derivatives5_assignment_2018

June 18, 2018

1 Instructions

1. Due: 11:59 PM 25/06/2018.
2. Answers must be
   - named as LastName_FirstName_5(.ipynb/.pdf)
   - uploaded directly to Google drive, under **assignments-student/YourName/**

2 Questions

Q1. (Short rate/Long rate) For the pricing of a ZC bond, we consider not only the usual short-term interest rate (short rate) r but also the long-term interest rate (long rate) ℓ. We suppose they follow the dynamics (for simplicity we drop the index t in the rate and Brownian processes)

$$\begin{aligned} \mathrm{d}r &= \beta_r(r,\ell,t)\,\mathrm{d}t + \eta_r(r,\ell,t)\,\mathrm{d}W_r \\ \mathrm{d}\ell &= \beta_\ell(r,\ell,t)\,\mathrm{d}t + \eta_\ell(r,\ell,t)\,\mathrm{d}W_\ell \\ \mathrm{d}\langle W_r, W_\ell\rangle &= \rho\,\mathrm{d}t. \end{aligned}$$

Let $B(r,\ell,t;T)$ be the price of a T-maturity ZC bond.

a. Use Ito's lemma to express the dynamic of B under the form

$$\frac{\mathrm{d}B}{B} = \mu(r,\ell,t)\,\mathrm{d}t + \sigma_r(r,\ell,t)\,\mathrm{d}W_r + \sigma_\ell(r,\ell,t)\,\mathrm{d}W_\ell.$$

Provide the explicit expression of $\mu, \sigma_r, \sigma_\ell$ as a function of B and its partial derivative, as well as $(\beta_r, \beta_\ell, \eta_r, \eta_\ell)$.

b. In order to remove the (two) risk exposures, we use bonds with three different maturities, V_1, V_2, V_3 units of bonds with maturity T_1, T_2, T_3, respectively. Show that the dynamic of the portfolio value P can be expressed under the form

$$\mathrm{d}P = \mu_P\,\mathrm{d}t + \sigma_{r,P}\,\mathrm{d}W_r + \sigma_{\ell,P}\,\mathrm{d}W_\ell,$$

with

$$\begin{aligned} \mu_P &= V_1\mu(T_1) + V_2\mu(T_2) + V_3\mu(T_3) \\ \sigma_{r,P} &= V_1\sigma_r(T_1) + V_2\sigma_r(T_2) + V_3\sigma_r(T_3) \\ \sigma_{\ell,P} &= V_1\sigma_\ell(T_1) + V_2\sigma_\ell(T_2) + V_3\sigma_\ell(T_3). \end{aligned}$$

In the above equations, we have simplied the notation in writing $\mu(T_i) = \mu(r,\ell,t;T_i)$, similarly for σ_r and σ_ℓ.

c. The NAO implies that, if we choose V_1, V_2, V_3 such that P is riskless, i.e., the stochastic terms $\sigma_{r,P}$ and $\sigma_{\ell,P}$ are all zero, then P must earn the riskless short rate. Show that

$$\begin{pmatrix} \sigma_r(T_1) & \sigma_r(T_2) & \sigma_r(T_3) \\ \sigma_\ell(T_1) & \sigma_\ell(T_2) & \sigma_\ell(T_3) \\ \mu(T_1) - r & \mu(T_2) - r & \mu(T_3) - r \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

d. We admit that the necessary condition for the above system has nontrivial solution is the last row is a linear combination of the first two rows. Since it holds true for arbitrary value of T_1, T_2, T_3, there must exist two multipliers $\lambda_r(r, \ell, t)$ and $\lambda_\ell(r, \ell, t)$ such that

$$\mu(r, \ell, t) - r = \lambda_r(r, \ell, t)\sigma_r(r, \ell, t) + \lambda_\ell(r, \ell, t)\sigma_\ell(r, \ell, t).$$

Provide an plausible interpretation for $\lambda_r(r, \ell, t)$ and $\lambda_\ell(r, \ell, t)$. Deduce the governing equation for the bond price B.

Q2. (Hull-White extended Vasisek) Hull-White proposed the following dynamics for r under $\mathbb{P}$, extended from the Vasicek model,

$$\mathrm{d}r_t = [\theta(t) + \alpha(t)(d - r_t)]\, \mathrm{d}t + \sigma(t)\, \mathrm{d}W_t.$$

a. Using the term-structure equation show that the T-maturity ZC-bond price $B(r; t; T)$ verifies

$$\frac{\partial B}{\partial t} + [\phi(t) - \alpha(t)r)]\frac{\partial B}{\partial r} + \frac{\sigma^2(t)}{2}\frac{\partial^2 B}{\partial r^2} - rB = 0 \ \ , \ \ B(\cdot, T; T) = 1.$$

Provide an explicit expression of $\phi(t)$, you can use $\lambda(t)$ for the time dependent market price of risk.

b. Show that the bond price $B(r, t; T) = e^{a(t,T) - b(t,T)r}$ where a, b verify

$$\frac{\partial a}{\partial t} - \phi(t)b + \frac{\sigma^2(t)}{2}b^2 = 0$$
$$\frac{\partial b}{\partial t} - \alpha(t)b + 1 = 0$$

with auxiliary conditions $a(T, T) = 0$ and $b(T, T) = 0$.

c. Solve for a, b in terms of α, ϕ and σ.

d. To avoid the estimation of α and ϕ in the dynamic of r, it is desirable to express $a(t, T)$ and $b(t, T)$ in terms of $a(0, t)$ and $b(0, t)$, show that the new set of governing equations is

$$\frac{\partial b}{\partial t}\frac{\partial b}{\partial T} - b\frac{\partial^2 b}{\partial t \partial T} + \frac{\partial b}{\partial T} = 0$$
$$b\frac{\partial^2 a}{\partial t \partial T} - \frac{\partial a}{\partial t}\frac{\partial b}{\partial T} + \frac{\sigma^2(t)b^2}{2}\frac{\partial b}{\partial T} = 0$$

with auxiliary conditions $a(T, T) = 0$ and $b(T, T) = 0$.

e. From the above equations, show that

$$b(t, T) = \frac{b(0, T) - b(0, t)}{\frac{\partial b}{\partial T}(0, T)\big|_{T=t}}$$

$$a(t, T) = a(0, T) - a(0, t) - b(t, T)\frac{\partial a}{\partial T}(0, T)\big|_{T=t} - \frac{1}{2}\left[b(t, T)\frac{\partial b}{\partial T}(0, T)\big|_{T=t}\right]^2 \int_0^t \left[\frac{\sigma(u)}{\frac{\partial b}{\partial T}(0, T)\big|_{T=u}}\right]^2 \mathrm{d}u.$$