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Sequences and Series

Sequence Convergence

The convergence of a sequence is one of the most fundamental concepts in mathematics. A sequence converges to a specific value, called the limit, if its terms progressively get closer to this value as the sequence grows (i.e., as n grows to infinity).

More specifically, let $\{a_n\}$ denote a sequence. The sequence $\{a_n\}$ converges to the limit a, if for every positive number ε, there exists an integer $N \in \mathbb{N}$, such that whenever $n \geq N$, then $|a_n - a| < \varepsilon$. This mathematical definition says that for sufficiently large n, the values of the sequence are confined to within an arbitrarily small distance ε of the limit a.

Algebraic and Order Limit Theorems

The algebraic and order limit theorems are important principles that govern the limits of sequences. The algebraic limit theorems focus on different arithmetic operations involving limits. For example, these theorems state that the limit of a sum or difference of sequences is just the sum or difference of their individual sequence limits. Similarly, the limit of a product or quotient of sequences is just the product or quotient of their respective limits (as long as the quotient denominator is not zero). The order limit theorems provide results regarding the ordering of sequences. For example, one result of the order limit theorems says that if sequence $\{a_n\}$ has limit a, and sequence $\{b_n\}$ has limit b, and every term of $\{a_n\}$ is less than or equal to every term of $\{b_n\}$, then $a \leq b$.

Monotone Convergence Theorem

The Monotone Convergence Theorem can be applied to either an increasing sequence (i.e. a sequence where each term is greater than or equal to the preceding term) or a decreasing sequence (i.e. a sequence where each term is less than or equal to the preceding term) whenever the sequence is bounded. For these bounded sequences, the Monotone Convergence Theorem says the sequences must converge to a limit. Note, this theorem doesn't explicitly state what the limit is, but it is a powerful result since it can be used to establish that sequence is convergent and thus its limit exists.

The Cauchy Criterion

The Cauchy criterion is another useful result for establishing that a sequence is convergent without explicitly finding what the sequence converges to. The Cauchy criterion states that a sequence converges if and only if the sequence is a Cauchy sequence. So, to establish that a sequence is convergent using this criterion, it is sufficient to show that the sequence is a Cauchy sequence. A Cauchy sequence is one in which adjacent terms get arbitrarily close to each other as the sequence progresses. More formally, a sequence is a Cauchy sequence if, for any value $\varepsilon > 0$, there exists an integer N, such that for all m, $n \geq N$, the absolute difference between terms in the sequence satisfies $|a_n - a_m| < \varepsilon$.