Peter Meyer-Nieberg

Banach Lattices

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Preface

This book is mainly concerned with the theory of Banach lattices and with linear operators defined on, or with values in Banach lattices. Moreover we will always consider more general classes of Riesz spaces so long as this does not involve more complicated constructions or proofs. In particular, we will not treat any phenomena which occur only in the non-Banach lattice situation.

Riesz spaces, also called vector lattices, K-lineals, are linear lattices which were first considered by F. Riesz, L. Kantorovič, and H. Freudenthal. Subsequently other important contributions came from the Soviet Union (L.V. Kantorovič, A.J. Judin, A.G. Pinsker, and B.Z. Vulikh), Japan (H. Nakano, T. Ogasawara, and K. Yosida), and the United States (G. Birkhoff, H.F. Bohnenblust, S. Kakutani, and M.M. Stone).

In the last twenty-five years the theory rapidly increased. Important contributions came from the Dutch school (W.A.J. Luxemburg, A.C. Zaanen) and the Tübinger school (H.H. Schaefer). In the middle seventies the research on this subject was essentially influenced by the books of H.H. Schaefer (1974) and W.A.J. Luxemburg and A.C. Zaanen (1971). More recently other important books concerning this subject appeared, A.C. Zaanen (1983), H.U. Schwarz (1984), and C.D. Aliprantis and O. Burkinshaw (1985).

It was the merit of H.H. Schaefer to present the theory of Banach lattices and positive operators as an inseparable part of the general Banach space and operator theory. In particular, deep results of the general theory and classical analysis were used to prove related properties in case of general Banach lattices.

The intentions for writing this present book were twofold. First there appeared in the literature many results completing the theory enormously. On the other hand, in the last fifteen years new techniques were developed. These new techniques are purely elementary and simple, but based on tricky computations involving Riesz space methods. See Sect. 2.3 for a unified and simplified approach. Once established, these disjoint sequence techniques lead to surprisingly simple and short proofs of many results originally known as deep. Thus these parts can now be viewed as the more trivial aspects of the theory. See Sect. 2.5 concerning weak compactness as a typical example. Originally these results were proved via representation by classical results, for instance of measure theory. The new methods systematically applied here for the first time have the advantage that they directly yield the Banach lattice theorems, including the classical versions in a trivial manner. Thus many classical theorems can now be proved by purely functional analytical methods in a much more elementary and simpler way. These methods enable us to give – with the exception of Chap. 5 – elementary approaches to many aspects of the theory as well as to include a lot of interesting new results without extending the volume of this book to much.

This book contains five chapters. Chap. 1 is concerned with the classical theory of general and normed Riesz spaces The first four sections of this chapter are the basis for the remainder of this book. Since this book is mainly concerned with the theory of Banach lattices, we do not treat general Riesz spaces in full detail. The interested reader will find more about this subject in Luxemburg and Zaanen (1971). In particular, this book contains many interesting applicatons of the general theory. Compare also Aliprantis and Burkinshaw (1985). Moreover we will treat the normed and the general situation at the same time because many results and proofs for the two cases are similar or even the same.

We start in Sect. 1.1 with the definition of ordered vector spaces, Riesz spaces, and normed Riesz spaces. Furthermore we will consider Dedekind completeness and order convergence of sequences and nets. Sect. 1.2 is concerned with the intrinsic structure of Riesz spaces; i.e. we will introduce bands, ideals, and sublattices and derive the classical theorems including Freudenthal's spectral theorem. Sect 1.3 deals with elementary properties of regular operators and order bounded functionals. Sect. 1.4 contains the elementary duality theory of Riesz spaces, in particular, Nakano's characterization of a Riesz space being an ideal in its bidual. Moreover we will characterize lattice homomorphisms on normed Riesz spaces by means of duality. Sect. 1.5 contains the Hahn-Banach extension theorems for the case where the range space is a Dedekind complete Riesz space. This theory is essentially due to Kantorovič. As in the classical situation, results and proofs are almost the same as in the real-valued situation. We will use here the 'sublinear method'.

Chap. 2 deals with classical Banach lattices as well as with properties of Banach lattices originally shown for special spaces. Moreover this chapter contains technical results being essential for the remainder of this book. In particular, the results of Sect. 2.3 will be used to prove many results concerning the 'topological aspects' of the theory.

Sect. 2.1 is concerned with $C(K)$- and M- spaces. Mainly we will treat those aspects of the theory which are closely related to the theory of Riesz spaces. For other aspects we refer to Semadeni (1971). We start with Kakutani's representation theorem for M- spaces with a unit. Furthermore this section contains characterizations of Dedekind complete $C(K)$- spaces, ideals and bands of $C(K)$, and M-spaces in terms of disjoint sequences. With a version of Borsuk and Dugundij's extension theorem, we will show Banach's theorem asserting that every separable Banach space isometrically embeds into $C(\Delta)$. In Sect. 2.2 we will introduce complexifications of uniformly complete Riesz spaces and complex regular operators. Sect. 2.3 concerning disjoint sequences is purely technical. Here we will gather in a simplified and unified form the tools which will be used in most of the remaining sections. These methods enables us to present many results as purely elementary and with short proofs. As a first application we will discuss in Sect. 2.4 order continuity of the norm. Furthermore we will treat KB-spaces, reflexive Banach lattices, and Fatou norms. In Sect 2.5 we will discuss weak compactness in Banach lattices. The main result of this section is a generalization of the classical Dunford-Pettis theorem concerning weak compactness in an L^1-space. With the methods of Sect. 2.3 we will here prove a version for KB-spaces. As simple consequences we will show classical theorems; for instance, Grothendieck's characterization of relatively weakly compact sets in the space of Radon measures and convergence theorems for sequences of measures. In Sect. 2.6 we will treat Riesz spaces of measurable functions. In particular, with the aid of the results of the previous sections we will show some of the well-known basic properties. This section is mainly included to demonstrate the consequences of the general theory to more concrete situations; the remainder of this book is not based on it. The classical L^p-spaces are considered in Sect. 2.7. First we will prove the representation theorem for abstract L^p-spaces, and give isometric classifications of separable L^p-spaces. Another important part of this section deals with the characterization of L^p-spaces in terms of positive contractive projections due to Ando. Sect. 2.8 is concerned with p-subadditive norm and cone p-absolute summing operators which will be characterized by means of disjoint sequences, factorization, and duality.

Chap. 3 is concerned with operators defined on Riesz spaces or with values in Riesz spaces. We will consider topological aspects of the theory as well as lattice theoretical ones.

We start in Sect. 3.1 with a discussion of regular operators on Riesz spaces being closely related to the lattice structure. In particular, we will introduce disjointness preserving operators, orthomorphisms, the center of a Riesz space, and f-algebras. This section contains the well-known material concerning these types of operators. Moreover, we will show that every bounded disjointness preserving operator is regular, and discuss the projection of all regular operators onto the center. In Sect. 3.2 first a characterization of L- and M- spaces in terms of regularity of all operators is shown. We continue with a discussion of injective Banach lattices, lattice homomorphisms on spaces of type $C(K)$, and norm-identities of positive operators on L- and M-spaces. Sect. 3.3 is devoted to kernel operators on ideals of measurable functions. The main tool here is Bukhvalov's characterization of kernel operators, which has a lot of interesting consequences. Sect. 3.4 is concerned with order weakly compact operators. This concept generalizes weakly compact operators on spaces of type $C(K)$. The methods of Sect. 2.3 enable us to give simple and short proofs for results which were known for the special case of operators on spaces of type $C(K)$ as deep. Furthermore we will introduce and discuss related types of operators. Based on the well-known interpolation method for operators due to Davies, Figuel, Johnson, and Pelczynski, we will discuss in Sect. 3.5 weakly compact operators into Banach lattices. Sect. 3.6 is concerned with approximately order bounded operators and related types of operators. The results here are again immediate consequences of the disjoint sequences technics of Sect. 2.3. In Sect. 3.7 we will discuss compact operators and Dunford-Pettis operators on Banach lattices. The results and proofs essentially are based on the previous sections. In Sect. 3.8 we will introduce the concept of regularly ordered tensor products. In particular, we will show that the regularly ordered tensor product of Banach lattices also is a Banach lattice. As an application of the theory of order weakly compact operators we will show in Sect. 3.9 the relevant properties of strongly additive vector measures.

Chap. 4 mainly is concerned with the spectral properties of positive operators on complex Banach lattices. As usual in spectral theory, we suppose the spaces under consideration to be complex. See Sect. 2.2 for definition of complex Banach lattices. Most sections of this chapter require only some basic facts concerning Banach lattice mostly to be found in Chap. 1 and Sect. 2.2.

Sect. 4.1 contains the classical material concerning this subject including the spectral behavior of lattice homomorphisms. Moreover we include a new result concerning lower estimates of positive operators. In Sect. 4.2 we will discuss positive irreducible operators basing on de Pagter's theorem which shows that a compact quasinilpotent operator fails to be irreducible. In Sect. 4.3 we will treat measures of non-compactness for operators on Banach lattices and give some relations to the essential spectral radius. This section is closely related to Sect. 3.6. In Sect. 4.4 we will treat the local spectral behavior of positive operators. In particular, we are interested in positive solutions of equations of the form $(\lambda I - T)x = z$ with $x \geq 0$. In Sect. 4.5 we will introduce the so-called order spectrum for regular operators, and derive some relevant properties. Based on Banach algebra methods, we will prove in Sect. 4.6 the 'zero two' law for operators on Banach lattices. Furthermore we will discuss here the spectral behavior of disjointness preserving operators.

In Chap. 5 we will treat Banach space properties of Banach lattices. The methods here are less elementary than in the other chapters. In particular, the proofs in Sects. 5.2 and 5.4 are much more complicated than the other ones.

Sect. 5.1 yields some Banach space properties of Banach lattices with methods introduced in Chapters 2 and 3. In particular, we will discuss property (u) for Banach lattices and discuss properties of subspaces of Banach lattices. Sect. 5.2 is concerned with the embedding of $C(0,1)$ and $C(\Delta)$ into Banach lattices. First we follow Rosenthal (1972), but include some additional material to make this section self-contained, and give some minor improvements. Applying this concept we will show some characterizations of properties of Banach lattices due to Lotz, Rosenthal and Ghoussoub, Johnson. Sect. 5.3 is concerned with Grothendieck spaces. Based on Sect. 5.2 and 2.3, we will characterize Grothendieck Banach lattices and discuss property (V). Sect. 5.4 is concerned with the Radon-Nikodym property of Banach lattices. To make this section self-contained, we start with preliminaries. Important parts of this section are a characterization of dual Banach lattices with the Radon-Nikodym property due to Lotz and order dentable Banach lattices. We will prove a characterization of separable order dentable Banach lattices due to Ghoussoub and Talagrand including in a trivial manner Talagrand's characterization of separable dual Banach lattices.

At the end of each section – with the exception of Sect. 3.9 – one can find several exercises. They are mainly included to make the reader more familiar with methods and theory. With the previously stated results many of them are not hard to be solved. On the other hand, none of them is required for an understanding of the text.

Finally I want to thank many colleagues for helpful discussions and comments on various versions of this book.

Osnabrück, April 1991 P. Meyer-Nieberg

Contents

1. Riesz Spaces

1.1 Basic Properties of Riesz Spaces and Banach Lattices

In this section we will investigate Riesz spaces and normed Riesz spaces, and deduce some of their basic properties. Most of the results we will present are well-known and go back to the origin of the theory of Riesz spaces. Many of them are due to G. Birkhoff, H. Freudenthal, L.V. Kantorovič, and F. Riesz. We will treat these basic results not very extensively as the intention is to give a self contained introductory part which is as short as possible. The interested reader can find more about this subject in the book of Luxemburg and Zaanen (1971).

A non-empty set M with a relation $\leq$ is said to be an *ordered set* whenever the following conditions are satisfied.

i) $x \leq x$ for every $x \in M$,
ii) $x \leq y$ and $y \leq x$ implies that $x = y$, and
iii) $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for any two elements $x, y \in M$ either $x \leq y$ or $y \leq x$, then M is called a *totally ordered set*. Let A be a subset of an ordered set M. $x \in M$ is called an *upper bound* of A if $y \leq x$ for every $y \in A$. $z \in M$ is called a *lower bound* of A if $y \geq z$ for all $y \in A$. Moreover, if there is an upper bound of A, then A is said to be *bounded from above*. If there is a lower bound of A, then A is called *bounded from below*. If A is bounded from above and from below, then we will briefly say that A is *order bounded*.

An ordered set $(M, \leq)$ is called a *lattice* if any two elements $x, y \in M$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

Similarly we denote the supremum and the infimum for arbitrary subsets. If v is the least upper bound of a subset $A \subset M$, then we will write

$$v = \sup(A) = \bigvee_{x \in A} x = \sup\{x : x \in A\}.$$

If u is the greatest lower bound of A, then we will write

$$u = \inf(A) = \bigwedge_{x \in A} x = \inf\{x : x \in A\}.$$

Of course, if $\sup(A)$ exists, then A is bounded from above. The converse need not to be true as the following example will show.

Example. We fix a non-empty set X.

i) If $\mathcal{M}$ is the set of all subsets of X ordered by inclusion, then $\mathcal{M}$ is a lattice which has X as the greatest element.

ii) Suppose that X is infinite. Let $\mathcal{N}$ be the collection of all subsets A of X such that either A is finite or the complement A^c of A is finite. It is easy to see that $\mathcal{N}$ is a lattice. Now consider a subset $Y \subset X$ such that $Y \notin \mathcal{N}$. $\mathcal{B} = \{\{x\} : x \in Y\}$ is a subset of $\mathcal{N}$ which does not have any least upper bound in $\mathcal{N}$.

iii) Suppose that X is infinite. Let $\mathcal{L}$ consist of all finite subsets $A \subset X$ such that the cardinality of A is even. Of course $\mathcal{L}$ is an ordered set, but it fails to be a lattice.

Let M be an ordered set and $x, y \in M$ such that $x \leq y$. We denote by

$$[x, y] = \{z \in M : x \leq z \leq y\}$$

the *order interval* between x and y. Obviously, a subset A is order bounded if and only if it is contained in some order interval.

A subset D of an ordered set M is called *upwards directed* if for any two $x, y \in D$ there exists $z \in D$ such that $z \geq x, y$.

A real vector space E which is also an ordered set is called an *ordered vector space* if the order and the vector space structure are compatible in the following sense:

If $x, y \in E$ such that $x \leq y$, then $x + z \leq y + z$ for all $z \in E$ and $ax \leq ay$ for all real a with $a \geq 0$.

If in addition $(E, \leq)$ is a lattice, then E is called a *Riesz space* (or *vector lattice*).

In the present book we are mainly interested in studying Riesz spaces. We will not treat the more general class of ordered vector spaces. The interested reader can find this material in the books of Peressini (1967), Schaefer (1971), and Jameson (1970).

In the remainder of this section let E be a Riesz space. Let the *positive cone* E_+ of E consist of all $x \in E$ such that $x \geq 0$. For every $x \in E$ let

$$x^+ = x \vee 0,\ x^- = (-x) \vee 0, \text{ and } \mid x \mid = x \vee (-x)$$

be the *positive part,* the *negative part*, and the *absolute value* of x, respectively.

$x, y \in E$ are called *disjoint* (denoted by $x \perp y$) if $|x| \wedge |y| = 0$.

Before we discuss some simple consequences of the definition, we will give an example.

Example. i) Let X be a non-empty set and let $B(X)$ be the collection of all bounded real valued functions defined on X. It is a simple and well-known fact that $B(X)$ is a vector space which is ordered by the positive cone

$$B(X)_+ = \{f \in B(X) : f(t) \geq 0 \text{ for all } t \in X\}.$$

Thus $f \geq g$ holds if and only if $f - g \in B(X)_+$. Obviously,

$$\begin{aligned}(f \vee g)(t) &= \max\{f(t), g(t)\} \text{ and}\\ (f \wedge g)(t) &= \min\{f(t), g(t)\}\end{aligned}$$

for every $t \in X$ and $f, g \in B(X)$. This shows that $B(X)$ is a Riesz space.

ii) Assume that X is infinite and assume that $\mathcal{L}$ is the collection of subsets of X as in the previous example. If E is the linear subspace of $B(X)$ generated by the characteristic functions of the sets in $\mathcal{L}$, then E is an ordered vector space but fails to be a Riesz space.

Theorem 1.1.1. *For all $x, y, z \in E$, and $a \in \mathbb{R}$ the following assertions hold.*

i) $x + y = x \vee y + x \wedge y$, $x \vee y = -(-x) \wedge (-y)$,
$x \vee y + z = (x + z) \vee (y + z)$, *and* $x \wedge y + z = (x + z) \wedge (y + z)$.

ii) $x = x^+ - x^-$.

iii) $|x| = x^+ + x^-$, $|ax| = |a|\,|x|$, *and* $|x + y| \leq |x| + |y|$.

iv) $x^+ \perp x^-$ *and the decomposition of x into the difference of two disjoint positive elements is unique.*

v) $x \leq y$ *is equivalent to* $x^+ \leq y^+$ *and* $y^- \leq x^-$.

vi) $x \perp y$ *is equivalent to* $|x| \vee |y| = |x| + |y|$.
In this case we have $|x + y| = |x| + |y|$.

vii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *and* $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

viii) E *has property* (D) (*the* decomposition property *of F. Riesz*). *If* $x, y, z \in E_+$, *and* $0 \leq z \leq x + y$, *then there exist* $u, v \in E_+$ *such that* $u \leq x$, $v \leq y$, *and* $z = u + v$.

ix) *For all* $x, y, z \in E_+$ *we have* $(x + y) \wedge z \leq x \wedge z + y \wedge z$.

x) $|x - y| = x \vee y - x \wedge y$ *and*
$|x - y| = |x \vee z - y \vee z| + |x \wedge z - y \wedge z|$.

Remark. For a moment, assume that E is as in the second example, $E = B(X)$ for some non-empty set X. Since the lattice operations on $B(X)$ are pointwise defined, all statements of the preceding theorem are true because they are true in the real numbers. This method of proving equalities and inequalities is possible in arbitrary Banach lattices: We will deduce that an equality or an inequality involving finitely many elements of E is valid if and only if it holds in the real numbers. See 2.1.20 for details. But in order to deduce this we need to know more about Riesz spaces. In particular, we will use some of the facts stated in 1.1.1.

Proof. i) Let $w = (-y) \vee (-x)$ and $v = x \vee y$. Obviously we have $w = -y \wedge x$. In view of $w \geq -y$ and $w \geq -x$ it follows that $w + x + y \geq x$ and $w + x + y \leq y$. Consequently, $w + x + y \geq x \vee y \; = \; v$.

ii) With $y \; = \; 0$ the assertion follows from i).

iii) By definition, it follows that $\mid x \mid = (-x) \vee (x)$. Thus i) implies

$$\begin{aligned} \mid x \mid = x + (-2x) \vee 0 &= x^+ - x^- + 2x^- = x^+ + x^- \text{ and} \\ \mid x + y \mid = (x + y) \vee (-x - y) &\leq (x^+ + y^+) \vee (x^- + y^-) \\ &\leq x^+ + y^+ + x^- + y^- = \mid x \mid + \mid y \mid . \end{aligned}$$

iv) We have $x^+ \wedge x^- = x^- + (x^+ - x^-) \wedge 0 = x^- - (-x) \vee 0 = 0$. Now, assume that $u, v \in E_+$ are disjoint such that $x = u - v$. Evidently, from $u \geq x$ and $v \geq x$ we conclude that $u \geq x^+$ and $v \geq x^-$. The decomposition is unique because $0 \leq u - x^+ = v - x^- = (u - x^+) \wedge (v - x^-) \leq u \wedge v = 0$.

v) This assertion is an immediate consequence of the definition.

ix) From $x, y, z \in E_+$ and from i) we conclude that

$$\begin{aligned} (x + y) \wedge z &= ((x + y) \wedge (z + y)) \wedge z = (y + (x \wedge z)) \wedge z \\ &\leq (y + (x \wedge z)) \wedge (z + (x \wedge z)) = (x \wedge z) + (y \wedge z). \end{aligned}$$

vi) The first equation of i) shows that the given two conditions are equivalent. Assume that x and y are disjoint. ix) implies that

$$\begin{aligned} (x^+ + y^+) \wedge (x^- + y^-) &\leq x^+ \wedge (x^- + y^-) + y^+ \wedge (x^- + y^-) \\ &\leq x^+ \wedge x^- + x^+ \wedge y^- y^+ \wedge x^- + y^+ \wedge y^- = 0. \end{aligned}$$

In view of the uniqueness of the decomposition into the difference of disjoint positive parts we see that $(x + y)^+ = x^+ - y^+$ and $(x + y)^- = x^- + y^-$. This shows the assertion.

vii) For the proof see the generalized version 1.1.2.

viii) Let $u = x \wedge z$ and $v = z - u$. Obviously, we have $u, v \in E_+$ and $u \leq x$. Now, $v \leq y$ follows from

$$y - v = y - z + u = (y - z) + z \wedge x = y \wedge (x + y - z) \geq 0.$$

x) For all $u, v \in E$ we have

$$\begin{aligned} \mid u - v \mid &= (u - v)^+ + (u - v)^- \\ &= u \vee v - v - (u \wedge v - v) = u \vee v - u \wedge v. \end{aligned}$$

With $u = x \vee z$ and $v = y \vee z$ (respectively with $u = x \wedge z$ and $v = y \wedge z$) it follows from vii) and i) that

$$\begin{aligned} |x - y| &= (x - y)^+ + (x - y)^- = x \vee y - x \wedge y \\ &= x \vee y + z - (x \wedge y + z) \\ &= (x \vee y) \vee z + (x \vee y) \wedge z - (x \wedge y) \vee z - (x \wedge y) \wedge z \\ &= |x \vee z - y \vee z| + |x \wedge z - y \wedge z|. \end{aligned}$$

This completes the proof. □

Proposition 1.1.2. *Assume that A is a non-empty subset of E such that* $\sup(A)$ *or* $\inf(A)$, *respectively, exists. For every $y \in E$ we have*

$$y + \sup(A) = \sup_{x\in A}(x+y),\ y + \inf(A) = \inf_{x\in A}(x+y),$$
$$y \wedge \sup(A) = \sup_{x\in A}(y \wedge x), \text{ and } y \vee \inf(A) = \inf_{x\in A}(y \vee x).$$

Proof. The first two assertion can easily be checked. We concentrate on the third one. First note that $y \wedge \sup(A) \geq y \wedge x$ for all $x \in A$. Pick $u \in E$ such that

$$y \wedge \sup(A) \geq u \geq y \wedge x \text{ for all } x \in A.$$

For every $x \in A$ we achieve

$$u \geq (y \wedge x) = (y - x) \wedge 0 + x \geq (y - \sup(A)) \wedge 0 + x.$$

Consequently,

$$u - (y - \sup(A)) \wedge 0 \geq \sup(A).$$

It follows that $u \geq \sup(A) + (y - \sup(A)) \wedge 0 = y \wedge \sup(A)$. This completes the proof. □

Proposition 1.1.3. *Assume that $x_1, \dots, x_n,\ y_1, \dots, y_m \in E_+$ satisfying $x_1 + \cdots + x_n = y_1 + \cdots + y_m$. There exist $z_{i,k} \in E_+$ ($i = 1, \dots, n$ and $k = 1, \dots, m$) such that*

$$x_i = \sum_{k=1}^{m} z_{i,k}\ (i = 1, \dots, n) \text{ and } y_k = \sum_{i=1}^{n} z_{i,k}\ (k = 1, \dots, m).$$

Proof. We will show the assertion by induction on m. For $m = 1$ the result easily follows from 1.1.1 viii) by induction on n. Assume that the result is true for a some $m \in \mathbb{N}$. Let

$$x_1 + \cdots + x_n = y_0 + \cdots + y_m \text{ where } x_i, y_k \geq 0.$$

In view of $y_0 \leq x_1 + \ldots + x_n$ there are $z_{1,0}, \dots, z_{n,0} \in E_+$ such that $z_{i,0} \leq x_i$ for every i and

$$y_0 = \sum_{i=1}^{n} z_{i,0}.$$

If $v_i = x_i - z_{i,0}$ for every i, then

$$\sum_{i=1}^{n} v_i = \sum_{k=1}^{m} y_k.$$

By induction hypothesis, there exist

$$z_{i,k} \geq 0 \quad (i = 1, \ldots, n,\ k = 1, \ldots, m) \text{ such that}$$

$$x_i - z_{i,0} = v_i = \sum_{k=1}^{m} z_{i,n}\ (i = 1, \ldots, n) \text{ and } y_k = \sum_{i=1}^{m} z_{i,k}\ (k = 1, \ldots, m).$$

This completes the proof. □

Proposition 1.1.4. *An ordered vector space E is a Riesz space provided that $E = E_+ - E_+$ and* $\sup(x, y) = x \vee y$ *exists for all $x, y \in E_+$.*

The proof of this proposition is straightforward, and is left to the reader.

Definition 1.1.5. *A seminorm ρ on E satisfying $\rho(x) \leq \rho(y)$ whenever $|\ x\ | \leq |\ y\ |$ is called a* lattice seminorm *and a* lattice norm *if, in addition, ρ is a norm. In the latter case $(E, \|\,.\,\|)$ is called a* normed Riesz space. *A normed Riesz space which is complete with respect to the norm is called a* Banach lattice.

Proposition 1.1.6. *For a normed Riesz space E the following assertions hold.*

i) *The lattice operations are continuous.*

ii) *The positive cone E_+ is closed.*

iii) $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ *for every increasing convergent sequence $(x_n)_1^\infty \subset E$.*

Proof. i) Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. By 1.1.2 x), for every $n \in \mathbb{N}$ we achieve

$$\begin{aligned} |x_n \wedge y_n - x \wedge y| &\leq |x_n \wedge y_n - x_n \wedge y| + |x_n \wedge y - x \wedge y| \\ &\leq |y_n - y| + |x_n - x|. \end{aligned}$$

Since $\|\,.\,\|$ is a lattice norm, the conclusion follows immediately.

ii) Assume that $x_n \in E_+$ for every $n \in \mathbb{N}$ and $x_n \to x \in E$ as $n \to \infty$. It follows from i) that

$$x_n = x_n \vee 0 \to x \vee 0 \text{ as } n \to \infty \text{ and } x = x \vee 0 \in E_+.$$

iii) Suppose that $(x_n)_1^\infty$ is an increasing convergent sequence and let $x = \lim_{n\to\infty} x_n$. In view of $x_m - x_n \in E_+$ for every $n \in \mathbb{N}$ and all $m \geq n$ it follows from ii) that $x - x_n \geq 0$ for all n. Assume that $u \in E$ such that $x_n \leq u \leq x$ for all $n \in \mathbb{N}$. i) implies that

$$x_n = x_n \wedge u \to x \wedge u = u \text{ as } n \to \infty.$$

Hence $u = x$. This completes the proof. □

Definition 1.1.7. i) *E is called* Archimedean *if $x \leq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above.*

ii) *E is called* Dedekind complete *if every non-empty order bounded set has a supremum and an infimum in E.*

iii) *E is called σ-*Dedekind complete *if every order bounded sequence has a supremum and an infimum in E.*

iv) *E has the* (interpolation) property (I) *if for all sequences $(x_n)_1^\infty$, $(y_m)_1^\infty \subset E$ such that $x_n \leq y_m$ for all m, n there exists $u \in E$ such that $x_n \leq u \leq y_m$ for all m, n.*

v) *E is called* uniformly complete *if $\sup\{\sum_{i=1}^n x_i : n \in \mathbb{N}\}$ exists for every uniformly bounded sequence $(x_n)_1^\infty \subset E_+$. Hereby, $(x_n)_1^\infty$ is called uniformly bounded if there exist $e \in E_+$ and $(a_n)_1^\infty \in \ell^1$ such that $x_n \leq a_n e$.*

Throughout this book we will assume that the Riesz spaces under consideration are Archimedean. The reason is that all classical function spaces and normed Riesz spaces are Archimedean and that non-Archimedean Riesz spaces have a somewhat pathological behavior. The other properties are much more stringent. There are many classical spaces which are not σ-Dedekind complete, for example the space $C[0,1]$ of continous functions on $[0,1]$. Sometimes the property as described in v) is called ℓ^1- complete (in the terminology of Schaefer (1971)). Since this property is equivalent to the fact that all suitable subspaces are complete with respect to some uniform norm, we prefer this notation.

Proposition 1.1.8. i) *Every σ-Dedekind complete Riesz space and every normed Riesz space is Archimedean.*

ii) *Every Dedekind complete Riesz space σ-Dedekind complete.*

iii) *Every σ-Dedekind complete Riesz space has property (I).*

iv) *Every Banach lattice and every Archimedean Riesz space with property (I) is uniformly complete.*

Proof. i) Consider $x, y \in E$ such that $nx \leq y$ for all $n \in \mathbb{N}$. Hence $x^+ \leq n^{-1}y^+$ for every $n \in \mathbb{N}$. If E is a normed Riesz space, then $\|x^+\| \leq n^{-1}\|y^+\|$. Hence $x^+ = 0$. On the other hand, assume that E is σ-Dedekind complete. With $z = \inf\{n^{-1}y^+ :\ n \in \mathbb{N}\}$ it follows that

$$x^+ \leq z \leq \inf\{(2n)^{-1}y^+ : n\} = 2^{-1}\inf\{n^{-1}y^+ : n\} = 2^{-1}z.$$

Consequently, $x^+ \leq 0$.

ii) and iii) are trivial.

iv) Let $(x_n)_1^\infty \subset E_+$ be a sequence such that $x_n \leq a_n e$ for a suitable sequence $(a_n)_1^\infty \in \ell_+^1$ and some $e \in E_+$. We set

$$y_n = x_1 + \cdots + x_n \text{ and } b_n = \sum_{j=n+1}^\infty a_j.$$

First assume that E is a Banach lattice. From

$$\|y_{n+p} - y_n\| \leq b_n\|e\| \text{ for all } p, n \in \mathbb{N}$$

and the completeness of E we conclude that there exists $y \in E$ such that $y_n \to y$ as $n \to \infty$. Now the assertion is a consequence of 1.1.6 iii). Finally, let E be Archimedean with property (I). For all $n \in \mathbb{N}$ we set

$$u_n = x_1 + \cdots + x_n + b_n e.$$

It follows that $y_m \leq z_n$ for all $m, n \in \mathbb{N}$. Hence there exists $y \in E$ such that

$$y_m \leq y \leq z_n \text{ for all } m \text{ and } n.$$

In order to show $y = \sup\{y_m : m \in \mathbb{N}\}$ we pick some $v \in E$ such that

$$0 \leq y_m \leq v \leq y \leq z_m = y_m + b_m e \text{ for all } m.$$

From

$$0 \leq y - v \leq y_m - v + b_m e \leq b_m e$$

it follows that $v = y$ since $b_m \to 0$ as $n \to \infty$ and E is Archimedean. This final remark completes the proof. □

Example. i) Assume that $n \geq 2$. We define the lexicographical order on $\mathbb{R}^n$ in the following way.

$$x = (x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) = y$$

if there exists

$$k \in \{0, \ldots, n\} \text{ such that } x_1 = y_1, \ldots, x_k = y_k \text{ and } x_{k+1} < y_{k+1}.$$

It can easily be checked that $\mathbb{R}^n$ equipped with this order is a Riesz space. Furthermore, it is totally ordered such that the order is non-Archimedean: if $x = (0, 1, 0, \ldots)$ and $y = (1, 0, \ldots, 0)$, then $nx \leq y$ for every $n \in \mathbb{N}$.

ii) Suppose that X is a compact Hausdorff space. We denote by $C(K)$ the Banach space of all real valued continuous functions on X. Let $\leq$ be the pointwise order on $C(K)$, $f \leq g$ if and only if $f(t) \leq g(t)$ for every $t \in K$. It is easy to see that $(C(K), \leq)$ is a Banach lattice. More about spaces of type $C(K)$ can be found in Sect. 2.1 .

iii) Assume that (Ω, Σ, μ) is a measure space such that $\Sigma \neq \mathcal{P}(\Omega)$ and Σ contains the points of Ω. For every $1 \leq p \leq \infty$ let $\mathcal{L}^p(\mu)$ consist of all measurable functions $f : \Omega \to \mathbb{R}$ such that $\int |f|^p d\mu < \infty$ if $p < \infty$ or such that $\operatorname{ess\,sup}|f| < \infty$ if $p = \infty$. Ordered pointwise, $\mathcal{L}^p(\mu)$ is σ-Dedekind complete, but fails to be Dedekind complete.

iv) ℓ^p $(1 \leq p \leq \infty)$ and c_0 are Dedekind complete Banach lattices where the order is defined pointwise.

v) Assume that (Ω, Σ, μ) is a measure space. Every $L^p(\mu)$ where $1 \leq p < \infty$ is a Dedekind complete Banach lattice. If μ is σ-finite, then $L^\infty(\mu)$ is Dedekind complete..

vi) Let (Ω, Σ) be a measurable space and let $\mathrm{ba}(\Sigma)$, ($\mathrm{ca}(\Sigma)$, respectively,) denote the space of all bounded realvalued finitely additive (countably additive) setfunctions $\mu : \Sigma \to \mathbb{R}$. $\nu \leq \mu$ is defined by $\nu(A) \leq \mu(A)$ for every $A \in \Sigma$. It is easy to see that $\mathrm{ba}(\Sigma)$ and $\mathrm{ca}(\Sigma)$ are Dedekind complete Banach lattices such that for every $A \in \Sigma$ and all $\nu, \mu \in \mathrm{ba}(\Sigma)$ we have

$$\nu \vee \mu(A) = \sup\{\nu(B) + \mu(A \setminus B) : B \in \Sigma,\ B \subset A\}$$

and $\|\nu\| = |\nu|(\Omega)$, see 1.1.E5.

Most assertions stated in this example easily follow from the definition. Except the Dedekind completeness of the spaces in iv), v), and vi). We give no direct proofs here because we will treat this subject in full generality in Sect. 2.4 .

A net $(x_i)_{i\in\Gamma}$ is called *increasing* (*decreasing*) whenever $x_i \leq x_j$ $(x_i \geq x_j)$ for all $i, j \in \Gamma$ such that $i \leq j$. If $(x_i)_{i\in\Gamma}$ is an increasing (a decreasing) net and $x = \sup\{x_i : i \in \Gamma\}$ $(x = \inf\{x_i : i \in \Gamma\})$, then we write $x_i \uparrow x$ as $i \in \Gamma$ and $x_i \downarrow x$ as $i \in \Gamma$, respectively. Similarly, we define increasing and decreasing sequences.

Definition 1.1.9. i) *A net $(y_i)_{i\in\Gamma}$ is called* order convergent *to y as $i \in \Gamma$ if there exists a net $(x_i)_{i\in\Gamma}$ satisfying $x_i \downarrow 0$ as $i \in \Gamma$ and $|y_i - y| \leq x_i$ for all $i \in \Gamma$. We will write $y = \text{o-}\lim_{i\in\Gamma} y_i$ or $y_i \to y$ in order as $i \in \Gamma$.*

ii) *A sequence $(y_n)_1^\infty$ is called* order convergent *to y as $n \to \infty$ if there exists a sequence $(x_n)_1^\infty$ such that $x_n \downarrow 0$ as $n \to \infty$ and $|\, y_n - y \,| \leq x_n$ for all $n \in \mathbb{N}$. We will write $x = \text{o-}\lim_{n\to\infty} x_n$ or $x_n \to x$ in order as $n \to \infty$.*

It should be mentioned that the order convergence in a Riesz space E does not necessarily correspond to a topology on E in the sense that order and topological convergent nets or sequences are the same. There exist spaces of type $C(K)$ without any Hausdorff vector topology such that a sequence is order convergent if and only if it is topologically convergent.

Example. i) If $E = C(0,1)$ and $f_n : f_n(t) = t^n$ for all $t \in [0,1]$ and $n \in \mathbb{N}$, then $f_n \downarrow 0$ as $n \to \infty$ and $\|f_n\| = 1$ for every $n \in \mathbb{N}$.

ii) Assume that (Ω, Σ, μ) is a measure space and let $E = L^p(\mu)$ for some $1 \leq p < \infty$. A sequence $(f_n)_1^\infty$ in E is order convergent to f as $n \to \infty$ if and only if there exists some $0 \leq g \in L^p(\mu)$ such that $|f_n| \leq g$ μ-almost everywhere and $f_n(t) \to f(t)$ as $n \to \infty$ for almost every $t \in \Omega$.

Proof. We have only to show ii). First assume that $f_n \to f$ in order as $n \to \infty$. There exists a sequence $g_n \downarrow 0$ such that $|f_n - f| \leq g_n$ for every $n \in \mathbb{N}$. Now $g_n \downarrow 0$ implies that $g_n(t) \to 0$ as $n \to \infty$. Thus $f_n(t) \to f(t)$ as $n \to \infty$ for almost every $t \in \Omega$. Furthermore, for all $n \in \mathbb{N}$ we obtain

$$|f_n| \le |f| + |f_n - f| \le |f| + g_n \le |f| + g_1 = g.$$

To prove the converse, we may assume that $f = 0$. Let

$$g_n = \sup\{|f_m| : m \ge n\}.$$

Obviously, the sequence $(g_n)_1^\infty$ is decreasing. We show that $\|g_n\|_p \to 0$ as $n \to \infty$. If this is done, then 1.1.7 iii) implies that $g = \inf g_n = 0$. Fix $\varepsilon > 0$ and pick $A \in \Sigma$ such that $\mu(A) < \infty$ and $\|\chi_{A^c} g\|_p < \varepsilon$. There exists $B \in \Sigma$ such that $B \subset A$ and $\|\chi_{A \setminus B} g\|_p < \varepsilon$ such that f_n converges to 0 uniformly on B as $n \to \infty$. Hence there exists n_0 such that $|f_n|^p \le \varepsilon^p/\mu(A)$ on B for all $n \ge n_o$. Consequently, $g_n^p \le \varepsilon^p/\mu(A)$ on B for all $n \ge n_0$. Thus

$$\|g_n\|_p \le \|\chi_B g_n\|_p + \|\chi_{A \setminus B} g_n\|_p + \|\chi_{A^c} g_n\|_p \le 3\varepsilon.$$

This completes the proof. $\square$

If the Riesz space is σ-Dedekind complete, then there is a useful characterization of order convergent sequences which is given in proposition. A short glance at the proof will convince the reader that a similar characterization for nets also holds in Dedekind complete Riesz spaces.

Proposition 1.1.10. *Assume that E is σ-Dedekind complete. For every order bounded sequence $(x_n)_1^\infty \subset E$ and $x \in E$ the following assertions are equivalent.*

i) $x = \text{o-}\lim_{n\to\infty} x_n$

ii) $x = \sup_{n\in\mathbb{N}} \{\inf\{x_m : m \ge n\}\} = \inf_{n\in\mathbb{N}} \{\sup\{x_m : m \ge n\}\}$.

Proof. For every $n \in \mathbb{N}$ let $u_n = \sup\{x_m : m \ge n\}$ and $v_n = \inf\{x_m : m \ge n\}$.

i) $\Rightarrow$ ii) We assume that $(y_n)_1^\infty$ is a sequence such that $y_n \downarrow 0$ as $n \to \infty$ and $|x_n - x| \le y_n$ for every n. Consequently, for all $m \ge n$ it follows that

$$-y_n \le -y_m \le x_m - x \le y_m \le y_n.$$

Thus for every $n \in \mathbb{N}$ we obtain

$$-y_n \le v_n - x \le u_n - x \le y_n.$$

It follows that

$$\begin{aligned} x = \sup\{x - y_n : n \in \mathbb{N}\} &\le \sup\{v_n : n \in \mathbb{N}\} \\ &\le \inf\{u_n : n \in \mathbb{N}\} \\ &\le \inf\{y_n + x : n \in \mathbb{N}\} = x. \end{aligned}$$

ii)$\Rightarrow$ i) For every $n \in \mathbb{N}$ let $y_n = u_n - v_n$. By definition, the sequence $(y_n)_1^\infty$ is decreasing such that

$$0 \le \inf\{y_n : n \in \mathbb{N}\} \le \inf\{u_m - v_n : n \in \mathbb{N}\} = u_m - x$$

for every $m \in \mathbb{N}$. It follows that

$$\inf\{y_n : n \in \mathbb{N}\} \leq \inf\{u_m - x : m \in \mathbb{N}\} = 0.$$

From $v_n \leq x \leq u_n$ for every $n \in \mathbb{N}$ we conclude that

$$\begin{aligned} |x_n - x| &= (x_n - x)^+ + (x - x_n)^- \\ &\leq (u_n - x)^+ + (x - v_n)^+ = u_n \vee x - v_n \wedge x = y_n. \end{aligned}$$

Thus the proof is complete. □

The following proposition shows that the vector space operations and the lattice operations are continuous with respect to order convergence. It is clear that the same assertion also holds for nets. The simple proof is based on 1.1.1 x), and is left to the reader.

Proposition 1.1.11. *Assume that $(x_n)_1^\infty$, $(y_n)_1^\infty$ are sequences in E such that $x_n \to x$ and $y_n \to y$ in order as $n \to \infty$ and suppose that $(a_n)_1^\infty$ is a real sequence satisfying $a_n \to a$ as $n \to \infty$. It follows that*

$$\text{o-lim}_{n\to\infty}(x_n + ay_n) = x + ay,$$
$$\text{o-lim}_{n\to\infty}\, x_n \vee y_n = x \vee y, \ \textit{and}$$
$$\text{o-lim}_{n\to\infty}(x_n \wedge y_n) = x \wedge y$$

If E is Archimedean, then $\text{o-lim}_{n\to\infty} a_n x_n = ax$.

Exercises

1.1.E1. Let $x, y, z \in E_+$. Show that
i) $(z - x)^+ + (y - x)^+ + (2x - x \vee y)^+ \geq x$.
ii) If $| z - y | = z + y$, then $z \perp y$.

1.1.E2. Show that $\mathbb{R}^n (n > 1)$ with the lexicographical order does not have property (I).

1.1.E3. Consider $C^1(0,1)$ the space of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$. Define an order by $f \geq g$ whenever $f(0) \geq g(0)$ and $f' \geq g'$ pointwise on $[0,1]$. Furthermore let $\|f\| = \|f'\|_\infty + |f(0)|$. Show that $(C^1(0,1), \geq, \|.\|)$ is a Banach lattice.

1.1.E4. Let X be a non-empty set and let $(f_n)_1^\infty \subset B(X)$ be a sequence which is uniformly convergent to some f. Show that $f_n \to f$ in order as $n \to \infty$.

1.1.E5. Let (Ω, Σ) be a measurable space.

i) Show that $\text{ba}(\Sigma)$ and $\text{ca}(\Sigma)$ are Dedekind complete Riesz spaces satisfying $\sup(\mathcal{R})(A) = \sup\{\nu(A) : \nu \in \mathcal{R}\}$ for every order bounded upwards directed set $\mathcal{R}$.

ii) Show that $\mu, \nu \in \text{ca}(\Sigma)$ are disjoint if and only if there exists $U \in \Sigma$ such that $\mu(A) = \mu(A \cap U)$ and $\nu(A) = \nu(A \cap U^c)$ for all $A \in \Sigma$.