Assignment 6

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2. Question

Question 1

(Monotonicity and convexity with respect to strike) Let S be a financial instrument, $C(K)$ be the price of a call option with payoff $g(S_T) = max(S_T - K, 0)$ where K is the strike-price. We suppose that $C(K)$ is twice differentiable, i.e., $\frac{\partial C}{\partial K}$ and $\frac{\partial^2 C}{\partial K^2}$ exist.

a. Suppose that the spot price at $t = 0$ is S_0 , provide the definition of in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM) call options.

Answer:

We have:

- In the money (ITM): $S_T > K$
- At the money (ATM): $S_T = K$
- Out the money (ATM): $S_T < K$

b. Which one is more expensive, an ITM or an OTM call option? Justify the answer by a rigorous proof using no-arbitrage argument, then use it to show $\frac{\partial C}{\partial K} \leq 0$.

Answer:

c. Let be given three call options, the first one is ITM, the second one ATM and the last one OTM, with strike K_1, K_2, K_3 , respectively. We consider a butterfly spread strategy which consists in buying 1 ITM call, selling 2 ATM calls and buying 1 OTM call with $K_1 + K_3 = 2K_2$. Show again by means of no-arbitrage argument that the initial cost of such strategy must be non-negative, then explain why we must have $\frac{\partial^2 C}{\partial K^2} \geq 0$.

Answer:

Question 2

(Call-Put symmetry) We consider the general Black-Scholes-Merton dynamics for an asset S under $\mathbb{Q}$

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$$

Let $C(t,x,K,T,r,q,\sigma)$ and $P(t,x,K,T,r,q,\sigma)$ be the price at t of an European call and put option with an initial spot price x, strike K, maturity T, risk-free interest-rate r, dividend rate q and volatility σ.

a. Provide the Black-Scholes-Merton formula for call/put option.

Answer:

We got the BSM formula:

$$C(t,x,K,T,r,q,\sigma) = xe^{-q(T-t)}N(d_+) - Ke^{-r(T-t)}N(d_-)$$

with

$$d_\pm = \frac{\ln\frac{xe^{(r-q)(T-t)}}{K} \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

Apply the put-call parity, we have:

$$\begin{aligned} C + Ke^{-r(T-t)} &= P + xe^{-q(T-t)} \\ &\Leftrightarrow P = C + Ke^{-r(T-t)} - xe^{-q(T-t)} \\ &\Leftrightarrow P = xe^{-q(T-t)}N(d_+) - Ke^{-r(T-t)}N(d_-) + Ke^{-r(T-t)} - xe^{-q(T-t)} \\ &\Leftrightarrow P = xe^{-q(T-t)}\Big(N(d_+) - 1\Big) - Ke^{-r(T-t)}\Big(N(d_-) - 1\Big) \end{aligned}$$

with

$$d_\pm = \frac{\ln\frac{xe^{(r-q)(T-t)}}{K} \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

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b. Using Ito's formula, provide an explicit expression for S_T .

Answer:

We have:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-q)dt + \sigma dW_t \\ \Leftrightarrow dS_t &= S_t(r-q)dt + S_t\sigma dW_t \end{aligned}$$

Consequence, we got $G = G(S_t, t)$ then

$$dG = \Big(\frac{\partial G}{\partial S_t}S_t(r-q) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S_t^2}(S_t\sigma)^2\Big)dt + \frac{\partial G}{\partial S_t}S_t\sigma dW_t$$

Let $G = ln(S_t)$, hence:

$$\begin{cases} \frac{\partial ln(S_t)}{\partial S_t} = \frac{1}{S_t} \Rightarrow \frac{\partial^2 G}{\partial S_t^2} = -\frac{1}{S_t^2} \\ \frac{\partial G}{\partial t} = \frac{\partial ln(S_t)}{\partial t} = 0 \end{cases}$$

Then:

$$
\begin{aligned}
dG = dln(S_t) &= \Big[\frac{1}{S_t}(r-q)S_t + 0 - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2\Big]dt + \frac{1}{S_t}\sigma S_t dW_t \\
&= (r-q-\frac{\sigma^2}{2})dt + \sigma dW_t \\
\Rightarrow \int_t^T dln(S_k) &= \int_t^T (r-q-\frac{\sigma^2}{2})dk + \int_t^T \sigma dW_k \\
\Leftrightarrow \int_t^T dln(S_k) &= (r-q-\frac{\sigma^2}{2})\int_t^T dk + \sigma \int_t^T dW_k \\
\Rightarrow ln\frac{S_T}{S_t} &= (r-\frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t) \\
\Leftrightarrow S_T &= S_t e^{(r-q-\frac{\sigma^2}{2})(T-t)+\sigma(W_T-W_t)}.
\end{aligned}
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c. In the case where $r = q = 0$ (only in this question), show that the following symmetry holds (For ease of reading we omit t, T, r, q and σ in this expression.)

$$C(x, K) = P(K, x).$$

Deduce from the previous questions that

$$\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(x - K\frac{S_T}{x}\right)^+\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{x}\left(\frac{x^2}{S_T} - K\right)^+\right]$$

Since it holds true for all positive K, we have just showed that for all positive payoff g

$$\mathbb{E}^{\mathbb{Q}}\left[g(S_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{x}g\left(\frac{x^2}{S_T}\right)\right]$$

Answer:

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d. Turning back to the general case, let $X = S^\gamma$. Use Ito's formula to derive the dynamics of X under $\mathbb{Q}$, then deduce without doing any calculation, the explicit price for an European put option with payoff $(K - X_T)^+$.

Answer:

We have:

$$dS_t = S_t(r-q)dt + S_t\sigma dW_t$$

Consequence, we got $X = S^\gamma$ then

$$dX = \Big(\frac{\partial X}{\partial S_t}S_t(r-q) + \frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial S_t^2}(S_t\sigma)^2\Big)dt + \frac{\partial X}{\partial S_t}S_t\sigma dW_t$$

Let $G = ln(S_t)$, hence:

$$\begin{cases} \frac{\partial X}{\partial S_t} = \gamma S_t^{\gamma-1} \Rightarrow \frac{\partial^2 X}{\partial S_t^2} = \gamma(\gamma-1)S_t^{\gamma-2} \\ \frac{\partial G}{\partial t} = \frac{\partial ln(S_t)}{\partial t} = 0 \end{cases}$$

Then:

$$\begin{aligned} dX &= \left[\gamma S_t^{\gamma-1}(r-q)S_t + 0 + \frac{1}{2}\gamma(\gamma-1)S_t^{\gamma-2}\sigma^2 S_t^2\right]dt + \gamma S_t^{\gamma-1}\sigma S_t dW_t \\ &= \left[\gamma S_t^{\gamma}(r-q) + \frac{1}{2}\gamma(\gamma-1)S_t^{\gamma}\sigma^2\right]dt + \gamma S_t^{\gamma}\sigma dW_t \\ &= \left[\gamma X(r-q) + \frac{1}{2}\gamma(\gamma-1)X\sigma^2\right]dt + \gamma X\sigma dW_t \\ &= X\left[\gamma(r-q) + \frac{1}{2}\gamma(\gamma-1)\sigma^2\right]dt + \gamma X\sigma dW_t \end{aligned}$$

Similar to question 2.b., we take intergral of both sides:

$$X_T = X_t e^{(\gamma(r-q)+\frac{1}{2}\gamma(\gamma-1)\sigma^2)(T-t)+\gamma\sigma(W_T-W_t)}$$

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e. Finally, show that for all positive payoff g and $\gamma = \frac{1-2(r-q)}{\sigma^2}$

$$\mathbb{E}^{\mathbb{Q}}\left[g(S_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{x}\right)^{\gamma} g\left(\frac{x^2}{S_T}\right)\right]$$

(Hint: use the variable X introduced in d.)

Answer:

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Question 3

(Schaeffer and Schwartz model) We consider a two-factor interest rate model consisting of the long rate l and the spread s between the short rate and long rate, i.e., $s = r - l$,

$$ds = \beta_s(s, l, t)dt + \eta_s(s, l, t)dW_s$$
$$d\ell = \beta_\ell(r, l, t)dt + \eta_\ell(r, l, t)dW_\ell$$

Moreover, empirical evidence shows that the long rate and the spread are almost uncorrelated, thus we suppose $d\langle W_s, W_\ell\rangle_t = 0$

A. Show that the price of a zero-coupon bond $B(s, \ell, t)$ verifies

$$\frac{\partial B}{\partial t} + \frac{\eta_s^2}{2}\frac{\partial^2 B}{\partial s^2} + \frac{\eta_\ell^2}{2}\frac{\partial^2 f}{\partial \ell^2} + (\beta_s - \lambda_s\eta_s)\frac{\partial B}{\partial s} + (\beta_\ell - \lambda_\ell\eta_\ell)\frac{\partial B}{\partial \ell} - (s+l)B = 0$$

where λ_s/λ_ℓ is the market price of the spread/long rate risk, respectively

Answer:

Applied Ito formula on $B(r, \ell, t)$ we have:

$$\begin{aligned} dB(s,\ell,t) = & \frac{\partial B}{\partial s}ds + \frac{\partial B}{\partial \ell}d\ell + \frac{\partial B}{\partial t}dt + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}d\langle s\rangle + \frac{1}{2}\frac{\partial^2 B}{\partial l^2}d\langle \ell\rangle + \frac{1}{2}\frac{\partial^2 B}{\partial t^2}d\langle t\rangle \\ & + \frac{1}{2}\frac{\partial^2 B}{\partial s\partial \ell}d\langle s,\ell\rangle + \frac{1}{2}\frac{\partial^2 B}{\partial s\partial t}d\langle s,t\rangle + \frac{1}{2}\frac{\partial^2 B}{\partial \ell\partial t}d\langle \ell,t\rangle \quad (1) \end{aligned}$$

On the other hand, we got:

$$\begin{cases} dW_t^2 = dt \\ dW_\ell^2 = dt \\ d\langle W_s, W_\ell\rangle = 0 \\ dt^i = 0 \quad \forall i > 1 \end{cases}$$

Then

$$\begin{cases} ds & = \beta_s(s,\ell,t)dt + \eta_s(s,\ell,t)dW_s \\ d\ell & = \beta_\ell(s,\ell,t)dt + \eta_\ell(s,\ell,t)dW_\ell \\ d\langle s,\ell\rangle = ds \times d\ell & = \Big(\beta_s(s,\ell,t)dt + \eta_s(s,\ell,t)dW_s\Big) \times \Big(\beta_\ell(s,\ell,t)dt + \eta_\ell(s,\ell,t)dW_\ell\Big) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} (ds)^2 & = (\beta_s(s,\ell,t)dt)^2 + 2 * \beta_s(s,\ell,t)dt * \eta_s(s,\ell,t)dW_s + (\eta_s(s,\ell,t)dW_s)^2 \\ (d\ell)^2 & = (\beta_\ell(s,\ell,t)dt)^2 + 2 * \beta_\ell(s,\ell,t)dt * \eta_\ell(s,\ell,t)dW_s + (\eta_\ell(s,\ell,t)dW_s)^2 \\ d\langle s,\ell\rangle & = \eta_s(s,\ell,t)\eta_\ell(s,\ell,t)\langle dW_s; dW_\ell\rangle \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} (ds)^2 & = \eta_s^2(s,\ell,t)dt \\ (d\ell)^2 & = \eta_\ell^2(s,\ell,t)dt \\ d\langle s,\ell\rangle & = \eta_s(s,\ell,t)\eta_\ell(s,\ell,t) \times 0 = 0 \end{cases}$$

Replace back into (1):

$$\begin{aligned} dB(s,\ell,t) = & \frac{\partial B}{\partial s}\Big[\beta_s(s,\ell,t)dt + \eta_s(s,\ell,t)dW_s\Big] + \frac{\partial B}{\partial \ell}\Big[\beta_\ell(s,\ell,t)dt + \eta_\ell(s,\ell,t)dW_\ell\Big] \\ & + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2(s,\ell,t)dt + \frac{1}{2}\frac{\partial^2 B}{\partial l^2}\eta_\ell^2(s,\ell,t)dt + \frac{\partial B}{\partial t}dt \\ = & \frac{\partial B}{\partial s}\beta_s(s,\ell,t)dt + \frac{\partial B}{\partial s}\eta_s(s,\ell,t)dW_s + \frac{\partial B}{\partial \ell}\beta_\ell(s,\ell,t)dt + \frac{\partial B}{\partial \ell}\eta_\ell(s,\ell,t)dW_\ell \\ & + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2(s,\ell,t)dt + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2(s,\ell,t)dt + \frac{\partial B}{\partial t}dt \\ = & \left[\frac{\partial B}{\partial s}\beta_s(s,\ell,t) + \frac{\partial B}{\partial \ell}\beta_\ell(s,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2(s,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2(s,\ell,t) + \frac{\partial B}{\partial t}\right]dt \\ & + \frac{\partial B}{\partial s}\eta_s(s,\ell,t)dW_s + \frac{\partial B}{\partial \ell}\eta_\ell(s,\ell,t)dW_\ell \end{aligned}$$

Then

$$\begin{aligned} \frac{dB}{B} = & \frac{1}{B}\left[\frac{\partial B}{\partial s}\beta_s(s,\ell,t) + \frac{\partial B}{\partial \ell}\beta_\ell(s,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2(s,\ell,t) + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2(s,\ell,t) + \frac{\partial B}{\partial t}\right]dt \\ & + \frac{1}{B}\frac{\partial B}{\partial s}\eta_s(s,\ell,t)dW_s + \frac{1}{B}\frac{\partial B}{\partial \ell}\eta_\ell(s,\ell,t)dW_\ell \\ = & \mu(s,\ell,t)dt + \sigma_s(s,\ell,t)dW_s + \sigma_\ell(s,\ell,t)dW_\ell \qquad (*) \end{aligned}$$

According to common formula of the Market Price of Risk: $\mu - r = \lambda.\sigma$. Then due to the two types for interest rate consisted in this model as long rate ℓ and short rate s. We have:

$$\mu - r = \lambda_s \times \sigma_s + \lambda_\ell \times \sigma_\ell$$
$$\Leftrightarrow \mu(s,\ell,t) - r = \lambda_s \times \sigma_s(s,\ell,t) + \lambda_\ell \times \sigma_\ell(s,\ell,t)$$

Then

$$\frac{1}{B}\left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial s}\beta_s + \frac{\partial B}{\partial \ell}\beta_\ell + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2 + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2\right] - r = \lambda_s \frac{1}{B}\frac{\partial B}{\partial s}\eta_s + \frac{1}{B}\lambda_\ell \frac{\partial B}{\partial \ell}\eta_\ell$$
$$\Leftrightarrow \left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial s}\beta_s + \frac{\partial B}{\partial \ell}\beta_\ell + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2 + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2\right] - Br = \lambda_s \frac{\partial B}{\partial s}\eta_s + \lambda_\ell \frac{\partial B}{\partial \ell}\eta_\ell$$
$$\Leftrightarrow \left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial s}\beta_s + \frac{\partial B}{\partial \ell}\beta_\ell + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2 + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2\right] - (s+\ell)B = \lambda_s \frac{\partial B}{\partial s}\eta_s + \lambda_\ell \frac{\partial B}{\partial \ell}\eta_\ell$$
$$\Leftrightarrow \frac{\partial B}{\partial t} + \frac{\partial B}{\partial s}\left[\beta_s - \lambda_s\eta_s\right] + \frac{\partial B}{\partial \ell}\left[\beta_\ell - \lambda_\ell\eta_\ell\right] + \frac{1}{2}\frac{\partial^2 B}{\partial s^2}\eta_s^2 + \frac{1}{2}\frac{\partial^2 B}{\partial \ell^2}\eta_\ell^2 - (s+\ell)B = 0$$

B. Let G be a *consol bond*, a perpetual bond (with infinite maturity) pays coupon at a continuous constant rate c. Let $G(\ell)$ denote the value of this bond, we admit that:

$$G(\ell) = \frac{c}{\ell}$$

B.1. Apply Ito's formula to express the dynamic of G under the form

$$\frac{dG}{G} = \mu_G dt + \sigma_G dW_\ell$$

Answer:

Apply Ito formula on $G(c,\ell)$ we have:

$$dG = \frac{\partial G}{\partial c}dc + \frac{\partial G}{\partial \ell}d\ell + \frac{1}{2}\frac{\partial^2 G}{\partial \ell^2}d\langle \ell \rangle$$

We also have:

$$\begin{cases} \frac{\partial G}{\partial \ell} = \frac{c}{\ell^2} \\ \frac{\partial^2 G}{\partial \ell^2} = \frac{c}{l^3} \\ d\ell = \beta_\ell dt + \eta_\ell dW_\ell \\ d\langle \ell \rangle = \eta_\ell^2 dt \\ dW_\ell^2 = dt \\ d\langle W_s, W_\ell \rangle = 0 \\ dt^i = 0 \quad \forall i > 1 \end{cases}$$

Then

$$
\begin{aligned}
dG &= 0 - \frac{c}{\ell^2}\left[\beta_\ell dt + \eta_\ell dW_\ell\right] + \frac{c}{\ell^3}\eta_\ell^2 dt \\
&= \frac{c}{\ell}\left[-\frac{1}{\ell}\beta_\ell + \frac{1}{\ell^2}\eta_\ell^2\right]dt - \frac{1}{\ell^2}\eta_\ell dW_\ell \\
&= G\left[-\frac{1}{\ell}\beta_\ell + \frac{1}{\ell^2}\eta_\ell^2\right]dt - G\frac{1}{\ell}\eta_\ell dW_\ell
\end{aligned}
$$

Finally:

$$
\begin{aligned}
\frac{dG}{G} &= \left[-\frac{1}{\ell}\beta_\ell + \frac{1}{\ell^2}\eta_\ell^2\right]dt - \frac{1}{\ell}\eta_\ell dW_\ell \\
&= \mu_G dt + \sigma_G dW_\ell
\end{aligned}
$$

With

$$
\begin{cases}
\mu_G = -\frac{1}{\ell}\beta_\ell + \frac{1}{\ell^2}\eta_\ell^2 \\
\sigma_G = -\frac{1}{\ell}\eta_\ell
\end{cases}
$$

B.2. The instantaneous rate of return of a consol bond is the sum of coupon rate ℓ and the drift rate of $G, \mu_c = \mu_G + \ell$, while the volatility is the same as the volatility of $G, \sigma_c = \sigma_G$. Show that

$$
\beta_\ell - \lambda_\ell \eta_\ell = \frac{\eta_\ell^2}{\ell} - s\ell
$$

Answer:

Consider λ_ℓ is the market price of risk, then we have:

$$
\begin{aligned}
\lambda_\ell &= \frac{\mu_c - r}{\sigma_c} \\
\Leftrightarrow \lambda_\ell &= \frac{\mu_G + \ell - s - \ell}{\sigma_G} \\
\Leftrightarrow \lambda_\ell &= \frac{-\frac{1}{\ell}\beta_\ell + \frac{1}{\ell^2}\eta_\ell^2 - s}{-\frac{1}{\ell}\eta_\ell} \\
\Leftrightarrow -\lambda_\ell \frac{1}{\ell}\eta_\ell &= -\frac{1}{\ell}\beta_\ell + \frac{1}{\ell^2}\eta_\ell^2 - s \\
\Leftrightarrow -\lambda_\ell \eta_\ell &= -\beta_\ell + \frac{1}{\ell}\eta_\ell^2 - s\ell \\
\Leftrightarrow \beta_\ell - \lambda_\ell \eta_\ell &= \frac{\eta_\ell^2}{\ell} - s\ell
\end{aligned}
$$