有效数字的计算

$$\varepsilon(x_1^* \pm x_2^*) = \varepsilon(x_1^*) + \varepsilon(x_2^*)$$

$$f(x^*) - f(x) \approx f'(x^*)(x^* - x) \approx f'(x)(x^* - x)$$

相对误差：$e_r^* = \frac{x^* - x}{x} \approx \frac{x^* - x}{x^*}$

x^* 为 x 的近似值

误差的关系

x 的相对误差为 δ
求 $\ln x$ 的误差

$$\ln x - \ln x^* = \ln \frac{x}{x^*} = \ln \frac{x - x^* + x^*}{x^*}$$

$$= \ln\left(\frac{x - x^*}{x^*} + 1\right) = \ln(\delta + 1)$$

x 的相对误差为 2%
x^n 的相对误差

设 $f(x) = x^n$

则 $f(x^*) - f(x) \approx f'(x)(x^* - x)$

$$= n \cdot x^{n-1}(x^* - x)$$

$$= n \cdot x^n \frac{x^* - x}{x} = 2\% \cdot n \cdot x^n$$

则 x^n 的相对误差 $= \frac{f(x^*) - f(x)}{f(x)}$

$$= \frac{2\% \, n \cdot x^n}{x^n} = 0.02n$$

拉格朗日插值. 共有 $n+1$ 个点 $(x_0, x_1, \cdots x_n)$

$$f(x) \approx L_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

其中 $$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

x 次数不超过 n 次的多项式.

$$w_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

其中.

$$w_{n+1}'(x_i) = (x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n).$$

误差: $R_n(x) = f(x) - L_n(x)$.

$$|R_n(x)| \le \frac{M_{n+1}}{(n+1)!}|w_{n+1}(x)|$$

某一点 a 的误差.

$$|R_n(a)| \le \frac{M_{n+1}}{(n+1)!}|w_{n+1}(a)|$$

曲线拟合: 最小二乘法:

有点 $x_0, x_1, \cdots x_n$. 要求拟合曲线为:

$y_0, y_1, \cdots y_n$

$$f(x) = (a_0\ a_1\ \cdots\ a_n)\begin{pmatrix} \varphi_0(x) = x^0 \\ \varphi_1(x) = x^1 \\ \vdots \\ \varphi_n(x) = x^n \end{pmatrix}$$

则:
$$\begin{bmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) & \cdots & (\varphi_0, \varphi_n) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) & \cdots & (\varphi_1, \varphi_n) \\ \vdots & \vdots & & \vdots \\ (\varphi_n, \varphi_0) & (\varphi_n, \varphi_1) & \cdots & (\varphi_n, \varphi_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (\varphi_0, f) \\ (\varphi_1, f) \\ \vdots \\ (\varphi_n, f) \end{bmatrix}$$

其中 $(\varphi_k, \varphi_j) = \sum_{i=0}^{n} \omega(x_i)\varphi_k(x_i)\varphi_j(x_i)$

$$(\varphi_k, y) = \sum_{i=0}^{n} \omega(x_i) y_i \varphi_k(x_i)$$

代数精度计算:

LU分解方程组.

解方程组 $Ax=b$ 化为解两个方程组.

$$\begin{cases} Ly = b \\ Ux = y \end{cases}$$

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix}$$

方程迭代法收敛性.