Contents

1 Modeling

1.1 Splines

1.1.1 Cubic Hermite Interpolation

Each point is defined by its position h_n and slope h_{m+n}, m being the number of control points. To simplify calculations, it is assumed that $t_0 = 0$ and $t_1 = 1$.

The goal is to convert from a monomial basis

$$\begin{aligned}
\phi_0(t) &= 1 \\
\phi_1(t) &= t \\
\phi_2(t) &= t^2 \\
\phi_3(t) &= t^3
\end{aligned}$$

to a hermite basis

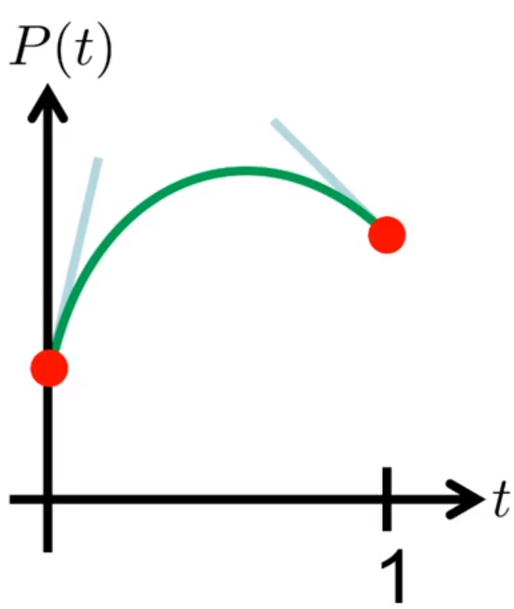
$$\begin{aligned}
H_0(t) &= 2t^3 - 3t^2 + 1 \\
H_1(t) &= -2t^3 + 3t^2 \\
H_2(t) &= t^3 - 2t^2 + t \\
H_3(t) &= t^3 - t^2
\end{aligned}$$

so that instead of having to manipulate polynomial coefficients

$$f(t) = a\phi_3(t) + b\phi_2(t) + c\phi_1(t) + d\phi_0(t)$$

an easier point slope method can be used:

$$f(t) = h_0H_0(t) + h_1H_1(t) + h_2H_2(t) + h_3H_3(t)$$



$$P(t) = at^3 + bt^2 + ct + d$$
$$P'(t) = 3at^2 + 2bt + c$$

$$\begin{aligned}
h_0 &= P(0) = d \\
h_1 &= P(1) = a + b + c + d \\
h_2 &= P'(0) = c \\
h_3 &= P'(1) = 3a + 2b + c
\end{aligned}$$

Unknowns in this equation are a, b, c, and d, so a matrix can be used to solve the systems of equations:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

a, b, c, and d can be obtained from h values by inverting the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

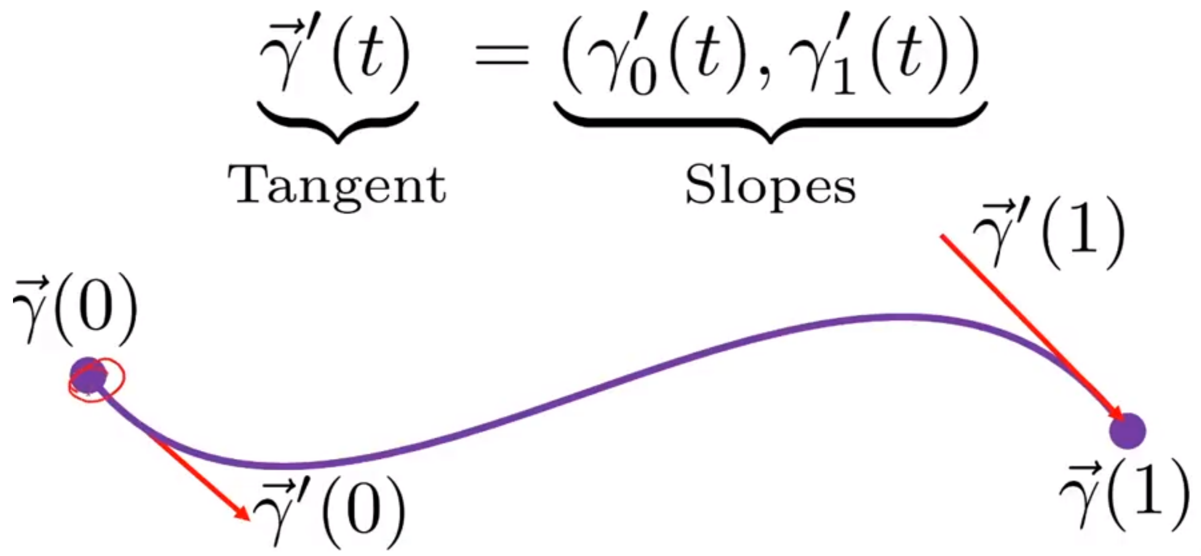
From these solved h values, $P(t)$ can now be converted to a form that is easier for a user to manipulate, in terms of h values:

$$\begin{aligned}
P(t) &= at^3 + bt^2 + ct + d \\
&= (2h_0 - 2h_1 + h_2 + h_3)t^3 \\
&\quad + (-3h_0 + 3h_1 - 2h_2 - h_3)t^2 \\
&\quad + h_2 t + h_0 \\
&= h_0(2t^3 - 3t^2 + 1) + h_1(-2t^3 + 3t^2) + \\
&\quad h_2(t^3 - 2t^2 + t) + h_3(t^3 - t^2)
\end{aligned}$$

Each equation in $P(t) = h_0(2t^3 - 3t^2 + 1) + h_1(-2t^3 + 3t^2) + h_2(t^3 - 2t^2 + t) + h_3(t^3 - t^2)$ that is multiplied by an h value is called a cubic hermite.

1.1.2 More than 1D

A parametric curve described by $\vec{\gamma}(t) = (\gamma_0(t), \gamma_1(t))$ can be converted into hermite basis like this:



where cubic hermite interpolation can be done for both dimensions.

1.1.3 Cubic blossom

The cubic blossom of a function $f(t)$ is $F(t_1, t_2, t_3)$.

Cubic blossoms have three properties:

1. Symmetric
   - $F(t_1, t_2, t_3) = F(t_1, t_3, t_2) = F(t_3, t_1, t_2) \cdots$
2. Affine
   - $F(\alpha u + (1-\alpha)v, t_2, t_3) = \alpha F(u, t_2, t_3) + (1-\alpha)F(v, t_2, t_3)$
3. Diagonal
   - $f(t) = F(t, t, t)$

Blossoming examples

- $f(t) = t^3 \mapsto F(t_1, t_2, t_3) = t_1 t_2 t_3$
- $f(t) = t^2 \mapsto F(t_1, t_2, t_3) = (t_1 t_2 + t_1 t_3 + t_2 t_3)/3$
- $f(t) = t \mapsto F(t_1, t_2, t_3) = (t_1 + t_2 + t_3)/3$
- $f(t) = 1 \mapsto F(t_1, t_2, t_3) = 1$
- $f(t) = 3t^3 - t + 1 = 3(t_1 t_2 t_3) - (t_1 + t_2 + t_3)/3 + 1$

Cubic curves can be blossomed by blossoming each coordinate function separately, which will give a function that maps 3 t variables to two dimensions x and y: $\vec{F}(t_1, t_2, t_3) : \mathbb{R}^3 \mapsto \mathbb{R}^2$. Some ways to obtain a blossom of a cubic curve: