Dynamic Programming

Chapter 3: Markov Dynamics

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2024

Topics

1. Markov chains
2. Stationarity and ergodicity
3. Approximation
4. Expectations
5. Job search revisited

Markov Models

In this chapter we review Markov models

- An essential workhorse
  - economics
  - finance
  - operations research
  - etc., etc.
- General
- Beautiful theory
- Natural fit for dynamic programming (Markov decisions)

Markov Chains

Let X be a finite set (the state space) with typical members x, x'

Consider a process $(X_t) = (X_0, X_1, \ldots)$ that jumps from state x to state x' according to given probabilities $P(x, x')$

Since $P(x, \cdot)$ is a distribution, we require that

$$P(x, x') \geqslant 0 \text{ for all } x' \text{ and } \textstyle\sum_{x' \in \mathsf{X}} P(x, x') = 1$$

In operator notation,

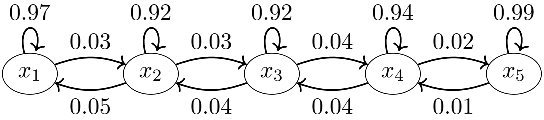
$$P \in \mathcal{L}(\mathbb{R}^{\mathsf{X}}) \text{ with } P \geqslant 0 \text{ and } P\mathbb{1} = \mathbb{1}$$

$$\iff \; P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$$

Example.

$$P = \begin{pmatrix} 0.97 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.05 & 0.92 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.04 & 0.92 & 0.04 & 0.00 \\ 0.00 & 0.00 & 0.04 & 0.94 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.01 & 0.99 \end{pmatrix}$$

Transition probabilities:



More formally, let

- $(X_t)_{t \geqslant 0}$ be a sequence of X-valued random variables and
- P be in $\mathcal{M}(\mathbb{R}^{\mathsf{X}})$

Def. We call $(X_t)_{t \geqslant 0}$ P-**Markov** if

$$\mathbb{P}\{X_{t+1} = x' \mid X_0, \ldots, X_t\} = P(X_t, x') \quad \text{for all} \quad t \geqslant 0,\ x' \in \mathsf{X}$$

Terminology:

- $(X_t)_{t \geqslant 0}$ is a **Markov chain**
- P is the **transition matrix** of $(X_t)_{t \geqslant 0}$
- X_0 and/or its distribution ψ_0 is the **initial condition**

This algorithm yields a P-Markov chain with initial condition ψ_0

$t \leftarrow 0$

$X_t \leftarrow$ a draw from ψ_0

while $t < \infty$ **do**

  $X_{t+1} \leftarrow$ a draw from the distribution $P(X_t, \cdot)$

  $t \leftarrow t + 1$

end

Application: Day Laborer

A worker is either

- unemployed ($X_t = 1$) or
- employed ($X_t = 2$) each day

Transitions:

- In state 1 he is hired with probability $\alpha \in (0, 1)$
- In state 2 he is fired with probability $\beta \in (0, 1)$

The corresponding state space and transition matrix are

$$\mathsf{X} = \{1, 2\} \quad \text{and} \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

```julia
function create_laborer_model(; α=0.3, β=0.2)
    return (; α, β)
end

function laborer_update(x, model)  # update X from t to t+1
    (; α, β) = model
    if x == 1
        x′ = rand() < α ? 2 : 1
    else
        x′ = rand() < β ? 1 : 2
    end
    return x′
end
```

Application: S-s Dynamics

Consider a firm whose inventory behavior follows S-s dynamics

Meaning:

- firm waits until its inventory falls below some level $s > 0$
- then replenishes by buying some fixed amount S

Reasonable if ordering inventory involves a fixed cost

(We will see this behavior later in a DP problem with fixed costs)

Inventory $(X_t)_{t \geqslant 0}$ obeys

$$X_{t+1} = \max\{X_t - D_{t+1}, 0\} + S\mathbb{1}\{X_t \leqslant s\}$$

where

- demand $(D_t)_{t \geqslant 1}$ is $\overset{\text{IID}}{\sim} \varphi \in \mathcal{D}(\mathbb{Z}_+)$
- $S =$ amount of stock ordered when inventory $\leqslant s$

We assume φ obeys the geometric distribution:

$$\varphi(d) = \mathbb{P}\{D_t = d\} = p(1-p)^d \text{ for } d \in \mathbb{Z}_+$$

We take $\mathsf{X} := \{0, \ldots, S + s\}$ to be the state space

Ex. Show that X satisfies

$$X_t \in \mathsf{X} \implies \mathbb{P}\{X_{t+1} \in \mathsf{X}\} = 1$$

Proof: Let $X_t = x \in S$, so that

$$X_{t+1} = \max\{x - D_{t+1}, 0\} + S\mathbb{1}\{x \leqslant s\}$$

Evidently $X_{t+1} \in \mathbb{Z}_+$. Also,

$$x \leqslant s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} + S \leqslant s + S$$

and

$$s < x \leqslant S + s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} \leqslant x \leqslant s + S$$

Ex. Show that X satisfies

$$X_t \in \mathsf{X} \implies \mathbb{P}\{X_{t+1} \in \mathsf{X}\} = 1$$

<u>Proof</u>: Let $X_t = x \in S$, so that

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and

$$s < x \leqslant S + s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} \leqslant x \leqslant s + S$$

If

$$h(x,d) = \max\{x - d, 0\} + S\mathbb{1}\{x \leqslant s\}$$

then

$$X_{t+1} = h(X_t, D_{t+1}) \quad \text{for all } t \geqslant 0$$

The transition matrix can be expressed as

$$P(x, x') = \mathbb{P}\{h(x, D_{t+1}) = x'\}$$

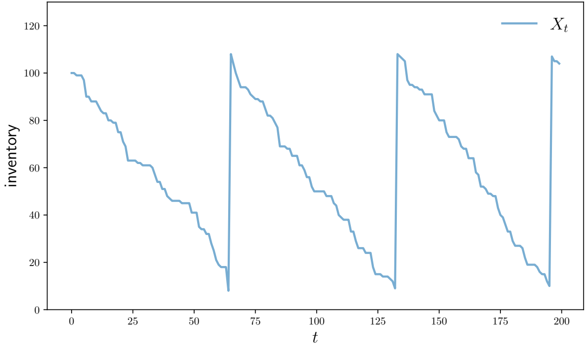
$$= \sum_{d \geqslant 0} \mathbb{1}\{h(x,d) = x'\}\varphi(d)$$

(In calculations we truncate the sum)

```julia
using Distributions, QuantEcon, IterTools

function create_inventory_model(; S=100,  # Order size
                                  s=10,   # Order threshold
                                  p=0.4)  # Demand parameter
    ϕ = Geometric(p)
    h(x, d) = max(x - d, 0) + S * (x <= s)
    return (; S, s, ϕ, h)
end

"Simulate the inventory process."
function sim_inventories(model; ts_length=200)
    (; S, s, ϕ, h) = model
    X = Vector{Int32}(undef, ts_length)
    X[1] = S  # Initial condition
    for t in 1:(ts_length-1)
        X[t+1] = h(X[t], rand(ϕ))
    end
    return X
end
```



X_t
inventory
t
0
20
40
60
80
100
120
0
25
50
75
100
125
150
175
200

Multistep transitions

Fix finite X and $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

- P^k (k-th power) is called the **k-step transition matrix**
- $P^k(x, x') :=$ the (x, x')-th element of P^k

Claim:

$$P^k(x, x') = \mathbb{P}\{X_{t+k} = x' \mid X_t = x\} \quad \text{for any } P\text{-chain } (X_t)_{t \geqslant 0}$$

Proof: Fix $x, x' \in \mathsf{X}$ and $t \geqslant 0$

True by definition when $k = 1$

Multistep transitions

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Proof: Fix $x, x' \in \mathsf{X}$ and $t \geqslant 0$

True by definition when $k = 1$

Now suppose true at k, so that

$$P^k(x, x') = \mathbb{P}\{X_{t+k} = x' \mid X_t = x\}$$

By the law of total probability, we have

$$\begin{aligned}
&\mathbb{P}\{X_{t+k+1} = x' \mid X_t = x\} \\
&\qquad = \sum_z \mathbb{P}\{X_{t+k+1} = x' \mid X_{t+k} = z\}\mathbb{P}\{X_{t+k} = z \mid X_t = x\}
\end{aligned}$$

Therefore

$$\mathbb{P}\{X_{t+k+1} = x' \mid X_t = x\} = \sum_z P^k(x, z)P(z, x') = P^{k+1}(x, x')$$

Irreducible Markov chains

Lemma. The following statements are equivalent:

1. $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is irreducible
2. For any P-chain (X_t) and any $x, x' \in \mathsf{X}$, $\exists\, k \geqslant 0$ s.t.

$$\mathbb{P}\{X_k = x' \,|\, X_0 = x\} > 0$$

Proof:

$$\sum_{k \geqslant 0} P^k \gg 0 \iff \forall\, x, x' \in \mathsf{X},\ \exists\, k \geqslant 0 \text{ s.t. } P^k(x, x') > 0$$

$$\iff \text{statement 2}$$

Marginals

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and P-chain (X_t), let $\psi_t \stackrel{d}{:=} X_t$ for all t

By the law of total probability, for all $x, x' \in \mathsf{X}$,

$$\mathbb{P}\{X_{t+1} = x'\} = \sum_{x \in \mathsf{X}} \mathbb{P}\{X_{t+1} = x' \mid X_t = x\} \mathbb{P}\{X_t = x\}$$

Equivalently,

$$\psi_{t+1}(x') = \sum_{x \in \mathsf{X}} P(x, x') \psi_t(x) \quad \text{for all } x \in \mathsf{X}$$

Treating each ψ_t as a <u>row</u> vector, we get $\psi_{t+1} = \psi_t P$

Stationarity

Distributions update via $\psi_{t+1} = \psi_t P$

Recall also that ψ^* is called **stationary** for P if $\psi^* = \psi^* P$

Now we can interpret this expression

If ψ^* is stationary for P, then

$$X_t \stackrel{d}{=} \psi^* \implies X_{t+1} \stackrel{d}{=} \psi^*$$

We recall that

- each $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ has at least one stationary distribution, and
- uniqueness in $\mathcal{D}(\mathsf{X})$ holds whenever P is irreducible

Ergodicity

Theorem. Let P be irreducible with stationary distribution ψ^*

For any P-Markov chain (X_t) and any $x \in \mathsf{X}$, we have

$$\mathbb{P}\left\{\lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\} = \psi^*(x)\right\} = 1$$

Meaning: For (almost) every P-Markov chain that we generate,

fraction of time chain in state $x \approx \psi^*(x)$

Markov chains with this property are said to be **ergodic**

Example. Recall the model

$$\mathsf{X} = \{1, 2\}, \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Ex. Assuming $0 < \alpha, \beta \leqslant 1$, show that

$$\psi^* := \frac{1}{\alpha + \beta} \begin{pmatrix} \beta & \alpha \end{pmatrix} = \text{ the unique stationary distribution}$$

Since P is irreducible, ergodicity holds:

$$\mathbb{P}\left\{ \lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\} = \psi^*(x) \right\} = 1$$

Ex. Using simulation, confirm that, for large k,

$$\frac{1}{k}\sum_{t=x}^{k} \mathbb{1}\{X_t = x\} \approx \psi^*(x) \qquad x = 1, 2$$

Check: this convergence does not depend on distribution of X_0

```julia
function sim_chain(k, p, model)
    X = Array{Int32}(undef, k)
    X[1] = rand() < p ? 1 : 2
    for t in 1:(k-1)
        X[t+1] = laborer_update(X[t], model)
    end
    return X
end

function test_convergence(; k=10_000_000, p=0.5)
    model = create_laborer_model()
    (; α, β) = model
    ψ_star = (1/(α + β)) * [β α]

    X = sim_chain(k, p, model)
    ψ_e = (1/k) * [sum(X .== 1)  sum(X .== 2)]
    error = maximum(abs.(ψ_star - ψ_e))
    approx_equal = isapprox(ψ_star, ψ_e, rtol=0.01)
    println("Sup norm deviation is $error")
    println("Approximate equality is $approx_equal")
 end
```

And now in Python

```
import numpy as np
from collections import namedtuple
from numba import njit, int32

Model = namedtuple("Model", ("α", "β"), defaults=(0.3, 0.2))

@njit
def laborer_update(x, model):  # update X from t to t+1
    α, β = model
    if x == 1:
        y = 2 if np.random.rand() < α else 1
    else:
        y = 1 if np.random.rand() < β else 2
    return y
```

```python
@njit
def sim_chain(k, p, model):
    X = np.empty(k, dtype=int32)
    X[0] = 1 if np.random.rand() < p else 2
    for t in range(k-1):
        X[t+1] = laborer_update(X[t], model)
    return X

@njit
def test_convergence(model, k=10_000_000, p=0.5):
    α, β = model
    ψ_star = (1/(α + β)) * np.array((β,  α))

    X = sim_chain(k, p, model)
    ψ_e = (1/k) * np.array((sum(X == 1), sum(X == 2)))
    error = np.max(np.abs(ψ_star - ψ_e))
    return error

model = Model()
error = test_convergence(model)
print(f"Sup norm deviation is {error}")
```

The next slide shows the results of an analogous simulation for the inventory model (slide 11)

The bar plot shows mean occupation time

$$M(x) := \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\}$$

for each inventory level x when k is large

The black line shows the stationary distribution ψ^*

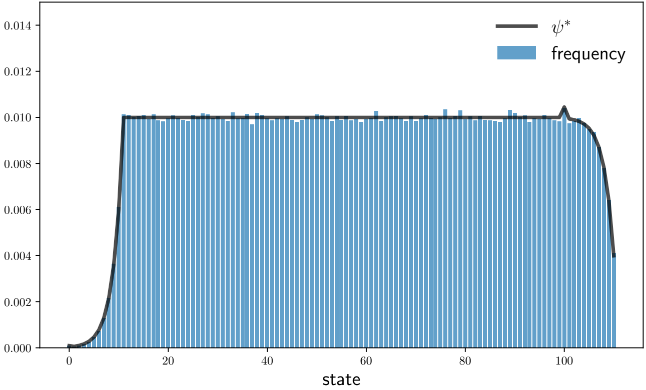

Figure: Ergodicity in the inventory model

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ with $P \gg 0$ and let ψ^* obey $\psi^* = \psi^* P$

Ex. Prove: $\psi P^t \to \psi^*$ as $t \to \infty$ for any $\psi \in \mathcal{D}(\mathsf{X})$

Proof: Since P is positive and $\rho(P) = 1$, PF theorem implies

$$P^t \to e \, \varepsilon \text{ as } t \to \infty,$$

where

- e is the dominant right eigenvector
- ε is the dominant left eigenvector
- the vectors are normalized s.t. $\langle e, \varepsilon \rangle = 1$

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where

- e is the dominant right eigenvector
- ε is the dominant left eigenvector
- the vectors are normalized s.t. $\langle e, \varepsilon \rangle = 1$

In the current setting,

- $\mathbb{1}$ is the dominant right eigenvector and
- ψ^* is the dominant left eigenvector

Hence

$$P^t \to \mathbb{1}\,\psi^* \quad \text{as} \quad t \to \infty$$

Thus, for any $\psi \in \mathcal{D}(\mathsf{X})$,

$$\psi P^t \to \psi \mathbb{1}\,\psi^* = \psi^*$$

Recall the model

$$\mathsf{X} = \{1, 2\}, \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \text{and} \quad \psi^* := \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \end{pmatrix}$$

Ex. Fix $\alpha = 0.3$ and $\beta = 0.2$

Compute the sequence (ψP^t) for different choices of ψ

Confirm that your results are consistent with the claim that

$$\psi P^t \to \psi^* \text{ as } t \to \infty \text{ for any } \psi \in \mathcal{D}(\mathsf{X})$$

Approximation

It can be helpful to reduce continuous state Markov models to finite state models

For example, suppose that $(X_t)_{t \geqslant 0}$ evolves in $\mathbb{R}$ according to

$$X_{t+1} = \rho X_t + b + \nu \varepsilon_{t+1}, \quad (\varepsilon_t) \overset{\text{IID}}{\sim} N(0,1). \tag{1}$$

This is a **linear Gaussian AR(1)** model

To approximate it we use **Tauchen's method**

We assume throughout that $|\rho| < 1$

Under this assumption, (1) has a unique **stationary distribution** ψ^* given by

$$\psi^* = N(\mu_x, \sigma_x^2) \quad \text{with} \quad \mu_x := \frac{b}{1-\rho} \quad \text{and} \quad \sigma_x^2 := \frac{\nu^2}{1-\rho^2}$$

This means that ψ^* has the following property:

$$X_t \stackrel{d}{=} \psi^* \text{ and } X_{t+1} = \rho X_t + b + \nu \varepsilon_{t+1} \text{ implies } X_{t+1} \stackrel{d}{=} \psi^*$$

Ex. Prove this. Hints: When $X_t \stackrel{d}{=} \psi^*$,

- is X_{t+1} normally distributed?
- what is its mean and variance?

Tauchen's discretization method

We start with the case $b = 0$

Fix $m, n \in \mathbb{N}$

Create state space $\mathsf{X} := \{x_1, \ldots, x_n\}$ via

- set $x_1 = -m\,\sigma_x$,
- set $x_n = m\,\sigma_x$ and
- set $x_{i+1} = x_i + s$ for $i \in \{1, \ldots, n-1\}$ where

$$s = \frac{x_n - x_1}{n - 1}$$

A grid that brackets the stationary mean on both sides by m standard deviations:

Create an $n \times n$ matrix P such that, For $i, j \in [n]$,

1. if $j = 1$, then set $P(x_i, x_j) = F(x_1 - \rho x_i + s/2)$
2. If $j = n$, then set $P(x_i, x_j) = 1 - F(x_n - \rho x_i - s/2)$
3. Otherwise, set

$$P(x_i, x_j) = F(x_j - \rho x_i + s/2) - F(x_j - \rho x_i - s/2)$$

Finally, if $b \neq 0$, then replace x_i with $x_i + \mu_x$ for each i

- shift the grid X to be centered on the stationary mean

```julia
using QuantEcon

ρ, b, ν = 0.9, 0.0, 1.0
μ_x = b/(1-ρ)
σ_x = sqrt(ν^2 / (1-ρ^2))

n = 15
mc = tauchen(n, ρ, ν)
approx_sd = stationary_distributions(mc)[1]

function psi_star(y)
    c = 1/(sqrt(2 * pi) * σ_x)
    return c * exp(-(y - μ_x)^2 / (2 * σ_x^2))
end
```

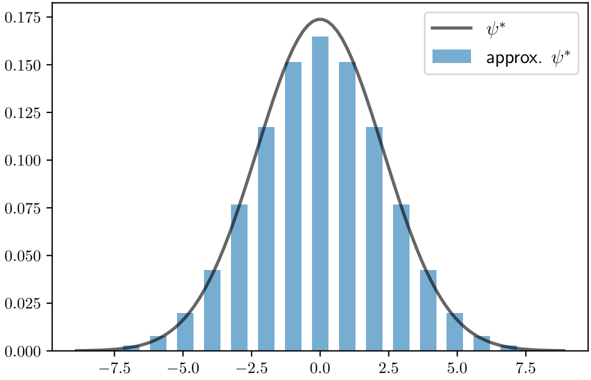

Figure: Comparison of $\psi^* = N(\mu_x, \sigma_x^2)$ and its discrete approximant

Conditional Expectations

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

For each $h \in \mathbb{R}^{\mathsf{X}}$ and $x \in \mathsf{X}$, we define

$$(Ph)(x) := \sum_{x' \in \mathsf{X}} h(x') P(x, x')$$

Equivalently

$$(Ph)(x) = \mathbb{E}[h(X_{t+1}) \,|\, X_t = x] \quad \text{when } (X_t) \text{ is } P\text{-Markov}$$

This interpretation extends to powers:

$$(P^k h)(x) = \sum_{x' \in \mathsf{X}} h(x') P^k(x, x') = \mathbb{E}[h(X_{t+k}) \,|\, X_t = x]$$

Note on conventions

When updating distributions we use row vectors:

$$\psi_{t+1}(x') = (\psi_t P)(x') = \sum_{x \in \mathsf{X}} P(x, x') \psi_t(x)$$

When taking conditional expectations we use column vectors:

$$(Ph)(x) := \sum_{x' \in \mathsf{X}} h(x') P(x, x')$$

The Law of Iterated Expectations

We now prove a version of the **law of iterated expectations**

Let (X_t) be P-Markov with $X_0 \stackrel{d}{=} \psi_0$

Fix $t, k \in \mathbb{N}$ and set $\mathbb{E}_t := \mathbb{E}[\cdot \mid X_t]$

We claim that

$$\mathbb{E}[\mathbb{E}_t[h(X_{t+k})]] = \mathbb{E}[h(X_{t+k})] \quad \text{for any } h \in \mathbb{R}^{\mathsf{X}}$$

(A special case of the general rule $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$)

To see this, recall that $\mathbb{E}[h(X_{t+k}) \mid X_t = x] = (P^k h)(x)$

Hence $\mathbb{E}[h(X_{t+k}) \mid X_t] = (P^k h)(X_t)$

Therefore,

$$\mathbb{E}[\mathbb{E}_t[h(X_{t+k})]] = \mathbb{E}[(P^k h)(X_t)]$$

$$= \sum_{x'} (P^k h)(x') \psi_t(x') = \sum_{x'} (P^k h)(x') (\psi_0 P^t)(x')$$

Since $\psi_0 P^t$ is a row vector, we can write the last expression as

$$\psi_0 P^t P^k h = \psi_0 P^{t+k} h = \psi_{t+k} h = \mathbb{E} h(X_{t+k})$$

Monotone Markov Chains

Let X be partially ordered by $\preceq$

$P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is called **monotone increasing** if

$$x, y \in \mathsf{X} \text{ and } x \preceq y \quad \implies \quad P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$$

Here $\preceq_{\mathrm{F}}$ is first order stochastic dominance, as defined in Ch. 2

Loosely speaking: when the current state goes up, the next period state is also likely to rise

Example. Consider the AR(1) model $X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}$

Apply Tauchen discretization, mapping to

- $n \times n$ stochastic matrix P on
- state space $\mathsf{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}$

Lemma. If $\rho \geqslant 0$ (+ve autocorrelation), then P is monotone increasing

Ex. Prove that P is monotone increasing if and only if P is invariant on $i\mathbb{R}^{\mathsf{X}}$

Proof of $\implies$

Suppose P is monotone increasing and fix $u \in i\mathbb{R}^{\mathsf{X}}$

We claim that $Pu \in i\mathbb{R}^{\mathsf{X}}$

To see this, pick any $x, y \in \mathsf{X}$ with $x \preceq y$

Since $P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$, we have

$$(Pu)(x) := \sum_{x'} u(x') P(x, x') \leqslant \sum_{x'} u(x') P(y, x') =: (Pu)(y)$$

Hence $Pu \in i\mathbb{R}^{\mathsf{X}}$, as was to be shown

Ex. Prove: If P is monotone increasing then so is P^t for all $t \in \mathbb{N}$

Proof: P is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Hence P^t is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Ex. Prove: If P is monotone increasing then so is P^t for all $t \in \mathbb{N}$

Proof: P is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Hence P^t is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Valuation

Solve optimization problems requires defining objectives clearly

Objective of dynamic programs = max **lifetime rewards**

Examples.

- lifetime wages for a worker
- lifetime utility for a consumer
- net present value for a firm

Let's start to look at computation of lifetime rewards

Fixed Discount Rates

Task: compute $\mathbb{E}$ of discounted future rewards

These sums take the form

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) := \mathbb{E} \left[\sum_{t \geqslant 0} \beta^t h(X_t) \,|\, X_0 = x \right]$$

Here

- $\beta \in \mathbb{R}_+$ and $h \in \mathbb{R}^{\mathsf{X}}$
- (X_t) is P-Markov on finite set X
- $\mathbb{E}_x$ indicates we are conditioning on $X_0 = x$

Example. Computing expected present value of a cash flow

Lemma. If $\beta \in (0, 1)$, then

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t)$$

is finite for all $x \in \mathsf{X}$, $I - \beta P$ is invertible and

$$v = \sum_{t \geqslant 0} (\beta P)^t h = (I - \beta P)^{-1} h \tag{2}$$

Proof: Observe that

$$\mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) = \sum_{t \geqslant 0} \beta^t \mathbb{E}_x h(X_t) = \sum_{t \geqslant 0} \beta^t (P^t h)(x)$$

Result (2) holds because $\rho(\beta P) < 1$ — why?

Application: A firm valuation problem

A firm receives profit stream $(\pi_t)_{t \geqslant 0}$ — a stochastic process

Valuation = expected present value of its profit stream:

$$V_0 = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t \quad \text{with} \quad \beta := \frac{1}{1+r}$$

Assume $\pi_t = \pi(X_t)$ where $(X_t)_{t \geqslant 0}$ is P-Markov, set

$$v(x) := \mathbb{E}_x \sum_{t=0}^{\infty} \beta^t \pi_t$$

Now $r > 0$ implies $v = (I - \beta P)^{-1} \pi$

Ex. Suppose

- X is partially ordered
- $\pi \in i\mathbb{R}^{\mathsf{X}}$ and P is monotone increasing

Prove that, under these conditions, v **is increasing on** X

Proof 1: Let π and P satisfy the stated conditions

Given $t \in \mathbb{N}$

- P monotone increasing implies P^t monotone increasing
- Since $\pi \in i\mathbb{R}^{\mathsf{X}}$, we see that $P^t \pi \in i\mathbb{R}^{\mathsf{X}}$

Hence $v = \sum_{t \geqslant 0} \beta^t P^t \pi$ is also increasing

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Proof 2: Let π and P satisfy the stated conditions

Rearranging $v = (I - \beta P)^{-1} \pi$ gives $v = \pi + \beta P v$

Hence v is the fixed point in $\mathbb{R}^{\mathsf{X}}$ of $Tv = \pi + \beta P v$

Since $\pi \in i\mathbb{R}^{\mathsf{X}}$ and P is monotone, T is invariant on $i\mathbb{R}^{\mathsf{X}}$

The operator T is a contraction of modulus β (Ex: check it)

Since $i\mathbb{R}^{\mathsf{X}}$ is closed, nonempty and invariant for T, the fixed point of T must lie in this set

In other words, v is increasing on X

Job Search Revisited

Now we return to the job search problem

Aims:

1. drop some of the restrictive assumptions we made earlier
2. analyze optimality

First extension: wage draws are correlated

- (W_t) is P-Markov on finite set $\mathsf{W} \subset \mathbb{R}_+$

The value function is denoted v^*

- $v^*(w) =$ max. lifetime value given current wage offer w

The value function satisfies the Bellman equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') P(w, w') \right\}$$

The corresponding Bellman operator is

$$(Tv)(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v(w') P(w, w') \right\}$$

Ex. Prove that T is an order-preserving self-map on $V := \mathbb{R}_+^{\mathsf{W}}$

Proof of the order-preserving property

Given $f, g \in V$ with $f \leqslant g$, we claim that $Tf \leqslant Tg$

Indeed, if $w \in \mathsf{W}$, then

$$\sum_{w'} f(w') P(w, w') \leqslant \sum_{w'} g(w') P(w, w')$$

It easily follows that $(Tf)(w) \leqslant (Tg)(w)$

Since w was arbitrary, we have $Tf \leqslant Tg$

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Set

$$\|f - g\|_\infty = \max_{w \in \mathsf{W}} |f(w) - g(w)|$$

Ex. Prove: T is a contraction of modulus β on V w.r.t. $\| \cdot \|_\infty$

Proof:

- Similar to the IID case
- Please complete as an exercise

Ex. Show that v^* is increasing on W whenever P is monotone increasing

Proof: Let $iV =$ all increasing functions in V

Since iV is closed, suffices to show that T is invariant on iV

Fix $v \in iV$

Then

- $h(w) := c + \beta(Pv)(w)$ is in iV and
- $e(w) := w/(1-\beta)$ is in iV

It follows that $Tv = e \vee h$ is in iV

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Fix $v \in V$

A policy $\sigma \colon \mathsf{W} \to \{0, 1\}$ is called v-**greedy** if

$$\sigma(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant c + \beta \sum_{w'} v(w')P(w, w')\right\}$$

- treat v as the value function, choose optimally

We use value function iteration (VFI) to solve the model

- Iterate from arbitrary $v \in V$ to get $v_k := T^k v$
- Compute the v_k-greedy policy

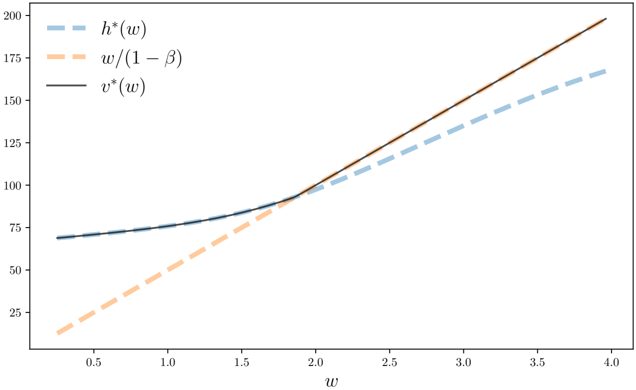
```julia
using QuantEcon, LinearAlgebra
include("s_approx.jl")

"Creates an instance of the job search model with Markov wages."
function create_markov_js_model(;
        n=200,       # wage grid size
        ρ=0.9,       # wage persistence
        ν=0.2,       # wage volatility
        β=0.98,      # discount factor
        c=1.0        # unemployment compensation
    )
    mc = tauchen(n, ρ, ν)
    w_vals, P = exp.(mc.state_values), mc.p
    return (; n, w_vals, P, β, c)
end
```

```julia
" The Bellman operator Tv = max{e, c + β P v} with e(w) = w / (1-β)."
function T(v, model)
    (; n, w_vals, P, β, c) = model
    h = c .+ β * P * v
    e = w_vals ./ (1 - β)
    return max.(e, h)
end

" Get a v-greedy policy."
function get_greedy(v, model)
    (; n, w_vals, P, β, c) = model
    σ = w_vals / (1 - β) .>= c .+ β * P * v
    return σ
end

"Solve the infinite-horizon Markov job search model by VFI."
function vfi(model)
    v_init = zero(model.w_vals)
    v_star = successive_approx(v -> T(v, model), v_init)
    σ_star = get_greedy(v_star, model)
    return v_star, σ_star
end
```

The **continuation value function** is given by

$$h^*(w) := c + \beta \sum_{w' \in \mathsf{W}} v^*(w') P(w, w')$$

- depends on w due to correlated wages



$h^*(w)$
$w/(1-\beta)$
$v^*(w)$
200
175
150
125
100
75
50
25
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
w

Ex. Explain why h^* is increasing in the last figure

Answer Since $\rho > 0$, P is monotone increasing

Hence $v^* \in iV$

Since $h^* = c + \beta P v^*$, it follows that $h^* \in iV$

Positive autocorrelation in wages means that

- high current wages predict high future wages
- value of waiting rises with current wages

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Positive autocorrelation in wages means that

- high current wages predict high future wages
- value of waiting rises with current wages

Job Search with Separation

Let's now allow for separation

- matches between workers and firms terminate with probability α every period

Other aspects of the problem are unchanged

Conditional on current offer w, let

- $v_u^*(w)$ = max lifetime value for unemployed worker
- $v_e^*(w)$ = max lifetime value for employed worker

We have

$$v_u^*(w) = \max \left\{ v_e^*(w),\ c + \beta \sum_{w'} v_u^*(w') P(w, w') \right\}$$

and

$$v_e^*(w) = w + \beta \left[\alpha \sum_{w'} v_u^*(w') P(w, w') + (1 - \alpha) v_e^*(w) \right]$$

Proposition If $0 < \alpha, \beta < 1$, then this system has a unique solution (v_u^*, v_e^*) in $V \times V$

Step one: solve for v_e^* as

$$v_e^*(w) = \frac{1}{1-\beta(1-\alpha)} \left(w + \alpha\beta (P v_u^*)(w)\right)$$

Substitute to get

$$v_u^*(w) = \max\left\{ \frac{1}{1-\beta(1-\alpha)} \left(w + \alpha\beta (P v_u^*)(w)\right),\, c + \beta \, (P v_u^*)(w) \right\}$$

Ex.

- Prove that $\exists$ a unique $v_u^* \in V$ that solves this equation
- Propose a convergent method for solving for both v_u^* and v_e^*

When unemployed, the stopping and continuation values are

$$s^*(w) := \frac{1}{1 - \beta(1 - \alpha)} (w + \alpha\beta(Pv_u^*)(w))$$

and

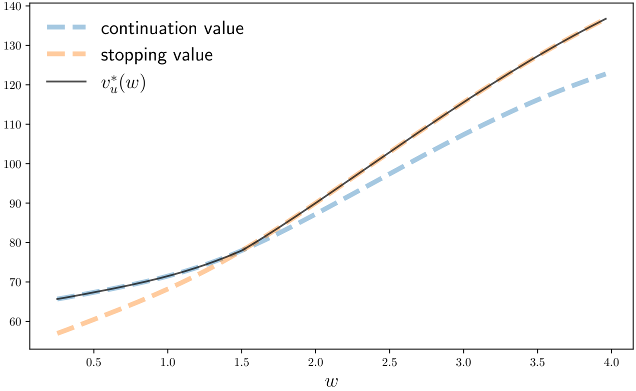
$$h^*(w) := c + \beta \, (Pv_u^*)(w)$$

Note $v_u^* = s^* \vee h^*$

Unemployed agent's optimal policy:

$$\sigma^*(w) := \mathbb{1}\{s^*(w) \geqslant h^*(w)\}$$

Reservation wage $w^* := \min\{w \in \mathsf{W} : s^*(w) \geqslant h^*(w)\}$

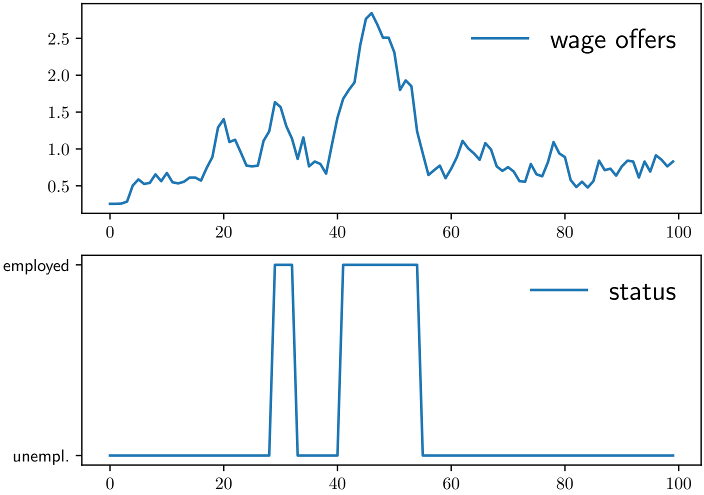


140
130
120
110
100
90
80
70
60
continuation value
stopping value
$v_u^*(w)$
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
w

```julia
include("markov_js_with_sep.jl")   # Code to solve model
using Distributions

# Create and solve model
model = create_js_with_sep_model()
(; n, w_vals, P, β, c, α) = model
v_star, σ_star = vfi(model)

# Create Markov distributions to draw from
P_dists = [DiscreteRV(P[i, :]) for i in 1:n]

function update_wages_idx(w_idx)
    return rand(P_dists[w_idx])
end
```

```
function sim_wages(ts_length=100)
    w_idx = rand(DiscreteUniform(1, n))
    W = zeros(ts_length)
    for t in 1:ts_length
        W[t] = w_vals[w_idx]
        w_idx = update_wages_idx(w_idx)
    end
    return W
end
```

```julia
function sim_outcomes(; ts_length=100)
    status = 0
    E, W = [], []
    w_idx = rand(DiscreteUniform(1, n))
    ts_length = 100
    for t in 1:ts_length
        if status == 0
            status = σ_star[w_idx] ? 1 : 0
        else
            status = rand() < α ? 0 : 1
        end
        push!(W, w_vals[w_idx])
        push!(E, status)
        w_idx = update_wages_idx(w_idx)
    end
    return W, E
end
```


wage offers
2.5
2.0
1.5
1.0
0.5
0
20
40
60
80
100
status
employed
unempl.
0
20
40
60
80
100

Ex. Here's an open-ended optional exercise

Let $E_t :=$ employment status

- Show that $X_t := (W_t, E_t)$ is a Markov chain
- Write down the state space and prove irreducibility

Let ψ^* be the unique stationary distribution

Ergodicity: fraction of time a worker spends unemployed should be equal to prob of unemployment under ψ^*

- Check it

Prob of unemployment under ψ^* equals unemployment rate

Adjust model parameters to match current unemployment rate