An impossibility theorem for wealth in heterogeneous-agent models with limited heterogeneity ☆

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Abstract

It has been conjectured that canonical Bewley–Huggett–Aiyagari heterogeneous-agent models cannot explain the joint distribution of income and wealth. The results stated below verify this conjecture and clarify its implications under very general conditions. We show in particular that if (i) agents are infinitely-lived, (ii) saving is risk-free, and (iii) agents have constant discount factors, then the wealth distribution inherits the tail behavior of income shocks (e.g., light-tailedness or the Pareto exponent). Our restrictions on utility require only that relative risk aversion is bounded, and a large variety of income processes are admitted. Our results show conclusively that it is necessary to go beyond standard models to explain the empirical fact that wealth is heavier-tailed than income. We demonstrate through examples that relaxing any of the above three conditions can generate Pareto tails.
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1. Introduction

When studying wealth inequality, one empirical feature stands out as striking and persistent over time and space: the wealth distribution exhibits a power law tail. This fact was first discovered by Pareto (1896, 1897) and has since been confirmed by many studies.[1] A closely related observation is that the income distribution is also heavy-tailed, although its Pareto exponent is significantly larger, implying a heavier tail for wealth than income.[2]

It is well known in the quantitative macroeconomics literature that canonical Bewley (1977, 1983, 1986)–Huggett (1993)–Aiyagari (1994) models have difficulty in explaining the joint distribution of income and wealth. For example, Aiyagari (1994) documents that the wealth Gini coefficient is 0.32 in the model, while it is 0.8 in the data. Huggett (1996) notes that the model-implied top 1% wealth share is half of that in the data. Krueger et al. (2016) argue that idiosyncratic unemployment risk and incomplete financial markets alone cannot generate a sufficiently dispersed wealth distribution,[3] even though such dispersion is crucial for the study of aggregate fluctuations. More specifically, Benhabib et al. (2017) show that, in a setting where income has a Pareto tail and agents use a linear consumption rule, the Pareto exponent of wealth is either entirely determined by the distribution of returns on wealth, or equal to the Pareto exponent of income. They argue that similar results must obtain with rational agents with constant relative risk aversion (CRRA) preferences because, in such settings, the policy rules are asymptotically linear.[4]

In this paper we confirm and significantly extend this conjecture by showing that, for canonical Bewley–Huggett–Aiyagari models, all attempts to explain the large skewness of the wealth distribution are bound to fail. By the qualification "canonical", we mean models in which (i) agents are infinitely-lived, (ii) saving is risk-free, and (iii) agents have constant discount factors. In our main result (Theorem 8), we prove that the equilibrium wealth distribution inherits the tail behavior of income shocks in *any* such model. This is an impossibility theorem in the following sense: the tail thickness of the model output (wealth) cannot exceed that of the input (income). If income is light-tailed (e.g., bounded, Gaussian, exponential, etc.), so is wealth. If income is heavy-tailed, so is wealth, but with the same Pareto exponent, contradicting the empirical relationship between the income and wealth distributions stated above. Thus, one cannot produce a model consistent with the data without relaxing at least one of the assumptions (i)–(iii).

Our findings can be understood via the following intuition. In infinite-horizon dynamic general equilibrium models, the discount factor $\beta > 0$ and the gross risk-free rate $R > 0$ must satisfy the "impatience" condition $\beta R < 1$, for otherwise individual consumption diverges to infinity according to results in Chamberlain and Wilson (2000), which violates market clearing. But under this impatience condition, we show that rational agents consume more than what is implied by

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[1] See Gabaix (2009, 2016) for introductions to power laws in economics.

[2] The Pareto exponent for wealth is about 1.5 (Pareto, 1897; Klass et al., 2006; Vermeulen, 2018), versus 2–3 for income (Atkinson, 2003; Nirei and Souma, 2007; Toda, 2012). Since the tail probability of a Pareto random variable satisfies $P(X > x) \sim x^{-\alpha}$, where $\alpha > 0$ is the Pareto exponent, a smaller value for α corresponds to higher tail probability, implying a heavier tail (more inequality).

[3] The literature has extended the "canonical" Bewley–Huggett–Aiyagari model to explain the heavy-tailed wealth distribution by introducing random discount factors (Krusell and Smith, 1998) and idiosyncratic investment risk (Benhabib et al., 2011). We discuss this literature in Section 4.

[4] In a similar vein, Wang (2007) shows that the constant absolute risk aversion (CARA) utility augmented with the Uzawa (1968) discounting function generates a stationary wealth distribution and that the wealth distribution cannot be more skewed than income.

the permanent income hypothesis $c(a) = (1 - 1/R)a$ (the interest income), and the accumulation equation for wealth a_t becomes a "contraction" in the sense that

$$a_{t+1} \leq \rho a_t + y_{t+1} \tag{1}$$

for large enough a_t, where y_{t+1} is income and ρ is some positive constant strictly less than 1. This inequality implies that the income shocks die out in the long run, and hence the wealth distribution inherits the tail behavior of income shocks. To obtain (1), we use the results from Li and Stachurski (2014), who show the validity of policy function iteration for solving income fluctuation problems. With the bound (1) in hand, we characterize the tail behavior of wealth using the properties of the moment generating function and applying several inequalities such as Markov, Hölder, and Minkowski.

Relative to the work of Benhabib et al. (2017) discussed above, our contributions are as follows: First, we provide a complete proof of the impossibility result stated above in the context of an equilibrium model with rational, optimizing agents, thereby confirming their conjecture on optimizing households with CRRA utility in a general equilibrium setting. Second, our results are established in a class of models where relative risk aversion need not be constant. We require only that relative risk aversion is asymptotically bounded. This means that minor deviations from standard utility functions cannot reverse our results. Similarly, our income process is required only to have a finite mean. Third, we provide a complete analytical framework on tail thickness that accommodates both light-tailed and heavy-tailed distributions, and connect it to the joint distribution of income and wealth. In the sense that we handle all classes of shocks and allow for nonstationary additive processes, our proofs extend what is contained in the related mathematical literature, such as the work of Grey (1994) on Pareto tails.[5] Moreover, our proofs are almost completely self contained, and hence can be readily adapted to subsequent research on income and wealth distributions that tackles extensions to our framework.

To tie up loose ends, we also show that the conditions of the impossibility theorem are tight. In Section 4, we show through examples that relaxing any of the three assumptions behind our main theorem (agents are infinitely-lived, saving is risk-free, and the discount factor is constant) can generate Pareto-tailed wealth distributions. In doing so, we draw on existing literature as it pertains to this topic and also provide a simple, exactly solved model that features heterogeneous discount factors and a Pareto-tailed wealth distribution.[6]

1.1. Other literature

Our work provides a logical converse to the findings of the growing literature that relies on idiosyncratic investment risk (as opposed to earnings risk) to explain the Pareto tail behavior in the wealth distribution.

As mentioned in the introduction, it has been known at least since Aiyagari (1994) that canonical Bewley–Huggett–Aiyagari models have difficulty in matching the empirical wealth distribution. While the vast majority of papers in this literature are numerical, several authors have theoretically shown that the wealth distribution is bounded under certain assumptions.

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[5] The extra generality means that our results are less sharp in other directions, although not in a way that impinges on our main findings. See the remark on page 7 for more details.

[6] This theory connects to several applied studies, such as Carroll et al. (2017) and McKay (2017), which use numerically solved heterogeneous-agent quantitative models with several agent types and different discount factors to generate skewed wealth distributions.

Schechtman and Escudero (1977, Theorems 3.8, 3.9) show the boundedness of wealth when income is independent and identically distributed (i.i.d.) with a bounded support and the utility function exhibits asymptotically constant relative risk aversion (CRRA). Aiyagari (1993, Proposition 4) relaxes the assumption on the utility function to bounded relative risk aversion (BRRA). Huggett (1993) proves the boundedness of wealth when utility is CRRA and income is a two-state Markov chain with a certain monotonicity property. To prove the existence of a stationary equilibrium in an Aiyagari economy, building on results in Li and Stachurski (2014) as we do, Açıkgöz (2018, Proposition 4) proves that wealth is bounded when the income process is a finite-state Markov chain and the absolute risk aversion coefficient converges to 0 as agents get richer (which is a weaker condition than BRRA). Achdou et al. (2017, Proposition 3) show a similar result in a continuous-time setting under the BRRA assumption.

Our contribution relative to this literature is that we do away with the boundedness assumption on income and prove, in a fully micro-founded general equilibrium setting, that wealth inherits the tail behavior of income, whether it be light-tailed or heavy-tailed. This is a significant contribution, for showing the boundedness alone does not tell us much about the tail behavior because any unbounded distributions (with potentially different tail properties) can be approximated by bounded ones.

The key to proving our main result is to show that under the impatience condition $\beta R < 1$, agents uniformly consume more than the interest income (Proposition 5), which implies the AR(1) upper bound (1) as a consequence of individual optimization and equilibrium considerations. This component of our paper is related to Carroll and Kimball (1996) and Jensen (2018), who prove the concavity of the consumption function when the utility function exhibits hyperbolic absolute risk aversion (HARA). The concavity of consumption implies a linear lower bound (though not necessarily as tight as $c(a) \geq ma$ with $m > 1 - 1/R$), which we exploit to obtain an AR(1) upper bound on wealth accumulation as in (1). In contrast to these papers, we obtain the linear lower bound on consumption under BRRA, which is a much weaker condition than HARA.

Finally, our paper is related to Benhabib et al. (2015), who show that a Bewley–Huggett–Aiyagari model with idiosyncratic investment risk can generate a Pareto-tailed wealth distribution. To obtain their possibility result, they derive a bound on wealth accumulation similar to (1) by assuming that agents have CRRA utility, and that earnings and investment risks are mutually independent and i.i.d. over time. Our paper is different in that (i) we focus on the impossibility result in the absence of financial risk, and (ii) we consider a less restrictive environment, requiring only bounded (rather than constant) relative risk aversion and minimal restrictions on nonfinancial income.

2. Tail thickness via moment generating function

In this section we define several notions of tail thickness of random variables using the moment generating function. To this end, recall that, for a random variable X defined on some probability space $(\Omega, \mathcal{F}, P)$, the moment generating function of X is defined at $s \in \mathbb{R}$ by $M_X(s) = E[e^{sX}] \in (0, \infty]$. We define light- and heavy-tailed random variables as follows:

Definition 1 *(Tail thickness).* We say that a random variable X has a *light* upper tail if $M_X(s) = E[e^{sX}] < \infty$ for some $s > 0$. Otherwise we say that X has a *heavy* upper tail.

Remark. One can justify this definition as follows. A random variable X is commonly referred to as having a heavy (Pareto) upper tail if there exist constants $A, \alpha > 0$ such that $P(X > x) \geq Ax^{-\alpha}$ for large enough x, where α is the Pareto exponent. Since for $y \geq 0$ we have $e^y \geq y^n/n!$ for any n by considering the Taylor expansion, for any $s, x > 0$ by Markov's inequality we obtain

$$E[e^{sX}] \geq E[e^{sX}1_{X>x}] \geq E[e^{sx}1_{X>x}] = e^{sx}P(X > x) \geq \frac{(sx)^n}{n!}Ax^{-\alpha}.$$

Taking $n > \alpha$ and letting $x \to \infty$, we obtain $E[e^{sX}] = \infty$.

The following lemma shows that the tail probability of a light-tailed random variable has an exponential upper bound.

Lemma 2. *If X is a light-tailed random variable, then for all $x \in \mathbb{R}$ we have*

$$P(X > x) \leq M_X(s)e^{-sx}. \tag{2}$$

Proof. This is immediate from Markov's inequality, since, for any x, we have

$$M_X(s) = E[e^{sX}] \geq E[e^{sX}1_{X>x}] \geq E[e^{sx}1_{X>x}] = e^{sx}P(X > x). \quad \square$$

The moment generating function $M_X(s)$ is convex and $M_X(0) = 1$. Therefore the set $\{s \in \mathbb{R} \mid M_X(s) < \infty\}$ is an interval containing 0, so

$$\lambda = \sup\{s \geq 0 \mid M_X(s) < \infty\} \in [0, \infty] \tag{3}$$

is well-defined. Taking the logarithm of (2) for $s \in (0, \lambda)$, dividing by $x > 0$, and letting $x \to \infty$, we obtain

$$\limsup_{x \to \infty} \frac{1}{x} \log P(X > x) \leq -s.$$

Therefore letting $s \uparrow \lambda$, it follows that

$$\limsup_{x \to \infty} \frac{1}{x} \log P(X > x) \leq -\lambda. \tag{4}$$

Using the argument in Widder (1941, pp. 42–43, Theorem 2.4a), one can easily show that the inequality (4) is actually an equality, although this fact plays no role in the subsequent discussion. Motivated by (4), we call the number λ in (3) the *exponential decay rate* of the random variable X.

Next we categorize heavy-tailed random variables. Since the logarithm of a Pareto random variable is exponential, it is convenient to define the tail thickness based on the logarithm. Let X be a heavy-tailed random variable and $X_+ = X1_{X \geq 0}$ be its positive part. The moment generating function of $\log X_+$ is

$$M_{\log X_+}(s) = E[e^{s \log X_+}] = E[X_+^s],$$

which is a convex function that is finite at $s = 0$.[7] By the same argument as above,

$$\alpha = \sup\{s \geq 0 \mid E[X_+^s] < \infty\} \in [0, \infty] \tag{5}$$

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[7] By convention, we set $0^0 = 1$. Therefore $E[X_+^s] \in (0, \infty]$ is well-defined for $s \geq 0$ and $E[X_+^0] = 1$.

is well-defined and we have the property

$$\limsup_{x \to \infty} \frac{\log P(X > x)}{\log x} = -\alpha.$$

We call α the *polynomial decay rate* of the random variable X.

So far we have defined the tail thickness of a single random variable X, but as we shall see below, it is convenient to define similar concepts for a class of random variables. Let $\mathcal{T}$ be some nonempty set and $(X_t)_{t \in \mathcal{T}}$ be a collection of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Then we say that $(X_t)_{t \in \mathcal{T}}$ is *uniformly light-tailed* if there exists $s > 0$ such that

$$M_{\mathcal{T}}(s) := \sup_{t \in \mathcal{T}} E[e^{s X_t}] < \infty.$$

If $(X_t)_{t \in \mathcal{T}}$ is uniformly light-tailed, then it immediately follows from Lemma 2 that

$$\sup_{t \in \mathcal{T}} P(X_t > x) \leq M_{\mathcal{T}}(s) e^{-sx}$$

for all x. Therefore the tail probabilities of $(X_t)_{t \in \mathcal{T}}$ can be uniformly bounded by an exponential function. By taking the supremum over such $s > 0$, we can define the exponential decay rate λ of $(X_t)_{t \in \mathcal{T}}$. We can similarly define uniformly heavy-tailed random variables and their polynomial decay rate α in the obvious way.

The following theorem, which is used in the proof of our main result, shows that a stochastic process that has a certain contraction property inherits the tail behavior of the shocks.

Theorem 3. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that (i) ϕ is bounded on any bounded set, and (ii) $\rho := \limsup_{x \to \infty} \phi(x)/x < 1$. Let $X_0 \geq 0$ be some real number and $\{X_t, Y_t\}_{t=1}^{\infty}$ be a nonnegative stochastic process such that*

$$X_t \leq \phi(X_{t-1}) + Y_t \tag{6}$$

for all $t \geq 1$. Then the following statements are true.

1. *If $\{Y_t\}_{t=1}^{\infty}$ has a compact support, then so does $\{X_t\}_{t=1}^{\infty}$.*
2. *If $\{Y_t\}_{t=1}^{\infty}$ is uniformly light-tailed with exponential decay rate λ, then $\{X_t\}_{t=1}^{\infty}$ is uniformly light-tailed with exponential decay rate $\lambda' \geq (1 - \rho)\lambda$.*
3. *If $\sup_t E[Y_t] < \infty$ and $\{Y_t\}_{t=1}^{\infty}$ is uniformly heavy-tailed with polynomial decay rate α, then $\{X_t\}_{t=1}^{\infty}$ has a polynomial decay rate $\alpha' \geq \alpha$.*

Remark. It could be the case that $\{Y_t\}_{t=1}^{\infty}$ is heavy-tailed but $\{X_t\}_{t=1}^{\infty}$ is light-tailed. An obvious example is $\phi(x) \equiv 0$ and $X_t \equiv 0$, in which case the polynomial decay rate of X_t is $\alpha' = \infty$.

Remark. The lower bounds on the tail exponents in Theorem 3 are sharp. To see this, suppose that $\rho \in [0, 1)$, $X_0 = 0$, $X_t = \rho X_{t-1} + Y_t$, and $\{Y_t\}_{t=1}^{\infty}$ is perfectly correlated, so $Y_t = Y_1$ for all t. By iteration, we obtain

$$X_t = Y_t + \rho Y_{t-1} + \cdots + \rho^{t-1} Y_1 = \frac{1 - \rho^t}{1 - \rho} Y_1.$$

Therefore $X_t \to X = \frac{1}{1-\rho}Y_1$ as $t \to \infty$ almost surely. Hence if Y_1 is exponentially-distributed with decay rate λ (i.e., Y_1 has density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$), then X is exponentially distributed with decay rate $(1-\rho)\lambda$. If Y_1 is Pareto-distributed with Pareto exponent α and minimum size 1 (i.e., Y_1 has density $f(x) = \alpha x^{-\alpha-1}$ for $x \geq 1$), then X is Pareto-distributed with Pareto exponent α and minimum size $\frac{1}{1-\rho}$.

Remark. Grey (1994) studies the Kesten (1973) process

$$X_t = A_t X_{t-1} + Y_t \tag{7}$$

when $\{A_t, Y_t\}_{t=1}^{\infty}$ is i.i.d. and shows under some assumptions that X_t and Y_t have the same Pareto exponent (implying the same polynomial decay rate). Ghosh et al. (2010) extend this result to the Markovian case. If we set $\phi(x) = \rho x$ in (6) and assume that Y_t is Markovian, then Theorem 3(3) can be strengthened by setting $A_t = \rho$ in (7). On the other hand, Theorem 3 allows us to (i) treat the bounded, light-tailed, and heavy-tailed cases simultaneously using elementary arguments and (ii) handle potential non-stationarity in $\{Y_t\}_{t=1}^{\infty}$.

Remark. Mirek (2011) studies the nonlinear recursion $X_t = \psi(X_{t-1}, \theta_t)$ when $\{\theta_t\}$ is i.i.d. and shows that X_t is heavy-tailed when ψ is asymptotically linear in the sense that $M(\theta) = \lim_{x \to \infty} \psi(x, \theta)/x$ exists and $|\psi(x, \theta) - M(\theta)x| \leq Y(\theta)$ for some $Y(\theta)$. Some key assumptions are (i) $M(\theta) > 1$ with positive probability (ensuring random growth, see Assumption H5) and (ii) the "additive" term $Y(\theta)$ does not dominate the "multiplicative" term $M(\theta)$ (see Assumption H7). Since our focus is impossibility of heavier tails without multiplicative risk, the assumption $\limsup_{x \to \infty} \phi(x)/x < 1$ in Theorem 3 suffices. Benhabib et al. (2015) apply the Mirek (2011) result to show that wealth is heavy-tailed.

3. Wealth accumulation and tail behavior

In this section we show that the wealth accumulation equation satisfies the AR(1) upper bound (1) under the impatience condition $\beta R < 1$ and other weak conditions, which allows us to prove that wealth inherits the tail behavior of income.

We consider the following income fluctuation problem (Schechtman and Escudero, 1977):

$$\text{maximize} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{8a}$$

$$\text{subject to} \quad a_{t+1} = R(a_t - c_t) + y_{t+1}, \tag{8b}$$

$$0 \leq c_t \leq a_t, \tag{8c}$$

where $u : \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is the utility function, $\beta > 0$ is the discount factor, $R > 0$ is the gross risk-free rate, $y_t \geq 0$ is income, a_t is financial wealth at the beginning of period t including current income, and the initial wealth $a_0 > 0$ is given. The constraint (8c) implies that consumption must be nonnegative and the agent cannot borrow. The no borrowing assumption is without loss of generality as discussed in Chamberlain and Wilson (2000) and Li and Stachurski (2014).

We impose standard assumptions on the utility function.

Assumption 1. The utility function is twice continuously differentiable on $\mathbb{R}_{++} = (0, \infty)$ and satisfies $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$.[8]

Regarding the income process, we introduce the following assumption.

Assumption 2. The income process $\{y_t\}$ takes the form $y_t = y(z_t)$, where $\{z_t\}$ is a Markov process on some set Z and $\sup_{z \in Z} E\left[y(z_{t+1}) \mid z_t = z\right] < \infty$.

Assumption 2 is relatively weak because we have not specified the state space Z. The income process can be very general: for example, it can accommodate nonstationary life-cycle features. The only important assumption is that income has a (uniformly) finite conditional mean, which is natural in a stationary general equilibrium environment.[9]

The reason why we assume that income is a function of a Markov process (not necessarily that income itself is Markovian) is for generality. For example, in the empirical literature on earnings dynamics, it is common to assume that income has a persistent-transitory decomposition

$$\log y_t = \xi_t + \eta_t,$$

where ξ_t and η_t are the persistent and transitory components, respectively (Ejrnæs and Browning, 2014). In this case $y_t = \exp(\xi_t + \eta_t)$ is a function of a Markov process $z_t = (\xi_t, \eta_t)$, but y_t itself may not be Markovian. More generally, Assumption 2 holds if $z_t = (\xi_t, \eta_t)$, $\{\xi_t\}$ follows a finite-state Markov chain, the distribution of η_t depends only on ξ_t, and $E\left[y(z_{t+1}) \mid \xi_t\right] < \infty$.

Due to strict concavity, the solution to the income fluctuation problem (8) is unique, if it exists. The following proposition shows that a solution exists under the "impatience" condition $\beta R < 1$ and that it can be computed by policy function iteration. This result is essentially due to Li and Stachurski (2014) (see Appendix B for details).

Proposition 4. *Suppose Assumptions 1 and 2 hold and $\beta R < 1$. Then there exists a unique consumption policy function $c(a, z)$ that solves the income fluctuation problem (8). Furthermore, we have $0 < c(a, z) \le a$, c is increasing in a, and $c(a, z)$ can be computed by policy function iteration.*[10]

Proof. Immediate from Lemmas 12–14 and the Banach fixed point theorem. □

To obtain the bound (1) for the wealth accumulation process, we assume that the utility function exhibits asymptotically bounded relative risk aversion (BRRA).[11]

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[8] These assumptions are stronger than necessary. It suffices that u is continuously differentiable on $(0, \infty)$, $u' > 0$, u' is strictly decreasing, and $u'(\infty) = 0$. In particular, we do not need the twice differentiability and the Inada condition $u'(0) = \infty$, although the proof becomes more involved.

[9] We use the uniform finiteness $\sup_{z \in Z} E\left[y(z_{t+1}) \mid z_t = z\right] < \infty$ to apply Theorem 3(3). Li and Stachurski (2014) prove the existence of a solution to the income fluctuation problem similar to (8) under a weaker finiteness assumption (see their Assumption 2.4).

[10] To be precise, $c(a, z)$ is the limit obtained by iterating the map $K : \mathcal{C} \to \mathcal{C}$ starting from any $c_0 \in \mathcal{C}$, where $\mathcal{C}$ is the set of candidate policy functions defined in Appendix B and $(Kc)(a, z)$ is the value $t \in (0, a]$ that satisfies the Euler equation (A.5).

[11] This assumption is stronger than necessary. It suffices to assume condition (A.6), which is slightly weaker than BRRA (see Lemma 11). However, we maintain BRRA because it is more intuitive and weak enough for all practical purposes.

Assumption 3. The utility function u satisfies $\limsup_{x \to \infty} \gamma(x) < \infty$, where

$$\gamma(x) := -\frac{xu''(x)}{u'(x)} > 0 \tag{9}$$

is the local relative risk aversion coefficient.

The widely used constant relative risk aversion (CRRA) utility clearly satisfies Assumption 3. More generally, let u be the hyperbolic absolute risk aversion (HARA) utility, so

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax + b}$$

for some $a, b \in \mathbb{R}$, where x takes values such that $ax + b > 0$. It is well known that the general functional form of HARA utility is

$$u(x) = \begin{cases} \frac{1}{a-1}(ax+b)^{1-1/a}, & (a \neq 0, 1) \\ -be^{-x/b}, & (a = 0, b > 0) \\ \log(x+b), & (a = 1) \end{cases} \tag{10}$$

up to an affine transformation. Since the relative risk aversion of u is

$$\gamma(x) = -\frac{xu''(x)}{u'(x)} = \frac{x}{ax+b},$$

we obtain

$$\limsup_{x \to \infty} \gamma(x) = \frac{1}{a} < \infty$$

if $a > 0$, so Assumption 3 holds. On the other hand, the constant absolute risk aversion (CARA) utility exhibits relative risk aversion $\gamma(x) = x/b \to \infty$ as $x \to \infty$, so it violates Assumption 3.

Remark. Schechtman and Escudero (1977) assume the following "asymptotic exponent" assumption:

$$\exists \gamma = \lim_{x \to \infty} -\frac{\log u'(x)}{\log x}. \tag{11}$$

Under twice continuous differentiability, the condition (11) is stronger than BRRA. To see this, by l'Hôpital's rule we have

$$\gamma = \lim_{x \to \infty} -\frac{\log u'(x)}{\log x} = \lim_{x \to \infty} -\frac{(\log u'(x))'}{(\log x)'} = \lim_{x \to \infty} -\frac{xu''(x)}{u'(x)} = \lim_{x \to \infty} \gamma(x),$$

so (11) implies that u is asymptotically CRRA (in particular, BRRA). Aiyagari (1993, Proposition 4) and Achdou et al. (2017, Proposition 3) assume that u is BRRA. On the other hand, BRRA is stronger than the assumption used in Rabault (2002), which is that the absolute risk aversion coefficient approaches 0:

$$\lim_{x \to \infty} -\frac{u''(x)}{u'(x)} = 0. \tag{12}$$

In fact, if u is BRRA, then

$$\lim_{x \to \infty} -\frac{u''(x)}{u'(x)} = \lim_{x \to \infty} \frac{\gamma(x)}{x} = 0$$

so (12) holds, but the converse is not true. (As a counterexample, take u such that $-u''(x)/u'(x) = x^{-\nu}$ for some $\nu \in (0, 1)$.)

The following proposition shows that under the impatience condition $\beta R < 1$, agents uniformly consume more than the interest income, which is related to the permanent income hypothesis.[12] We use this result to bound the wealth accumulation process from above.

Proposition 5. *Suppose Assumptions 1–3 hold and $1 \le R < 1/\beta$. Let*

$$\bar{\gamma} = \limsup_{x \to \infty} -\frac{x u''(x)}{u'(x)} \in [0, \infty)$$

be the asymptotic relative risk aversion coefficient and $c(a, z)$ be the optimal consumption rule for the income fluctuation problem established in Proposition 4. Then for all $m \in (1 - 1/R, 1 - \beta^{1/\bar{\gamma}} R^{1/\bar{\gamma}-1})$, there exists an asset level $A \ge 0$ such that, for all $a \ge A$ and $z \in Z$, we have

$$c(a, z) \ge ma. \tag{13}$$

The reason we need Assumption 3 is because we are working with arbitrary (potentially unbounded) income processes. Under Assumption 1, $\beta R < 1$, and (12), Rabault (2002, Lemma C.1) shows that the next period's wealth in the consumption-saving problem becomes infinitely smaller than the current wealth a as $a \to \infty$, which implies that a is bounded. Açıkgöz (2018, Proposition 2) proves a similar result by relaxing the limit in (12) to liminf. Although such arguments are enough for obtaining an upper bound for wealth when income is bounded, for the unbounded case we need $m > 1 - 1/R$, for which a stronger assumption such as BRRA is necessary. Proposition 5 not only tells us that we can take some such number m, but it also gives an explicit choice: any number $m \in (1 - 1/R, 1 - \beta^{1/\bar{\gamma}} R^{1/\bar{\gamma}-1})$ will do, where the bounds depend only on the discount factor β, the gross risk-free rate R, and the asymptotic relative risk aversion coefficient $\bar{\gamma}$.

Using Proposition 5, we can show that the wealth dynamics arising from the income fluctuation problem has the contraction property shown in (1) under the impatience condition $\beta R < 1$.

Proposition 6. *Suppose Assumptions 1–3 hold and $\beta R < 1$. Let $\{a_t\}$ be the wealth arising from the solution to the income fluctuation problem (8). Then the contraction property (1) holds for sufficiently high wealth level, and consequently the following statements are true:*

1. *If $\{y_t\}$ is uniformly light-tailed, then so is $\{a_t\}$.*
2. *If $\{y_t\}$ is uniformly heavy-tailed with polynomial decay rate α, then $\{a_t\}$ has polynomial decay rate $\alpha' \ge \alpha$.*

Furthermore, the coefficient $\rho \in (0, 1)$ in (1) can be chosen as follows:

- *If $R < 1$, then $\rho = R$.*

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[12] Interestingly, Wang (2003) and Toda (2017) show that the permanent income hypothesis holds in general equilibrium when the utility function is CARA, which is ruled out by Assumption 3.

- If $R \geq 1$, then ρ is any number in $((\beta R)^{1/\bar{\gamma}}, 1)$, where $\bar{\gamma}$ is the asymptotic relative risk aversion coefficient defined in Proposition 5.

Proof. If $R < 1$, by the budget constraint (8b) we obtain

$$a_{t+1} \leq R a_t + y_{t+1} = \rho a_t + y_{t+1}, \tag{14}$$

where $\rho = R < 1$. Hence (1) holds. If $1 \leq R < 1/\beta$, by the budget constraint and Proposition 5, we can take any $m \in (1 - 1/R, 1 - \beta^{1/\bar{\gamma}} R^{1/\bar{\gamma}-1})$ and some $A \geq 0$ such that

$$a_{t+1} \leq R(a_t - m a_t) + y_{t+1} = \rho a_t + y_{t+1} \tag{15}$$

for $\rho = R(1 - m) \in ((\beta R)^{1/\bar{\gamma}}, 1)$ whenever $a_t \geq A$. Once again, the bound in (1) holds.

By (14) and (15), letting $\phi(x) = \max\{\rho x, RA\}$, we obtain

$$a_{t+1} \leq \phi(a_t) + y_{t+1}.$$

Therefore the claims follow from Theorem 3.   □

The intuition for this result is as follows. Under the impatience condition $\beta R < 1$, by Proposition 5, agents uniformly consume more than the interest income at high wealth level. Since agents draw down their assets, wealth behaves similarly to income.

We now apply Proposition 6 to show that in canonical heterogeneous-agent models, the wealth distribution cannot have a heavier tail than income, which is our main result. We formally define a Bewley–Huggett–Aiyagari model as follows.

Definition 7. A *Bewley–Huggett–Aiyagari model* is any dynamic general equilibrium model such that ex ante identical, infinitely-lived agents solve an income fluctuation problem (8).

Remark. By requiring that the gross risk-free rate R is constant over time in (8b), we are excluding models with aggregate shocks. Thus, we are focusing on a stationary environment at the aggregate level, although the individual income processes may be nonstationary according to Assumption 2. We conjecture that our results extend to models with aggregate risk, although it is beyond the scope of this paper.

While the preceding argument assumes the impatience condition $\beta R < 1$, in general equilibrium models this condition is necessarily satisfied. To prove this impatience condition, we introduce an additional assumption on the income process.

Assumption 4. Let $\{y_t\}_{t=0}^{\infty}$ be the income process. There is an $\epsilon > 0$ such that for any $x \in \mathbb{R}$, we have

$$\Pr\left(x \leq \sum_{s=0}^{\infty} \beta^s y_{t+s} \leq x + \epsilon \,|\, z^t\right) < 1 - \epsilon \tag{16}$$

for all $t \geq 0$ and history $z^t = (z_0, \ldots, z_t)$.

Assumption 4 is identical to condition $(U\gamma)$ in Chamberlain and Wilson (2000). Note that because $y_t \geq 0$ by assumption, the discounted sum of future income $\sum_{s=0}^{\infty} \beta^s y_{t+s}$ exists in $[0, \infty]$. Condition (16) says that the conditional distribution of this discounted sum is not concentrated

on a small enough interval. This condition is relatively weak and it roughly says that the discounted sum of income is stochastic. It holds, for example, if income is stationary and stochastic (nondeterministic).

Theorem 8 *(Impossibility of heavy/heavier tails). Consider a Bewley–Huggett–Aiyagari model such that Assumptions 1–3 hold. Suppose that an equilibrium with a wealth distribution with a finite mean exists and let $R > 0$ be the equilibrium gross risk-free rate. If $R \neq 1/\beta$, then the following statements are true:*

1. *If the income process is light-tailed, then so is the wealth distribution.*
2. *If the income process is heavy-tailed with polynomial decay rate α, then the wealth distribution has a polynomial decay rate $\alpha' \geq \alpha$.*

If in addition Assumption 4 holds, the condition $R \neq 1/\beta$ can be dropped.

Remark. Although we present Theorem 8 as if the economy consists of ex ante identical agents, the result trivially generalizes to models with multiple agent types with heterogeneous preferences and income processes as long as each type satisfies the assumptions (in particular, the infinite horizon setting).

The proof is an immediate consequence of Proposition 6 combined with the convergence results in Chamberlain and Wilson (2000). Theorem 8 is valuable since it places few assumptions. The only important assumption is that an equilibrium exists, which gives us the impatience condition $\beta R < 1$ to apply Proposition 6. This assumption is minimal, for it is vacuous to study the wealth distribution unless an equilibrium exists. Regarding the income shocks, persistence and/or stationarity are irrelevant.

Theorem 8 has two important implications on the wealth distribution. (i) It is impossible to generate heavy-tailed wealth distributions from light-tailed income shocks. (ii) If the income shock has a Pareto tail with exponent α, the wealth distribution can have a Pareto tail, but its tail exponent α' can never fall below that of income shocks. Noting that smaller tail exponent means heavier tail, it follows that the wealth distribution cannot have a heavier tail than income.

Below, we discuss several applications of Theorem 8.

Example 1. Aiyagari (1994) uses the CRRA utility and a finite-state Markov chain for income. Hence by Theorem 8, the wealth distribution is light-tailed. (In fact, it is bounded by applying Theorem 3(1).[13])

Example 2. In Quadrini (2000), even though there is idiosyncratic investment risk (stochastic returns), agents are restricted to only three levels of investment $\{k_1, k_2, k_3\}$ (see p. 25). Therefore the only investment vehicle that allows for unbounded investment is the risk-free asset, and the budget constraint reduces to one with stochastic income only. Since utility is CRRA, the wealth distribution is light-tailed.

---

[13] The boundedness result for the case with finite-state Markov chain is already known from Aiyagari (1993, Proposition 4) and Açıkgöz (2018, Proposition 4). The case when the income process is unbounded but light-tailed is new.

Example 3. In Castañeda et al. (2003), the utility function is additively separable between consumption and leisure and the consumption part is CRRA. Since shocks follow a finite-state Markov chain, the wealth distribution is light-tailed.

Example 4. The budget constraint in Cagetti and De Nardi (2006) is

$$a' = (1 - \delta)k + \theta k^\nu - (1 + r)(k - a) - c,$$

where a is risk-free asset and k is capital (see their Equation (4) on p. 846). Here $\nu \in (0, 1)$ is a parameter, r is the net interest rate, $\delta \in (0, 1)$ is capital depreciation rate, and θ is a random variable for productivity that has bounded support. Although there is some restriction on k, by ignoring the constraint and maximizing, we can bound the right-hand side by $(1 + r)a + Y - c$, where Y is some random variable with bounded support. Since utility is CRRA, the wealth distribution is light-tailed.

4. Possibility results

Our Theorem 8 states that in canonical Bewley–Huggett–Aiyagari models in which (i) agents are infinitely-lived, (ii) agents have constant discount factors, and (iii) the only financial asset is risk-free, the wealth distribution necessarily inherits the tail property of income. Thus, it is necessary to go beyond standard models to explain the empirical fact that wealth is heavier-tailed than income. A natural question is whether relaxing any of these assumptions can generate heavy-tailed wealth distributions from light-tailed income shocks. The answer is yes.

First, consider relaxing condition (iii) (saving is risk-free). In this case the return to investment is stochastic, and it is well-known that the wealth distribution can be heavy-tailed. (See, for example, Nirei and Souma (2007), Benhabib et al. (2011, 2015), Toda (2014), Cao and Luo (2017), and the references in Benhabib and Bisin (2018).) Next, consider relaxing condition (ii) (constant discounting). Krusell and Smith (1998) have numerically shown that when agents have random discount factors (i.e., they are more patient in some states than in others), the wealth distribution can be more skewed than the income distribution. Recently, Toda (2019) has theoretically proved that the wealth distribution can have a Pareto tail under random discounting even if there is no income risk, although the numerical value of the Pareto exponent is highly sensitive to the calibration of the discount factor process. Finally, consider relaxing condition (i) (infinite horizon). Carroll et al. (2017) and McKay (2017) numerically solve heterogeneous-agent quantitative models with several agent types with different discount factors to generate skewed wealth distributions.[14] Below, we provide a simplified version of such heterogeneous discount factor models and theoretically show that its wealth distribution is Pareto-tailed.[15]

Consider an infinite-horizon endowment economy consisting of several agent types indexed by $j = 1, \ldots, J$. Let $\pi_j \in (0, 1)$ be the fraction of type j agents, where $\sum_{j=1}^J \pi_j = 1$. Type j agents are born and die with probability $p_j \in (0, 1)$ every period, are endowed with constant endowment $y_j > 0$ every period, and have CRRA utility

---

[14] To our knowledge, Samwick (1998) is the first paper that uses a model with constant but heterogeneous discount factors to explain the wealth distribution.

[15] This example is essentially just a discrete-time, micro-founded, general equilibrium version of Wold and Whittle (1957), so there is nothing surprising in the results. Nevertheless, we think that it has a pedagogical value since Carroll et al. (2017) and McKay (2017) do not theoretically characterize the wealth distribution and the assumptions in Wold and Whittle (1957) are endogenously satisfied in equilibrium.

$$E_0 \sum_{t=0}^{\infty} \tilde{\beta}_j^t \frac{c_t^{1-\gamma_j}}{1-\gamma_j}, \tag{17}$$

where $\tilde{\beta}_j = \beta_j(1 - p_j)$ is the effective discount factor ($\beta_j \in (0, 1)$ is the discount factor) and $\gamma_j > 0$ is the relative risk aversion coefficient. There is a risk-free asset in zero net supply, and let $R > 0$ be the gross risk-free rate determined in equilibrium. We assume that there is a perfectly competitive annuity market and let $\tilde{R}_j = \frac{R}{1-p_j}$ be the effective gross risk-free rate faced by type j agents. We can show that a type j agent maximizes utility (17) subject to the budget constraint

$$w_{t+1} = \tilde{R}_j(w_t - c_t), \tag{18}$$

where $w_t > 0$ is wealth (financial wealth plus the annuity value of future income; see Toda (2019) for a rigorous discussion). A stationary equilibrium consists of a gross risk-free rate $R > 0$, optimal consumption rules, and wealth distributions such that (i) agents optimize, (ii) the commodity and risk-free asset markets clear, and (iii) the wealth distributions are invariant. The following theorem shows that a stationary equilibrium always exists and the wealth distribution exhibits a Pareto tail if and only if discount factors are heterogeneous across agent types.

Theorem 9. *A stationary equilibrium exists. Letting $R > 0$ be the equilibrium gross risk-free rate, the following statements are true.*

1. *If $\{\beta_j\}_{j=1}^J$ take at least two distinct values, then $\beta_j R > 1$ for at least one j and the stationary wealth distribution has a Pareto upper tail with exponent*

$$\alpha = \min_{j:\beta_j R > 1} \left[-\gamma_j \frac{\log(1 - p_j)}{\log(\beta_j R)} \right] > 1. \tag{19}$$

2. *If $\beta_1 = \cdots = \beta_J = \beta$, then $R = 1/\beta$ and the wealth distribution of each type is degenerate.*

Theorem 9 shows that neither idiosyncratic investment risk nor random discounting are necessary for Pareto tails. Random birth/death is sufficient, although discount factor heterogeneity is necessary in this case.

5. Concluding remarks

In this paper we rigorously prove under weak conditions that, in canonical Bewley–Huggett–Aiyagari models in which (i) agents are infinitely-lived, (ii) saving is risk-free, and (iii) agents have constant discount factors, the wealth distribution always inherits the tail behavior of income shocks. The key intuition is that (i) equilibrium considerations (market clearing) combined with the convergence results in Chamberlain and Wilson (2000) require the "impatience" condition $\beta R < 1$, but (ii) under this condition agents draw down their assets (Proposition 5) and income shocks die out in the long run (Theorem 3).

The impossibility of heavier-tailed wealth distributions in canonical models comes from the fact that individual wealth shrinks with probability 1 as in (1). The literature has long considered mechanisms to break the tight link between income and wealth, including random birth/death, random discount factors, and idiosyncratic investment risk. In all of these cases, individual wealth can grow with positive probability, which essentially makes the wealth accumulation a random growth model (which is known to generate Pareto tails). Which mechanism is more important is an empirical question.

Appendix A. Proofs

A.1. Proof of Theorem 3

We first show that we may assume $\phi(x) = \rho x$ without loss of generality. To this end, take $\rho' \in (\rho, 1)$. By assumption, $\rho = \limsup_{x \to \infty} \phi(x)/x < 1$, so we can choose $\bar{x}$ such that $\phi(x) \le \rho' x$ for $x \ge \bar{x}$. Since ϕ is bounded on bounded sets, we can choose $M \ge 0$ such that $\phi(x) \le M$ for $x \in [0, \bar{x}]$. Therefore $\phi(x) \le \max\{M, \rho' x\} \le \rho' x + M$ for any $x \ge 0$, so (6) implies $X_t \le \phi(X_{t-1}) + Y_t \le \rho' X_{t-1} + M + Y_t$. If we define $Y_t' = Y_t + M$, then (6) holds for $\phi(x) = \rho' x$ and $Y_t = Y_t'$. Since adding a constant M to Y_t does not change its tail behavior (e.g., boundedness, exponential decay rate, polynomial decay rate), setting $\phi(x) = \rho x$ in (6) costs no generality.

Iterating on (6) with $\phi(x) = \rho x$ yields

$$X_t \le Y_t + \rho Y_{t-1} + \cdots + \rho^{t-1} Y_1 + \rho^t X_0. \tag{A.1}$$

Case 1 (${\{Y_t\}}_{t=1}^{\infty}$ *has a compact support*). Take $Y \ge 0$ such that $Y_t \in [0, Y]$ for all t. Then it follows from (A.1) that

$$X_t \le (1 + \rho + \cdots + \rho^{t-1}) Y + \rho^t X_0 = \frac{1 - \rho^t}{1 - \rho} Y + \rho^t X_0 \le \frac{1}{1 - \rho} Y + X_0,$$

so ${\{X_t\}}_{t=1}^{\infty}$ is bounded.

Case 2 (${\{Y_t\}}_{t=1}^{\infty}$ *is uniformly light-tailed*). Let $\lambda > 0$ be the exponential decay rate. By definition, for any $s \in [0, \lambda)$, we have

$$f(s) := \sup_t E[e^{sY_t}] < \infty. \tag{A.2}$$

In general, for any random variables Z_1, Z_2 and $\theta \in (0, 1)$, by Hölder's inequality we have

$$M_{(1-\theta)Z_1 + \theta Z_2}(s) = E[e^{s((1-\theta)Z_1 + \theta Z_2)}] = E\left[\left(e^{sZ_1}\right)^{1-\theta} \left(e^{sZ_2}\right)^{\theta}\right]$$

$$\le E[e^{sZ_1}]^{1-\theta} E[e^{sZ_2}]^{\theta} = M_{Z_1}(s)^{1-\theta} M_{Z_2}(s)^{\theta}.$$

Multiplying both sides of (A.1) by $1 - \rho > 0$, we get

$$(1 - \rho) X_t \le \sum_{k=0}^{t} \theta_k Y_k, \tag{A.3}$$

where $Y_0 \equiv (1 - \rho) X_0$, $\theta_0 = \rho^t$, and $\theta_k = (1 - \rho)\rho^{t-k}$ for $k \ge 1$. Noting that $\theta_k \ge 0$ for all k and $\sum_{k=0}^{t} \theta_k = 1$, multiplying (A.3) by $s > 0$, taking the exponential, taking the expectation, and applying Hölder's inequality, it follows that

$$E[e^{(1-\rho)sX_t}] \le \prod_{k=0}^{t} E[e^{sY_k}]^{\theta_k} = e^{(1-\rho)\rho^t sX_0} \prod_{k=1}^{t} E[e^{sY_k}]^{\theta_k} \le e^{(1-\rho)\rho^t sX_0} f(s)^{1-\rho^t},$$

where $f(s)$ is as in (A.2). Redefining $(1 - \rho)s$ as s and noting that $0 < \rho^t < 1$ and $X_0 \ge 0$, we obtain

$$E[e^{sX_t}] \le e^{sX_0} \max\left\{1, f\left(\frac{s}{1-\rho}\right)\right\}.$$

By the definition of the exponential decay rate $\lambda > 0$, the right-hand side is finite if $\frac{s}{1-\rho} < \lambda \iff s < (1-\rho)\lambda$. Therefore by definition $\{X_t\}_{t=1}^{\infty}$ is uniformly light-tailed, and the exponential decay rate satisfies $\lambda' \geq (1-\rho)\lambda$.

Case 3 ($\{Y_t\}_{t=1}^{\infty}$ *is uniformly integrable and heavy-tailed*). Since by assumption $\sup_t E[Y_t^1] = \sup_t E[Y_t] < \infty$, the polynomial decay rate satisfies $\alpha \geq 1$. Let $s \in [1, \alpha]$. Applying Minkowski's inequality to both sides of (6) yields $E[X_t^s]^{1/s} \leq \rho E[X_{t-1}^s]^{1/s} + E[Y_t^s]^{1/s}$. Letting $f(s) = \sup_t E[Y_t^s]$ and iterating, we get

$$E[X_t^s]^{1/s} \leq \frac{1-\rho^t}{1-\rho} f(s)^{1/s} + \rho^t X_0 \leq \frac{1}{1-\rho} f(s)^{1/s} + X_0.$$

Therefore

$$E[X_t^s] \leq \left(\frac{1}{1-\rho} f(s)^{1/s} + X_0 \right)^s.$$

Since the right-hand side does not depend on t and $f(s) < \infty$ for $s = 1$ and $s < \alpha$ by definition, it follows that the polynomial decay rate of $\{X_t\}_{t=1}^{\infty}$ satisfies $\alpha' \geq \alpha \geq 1$. $\square$

A.2. Proof of Proposition 5

To prove Proposition 5, we first use the fact that the optimal consumption rule in the original problem can be bounded below by that with zero income.

Lemma 10. *Suppose Assumptions 1 and 2 hold and $\beta R < 1$. Given asset $a > 0$ and state z, let $c(a, z), c_0(a)$ be the optimal consumption rules for the income fluctuation problem (8) with and without income (which are established in Proposition 4). Then $c(a, z) \geq c_0(a)$.*

Proof. If an optimal consumption rule exists, by considering whether the no borrowing constraint $c \leq a$ binds or not, it must satisfy the Euler equation

$$u'(c(a, z)) = \max \left\{ \beta R E \left[u'(c(R(a - c(a, z)) + y', z')) \mid z \right], u'(a) \right\}. \tag{A.4}$$

We use policy function iteration as discussed in Appendix B to characterize properties of $c(a, z)$.

Given a candidate policy function $c(a, z)$, define the policy function (Coleman) operator $(Kc)(a, z)$ as the unique value $0 < t \leq a$ that solves the equation

$$u'(t) = \max \left\{ \beta R E \left[u'(c(R(a - t) + y', z')) \mid z \right], u'(a) \right\}. \tag{A.5}$$

Lemma 13 shows that K is a well-defined monotone self map. Proposition 4 shows that K has a unique fixed point and $K^n c$ converges to this fixed point as $n \to \infty$. Because the optimal consumption policy is a fixed point of K, which is unique, it suffices to show $K c_0(a) \geq c_0(a)$, for if that is the case we obtain $c_0(a) \leq (K^n c_0)(a) \to c(a, z)$.

Let $t = (K c_0)(a)$ solve (A.5) for $c(a, z) = c_0(a)$. To show $t \geq c_0(a)$, suppose on the contrary that $t < c_0(a)$. Since by Proposition 4 $c_0(a)$ is increasing in a, $y' \geq 0$, and $t < c_0(a)$, we have

$$c_0(R(a - t) + y') \geq c_0(R(a - c_0(a))).$$

Since u' is strictly decreasing and $c_0(a)$ satisfies (A.4) for $y' \equiv 0$, we obtain

$$u'(t) > u'(c_0(a)) = \max\left\{\beta RE\left[u'(c_0(R(a - c_0(a)))) \mid z\right], u'(a)\right\}$$
$$\geq \max\left\{\beta RE\left[u'(c_0(R(a - t) + y')) \mid z\right], u'(a)\right\} = u'(t),$$

which is a contradiction. Therefore $(Kc_0)(a) = t \geq c_0(a)$. $\quad\square$

Next, we derive a useful implication of BRRA. Consider the following condition:

For any constant $\kappa \in (0, 1)$, we have $\quad \liminf_{x \to \infty} \dfrac{(u')^{-1}(\kappa u'(x))}{x} > 1.$ $\qquad$ (A.6)

Condition (A.6) is relatively weak. To see this, since $u' > 0$, $\kappa \in (0, 1)$, and $u'' < 0$ (hence u' and $(u')^{-1}$ are decreasing), we always have

$$\frac{(u')^{-1}(\kappa u'(x))}{x} > \frac{(u')^{-1}(u'(x))}{x} = 1.$$

Condition (A.6) adds a degree of uniformity to this bound at infinity. The following lemma shows that bounded relative risk aversion (BRRA, Assumption 3) is sufficient for condition (A.6) to hold, and almost necessary.

Lemma 11. *Let $\gamma(x) = -xu''(x)/u'(x)$ be the local relative risk aversion coefficient. Then the following statements are true.*

1. *If $\limsup_{x \to \infty} \gamma(x) < \infty$, then condition (A.6) holds.*
2. *If $\lim_{x \to \infty} \gamma(x) = \infty$, then condition (A.6) fails.*

Proof. Take any $\kappa \in (0, 1)$. For any $x > 0$, define $y = (u')^{-1}(\kappa u'(x))$. By definition, $u'(y)/u'(x) = \kappa \in (0, 1)$. Since $u'' < 0$, we have $y > x$. By the Fundamental Theorem of Calculus and (9), we obtain

$$-\log \kappa = \log u'(x) - \log u'(y) = -\int_1^{y/x} \frac{\partial}{\partial s} \log u'(xs)\,ds$$

$$= -\int_1^{y/x} \frac{xu''(xs)}{u'(xs)}\,ds = \int_1^{y/x} \frac{\gamma(xs)}{s}\,ds. \qquad (A.7)$$

1. If $\limsup_{x \to \infty} \gamma(x) < \infty$, then there exists $M < \infty$ such that $\gamma(x) \leq M$ for large enough x. Then (A.7) implies

$$-\log \kappa \leq \int_1^{y/x} \frac{M}{s}\,ds = M \log \frac{y}{x} \iff \frac{y}{x} \geq \kappa^{-1/M}.$$

Since this inequality holds for large enough x and $y = (u')^{-1}(\kappa u'(x))$, we obtain

$$\liminf_{x \to \infty} \frac{(u')^{-1}(\kappa u'(x))}{x} \geq \kappa^{-1/M} > 1, \qquad (A.8)$$

which is (A.6).

2. If $\lim_{x \to \infty} \gamma(x) = \infty$, take any $M > 0$ and choose $\bar{x} > 0$ such that $\gamma(x) \geq M$ for all $x \geq \bar{x}$. Then for $x \geq \bar{x}$, by (A.7) we obtain

$$
-\log \kappa \geq \int_1^{y/x} \frac{M}{s}\, ds = M \log \frac{y}{x} \iff \frac{y}{x} \leq \kappa^{-1/M}.
$$

Since this inequality holds for large enough x and $y = (u')^{-1}(\kappa u'(x))$, we obtain

$$
\liminf_{x \to \infty} \frac{(u')^{-1}(\kappa u'(x))}{x} \leq \kappa^{-1/M} \to 1
$$

as $M \to \infty$, so (A.6) fails. □

We now use condition (A.6) to prove Proposition 5.

Proof of Proposition 5. Let K be the policy function operator defined in (A.5) associated with the zero income model, so for a consumption policy $c = c(a)$, the number $t = (Kc)(a)$ solves

$$
u'(t) = \max \left\{ \beta R u'(c(R(a - t))),\, u'(a) \right\}. \tag{A.9}
$$

Fix some $m \in (1 - 1/R, 1)$, $A > 0$, and consider the candidate policy

$$
c(a) = \begin{cases} \epsilon c_0(a), & (0 < a < A) \\ ma, & (a \geq A) \end{cases} \tag{A.10}
$$

where $\epsilon \in (0, 1)$ is a number such that $\epsilon c_0(A) \leq mA$. Clearly $c(a)$ in (A.10) is increasing and satisfies $0 < c(a) \leq a$. If we can show $(Kc)(a) \geq c(a)$ for all a, then by Proposition 4 and Lemma 10, we obtain

$$
c(a) \leq (K^n c)(a) \to c_0(a) \leq c(a, z),
$$

which implies (13) for $a \geq A$.

Step 1. *Let $\bar{\gamma} = \limsup_{x \to \infty} -x u''(x)/u'(x) < \infty$ be the asymptotic relative risk aversion coefficient and define $\bar{m}$ by $(\beta R)^{1/\bar{\gamma}} = R(1 - \bar{m})$. Then $\bar{m} \in (1 - 1/R, 1]$. Furthermore, for any $m \in (1 - 1/R, \bar{m})$, we have*

$$
\liminf_{x \to \infty} \frac{(u')^{-1}(\beta R u(x))}{x} > \frac{1}{R(1 - m)}. \tag{A.11}
$$

Since by assumption $\beta R < 1$ and $\bar{\gamma} \in [0, \infty)$, we have $(\beta R)^{1/\bar{\gamma}} \in [0, 1)$. Since $(\beta R)^{1/\bar{\gamma}} = R(1 - \bar{m})$ and $R \geq 1$ by assumption, we have $\bar{m} \in (1 - 1/R, 1]$. Take any $m \in (1 - 1/R, \bar{m})$. Then $(\beta R)^{1/\bar{\gamma}} = R(1 - \bar{m}) < R(1 - m)$. Since $\beta R < 1$, we can take sufficiently small $M > \bar{\gamma}$ such that $(\beta R)^{1/M} < R(1 - m)$. Letting $\kappa = \beta R < 1$ in (A.8), we obtain (A.11).

Step 2. *The following statement is true:*

$$
(\forall m \in (1 - 1/R, \bar{m}))(\exists A > 0)(\forall a \geq A)(t = (Kc)(a) \geq ma). \tag{A.12}
$$

In seeking a contradiction, suppose

$$(\exists m \in (1 - 1/R, \bar{m}))(\forall A > 0)(\exists a \geq A)(t = (Kc)(a) < ma). \tag{A.13}$$

If $\beta R u'(c(R(a - t))) < u'(a)$, by (A.9) we have $u'(t) = u'(a) \iff t = a \geq ma$, which contradicts (A.13). Therefore $\beta R u'(c(R(a - t))) \geq u'(a)$. Noting that we are considering the candidate policy (A.10), it follows from (A.9) that

$$g(t) := u'(t) - \beta R u'(mR(a - t)) = 0.$$

Since $g'(t) = u''(t) + \beta m R^2 u''(mR(a - t)) < 0$, g is strictly decreasing. Since $t < ma$ by (A.13), we obtain

$$0 = g(t) > g(ma) = u'(ma) - \beta R u'(mR(1 - m)a)$$

$$\implies \frac{(u')^{-1}(\beta R u'(mR(1 - m)a))}{mR(1 - m)a} < \frac{1}{R(1 - m)}.$$

By (A.13), $a > 0$ can be taken arbitrarily large. Therefore letting $a \to \infty$ and $x = mR(1 - m)a$, we obtain

$$\liminf_{x \to \infty} \frac{(u')^{-1}(\beta R u'(x))}{x} \leq \frac{1}{R(1 - m)},$$

which contradicts (A.11). Therefore (A.12) holds.

Step 3. *The bound* (13) *holds.*

Take any $m \in (1 - 1/R, \bar{m})$ and $A > 0$ such that (A.12) holds. Take $\epsilon \in (0, 1)$ such that $\epsilon c_0(A) \leq mA$ and define $c(a)$ as in (A.10). By (A.12), we have $(Kc)(a) \geq ma = c(a)$ for $a \geq A$. Therefore it remains to show $(Kc)(a) \geq c(a)$ for $a < A$. Suppose on the contrary that $(Kc)(a) < c(a)$ for some $a < A$. Since $c(a) = \epsilon c_0(a)$ and $\epsilon \in (0, 1)$, we have $(Kc)(a) < c_0(a)$. Applying K and using monotonicity and Proposition 4, it follows that

$$c_0(a) > (Kc)(a) \geq (K^n c)(a) \to c_0(a),$$

which is a contradiction. Therefore $(Kc)(a) \geq c(a)$. The rest of the proof follows from the discussion at the beginning of the proof.   □

A.3. Proof of Theorem 8

First let us show $\beta R \leq 1$. By considering whether the borrowing constraint binds or not, the Euler equation becomes

$$u'(c_t) = \max \left\{ \beta R E_t[u'(c_{t+1})], u'(a_t) \right\}.$$

Therefore $u'(c_t) \geq \beta R E_t[u'(c_{t+1})]$. Multiplying both sides by $(\beta R)^t > 0$ and letting $M_t = (\beta R)^t u'(c_t) > 0$, we obtain $M_t \geq E_t[M_{t+1}]$. Since $M_0 = u'(c_0) < \infty$, the process $\{M_t\}_{t=0}^\infty$ is a supermartingale. By the Martingale Convergence Theorem (Pollard, 2002, p. 148), there exists an integrable random variable $M \geq 0$ such that $\lim_{t \to \infty} M_t = M$ almost surely. Since $E[M] < \infty$, we have $M < \infty$ (*a.s.*). If $\beta R > 1$, then

$$u'(c_t) = (\beta R)^{-t} M_t \to 0 \cdot M = 0 \quad (a.s.),$$

so $c_t \to \infty$ $(a.s.)$ because $u' > 0$ and $u'(\infty) = 0$. (This argument is exactly the same as Theorem 2 of Chamberlain and Wilson (2000).) Since this is true for any agent, the aggregate consumption diverges to infinity, which is impossible in a model with a wealth distribution with a finite mean. Therefore $\beta R \le 1$.

If $R \ne 1/\beta$, then $\beta R < 1$, so the conclusion follows from Proposition 6.

Finally, suppose Assumption 4 holds. If $\beta R = 1$, then by Chamberlain and Wilson (2000, Theorem 4) we have $c_t \to \infty$ $(a.s.)$, which is a contradiction. $\square$

A.4. Proof of Theorem 9

Since the proof is similar to Toda (2019, Theorems 3, 4), we only provide a sketch.

It is well-known that the optimal consumption rule for maximizing the CRRA utility (17) subject to the budget constraint (18) is

$$c = \left(1 - \tilde{\beta}_j^{1/\gamma_j} \tilde{R}_j^{1/\gamma_j - 1}\right) w.$$

(See Levhari and Srinivasan (1969), Samuelson (1969), or more generally, Toda (2014, 2019) for the solution in a Markovian environment.) Then the law of motion for wealth becomes

$$w_{t+1} = \left(\tilde{\beta}_j \tilde{R}_j\right)^{1/\gamma_j} w_t = (\beta_j R)^{1/\gamma_j} w_t. \tag{A.14}$$

Let W_j be the average wealth of type j agents. Since agents are born and die with probability p_j, by accounting we obtain

$$W_j = (1 - p_j)(\beta_j R)^{1/\gamma_j} W_j + p_j w_{j0}, \tag{A.15}$$

where

$$w_{j0} = \sum_{t=0}^{\infty} \tilde{R}_j^{-t} y_j = \frac{\tilde{R}_j}{\tilde{R}_j - 1} y_j \tag{A.16}$$

is the initial wealth (present discounted value of future income) of type j agents. Assuming $(1 - p_j)(\beta_j R)^{1/\gamma_j} < 1$, we can solve (A.15) to obtain

$$W_j = \frac{p_j w_{j0}}{1 - (1 - p_j)(\beta_j R)^{1/\gamma_j}}. \tag{A.17}$$

Combining (A.16) and (A.17), the market clearing condition becomes

$$0 = \sum_{j=1}^{J} \pi_j (W_j - w_{j0}) = \sum_{j=1}^{J} \frac{R \pi_j y_j \left((\beta_j R)^{1/\gamma_j} - 1\right)}{\left(\frac{R}{1-p_j} - 1\right)\left(1 - (1 - p_j)(\beta_j R)^{1/\gamma_j}\right)}. \tag{A.18}$$

Let $f(R)$ be the right-hand side of (A.18) and $\bar{R} = \min_j [\beta_j(1 - p_j)^{\gamma_j}]^{-1}$, which is greater than 1 since $\beta_j, p_j \in (0, 1)$ and $\gamma_j > 0$. Since $\frac{R}{1-p_j} > 1$ and $(1 - p_j)(\beta_j R)^{1/\gamma_j} < 1$ on $R \in [1, \bar{R})$, the function $f(R)$ is well-defined in this range and the denominators are positive. Since $\beta_j < 1$, we have $f(1) < 0$. Since $(\beta_j R)^{1/\gamma_j} \to \frac{1}{1-p_j} > 1$ as $R \uparrow \bar{R}$ for j that achieves the minimum in the definition of $\bar{R}$, we have $f(R) \uparrow \infty$ as $R \uparrow \bar{R}$. Clearly f is continuous on $[1, \bar{R})$. Therefore there exists $R \in (1, \bar{R})$ such that $f(R) = 0$, so an equilibrium exists.

To show the implications for the wealth distribution, first consider the case $\beta_1 = \cdots = \beta_J = \beta$. Then

$$f(R) = \sum_{j=1}^{J} \frac{R \pi_j y_j \left((\beta R)^{1/\gamma_j} - 1 \right)}{\left(\frac{R}{1-p_j} - 1 \right) \left(1 - (1-p_j)(\beta R)^{1/\gamma_j} \right)}.$$

Since the denominators are positive on $[1, \bar{R}]$ and the sign of the numerators depends only on the magnitude of βR relative to 1, we have $f(R) \gtrless 0$ according as $R \gtrless 1/\beta$. Therefore the unique equilibrium gross risk-free rate is $R = 1/\beta$. Then the dynamics of wealth (A.14) becomes $w_{t+1} = w_t$, so individual wealth is constant over time and the wealth distribution of each type is degenerate.

Finally, suppose $\{\beta_j\}_{j=1}^{J}$ take at least two distinct values. Let $G_j = (\beta_j R)^{1/\gamma_j}$ be the gross growth rate of wealth in (A.14). Since the numerator in (A.18) has the same sign as $G_j - 1$ and $G_j \gtrless 1$ according as $\beta_j \gtrless 1/R$, the fact that $f(R) = 0$ and $\{\beta_j\}_{j=1}^{J}$ are not all equal implies that there is some j with $G_j > 1$ and others with $G_j < 1$.

An agent type with $G_j \leq 1$ does not affect the upper tail of the wealth distribution since the wealth does not grow. Consider any type j with $G_j > 1$. Because type j agents die with probability p_j every period, the probability that an agent survives at least n periods is $(1 - p_j)^n$. Then wealth is at least $G_j^n w_{j0}$. Therefore we have

$$\Pr(w \geq G_j^n w_{j0}) = (1 - p_j)^n,$$

so letting $x = G_j^n w_{j0}$, we obtain $\Pr(w \geq x) = (x/w_{j0})^{-\alpha_j}$ for

$$\alpha_j = -\frac{\log(1-p_j)}{\log G_j} = -\gamma_j \frac{\log(1-p_j)}{\log(\beta_j R)} > 0.$$

Therefore the wealth distribution of type j has a Pareto upper tail with exponent α_j. Noting that $(1 - p_j)(\beta_j R)^{1/\gamma_j} < 1$ and $\beta_j R > 1$, taking the logarithm we obtain $\alpha_j > 1$. Since the Pareto exponent of the entire cross-sectional distribution is the smallest exponent among all types, we obtain (19). $\square$

Appendix B. Policy iteration in income fluctuation problem

In this appendix we adopt the arguments in Li and Stachurski (2014) to characterize the solution to the income fluctuation problem by policy function iteration.[16] Throughout this appendix, we maintain Assumptions 1 and 2 but not Assumption 3.

Let $S = \mathbb{R}_{++} \times Z$ be the state space, where Z is as in Assumption 2. To identify a solution, let $\mathcal{C}$ be the set of functions $c : S \to \mathbb{R}$ such that $c(a, z)$ is increasing in a, $0 < c(a, z) \leq a$, and $\|u' \circ c - \phi\| < \infty$ (sup norm), where $\phi(a, z) \equiv u'(a)$. The set $\mathcal{C}$ identifies a set of candidate consumption functions. For $c, d \in \mathcal{C}$, define the distance

$$\rho(c, d) = \|u' \circ c - u' \circ d\|.$$

Lemma 12. $(\mathcal{C}, \rho)$ *is a complete metric space.*

---

[16] Our discussion is slightly different from Li and Stachurski (2014) due to the timing convention. In Li and Stachurski (2014), a is savings (end-of-period asset holdings) and the budget constraint is $c + a' = Ra + y$. In our framework, a is beginning-of-period wealth and the budget constraint is $a' = R(a - c) + y'$. This change in the notation allows us to weaken the monotonicity requirement in Assumption 2.3 of Li and Stachurski (2014).

Proof. Take any $c, d, e \in \mathcal{C}$. Clearly $\rho(c, d) \geq 0$, $\rho(c, d) = \rho(d, c)$, and

$$\rho(c, d) = 0 \iff (\forall a, z) u'(c(a, z)) = u'(d(a, z)) \iff c = d.$$

Furthermore, by the triangle inequality for the sup norm, we have

$$\rho(c, d) = \left\| u' \circ c - u' \circ d \right\| \leq \left\| u' \circ c - \phi \right\| + \left\| u' \circ c - \phi \right\| < \infty$$

and

$$\rho(c, d) = \left\| u' \circ c - u' \circ d \right\| \leq \left\| u' \circ c - u' \circ e \right\| + \left\| u' \circ e - u' \circ d \right\| = \rho(c, e) + \rho(e, d).$$

Therefore $(\mathcal{C}, \rho)$ is a metric space.

To show completeness, suppose $\{c_n\}$ is a Cauchy sequence in $(\mathcal{C}, \rho)$. Then $\{u'(c_n(a, z))\}$ is a Cauchy sequence in $\mathbb{R}$, so it has a limit μ. Since $c_n(a, z) \leq a$ and u' is strictly decreasing, we have $u'(c_n(a, z)) \geq u'(a)$ and hence $\mu \geq u'(a)$. Since u' is continuous, strictly decreasing, and $\mu < \infty$, there exists a unique $c(a, z) \in (0, a]$ such that $u'(c(a, z)) = \mu$. Since u' is continuous, we have $c_n(a, z) \to c(a, z)$ pointwise. Since $c_n(a, z)$ is increasing in a, so is $c(a, z)$. Therefore $(\mathcal{C}, \rho)$ is complete. $\quad\square$

For $c \in \mathcal{C}$, define $(Kc)(a, z)$ by the value $t \in (0, a]$ that satisfies the Euler equation (A.5).

Lemma 13. *For any $c \in \mathcal{C}$ and $(a, z) \in S$, $(Kc)(a, z)$ is well-defined. Furthermore, $K : \mathcal{C} \to \mathcal{C}$ and $c \leq d \implies Kc \leq Kd$.*

Proof. Fix any $c \in \mathcal{C}$ and $(a, z) \in S$. For $t \in (0, a]$, define

$$g(t) = u'(t) - \max \left\{ \beta R E \left[u'(c(R(a - t) + y', z')) \mid z \right], u'(a) \right\}.$$

The second term is finite because of the max operator and $u'(a) < \infty$. Therefore g is finite on $(0, a]$. Since $u'' < 0$, g is continuous and strictly decreasing. Furthermore, $g(0) = \infty$ and

$$g(a) = u'(a) - \max \left\{ \beta R E \left[u'(c(y', z')) \mid z \right], u'(a) \right\} \leq 0.$$

Therefore there exists a unique $t \in (0, a]$ such that $g(t) = 0$, so (A.5) holds.

To show $K : \mathcal{C} \to \mathcal{C}$, it remains to show that $(Kc)(a, z)$ is increasing in a. To show this, let $a_1 \leq a_2$ and $t_j = (Kc)(a_j, z)$ for $j = 1, 2$. Suppose on the contrary that $t_1 > t_2$. Then using the fact that c is increasing and u' is strictly decreasing, we obtain

$$u'(t_2) > u'(t_1) = \max \left\{ \beta R E \left[u'(c(R(a_1 - t_1) + y', z')) \mid z \right], u'(a_1) \right\}$$
$$\geq \max \left\{ \beta R E \left[u'(c(R(a_2 - t_2) + y', z')) \mid z \right], u'(a_2) \right\} = u'(t_2),$$

which is a contradiction. Therefore $t_1 \leq t_2$. The proof of $c \leq d \implies Kc \leq Kd$ is similar. $\quad\square$

Lemma 14. *If $\beta R < 1$, then $\rho(Kc, Kd) \leq \beta R \rho(c, d)$ for all $c, d \in \mathcal{C}$.*

Proof. Similar to the proof of Lemma A.5 in Li and Stachurski (2014). $\quad\square$

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Corrigendum

Corrigendum to "An impossibility theorem for wealth in heterogeneous-agent models with limited heterogeneity" [Journal of Economic Theory 182 (2019) 1–24]

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Abstract

This note corrects the proof of Proposition 5 in Stachurski and Toda (2019), which shows that the consumption function has an explicit linear lower bound and is used to prove their main result that wealth inherits the tail behavior of income in Bewley–Huggett–Aiyagari models.
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1. Introduction

It has been a 'folk theorem' in the quantitative macroeconomics literature that heterogeneous-agent models that feature infinitely-lived agents, constant discount factors, and risk-free asset returns have difficulty in explaining the empirically observed heavy-tailed behavior of the wealth distribution. Stachurski and Toda (2019) (henceforth ST) provide a theoretical explanation by proving that the wealth accumulation process in such models has an AR(1) upper bound (Proposition 6), which implies that the wealth inherits the tail behavior of income (Theorems 3 and 8).

However, their proof of Proposition 5, which is central to their analysis, contains errors. This note provides a correct proof after slightly strengthening the assumptions.

Although the assumptions used here are stricter than those in ST, they do not exclude the applications that follow the main impossibility theorem (Theorem 8).

2. Assumptions and corrected proof

We consider the following income fluctuation problem:

$$\text{maximize} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1a}$$

$$\text{subject to} \quad a_{t+1} = R(a_t - c_t) + y_{t+1}, \tag{1b}$$

$$0 \leq c_t \leq a_t, \tag{1c}$$

where $u \colon \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is the utility function, $\beta > 0$ is the discount factor, $R > 0$ is the gross risk-free rate, $y_t \geq 0$ is income, a_t is financial wealth at the beginning of period t including current income, and initial wealth $a_0 > 0$ is given.

Following Assumption 1 in ST, the utility function is twice continuously differentiable on $(0, \infty)$ and satisfies $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. We slightly strengthen Assumption 2 in ST by adding some regularity conditions as in Ma et al. (2020):

Assumption 2'. The income process $\{y_t\}$ takes the form $y_t = y(z_t, \eta_t)$, where $\{z_t\}$ is a Markov chain taking values in a finite set Z with transition probability matrix Π,[1] $\{\eta_t\}$ is IID, y is a nonnegative measurable function, and

$$\sup_{z \in Z} E\left[y(z', \eta') \mid z\right] < \infty \quad \text{and} \quad \sup_{z \in Z} E\left[u'(y(z', \eta')) \mid z\right] < \infty.$$

Note that the transitory shock $\{\eta_t\}$ could be unbounded or heavy-tailed. The condition $\sup_{z \in Z} E\left[u'(y(z', \eta')) \mid z\right] < \infty$, which is not mentioned in ST, is required so that the policy functions remain in the space $\mathcal{C}$ defined below. See Proposition B.4 of Ma et al. (2020) for more details. In all of what follows, Assumption 1 of ST and Assumption 2' above are taken to be in force. We let $S = \mathbb{R}_{++} \times Z$ and take $\mathcal{C}$ to be the set of continuous functions from S to $(0, \infty)$ such that c is increasing in its first argument, $c(a, z) \leq a$ for all $(a, z) \in S$, and $\|u' \circ c - u'\| < \infty$, where $\|\cdot\|$ is the supremum norm on S. Given a candidate policy function $c \in \mathcal{C}$, define $(Kc)(a, z)$ to be the unique t in $(0, a]$ that solves the Euler equation

$$u'(t) = \max\{\beta R E\left[u'(c(R(a - t) + y', z')) \mid z\right], u'(a)\}. \tag{2}$$

Under these conditions and $\beta R < 1$, K is a contraction mapping with respect to a complete metric on $\mathcal{C}$. Moreover, its fixed point in $\mathcal{C}$ is the unique optimal consumption policy.[2]

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[1] The results presented here can be extended to the case where Z is an abstract metric space under suitable regularity conditions on the stochastic kernel Π, such as those adopted in Li and Stachurski (2014). The case of finite Z is sufficient for the applications we consider.

[2] These facts are proved in Theorem 2.2 of Ma et al. (2020). Although Ma et al. (2020) further assume that Π is irreducible, this assumption is required only for ergodicity and not for optimality. The convergence of a sequence in the metric in question implies pointwise convergence, a fact that we make use of in the proofs below.

Proposition 5 of ST obtains a linear lower bound $c(a, z) \geq ma$ for c with $m > 1 - 1/R$, so that the budget constraint (1b) implies an AR(1) upper bound $a_{t+1} \leq \rho a_t + y_{t+1}$ with $\rho = R(1 - m) \in [0, 1)$ for the wealth accumulation process. However, the proof of Proposition 5 in ST contains errors. First, the candidate policy in (A.10) is potentially discontinuous, which is not allowed when applying policy function iteration (in particular, the intermediate value theorem). Second, the argument in Step 3 of the proof requires the inequality to hold pointwise, which is not necessarily true under the stated assumptions.

To correct these errors, we first present a simple lemma, which is essentially a special case of Proposition 2.6 of Ma et al. (2020).

Lemma 1. *If $\beta R < 1$ and there exists an $m \in (0, 1)$ such that*

$$u'(a) \geq \beta R u'(R(1 - m)a) \ \text{for all } a > 0, \tag{3}$$

then the optimal consumption policy c satisfies $c(a, z) \geq ma$ for all $(a, z) \in S$.

Proof. Let (3) hold for some $m \in (0, 1)$. Define

$$\mathcal{C}_0 := \{c \in \mathcal{C} \mid c(a, z) \geq ma \text{ for all } (a, z) \in S\}.$$

Clearly $\mathcal{C}_0$ is a closed subset of $\mathcal{C}$. Let us show that $K\mathcal{C}_0 \subset \mathcal{C}_0$. To this end, suppose to the contrary that $K\mathcal{C}_0 \not\subset \mathcal{C}_0$. Then by definition there exist $c_0 \in \mathcal{C}_0$ and $(a, z) \in S$ such that $t := Kc_0(a, z) < ma < a$. Since $t < a$, the Euler equation (2) implies

$$u'(t) = \beta R E \left[u'(c_0(R(a - t) + y', z')) \mid z \right].$$

Since $t < ma$, u' is strictly decreasing, $c_0 \in \mathcal{C}_0$, and $y' \geq 0$, we obtain

$$\begin{aligned}
u'(ma) < u'(t) &= \beta R E \left[u'(c_0(R(a - t) + y', z')) \mid z \right] \\
&\leq \beta R E \left[u'(m(R(a - t) + y')) \mid z \right] \\
&\leq \beta R E \left[u'(mR(1 - m)a) \mid z \right] = \beta R u'(R(1 - m)ma).
\end{aligned}$$

This contradicts (3) after replacing ma by a. Therefore $K\mathcal{C}_0 \subset \mathcal{C}_0$.

Since clearly the function $c_0(a, z) = a$ is in $\mathcal{C}_0$ and $K : \mathcal{C}_0 \to \mathcal{C}_0$ is a contraction mapping, we have $\mathcal{C}_0 \ni K^n c_0 =: c_n \to c$. Therefore $c \in \mathcal{C}_0$ and $c(a, z) \geq ma$ for all $(a, z) \in S$. $\quad\square$

To apply Lemma 1, we slightly strengthen another assumption in ST. Let

$$\gamma := \sup_{x > 0} -\frac{x u''(x)}{u'(x)}. \tag{4}$$

Assumption 3'. The utility function u exhibits bounded relative risk aversion. In particular, γ in (4) is finite.

Proposition 5'. *If Assumption 3' holds and $1 \leq R < 1/\beta$, then the optimal consumption rule satisfies $c(a, z) \geq ma$ for all $(a, z) \in S$, where $m := 1 - \beta^{1/\gamma} R^{1/\gamma - 1} > 1 - 1/R \geq 0$.*

Proof. Since $\beta R < 1$ and $R \geq 1$, we have

$$1 > m = 1 - (\beta R)^{1/\gamma}/R > 1 - 1/R \geq 0.$$

Let $\kappa = \beta R < 1$. The proof of Lemma 11 in ST implies that, if $x > 0$ and $y = (u')^{-1}(\kappa u'(x))$, then $y/x \geq \kappa^{-1/\gamma}$. Since u' is decreasing, we obtain

$$u'(y) = \kappa u'(x) \geq \kappa u'(\kappa^{1/\gamma} y).$$

Setting $y = a$, $\kappa = \beta R < 1$, and noting that $R(1 - m) = (\beta R)^{1/\gamma}$, we obtain

$$u'(a) \geq \beta R u'((\beta R)^{1/\gamma} a) = \beta R u'(R(1 - m)a).$$

Hence (3) holds. An application of Lemma 1 now yields $c(a, z) \geq ma$. $\square$

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