derivatives6_assignment_2018

July 28, 2018

1 Instructions

1. Due: 11:59 AM 04/08/2018.
2. Answers must be
   - named as LastName_FirstName_6(.ipynb/.pdf)
   - uploaded directly to Google drive, under **assignments-student/YourName/**

2 Questions

Q1. (Monotonicity and convexity with respect to strike) Let S be a financial instrument, $C(K)$ be the price of a call option with payoff $g(S_T) = \max(S_T - K, 0)$ where K is the strike-price. We suppose that $C(K)$ is twice differentiable, i.e., $\partial C/\partial K$ and $\partial^2 C/\partial K^2$ exist.

a. Suppose that the spot price at $t = 0$ is S_0, provide the definition of in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM) call options.

b. Which one is more expensive, an ITM or an OTM call option? Justify the answer by a rigorous proof using no-arbitrage argument, then use it to show $\partial C/\partial K \leq 0$.

c. Let be given three call options, the first one is ITM, the second one ATM and the last one OTM, with strike K_1, K_2, K_3, respectively. We consider a *butterfly spread* strategy which consists in buying 1 ITM call, selling 2 ATM calls and buying 1 OTM call with $K_1 + K_3 = 2K_2$. Show again by means of no-arbitrage argument that the initial cost of such strategy must be non-negative, then explain why we must have $\partial^2 C/\partial K^2 \geq 0$.

Q2. (Call-Put symmetry) We consider the general Black-Scholes-Merton dynamics for an asset S under $\mathbb{Q}$

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

Let $C(t, x, K, T, r, q, \sigma)$ and $P(t, x, K, T, r, q, \sigma)$ be the price at t of an European call and put option with an initial spot price x, strike K, maturity T, risk-free interest-rate r, dividend rate q and volatility σ.

a. Provide the Black-Scholes-Merton formula for call/put option.

b. Using Ito's formula, provide an explicit expression for S_T.

c. In the case where $r = q = 0$ (only in this question), show that the following symmetry holds (For ease of reading we omit t, T, r, q and σ in this expression.)

$$C(x, K) = P(K, x).$$

Deduce from the previous questions that

$$\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(x - K\frac{S_T}{x}\right)^+\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{x}\left(\frac{x^2}{S_T} - K\right)^+\right].$$

Since it holds true for all positive K, we have just showed that for all positive payoff g

$$\mathbb{E}^{\mathbb{Q}}\left[g(S_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{x} g\left(\frac{x^2}{S_T}\right)\right].$$

d. Turning back to the general case, let $X = S^\gamma$. Use Ito's formula to derive the dynamics of X under $\mathbb{Q}$, then deduce without doing any calculation, the explicit price for an European put option with payoff $(K - X_T)^+$.

e. Finally, show that for all positive payoff g and $\gamma = 1 - 2(r-q)/\sigma^2$

$$\mathbb{E}^{\mathbb{Q}}\left[g(S_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{x}\right)^\gamma g\left(\frac{x^2}{S_T}\right)\right].$$

(Hint: use the variable X introduced in d.)

Q3. (Schaeffer and Schwartz model) We consider a two-factor interest rate model consisting of the long rate ℓ and the *spread* s between the short rate and long rate, i.e., $s = r - l$,

$$\begin{aligned} \mathrm{d}s &= \beta_s(s, l, t)\,\mathrm{d}t + \eta_s(s, l, t)\,\mathrm{d}W_s \\ \mathrm{d}\ell &= \beta_\ell(s, l, t)\,\mathrm{d}t + \eta_\ell(s, l, t)\,\mathrm{d}W_\ell\ . \end{aligned}$$

Moreover, empirical evidence shows that the long rate and the spread are almost uncorrelated, thus we suppose $\mathrm{d}\langle W_s, W_\ell\rangle_t = 0$.

A. Show that the price of a zero-coupon bond $B(s, \ell, t)$ verifies

$$\frac{\partial B}{\partial t} + \frac{\eta_s^2}{2}\frac{\partial^2 B}{\partial s^2} + \frac{\eta_\ell^2}{2}\frac{\partial^2 B}{\partial \ell^2} + (\beta_s - \lambda_s\eta_s)\,\frac{\partial B}{\partial s} + (\beta_\ell - \lambda_\ell\eta_\ell)\,\frac{\partial B}{\partial \ell} - (s + \ell)B = 0\,,$$

where λ_s/λ_ℓ is the market price of the spread/long rate risk, respectively.

B. Let G be a *consol bond*, a perpetual bond (with infinite maturity) pays coupon at a continuous constant rate c. Let $G(\ell)$ denote the value of this bond, we admit that

$$G(\ell) = \frac{c}{\ell}.$$

B.1. Apply Ito's formula to express the dynamic of G under the form

$$\frac{\mathrm{d}G}{G} = \mu_G\,\mathrm{d}t + \sigma_G\,\mathrm{d}W_\ell.$$

B.2. The instantaneous rate of return of a consol bond is the sum of coupon rate ℓ and the drift rate of G, $\mu_c = \mu_G + \ell$, while the volatility is the same as the volatility of G, $\sigma_c = \sigma_G$. Show that

$$\beta_\ell - \lambda_\ell\eta_\ell = \frac{\eta_\ell^2}{\ell} - s\ell\ .$$