The Fundamental Theorem of Calculus

Gary Hobson

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Introduction

- The Fundamental Theorem of Calculus (FTC) establishes the relationship between differentiation and integration.
- It has two main parts:
  1. The first part states that an antiderivative can be obtained through integration.
  2. The second part states that differentiation and integration are inverse processes.

Statement of the Theorem

Fundamental Theorem of Calculus (FTC):
Part 1: If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then F is differentiable, and

$$F'(x) = f(x) \quad \text{for all } x \in [a, b].$$

Part 2: If F is any antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof of Part 1

Goal: Show that if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.
Proof:

- Consider the definition of the derivative:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

- Expanding $F(x+h)$:

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

- By the properties of definite integrals:

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

- Using the Mean Value Theorem for Integrals:

$$\int_x^{x+h} f(t)\,dt = f(c) \cdot h, \quad \text{for some } c \in [x, x+h].$$

- Taking the limit:

$$\lim_{h \to 0} \frac{f(c) \cdot h}{h} = f(x).$$

So, $F'(x) = f(x)$.

Proof of Part 2

Goal: Show that if F is an antiderivative of f, then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof:

- Define a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with subintervals $\Delta x_i = x_i - x_{i-1}$.
- Using the Riemann sum:
$$S = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$
- By the Mean Value Theorem, there exists $c_i \in [x_{i-1}, x_i]$ such that:
$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i.$$
- Summing over all intervals:
$$\sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a).$$
- Taking the limit as $\max \Delta x_i \to 0$:
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So, the theorem is proved.

Applications of the FTC

- The FTC allows us to compute definite integrals using antiderivatives.
- Used in physics, engineering, and economics for solving area, velocity, and accumulation problems.
- Example:

$$\int_0^1 2x\,dx = x^2\Big|_0^1 = 1 - 0 = 1.$$

Conclusion

- The Fundamental Theorem of Calculus bridges differentiation and integration.
- Part 1 states that differentiation undoes integration.
- Part 2 provides an efficient way to evaluate definite integrals.
- This theorem is foundational in calculus and real analysis.