648	3.52141	3.21017	2.91984	2.63006	2.32067	1.95939	1.45983
15	5.55098	4.76862	4.35291	3.85082	3.48966	3.18166	2.89428	2.60739	2.30098	1.94309	1.44802
16	5.50351	4.72912	4.31749	3.82018	3.46235	3.15712	2.87226	2.58782	2.28397	1.92898	1.43777
17	5.46247	4.69491	4.28678	3.79357	3.43861	3.13576	2.85308	2.57076	2.26913	1.91666	1.42880
18	5.42663	4.66499	4.25990	3.77025	3.41778	3.11701	2.83622	2.55576	2.25606	1.90580	1.42089
19	5.39507	4.63860	4.23616	3.74964	3.39935	3.10041	2.82129	2.54246	2.24447	1.89615	1.41385
20	5.36708	4.61517	4.21507	3.73130	3.38295	3.08562	2.80798	2.53060	2.23413	1.88754	1.40755
21	5.34208	4.59421	4.19620	3.71488	3.36825	3.07236	2.79604	2.51995	2.22483	1.87979	1.40188
22	5.31962	4.57536	4.17921	3.70009	3.35500	3.06040	2.78527	2.51034	2.21644	1.87279	1.39676
23	5.29934	4.55832	4.16384	3.68670	3.34300	3.04956	2.77549	2.50162	2.20882	1.86643	1.39210
24	5.28092	4.54283	4.14987	3.67451	3.33207	3.03969	2.76659	2.49366	2.20186	1.86062	1.38784
25	5.26413	4.52870	4.13711	3.66338	3.32209	3.03066	2.75845	2.48639	2.19551	1.85531	1.38394
26	5.24875	4.51574	4.12541	3.65316	3.31291	3.02237	2.75096	2.47970	2.18965	1.85041	1.38034
27	5.23463	4.50384	4.11466	3.64377	3.30448	3.01474	2.74407	2.47354	2.18426	1.84591	1.37703
28	5.22162	4.49286	4.10473	3.63509	3.29669	3.00769	2.73771	2.46785	2.17927	1.84173	1.37396
29	5.20958	4.48270	4.09555	3.62706	3.28947	3.00116	2.73181	2.46257	2.17465	1.83787	1.37111
30	5.19842	4.47327	4.08701	3.61960	3.28276	2.99509	2.72632	2.45766	2.17035	1.83427	1.36846

TABLE V. (*Continued*)

f	β=Type II Error										
	.01	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90
35	5.15286	4.43472	4.05212	3.58904	3.25527	2.97018	2.70380	2.43750	2.15267	1.81945	1.35754
40	5.11943	4.40637	4.02642	3.56649	3.23497	2.95177	2.68714	2.42256	2.13956	1.80845	1.34942
45	5.09386	4.38465	4.00671	3.54919	3.21937	2.93762	2.67432	2.41106	2.12947	1.79997	1.34315
50	5.07366	4.36747	3.99111	3.53547	3.20700	2.92638	2.66414	2.40193	2.12143	1.79322	1.33815
55	5.05732	4.35355	3.97847	3.52436	3.19697	3.91727	2.65587	2.39451	2.11491	1.78774	1.33409
60	5.04381	4.34204	3.96800	3.51514	3.18865	2.90971	2.64901	2.38835	2.10949	1.78318	1.33071
65	5.03247	4.33236	3.95920	3.50739	3.18165	2.90334	2.64324	2.38316	2.10492	1.77933	1.32786
70	5.02281	4.32411	3.95170	3.50078	3.17568	2.89791	2.63831	2.37873	2.10103	1.77605	1.32542
75	5.01448	4.31700	3.94522	3.49508	3.17052	2.89322	2.63405	2.37490	2.09766	1.77322	1.32331
80	5.00724	4.31081	3.93958	3.49011	3.16603	2.88913	2.63034	2.37157	2.09472	1.77074	1.32148
85	5.00086	4.30535	3.93462	3.48573	3.16207	2.88553	2.62707	2.36862	2.09213	1.76856	1.31985
90	4.99522	4.30053	3.93023	3.48186	3.15857	2.88235	2.62418	2.36602	2.08984	1.76663	1.31842
95	4.99019	4.29623	3.92631	3.47840	3.15544	2.87950	2.62159	2.36370	2.08779	1.76490	1.31714
100	4.98567	4.29236	3.92278	3.47528	3.15262	2.87693	2.61926	2.36160	2.08594	1.76334	1.31598
110	4.97789	4.28570	3.91671	3.46993	3.14777	2.87252	2.61525	2.35799	2.08276	1.76066	1.31398
120	4.97144	4.28018	3.91168	3.46548	3.14375	2.86886	2.61192	2.35500	2.08012	1.75843	1.31233
130	4.96601	4.27552	3.90744	3.46173	3.14036	2.86577	2.60911	2.35247	2.07789	1.75656	1.31093
140	4.96136	4.27153	3.90380	3.45852	3.13745	2.86311	2.60670	2.35030	2.07598	1.75494	1.30973
150	4.95735	4.26809	3.90066	3.45575	3.13494	2.86083	2.60462	2.34842	2.07433	1.75355	1.30869
160	4.95385	4.26509	3.89793	3.45333	3.13275	2.85883	2.60281	2.34679	2.07289	1.75233	1.30779
170	4.95076	4.26244	3.89551	3.45119	3.13081	2.85707	2.60121	2.34535	2.07161	1.75126	1.30699
180	4.94802	4.26009	3.89336	3.44930	3.12910	2.85550	2.59978	2.34407	2.07048	1.75030	1.30628
190	4.94557	4.25799	3.89144	3.44760	3.12756	2.85410	2.59851	2.34292	2.06947	1.74945	1.30564
200	4.94337	4.25610	3.88972	3.44607	3.12618	2.85284	2.59736	2.34189	2.06856	1.74868	1.30507
∞	4.90218	4.22068	3.85738	3.41745	3.10023	2.82918	2.57583	2.32248	2.05143	1.73421	1.29428
	Two-Sided α=0.010										
max	.00001	.00001	.00001	.00001	.00001	.00001	.00001	.00001	.00001	.00004	.00034
min	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

The values of δ were computed corresponding to $\alpha=0.10$, 0.05, 0.02 and 0.01 for two-sided tests and β equal to the same values as for the one-sided tests. The differences between the δ's (two-sided with $\alpha=0.10$, 0.05, 0.02 and 0.01) and the δ's (one-sided with $\alpha=0.05$, 0.025, 0.01 and 0.005, respectively) was taken. The two-sided δ's are always smaller than the one-sided. The maximum value and the minimum value for all degrees of freedom are recorded at the end of Tables II, III, IV and V for each β and α. Note that for $\alpha=0.01$ and 0.02 (two-sided), these differences are very small. For $\alpha=0.01$, $\beta=0.70$ and 0.80, there are differences which could be significant in some problems. When $\alpha=0.10$ (two-sided) and $\beta=0.90$, $\delta=0$ for all degrees of freedom. In all other cases, the value of δ for two-sided tests can be obtained to a known precision by looking up the value for one-sided tests in Tables II, III, IV and, V (with one-half the two-sided α-value) and subtracting the maximum and minimum differences. The true δ will be between (or equal to) these two values. Observation of the complete list of differences discloses that most of the differences are near the maximum value given, and hence if only one value is to be chosen for δ one should use that value obtained from the maximum. The minimum differences usually (but not always) occur with $f=1$ and may persist for $f=2$.

For example with $f=3$, $\alpha=.050$ (two-sided), and $\beta=0.90$, enter Table IV and obtain δ (one-sided) $=0.93625$. The maximum correction to δ (two-sided) is 0.03414 giving δ (two-sided) $=0.90211$ and the minimum correction is 0.02617 giving δ (two-sided) $=0.91008$. In other words the value of δ (two-sided) is between 0.90211 and 0.91008. The value obtained by direct calculation is 0.90333.

4. DETAILS OF THE CALCULATION OF THE TABLES

The calculations for t_α were carried out as follows. First, we iterated on trial values of t_α until we found one which gave α to within 10^{-9} of the stated value. This value was then rounded to 5 decimal places and printed. Then each of these values and two values on either side, i.e. the value$+0.00002$, the value $+0.00001$, the value, the value-0.00001, the value-0.00002, were run through the program again and that value selected which corresponded to a computed α closest to that required. Note that this corresponds to a rounding procedure which can give slightly different results than the first procedure, and in a few cases we did get different answers. This second rounding procedure was then used on the numerical integration calculation and the calculations using the formulas given by Amos. Discrepancies were resolved by refining the calculations wherever that was necessary.

The same procedure was used in Tables II through V, except, of course, the noncentral t-distribution instead of the t-distribution was used. That is, a trial value of δ was calculated and then two values on either side of this value and the value itself were checked to see which gave the nearest value to the indicated β. The formulas for the checks were run in this latter manner also and all discrepancies were resolved except for $\alpha=0.005$, $\beta=0.01$, $f=7$ where one procedure gave $\delta=6.62076$ and another gave 6.62077. In this case it appears that the correct value of δ is so close to 6.620765 that it is impossible to determine whether this should be rounded up or down with the number of digits contained in the computer. We thought the evidence tipped the scales toward rounding up, but this evidence was not conclusive. We had some difficulty getting the case $f=1$ to check, but finally got the needed check by preparing special programs just for the case $f=1$.

5. ACKNOWLEDGMENT

The computing on the tables given here was programmed by Mrs. Marjorie Endres of Sandia Corporation. In order to compute, check and cross check these tables, approximately 1,000 pages of 11-inch by 12-inch output was printed. Mrs. Endres conscientiously examined all of this output for discrepancies which would indicate any possible errors in the original tables. In addition a great deal of the checking was done internal in the machine. These figures are only mentioned to give the reader some idea of the magnitude of a project calling for only a few pages of tables with high accuracy. There is certainly a great deal of redundancy in what was done here, but it is this redundancy that builds up confidence in the accuracy of the tables.

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