MODULE SIX PROBLEM SET

Gary Hobson

Problem 1

Text Problem 5.2.3

(a) Compute the derivative of $h(x) = \frac{1}{x}$ using Definition 5.2.1
The definition of the derivative states that:

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}.$$

Substituting $h(x) = \frac{1}{x}$:

$$h'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}.$$

Simplify the numerator:

$$\frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)}.$$

So, the difference quotient becomes:

$$\frac{-h}{h \cdot x(x+h)} = \frac{-1}{x(x+h)}.$$

Taking the limit as $h \to 0$, we get:

$$h'(x) = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$

So, the derivative of $h(x) = \frac{1}{x}$ is:

$$h'(x) = -\frac{1}{x^2}.$$

(b) Use the Chain Rule to Prove Part (iv) of Theorem 5.2.4
The quotient rule is:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad \text{for } g(x) \neq 0.$$

To prove this using the Chain Rule, So:

$$\frac{1}{g(x)} = (g(x))^{-1}.$$

Applying the Chain Rule:

$$\left(g(x)^{-1}\right)' = -g(x)^{-2}g'(x).$$

Using the result from part (a):

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}.$$

Now for the general case $f(x)/g(x)$:

$$\frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}.$$

Product rule:

$$\left(\frac{f}{g}\right)' = f'g^{-1} + f \cdot (-g^{-2}g').$$
$$= \frac{f'g - fg'}{g^2}.$$

which is the quotient rule.

(c) Prove Theorem 5.2.4(iv) Directly
The definition of the derivative is:

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}.$$

Rewriting the numerator:

$$\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}.$$

Using the difference quotient:

$$\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}.$$

Rewriting:

$$= \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{hg(x+h)g(x)}.$$

Splitting the terms:

$$= \frac{g(x)\cdot \frac{f(x+h)-f(x)}{h} - f(x)\cdot \frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}.$$

Taking the limit:

$$= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

which is the quotient rule.

Problem 2

Text Problem 5.2.9 (a)

- The function f is differentiable on some interval I, meaning f' exists at every point in I.
- The function f' is *not constant*, meaning there exist points $x_1, x_2 \in I$ where $f'(x_1) \neq f'(x_2)$.
- We need to determine whether f' necessarily takes an irrational value somewhere in I.

- The key observation is that f' is a real-valued function.
- The set of rational numbers $\mathbb{Q}$ is countable, whereas the set of real numbers $\mathbb{R}$ is uncountable.
- If f' is nonconstant, it varies over some interval, and if it were to take only rational values, it would be a function with a countable range.
- However, differentiability typically implies a function is well behaved, and it is difficult (though not impossible) for a differentiable function to take only rational values.

To disprove the conjecture, I will construct a differentiable function whose derivative is nonconstant yet always rational. Consider:

$$f(x) = \frac{x^2}{2}$$

- The derivative is $f'(x) = x$.
- This function is differentiable everywhere, and $f'(x)$ is nonconstant.
- If we restrict $f'(x)$ to an interval where x is always rational (say, $I = \mathbb{Q} \cap (0, 1)$), then f' takes only rational values on I.
- However, f' still varies, meaning it is nonconstant.

Thus, we have found a counterexample where f' is nonconstant but does not necessarily take any irrational values.

Problem 3

Text Problem 5.3.3

(a) Is there a $d \in [0, 3]$ where $h(d) = d$

Define the function:

$$g(x) = h(x) - x.$$

We want to show that there exists some $d \in [0, 3]$ such that $g(d) = 0$, i.e., $h(d) = d$.

- Compute $g(0)$ and $g(3)$:

$$g(0) = h(0) - 0 = 1 - 0 = 1.$$

$$g(3) = h(3) - 3 = 2 - 3 = -1.$$

- Since $g(3) = -1$ and $g(0) = 1$, we see that $g(x)$ changes sign on $[0, 3]$.
- By the Intermediate Value Theorem, because $g(x)$ is continuous, there must exist some $d \in (0, 3)$ where $g(d) = 0$, meaning:

$$h(d) = d.$$

(b) Is there a c where $h'(c) = \frac{1}{3}$

Define the function:

$$p(x) = h(x) - 2.$$

From the given values, we have:

$$p(1) = h(1) - 2 = 0, \quad p(3) = h(3) - 2 = 0.$$

So, since $p(x)$ is differentiable, we apply Rolle's Theorem, which guarantees that there exists some $c \in (1, 3)$ where:

$$p'(c) = 0.$$

Since $p'(x) = h'(x)$, I can conclude:

$$h'(c) = 0 \quad \text{for some } c \in (1, 3).$$

Then, using the Mean Value Theorem on $h(x)$ over the interval $[0, 3]$, we get:

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}.$$

Thus, there must be some $c \in (0, 3)$ where $h'(c) = \frac{1}{3}$.

(c) Existence of x where $h'(x) = \frac{1}{4}$

Applying the Mean Value Theorem again, I have two subintervals: $[0, 1]$ and $[1, 3]$. Since $h(x)$ is differentiable, MVT guarantees the existence of some points $c_1 \in (0, 1)$ and $c_2 \in (1, 3)$ where:

$$h'(c_1) = \frac{h(1) - h(0)}{1 - 0} = \frac{2 - 1}{1} = 1.$$

$$h'(c_2) = \frac{h(3) - h(1)}{3 - 1} = \frac{2 - 2}{2} = 0.$$

Since $h'(x)$ is differentiable on $[0, 3]$, the Intermediate Value Theorem for derivatives guarantees that $h'(x)$ takes every value between 0 and 1 within this interval. In particular, there must be some $x \in (0, 3)$ such that:

$$h'(x) = \frac{1}{4}.$$

Problem 4

Text Problem 5.3.10

Since $\lim_{x\to 0} f(x)$ Since $\sin(1/x^4)$ is always bounded between -1 and 1. So,

$$-xe^{-1/x^2} \leq f(x) \leq xe^{-1/x^2}.$$

Since $xe^{-1/x^2} \to 0$ as $x \to 0$, we apply the squeeze theorem to conclude:

$$\lim_{x\to 0} f(x) = 0.$$

Since $\lim_{x\to 0} g(x)$ is

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} e^{-1/x^2} = 0.$$

and Since $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is

$$\frac{f(x)}{g(x)} = x \sin\left(\frac{1}{x^4}\right).$$

Since $\sin(1/x^4)$ oscillates between -1 and 1, it is bound:

$$-x \leq x\sin(1/x^4) \leq x.$$

Using the squeeze theorem:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0.$$

To compute $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ you must differentiate $f(x)$ and $g(x)$ using the product and chain rules.

Derivative of $g(x)$:

$$g'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}.$$

Derivative of $f(x)$:

$$f'(x) = \left[x\sin(1/x^4) \cdot e^{-1/x^2}\right]'.$$

Using the product rule:

$$f'(x) = e^{-1/x^2}\left(\sin(1/x^4) + x\cos(1/x^4) \cdot \left(-\frac{4}{x^5}\right)\right).$$

Since $\sin(1/x^4)$ is bounded and the second term oscillates while decaying, the dominant term in both derivatives is e^{-1/x^2}. This results in:

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0.$$

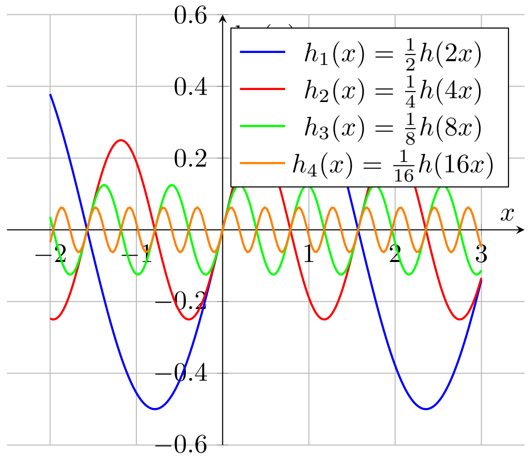
This result is surprising because I expect the ratio of derivatives to behave similarly to the function ratio (per L'Hpital's rule). However, due to the oscillatory behavior of $\sin(1/x^4)$ and the exponential damping from e^{-1/x^2}, both the function and its derivative approach zero at the same rate.

This does **not** contradict Theorem 5.3.6.1 because the conditions of L'Hpital's rule hold, and I simply found that the limit of their derivative ratio also vanishes.

Problem 5

Text Problem 5.4.1

Graph of $\frac{1}{2}h(2x)$ on $[-2, 3]$ I let $h(x) = \sin(x)$ for demonstration purposes.



Qualitative description of $h_n(x)$ as n increases

The function sequence is given by:

$$h_n(x) = \frac{1}{2^n}h(2^n x).$$

As n increases:

- The factor 2^n compresses $h(x)$ horizontally, meaning oscillations become more frequent.
- The factor $\frac{1}{2^n}$ scales $h(x)$ vertically, making the function shrink closer to the x-axis.
- The functions $h_n(x)$ oscillate faster but with decreasing amplitude.

The function $g(x)$ is defined as:

$$g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}h(2^n x).$$

Since $g(x)$ is defined as a sum of continuous functions where each term gets smaller, the Weierstrass M-test ensures uniform convergence, which implies $g(x)$ is continuous everywhere.

The oscillations of $h_n(x)$ at finer and finer scales cause the function $g(x)$ to behave erratically. Since the difference quotients do not converge uniformly, $g(x)$ is not differentiable at any point.

Problem 6

Text Problem 5.4.5

To determine whether $g'(0)$ exists, we use the definition of the derivative:

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x}.$$

If this limit does not exist, then $g'(0)$ does not exist.

We are given the difference quotient:

$$\frac{g(x_m) - g(0)}{x_m} = m + 1.$$

Taking the limit as $m \to \infty$, we observe that this expression grows arbitrarily large, meaning the right-hand limit approaches $+\infty$.

Now, consider a different sequence $x_m = -\frac{1}{2^m}$ approaching 0. For this sequence:

$$\frac{g(x_m) - g(0)}{x_m} \to -\infty.$$

Since the difference quotient approaches $+\infty$ along one sequence and $-\infty$ along another, the left-hand and right-hand limits do not agree.

Thus, we conclude that:

$$g'(0) \text{ does not exist.}$$