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Module 3 Homework

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5.4 - 5

$$x_{n+2} = c_0x_n + c_1x_{n+1} \pmod{3}$$

this gives:

$$1, 1, 0, 2, 2, 0, 1, 1$$

n	x_n	x_{n+1}	x_{n+2}	Equation
0	1	1	0	$0 = c_0(1) + c_1(1) \pmod{3}$
1	1	0	2	$2 = c_0(1) + c_1(0) \pmod{3}$

$$c_0 + c_1 \equiv 0 \pmod{3}$$

$$c_0 \equiv 2 \pmod{3}$$

So,

$$c_0 = 2$$

Substitute,

$$2 + c_1 \equiv 0 \pmod{3} \Rightarrow c_1 \equiv -2 \equiv 1 \pmod{3}$$

$$c_0 = 2, \quad c_1 = 1 \pmod{3}$$

6.6 - 2

$$\det(A) = (1)(1) - (1)(1) = 0$$

So,

$$\det(A) \equiv 0 \pmod{26}$$

If $\det(A) \equiv 0 \pmod{26}$, then:

$$\gcd(0, 26) = 26 \neq 1$$

Since the GCD is not 1, the matrix is not invertible modulo 26.

So, decryption is impossible.

6.6 - 6

- aaa \rightarrow vector $(0, 0, 0)$
- baa \rightarrow vector $(1, 0, 0)$
- aba \rightarrow vector $(0, 1, 0)$
- aab \rightarrow vector $(0, 0, 1)$

The last three are needed to form a 3×3 identity matrix when stacked:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, the corresponding ciphertext blocks will become the rows of the matrix M :

$$C = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \Rightarrow M = C$$

Choose three plaintexts:

"baa", "aba", "aab" \rightarrow these convert to vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Encrypt each using the unknown Hill cipher.

Stack the resulting ciphertexts as rows to obtain the matrix M .

6.6 - 8

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \pmod{26}$$

Let $P = \begin{pmatrix} a \\ b \end{pmatrix}$ be a plaintext vector. Encryption is:

$$C = M \cdot P \pmod{26}$$

Find two vectors $P_1 \neq P_2$ such that:

$$M \cdot P_1 \equiv M \cdot P_2 \pmod{26}$$

This implies:

$$M \cdot (P_1 - P_2) \equiv 0 \pmod{26}$$

Let's try:

$$P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 14 \\ 13 \end{pmatrix}$$

Compute:

$$M \cdot P_1 = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 1 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$M \cdot P_2 = \begin{pmatrix} 1 \cdot 14 + 2 \cdot 13 \\ 3 \cdot 14 + 4 \cdot 13 \end{pmatrix} = \begin{pmatrix} 14 + 26 \\ 42 + 52 \end{pmatrix} = \begin{pmatrix} 40 \\ 94 \end{pmatrix} \pmod{26} = \begin{pmatrix} 40 \pmod{26} \\ 94 \pmod{26} \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

Since these are not equal, let's try a simpler approach.

Compute the null space (kernel) of M modulo 26. That is, find $v = \begin{pmatrix} x \\ y \end{pmatrix}$ such that:

$$M \cdot v \equiv 0 \pmod{26} \quad \Rightarrow \quad \begin{cases} x + 2y \equiv 0 \pmod{26} \\ 3x + 4y \equiv 0 \pmod{26} \end{cases}$$

Solve:

From the first equation:

$$x \equiv -2y \pmod{26}$$

Substitute into the second:

$$3(-2y) + 4y \equiv -6y + 4y = -2y \equiv 0 \pmod{26} \quad \Rightarrow \quad y \equiv 0 \pmod{13}$$

Try $y = 13$, then:

$$x = -2 \cdot 13 = -26 \equiv 0 \pmod{26}$$

Thus, $v = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$.

For any plaintext P , $P + v$ will encrypt to the same ciphertext. Let's test this:

Let:

$$P_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Then:

$$P_2 = \begin{pmatrix} 1 \\ 2 + 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \end{pmatrix}$$

Compute:

$$M \cdot P_1 = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 \\ 3 \cdot 1 + 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 + 4 \\ 3 + 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

$$M \cdot P_2 = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 15 \\ 3 \cdot 1 + 4 \cdot 15 \end{pmatrix} = \begin{pmatrix} 1 + 30 \\ 3 + 60 \end{pmatrix} = \begin{pmatrix} 31 \\ 63 \end{pmatrix} \pmod{26} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

Confirmed: $P_1 = (1, 2)$ and $P_2 = (1, 15)$ encrypt to the same ciphertext.