derivatives3_assignment_2018

April 28, 2018

1 Instructions

1. Due: 23:59 07/05/2018.
2. Answers must be
   - named as assignment3.(pdf/ipynb)
   - uploaded directly to Google drive, under **students/YourName** folder.

2 Questions

We consider the problem of pricing an option with payoff $\phi(S_T)$ where the underlying asset S follows the dynamics (under $\mathbb{Q}$):

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma \mathrm{d}W_t$$

where the risk-free rate and the volatility coefficient are all deterministic.

For all numerical applications in this assignment we take

$$x = 100, r = 0.2, \sigma = 0.1, T = 1.$$

Q1. Justify with a short argument why the price of such an option when $S_t = x$ is given by

$$u(t,x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\phi(S_T)\,\middle|S_t = x\right].$$

Q2. Suppose we are able to simulate n values of S_T, $\left\{S_T^{(1)}, \ldots, S_T^{(n)}\right\}$, cite the theorem allowing us to approximate the price of this option by

$$u(t,x) \approx u^{MC}(t,x) := e^{-r(T-t)}\frac{\sum_{i=1}^{n}\phi(S_T^{(i)})}{n}.$$

Give an estimation of the expected value and the standard-deviation of $u^{MC}(t,x)$ from those of $\phi(S_T)$. Provide the 95%-confidence interval of the estimated price.

Q3. We consider the simulation of S_T given S_t=x, show (with help of Ito's lemma) that

$$S_T = S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T - W_t)}.$$

Using the property of the standard Brownian motion, describe how to simulate S_T from a random normal distribution.

Q4. In the previous question we directly simulate the normal distribution, show that if we only have the uniform $[0,1]$ distribution, we would have

$$Z = \mathcal{N}^{-1}(U) \sim N(0,1) \ \ , U \sim \mathcal{U}(0,1),$$

where $\mathcal{N}^{-1}$ is the inverse of the standard normal distribution c.d.f.

Q5. Instead of simulate the true uniform variable, one is able also to use a *low discrepancy sequence*, for instance, the Sobol sequence (see this link). Apply the previous method to generate n values ($n \in \{1000, 10000, 100000\}$) for the normal distribution, verify indeed that when n becomes large the distribution of generated values converge to the standard normal distribution.

Q6. For the purpose of *hedging*, we also need (among many greeks) the *delta*, i.e., $\Delta = \partial u / \partial x$. Provide two schemes to approximate $\Delta(t,x)$ from $u(t,x)$, $u(t, x \pm \varepsilon)$.

Q7. When we approximate $u(t, x \pm \varepsilon)$, we can either simulate new random values or reuse the same random values as when we approximate $u(t,x)$. Verify numerically indeed that it's better to reuse the same random sequences (in comparing for example the rate of convergence to the true value) in the case of an European call option $\phi(S_T) = \max(S_T - K, 0)$, For all numerical applications we take $t = 0$, $K = 100$ and $\epsilon = 0.001$, we also make use of the Sobol sequences.

Q8. The FDM scheme to calculate Δ is useful only when the payoff function is smooth (as in European call/put case). When it's not the case, for example a *binary option*, then FDM method converges very slow to the exact value. Verify that in the case of a *binary call* option $\phi(S_T) = \mathbb{1}_{S_T \geq K}$ the estimation error is very large even with $n = 10^6$. The estimation error is calculated as

$$\text{err in percentage} = 100 \times \left| \frac{\text{MC value} - \text{exact value}}{\text{exact value}} \right|.$$

Q9. In such cases, we are able to use an advanced method, the *Malliavin calculus*, to calculate the delta (as well as the other greeks). We admit the following result

$$\Delta(0,x) = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[\pi_\Delta \cdot \phi(S_T) \, \middle| S_0 = x \right] \ \ , \ \ \pi_\Delta = \frac{W_T}{x \sigma T}.$$

Compare the estimation error between this case and the previous one.