The Fundamental Theorem of Calculus

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Introduction

- The Fundamental Theorem of Calculus (FTC) establishes the relationship between differentiation and integration.
- It has two main parts:
    1. The first part states that an antiderivative can be obtained through integration.
    2. The second part states that differentiation and integration are inverse processes.

Proof of Part 1 - Derivative Definition

Proof:

- Applying the Definition of the Derivative:
  To prove this formally, we use the definition of the derivative:

  $$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

  This definition expresses the rate of change of $F(x)$ as the limit of the difference quotient.

Expressing $F(x+h) - F(x)$ Using the Integral Definition:

Since $F(x)$ is defined as:

$$F(x) = \int_a^x f(t)\,dt,$$

- Substituting $x + h$ gives:

$$F(x+h) = \int_a^{x+h} f(t)\,dt.$$

- Now, computing the difference:

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

- Using the additivity property of definite integrals:

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt.$$

- Thus, subtracting $\int_a^x f(t)\,dt$ (ie.. $F(x)$) from both sides, we get:

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt.$$

Applying the Mean Value Theorem for Integrals:

The Mean Value Theorem for Integrals states that if f is continuous on an interval $[a, b]$, then there exists some point $c \in [a, b]$ such that:

$$\int_a^b f(t)\,dt = f(c) \cdot (b - a).$$

Applying this theorem to the integral $\int_x^{x+h} f(t)\,dt$, we conclude that there exists some $c \in [x, x+h]$ such that:

$$\int_x^{x+h} f(t)\,dt = f(c) \cdot h.$$

This equation tells us that the integral over the small interval $[x, x+h]$ can be approximated as $f(c)$ times the width of the interval h.

Taking the Limit:

Dividing both sides by h, we obtain:

$$\frac{F(x+h)-F(x)}{h} = f(c).$$

Now, we take the limit as $h \to 0$. Since c is in the interval $[x, x+h]$ and $f(x)$ is continuous, we know that $c \to x$ as $h \to 0$, so $f(c) \to f(x)$.
So we get:

$$\lim_{h \to 0} \frac{F(x+h)-F(x)}{h} = f(x).$$

Since this is exactly the definition of the derivative, we conclude:

$$F'(x) = f(x).$$

Conclusion of Part 1:

- This completes the proof of Part 1 of the Fundamental Theorem of Calculus.
- The key takeaway is that if we construct a function by integrating another function, then the derivative of that integral function brings us back to the original function.
- This establishes the fundamental inverse relationship between differentiation and integration.

Proof of Part 2 of the Fundamental Theorem of Calculus

Step 1: Understanding the Goal

The second part of the Fundamental Theorem of Calculus states that:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$, meaning:

$$F'(x) = f(x).$$

This result is crucial because it tells us that we can evaluate a definite integral simply by finding an antiderivative rather than computing an infinite sum of areas.

Step 2: Partitioning the Interval

To prove this, we divide the interval $[a, b]$ into n subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Each subinterval has a small width Δx_i, which is given by:

$$\Delta x_i = x_i - x_{i-1}.$$

The goal is to approximate the integral

$$\int_a^b f(x)\,dx$$

using a Riemann sum and then take the limit as the partition gets finer.

Step 3: Applying the Mean Value Theorem

Since $F(x)$ is differentiable, we apply the Mean Value Theorem (MVT) to each subinterval $[x_{i-1}, x_i]$. The MVT states that for each subinterval, there exists some point $c_i \in [x_{i-1}, x_i]$ such that:

$$F(x_i) - F(x_{i-1}) = F'(c_i) \cdot \Delta x_i.$$

Since we know that $F'(x) = f(x)$, this simplifies to:

$$F(x_i) - F(x_{i-1}) = f(c_i) \cdot \Delta x_i.$$

This equation expresses the small change in $F(x)$ over each subinterval in terms of the function $f(x)$.

Step 4: Summing Over All Subintervals

Now, we sum this equation over all subintervals:

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

On the left-hand side, notice that it forms a telescoping sum:

$$[F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \cdots + [F(x_n) - F(x_{n-1})].$$

Since all the intermediate terms cancel out, we are left with:

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

The right-hand side is a Riemann sum, which approximates the integral:

$$\int_a^b f(x)\,dx.$$

Step 5: Taking the Limit

To obtain the exact value of the integral, we take the limit as the partition gets infinitely fine (i.e., as the maximum subinterval width approaches zero):

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$$

Since the left-hand side is already equal to $F(b) - F(a)$, we conclude:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This proves that the definite integral can be computed simply by evaluating the antiderivative at the endpoints.

Part 2 Conclusion

- This proof shows that instead of manually computing an infinite sum of areas, we can evaluate a definite integral just by finding an antiderivative and subtracting its values at the limits of integration.
- The Fundamental Theorem of Calculus (Part 2) is powerful because it transforms the problem of summing an infinite number of tiny areas into a simple subtraction problem.

Applications of the Fundamental Theorem of Calculus

- **Simplifies Computation:** The FTC allows us to compute definite integrals using antiderivatives, avoiding the need for summing infinite small areas. Such as:

$$\int_0^1 2x\,dx = x^2\Big|_0^1 = 1 - 0 = 1.$$

- **Physics & Motion:** Used to calculate displacement, velocity, and acceleration in kinematics.
  - Example: If a particle moves with velocity $v(t)$, its displacement over time $[a, b]$ is given by:

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

  where $s(t)$ is the position function.

- **Engineering & Signal Processing:** Used in control systems, fluid dynamics, and electrical circuits.
  - Example: Finding the charge Q stored in a capacitor by integrating the current over time:

$$Q = \int_0^T I(t)\,dt$$

- **Economics & Finance:** Used to model accumulated profit, cost, and consumer/producer surplus.
  - Example: If $M(x)$ is the marginal cost function, the total cost over an interval is:

$$\int_{x_1}^{x_2} M(x)\,dx = C(x_2) - C(x_1)$$

- **Probability & Statistics:** Helps compute probabilities from probability density functions (PDFs).
  - Example: The probability that a continuous random variable X lies in an interval $[a, b]$ is:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

  where $f(x)$ is the probability density function.

Conclusion

- **Bridge Between Differentiation and Integration:** The Fundamental Theorem of Calculus (FTC) establishes the deep connection between these two fundamental operations.
- **Part 1 - Differentiation of Integrals:** States that the derivative of an integral of a function recovers the original function, confirming that differentiation undoes integration.
- **Part 2 - Evaluating Definite Integrals:** Provides an efficient method for computing definite integrals by evaluating an antiderivative at the limits of integration, avoiding the complexity of Riemann sums.
- **Practical Significance:** The FTC simplifies complex computations and has widespread applications in physics, engineering, economics, and probability theory.
- **Core Principle of Calculus:** Serves as a foundational result in real analysis and advanced mathematics, enabling the development of further concepts such as differential equations, multivariable calculus, and integral transformations.
- **Conclusion:** The FTC is not only a fundamental result in pure mathematics but also a powerful tool that simplifies problem-solving across various scientific and engineering disciplines.