MODULE SIX QUIZ

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Problem 1

Text Problem 4.2.5 (b,d)

PROBLEM 4.2.5

Definition 4.2.1 (Functional Limit). A function f has a limit L at $x = c$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \epsilon$.

(b) $\lim_{x \to 0} x^3 = 0$**. Proof:**

- Let $\epsilon > 0$.
- We need to find $\delta > 0$ such that if $0 < |x - 0| < \delta$, then:

$$|x^3 - 0| < \epsilon.$$

- This simplifies to $|x^3| < \epsilon$.
- Since $|x^3| = |x|^3$, choose $\delta = \epsilon^{1/3}$.
- Then, if $0 < |x| < \delta$, we get:

$$|x^3| < \delta^3 = (\epsilon^{1/3})^3 = \epsilon.$$

Thus, $\lim_{x \to 0} x^3 = 0$.

(d) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$**. Proof:**

- Let $\epsilon > 0$.
- We need to find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then:

$$\left| \frac{1}{x} - \frac{1}{3} \right| < \epsilon.$$

- Simplify:

$$\left| \frac{1}{x} - \frac{1}{3} \right| = \left| \frac{3 - x}{3x} \right|.$$

- Assume $|x - 3| < 1$, so $2 < x < 4$, giving $6 < 3x < 12$ and $|3x| \geq 6$.
- Choose $\delta = \min\left(1, \frac{6\epsilon}{1}\right) = \min(1, 6\epsilon)$.
- Then, if $0 < |x - 3| < \delta$, we have:

$$\left| \frac{3 - x}{3x} \right| \leq \frac{|x - 3|}{6} < \frac{\delta}{6} \leq \epsilon.$$

Thus, $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

Problem 2

Text Problem 4.2.8 (b,d)

(b) $\lim_{x \to 7/4} \frac{|x-2|}{x-2}$**. Solution:**

- This is similar to part (a), but with a different point of approach:
  - If $x > 2$, then $|x-2| = x-2$, so:
  $$\frac{|x-2|}{x-2} = 1.$$
  - If $x < 2$, then $|x-2| = -(x-2)$, so:
  $$\frac{|x-2|}{x-2} = -1.$$
- Since $7/4 = 1.75 < 2$, we are always in the case where $|x-2| = -(x-2)$, so:

$$\lim_{x \to 7/4} \frac{|x-2|}{x-2} = -1.$$

(d) $\lim_{x \to 0} \sqrt{3x}(-1)^{\lfloor 1/x \rfloor}$**. Solution:**

- This function is a product of $\sqrt{3x}$ and $(-1)^{\lfloor 1/x \rfloor}$.
- **Behavior of** $\sqrt{3x}$:
  - If $x > 0$, then $\sqrt{3x} \to 0$.
  - If $x < 0$, then $\sqrt{3x}$ is not defined in the real numbers because $3x < 0$.
- **Behavior of** $(-1)^{\lfloor 1/x \rfloor}$:
  - As seen in part (c), this function oscillates indefinitely between 1 and -1.
- Since $\sqrt{3x} \to 0$ as $x \to 0^+$, while $(-1)^{\lfloor 1/x \rfloor}$ oscillates, the product also goes to 0 because an oscillating function multiplied by something tending to 0 results in a limit of 0.

$$\lim_{x \to 0} \sqrt{3x}(-1)^{\lfloor 1/x \rfloor} = 0.$$

Problem 3

Text Problem 4.3.6 (c,d)

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0

if,

$$f(x) = 0, \quad \text{which is continuous at } x = 0.$$

and $g(x)$ be any function that is not continuous at $x = 0$, such as:

$$g(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

then,

$$f(x)g(x) = 0 \cdot g(x) = 0,$$

which is constant and continuous everywhere, including at $x = 0$.

So, this is possible.

(d) A function $f(x)$ is not continuous at 0 so that $f(x) + \frac{1}{f(x)}$ is continuous at 0

- If such a function $f(x)$ exists and $f(x)$ is not continuous at $x = 0$, it must have a jump, oscillation, or an undefined point.
- If $f(x) \to 0$, then $1/f(x)$ would diverge to $\pm\infty$, making $f(x) + 1/f(x)$ undefined at $x = 0$.
- If $f(x)$ jumps between two values, then $1/f(x)$ will also jump, making $f(x) + 1/f(x)$ discontinuous.

So, no such function can exist, making this **impossible**.

Problem 4

Text Problem 4.4.2 (c)

(c) Is $h(x) = x\sin(1/x)$ uniformly continuous on $(0,1)$?

So,

$$h'(x) = \sin(1/x) + x\cos(1/x)\cdot(-1/x^2) = \sin(1/x) - \frac{\cos(1/x)}{x}.$$

$\frac{\cos(1/x)}{x}$, grows larger as $x \to 0$. So, the function is going to have large oscillations near 0. This will prevent uniform continuity.

Because:

$$x_n = \frac{1}{\pi n}, \quad y_n = \frac{1}{\pi n + \pi/2}.$$

For these choices,

$$|x_n - y_n| \to 0 \quad \text{but} \quad |h(x_n) - h(y_n)| \to 1.$$

Since $|h(x_n) - h(y_n)|$ does not get arbitrarily small, $h(x)$ fails the uniform continuity condition.

So, $h(x)$ is **not** uniformly continuous on $(0,1)$.

Problem 5

Text Problem 5.2.5

(a) A function is continuous at $x = 0$ if:

$$\lim_{x \to 0} f(x) = f(0).$$

Since $f(0) = 0$, we need to check:

$$\lim_{x \to 0^+} x^a = 0.$$

- If $a > 0$, then $x^a \to 0$ as $x \to 0^+$, so $f(x)$ is continuous at 0.
- If $a \leq 0$, then $\lim_{x \to 0^+} x^a$ does not necessarily approach 0 (it diverges to ∞ if $a < 0$), meaning $f(x)$ is not continuous.

So, $f(x)$ is **continuous at** $x = 0$ **for** $a > 0$.

(b) The derivative at $x = 0$ is given by the definition:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

Since $f(0) = 0$, this simplifies to:

$$f'(0) = \lim_{h \to 0} \frac{h^a}{h} = \lim_{h \to 0} h^{a-1}.$$

- If $a - 1 > 0$ (i.e., $a > 1$), then $h^{a-1} \to 0$ as $h \to 0$, so $f(x)$ is differentiable at $x = 0$.
- If $a - 1 < 0$ (i.e., $a < 1$), then $h^{a-1} \to \infty$, so $f(x)$ is not differentiable.
- If $a = 1$, then $h^{a-1} = h^0 = 1$, so $f'(0) = 1$, and differentiability holds.

Thus, $f(x)$ is **differentiable at** $x = 0$ **for** $a > 1$.

Continuity of the derivative: The derivative for $x > 0$ is:

$$f'(x) = ax^{a-1}.$$

For continuity at $x = 0$:

$$\lim_{x \to 0^+} ax^{a-1} = 0.$$

This is only true when $a - 1 > 0$, or $a > 1$.
Thus, $f'(x)$ is **continuous at** $x = 0$ **when** $a > 1$.

(c) To check second-order differentiability, compute:

$$f''(x) = a(a-1)x^{a-2}, \quad x > 0.$$

For differentiability at $x = 0$:

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{ah^{a-1}}{h} = \lim_{h \to 0} ah^{a-2}.$$

- If $a - 2 > 0$ (i.e., $a > 2$), then $h^{a-2} \to 0$, so $f''(x)$ exists at $x = 0$.
- If $a - 2 \leq 0$, then h^{a-2} does not go to 0, meaning the second derivative does not exist.

So, $f(x)$ is **twice differentiable at** $x = 0$ **for** $a > 2$.