Understanding the Symbols in the Kalman Filter

1 Discrete-Time Instant (k)

Symbol: k
What it Represents: A discrete time step in the system.
Role in Kalman Filter: The Kalman filter operates recursively, updating the state estimate at each time step k.
Example: If you are tracking an object moving every 0.1s:

- $k = 0$ (initial time step)
- $k = 1$ (after 0.1s)
- $k = 2$ (after 0.2s), etc.

2 State Vector (x_k)

Symbol: $x_k \in \mathbb{R}^n$, where n is the number of state variables.
What it Represents: The state of the system at time k.
Role in Kalman Filter: The quantity we estimate at each time step.
Example: If tracking an object's position and velocity in 1D:

$$x_k = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix} \in \mathbb{R}^2$$

3 Control Input Vector (u_{k-1})

Symbol: $u_{k-1} \in \mathbb{R}^m$, where m is the number of control inputs.
What it Represents: The external control inputs applied at time $k - 1$.
Example: If an external acceleration affects an object's motion:

$$u_{k-1} = \text{applied acceleration}$$

4 Process Noise Vector (w_{k-1})

Symbol: $w_{k-1} \in \mathbb{R}^n$
What it Represents: Random disturbances in the system model (process

noise).
Role in Kalman Filter: Accounts for uncertainty in the system dynamics.
Example: Wind force affecting an object's motion.
Assumptions: White noise, zero mean, and covariance matrix:

$$Q_k = E[w_k w_k^T]$$

5 State Transition Matrix (A_{k-1})

Symbol: $A_{k-1} \in \mathbb{R}^{n \times n}$
What it Represents: Defines how the state evolves from time step $k-1$ to k.
Example: If tracking position and velocity:

$$A_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

6 Control Input Matrix (B_{k-1})

Symbol: $B_{k-1} \in \mathbb{R}^{n \times m}$
Example: If acceleration is the control input:

$$B_k = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$$

7 Output Matrix (C_k)

Symbol: $C_k \in \mathbb{R}^{r \times n}$, where r is the number of measurements.
Example: If we measure only position:

$$C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

8 Measurement Noise Vector (v_k)

Symbol: $v_k \in \mathbb{R}^r$
Covariance Matrix:

$$R_k = E[v_k v_k^T]$$

9 Kalman Filter Steps

9.1 Prediction Step

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}$$
$$P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^T + Q_k$$

9.2 Update Step

$$\begin{aligned} K_k &= P_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R_k)^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1}) \\ P_{k|k} &= (I - K_k C_k) P_{k|k-1} \end{aligned}$$

State Estimates in the Kalman Filter

Predicted State Estimate ($\hat{x}_k^-$)

This is the state prediction based on the motion model before incorporating a new measurement.

It is computed using the state transition equation:

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$$

It represents where we think the system should be, given previous motion.

Updated State Estimate ($\hat{x}_k$)

This is the corrected state estimate after incorporating the sensor measurement y_k.

It is computed using the Kalman update equation:

$$\hat{x}_k = \hat{x}_k^- + K_k(y_k - H\hat{x}_k^-)$$

It represents the best estimate of the state after blending the prediction with the new measurement.