Assignment 4

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2. Question

Let us suppose the underlying asset price, S_t , verifies the following dynamics under $\mathbb{Q}$

$$dS_t = S_t(rdt + \sigma dW_t),$$

where W. is a standard Brownian motion under $(\mathbb{Q}, \mathcal{F}\cdot)$.

We consider the pricing of an Asian option where the payoff function g depends not only on the final but also on the average price,

$$A_t = \frac{1}{t}\int_0^t S_u du$$

The price of such an option at t is $V(t, S, A)$. We recall that, under $\mathbb{Q}$, $(e^{-rt}V(t, S_t, A_t))_t$ is a martingale. For all numerical applications we take $g(S, A) = max(A - K, 0)$ (Asian call option with fixed strike), and $S_0 = 100, K = 100, \sigma = 0.3, r = 0.02, T = 1.$

2.1 Monte-Carlo simulation

Question 1.

Provide an expression allowing one to use the Monte-Carlo to estimate $V(t, S, A)$ from $g(S_T, A_T)$.

Answer:

We got the price of an option at time t is the expected payoff value which is discounted by the risk free rate:
$$V(t,S,A) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[g(S_T,A_T)\right]$$

By using Monte-Carlo method to estimate $V(t,S,A)$ from $g(S_T,A_T)$, we got
$$\mathbb{E}^{\mathbb{Q}}\left[g(S_T,A_T)\right] \approx \frac{\sum_{i=1}^{N} g(S_T^{(i)},A_T^{(i)})}{n}$$

Then
$$V(t,S,A) \approx V^{MC}(t,S,A) \approx e^{-r(T-t)}\frac{\sum_{i=1}^{N} g(S_T^{(i)},A_T^{(i)})}{n}$$

Question 2.

In this case, we need to simulate not only S_T but also S_t for all $t < T$ to have A_T. </br> Show that
$$S_t = S_0 exp^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$$
Describe how to simulate directly $\{S_t, t = t_k = k.T/N = k.\delta t\}_{k=1,\ldots,N}$

Answer:

Let $f(S_t) = ln(S_t)$, we apply Itô Lemma as follow:
$$df(S_t,t) = rdt + \sigma dW_t$$
We also have
$$\begin{cases} \frac{\partial f}{\partial S_t} = \frac{1}{S_t} \\ \frac{\partial f}{\partial t} = 0 \\ \frac{\partial^2 f}{\partial^2 {S_t}^2} = \frac{-1}{S_t^2} \\ d\langle S_t\rangle = \sigma^2 {S_t}^2 dt \end{cases}$$
Then

$$
\begin{aligned}
df(S_t,t) = d\big(ln(S_t)\big) &= \left(\frac{\partial f}{\partial S_t} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial^2 {S_t}^2}\right)dt + \left(\frac{\partial f}{\partial S_t} b\right)W_t \\
&= \frac{1}{S_t}dS_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)d\langle S_t\rangle \\
&= \frac{1}{S_t}.\,S_t(rdt+\sigma dW_t) - \frac{1}{2}\left(\frac{1}{S_t^2}\right).\,S_t^2\sigma^2 dt \text{ (As } S_t \text{ follow an Itô process, then } dS_t = \text{ ı} \\
&= \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \int_0^t dln(S_u) &= \left(r - \frac{\sigma^2}{2}\right)\int_0^t du + \sigma\int_0^t dW_u \\
\Leftrightarrow ln\left(\frac{S_t}{S_0}\right) &= \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t \\
\Leftrightarrow \frac{S_t}{S_0} &= exp^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} \\
\Leftrightarrow S_t &= S_0 exp^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} \text{ (proved)}
\end{aligned}
$$

The interval $[0, T]$ will be divided into N steps. The step size will be noted $\delta_t = T/N$ and we define the times $t = t_k = k \cdot T/N = k \cdot \delta_t$ with k=1,2,...,N.

To simulate S_t, we can simulate the brownian motion W_t and then computing S_t with the previous formula.

To simulate W_t:

- we draw $(G_i)_{i=1,\ldots,n} \sim \mathcal{N}(0,1)$ and $i \in \{1,..,n\} i.i.d$
- $W_0 = 0$
- for $t = t_k = k \cdot T/N = k \cdot \delta_t$ do $W_t = W_{t-1} + \sqrt{t_i - t_{i-1}} * G_i$

Then, we can stimulate S_T with all the known variable: $S_0, r, sigma, t, W_t$.

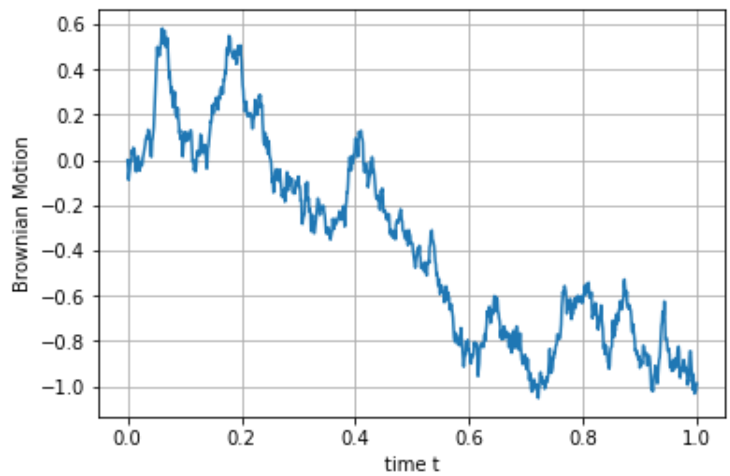
$S_t = S_0 exp^{(r-\sigma^2)t+\sigma W_t}$

In [89]:
```
S_0 = 100
K = 100
sigma = 0.3
r = 0.02
T = 1
```

In [51]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

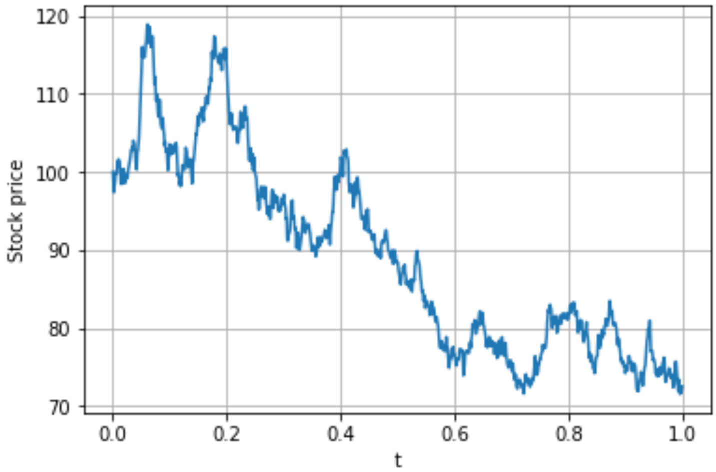
In [11]:
```
# Brownian motion simulation function
def BMS(N,T):
    G = np.random.normal(0,1,N)

    W=np.empty(N)
    W[0]=0

    for i in range (1,N,1):
        W[i]=W[i-1]+np.sqrt(T/N)*G[i]
    return W
```

In [23]:
```
#Plotting
N=1000
T=1
BM = BMS(N,T)
t = np.linspace(0.0, T, N)
plt.plot(t, BM)
plt.xlabel('time t')
plt.ylabel('Brownian Motion')
plt.grid(True)
plt.show()
```



In [37]:
```
#Apply to S
S=np.empty(N)
S[0]=S_0
for i in range(len(BM)):
    S[i] = S[0] * np.exp ((r-(sigma**2)/2)* i*T/N + BM[i]*sigma)
```

In [38]:
```
#plotting
plt.plot(t, S)
plt.xlabel('t')
plt.ylabel('Stock price')
plt.grid(True)
plt.show()
```



Question 3.

Turning to A_T, an easy way is to approximate it by (scheme 1)

$$A_T = \frac{1}{T}\int_0^T S_u du \approx \frac{1}{N}\sum_{k=1}^{N} S_{t_k}$$

We provide another method to simulate A_T , we first remark that, from the simulation of S_T we also obtain W_{t_k} . Suppose we have $\left\{W_{t_k} = w_k\right\}\ k = 0, \ldots, N$, we seek to simulate exactly the distribution of

$$I_k := \int_{t_k}^{t_{k+1}} W_u du$$

We admit (but it can be showed easily) that

$$I_k \mid (W_{t_k} = w_k, W_{t_{k+1}} = w_{k+1}) \sim \mathcal{N}\left(\frac{w_k + w_{k+1}}{2}\delta t, \frac{(\delta t)^3}{3}\right)$$

Now we rewrite

$$T \cdot A_T = \int_0^T S_u du = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} S_u du = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} S_{t_k} exp^{(r-\frac{\sigma^2}{2})(u-t_k)+\sigma(W_u - W_{t_k})} du$$

Finally, using the first order development $e^x = 1 + x + o(x)$, we have (scheme 2)

$$A_T \approx \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\left(\delta t + \left(r - \frac{\sigma^2}{2}\right)\frac{(\delta t)^2}{2} + \sigma I_k - \sigma \delta t W_t\right)$$

Implement the above two schemes and compute the option price, compare their quality (in term of the estimation variance). For all numerical applications we take N = 1000 (time step $\delta t = 0.001$) and $M \in \{10^2, 10^3, \ldots, 10^6\}$ (number of simulations).

Answer:

In [114]:
```
delta = T/N
W = BMS(N+1,T)
```

In [100]:
```
# Scheme 1 calculation
def AT1(M):
    S=np.empty(N)
    A=np.empty(M)
    S[0]=S_0
    poff=np.empty(M)
    for j in range(0,M,1):
        total=0
        for i in range (1,N):
            S[i] = S[0] * np.exp ((r-(sigma**2)/2)* i*T/N + W[i]*sigma)
        A[j]=np.sum(S)/N
        poff[j]=np.maximum(A[j]-K,0)
    return np.exp(-r*T)*np.sum(poff)/M
```

In [84]:
```
# Scheme 2 calculation
def AT2(M):
    S=np.empty(N)
    S[0]=S_0
    A=np.empty(M)
    Y=np.empty(M)
    poff=np.empty(M)

    for h in range (0,M,1):
        for i in range (1,N):
            S[i] = S[0] * np.exp ((r-(sigma**2)/2)* i*T/N + W[i]*sigma)

        I=np.empty(N-1)
        for i in range(0,N-1,1):
            mean=0.5*(W[i]+W[i+1])*delta
            mu = delta**3/3
            I[i]=np.random.normal(mean,mu)

        Y=np.empty(N-1)
        for i in range (0,N-1,1):
            Y[i]=S[i]*(delta+(r-0.5*sigma*sigma)*0.5*delta**2+sigma*I[i]
-sigma*W[i]*delta)

        A[h]=np.sum(Y)/T
        poff[h]=np.maximum(A[h]-K,0)
    return np.exp(-r*T)*np.mean(poff)
```

In [120]:
```
#Create table for comparision
MM =[10**2,10**3,10**4,10**5]
ATT1 = np.empty(len(MM))
ATT2 = np.empty(len(MM))

for i in range (0,len(MM),1):
    ATT1[i] = AT1(MM[i])
    ATT2[i] = AT2(MM[i])
```

In [136]:
```
MM
```

Out[136]:
```
[100, 1000, 10000, 100000]
```

In [137]:
```
ATT1
```

Out[137]:
```
array([14.79307016, 14.79307016, 14.79307016, 14.79307016])
```

In [138]:
```
ATT2
```

Out[138]:
```
array([14.68901463, 14.68901469, 14.68901468, 14.68901468])
```

2.2 PDE

Question 1.

Show that V verifies the following PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{S-A}{t}\frac{\partial V}{\partial A} - rV = 0,\ V(t,S,A) = g(S,A)$$

Answer:

Applying **Itô formula** to $V(t,S,A)$, we got:

$$d\Big(V(t,S,A)\Big) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial A}dA + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d\langle S\rangle_t + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}d\langle A\rangle_t + \frac{\partial^2 V}{\partial A\partial S}d\langle S,A\rangle_t\ (1)$$

We also have

$$\begin{cases} dS = rSdt + \sigma SdW_t \\ A_T = \frac{1}{T}\int_0^T S_u du \Rightarrow dA = \frac{S_t - S_0}{t}dt\ (\ \lim_{t\to 0} A_t = S_0\ ) \Rightarrow dA = \frac{S_t - A}{t}dt \\ dS \text{ got stochastic term: } \sigma SdW_t \Rightarrow d\langle S\rangle_t = \sigma^2 S^2 dt \\ dA \text{ doesnot have stochastic term } \Rightarrow d\langle A\rangle_t = 0 \text{ and } d\langle S,A\rangle_t = 0 \end{cases}$$

Replace into (1) we got

$$d\Big(V(t,S,A)\Big) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\Big(rSdt + \sigma SdW_t\Big) + \frac{\partial V}{\partial A}\Big(\frac{S_t - A}{t}\Big)dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt$$

According to the "No arbitrage strategy" rule, we have

$$\begin{aligned}
&\Rightarrow dV && = rVdt \\
&\Leftrightarrow \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\Big(rSdt + \sigma SdW_t\Big) + \frac{\partial V}{\partial A}\Big(\frac{S_t - A}{t}\Big)dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - rVdt && = 0 \\
&\Leftrightarrow \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}rSdt + \frac{\partial V}{\partial S}\sigma SdW_t + \frac{\partial V}{\partial A}\Big(\frac{S_t - A}{t}\Big)dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - rVdt && = 0 \\
&\Leftrightarrow dt\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial A}\Big(\frac{S_t - A}{t}\Big) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - rV\right) + \frac{\partial V}{\partial S}\sigma SdW_t && = 0 \\
&\Rightarrow \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial A}\Big(\frac{S_t - A}{t}\Big) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - rV && = 0
\end{aligned}$$

proved

Question 2.

For some specific payoff (as in our case), we make the following change of variables

$$\xi = \frac{K - tA/T}{S}, V(t,S,A) = Sf(t,\xi)$$

Verify that f solves

$$\frac{\partial f}{\partial t} + \frac{\sigma^2 \xi^2}{2}\frac{\partial^2 f}{\partial \xi^2} - \left(\frac{1}{T} + r\xi\right)\frac{\partial f}{\partial \xi} = 0,\ f(T,\xi) = \phi(\xi) := max(-\xi, 0)$$

We also admit that

$$f(t,\xi) \approx \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) - \xi e^{-r(T-t)} \text{ when } \xi \to -\infty,\ f(t,\xi) = 0 \text{ when } \xi \to \infty$$

Answer:

We have:

$$\xi = \frac{K - tA/T}{S} = (K - \frac{tA}{T})S^{-1} = KS^{-1} - tAT^{-1}S^{-1}$$

Apply **Ito Lemma** on $\xi(t,S)$, we get:

$$\begin{aligned} d\xi &= \frac{\partial \xi}{\partial t}dt + \frac{\partial \xi}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \xi}{\partial S^2}d\langle S\rangle \\ &= \frac{-A}{TS}dt + \frac{-\xi}{S}(Srdt + S\sigma dW_t) + \frac{1}{2}\frac{2\xi}{S^2}(S^2\sigma^2 dt) \\ &= \frac{-A}{TS}dt + \xi\left(-rdt - \sigma dW_t + \sigma^2 dt\right) \\ &= \left(\frac{-A}{TS} - r\xi + \sigma^2\xi\right)dt - \sigma\xi dW_t \end{aligned}$$

As

$$\begin{cases} dS = rSdt + \sigma S dW_t \\ dS \text{ got stochastic term: } \sigma S dW_t \Rightarrow d\langle S\rangle = \sigma^2 S^2 dt \\ \frac{\partial \xi}{\partial t}dt = \frac{-A}{TS}dt \\ \frac{\partial \xi}{\partial S}dS = (K - \frac{tA}{T})S^{-2} = \frac{-\xi}{S}dS \\ \frac{\partial^2 \xi}{\partial S^2}d\langle S\rangle = 2(K - \frac{tA}{T})S^{-3} = \frac{2\xi}{S^2}d\langle S\rangle \end{cases}$$

Then

$$d\langle \xi\rangle = \sigma\xi dW_t$$

On the other hand, we apply **Itô Lemma** on $f(t,\xi)$:

$$\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}d\langle \xi \rangle \\
\Leftrightarrow df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \xi}\left(\left(\frac{-A}{TS} - r\xi + \sigma^2\xi\right)dt - \sigma\xi dW_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\left(\sigma^2\xi^2 dt\right) \\
\Leftrightarrow df &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\left(\frac{-A}{TS} - r\xi + \sigma^2\xi\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\sigma^2\xi^2\right)dt - \frac{\partial f}{\partial \xi}\sigma\xi dW_t
\end{aligned}$$

We also got:

$$\begin{cases}
d\langle S,f\rangle = (\sigma S)\left(-\frac{\partial f}{\partial \xi}\sigma\xi\right)dW_t = -\frac{\partial f}{\partial \xi}S\sigma^2\xi dt \\
df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\left(\frac{-A}{TS} - r\xi + \sigma^2\xi\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\sigma^2\xi^2\right)dt - \frac{\partial f}{\partial \xi}\sigma\xi dW_t \\
dS = rSdt + \sigma S dW_t
\end{cases}$$

Let apply **Itô Lemma** again on $V(t,S,A)$:

$$\begin{aligned}
& V(t,S,A) = Sf(t,\xi) \\
\Leftrightarrow \quad & dV = fdS + Sdf + d\langle S,f\rangle \\
\Leftrightarrow \quad & dV = f.S(rdt + \sigma dW_t) + S.\left(\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\left(\frac{-A}{TS} - r\xi + \sigma^2\xi\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\sigma^2\xi^2\right)dt - \frac{\partial f}{\partial \xi}\sigma\xi dW_t\right) - \\
\Leftrightarrow \quad & dV = S.\left(fr + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\left(\frac{-A}{TS} - r\xi\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\sigma^2\xi^2\right)dt + \left(fS\sigma - \frac{\partial f}{\partial \xi}\sigma\xi\right)dW_t
\end{aligned}$$

According to the "No arbitrage strategy" rule, we have

$$\begin{aligned}
& dV = rVdt \\
\Leftrightarrow \quad & dV - rSfdt = 0 \\
\Leftrightarrow \quad & S.\left(fr + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\left(\frac{-A}{TS} - r\xi\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\sigma^2\xi^2\right)dt + \left(fS\sigma - \frac{\partial f}{\partial \xi}\sigma\xi\right)dW_t - rSfdt = 0 \\
\Leftrightarrow \quad & S.\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\left(\frac{-A}{TS} - r\xi\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\sigma^2\xi^2\right)dt + \left(fS\sigma - \frac{\partial f}{\partial \xi}\sigma\xi\right)dW_t = 0
\end{aligned}$$

Then

$$\frac{\partial f}{\partial t} + \frac{\sigma^2\xi^2}{2}\frac{\partial^2 f}{\partial \xi^2} - \left(\frac{1}{T} + r\xi\right)\frac{\partial f}{\partial \xi} = 0 \text{ (proved!)}$$

Question 3.

In order to solve the above PDE, we need to approximate $\partial/\partial t$, $\partial/\partial\xi$ and $\partial^2/\partial\xi^2$, suggest some finite difference schemes to approximate those operators. We use the usual notation δt and $\delta\xi$ for t and ξ, respectively.

Answer:

Question 4.

Discretize the PDE with the Crank-Nicolson scheme and solve for $f(t,\xi)$ on $[0,T]\times[-6,0]$. Compare the option values given by the PDE approach and the Monte-Carlo estimation when σ ∈ { 0.01, 0.05, 0.1, 0.2, 0.3 } . For all numerical applications we take N ∈ { 200, 500, 1000 } (discretization in t space) and M ∈ { 100, 300, 600 } (discretization in ξ space). What are your remarks regarding the precision of the PDE values?

Answer: