derivatives4_assignment_2018

May 28, 2018

1 Instructions

1. Due: 12:00 PM 04/06/2018.
2. Answers must be
   - named as LastName_FirstName_4(.ipynb/.pdf)
   - uploaded directly to Google drive, under **assignments-student/YourName/**

2 Questions

Let us suppose the underlying asset price, S_t, verifies the following dynamics under $\mathbb{Q}$

$$\mathrm{d}S_t = S_t(r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t),$$

where $W_.$ is a standard Brownian motion under $(\mathbb{Q}, \mathcal{F}_.)$.

We consider the pricing of an Asian option where the payoff function g depends not only on the final but also on the *average price*,

$$A_t = \frac{1}{t}\int_0^t S_u\,\mathrm{d}u.$$

The price of such an option at t is $V(t, S, A)$. We recall that, under $\mathbb{Q}$, $(e^{-rt}V(t, S_t, A_t))_t$ is a martingale. For all numerical applications we take $g(S, A) = \max(A - K, 0)$ (Asian call option with fixed strike), and

$$S_0 = 100, K = 100, \sigma = 0.3, r = 0.02, T = 1.$$

2.1 Monte-Carlo simulation

1. Provide an expression allowing one to use the Monte-Carlo to estimate $V(t, S, A)$ from $g(S_T, A_T)$.
2. In this case, we need to simulate not only S_T but also S_t for all $t < T$ to have A_T. Show that

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

   Describe how to simulate directly $\{S_t, t = t_k = k \cdot T/N = k \cdot \delta t\}_{k=1,\ldots,N}$.

3. Turning to A_T, an easy way is to approximate it by (scheme 1)

$$A_T = \frac{1}{T}\int_0^T S_u \,\mathrm{d}u \approx \frac{1}{N}\sum_{k=1}^{N} S_{t_k}.$$

We provide another method to simulate A_T, we first remark that, from the simulation of S_T we also obtain W_{t_k}. Suppose we have $\{W_{t_k} = w_k\}_{k=0,\ldots,N}$, we seek to simulate exactly the distribution of

$$I_k := \int_{t_k}^{t_{k+1}} W_u \,\mathrm{d}u.$$

We admit (but it can be showed easily) that

$$I_k \,\Big|\, \left(W_{t_k} = w_k, W_{t_{k+1}} = w_{k+1}\right) \sim \mathcal{N}\left(\frac{w_k + w_{k+1}}{2}\delta t, \frac{(\delta t)^3}{3}\right).$$

Now we rewrite

$$T \cdot A_T = \int_0^T S_u \,\mathrm{d}u = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} S_u \,\mathrm{d}u = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} S_{t_k} \exp\left(\left(r - \frac{\sigma^2}{2}\right)(u - t_k) + \sigma\left(W_u - W_{t_k}\right)\right) \mathrm{d}u.$$

Finally, using the first order development $e^x = 1 + x + o(x)$, we have (scheme 2)

$$A_T \approx \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\left(\delta t + \left(r - \frac{\sigma^2}{2}\right)\frac{(\delta t)^2}{2} + \sigma I_k - \sigma \delta t\, W_{t_k}\right).$$

Implement the above two schemes and compute the option price, compare their quality (in term of the estimation variance). For all numerical applications we take $N = 1000$ (time step $\delta t = 0.001$) and $M \in \left\{10^2, 10^3, \ldots, 10^6\right\}$ (number of simulations).

2.2 PDE

1. Show that V verifies the following PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{t}(S - A)\frac{\partial V}{\partial A} - rV = 0 \;\; , \;\; V(T, S, A) = g(S, A).$$

2. For some specific payoff (as in our case), we make the following change of variables

$$\xi = \frac{K - tA/T}{S} \; , \; V(t, S, A) = Sf\left(t, \xi\right).$$

Verify that f solves

$$\frac{\partial f}{\partial t} + \frac{\sigma^2 \xi^2}{2}\frac{\partial^2 f}{\partial \xi^2} - \left(\frac{1}{T} + r\xi\right)\frac{\partial f}{\partial \xi} = 0 \;\; , \;\; f(T, \xi) = \phi(\xi) := \max(-\xi, 0).$$

We admit also that

$$f(t, \xi) \approx \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) - \xi e^{-r(T-t)} \text{ when } \xi \to -\infty \;\; , \;\; f(t, \xi) = 0 \text{ when } \xi \to \infty.$$

3. In order to solve the above PDE, we need to approximate $\partial/\partial t$, $\partial/\partial\xi$ and $\partial^2/\partial\xi^2$, suggest some finite difference schemes to approximate those operators. We use the usual notation δt and $\delta\xi$ for t and ξ, respectively.

4. Discretize the PDE with the Crank-Nicolson scheme and solve for $f(t,\xi)$ on $[0,T] \times [-6,0]$. Compare the option values given by the PDE approach and the Monte-Carlo estimation when $\sigma \in \{0.01, 0.05, 0.1, 0.2, 0.3\}$. For all numerical applications we take $N \in \{200, 500, 1000\}$ (discretization in t space) and $M \in \{100, 300, 600\}$ (discretization in ξ space). What are your remarks regarding the precision of the PDE values?