Assignment 5

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2. Question

Question 1

(Short rate/Long rate) For the pricing of a ZC bond, we consider not only the usual short-term interest rate (short rate) r but also the long-term interest rate (long rate) ℓ. We suppose they follow the dynamics (for simplicity we drop the index t in the rate and Brownian processes)

$$dr = \beta_r(r, \ell, t)dt + \eta_r(r, \ell, t)dW_r$$
$$d\ell = \beta_\ell(r, \ell, t)dt + \eta_\ell(r, \ell, t)dW_\ell$$
$$d\langle W_r, W_\ell \rangle = \rho dt$$

Let $B(r, \ell, t; T)$ be the price of a T-maturity ZC bond.

a. Use Ito's lemma to express the dynamic of B under the form

$$\frac{dB}{B} = \mu(r, \ell, t)dt + \sigma_r(r, \ell, t)dW_r + \sigma_\ell(r, \ell, \ell, t)dW_\ell$$

Provide the explicit expression of $\mu, \sigma_r, \sigma_\ell$ as a function of B and its partial derivative, as well as $(\beta_r, \beta_\ell, \eta_r, \eta_\ell)$.

Answer:

Applied Ito formula on $B(r, \ell, t; T)$ we have:

$$dB(r, \ell, t; T) = \frac{\partial B}{\partial r}dr + \frac{\partial B}{\partial \ell}d\ell + \frac{\partial B}{\partial t}dt + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}d\langle r \rangle + \frac{1}{2}\frac{\partial^2 B}{\partial l^2}d\langle \ell \rangle + \frac{1}{2}\frac{\partial^2 B}{\partial t^2}d\langle t \rangle + \frac{1}{2}\frac{\partial^2 B}{\partial r \partial \ell}d\langle r, \ell \rangle + \frac{1}{2}$$

On the other hand, we got:

$$\begin{cases} dW_t^2 = dt \\ dW_\ell^2 = dt \\ d\langle W_r, W_\ell \rangle = \rho dt \\ dt^i = 0 \quad \forall i > 1 \end{cases}$$

Then

$$\begin{cases} dr & = d\langle r\rangle & = \beta_r(r,\ell,t)dt + \eta_r(r,\ell,t)dW_r \\ d\ell & = d\langle \ell\rangle & = \beta_\ell(r,\ell,t)dt + \eta_\ell(r,\ell,t)dW_\ell \\ d\langle r,\ell\rangle & = dr\times d\ell & = \left(\beta_r(r,\ell,t)dt + \eta_r(r,\ell,t)dW_r\right)\times\left(\beta_\ell(r,\ell,t)dt + \eta_\ell(r,\ell,t)dW_\ell\right) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} (dr)^2 & = (\beta_r(r,\ell,t)dt)^2 + 2*\beta_r(r,\ell,t)dt*\eta_r(r,\ell,t)dW_r + (\eta_r(r,\ell,t)dW_r)^2 \\ (d\ell)^2 & = (\beta_\ell(r,\ell,t)dt)^2 + 2*\beta_\ell(r,\ell,t)dt*\eta_\ell(r,\ell,t)dW_r + (\eta_\ell(r,\ell,t)dW_r)^2 \\ d\langle r,\ell\rangle & = \eta_r(r,\ell,t)\eta_\ell(r,\ell,t)\langle dW_r;dW_\ell\rangle \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} (dr)^2 & = \eta_r^2(r,\ell,t)dt \\ (d\ell)^2 & = \eta_\ell^2(r,\ell,t)dt \\ d\langle r,\ell\rangle & = \eta_r(r,\ell,t)\eta_\ell(r,\ell,t)\rho dt \end{cases}$$

Replace back into (1):

$$\begin{aligned} dB(r,\ell,t;T) = & \frac{\partial B}{\partial r}\left[\beta_r(r,\ell,t)dt + \eta_r(r,\ell,t)dW_r\right] + \frac{\partial B}{\partial \ell}\left[\beta_\ell(r,\ell,t)dt + \eta_\ell(r,\ell,t)dW_\ell\right] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\eta_r^2(r,\ell,t) \\ & + \frac{1}{2}\frac{\partial^2 B}{\partial r\partial \ell}\eta_r(r,\ell,t)\eta_\ell(r,\ell,t)\rho dt \\ = & \frac{\partial B}{\partial r}\beta_r(r,\ell,t)dt + \frac{\partial B}{\partial r}\eta_r(r,\ell,t)dW_r + \frac{\partial B}{\partial \ell}\beta_\ell(r,\ell,t)dt + \frac{\partial B}{\partial \ell}\eta_\ell(r,\ell,t)dW_\ell + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\eta_r^2(r,t \\ & + \frac{1}{2}\frac{\partial^2 B}{\partial r\partial \ell}\eta_r(r,\ell,t)\eta_\ell(r,\ell,t)\rho dt \\ = & \Bigg[\frac{\partial B}{\partial r}\beta_r(r,\ell,t) + \frac{\partial B}{\partial \ell}\beta_\ell(r,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\eta_r^2(r,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial l^2}\eta_\ell^2(r,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial r\partial \ell}\eta_r(r,\ell,t)\eta \\ & + \frac{\partial B}{\partial r}\eta_r(r,\ell,t)dW_r + \frac{\partial B}{\partial \ell}\eta_\ell(r,\ell,t)dW_\ell \end{aligned}$$

Then

$$\begin{aligned} \frac{dB}{B} = & \frac{1}{B}\Bigg[\frac{\partial B}{\partial r}\beta_r(r,\ell,t) + \frac{\partial B}{\partial \ell}\beta_\ell(r,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\eta_r^2(r,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial l^2}\eta_\ell^2(r,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial r\partial \ell}\eta_r(r,\ell,t)\eta_\ell(r,\ell, \\ & + \frac{1}{B}\frac{\partial B}{\partial \ell}\eta_\ell(r,\ell,t)dW_\ell \\ = & \mu(r,\ell,t)dt + \sigma_r(r,\ell,t)dW_r + \sigma_\ell(r,\ell,\ell,t)dW_\ell \end{aligned}$$

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b. In order to remove the (two) risk exposures, we use bonds with three different maturities, V_1, V_2, V_3 units of bonds with maturity T_1, T_2, T_3 , respectively. Show that the dynamic of the portfolio value P can be expressed under the form

$$dP = \mu_P dt + \sigma_{r,P}dW_r + \sigma_{\ell,P}dW_\ell,$$

with

$$\begin{aligned} \mu_P &= V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3) \\ \sigma_{r,P} &= V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3) \\ \sigma_{\ell,P} &= V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3) \end{aligned}$$

In the above equations, we have simplied the notation in writing $\mu(T_i) = \mu(r,\ell,t;T_i)$, similarly for σ_r and σ_ℓ

Answer:

Let P is a portfolio consist of three bonds V_1, V_2, V_3 units of bonds with maturity T_1, T_2, T_3 , respectively:

$$P = V_1 + V_2 + V_3 \quad (1)$$

Then, according to Lecture 5 - page 12, we got the Variation of P as below:

$$\frac{DP}{P} = \frac{V_1}{V_1 + V_2 + V_3}\frac{dB(T_1)}{B(T_1)} + \frac{V_2}{V_1 + V_2 + V_3}\frac{dB(T_2)}{B(T_2)} + \frac{V_3}{V_1 + V_2 + V_3}\frac{dB(T_3)}{B(T_3)}$$

$$\Leftrightarrow \qquad DP = V_1\frac{dB(T_1)}{B(T_1)} + V_2\frac{dB(T_2)}{B(T_2)} + V_3\frac{dB(T_3)}{B(T_3)} \qquad \text{as (1)}$$

Apply the conclusion from Question a, we have:

$$\begin{cases} \frac{dB(T_1)}{B(T_1)} = \mu(T_1)dt + \sigma_r(T_1)dW_r + \sigma_\ell(T_1)dW_\ell \\ \frac{dB(T_2)}{B(T_2)} = \mu(T_2)dt + \sigma_r(T_2)dW_r + \sigma_\ell(T_2)dW_\ell \\ \frac{dB(T_3)}{B(T_3)} = \mu(T_3)dt + \sigma_r(T_3)dW_r + \sigma_\ell(T_3)dW_\ell \end{cases}$$

So

$$\begin{aligned} DP &= V_1\Big[\mu(T_1)dt + \sigma_r(T_1)dW_r + \sigma_\ell(T_1)dW_\ell\Big] + V_2\Big[\mu(T_2)dt + \sigma_r(T_2)dW_r + \sigma_\ell(T_2)dW_\ell\Big] \\ &+ V_3\Big[\mu(T_3)dt + \sigma_r(T_3)dW_r + \sigma_\ell(T_3)dW_\ell\Big] \\ &= \Big[V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3)\Big]dt + \Big[V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3)\Big]dW_r \\ &+ \Big[V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3)\Big]dW_\ell \end{aligned}$$

Let

$$\begin{cases} \mu_P = V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3) \\ \sigma_{r,P} = V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3) \\ \sigma_{\ell,P} = V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3) \end{cases}$$

We finally got

$$dP = \mu_P dt + \sigma_{r,P}dW_r + \sigma_{\ell,P}dW_\ell$$

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c. The NAO implies that, if we choose V_1, V_2, V_3 such that P is riskless, i.e., the stochastic terms $\sigma_{r,P}$ and $\sigma_{\ell,P}$ are all zero, then P must earn the riskless short rate. Show that:

$$\begin{pmatrix} \sigma_r(T_1) & \sigma_r(T_2) & \sigma_r(T_3) \\ \sigma_\ell(T_1) & \sigma_\ell(T_2) & \sigma_\ell(T_3) \\ \mu(T_1) - r & \mu(T_2) - r & \mu(T_3) - r \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Answer:

We got:

$$\begin{pmatrix} \sigma_r(T_1) & \sigma_r(T_2) & \sigma_r(T_3) \\ \sigma_\ell(T_1) & \sigma_\ell(T_2) & \sigma_\ell(T_3) \\ \mu(T_1)-r & \mu(T_2)-r & \mu(T_3)-r \end{pmatrix} \times \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3) \\ V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3) \\ V_1\left[\mu(T_1)-r\right] + V_2\left[\mu(T_2)-r\right] + V_3\left[\mu(T_3)-r\right] \end{pmatrix}$$
$$= \begin{pmatrix} V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3) \\ V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3) \\ V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3) - r(V_1 + V_2 + V_3) \end{pmatrix}$$

First of all, we have all the stochastic term will equal to 0 due to the riskless condition of P. Then

$$\begin{cases} V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3) = 0 \\ V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3) = 0 \end{cases}$$

On the other hand, P is riskless also means $dP = rP$. But we also got:

$$\begin{cases} dP = \mu_P dt + \sigma_{r,P} dW_r + \sigma_{\ell,P} dW_\ell = \mu_p dt \\ rP = r(V_1 + V_2 + V_3) \end{cases}$$

$$\begin{aligned} \Leftrightarrow \quad & \mu_P dt = r(V_1 + V_2 + V_3) \\ \Leftrightarrow \quad & V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3) = r(V_1 + V_2 + V_3) \\ \Leftrightarrow \quad & V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3) - r(V_1 + V_2 + V_3) = 0 \end{aligned}$$

Then $(*)$:

$$\begin{pmatrix} \sigma_r(T_1) & \sigma_r(T_2) & \sigma_r(T_3) \\ \sigma_\ell(T_1) & \sigma_\ell(T_2) & \sigma_\ell(T_3) \\ \mu(T_1)-r & \mu(T_2)-r & \mu(T_3)-r \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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d. We admit that the necessary condition for the above system has nontrivial solution is the last row is a linear combination of the first two rows. Since it holds true for arbitrary value of T_1, T_2, T_3 , there must exist two multipliers $\lambda_r(r, \ell, t)$ and $\lambda_\ell(r, \ell, t)$ such that

$$\mu(r, \ell, t) - r = \lambda_r(r, \ell, t)\sigma_r(r, \ell, t) + \lambda_\ell(r, \ell, t)\sigma_\ell(r, \ell, t).$$

Provide an plausible interpretation for $\lambda_r(r, \ell, t)$ and $\lambda_\ell(r, \ell, t)$. Deduce the governing equation for the bond price B.

Answer:

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Question 2

(Hull-White extended Vasisek) Hull-White proposed the following dynamics for r under $\mathbb{P}$, extended from the Vasicek model,

$$dr_t = \left[\theta(t) + \alpha(t)(d - r_t)\right]dt + \sigma(t)dW_t.$$

a. Using the term-structure equation show that the T-maturity ZC-bond price $B(r; t; T)$ verifies

$$\frac{\partial B}{\partial t} + \left[\phi(t) - \alpha(t)r\right]\frac{\partial B}{\partial r} + \frac{\sigma^2(t)}{2}\frac{\partial^2 B}{\partial r^2} - rB = 0, \quad B(.,T;T) = 1$$

Provide an explicit expression of $\phi(t)$, you can use $\lambda(t)$ for the time dependent market price of risk.

Answer:

Apply Ito formula into $B(r,t)$ we have:

$$dB(r,t) = \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial r}dr + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}d\langle r\rangle$$

As:

$$dr_t = \left[\theta(t) + \alpha(t)(d - r_t)\right]dt + \sigma(t)dW_t \Leftrightarrow d\langle r\rangle = (dr_t)^2 = \sigma^2(t)dt$$

Then

$$\begin{aligned} dB(r,t) &= \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial r}\left[\left[\theta(t) + \alpha(t)(d - r_t)\right]dt + \sigma(t)dW_t\right] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(t)dt \\ &= \left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\left[\theta(t) + \alpha(t)(d - r_t)\right] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(t)\right]dt + \frac{\partial B}{\partial r}\sigma(t)dW_t \end{aligned}$$

We also have SDE:

$$dB(r,t) = rBdt + \partial BdW_t \Leftrightarrow \begin{cases} rB = \frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\left[\theta(t) + \alpha(t)(d - r_t)\right] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(t) \\ \partial B = \frac{\partial B}{\partial r}\sigma(t) \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\left[\theta(t) + \alpha(t)(d - r_t)\right] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(t) - rB = 0 \\ \frac{\partial B}{\partial r}\sigma(t) - \partial B = 0 \end{cases} \Rightarrow \text{proved!}$$

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b. Show the bond prince $B(r,t;T) = e^{a(t,T) - b(t,T)r}$ where a, b verify

$$\frac{\partial a}{\partial t} - \phi(t)b + \frac{\sigma^2(t)}{2}b^2 = 0$$

$$\frac{\partial b}{\partial t} - \alpha(t)b + 1 = 0$$

with auxiliary conditions $a(T,T) = 0$ and $b(T,T) = 0$.

Answer:

Let look at the $dB(r,t)$, we have:

$$dB(r,t) = \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial r}dr + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}d\langle r\rangle$$

Then:

$$\begin{cases} \frac{\partial B}{\partial t} = \left(\frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}r\right)e^{a(t,T)-b(t,T)r} = \left(\frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}r\right)B \\ \frac{\partial B}{\partial r} = -be^{a(t,T)-b(t,T)r} = bB \\ \frac{\partial^2 B}{\partial r^2} = b^2 B \end{cases}$$

In Question 2.a. we also got:

$$\begin{aligned} & \frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\Big[\theta(t) + \alpha(t)(d-r)\Big] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(t) - rB = 0 \\ \Leftrightarrow & \left(\frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}r\right)B + bB\Big[\theta(t) + \alpha(t)(d-r)\Big] + \frac{1}{2}b^2 B\sigma^2(t) - rB = 0 \\ \Leftrightarrow & \left(\frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}r\right) + b\Big[\theta(t) + \alpha(t)(d-r)\Big] + \frac{1}{2}b^2\sigma^2(t) - r = 0 \\ \Leftrightarrow & \left(\frac{\partial a}{\partial t} + b\theta(t) + b\alpha(t)d + \frac{1}{2}b^2\sigma^2(t)\right) + r\left(\frac{\partial b}{\partial t} + b\alpha(t) - 1\right) = 0 \\ \Leftrightarrow & \begin{cases} \frac{\partial a}{\partial t} + b\theta(t) + b\alpha(t)d + \frac{1}{2}b^2\sigma^2(t) = 0 \\ \frac{\partial b}{\partial t} + b\alpha(t) - 1 = 0 \quad \text{As } r \neq 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \frac{\partial a}{\partial t} + \phi(t)b + \frac{1}{2}b^2\sigma^2(t) = 0 \quad \text{let } \phi(t) = \theta(t) + \alpha(t)d \\ \frac{\partial b}{\partial t} + b\alpha(t) - 1 = 0 \end{cases} \\ \Leftrightarrow & \quad \text{Proved!} \end{aligned}$$

c. Solve for a, b in terms of α, ϕ and σ.

Answer:

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d. To avoid the estimation of α and ϕ in the dynamic of r, it is desirable to express $a(t,T)$ and $b(t,T)$ in terms of $a(0,t)$ and $b(0,t)$, show that the new set of governing equations is

$$\frac{\partial b}{\partial t}\frac{\partial b}{\partial T} - b\frac{\partial^2 b}{\partial t \partial T} + \frac{\partial b}{\partial T} = 0$$
$$b\frac{\partial^2 a}{\partial t \partial T} - \frac{\partial a}{\partial t}\frac{\partial b}{\partial T} + \frac{\sigma^2(t)b^2}{2}\frac{\partial b}{\partial T} = 0$$

with auxiliary conditions $a(T,T) = 0$ and $b(T,T) = 0$.

Answer:

e. From the above equations, show that:

$$b(t,T) = \frac{b(0,T) - b(0,t)}{\frac{\partial b}{\partial T}\left(0,T\right)|_{T=t}}$$

$$a(t,T) = a(0,T) - a(0,t) - b(t,T)\frac{\partial a}{\partial T}\left(0,T\right)\ |_{T=t} - \frac{1}{2}\Big[b(t,T)\frac{\partial b}{\partial T}\left(0,T\right)\ |_{T=t}\ \Big]^2 \int_0^t \left[\frac{\sigma(u)}{\frac{\partial b}{\partial T}\left(0,T\right)|_{T=0}}\right]^2 \mathrm{d}u$$

Answer: