MODULE EIGHT PROBLEM SET

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Problem 1

Text Problem 7.2.2

(a) Compute $L(f,P)$, $U(f,P)$, and $U(f,P) - L(f,P)$

The lower sum $L(f,P)$ and upper sum $U(f,P)$ are given by:

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$$

where m_i is the minimum function value in each subinterval, and M_i is the maximum function value.

The subintervals of the partition are:

(1) $[1, 3/2]$ with width $\Delta x_1 = 3/2 - 1 = 1/2$.
(2) $[3/2, 2]$ with width $\Delta x_2 = 2 - 3/2 = 1/2$.
(3) $[2, 4]$ with width $\Delta x_3 = 4 - 2 = 2$.

For each subinterval, the function values are:

$$m_1 = f(3/2) = \frac{2}{3}, \quad M_1 = f(1) = 1,$$
$$m_2 = f(2) = \frac{1}{2}, \quad M_2 = f(3/2) = \frac{2}{3},$$
$$m_3 = f(4) = \frac{1}{4}, \quad M_3 = f(2) = \frac{1}{2}.$$

Lower Sum:

$$\begin{aligned} L(f,P) &= \left(\frac{2}{3} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{4} \times 2\right) \\ &= \frac{1}{3} + \frac{1}{4} + \frac{1}{2} = \frac{4}{12} + \frac{3}{12} + \frac{6}{12} = \frac{13}{12}. \end{aligned}$$

Upper Sum:

$$\begin{aligned} U(f,P) &= \left(1 \times \frac{1}{2}\right) + \left(\frac{2}{3} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times 2\right) \\ &= \frac{1}{2} + \frac{1}{3} + 1 = \frac{3}{6} + \frac{2}{6} + \frac{6}{6} = \frac{11}{6}. \end{aligned}$$

Difference:

$$U(f,P) - L(f,P) = \frac{11}{6} - \frac{13}{12} = \frac{22}{12} - \frac{13}{12} = \frac{9}{12} = \frac{3}{4}.$$

(b) Effect of Adding $x = 3$ to the Partition
If we add $x = 3$ to the partition, the subintervals become:

$$[1, 3/2], \quad [3/2, 2], \quad [2, 3], \quad [3, 4].$$

Since the function $f(x) = 1/x$ is decreasing, refining the partition further reduces the difference between the upper and lower sums. Thus,

$$U(f, P) - L(f, P) \text{ decreases.}$$

(c) Finding P' such that $U(f, P') - L(f, P') < 2/5$
To further reduce the difference, we refine the partition by adding points where the function changes rapidly (i.e., smaller x values). A possible choice for a finer partition is:

$$P' = \{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, 3, 4\}.$$

This finer partition ensures:

$$U(f, P') - L(f, P') < \frac{2}{5}.$$

Problem 2

Text Problem 7.2.7

Part 1: Definition of Upper and Lower Sums For any partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$, the lower sum and upper sum are defined as:

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

where:

- $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ (minimum function value in each subinterval),
- $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ (maximum function value in each subinterval),
- $\Delta x_i = x_i - x_{i-1}$ is the width of each subinterval.

Part 2: Behavior of an Increasing Function Since f is increasing, we know that:

$$m_i = f(x_{i-1}), \quad M_i = f(x_i).$$

Thus, the difference between the upper and lower sum simplifies to:

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \Delta x_i.$$

Part 3: Convergence to Zero As we refine the partition by making $\max \Delta x_i \to 0$, the difference $f(x_i) - f(x_{i-1})$ within each subinterval becomes small. The sum of these differences over all subintervals approximates the total variation of f, which is finite since f is increasing.

Thus, as the partition gets finer, the difference between the upper and lower sums can be made arbitrarily small:

$$\lim_{\|P\| \to 0} (U(f, P) - L(f, P)) = 0.$$

By the definition of Riemann integrability, this confirms that f is integrable on $[a, b]$.

Problem 3

Text Problem 7.4.6

(a) if $|f(x)| \leq M$ on $[a, b]$, then

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

The identity

$$a^2 - b^2 = (a - b)(a + b),$$

setting $a = f(x)$ and $b = f(y)$, we obtain

$$(f(x))^2 - (f(y))^2 = (f(x) - f(y))(f(x) + f(y)).$$

Taking absolute values,

$$|(f(x))^2 - (f(y))^2| = |f(x) - f(y)| \cdot |f(x) + f(y)|.$$

Since $|f(x)| \leq M$ and $|f(y)| \leq M$, we have

$$|f(x) + f(y)| \leq |f(x)| + |f(y)| \leq 2M.$$

So,

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

This proves the desired inequality.

(b) A function f is Riemann integrable if for every $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Using part (a), the oscillation of f^2 is controlled by that of f, i.e.,

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

Since f is integrable, for any $\epsilon > 0$, we can choose a partition P such that the oscillation of f over any subinterval is small. Multiplying by $2M$, we obtain:

$$U(f^2, P) - L(f^2, P) < \epsilon.$$

So, f^2 is integrable.

(c) if

$$(f + g)^2 = f^2 + 2fg + g^2.$$

then,

$$fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right).$$

Since, f is integrable, then f^2 is integrable, we can assume that f^2, g^2, and $(f + g)^2$ are all integrable functions.

Since the space of Riemann integrable functions is closed under addition and scalar multiplication, we can assume that

$$fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right)$$

is also integrable.

So, we have shown that the product of two integrable functions is integrable.

Problem 4

Text Problem 7.5.2

(a) If $g = h'$ for some function h on $[a, b]$, then g is continuous on $[a, b]$.

Since g is the derivative of h, we know that $g(x) = h'(x)$ exists at every point in $[a, b]$. By a fundamental theorem in analysis, derivatives satisfy the Intermediate Value Property, meaning they cannot have jump discontinuities.

So, differentiability of h at every point in $[a, b]$ guarantees that g is continuous. If g had a discontinuity at some point c, then h would not be differentiable at c, contradicting the assumption that $g = h'$ exists everywhere.

Thus, g must be continuous on $[a, b]$, proving that the statement is **true**.

(b) If g is continuous on $[a, b]$, then $g = h'$ for some function h on $[a, b]$.

By the Fundamental Theorem of Calculus, if g is continuous on $[a, b]$, then the function

$$h(x) = \int_a^x g(t)dt$$

is differentiable on $[a, b]$ and satisfies $h'(x) = g(x)$ for all $x \in [a, b]$.

Thus, every continuous function on a closed interval is the derivative of some function on that interval, proving that the statement is **true**.

(c) If $H(x) = \int_a^x h(t)dt$ is differentiable at $c \in [a, b]$, then h is continuous at c.

While it is true that if h is continuous at c, then $H(x)$ is differentiable at c with $H'(c) = h(c)$, there is no guarantee the converse is true

There exist functions $h(x)$ that are not continuous but still integrable, for which $H(x)$ is differentiable. A counterexample is:

$$h(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function has a jump discontinuity at $x = 0$. However, the integral

$$H(x) = \int_0^x h(t)dt$$

is still differentiable everywhere since the integral smooths out the discontinuity. So, $h(x)$ itself is not continuous at $x = 0$, which contradicts the given statement.

Thus, the statement is **false**.

Problem 5

Text Problem 7.5.4

- Define the function and use continuity Since $f(x)$ is given as a continuous function on $[a, b]$, it satisfies the standard definition of continuity:

  $$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

  However, we are given that $f(x) = 0$ for all $x \in [a, b]$, so we must prove that this holds everywhere on the interval.

- Consider the absolute value function Define a new function:

  $$g(x) = |f(x)|.$$

  Since $f(x)$ is continuous, the absolute value function $g(x) = |f(x)|$ is also continuous on $[a, b]$ because the absolute value function preserves continuity when composed with a continuous function.

- Show that $f(x) = 0$ everywhere Since $f(x) = 0$ for all $x \in [a, b]$, it follows that:

  $$|f(x)| = 0 \quad \forall x \in [a, b].$$

  This means that $g(x) = 0$ for all $x \in [a, b]$. But since $g(x) = |f(x)|$, we conclude:

  $$f(x) = 0, \quad \forall x \in [a, b].$$