Chapter 1
Elementary analysis

Advanced algorithms on March 16, 2019

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Contents

Algorithm

What is an algorithm?

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Properties of algorithms

- Input from a specified set,
- Output from a specified set (solution),
- Definiteness of every step in the computation,
- Correctness of output for every possible input,
- Finiteness of the number of calculation steps,
- Effectiveness of each calculation step and
- Generality for a class of problems.

Complexity

- Generally, not much interested in time and space complexity for small inputs.
- Given two algorithms A and B for solving problem P.

Input size	Algorithm A	Algorithm B
n	5000 n	1.2^n
10	50,000	6
100	500,000	2,817,975
$1,000$	5,000,000	1.5×10^{79}
$100,000$	5×10^8	1.3×10^{7918}

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- B cannot be used for large inputs, while A is still feasible.
- So what is important is the growth of the complexity functions.
- Growth of time and space complexity with increasing input size n is a suitable measure for the comparison of algorithms.

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- Most important: Order of growth within a constant multiple as $n \to \infty$

Asymptotic growth rate

A way of comparing functions that ignores constant factors and small input sizes

- $O(g(n))$: class of functions $f(n)$ that grow no faster than $g(n)$
- $\Theta(g(n))$: class of functions $f(n)$ that grow at the same rate as $g(n)$
- $\Omega(g(n))$: class of functions $f(n)$ that grow at least as fast as $g(n)$

Complexity classes - a small vocabulary

- Constant: $O(1)$ (independing on the input size)
- Sub-linear or logarithmic: $O(\log n)$
- Linear: $O(n)$
- Quasi-linear: $O(n \log n)$
- Quadratic: $O(n^2)$
- Cubic: $O(n^3)$
- Polynomial: $O(n^p)$ ($O(n^2)$, $O(n^3)$, etc)
- Quasi-polynomial: $O(n^{\log(n)})$
- Exponential: $O(2^n)$
- Factorial: $O(n!)$

Asymptotic upper bound - worst case

Asymptotic upper bound “big O”

$T(n) = O(f(n))$ **iif** $\exists c \in R^+$, $c > 0$ and $\exists n_0 \in N$, $n_0 > 0$ such that $\forall n > n_0$: $T(n) \leq c \times f(n)$

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Example

Let $T(n) = 2n + 3n^3 + 5$. $T(n)$ is in $O(n^3)$ with:

- ($c = 8$ and $n_0 = 1$) or ($c = 5$ and $n_0 = 2$)

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- ($c = 8$ and $n_0 = 1$) or ($c = 5$ and $n_0 = 2$)

Principle: the lower-order terms are negligible.

Asymtotic lower bound - best case

"big Omega"

$T(n) = \Omega(f(n))$ **iif** $\exists c \in R^+$, $c > 0$ and $\exists n_0 \in N$, $n_0 > 0$ such that $\forall n > n_0$: $T(n) \geq c \times f(n)$

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Example

Let $T(n) = 2n + 3n^3 + 5$. $T(n)$ is in $\Omega(n^3)$ with:

- ($c = 1$ and $n_0 = 1$)

Asymtotic approximating bound - average case

"big Theta"

$T(n) = \Theta(f(n))$ **iif** $\exists c_1, c_2 \in R^+$, $c_1 > 0$, $c_2 > 0$ and $\exists n_0 \in N$, $n_0 > 0$
such that $\forall n > n_0$: $c_1 \times f(n) \leq T(n) \leq c_2 \times f(n)$

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Property

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Property

$T(n) = O(f(n))$ and $T(n) = \Omega(f(n)) \Longrightarrow T(n) = \Theta(f(n))$

Example

Let $T(n) = 2n + 3n^3 + 5$.
So, $T(n)$ is in $O(n^3)$ and in $\Omega(n^3)$.
Consequently, $T(n)$ is in $\Theta(n^3)$.

Other properties

Not transitive

- $f(n) = n^2$; $g(n) = n$
- $\Rightarrow f(n) = O(n^2) = g(n)$ but $f(n) \neq g(n)$

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- $f(n) = O(g(n))$ & $g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
- $f(n) = \Omega(g(n))$ & $g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$
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Additivity

- $f(n) = O(h(n))$ & $g(n) = O(h(n)) \Rightarrow f(n) + g(n) = O(h(n))$
- $f(n) = \Omega(h(n))$ & $g(n) = \Omega(h(n)) \Rightarrow f(n) + g(n) = \Omega(h(n))$
- $f(n) = \Theta(h(n))$ & $g(n) = \Theta(h(n)) \Rightarrow f(n) + g(n) = \Theta(h(n))$

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- if $T(n) = \begin{cases} 2n+5 & if \quad n \quad \text{is even} \\ n^2 - n + 1 & if \quad n \quad \text{is odd} \end{cases}$, then $T(n) = O(?)$ and $T(n) = \Omega(?)$.

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Compare the asymptotic behaviours of

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2. $\log_2 n$ and $\log_{10} n$

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1. Prove that for any positive functions f and g, $f(n) + g(n)$ and $max(f(n); g(n))$ are asymptotically equivalent.
2. Give a (necessary and sufficient) condition on positive functions f and g to ensure that $f(n) + g(n)$ and $f(n)$ are asymptotically equivalent.

Common asymptotic behaviours

Size	Approximate computational time					
n	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n \log n)$	$\Theta(n^2)$	$\Theta(2^n)$	$\Theta(n!)$
10	3.10^{-9}s	10^{-8}s	3.10^{-8}s	10^{-7}s	10^{-6}s	3.10^{-3}s
10^2	7.10^{-9}s	10^{-7}s	7.10^{-7}s	10^{-5}s	4.10^{13}y	*
10^3	10^{-8}s	10^{-6}s	10^{-5}s	10^{-3}s	*	*
10^4	$1,3.10^{-8}$s	10^{-5}s	10^{-4}s	10^{-1}s	*	*
10^5	$1,7.10^{-8}$s	10^{-4}s	2.10^{-3}s	10s	*	*
10^6	2.10^{-8}s	10^{-3}s	2.10^{-2}s	17m	*	*

Example

1. **Var** int: $d = 0$
2. **For** i from 1 to n **do**
   1. $d = d + 1$
   2. $a[i] = a[i] \times a[i] + d \times d$
3. **Endfor**

Example

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3. **Endfor**

Number of elementary operations: $1 + n \times (1 + 1) = 2n + 1$.

Linear loop example

1. **Var** int: $i = 1$
2. **While** $i \leq n$ **do**
    1. Write "Bonjour"
    2. $i = i + 1$
3. **EndWhile**

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Number of elementary operations: $2n + 1$.

Logarithmic loop example

1. **Var** int: $i = 1$
2. **While** $i \leq n$ **do**
   1. Write "Bonjour"
   2. $i = i \times 2$
3. **EndWhile**

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3. **EndWhile**

1. **Var** int: $i = n$
2. **While** $i \geq 1$ **do**
    1. Write "Bonjour"
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Number of elementary operations: $1 + \log_2(n)$.

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Nb of iterations = nb of iterations of external loop $\times$ nb of iterations of internal loop

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    1. **Var** int: $j = 1$
    2. **While** $j \leq n$ **do**
        1. Write "Bonjour"
        2. $j = j \times 3$
    3. **EndWhile**
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Number of elementary operations: $1 + n + n \times \log_3(n)$.

Exercise

Function XYZ(array: $a[]$)

1. **Var** int: i
2. **For** i from 1 to n **do**
    1. **Var** int: $t = a[i]$
    2. **Var** int: j
    3. **For** j from $i - 1$ to 0 **do**
        1. $a[j+1] = a[j]$
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    5. $a[j+1] = t$
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Homeworks

Give algorithms having number of elementary operations as below.

- $T_1(n) = 3 + 5n$
- $T_2(n) = n \log_2 n$
- $T_3(n) = n^3$
- $T_4(n) = (3n)!$
- $T_5(n) = \log_2(3n)$
- $T_6(n) = 2\log_3(2n)$
- $T_7(n) = n^2 \log_4 n$
- $T_8(n) = \sqrt{n}$
- $T_9(n) = \sqrt[3]{n^2}$
- $T_{10}(n) = 2^n$
- $T_{11}(n) = n!$