Dynamic Programming

Chapter 10: Continuous Time

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Background

Earlier chapters treated dynamics in discrete time

Now we switch to continuous time

We restrict ourselves to finite state spaces (pure jump processes)

- permits a rigorous and self-contained treatment
- covers useful models
- lays foundations for a treatment of general state problems

Our first step is to review continuous time Markov chains

Recall: if $(X_t) = (X_0, X_1, \ldots)$ is P-Markov and $\psi_t \stackrel{d}{=} X_t$, then

$$\psi_{t+1} = \psi_t P \quad \text{for all } t$$

This rule is a linear difference equation in distribution space

How to shift to continuous time?

Answer: distributions follow a linear **differential equation** in distribution space

Hence we recall some facts about linear differential equations

- start with scalar case
- then shift to vector valued linear ODEs

The exponential function

The real-valued **exponential function** can be defined by

$$\mathrm{e}^x =: \exp(x) := \sum_{k \geqslant 0} \frac{x^k}{k!} \qquad (x \in \mathbb{R})$$

Properties: For $a, b \in \mathbb{R}$,

- $\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$
- $t \mapsto \mathrm{e}^{ta}$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{ta} = a\mathrm{e}^{ta}$$

Example. Let $u_t =$ balance of a savings account paying continuously compounded interest rate r

Then

$$u'_t := \frac{\mathrm{d}}{\mathrm{d}t} u_t = r u_t \quad \text{for all } \; t \geqslant 0, \quad u_0 \text{ given} \tag{1}$$

Ex. Show that $u_t := \mathrm{e}^{rt} u_0$ is the only solution to $u'_t = r u_t$

Proof: This function is a solution because

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{rt} u_0 = r\mathrm{e}^{rt} u_0 = r u_t$$

Why is it the only solution?

Suppose $t \mapsto y_t$ also satisfies $y_t' = r y_t$ and $y_0 = u_0$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(y_t \, \mathrm{e}^{-rt} \right) = y_t' \, \mathrm{e}^{-rt} - r y_t \, \mathrm{e}^{-rt} = r y_t \, \mathrm{e}^{-rt} - r y_t \, \mathrm{e}^{-rt} = 0$$

Hence $y_t \, \mathrm{e}^{-rt}$ is constant in t on $\mathbb{R}_+$

In other words, $y_t = c \, \mathrm{e}^{rt}$ for some c

Setting $t = 0$ and using the initial condition gives $c = u_0$

$$\therefore \quad y_t = \mathrm{e}^{rt} u_0 = u_t$$

Complex exponentials

The exponential e^{λ} of $\lambda \in \mathbb{C}$ is defined analogously:

$$\mathrm{e}^{\lambda} =: \exp(\lambda) := \sum_{k \geqslant 0} \frac{\lambda^k}{k!}$$

From the identity $\mathrm{e}^{ib} = \cos(b) + i\sin(b)$

- i is the imaginary unit

Using this identity and $\lambda = a + ib$ gives

$$\mathrm{e}^{\lambda} = \mathrm{e}^{a+ib} = \mathrm{e}^{a}(\cos(b) + i\sin(b))$$

This equation will soon prove useful

Extension to matrices

The real exponential formula extends to the **matrix exponential** via

$$\mathrm{e}^A := I + A + \frac{A^2}{2!} + \cdots = \sum_{k \geqslant 0} \frac{A^k}{k!}$$

- A is any square matrix (or linear operator)
- the series always converges in norm

In the next slide, $\sigma(A) :=$ all eigenvalues (**spectrum**) of A

Lemma. Let A and B be square matrices

1. If A is diagonalizable with $A = PDP^{-1}$, then $\mathrm{e}^{A} = P\mathrm{e}^{D}P^{-1}$
2. If $AB = BA$, then $\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}$
3. $\mathrm{e}^{A^\top} = (\mathrm{e}^{A})^\top$ and $\mathrm{e}^{mA} = (\mathrm{e}^{A})^m$ for all $m \in \mathbb{N}$
4. $\lambda \in \sigma(A)$ iff $\mathrm{e}^{\lambda} \in \sigma(\mathrm{e}^{A})$
5. $t \mapsto \mathrm{e}^{tA}$ is differentiable and
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA} = \mathrm{e}^{tA}A$$
6. The fundamental theorem of calculus holds:
$$\mathrm{e}^{tA} = \mathrm{e}^{sA} + \int_s^t \mathrm{e}^{\tau A}A\,\mathrm{d}\tau \quad \text{for all } s \leqslant t$$

In the last slide, differentiation and integration are element-by-element

Example.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} t^2 \\ \ln t \end{pmatrix} = \begin{pmatrix} (1/2)t \\ (1/t) \end{pmatrix}$$

and

$$\int \begin{pmatrix} f(t) & g(t) \\ u(t) & v(t) \end{pmatrix} \mathrm{d}t = \begin{pmatrix} \int f(t)\,\mathrm{d}t & \int g(t)\,\mathrm{d}t \\ \int u(t)\,\mathrm{d}t & \int v(t)\,\mathrm{d}t \end{pmatrix}$$

Ex. Confirm that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$

Proof: Observe that, for any $t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = \lim_{h\to 0} \frac{\mathrm{e}^{tA+hA} - \mathrm{e}^{tA}}{h} = \mathrm{e}^{tA} \lim_{h\to 0} \frac{\mathrm{e}^{hA} - I}{h}$$

By definition,

$$\frac{\mathrm{e}^{hA} - I}{h} = A + \frac{1}{2!}hA^2 + \frac{1}{3!}h^2A^3 + \cdots$$

This converges to A as $h \to 0$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = \mathrm{e}^{tA}A$$

Ex. Using the lemma, show that e^{A} is invertible with inverse e^{-A}

Fix $n \times n$ matrix A and let $B = -A$

Evidently A and B commute (i.e., $AB = BA$), so

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{A+B} = \mathrm{e}^{A-A} = \mathrm{e}^{0}$$

Moreover,

$$\mathrm{e}^{0} = I + \sum_{k \geqslant 0} \frac{0^k}{k!} = I$$

Hence $\mathrm{e}^{A}\mathrm{e}^{-A} = I$, which proves the claim

Continuous time dynamical systems

Recall:

- a discrete dynamical system is a pair (U, S), where U is a set and S is a self-map on U
- trajectories are sequences $(S^t u)_{t \geqslant 0} = (u, Su, S^2 u, \ldots)$, where $u \in U$ is the initial condition

What is the continuous time equivalent?

We consider a pair $(U, (S_t)_{t \geqslant 0})$ where U is any set and S_t is a self-map on U for each $t \in \mathbb{R}_+$

The interpretation is that if $u \in U$ is the current state of the system, then $S_t u$ will be the state after t units of time

The map $t \mapsto S_t u$ is the trajectory from u

Example. For the savings balance $u_t = \mathrm{e}^{rt} u_0$, we take $U = \mathbb{R}$ and $S_t u = \mathrm{e}^{rt} u$

Then $S_t u$ is the state at time t given initial deposit u

To understand the pair $(U, (S_t)_{t \geqslant 0})$ as a continuous time dynamical system, we require

1. that S_0 is the identity map and
2. the **semigroup property**: for all $t, t' \geqslant 0$,

$$S_{t+t'} = S_{t'} \circ S_t$$

Meaning: if we

- start at u
- move forward to $u_t := S_t u$ and
- move again to $S_{t'} u_t$ after another t' units of time

the outcome is the same as moving from u to $S_{t+t'}\, u$ in one step

Linear initial value problems

Let A be $n \times n$ and u_t', u_t be column vectors in $\mathbb{R}^n$

Proposition. The unique solution of the n-dimensional IVP

$$u_t' = Au_t, \qquad u_0 \in \mathbb{R}^n \text{ given} \tag{2}$$

in the set of continuous functions $t \mapsto u_t$ mapping $\mathbb{R}_+$ to $\mathbb{R}^n$ is

$$u_t = \mathrm{e}^{tA} u_0 \qquad (t \geqslant 0). \tag{3}$$

Proof: That $u_t := \mathrm{e}^{tA} u_0$ solves (2) follows from slide 9

Uniqueness can be proved using an argument similar to that used to solve the exercise on slide 4

The last proposition motivates us to study flows of the form

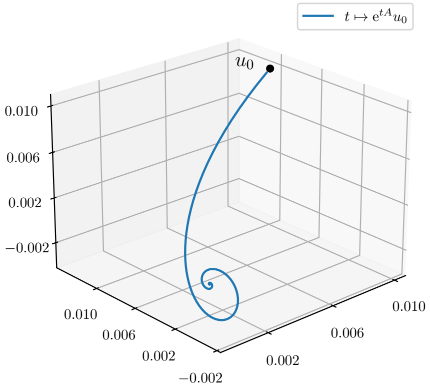
$$t \mapsto u_t, \quad u_t = \mathrm{e}^{tA} u_0 \qquad (t \geqslant 0) \tag{4}$$

where

- A is $n \times n$
- u_0 is a vector in $\mathbb{R}^n$ (initial condition)
- u_t is the "state" of the system at time t

The next slide illustrates for

$$A := \begin{pmatrix} -2.0 & -0.4 & 0 \\ -1.4 & -1.0 & 2.2 \\ 0.0 & -2.0 & -0.6 \end{pmatrix} \tag{5}$$



$t \mapsto e^{tA} u_0$
u_0
0.010
0.006
0.002
−0.002
0.010
0.006
0.002
−0.002
0.010
0.006
0.002

Stability

How do these exponential flows depend on A?

For example, when do we have

$$u_t := e^{tA} u_0 \to 0 \quad \text{as } t \to \infty$$

Two options

1. analyze this flow at every u_0
2. directly consider the matrix-valued flow $t \mapsto e^{tA}$

Below we take the second option, ask when $e^{tA} \to 0$

Suppose first that A is diagonalizable with $A = P^{-1}DP$

- $D = \mathrm{diag}_j(\lambda_j)$ contains the eigenvalues of A

Recall from slide 9 that for any $t \geqslant 0$,

$$\mathrm{e}^{tA} = \mathrm{e}^{tP^{-1}DP} = P^{-1}\mathrm{e}^{tD}P \tag{6}$$

and, moreover,

$$\mathrm{e}^{tD} = \mathrm{diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})$$

Hence long run dynamics of e^{tA} fully determined by

$$t \mapsto \mathrm{e}^{t\lambda_j} \text{ for } j = 1, \ldots, n$$

So how does $\mathrm{e}^{t\lambda}$ evolve over time when $\lambda \in \mathbb{C}$?

To answer this question we write $\lambda = a + ib$ to obtain

$$\mathrm{e}^{t\lambda} = \mathrm{e}^{ta}(\cos(tb) + i\sin(tb)).$$

Hence

$$\mathrm{e}^{t\lambda} \to 0 \text{ as } t \to \infty \quad \iff \quad \operatorname{Re}\lambda < 0$$

$$\therefore \quad \mathrm{e}^{tA} \to 0 \text{ as } t \to \infty \quad \iff \quad \operatorname{Re}\lambda_j < 0 \text{ for all } \lambda_j \in \sigma(A)$$

Equivalently, $\mathrm{e}^{tA} \to 0$ if and only if $s(A) < 0$, where

$$s(A) := \max_{\lambda \in \sigma(A)} \operatorname{Re}\lambda$$

is called the **spectral bound** of A

The last result illustrated the importance of the spectral bound

Letting $\|\cdot\|$ be the matrix norm, we have

Lemma. For each $n \times n$ matrix A and $\tau > 0$ we have

$$\tau s(A) = s(\tau A)$$

Moreover,

$$\mathrm{e}^{s(A)} = \rho(\mathrm{e}^{A}) \quad \text{and} \quad s(A) = \lim_{t \to \infty} \frac{1}{t} \ln \|\mathrm{e}^{tA}\|$$

Ex. Confirm that $\rho(\mathrm{e}^A) = \mathrm{e}^{s(A)}$

<u>Proof</u>: Recall from slide 9 that

$$\lambda \in \sigma(A) \text{ if and only if } \mathrm{e}^\lambda \in \sigma(\mathrm{e}^A)$$

From the definition $s(A) := \max_{\lambda \in \sigma(A)} \operatorname{Re} \lambda$, we have

$$\rho(\mathrm{e}^A) = \max_{\lambda \in \sigma(\mathrm{e}^A)} |\lambda| = \max_{\lambda \in \sigma(A)} |\mathrm{e}^\lambda| = \max_{\lambda \in \sigma(A)} \mathrm{e}^{\operatorname{Re} \lambda} = \mathrm{e}^{s(A)}$$

The next theorem extends our stability result for the diagonal case

Theorem. For any square matrix A, the following statements are equivalent:

1. $s(A) < 0$
2. $\|\mathrm{e}^{tA}\| \to 0$ as $t \to \infty$
3. $\exists\ M, \omega > 0$ such that $\|\mathrm{e}^{tA}\| \leqslant M\mathrm{e}^{-t\omega}$ for all $t \geqslant 0$
4. $\int_0^\infty \|\mathrm{e}^{tA}u_0\|^p \,\mathrm{d}t < \infty$ for all $p \geqslant 1$ and $u_0 \in \mathbb{R}^n$

Let's sketch the proof that $s(A) < 0$ implies $\mathrm{e}^{tA} \to 0$ as $t \to \infty$

Suppose $s(A) < 0$

Fix $\varepsilon > 0$ such that $s(A) + \varepsilon < 0$ and

Recall that

$$s(A) = \lim_{t \to \infty} \frac{1}{t} \ln \|\mathrm{e}^{tA}\|$$

Hence $\exists$ a $T < \infty$ such that

$$\frac{1}{t} \ln \|\mathrm{e}^{tA}\| \leqslant s(A) + \varepsilon \text{ for all } t \geqslant T$$

Equivalently, for t large, we have $\|\mathrm{e}^{tA}\| \leqslant \mathrm{e}^{t(s(A)+\varepsilon)}$

The claim follows

Semigroup terminology

Advanced treatments of continuous time systems often begin with operator semigroups

Let's briefly describe these and connect them to things we have studied earlier

Let X be a finite set and let $(S_t)_{t \geqslant 0}$ be a subset of $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$ indexed by $t \in \mathbb{R}_+$

The family $(S_t)_{t \geqslant 0}$ is called an **operator semigroup** on $\mathbb{R}^{\mathsf{X}}$ if

1. $S_0 = I$, where I is the identity,
2. $S_{t+t'} = S_t \circ S_{t'}$, and
3. $t \mapsto S_t$ is continuous as a map from $\mathbb{R}_+$ to $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$

Given an operator semigroup $(S_t)_{t \geqslant 0}$ on $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$, does there always exist a "vector field" type object that "generates" $(S_t)_{t \geqslant 0}$?

When X is finite, the answer is affirmative

This object is called the **infinitesimal generator** of the semigroup and is defined by

$$A = \lim_{t \downarrow 0} \frac{S_t - S_0}{t} = \lim_{t \downarrow 0} \frac{S_t - I}{t} \tag{7}$$

At $u \in U$, the vector Au indicates the instantaneous change in the state

Example. Fix A in $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$ and let $(S_t)_{t \geqslant 0}$ be defined by $S_t = \mathrm{e}^{tA}$

Then $(S_t)_{t \geqslant 0}$ is an operator semigroup on $\mathbb{R}^{\mathsf{X}}$

To verify this we take $X = \{x_1, \ldots, x_n\}$ and S_t and A as $n \times n$ matrices

The operator semigroup properties now follow directly the lemma on slide 9

For example, S_t is continuous in t because it is differentiable in t

The infinitesimal generator is

$$\lim_{t \downarrow 0} \frac{S_t - S_0}{t} = \lim_{t \downarrow 0} \frac{\mathrm{e}^{tA} - \mathrm{e}^0}{t} = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{tA} \Big|_{t=0} = A\mathrm{e}^{0A} = A$$

The next slide shows that this is the only example of an operator semigroup on $\mathbb{R}^{\mathsf{X}}$ when $|\mathsf{X}| < \infty$

Proposition. If $(S_t)_{t\geqslant 0}$ is an operator semigroup on $\mathbb{R}^{\mathsf{X}}$ and X is finite, then

1. there exists an $A \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ such that $S_t = \mathrm{e}^{tA}$ for all $t \geqslant 0$, and
2. A is the infinitesimal generator of $(S_t)_{t\geqslant 0}$.

Semigroups of this form are called **exponential semigroups**

Put differently: in finite dimensions, the only operator semigroups are exponential semigroups

Markov Semigroups

We are now ready to specialize to the Markov case, where dynamics evolve in distribution space

Let $|\mathsf{X}| = n$ and let $(X_t)_{t \geqslant 0}$ be P-Markov on X for some $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

The marginal distributions of $(X_t)_{t \geqslant 0}$ evolve according to the linear difference system $\psi_{t+1} = \psi_t P$

We now seek a continuous time analog in the form of linear differential equations that drive the evolution of distributions

To this end we define an $n \times n$ matrix Q to be an **intensity matrix** when

$$Q(x, x') \geqslant 0 \text{ whenever } x \neq x' \quad \text{and} \quad \sum_{x'} Q(x, x') = 0 \text{ for all } x \in \mathsf{X}$$

Example. The matrix

$$Q := \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

is an intensity matrix, since

- off-diagonal terms are nonnegative and
- rows sum to zero

We call $\mathcal{D}(\mathsf{X})$ **invariant** for

$$\psi_t' = \psi_t \, Q, \qquad \psi_0 \in \mathcal{D}(\mathsf{X}) \text{ given.} \tag{8}$$

if the solution $(\psi_t)_{t \geqslant 0}$ remains in $\mathcal{D}(\mathsf{X})$ for all $t \geqslant 0$

- ψ_t and ψ_t' are understood to be row vectors

By the result on slide 16, we can rephase by stating that $\mathcal{D}(\mathsf{X})$ is invariant for (8) whenever

$$\psi_0 \in \mathcal{D}(\mathsf{X}) \quad \Longrightarrow \quad \psi_0 \, \mathrm{e}^{tQ} \in \mathcal{D}(\mathsf{X}) \text{ for all } t \geqslant 0 \tag{9}$$

Proposition. Let Q be $n \times n$ and set $P_t := \mathrm{e}^{tQ}$ for each $t \geqslant 0$ The following statements are equivalent:

1. Q is an intensity matrix.
2. P_t is a stochastic matrix for all $t \geqslant 0$.
3. the set of distributions $\mathcal{D}(\mathsf{X})$ is invariant for the IVP (8).

Meaning: the set of $n \times n$ intensity matrices coincides with the set of continuous time Markov models on X

Any specification outside this class fails to generate flows in distribution space.

Proof: See the book

Markov Semigroups

The family $(P_t)_{t \geqslant 0} = (\mathrm{e}^{tQ})_{t \geqslant 0}$ that solves $\psi_t' = \psi_t Q$ is an exponential semigroup

When Q is an intensity matrix, it is also called the **Markov semigroup** generated by Q

- Q is also called the infinitesimal generator of $(P_t)_{t \geqslant 0}$

$(P_t)_{t \geqslant 0}$ satisfies the semigroup property

$$P_{s+t} = P_s P_t \quad \text{for all } s, t \geqslant 0$$

This can be written more explicitly as

$$P_{s+t}(x, x') = \sum_{z \in \mathsf{X}} P_s(x, z) P_t(z, x')$$

for $s, t \geqslant 0$ and $x, x' \in \mathsf{X}$

- called the **Chapman–Kolmogorov equation**

The probability of moving from x to x' over $s + t$ units of time equals

1. the probability of moving from x to z over s units of time
2. and then z to x' over t units of time

summed over all z

Let $C(\mathbb{R}_+, \mathsf{X})$ be the set of right-continuous functions from $\mathbb{R}_+$ to X and let $(P_t)_{t \geqslant 0}$ be a Markov semigroup in $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$

A **continuous time Markov chain** generated by $(P_t)_{t \geqslant 0}$ is a $C(\mathbb{R}_+, \mathsf{X})$-valued random element $(X_t)_{t \geqslant 0}$ that satisfies

$$\mathbb{P}\{X_{s+t} = x' \,|\, \mathcal{F}_s\} = P_t(X_s, x') \qquad \text{for all } s, t \geqslant 0 \text{ and } x' \in \mathsf{X} \tag{10}$$

where $\mathcal{F}_s := (X_\tau)_{0 \leqslant \tau \leqslant s}$ is the history of the process up to time s

We will call a continuous time Markov chain $(X_t)_{t \geqslant 0}$ **Q-Markov** when (10) holds and Q is the infinitesimal generator of $(P_t)_{t \geqslant 0}$

Let $(X_t)_{t \geqslant 0}$, Q and P_t be as above

Conditioning on $X_s = x$, we get

$$P_t(x, x') = \mathbb{P}\{X_{s+t} = x' \mid X_s = x\} \qquad (s, t \geqslant 0,\ x, x' \in \mathsf{X})$$

In what follows, $\mathbb{P}_x$ and $\mathbb{E}_x$ denote probabilities and expectations conditional on $X_0 = x$

Given $h \in \mathbb{R}^{\mathsf{X}}$, we have

$$\mathbb{E}_x\, h(X_t) = \sum_{x'} P_t(x, x') h(x') =: (P_t h)(x)$$

This expression mirrors the discrete time case

A jump chain construction

We now describe a standard method for constructing continuous time Markov chains by using three components:

1. an initial condition $\psi \in \mathcal{D}(\mathsf{X})$,
2. a **jump matrix** $\Pi \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$, and
3. a **rate function** λ mapping X to $(0, \infty)$.

The process (X_t)

- starts at state x, which is drawn from ψ
- waits there for an exponential time W with rate $\lambda(x)$ and
- updates to a new state x' drawn from $\Pi(x, \cdot)$

We take x' as the new state for the process and repeat

Algorithm 1: Jump chain algorithm

draw Y_0 from ψ, set $J_0 = 0$ and $k = 1$
while $t < \infty$ **do**
  draw W_k independently from Exp$(\lambda(Y_{k-1}))$
  $J_k \leftarrow J_{k-1} + W_k$
  $X_t \leftarrow Y_{k-1}$ for all t in $[J_{k-1}, J_k)$
  draw Y_k from $\Pi(Y_{k-1}, \cdot)$
  $k \leftarrow k + 1$
end

- (W_k) is called the sequence of **wait times**
- the sums $J_k = \sum_{i=1}^{k} W_i$ are called the **jump times** and
- (Y_k) is called the **embedded jump chain**

Let $I \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ be the identity matrix ($I(x, x') = \mathbb{1}\{x = x'\}$)

Define $Q \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ via

$$Q(x, x') = \lambda(x)(\Pi(x, x') - I(x, x')) \qquad (x, x' \in \mathsf{X})$$

Ex. Check that Q is an intensity matrix

Proposition. The process $(X_t)_{t \geqslant 0}$ generated by the jump chain algorithm is Q-Markov

Proof uses the Kolmogorov backward equation

- see the book for details

Some intuition for

$$Q(x, x') = \lambda(x)(\Pi(x, x') - I(x, x'))$$

If $x \neq x'$, the rate of flow from x to x' is

$$\lambda(x)\Pi(x, x') = Q(x, x')$$

What about $x = x'$?

The jump matrix Π is constructed s.t. $\Pi(x, x) = 0$

- at jump times, we actually jump (don't stay at x)

Rate of flow out of x is $\lambda(x)$

Hence the rate of flow from x to x is

$$-\lambda(x) = Q(x, x)$$

Application: inventory dynamics

Let X_t be a firm's inventory at time t

When current stock is $x > 0$, customers arrive at rate $\lambda(x) > 0$

- wait time for the next customer is $\mathrm{Exp}(\lambda(x))$

The k-th customer demands U_k units, where each U_k is an independent draw from a fixed distribution φ on $\mathbb{N}$

Inventory falls by $U_k \wedge X_t$

When inventory hits zero the firm orders b units of new stock

The wait time for new stock is $\mathrm{Exp}(\lambda(0))$

Let Y = inventory size after the next jump, given current stock x

If $x > 0$, then Y is a draw from the distribution of $x - U \wedge x$ where $U \sim \varphi$

If $x = 0$, then $Y \equiv b$

Hence Y is a draw from $\Pi(x, \cdot)$, where $\Pi(0, y) = \mathbb{1}\{y = b\}$ and, for $0 < x \leqslant b$,

$$\Pi(x, y) = \begin{cases} 0 & \text{if } x \leqslant y \\ \mathbb{P}\{x - U = y\} & \text{if } 0 < y < x \\ \mathbb{P}\{U \geqslant x\} & \text{if } y = 0 \end{cases} \tag{11}$$

Ex. Prove that Π is a stochastic matrix on $\mathsf{X} := \{0, 1, \ldots, b\}$

We can simulate the inventory process $(X_t)_{t \geqslant 0}$ via the jump chain algorithm

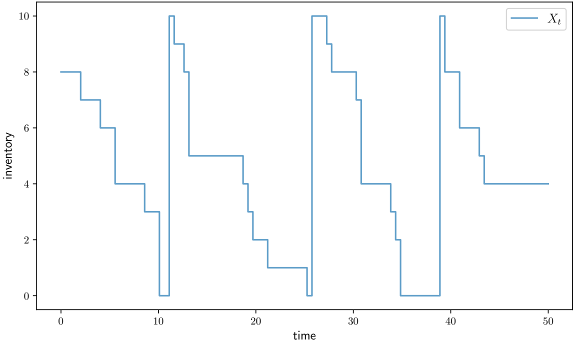
- (W_k) is the wait time for customers / new inventory and
- (Y_k) is the level of inventory immediately after each jump

By the proposition on slide 40, the process (X_t) is Q-Markov with

$$Q(x, x') = \lambda(x)(\Pi(x, x') - I(x, x'))$$

The next slide shows a simulation when orders are geometric, so that

$$\varphi(k) = \mathbb{P}\{U = k\} = (1 - \alpha)^{k-1}\alpha \qquad (k \in \mathbb{N}, \ \alpha \in (0, 1)).$$



X_t
inventory
time
0
2
4
6
8
10
0
10
20
30
40
50

Valuation with constant discounting

Consider

$$v(x) := \mathbb{E}_x \int_0^\infty \mathrm{e}^{-t\delta} h(X_t)\,\mathrm{d}t \qquad (x \in \mathsf{X})$$

for some $\delta \in \mathbb{R}$ and $h \in \mathbb{R}^{\mathsf{X}}$

- $(X_t)_{t \geqslant 0}$ is Q-Markov on X and $P_t = \mathrm{e}^{tQ}$

Interpretation:

- $h(X_t)$ is an instantaneous reward at t
- δ is a fixed discount rate
- $v(x)$ is lifetime value conditional on starting at x

Proposition. If $\delta > 0$, then v is finite, $\delta I - Q$ is bijective,

$$(\delta I - Q)^{-1} \geqslant 0 \quad \text{and} \quad v = (\delta I - Q)^{-1} h$$

In addition, v is the unique fixed point of

$$Uw = h + (Q + (1 - \delta) I) w \qquad \left(w \in \mathbb{R}^{\mathsf{X}} \right)$$

and U is order stable on $\mathbb{R}^{\mathsf{X}}$

Proof: Letting $A := Q - \delta I$, we claim that $s(A) < 0$

Using the result for spectral bounds in slide 22, we have

$$\begin{aligned} e^{s(Q-\delta I)} = \rho(e^{Q-\delta I}) &= \rho(e^{Q} e^{-\delta I}) \\ &= \rho(e^{Q} e^{-\delta} I) \\ &= e^{-\delta}\rho(e^{Q}) = e^{-\delta}\rho(P_1) = e^{-\delta} \end{aligned}$$

Therefore $s(Q - \delta I) = -\delta$

$$\therefore \quad s(A) = s(Q - \delta I) < 0$$

We have just shown that $s(A) = s(Q - \delta I) < 0$

Hence A has nonzero determinant and is therefore nonsingular

$$\therefore \quad -A = \delta I - Q \text{ is nonsingular / bijective}$$

Also, $s(A) < 0$ and the stability result on slide 24 yield

$$v(x) = \int_0^\infty \mathrm{e}^{-t\delta}\, \mathbb{E}_x\, h(X_t)\, \mathrm{d}t$$

$$= \int_0^\infty \mathrm{e}^{-t\delta}\, (P_t\, h)(x)\, \mathrm{d}t$$

$$= \int_0^\infty \mathrm{e}^{-t\delta} (\mathrm{e}^{tQ}\, h)(x)\, \mathrm{d}t = \int_0^\infty (\mathrm{e}^{tA}\, h)(x)\, \mathrm{d}t < \infty$$

We have

$$v = \int_0^\infty \mathrm{e}^{\tau A} h \, \mathrm{d}\tau = \int_0^t \mathrm{e}^{\tau A} h \, \mathrm{d}\tau + \int_t^\infty \mathrm{e}^{\tau A} h \, \mathrm{d}\tau$$

But

$$\begin{aligned} \int_t^\infty \mathrm{e}^{\tau A} h \, \mathrm{d}\tau &= \int_0^\infty \mathrm{e}^{(t+\tau) A} h \, \mathrm{d}\tau \\ &= \int_0^\infty \mathrm{e}^{tA} \mathrm{e}^{\tau A} h \, \mathrm{d}\tau = \mathrm{e}^{tA} \int_0^\infty \mathrm{e}^{\tau A} h \, \mathrm{d}\tau = \mathrm{e}^{tA} v \end{aligned}$$

$$\therefore \quad v = \int_0^t \mathrm{e}^{\tau A} h \, \mathrm{d}\tau + \mathrm{e}^{tA} v$$

Rearranging $v = \int_0^t \mathrm{e}^{\tau A} h \, \mathrm{d}\tau + \mathrm{e}^{tA} v$ and dividing by $t > 0$ yields

$$-\frac{\mathrm{e}^{tA} - I}{t} v = \frac{1}{t} \int_0^t \mathrm{e}^{\tau A} h \, \mathrm{d}\tau \tag{12}$$

By the fundamental theorem of calculus,

$$\lim_{t \to 0} \frac{1}{t} \int_0^t \mathrm{e}^{\tau A} h \, \mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{e}^{\tau A} h \, \mathrm{d}\tau \, \Big|_{t=0} = \mathrm{e}^{0A} h = I \, h = h$$

As a result, taking $t \to 0$ in (12)

$$-Av = -A\mathrm{e}^{0A} v = -\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{tA} v \, \Big|_{t=0} = \lim_{t \to 0} -\frac{\mathrm{e}^{tA} - I}{t} v = h$$

We have shown that

1. $A := Q - \delta I$ is bijective and
2. $-Av = h$

Hence

$$v = -A^{-1}h = (-A)^{-1}h = (\delta I - Q)^{-1}h$$

From $v(x) = \mathbb{E}_x \int_0^\infty \mathrm{e}^{-t\delta} h(X_t) \,\mathrm{d}t$ we have

$$h \geqslant 0 \implies (\delta I - Q)^{-1}h \geqslant 0$$

Hence $(\delta I - Q)^{-1} \geqslant 0$, as claimed

It remains only to show that v is the unique fixed point of

$$Uw = h + (Q + (1 - \delta)I)w$$

and U is order stable on $\mathbb{R}^{\mathsf{X}}$

The first claim is true because

$$\begin{aligned}
Uw = w \iff & h + Qw + w - \delta w = w \\
\iff & h + Qw - \delta w = 0 \\
\iff & (\delta I - Q)w = h \\
\iff & w = (\delta I - Q)^{-1} h = v
\end{aligned}$$

To prove that U is order stable, we need to show that U is upward and downward stability on $\mathbb{R}^{\mathsf{X}}$

For upward stability, suppose that $w \in \mathbb{R}^{\mathsf{X}}$ and $Uw \geqslant w$

Then $h + Aw \geqslant 0$, or $-Aw \leqslant h$

But $-A^{-1} \geqslant 0$, so $w \leqslant -A^{-1}h = v$ and upward stability holds

The proof of downward stability is similar

Continuous time Markov decision processes

Fix two finite sets A and X, called the state and action spaces respectively

Informally, a continuous time Markov decision process is an optimization problem where the aim is to maximize

$$v(x) := \mathbb{E}_x \int_0^\infty \mathrm{e}^{-t\delta} r(X_t, A_t) \, \mathrm{d}t$$

where

- $X_t \in \mathsf{X}$ is the state
- $A_t \in \Gamma(X_t) \subset \mathsf{A}$ is the action

Formally...

A **continuous time Markov decision process** is a tuple $\mathcal{C} = (\Gamma, \delta, r, Q)$ consisting of

1. a nonempty **feasible correspondence** Γ from $\mathsf{X} \to \mathsf{A}$, which in turn defines the **feasible state-action pairs**

$$\mathsf{G} := \{(x, a) \in \mathsf{X} \times \mathsf{A} : a \in \Gamma(x)\}$$

2. a constant $\delta > 0$, referred to as the **discount rate**
3. a function r from G to $\mathbb{R}$, referred to as the **reward function** and
4. an **intensity kernel** Q from G to X; that is, a map Q from $\mathsf{G} \times \mathsf{X}$ to $\mathbb{R}$ satisfying

$$\sum_{x' \in \mathsf{X}} Q(x, a, x') = 0 \quad \text{for all } (x, a) \text{ in } \mathsf{G}$$

and $Q(x, a, x') \geqslant 0$ whenever $x \neq x'$

Intuition: at state x with action a over the short interval from t to $t + h$,

- the controller receives instantaneous reward $r(x, a)h$ and
- the state transitions to state x' with probability $Q(x, a, x')h + o(h)$

The set of **feasible policies** is

$$\Sigma := \{\sigma \in \mathsf{A}^{\mathsf{X}} : \sigma(x) \in \Gamma(x) \text{ for all } x \in \mathsf{X}\} \tag{13}$$

Choosing policy σ from Σ means that we respond to state X_t with action $A_t := \sigma(X_t)$ at every $t \in \mathbb{R}_+$

Lifetime Values

Under policy σ, the state evolves according to the intensity matrix

$$Q_\sigma(x, x') := Q(x, \sigma(x), x')$$

Letting

$$r_\sigma(x) := r(x, \sigma(x))$$

the **lifetime value** of following σ starting from state x is defined as

$$v_\sigma(x) = \mathbb{E}_x \int_0^\infty \mathrm{e}^{-\delta t} r_\sigma(X_t) \, \mathrm{d}t$$

where $(X_t)_{t \geqslant 0}$ is Q_σ-Markov with $X_0 = x$

We call v_σ the **σ-value function**

Lemma. The σ-value function associated with $\sigma \in \Sigma$ obeys

$$v_\sigma = (\delta I - Q_\sigma)^{-1} r_\sigma$$

In addition, v_σ is the unique fixed point of

$$T_\sigma \, v = r_\sigma + (Q_\sigma + (1 - \delta) I) v.$$

and T_σ is order stable on $\mathbb{R}^{\mathsf{X}}$

This follows directly from

- $\delta > 0$
- the result on slide 47

Provides a straightforward method for computing v_σ

A policy $\sigma \in \Sigma$ is called v-**greedy** for $\mathcal{C}$ if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} \left\{ r(x, a) + \sum_{x'} v(x') Q(x, a, x') \right\} \quad \text{for all } x \in \mathsf{X}. \tag{14}$$

A v-greedy policy chooses actions optimally to trade off

- high current rewards versus
- high rate of flow into future states with high values

The discount factor does not appear in (14) because the trade-off is instantaneous

Algorithm 2: Continuous time Howard policy iteration

input $\sigma_0 \in \Sigma$, an initial guess of σ^*

$k \leftarrow 0$

$\varepsilon \leftarrow 1$

while $\varepsilon > 0$ **do**

- $v_k \leftarrow (\delta I - Q_{\sigma_k})^{-1} r_{\sigma_k}$
- $\sigma_{k+1} \leftarrow$ a v_k-greedy policy
- $\varepsilon \leftarrow \mathbb{1}\{\sigma_k \neq \sigma_{k+1}\}$
- $k \leftarrow k + 1$

end

return σ_k

Optimality

For a continuous time MDP $\mathcal{C} = (\Gamma, \delta, r, Q)$ with σ-value functions $\{v_\sigma\}$,

- the **value function** generated by $\mathcal{C}$ is $v^* := \bigvee_\sigma v_\sigma$, and
- a policy is called **optimal** for $\mathcal{C}$ if $v_\sigma = v^*$.

A function $v \in \mathbb{R}^{\mathsf{X}}$ is said to satisfy a **Hamilton–Jacobi–Bellman** (**HJB**) equation if

$$\delta v(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \sum_{x'} v(x') Q(x, a, x') \right\} \tag{15}$$

for all $x \in \mathsf{X}$

We say that $\mathcal{C}$ obeys **Bellman's principle of optimality** if

$$\sigma \in \Sigma \text{ is optimal for } \mathcal{C} \quad \iff \quad \sigma \text{ is } v^*\text{-greedy}$$

Theorem. If $\mathcal{C} = (\Gamma, \delta, r, Q)$ is a continuous time MDP, then

1. the value function v^* is the unique solution to the HJB equation in $\mathbb{R}^{\mathsf{X}}$,
2. $\mathcal{C}$ obeys Bellman's principle of optimality, and
3. $\mathcal{C}$ has at least one optimal policy.

In addition, continuous time HPI converges to an optimal policy in finitely many steps

Proof: Let $\mathcal{C} = (\Gamma, \delta, r, Q)$ be a fixed continuous time MDP with lifetime values $\{v_\sigma\}$ and value function v^*

Consider the order stable ADP $\mathcal{A} := (\mathbb{R}^{\mathsf{X}}, \{T_\sigma\})$ with

$$T_\sigma \, v = r_\sigma + (Q_\sigma + (1 - \delta)I)v.$$

The ADP Bellman max-operator is $T := \bigvee_\sigma T_\sigma$, which can be written more explicitly as

$$(Tv)(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \sum_{x'} v(x')Q(x, a, x') \right\} + (1 - \delta)v(x) \tag{16}$$

For each $v \in \mathbb{R}^{\mathsf{X}}$, the set of v-max-greedy policies is nonempty

Since Σ is finite, it follows that $\mathcal{A}$ is max-stable

Hence an optimal policy always exists and the value function v^* is the unique fixed point of T in $\mathbb{R}^{\mathsf{X}}$

The last statement is equivalent to the assertion that v^* is the unique element of $\mathbb{R}^{\mathsf{X}}$ satisfying

$$v^*(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \sum_{x'} v^*(x') Q(x,a,x') \right\} + (1-\delta) v^*(x)$$

Rearranging this expression confirms that v^* is the unique solution to the HJB equation in $\mathbb{R}^{\mathsf{X}}$.

A policy is optimal for $\mathcal{A}$ (and hence $\mathcal{C}$) if and only if $T_\sigma v^* = T v^*$

This proves the claim that $\mathcal{C}$ obeys Bellman's principle of optimality

The continuous time HPI routine in slide 61 is just ADP max-HPI specialized to the current setting

Hence, continuous time HPI converges to an optimal policy in finitely many steps

Application: job search

We study continuous time job search with separation

A worker can be either unemployed (state 0) or employed (state 1)

When the worker is employed, she can be fired at any time

Firing occurs at rate $\alpha > 0$

- for $h \approx 0$, probability of being fired over $[t, t+h]$ is $\approx \alpha h$

When unemployed, the worker receives

- flow unemployment compensation c and
- job offers at rate κ

She discounts the future at rate $\delta > 0$

Job offers are at wage w in finite set W

Conditional on current w, the next offer is drawn from $P(w, \cdot)$

For the state space we set

$$\mathsf{X} = \{0, 1\} \times \mathsf{W} \text{ with typical state } x = (s, w)$$

Here

- s is binary and indicates current employment status
- w is the current wage

Let

$$\lambda(x) = \lambda(s, w) = \mathbb{1}\{s = 0\}\kappa + \mathbb{1}\{s = 1\}\alpha$$

denote the state-dependent jump rate

Let $a \in \mathsf{A} := \{0, 1\}$ indicate the action (reject, accept)

Let $\Pi(x, a, x')$ represent the jump probabilities, with

$$\Pi((0, w), a, (0, w')) = P(w, w')(1 - a) \qquad \text{(unemployed to unemployed)}$$

$$\Pi((0, w), a, (1, w')) = P(w, w')a \qquad \text{(unemployed to employed)}$$

$$\Pi((1, w), a, (0, w')) = P(w, w') \qquad \text{(employed to unemployed)}$$

$$\Pi((1, w), a, (1, w')) = 0 \qquad \text{(employed to employed)}$$

The probability assigned to the last line is zero because a jump from $s = 1$ occurs when the worker is fired

Motivated by the jump chain construction of intensity matrices in, we set

$$Q(x, a, x') = \lambda(x)(\Pi(x, a, x') - I(x, x'))$$

Fix $\sigma \in \Sigma := \{0, 1\}^{\mathsf{X}}$

The operator

$$Q_\sigma(x, x') := \lambda(x)(\Pi(x, \sigma(x), x') - I(x, x'))$$

is an intensity matrix for the jump chain under policy σ

- inventory is Q_σ-Markov under policy σ

If we define

$$r(x, a) = r((s, w), a) = c\mathbb{1}\{s = 0\} + w\mathbb{1}\{s = 1\},$$

then lifetime value is given by

$$v_\sigma(x) = \mathbb{E}_x \int_0^\infty \mathrm{e}^{-\delta t} r_\sigma(X_t) \, \mathrm{d}t,$$

where $(X_t)_{t \geqslant 0}$ is Q_σ-Markov and $X_0 = x$

With Γ defined by $\Gamma(x) = \mathsf{A}$ for all $x \in \mathsf{X}$, the tuple $\mathcal{C} = (\Gamma, \delta, r, Q)$ is a continuous time MDP

By the result on slide 63, An optimal policy exists and can be computed with HPI in a finite number of iterations

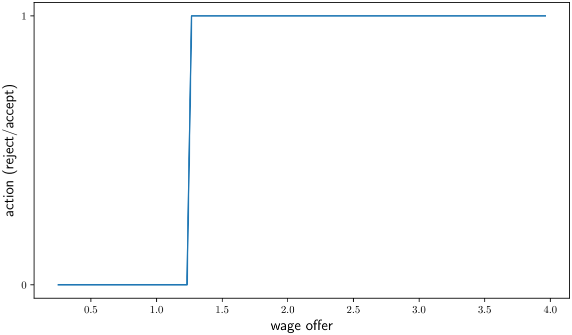


Figure: Continuous time job search policy

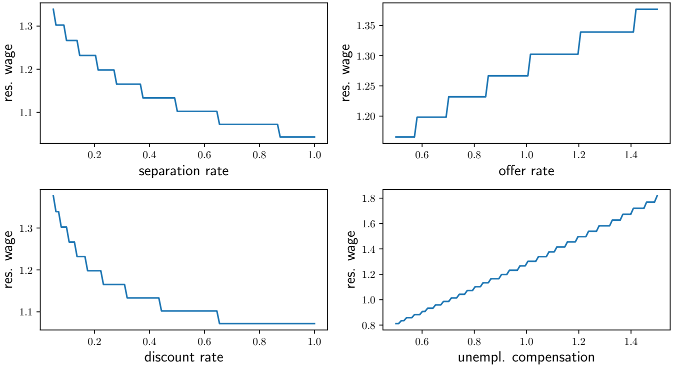


Figure: Continuous time job search reservation wage