Contents

1 Modeling

1.1 Splines

1.1.1 Cubic Hermite Interpolation

Each point is defined by its position h_n and slope h_{m+n}, m being the number of control points. To simplify calculations, it is assumed that $t_0 = 0$ and $t_1 = 1$.

The goal is to convert from a monomial basis

$$\begin{aligned}\phi_0(t) &= 1\\ \phi_1(t) &= t\\ \phi_2(t) &= t^2\\ \phi_3(t) &= t^3\end{aligned}$$

to a hermite basis

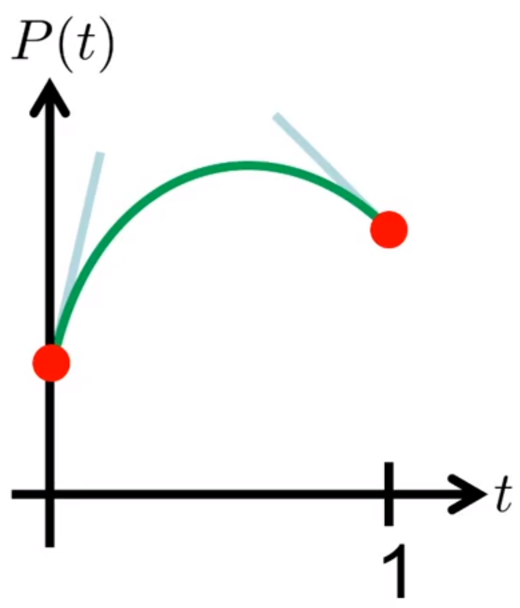
$$\begin{aligned}H_0(t) &= 2t^3 - 3t^2 + 1\\ H_1(t) &= -2t^3 + 3t^2\\ H_2(t) &= t^3 - 2t^2 + t\\ H_3(t) &= t^3 - t^2\end{aligned}$$

so that instead of having to manipulate polynomial coefficients

$$f(t) = a\phi_3(t) + b\phi_2(t) + c\phi_1(t) + d\phi_0(t)$$

an easier point slope method can be used:

$$f(t) = h_0H_0(t) + h_1H_1(t) + h_2H_2(t) + h_3H_3(t)$$



$$P(t) = at^3 + bt^2 + ct + d$$
$$P'(t) = 3at^2 + 2bt + c$$

$$\begin{aligned}h_0 &= P(0) = d\\ h_1 &= P(1) = a + b + c + d\\ h_2 &= P'(0) = c\\ h_3 &= P'(1) = 3a + 2b + c\end{aligned}$$

Unknowns in this equation are a, b, c, and d, so a matrix can be used to solve the systems of equations:

$$\begin{pmatrix}0&0&0&1\\1&1&1&1\\0&0&1&0\\3&2&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}h_0\\h_1\\h_2\\h_3\end{pmatrix}$$

a, b, c, and d can be obtained from h values by inverting the matrix:

$$\begin{pmatrix}0&0&0&1\\1&1&1&1\\0&0&1&0\\3&2&1&0\end{pmatrix}^{-1}\begin{pmatrix}h_0\\h_1\\h_2\\h_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

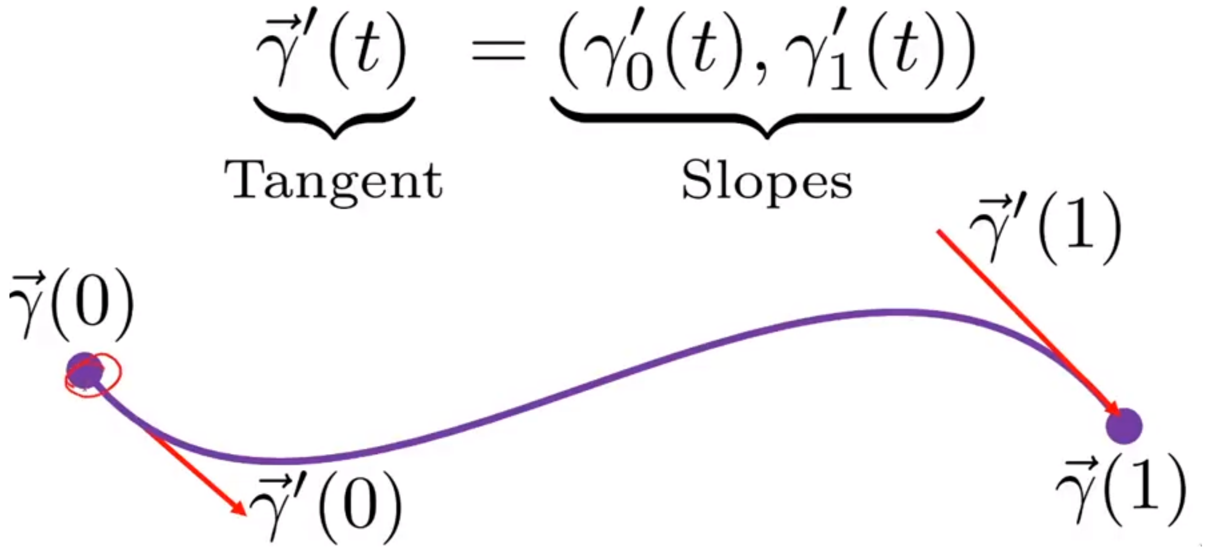
From these solved h values, $P(t)$ can now be converted to a form that is easier for a user to manipulate, in terms of h values:

$$\begin{aligned}P(t) &= at^3 + bt^2 + ct + d \\ &= (2h_0 - 2h_1 + h_2 + h_3)t^3 \\ &\quad + (-3h_0 + 3h_1 - 2h_2 - h_3)t^2 \\ &\quad + h_2 t + h_0 \\ &= h_0(2t^3 - 3t^2 + 1) + h_1(-2t^3 + 3t^2) + \\ &\quad h_2(t^3 - 2t^2 + t) + h_3(t^3 - t^2)\end{aligned}$$

Each equation in $P(t) = h_0(2t^3 - 3t^2 + 1) + h_1(-2t^3 + 3t^2) + h_2(t^3 - 2t^2 + t) + h_3(t^3 - t^2)$ that is multiplied by an h value is called a cubic hermite.

1.1.2 More than 1D

A parametric curve described by $\vec{\gamma}(t) = (\gamma_0(t), \gamma_1(t))$ can be converted into hermite basis like this:


$\underbrace{\vec{\gamma}'(t)}_{\text{Tangent}} = \underbrace{(\gamma_0'(t), \gamma_1'(t))}_{\text{Slopes}}$
$\vec{\gamma}'(1)$
$\vec{\gamma}(0)$
$\vec{\gamma}'(0)$
$\vec{\gamma}(1)$

where cubic hermite interpolation can be done for both dimensions.

1.1.3 Cubic blossom

The cubic blossom of a function $\vec{f}(t)$ is $\vec{F}(t_1, t_2, t_3)$.

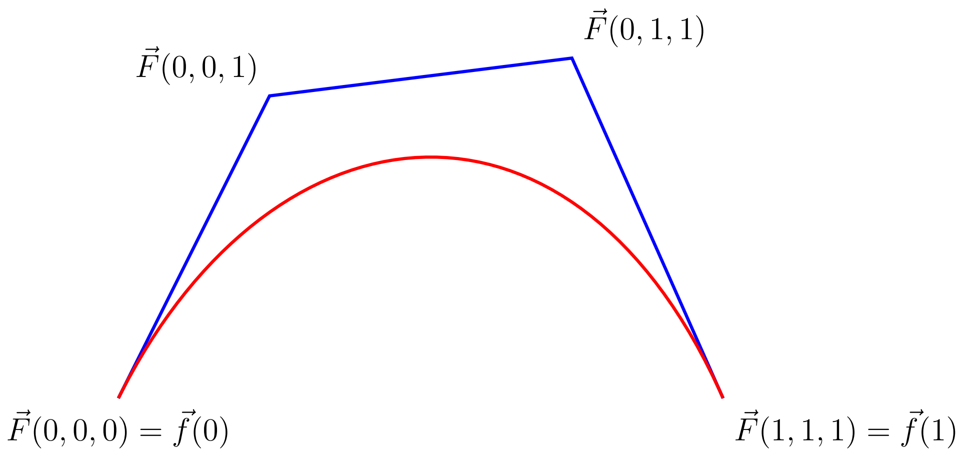
Cubic blossoms have three properties:

1. Symmetric
   - $\vec{F}(t_1, t_2, t_3) = \vec{F}(t_1, t_3, t_2) = \vec{F}(t_3, t_1, t_2) \cdots$
2. Affine
   - $\vec{F}(\alpha u + (1-\alpha)v, t_2, t_3) = \alpha\vec{F}(u, t_2, t_3) + (1-\alpha)\vec{F}(v, t_2, t_3)$
   - If only one of $\vec{F}$'s arguments $t_c = \alpha u + (1-\alpha)v$ is changing between different points on $\vec{F}$, then any value $\vec{F}(t_c, t_2, t_3)$ in between $\vec{F}(u, t_2, t_3)$ and $\vec{F}(v, t_2, t_3)$ is scaled equivalently with α like t_c is scaled between u and v.
3. Diagonal
   - $\vec{f}(t) = \vec{F}(t, t, t)$

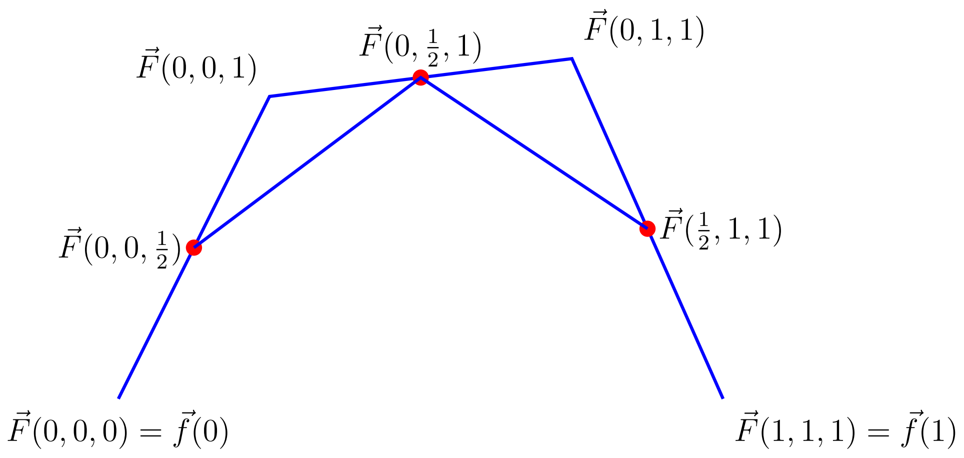
Blossoming examples

- $\vec{f}(t) = t^3 \mapsto \vec{F}(t_1, t_2, t_3) = t_1 t_2 t_3$
- $\vec{f}(t) = t^2 \mapsto \vec{F}(t_1, t_2, t_3) = (t_1 t_2 + t_1 t_3 + t_2 t_3)/3$
- $\vec{f}(t) = t \mapsto \vec{F}(t_1, t_2, t_3) = (t_1 + t_2 + t_3)/3$
- $\vec{f}(t) = 1 \mapsto \vec{F}(t_1, t_2, t_3) = 1$
- $\vec{f}(t) = 3t^3 - t + 1 = 3(t_1 t_2 t_3) - (t_1 + t_2 + t_3)/3 + 1$

Cubic curves can be blossomed by blossoming each coordinate function separately, which will give a function that maps 3 t variables to two dimensions x and y: $\vec{F}(t_1, t_2, t_3) : \mathbb{R}^3 \mapsto \mathbb{R}^2$. A cubic curve can be obtained from a blossom by specifying four points $\vec{F}(0,0,0), \vec{F}(0,0,1), \vec{F}(0,1,1), \vec{F}(1,1,1)$ (which form a cubic control polygon) and subdividing the surface given by the selected points (known as the De Castelijau's Algorithm). Only these four points are required because of the symmetry property of a blossom.

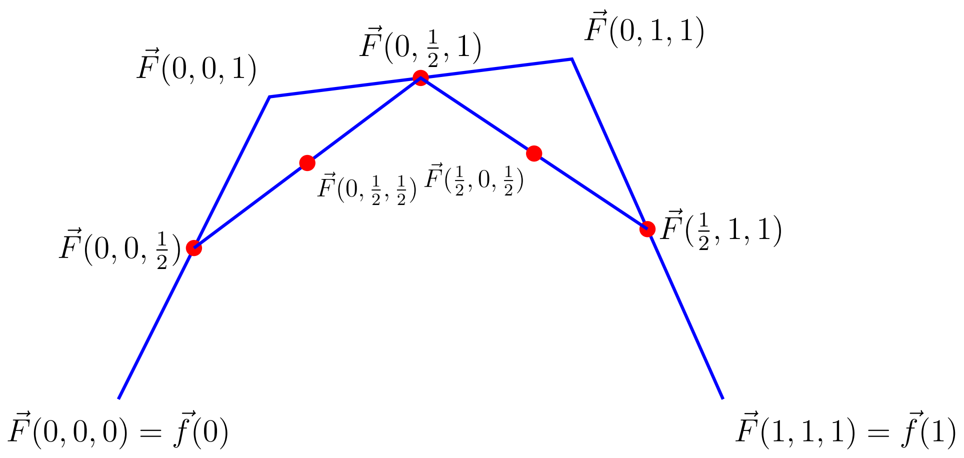


Subdividing the polygon:



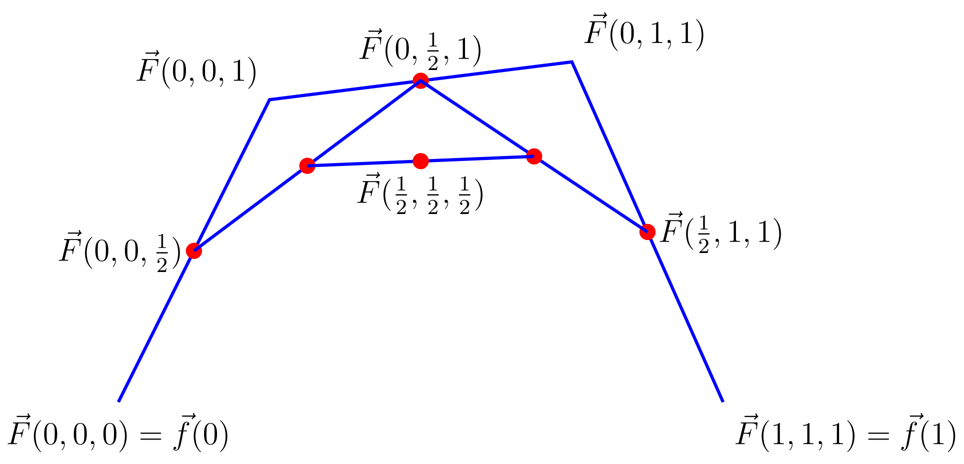
$\vec{F}(0,0,1)$
$\vec{F}(0,\frac{1}{2},1)$
$\vec{F}(0,1,1)$
$\vec{F}(0,0,\frac{1}{2})$
$\vec{F}(\frac{1}{2},1,1)$
$\vec{F}(0,0,0) = \vec{f}(0)$
$\vec{F}(1,1,1) = \vec{f}(1)$

$\vec{F}$'s arguments at the subdivision points can be determined due to the affine property of a blossom. Further subdivision



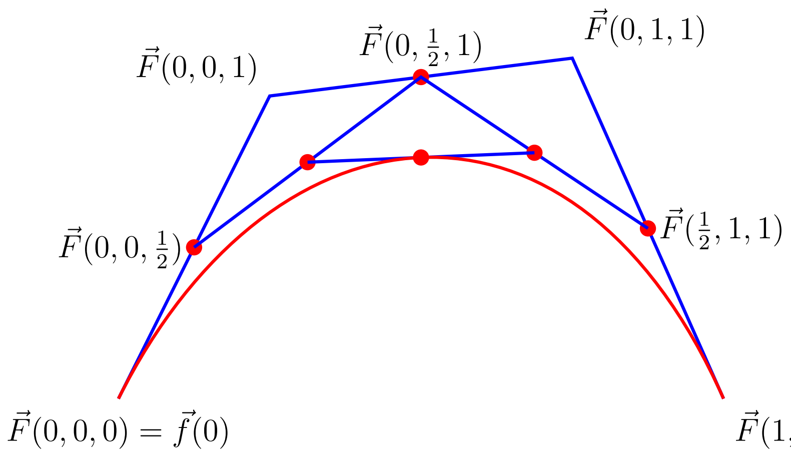
$\vec{F}(0,0,1)$
$\vec{F}(0,\frac{1}{2},1)$
$\vec{F}(0,1,1)$
$\vec{F}(0,\frac{1}{2},\frac{1}{2})$
$\vec{F}(\frac{1}{2},0,\frac{1}{2})$
$\vec{F}(0,0,\frac{1}{2})$
$\vec{F}(\frac{1}{2},1,1)$
$\vec{F}(0,0,0) = \vec{f}(0)$
$\vec{F}(1,1,1) = \vec{f}(1)$

One more subdivision (most recent two subdivision point values omitted because there's no room left)



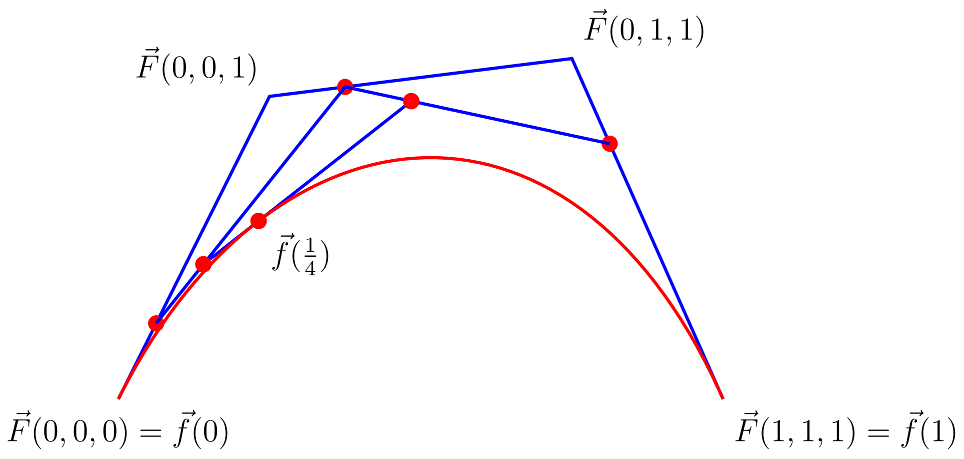
$\vec{F}(0,0,1)$
$\vec{F}(0,\frac{1}{2},1)$
$\vec{F}(0,1,1)$
$\vec{F}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
$\vec{F}(0,0,\frac{1}{2})$
$\vec{F}(\frac{1}{2},1,1)$
$\vec{F}(0,0,0) = \vec{f}(0)$
$\vec{F}(1,1,1) = \vec{f}(1)$

The final subdivision point can be shown to equal $\vec{f}(\frac{1}{2})$ because of the diagonal property of blossoms, meaning the $\vec{f}$ curve will intersect this third subdivision point:



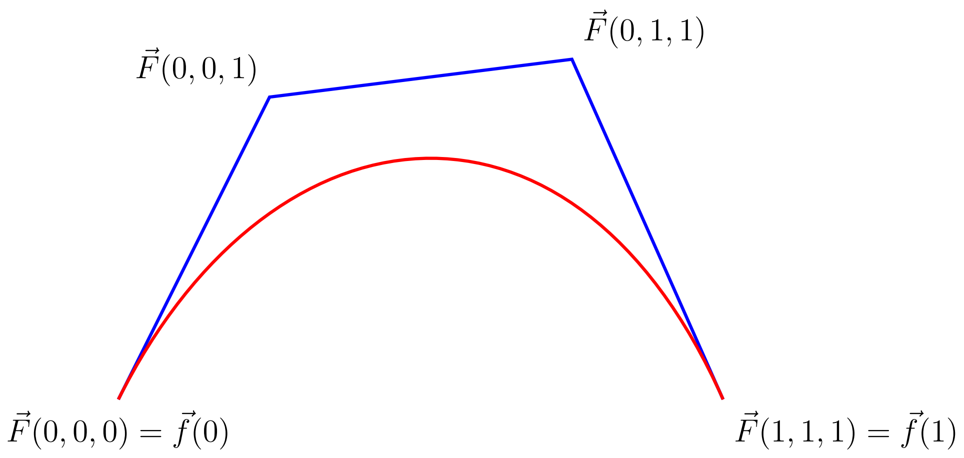
$\vec{F}(0,\frac{1}{2},1)$
$\vec{F}(0,1,1)$
$\vec{F}(0,0,1)$
$\vec{F}(0,0,\frac{1}{2})$
$\vec{F}(\frac{1}{2},1,1)$
$\vec{F}(0,0,0) = \vec{f}(0)$
$\vec{F}(1,1,1) = \vec{f}(1)$

Other values of $\vec{f}$ can be found with subdividing the cubic control polygon on different values. For example, to find $\vec{f}(\frac{1}{4})$, each edge would be divided $\frac{1}{4}$ of the way along it.



1.1.4 Bernstein polynomials

Another basis that can represent cubic curves. Bernstein basis is canonical for Bezier.



The red curve above can be represented as $\vec{f}(t) = \vec{F}(0,0,0)B_0(t) + \vec{F}(0,0,1)B_1(t) + \vec{F}(0,1,1)B_2(t) + F(1,1,1)B_3(t)$, knowing

$$\begin{aligned}
B_0(t) &= (1-t)^3 = 1 - 3t + 3t^2 - t^3 \\
B_1(t) &= 3t(1-t)^2 = 3t - 6t^2 + 3t^3 \\
B_2(t) &= 3t^2(1-t) = 3t^2 - 3t^3 \\
B_3(t) &= t^3
\end{aligned}$$

Changing basis from monomial to Bernstein:

$$\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} B_0(t) \\ B_1(t) \\ B_2(t) \\ B_3(t) \end{pmatrix}$$

And to change from Bernstein to monomial, the inverse of the matrix can be used.

1.1.5 General spline formulation

$\vec{\gamma}(t)$ = Geometry · Spline basis · Monomial basis

- Geometry contains control point coordinates
- Spline basis defines the type of spline (Hermite, Bernstein, etc)
- Monomial basis is a column vector $(1, t, t^2, \cdots, t^n)$

Example of a spline represented in the Bernstein basis:

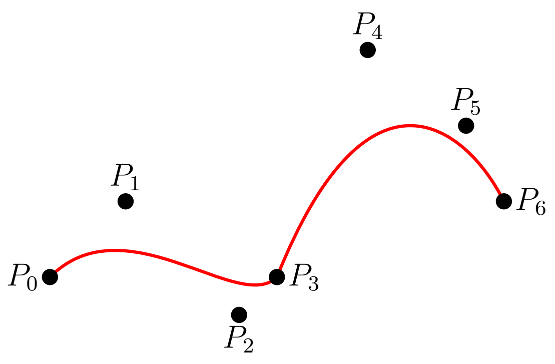
$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

1.1.6 Orders of continuity

C^0 = continuous (seam can be sharp)
G^1 = geometric continuity (Tangent vectors align at the seam)
C^1 = parametric continuity (Same velocity at the seam)
C^2 = curvature continuity (Tangents and tangent derivatives are the same)

1.1.7 Cubic B-splines

≥ 4 control points
Chain together splines in fours, popping the back control point off and pushing the next control point on



The full curve is composed of many smaller splines, in which the nth spline is formed by points $P_{n..n+3}$. The benefit of this method is that it guarantees C^1 continuity since every spline shares three control points with the one that comes before and after it.

The end points won't connect with this method, but repeating endpoints will fix it.

The basis functions as well as the basis conversion matrix for cubic b-spline are listed here:

$$B_1(t) = \frac{1}{6}(1-t)^3$$

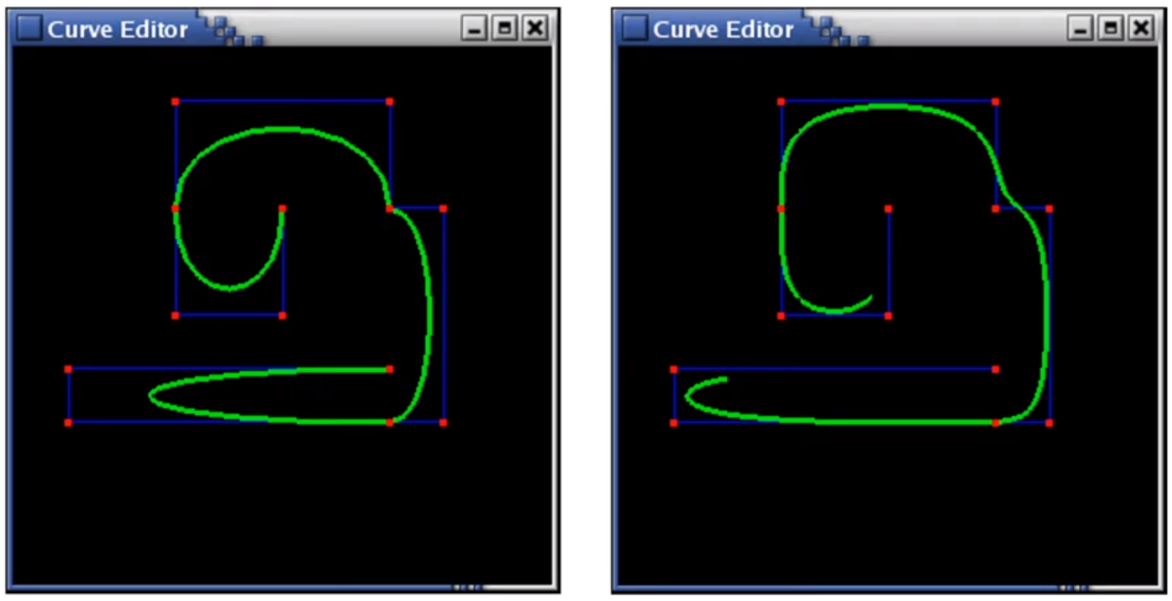
$$B_2(t) = \frac{1}{6}(3t^3 - 6t^2 + 4)$$

$$B_3(t) = \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)$$

$$B_4(t) = \frac{1}{6}t^3$$

$$B_{B-spline} = \frac{1}{6} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.1.8 Bezier vs B-spline



Bézier B-Spline

Bezier is derived from Bernstein polynomials, while B-spline uses a different set of basis functions. Additionally, Bezier will try to intersect with control points and only guarantees C^0 continuity, while B-spline does not and guarantees C^1 continuity.

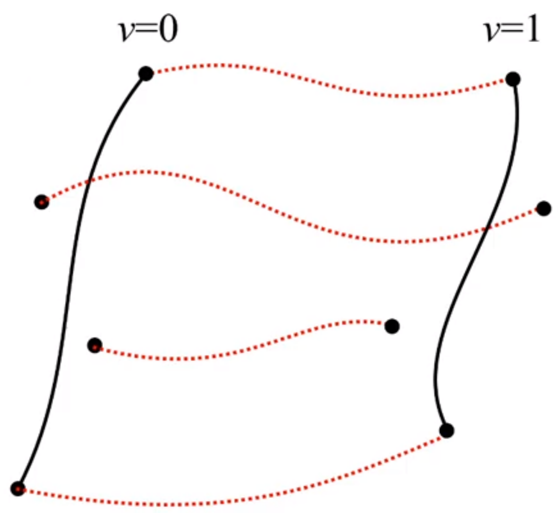
Converting between bezier and b-spline, given G = geometry, B_0 = current basis matrix, $T(t)$ = monomial basis, and B_1 = the basis matrix to convert to:

$$\begin{aligned}\vec{\gamma}(t) &= G \cdot B_0 \cdot T(t) \\ &= G \cdot B_0 \cdot B_1^{-1} \cdot B_1 \cdot T(t) \\ &= (G \cdot B_0 \cdot B_1^{-1}) \cdot B_1 \cdot T(t)\end{aligned}$$

The new geometry matrix is then represented as $G \cdot B \cdot B_1^{-1}$, which shows that to convert between bezier and b-spline, a different set of data is required for the same curve.

1.2 2D Surfaces

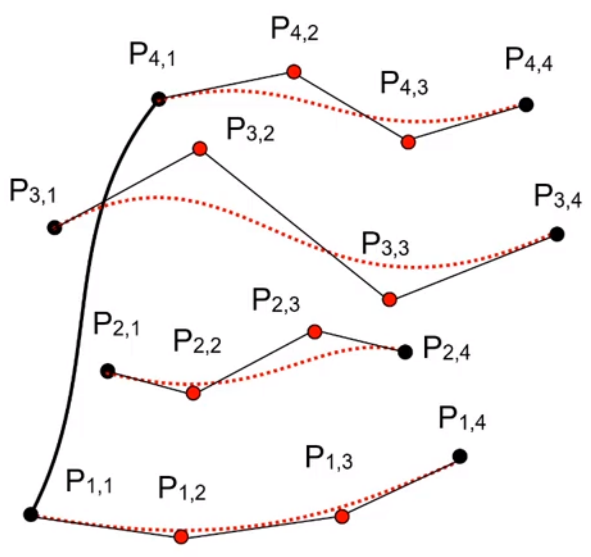
1.2.1 Bicubic Bezier surfaces



Given u changes vertically and v changes horizontally,

$$\begin{aligned}P(u,v) = (1-u)^3 P_1(v) \\ + 3u(1-u)^2 P_2(v) \\ + 3u^2(1-u) P_3(v) \\ + u^3 P_4(v)\end{aligned}$$

For any constant u for $0 \leq u \leq 1$ there is a bezier curve as a function of v for $0 \leq v \leq 1$ at that u, and vice versa.



$$P(u,v) = \sum_{i=1}^{4} B_i(u) P_i(v)$$

$$P_i(v) = \sum_{j=1}^{4} B_j(v) P_{i,j}$$

Which can be compactly expressed as:

$$P(u,v) = \sum_{i=1}^{4} B_i(u) \left[\sum_{j=1}^{4} P_{i,j} B_j(v) \right]$$
$$= \sum_{i=1}^{4} \sum_{j=1}^{4} P_{i,j} B_{i,j}(u,v)$$

where $B_{i,j}(u,v) = B_i(u)B_j(v)$.

1.2.2 Tangents and Normals for patches

The partial derivatives $\partial P/\partial u$ and $\partial P/\partial v$ are tangent vectors.
Normal vector is equal to the cross product of the tangent vectors:

$$\vec{N} = \frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v}$$

$\vec{N}$ is typically not unit, so it needs to be normalized

1.2.3 Matrix notation for patches

$$P(u,v) = \begin{pmatrix} B_1(u) & \cdots & B_4(u) \end{pmatrix} \begin{pmatrix} P_{1,1} & \cdots & P_{1,4} \\ \vdots & \ddots & \vdots \\ P_{4,1} & \cdots & P_{4,4} \end{pmatrix} \begin{pmatrix} B_1(v) \\ \vdots \\ B_4(v) \end{pmatrix}$$

1.2.4 Implicit surfaces

$f(x,y,z) = 0$ is on the surface
$f(x,y,z) < 0$ is inside the surface
$f(x,y,z) > 0$ is outside the surface

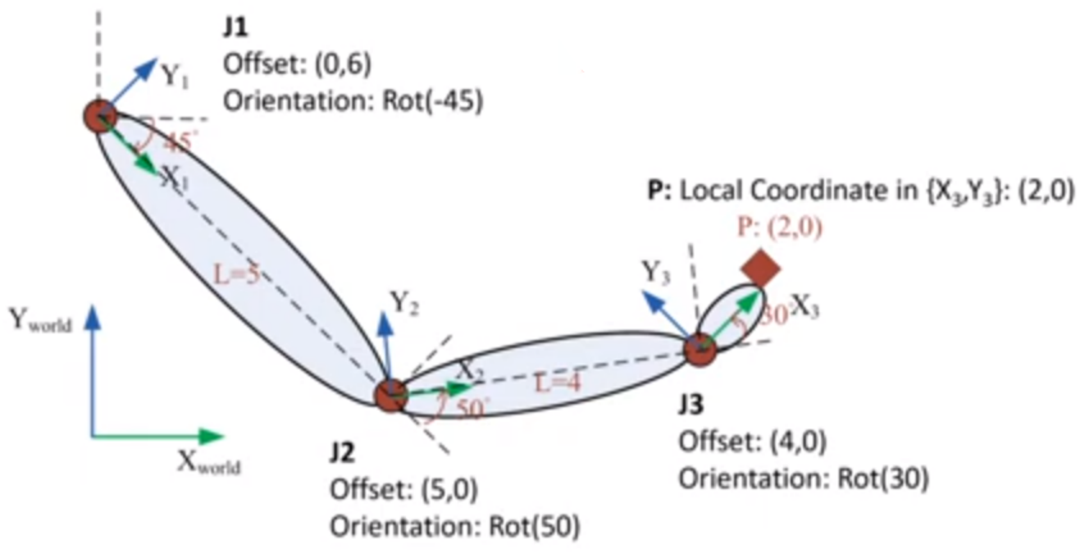
1.3 Hierarchical modeling

1.3.1 Forward kinematics

Specifies base position and joint angles. To compute the final position and orientation of a joint in world coordinates, the motion of all its ancestor joints must be computed as well.

Given values for joint dof (degree of freedom) $\vec{\theta} = [\theta_1, \theta_2, \cdots, \theta_n]^T$, compute the end effector in world space $\vec{\boldsymbol{e}} = [\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m]^T$.

$\vec{\boldsymbol{e}} = F(\boldsymbol{\theta})$



In the diagram above, P in world coordinates can be calculated with $P_{world} = M_{01}M_{12}M_{23}P_{local}$, or $T(0,6)R(-45)T(5,0)R(50)T(4,0)R(30)P_{local}P_{local}$, T being a transformation matrix and R being a rotation matrix.

1.3.2 Inverse kinematics

Forward kinematics has some issues such as determining what orientation the arm should be in to interact with other objects, which is what inverse kinematics aims to solve.

In forward kinematics, $(\boldsymbol{e}_1, \boldsymbol{e}_2) = F(\theta_1, \theta_2, \theta_3)$; however, in inverse kinematics, $(\theta_1, \theta_2, \theta_3) = G(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

Inverse kinematics is difficult because sometimes there is no solution / multiple solutions to a problem. Typically it can't be analytically solved and requires numerical methods.

- Jacobian iterative method
- Optimization based methods

Given position of a point in local coordinates $\boldsymbol{v}_s$ and desired position in world coordinates $\boldsymbol{v}_w$, find skeleton parameters $\boldsymbol{p}$.

$$\boldsymbol{v}_w = S(\boldsymbol{p})\boldsymbol{v}_s = S(x_h, y_h, z_h, \theta_h, \phi_h, \sigma_h, \theta_t, \phi_t, \sigma_t, \theta_c, \theta_f, \phi_f)\boldsymbol{v}_s$$

1.3.3 Jacobian

$\boldsymbol{e} = F(\boldsymbol{\theta})$ where $\boldsymbol{e}$ is known and $\boldsymbol{\theta}$ is to be solved for.

The Jacobian is the matrix of partial derivatives of $\boldsymbol{F}$: $J = \mathrm{d}F/\,\mathrm{d}\boldsymbol{\theta}$

For $\boldsymbol{F} = [\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_m]^T$ and $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_n]^T$:

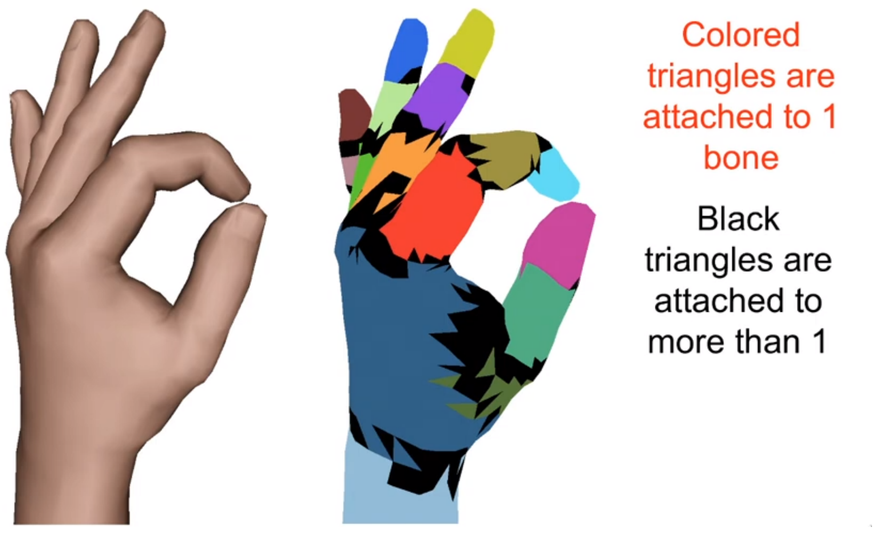
$$J = \begin{bmatrix} \frac{\partial \boldsymbol{F}_1}{\partial \theta_1} & \cdots & \frac{\partial F_1}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial \theta_1} & \cdots & \frac{\partial F_m}{\partial \theta_n} \end{bmatrix}$$

1.4 Skinning

1.4.1 SSD

Skin is made up of vertices, some of which are attached to a single bone, while others are attached to multiple.

Example


Colored triangles are attached to 1 bone
Black triangles are attached to more than 1

1.4.2 Vertex weights

Weight w_{ij} is assigned for each vertex $\boldsymbol{p}_i$ for each bone $\boldsymbol{b}_j$. ("How much will vertex i move with bone j?")

$w_{ij} = 1$ means $\boldsymbol{p}_i$ is rigidly attached to bone $\boldsymbol{b}_j$. Weights should not be negative, and the sum of weights across all bones for each vertex should equal 1.

The number of bones that can influence a single vertex is typically limited to $N = 4$ bones/vertex.

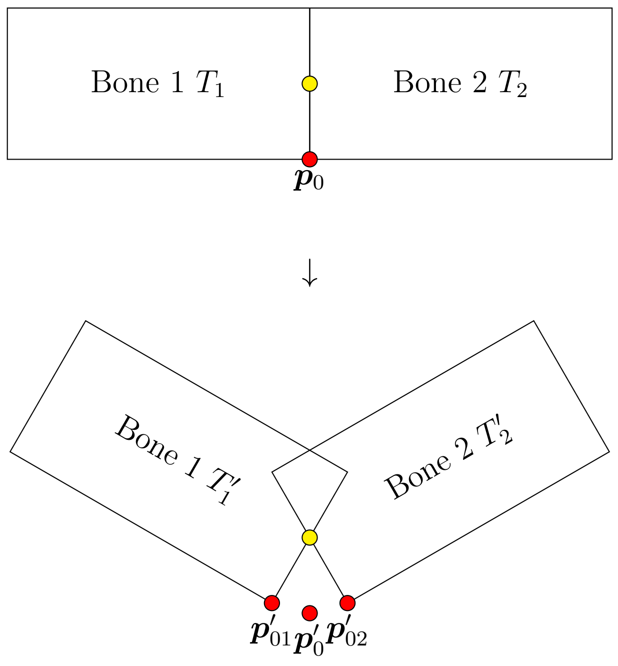
1.4.3 Linear blend skinning

First transform each vertex $\boldsymbol{p}_i$ with each bone as if it were tied to it rigidly, and then blend the results using weights.

Given $\boldsymbol{p}'_{ij}$ is vertex i transformed by bone j, T_j is the current transformation of bone j, and $\boldsymbol{p}'_i$ is the new position of vertex i:

$$\boldsymbol{p}'_{ij} = T_j \boldsymbol{p}_i$$
$$\boldsymbol{p}'_i = \sum_j w_{ij} \boldsymbol{p}'_{ij}$$

Example



Vertex $\boldsymbol{p}_0$ has weights $w_{01} = 0.5$ and $w_{02} = 0.5$. Points $\boldsymbol{p}_{01}'$ and $\boldsymbol{p}_{02}'$ are generated by transforming point $\boldsymbol{p}_0$ using T_1' and T_2'. $\boldsymbol{p}_0'$ can be determined using $\boldsymbol{p}_0' = w_{01}\boldsymbol{p}_{01}' + w_{02}\boldsymbol{p}_{02}' = 0.5\boldsymbol{p}_{01}' + 0.5\boldsymbol{p}_{02}'$.

2 Raytracing

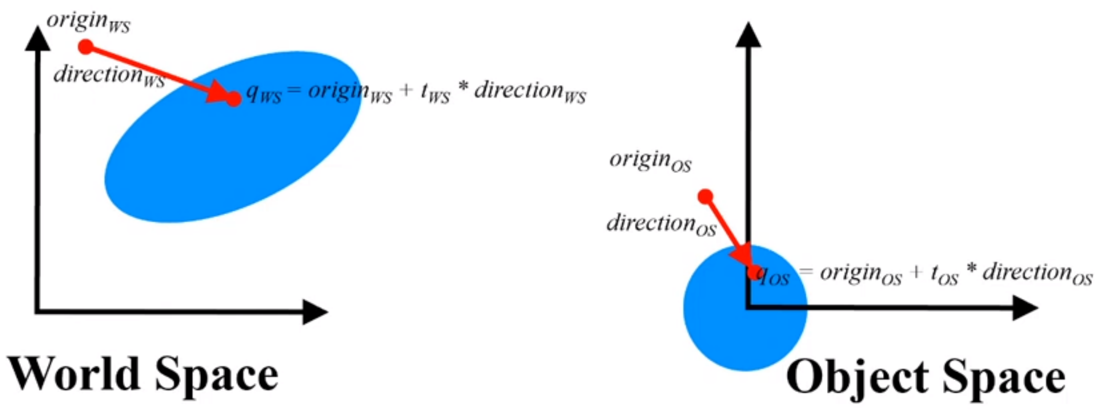
2.1 Geometry representation

A ray is represented as $\boldsymbol{p}(t) = \boldsymbol{R}_o + \boldsymbol{R}_d t$, $\boldsymbol{R}_o$ being the ray origin and $\boldsymbol{R}_d$ being the ray direction. All solutions for t under section 2.1 refer to the t parameter for $\boldsymbol{p}(t)$.

Implicit equations $H(\boldsymbol{P})$ are boolean functions. For example, given the implicit equation $H(\boldsymbol{P}) = \boldsymbol{a} \cdot \boldsymbol{P} = 0$, $H(\boldsymbol{P})$ returns $\boldsymbol{a} \cdot \boldsymbol{P} \leq 0$.

For the computation of intersections with any object, the object should be assumed to be at the origin, with the ray being transformed by the inverse of the object transformation matrix, as well as the ray direction. For point transformation, use $w = 1$, and direction transformation $w = 0$ because direction transformation doesn't include translation.

If ray direction is normalized, $t_{ws} \neq t_{os}$. and must be rescaled after the intersection. If ray direction is not normalized, $t_{ws} = t_{os}$, which is why it's preferred to not normalize transformed ray direction.


$origin_{WS}$
$direction_{WS}$
$q_{WS} = origin_{WS} + t_{WS} * direction_{WS}$
World Space
$origin_{OS}$
$direction_{OS}$
$q_{OS} = origin_{OS} + t_{OS} * direction_{OS}$
Object Space

To transform object normals properly, use $\boldsymbol{n}_{ws} = (M^{-1})^T \boldsymbol{n}_{os}$.

2.1.1 Sphere

Implicit equation: $H(\boldsymbol{P}) = \boldsymbol{P} \cdot \boldsymbol{P} - r^2 = 0$

Intersection detection (r is sphere radius)

$$\begin{aligned} a &= \boldsymbol{R}_d \cdot \boldsymbol{R}_d \\ b &= 2(-\boldsymbol{R}_o \cdot \boldsymbol{R}_d) \\ c &= \boldsymbol{R}_o \cdot \boldsymbol{R}_o - r^2 \\ d &= b^2 - 4ac \\ t &= \frac{b \pm \sqrt{d}}{2a} \end{aligned}$$

If $d < 0$, there is no intersection. If both solutions for t are negative, there is no intersection. If only one solution for t is positive, that is where the intersection occurs. If both solutions for t are positive, the intersection occurs at the closest t.

2.1.2 Infinite plane

Implicit equation: $H(\boldsymbol{P}) = \boldsymbol{n} \cdot \boldsymbol{P} + D = 0$, with $\boldsymbol{n}$ = plane normal, $D = -(\boldsymbol{n} \cdot \boldsymbol{P}_0)$, $\boldsymbol{P}_0$ = some point on the plane.

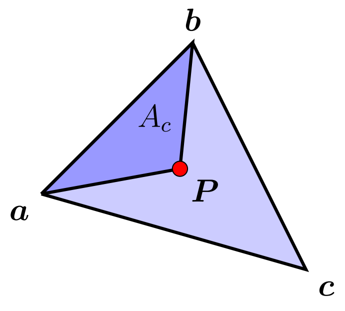
If there is an intersection, $\boldsymbol{n} \cdot \boldsymbol{P} + D$ gives the distance from $\boldsymbol{P}$ to $\boldsymbol{P}_0$.

2.1.3 Triangle

Any point $\boldsymbol{P}$ on a triangle $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ can be represented as $\boldsymbol{P}(\alpha, \beta, \gamma) = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$, with $\alpha + \beta + \gamma = 1$.

If α, β, $\gamma \geq 0$ then the barycentric coordinates are inside the triangle.

Computation of α, β, γ - geometric approach (A = area of entire triangle, A_x = area of subsection of triangle opposite to triangle vertex $\boldsymbol{x}$, $\boldsymbol{P}$ = random point on triangle)



$$\alpha = \frac{A_a}{A}$$
$$\beta = \frac{A_b}{A}$$
$$\gamma = \frac{A_c}{A}$$

Computation of α, β, γ - algebraic approach

Since α, β, and γ sum to 1, $\alpha = 1 - \beta - \gamma$, allowing for $\boldsymbol{P}(\alpha, \beta, \gamma)$ to be written as $\boldsymbol{P}(\beta, \gamma) = (1 - \beta - \gamma)\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$, or $\boldsymbol{P}(\beta, \gamma) = \boldsymbol{a} + \beta(\boldsymbol{b} - \boldsymbol{a}) + \gamma(\boldsymbol{c} - \boldsymbol{a})$. To simplify, $\boldsymbol{P}(\beta, \gamma)$ can also be written as $\boldsymbol{a} + \beta\vec{e}_1 + \gamma\vec{e}_2$, where $\vec{e}$ is an edge vector.

$$\boldsymbol{P} = \boldsymbol{a} + \beta\vec{e}_1 + \gamma\vec{e}_2$$
$$\vec{e}_1 \cdot (\boldsymbol{P} - \boldsymbol{a}) = \vec{e}_1 \cdot (\beta\vec{e}_1 + \gamma\vec{e}_2)$$
$$\vec{e}_2 \cdot (\boldsymbol{P} - \boldsymbol{a}) = \vec{e}_2 \cdot (\beta\vec{e}_1 + \gamma\vec{e}_2)$$
$$\begin{pmatrix} \vec{e}_1 \cdot (\boldsymbol{P} - \boldsymbol{a}) \\ \vec{e}_2 \cdot (\boldsymbol{P} - \boldsymbol{a}) \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$$

Intersection with barycentric triangle - set ray equal to barycentric equation

$$\boldsymbol{P}(t) = \boldsymbol{P}(\beta, \gamma)$$
$$\boldsymbol{R}_0 + t\boldsymbol{R}_d = \boldsymbol{a} + \beta(\boldsymbol{b} - \boldsymbol{a}) + \gamma(\boldsymbol{c} - \boldsymbol{a})$$

Intersection if $\beta + \gamma \leq 1$, $\beta \geq 0$, $\gamma \geq 0$

Separate equation into x, y, z components and turn it into matrix form:

$$\begin{pmatrix} \boldsymbol{a}_x - \boldsymbol{R}_{ox} \\ \boldsymbol{a}_y - \boldsymbol{R}_{oy} \\ \boldsymbol{a}_z - \boldsymbol{R}_{oz} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_x - \boldsymbol{b}_x & \boldsymbol{a}_x - \boldsymbol{c}_x & \boldsymbol{R}_{dx} \\ \boldsymbol{a}_y - \boldsymbol{b}_y & \boldsymbol{a}_y - \boldsymbol{c}_y & \boldsymbol{R}_{dy} \\ \boldsymbol{a}_z - \boldsymbol{b}_z & \boldsymbol{a}_z - \boldsymbol{c}_z & \boldsymbol{R}_{dz} \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ t \end{pmatrix}$$

Cramer's rule can be used to solve for one variable at a time:

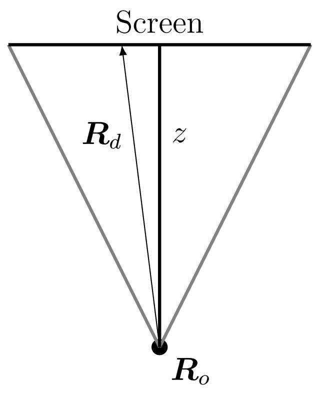
$$A = \begin{pmatrix} a_x - b_x & a_x - c_x & R_{dx} \\ a_y - b_y & a_y - c_y & R_{dy} \\ a_z - b_z & a_z - c_z & R_{dz} \end{pmatrix}$$
$$\beta = \frac{\begin{vmatrix} a_x - R_{ox} & a_x - c_x & R_{dx} \\ a_y - R_{oy} & a_y - c_y & R_{dy} \\ a_z - R_{oz} & a_z - c_z & R_{dz} \end{vmatrix}}{|A|}$$
$$\gamma = \frac{\begin{vmatrix} a_x - b_x & a_x - R_{ox} & R_{dx} \\ a_y - b_y & a_y - R_{oy} & R_{dy} \\ a_z - b_z & a_z - R_{oz} & R_{dz} \end{vmatrix}}{|A|}$$
$$t = \frac{\begin{vmatrix} a_x - b_x & a_x - c_x & a_x - R_{ox} \\ a_y - b_y & a_y - c_y & a_y - R_{oy} \\ a_z - b_z & a_z - c_z & a_z - R_{oz} \end{vmatrix}}{|A|}$$

2.2 Fundamentals

2.2.1 Raycasting

Raycasting is the process of casting a ray for every pixel on the screen, which means $\boldsymbol{R}_d$ needs to be determined for each pixel.

With z = distance from ray origin to screen, f = fov, p_x and p_y = pixel x and y, w and h = screen width and height:



With θ = x angle, ϕ = y angle:

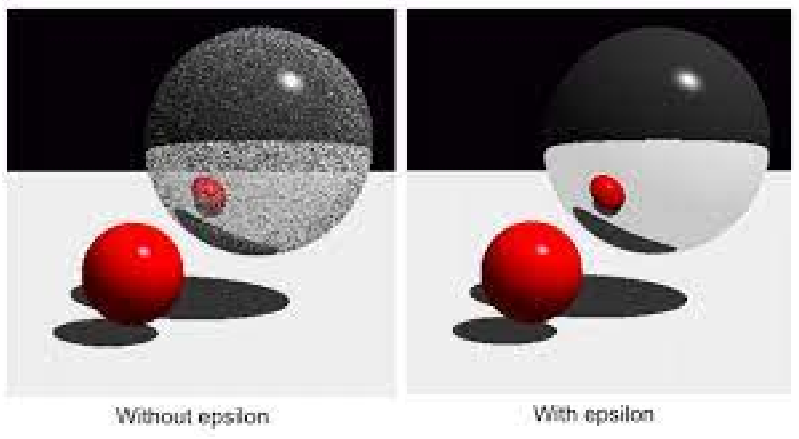
$$\begin{aligned}
\theta &= \frac{x}{w}f - \frac{f}{2} \\
\phi &= \frac{y}{h}f - \frac{f}{2} \\
p_x &= \sin(\theta) \\
p_y &= \sin(\phi) \\
\boldsymbol{R}_d &= [p_x, p_y, 1, 0]^T
\end{aligned}$$

2.2.2 Shadows

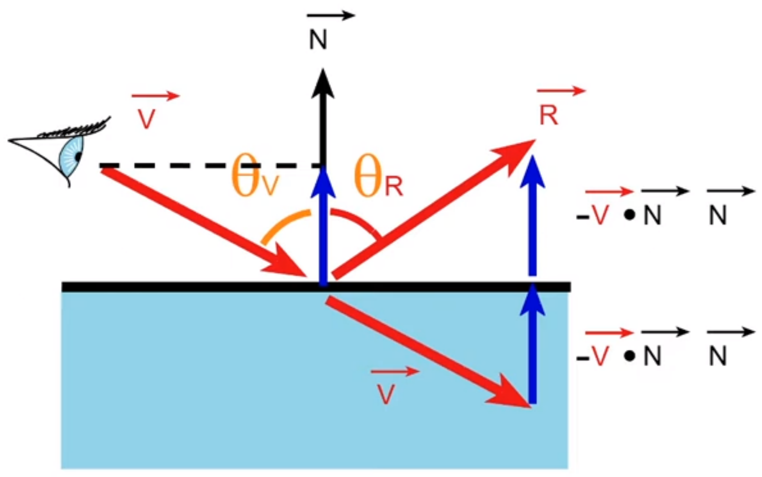
Shadows can be implemented by casting a ray from the original ray intersection point towards each light, and determining if they contribute to the light at that point.

Rays will often self-intersect when checking for shadows on the surface they start off of, which is why the new $\boldsymbol{R}_o$ should be equal to $\boldsymbol{R}_{hit} + \epsilon\boldsymbol{n}$, with $\boldsymbol{R}_{hit}$ = ray object intersection point, $\boldsymbol{n}$ = object surface normal, ϵ = some small value (ex. 0.001), depending on how large objects typically are.

For light sources that have area, cast multiple rays to different spots on the light source and take the fraction of the rays that hit the light.



2.2.3 Reflection



$\vec{N}$
$\vec{V}$
$\vec{R}$
θ_V
θ_R
$-\vec{V} \bullet \vec{N} \quad \vec{N}$
$-\vec{V} \bullet \vec{N} \quad \vec{N}$
$\vec{V}$

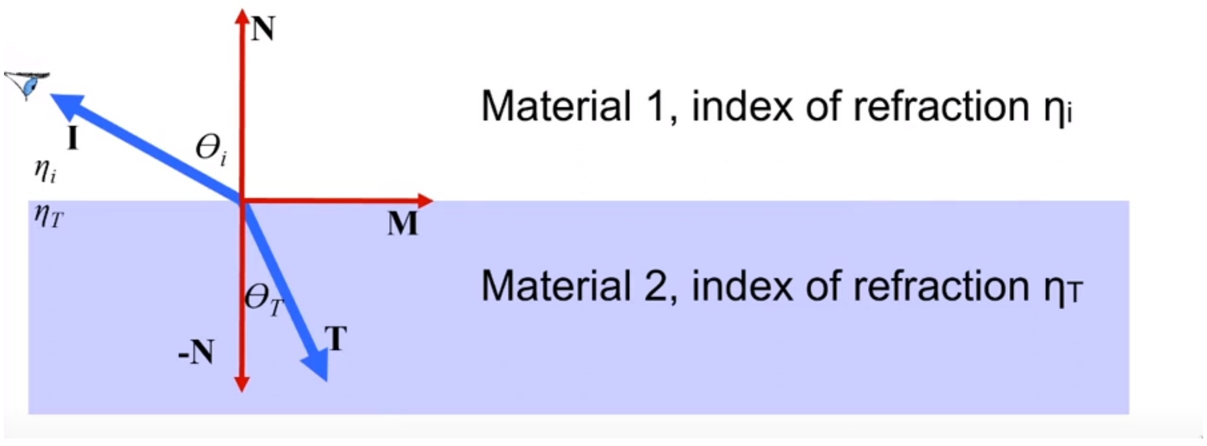
For reflection, the result ray $\boldsymbol{R}$ is expressed as $\boldsymbol{R} = \boldsymbol{V} - 2(\boldsymbol{V} \cdot \boldsymbol{N})\boldsymbol{N}$.

Amount of reflection in traditional raytracing is determined by some constant k_s which gave the final pixel color as $c_p = k_s c_m + (1 - k_s) c_r$, with c_p = final pixel color, c_m = material color, c_r = color the reflected ray returns.

Fresnel discovered that reflections at a grazing angle reflected more, and Schlick approximated $R(\theta) = R_0 + (1 - R_0)(1 - \cos(\theta))^5$, with $R(\theta)$ = reflectiveness, R_0 = base reflectiveness.

Ideally a raytracer should cast multiple rays with slight variation in the same general direction and take the average of their resulting colors when reflecting to render a more realistic surface.

2.2.4 Refraction


N
Material 1, index of refraction ηi
I
θ_i
η_i
η_T
M
Material 2, index of refraction ηT
θ_T
-N
T

Snell-Descartes law states $n_i \sin(\theta_i) = n_T \sin(\theta_T)$

$$\frac{\sin(\theta_T)}{\sin(\theta_i)} = \frac{n_i}{n_T} = n_r$$

n_r is called the relative index of refraction

The vector $\boldsymbol{T}$ can be calculated using $\boldsymbol{T} = \left(n_r(\boldsymbol{N} \cdot \boldsymbol{I}) - \sqrt{1 - n_r^2(1 - (\boldsymbol{N} \cdot \boldsymbol{I})^2)}\right) \boldsymbol{N} - n_r \boldsymbol{I}$

Similar to reflection, refraction should use the same technique of average of multiple rays to render more realistic surfaces.

2.3 Optimization

2.3.1 Conservative bounding volume