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Solving the income fluctuation problem with unbounded rewards

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ABSTRACT

This paper studies the income fluctuation problem without imposing bounds on utility, assets, income or consumption. We prove that the Coleman operator is a contraction mapping over the natural class of candidate consumption policies when endowed with a metric that evaluates consumption differences in terms of marginal utility. We show that this metric is complete, and that the fixed point of the operator coincides with the unique optimal policy. As a consequence, even in this unbounded setting, policy function iteration always converges to the optimal policy at a geometric rate.

1. Introduction

The income fluctuation problem refers to a classic decision problem that lies at the heart of modern macroeconomic theory. In the problem, agents choose a state-contingent path for savings and consumption in order to maximize expected lifetime utility, taking as given the rate of return on assets, an exogenous stream of non-capital income, and, in many cases, a borrowing constraint. The model has been used to analyze household behavior in many fundamental economic and financial applications. The literature is too large to enumerate, but some broadly representative examples include Schechtman and Escudero (1977), Deaton (1991), Huggett (1993), Aiyagari (1994), Krusell and Smith (1998), Deaton and Laroque (1992, 1996) and Angeletos (2007).

Early work on consumption behavior focused on highly simplified problems with closed-form solutions. It turned out that these models have only limited ability to fit consumption data (see, e.g., Carroll, 2001). Adding more realistic features has led to better models, but in these settings computation cannot be avoided. The computational problem remains a nontrivial one because in most modeling exercises the consumer problem is embedded in a larger equilibrium or estimation problem, and needs to be solved quickly, accurately and reliably for many different parameter values.

A variety of solution techniques have been proposed for the income fluctuation problem specifically or for optimization problems that subsume the income fluctuation problem. The literature now contains many numerical studies presenting

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simulation results for particular solution methods according to particular criteria in particular applications and at particular sets of parameter values. While such studies can certainly complement theoretical analysis, they cannot substitute for it, and there remains a lack of clear analytical results proving convergence at given rates for a given method over a continuum of standard applications, parameter values and initial conditions.

In this paper, we provide analytical results on convergence for the common solution method known as policy function iteration (or, in some circles, time iteration). The basic ideas behind policy function iteration were illuminated by Coleman (1990). As is well-known, when the utility function is bounded, policy iteration is globally convergent. The reason is that the operator that implements policy iteration—the Coleman operator—is essentially conjugate to the Bellman operator (Coleman, 1990). When rewards are bounded, global geometric stability of the Bellman operator is guaranteed by classical dynamic programming theory. By applying this conjugacy between the two operators, one can then show that the Coleman operator has all of the same properties. Rendahl (2013) makes use of these ideas to provide a detailed treatment of policy iteration in the bounded reward case, working with an abstract optimization problem that permits occasionally binding constraints.

For standard income fluctuation models, however, utility is unbounded, and consumption can become either arbitrarily small or arbitrarily large. In these settings, the Bellman operator is not a contraction mapping in the usual metric, and we cannot claim that iterates of the Bellman operator converge uniformly to the value function. In fact the uniform deviation is typically infinite, regardless of how many iterations are performed. Thus the standard dynamic programming theory does not apply.

In response to these issues, the present paper develops an alternative approach to the income fluctuation optimization problem that delivers sharper results than previously obtained—even when rewards are bounded. Our focus is directly on the Coleman operator, rather than drawing connections to the Bellman operator. Because we work with the Coleman operator and policy function iteration, our main results are formulated in policy function space rather than value function space, and unbounded rewards cause no difficulties for the analysis.

As our most significant theoretical result, we show that a version of the Coleman operator adapted to the income fluctuation problem is in fact a contraction mapping in a complete metric space of candidate consumption policies, even when rewards are unbounded. We also prove that the asset-consumption path associated with the fixed point of Coleman's operator satisfies the sequential Euler equation and transversality conditions, and that the Euler equation and transversality conditions are sufficient for optimality. Putting these facts together, we show that a unique optimal consumption policy exists, and that, for any well-behaved initial condition, policy function iteration converges to this optimal policy at a geometric rate. In particular, we prove that the pointwise deviation between the n-th iterate and the optimal policy converges to zero at a geometric rate, and the same is true for the uniform deviation over any bounded set. (As will be discussed later, this is in a sense the best possible result for policy function iteration in the unbounded setting.) Moreover, we give a computable upper bound on the deviation in terms of observable quantities.

All of these results are obtained in a setting that can accommodate a broad range of standard applications. In particular, no specific structure is imposed on utility beyond differentiability, concavity and the usual slope conditions. Utility can be unbounded both above and below. In addition, non-capital income and the asset space are allowed to be unbounded. The income process is permitted to be nonstationary, as is required in certain applications.[1]

In terms of connections to the existing literature, perhaps the most closely related results are those found in a recent paper on heterogeneous agent incomplete market economies by Kuhn (2013). Like us, Kuhn permits unbounded rewards and unbounded asset and shock spaces. As one component of his investigation into decentralized equilibria, he studies the same consumer problem considered in this paper. By applying an order-theoretic approach to the analysis of the Coleman operator, he establishes existence of a fixed point, which corresponds to an optimal consumption policy, and provides some convergence results for policy function iteration. On one hand, the present paper is much narrower than Kuhn's paper, in the sense that we concern ourselves only with the consumer's problem. On the other hand, our results on the consumer problem's are considerably sharper. We obtain not only the existence of a fixed point but also uniqueness, as well as geometric rates of convergence of policy function iteration.

Regarding earlier literature, the Coleman operator was originally introduced as a constructive iterative method for solving stochastic optimal growth models (Coleman, 1990). It has often been used to establish existence of equilibrium in economies with distortions, notably by Coleman (1991), Greenwood and Huffman (1995), Datta et al. (2002), Morand and Reffett (2003), Datta et al. (2005) and Mirman et al. (2008). In these papers, fixed points of the Coleman operator were analyzed using a variety of methods related to order preserving structures, continuity, compactness and concavity. The last four papers derive fixed point results in very general settings, but always with either bounded utility, compact state spaces or both.

There are other approaches to the optimization problem treated in this paper besides analysis of the Coleman operator, even in the unbounded setting. One such alternative is value function iteration paired with weighted supremum norms rather that standard supremum norms. While the weighted supremum norm strategy is well suited to convergence analysis,

[1] Predictions of this class of problems can be highly sensitive to the persistence and stationarity of the shock process—hence the need to include the possibility of nonstationary income dynamics. Recent papers addressing this point include Kaplan and Violante (2010), Blundell et al. (2008), Moll (2012) and Kuhn (2013).

it is also very challenging when utility can be unbounded both above *and* below. Some success in this direction has been obtained by Carroll (2004), who considers a related buffer stock savings problem. He develops an ingenious weighted supremum norm approach to optimization via the Bellman operator. However, his results are more specialized, as they are tied to a particular class of utility functions. Moreover, the value function iteration approach tends to give weaker results in terms of convergence of policy functions. For example, Santos (2000) provides a way to establish bounds on policy function deviation from value function deviation. Those bounds do not apply here, but even if they did they would give worse convergence rates than those obtained below. The reason is that moving from value function results to policy function results involves a loss. Indeed, when the uniform deviation between the approximate and true value functions is $O(\gamma^n)$ for some $\gamma < 1$, the rate obtained by Santos (2000, Theorem 3.3) for policies is only $O(\gamma^{n/2})$.

As a final remark on the computational literature, we also note the work of Moreira and Maldonado (2003), which also relates to the contractiveness of policy iteration. This work is interesting and merits further investigation but it is not closely related to our research as it analyzes only local convergence in a neighborhood of a "stationary point" of the policy function, and requires interiority. As for our optimality results, relatively similar findings were obtained by Rabault (2002) using a less standard optimality criterion. Some of our underlying optimality results could also potentially be obtained by modifying the arguments found in Deaton (1991), Deaton and Laroque (1992, 1996), Chambers and Bailey (1996), Kamihigashi (2007) or Le Van and Vailakis (2012).

The paper proceeds as follows. Section 2 describes the model. Initial optimality results are given in Section 3. Section 4 contains our main results on policy function iteration. Numerical issues are discussed in Section 5. Section 6 concludes. All proofs can be found in Section Appendix A.

2. Set up

We consider a standard optimal savings problem, also known as an income fluctuation problem. In the problem, an agent chooses a consumption plan $\{c_t\}_{t \geq 0}$ to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the constraints

$$c_t + a_{t+1} \leq R a_t + y_t, \quad c_t \geq 0, \quad a_t \geq -b, \quad t = 0, 1, \ldots \tag{1}$$

Here $\beta \in (0,1)$ is the discount factor, a_t is asset holdings at time t, c_t is consumption, $R := 1+r$ where r is the interest rate, b is an exogenous borrowing constraint, and y_t is non-capital income.

Assumption 2.1. $R > 1$, $\beta R < 1$ and $b = 0$.

Assumption 2.2. The utility function $u: (0,\infty) \to \mathbb{R}$ is continuously differentiable, strictly increasing and strictly concave, with $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$.

Assumption 2.3. The income process $\{y_t\}$ takes the form $y_t = y(z_t)$, where $y: \mathbb{R}^d \to (0,\infty)$ is continuous and increasing, and $\{z_t\}$ is a vector-valued Markov process on $\mathbb{R}^d$ with Feller and increasing Markov kernel Π.[2]

Assumption 2.4. $M(z) := \sum_{t=0}^{\infty} R^{-t} \mathbb{E}[y_t | z_0 = z]$ is finite for all $z \in \mathbb{R}^d$.

Assumption 2.5. The supremum of $\mathbb{E}[u'(y_{t+1}) | z_t = z]$ over $z \in \mathbb{R}^d$ is finite.

Regarding Assumption 2.1, we assume that $b = 0$ because it simplifies the exposition and costs no generality. To see that this is so, observe that, if $b > 0$, then, since $c_t + a_{t+1} \leq R a_t + y_t$ is equivalent to $c_t + a_{t+1} + b \leq R(a_t + b) - rb + y_t$, we can rewrite the constraints as $\hat{a}_t \geq 0$ and $c_t + \hat{a}_{t+1} \leq R \hat{a}_t + \hat{y}_t$ where $\hat{a}_t := a_t + b$ and $\hat{y}_t := y_t - rb$. However, for implementations with $b > 0$, care must be taken to reinterpret the assumptions placed on y_t as assumptions on $\hat{y}_t$.

Assumption 2.4 is a growth restriction on non-capital income that generalizes the common assumption of a bounded income process (in which case finiteness of $M(z)$ is trivial). It also holds if $\{y_t\}$ is either mean-reverting or nonstationary but does not grow too fast. Assumption 2.5 states that expected marginal utility of next period consumption for an agent without assets is still finite. An assumption along these lines cannot be avoided in the current setting, where income is stochastic and can be arbitrarily small (otherwise finiteness of the expectation in the Euler equation cannot be guaranteed).

A variety of income processes satisfy our assumptions, including models with transitory and permanent components and shocks with heavy tails. For example, suppose that income is the sum of permanent and transitory components ξ_t and η_t, where $\{\xi_t\}$ follows a martingale and $\{\eta_t\}$ is uncorrelated. In particular, suppose that $\xi_{t+1} = \xi_t u_{t+1}$ where $\{u_t\}$ is IID and lognormal with unit mean, and that $\{\eta_t\}$ is IID with $\mathbb{E}[u'(\eta_t)] < \infty$. This can be placed in our framework by setting $z_t = (\xi_t, \eta_t)$ and $y_t = y(z_t) = \xi_t + \eta_t$. All of the preceding assumptions are satisfied. Assumption 2.4 holds because permanent income is

[2] In particular, $\int h(z') \Pi(z, dz')$ is continuous in z whenever $h: \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous, and increasing in z whenever h is bounded and increasing.

a martingale, and hence $\mathbb{E}[y_t | z_0 = z]$ is constant. Assumption 2.5 holds because $y_{t+1} \geq \eta_{t+1}$, implying

$$\mathbb{E}[u'(y_{t+1}) | z_t = z] \leq \mathbb{E}[u'(\eta_{t+1})] < \infty.$$

For the remainder of the paper, Assumptions 2.1–2.5 are all assumed to hold.

3. Optimality

The asset space is $\mathbb{R}_+ := [0, \infty)$ and the state space is $\mathbb{S} := \mathbb{R}_+ \times \mathbb{R}^d$. Let $\mathcal{F}_t$ be the σ-algebra generated by $\{z_1, \ldots, z_t\}$. A *feasible consumption path* from $(a, z) \in \mathbb{S}$ is a consumption sequence $\{c_t\}$ such that c_t is $\mathcal{F}_t$ measurable for all t, the constraints in (1) are satisfied, and $(a_0, z_0) = (a, z)$. The *value function* $V : \mathbb{S} \to \mathbb{R}$ is defined by

$$V(a, z) := \sup \mathbb{E}\left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \quad ((a, z) \in \mathbb{S}) \tag{2}$$

where the supremum is over all feasible consumption paths from (a,z). Assumptions 2.1–2.5 imply that $V(a, z) < \infty$ for any initial conditions in $(a, z) \in \mathbb{S}$, as shown in the appendix. An *optimal consumption path* from (a,z) is a feasible consumption path from (a,z) that attains the supremum in (2). Given strict concavity of u, if an optimal consumption path from (a,z) exists then it must be unique.[3]

As is well-known, the bound $c_t \leq Ra_t + y_t$ can be binding in this model (cf., e.g., Deaton, 1991), and, as a result, the intertemporal first order condition is $u'(c_t) \geq \beta R \mathbb{E}_t[u'(c_{t+1})]$ with equality when $c_t < Ra_t + y_t$. Since $u'(c_t) \geq u'(Ra_t + y_t)$ always holds, this restriction can also be expressed by the single equality

$$u'(c_t) = \max\{\beta R \mathbb{E}_t[u'(c_{t+1})], u'(Ra_t + y_t)\}, \tag{3}$$

where the expectation $\mathbb{E}_t$ conditions on $\mathcal{F}_t$. Condition (3) is the (sequential) Euler equation for our problem. The transversality condition is

$$\lim_{t \to \infty} \beta^t \mathbb{E}[u'(c_t) a_{t+1}] = 0. \tag{4}$$

It is generally recognized that for concave problems such as this one, the Euler equation and the transversality condition are sufficient for optimality. We could not locate such a result in the existing literature that covers the present case, where shocks are Markovian and constraints are occasionally binding.[4] Hence, for completeness, we prove the following theorem.

Theorem 3.1. *Let $(a, z) \in \mathbb{S}$ and let $\{c_t\}$ be a feasible consumption path from (a,z). If $\{c_t\}$ and the corresponding asset path $\{a_t\}$ satisfy the Euler equation (3) and the transversality condition (4), then $\{c_t\}$ is the unique optimal path from (a,z).*

The proof can be found in Section Appendix A. In the next section we turn to our main results, which pertain to identification, characterization and computation of optimal paths.

4. Existence and computation

In what follows, $\| \cdot \|$ always represents the supremum norm, "increasing" is synonymous with "nondecreasing" and "decreasing" is synonymous with "nonincreasing". A derivative or partial derivative of a function of $a \in \mathbb{R}_+$ evaluated at the lower limit $a = 0$ is just a right-hand derivative. All proofs are deferred to Section Appendix A.

4.1. *Coleman's operator*

The Euler equation (3) can be expressed in terms of policy functions as

$$u'(c(a, z)) = \max\left\{ \beta R \int u'\{c[Ra + y(z) - c(a, z), \hat{z}]\} \Pi(z, d\hat{z}), \quad \varphi(a, z) \right\}, \tag{5}$$

where

$$\varphi(a, z) := u'(Ra + y(z)), \quad ((a, z) \in \mathbb{S}).$$

Here $c(a, z)$ is understood to be consumption at state $(a, z) \in \mathbb{S}$. Eq. (5) is a functional equation in c. In order to identify a solution, let $\mathcal{C}$ be the set of continuous increasing functions $c : \mathbb{S} \to \mathbb{R}$ such that $0 < c(a, z) \leq Ra + y(z)$ for all $(a, z) \in \mathbb{S}$ and $\| u' \circ c - \varphi \| < \infty$. The set $\mathcal{C}$ identifies a set of candidate consumption functions. In order to compare two policies, we pair $\mathcal{C}$ with the distance

$$\rho(c, d) := \| u' \circ c - u' \circ d \| \quad (c, d \in \mathcal{C})$$

[3] Given our assumptions, the function $\{c_t\} \mapsto \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$ is strictly concave over the set of feasible consumption paths from (a,z), and the latter is convex.

[4] The closest result of which we are aware is Theorem 1.2 of Schechtman and Escudero (1977), which treats the income fluctuation problem with bounded IID shocks. Note however that the concept of optimality used in that paper is the "overtaking" criterion.

that evaluates their maximal difference in terms of marginal utility. Note that ρ is finite on $\mathcal{C}$, since $\|u' \circ c - u' \circ d\| \leq \|u' \circ c - \varphi\| + \|u' \circ d - \varphi\|$, and the last two terms are finite by our definition of $\mathcal{C}$.

We let K denote Coleman's policy function operator (Coleman, 1990), slightly modified to incorporate occasionally binding constraints. For given $c \in \mathcal{C}$, the value $Kc(a,z)$ is the unique value t such that $0 < t \leq Ra + y(z)$ and

$$u'(t) = \max\left\{\beta R \int u'\{c[Ra + y(z) - t, \hat{z}]\} \Pi(z, d\hat{z}),\ \varphi(a,z)\right\}. \tag{6}$$

It is immediate from the definition of K that any fixed point of K in $\mathcal{C}$ solves the functional Euler equation (5), and, conversely, any solution to (5) is a fixed point of K.

Proposition 4.1. *The following statements are true*:

1. *The pair $(\mathcal{C}, \rho)$ is a complete metric space.*
2. *The operator K is a well-defined mapping from $\mathcal{C}$ into itself.*
3. *On $(\mathcal{C}, \rho)$, the operator K is a contraction of modulus βR.*

Proposition 4.1 and Banach's contraction mapping theorem imply that there exists a unique fixed point of K in $\mathcal{C}$, and that this fixed point can be obtained by iteration of K on any initial $c \in \mathcal{C}$. These results are summarized in Corollary 4.1.

Corollary 4.1. *There exists a unique solution c^* to the functional equation (5) in $\mathcal{C}$, and, moreover, $\rho(K^n c, c^*) = O((\beta R)^n)$ for any $c \in \mathcal{C}$.*

Since $c^* \in \mathcal{C}$, we see that c^* is continuous and increasing in both arguments. Regarding dynamics, fix initial condition $(a,z) \in \mathbb{S}$. The consumption path from (a,z) generated by c^* is the path $\{c_t\}$ defined recursively by $(a_0, z_0) = (a,z)$, $c_t = c^*(a_t, z_t)$ and $a_{t+1} = Ra_t + y(z_t) - c_t$. We can now state our main optimality result.

Theorem 4.1. *For any $(a,z) \in \mathbb{S}$, the consumption path from (a,z) generated by c^* is optimal.*

Together, Theorem 4.1 and Corollary 4.1 indicate that by iterating with K on arbitrary $c \in \mathcal{C}$, we can compute a policy c_n such that the deviation from the optimal policy c^* as measured by the distance ρ is arbitrarily small—modulo approximation and numerical error. Since $\rho(c_n, c^*) = \|u' \circ c_n - u' \circ c^*\|$, this translates to uniformly accurate approximation of marginal utility of consumption at the optimum. Accurate computation of marginal utility of consumption can be of interest in and of itself—for example, when computing stochastic discount factors. However, there will be situations where the primary interest is in accurate computation of c^*. In this connection, note that in general we cannot hope to obtain the global uniform convergence

$$\sup_{s \in \mathbb{S}} |c_n(s) - c^*(s)| \to 0$$

as $n \to \infty$. The reason is that $\mathbb{S}$ is unbounded, and, in general, so are c_n and c^*. As a result, the supremum is infinite. Thus, the most we can aim to prove is uniform convergence on compacts, or, more generally, uniform convergence on bounded sets. The next result shows that this convergence does hold, at least when z_t is bounded below. Moreover, it gives us an error bound on the uniform deviation in terms of observable quantities.

To state the result, suppose that Assumption 2.5 is strengthened as follows:

Assumption 4.1. There exists a vector $Z \in \mathbb{R}^d$ such that $z_t \geq Z$ with probability one.

Let $F \subset \mathbb{S}$, let $r_1 := \inf_{(a,z) \in F} u'(Ra + y(z))$, let $r_2 := u'(y(Z))$, let m denote the inverse of u', and let

$$L(R, y, u, F) := \max_{r_1 \leq r \leq r_2} \frac{1}{-u''(m(r))}. \tag{7}$$

Under Assumptions 2.1–2.5 and 4.1 we have the following result:

Theorem 4.2. *If u is twice continuously differentiable and F is bounded, then $L := L(R, y, u, F)$ is a finite positive constant, and, for any $c \in \mathcal{C}$,*

$$\sup_{s \in F} |K^n c(s) - c^*(s)| \leq \frac{L\beta R}{1 - \beta R} \rho(K^n c, K^{n-1} c). \tag{8}$$

Note that all of the quantities on the right-hand size of (8) are observable when generating the sequence $\{K^n c\}$ by iterating on c with K. In view of Proposition 4.1, the right-hand size converges to zero at rate $O((\beta R)^n)$. An immediate corollary is that if $\mathbb{S}$ itself is bounded, then $K^n c$ converges uniformly to c^* at rate $O((\beta R)^n)$ for all $c \in \mathcal{C}$.

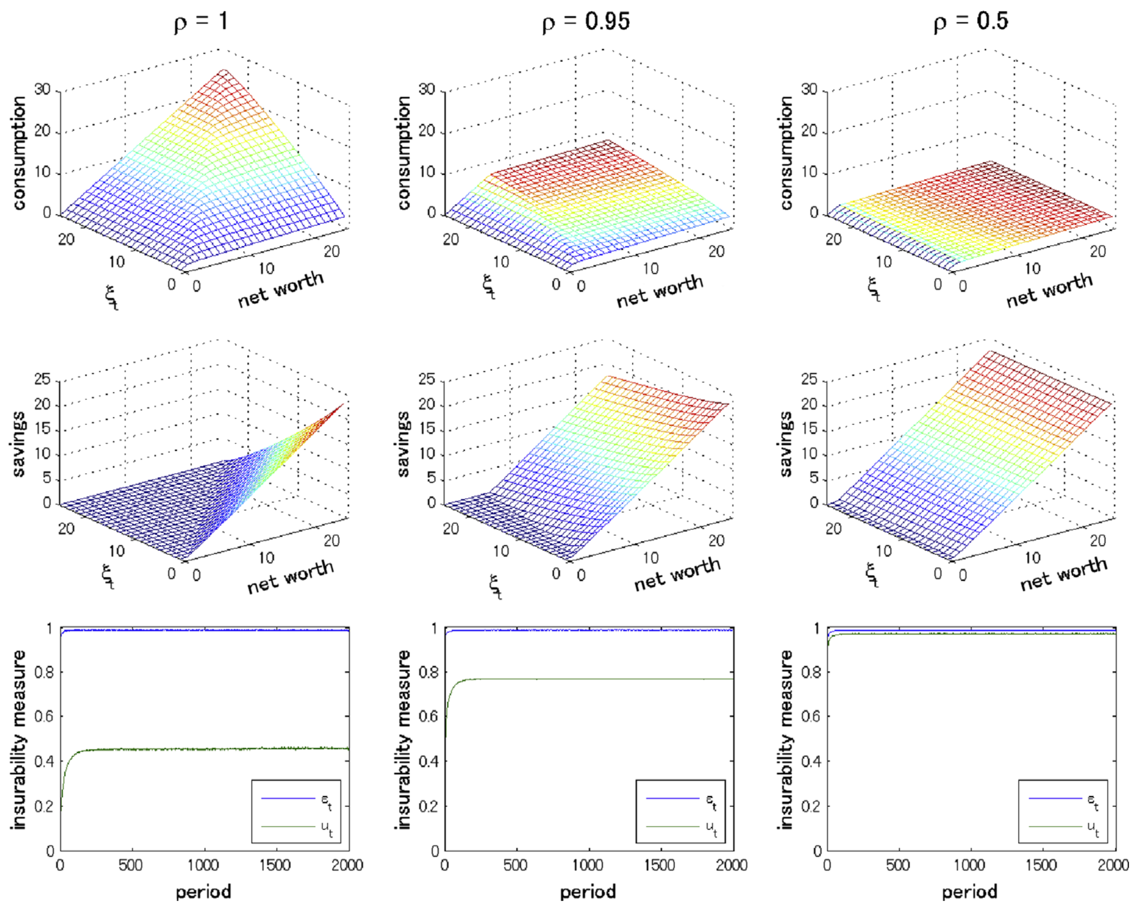

Fig. 1. Baseline exercise, with $b=0$ and $\mu_\eta=-5$.

5. A numerical example

There is a large literature on the estimation of the household income process, in particular, distinguishing between permanent and transitory components of the income process.[5] One contribution of our results is to facilitate calculation of consumption behavior in the infinite horizon income fluctuation model by guaranteeing geometric convergence of policy function iteration from a wide set of initial conditions, and for potentially unbounded and nonstationary income processes. In what follows we implement policy function iteration for a variety of income scenarios, including those where income is nonstationary. Through this exercise, we highlight features of consumption behavior that appear to arise from nonstationarity of the income processes. Our analysis is by no means exhaustive. Understanding consumption behavior under permanent income shocks in an infinite horizon setting would require more detailed analysis.

5.1. Implementation

Suppose that income follows the process

$$y_t = \xi_t \varepsilon_t + \eta_t,$$
$$\ln \xi_t = \rho \ln \xi_{t-1} + \ln u_t,$$

where ε_t is a multiplicative transitory shock and u_t is a shock to the persistent component of income; and that ξ_0 is drawn from a log normal distribution $LN(0,\sigma_0^2)$. The sequences $\{\varepsilon_t\}$ and $\{u_t\}$ are IID, with $\varepsilon_t \sim LN(0,\sigma_\varepsilon^2)$ and $u_t \sim LN(0,\sigma_u^2)$. As in Kaplan and Violante (2010), we take $\sigma_\varepsilon=0.05$, $\sigma_u=0.01$, $\sigma_0=0.15$. The additive component η_t can be interpreted as social

[5] Carroll (2009) and Heathcote et al. (2014) provide characterizations for specific settings. See Jappelli and Pistaferri (2010) for a survey of results.

security, gifts, etc. We assume that $\{\eta_t\}$ is IID and $LN(\mu_\eta, \sigma_\eta^2)$, with $\sigma_\eta = 0.001$. We take $u(c) = c^{1-\theta}/(1-\theta)$ with $\theta = 2$, and set $\beta = 0.95$ and $R = 1.02$. For the remaining parameters we consider the following two scenarios:

1. *Baseline*: $\mu_\eta = -5$, $b = 0$ and $\rho \in \{0.5, 0.95, 1\}$.
2. *Higher expected minimum level of income*: $\mu_\eta = 0$, $b = 0$ and $\rho \in \{0.95, 1\}$.

Scenario 1 is used to illustrate how the policy function changes with the persistence of income shocks when the borrowing constraint is relatively loose.[6] Scenario 2 illustrates how the policy functions change with the persistence of income shocks when consumers expect a higher minimum level of income (e.g., increased social security benefits).

In all cases we combine piecewise linear interpolation to approximate policies with iteration of the Coleman operator. Expectations in the Euler equation are evaluated via Monte Carlo with 1000 draws. When $\rho = 1$ there is no reasonable upper bound on the state, and hence in our computations we set the policy function outside of the grid to its value at the closest grid point. We also simplify the state space by using net worth $w_t := a_t(1+r) + y_t$ and ξ_t as the state variables. The grid points (w_n, ξ_n) lie in $[-bR + 10^{-4}, 100] \times [10^{-4}, 25]$, with 100 points for the w grid and 50 for the ξ grid. The grid is scaled to be more dense when w and ξ are small.[7]

In addition to the policy functions, we also report the insurability measure of ε_t and u_t. The insurability measure of a shock to income is a statistic commonly used to capture the fraction of the shock that does not translate into changes in consumption. For shock x_t it is defined as

$$\varphi_t^x := 1 - \frac{\text{cov}(\Delta \ln c_t, \ln x_t)}{\text{var}(\ln x_t)}.$$

When the permanent income hypothesis holds, the measure is close to zero for permanent shocks and close to one for transitory shocks. We calculate φ_t^x by simulating a panel of 10,000 household and tracking them for 2000 periods, computing φ_t^x for each t. All households start with $a_0 = 0$ and $\xi_0 \sim LN(0, \sigma_0^2)$.

5.2. Results

Fig. 1 corresponds to the baseline exercise, displaying the consumption policy, savings policy and insurability measure for different degrees of income persistence. Note that the change in consumption and savings policies from $\rho = 0.95$ to $\rho = 1$ is much larger than from $\rho = 0.5$ to $\rho = 0.95$. First, consumption level is much higher when $\rho = 1$. Second, when $\rho = 1$, the borrowing constraint binds on a much larger region. In particular, it binds even in the low net worth and high ξ region. Since temporary income shocks average out, this region is where the consumer has low asset and high income relative to asset. When $\rho = 1$, the consumer sees the high income as permanent and consumes. When $\rho = 0.95$, even though the income process is highly persistent, the consumer still sees the high income as temporary and saves for the future.

Another way of looking at this is through the insurability measure for u_t. A much larger fraction of u_t translates into consumption changes when $\rho = 1$ than when $\rho = 0.95$. (When $\rho = 0.5$, consumption responds to u_t as if it is a transitory shock.) In all cases, almost all of the transitory shock ε_t is self-insured.

In scenario two we look at the effect on the results in scenario one caused by an increase in μ_η. Fig. 2 shows that when $\mathbb{E}\eta$ increases from 0.006 to 1, the region where the borrowing constraint binds expands somewhat, but the change is not strikingly large. This suggests that the additive shock η_t is not the cause of the difference in the policy functions in scenario one. While the policy functions do not change much from scenario one, the insurability measure is noisier for u_t and lower for ε_t. The difference in the insurability measure between $\rho = 1$ and $\rho = 0.95$ also disappears. This is perhaps because η_t is an additive shock that acts like a noise term. When μ_η increases, a larger fraction of total income becomes transitory so that even when the persistence of ξ_t increases, the persistence of total income does not increase as much.

6. Conclusion

This paper studies the income fluctuation problem with unbounded utility, assets, income and consumption. We show that the Coleman operator is a contraction mapping over a set candidate consumption policies when endowed with a metric measuring marginal utility, and that this metric is complete. We prove that its fixed point is the unique optimal policy, and that, even though rewards and marginal utility are unbounded, policy function iteration always converges to the optimal policy at a geometric rate. In addition, we obtain computable error bounds on the supremum deviation between the current iterate and the optimal policy c^*. Some numerical examples are presented.

In the context of the income fluctuation problem, it may be possible to vary our assumptions while obtaining the same conclusions, or to develop tighter bounds using structure available in particular applications. It might also be possible to connect our results to numerical techniques involving endogenous grids, or to apply our methods to other dynamic

[6] In Aiyagari (1994), the natural debt limit is $\min y/r$, where $\min y > 0$. At this limit the consumer is never borrowing constrained. Here, since y_t is not bounded away from zero, we approximate the natural debt limit by setting $\mathbb{E}[\eta_t] \approx 0$ and taking $b = 0$.

[7] Varying the number of draws used for integration, the grid range and grid density gives essentially the same results. A more detailed description plus code for running the simulations can be found at https://github.com/jstac/policy_iteration.

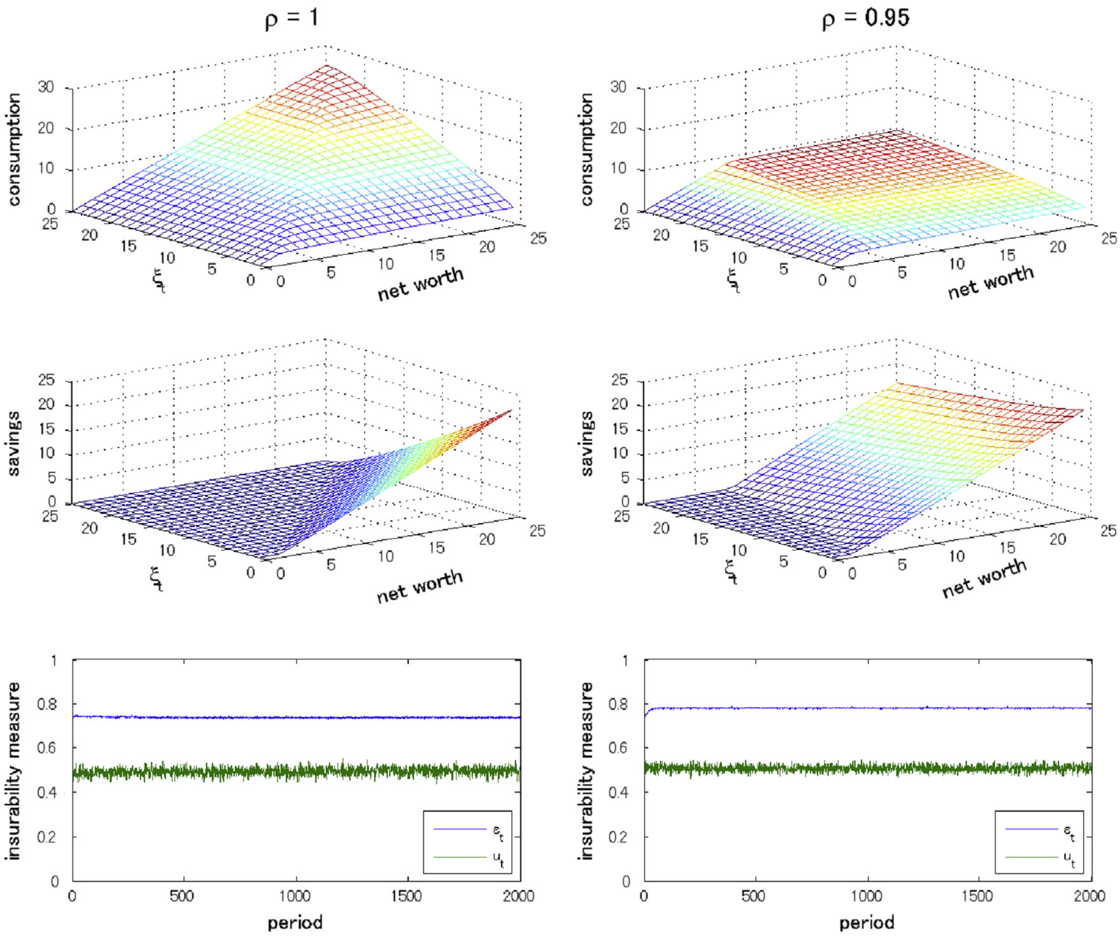

Fig. 2. Higher minimum income, with $b=0$ and $\mu_\eta=0$.

programming problems. For example, the Coleman operator may turn out to be a contraction mapping for other programming problems characterized by Euler equations once the right metric is obtained. Furthermore, one can potentially embed the current model with permanent shocks in a perpetual youth framework and solve for a stationary competitive equilibrium. These ideas are left for future research.

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Appendix A

A.1. Proofs from Section 3

We begin with an auxiliary finiteness result, which shows that the consumer problem is always well defined. Given our assumptions on u, there exists an $L<\infty$ with $u(c)\le c+L$. Using this L, we have the following:

Lemma A.1. *For any initial conditions* $(a_0, z_0)\in\mathbb{S}$ *and feasible path* $\{c_t\}$ *we have*

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t u(c_t)\right\}\le\frac{Ra_0+M(z_0)}{1-\beta R}+\frac{L}{1-\beta}.$$

Proof of Lemma A.1. Recall that L is a constant satisfying $u(c) \le c + L$, and hence

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) \le \mathbb{E} \sum_{t=0}^{\infty} \beta^t c_t + \frac{L}{1-\beta}. \tag{9}$$

Regarding assets, it is elementary to show that

$$a_t \le R^t a_0 + R^t \sum_{j=0}^{t-1} R^{-j-1} y_j, \quad t = 1, 2, \ldots \tag{10}$$

Since $c_t \le Ra_t + y_t$, we then have $c_t \le R^{t+1} a_0 + R^t \sum_{j=0}^{t} R^{-j} y_j$. Combining this bound with (9) and using the definition of $M(z_0)$ gives the desired inequality. □

Proof of Theorem 3.1. As discussed in Section 2, at most one optimal path exists in this model, so we need only prove that the path in question is maximal. To this end, fix $(a, z) \in \mathbb{S}$ and let $\{c_t\}$ be a feasible consumption path from (a,z) such that $\{c_t\}$ and the corresponding asset path $\{a_t\}$ satisfy the Euler equation (3) and the transversality condition (4). Let $\{\hat{c}_t\}$ be another feasible consumption path from (a,z) with corresponding asset path $\{\hat{a}_t\}$. Let

$$Q_T := \sum_{t=0}^{T} \beta^t \mathbb{E}\{u(\hat{c}_t) - u(c_t)\}.$$

It suffices to prove that $\lim_{T\to\infty} Q_T \le 0$. To this end, observe that, by concavity of u,

$$\begin{aligned} Q_T &= \sum_{t=0}^{T} \beta^t \mathbb{E}\{u(R\hat{a}_t + y_t - \hat{a}_{t+1}) - u(Ra_t + y_t - a_{t+1})\} \\ &\le \sum_{t=0}^{T} \beta^t \mathbb{E}\{Ru'(Ra_t + y_t - a_{t+1})(\hat{a}_t - a_t) - u'(Ra_t + y_t - a_{t+1})(\hat{a}_{t+1} - a_{t+1})\} \\ &= \sum_{t=0}^{T} \beta^t \mathbb{E}\{Ru'(c_t)(\hat{a}_t - a_t) - u'(c_t)(\hat{a}_{t+1} - a_{t+1})\}. \end{aligned}$$

By simple rearrangement, and using the fact that $\hat{a}_0 = a_0$ and hence $Ru'(c_0)(\hat{a}_0 - a_0) = 0$, we can write this bound as

$$Q_T \le -\sum_{t=0}^{T-1} \beta^t \mathbb{E}\{[u'(c_t) - \beta R u'(c_{t+1})](\hat{a}_{t+1} - a_{t+1})\} - \beta^T \mathbb{E} u'(c_T)(\hat{a}_{T+1} - a_{T+1}).$$

Suppose for the moment that

$$\mathbb{E}\{[u'(c_t) - \beta R u'(c_{t+1})](\hat{a}_{t+1} - a_{t+1})\} \ge 0, \quad \forall\, t \ge 0. \tag{11}$$

In this case we have

$$Q_T \le -\beta^T \mathbb{E} u'(c_T)(\hat{a}_{T+1} - a_{T+1}) \le \beta^T \mathbb{E} u'(c_T) a_{T+1} \to 0,$$

where the second inequality is by $\hat{a}_{T+1} \ge 0$ and the final convergence is by transversality. Hence it remains only to show that (11) holds. Using the law of iterated expectations and the fact that, by feasibility, the random variables c_t, a_{t+1} and $\hat{a}_{t+1}$ are all $\mathcal{F}_t$−measurable, we can write the inequality in (11) as

$$\mathbb{E}\{[u'(c_t) - \beta R \mathbb{E}_t u'(c_{t+1})](\hat{a}_{t+1} - a_{t+1})\} \ge 0. \tag{12}$$

By subtracting $\beta R \mathbb{E}_t u'(c_{t+1})$ from both sides of the Euler equation (3), we can write the expression inside the outer expectation as

$$Y_t := [u'(Ra_t + y_t) - \beta R \mathbb{E}_t u'(c_{t+1})]_+ (\hat{a}_{t+1} - a_{t+1}),$$

where $[x]_+ := 0 \vee x$. We claim that $Y_t \ge 0$ almost surely. By writing Y_t as $Y_t \mathbb{1}\{c_t = Ra_t + y_t\} + Y_t \mathbb{1}\{c_t < Ra_t + y_t\}$ we can consider the cases $c_t = Ra_t + y_t$ and $c_t < Ra_t + y_t$ in turn. First suppose that $c_t = Ra_t + y_t$. In this case we have $a_{t+1} = 0$, and hence $\hat{a}_{t+1} - a_{t+1} \ge 0$. It follows that $Y_t \ge 0$. Next suppose that $c_t < Ra_t + y_t$. Then $\beta R \mathbb{E}_t u'(c_{t+1}) = u'(c_t) > u'(Ra_t + y_t)$, from which it is immediate that Y_t=0. Thus, in either case we have $Y_t \ge 0$, and hence (12) is valid. The proof is now complete. □

A.2. Proofs from Section 4: preliminaries

We now turn to some ideas that will be used as stepping stones to proving our main results. They concern evolution of the pricing functionals, and extend results found in Deaton (1991), Deaton and Laroque (1992) and Chambers and Bailey (1996) to a more general setting. In the discussion, we repeatedly use the following simple fact: If $h_1 \le h_2$ are decreasing functions on $\mathbb{R}$ with (necessarily unique) fixed points x_1 and x_2, then $x_1 \le x_2$. We let $\mathcal{P}$ be the set of continuous decreasing functions p from $\mathbb{S}$ to $\mathbb{R}$ such that $p \ge \varphi$ and $\| p - \varphi \| < \infty$, where $\varphi$is as defined immediately below (5). Heuristically, $\mathcal{P}$ is the set of pricing functionals corresponding to the consumption policies in $\mathcal{C}$. Consider the pricing functional equation:

$$p(a,z) = \max\left\{ \beta R \int p[Ra + y(z) - (u')^{-1}(p(a,z)), \hat{z}] \Pi(z, d\hat{z}),\ \varphi(a,z) \right\}, \tag{13}$$

which is a functional equation in p. Let T be the operator from $\mathcal{P}$ to $\mathcal{P}$ defined as follows: For $p \in \mathcal{P}$, the function Tp is defined such that $Tp(a,z)$ is the unique $r \geq \varphi(a,z)$ solving

$$r = \max\left\{\beta R \int p[Ra + y(z) - (u')^{-1}(r), \hat{z}]\ \Pi(z, d\hat{z}),\ \varphi(a,z)\right\}. \tag{14}$$

We now state four lemmas and then give their proofs: Lemmas A.2 and A.3 show that Tp is well-defined for $p \in \mathcal{P}$, and, moreover, that $p \in \mathcal{P}$ implies $Tp \in \mathcal{P}$. Lemmas A.4 and A.5 give useful properties of T. Lemma A.5 is similar to Theorem 2 of Chambers and Bailey (1996), who studied the operator T in the context of commodity pricing models.

Lemma A.2. *Given $p \in \mathcal{P}$ and $(a,z) \in \mathbb{S}$, there exists a unique $r \geq \varphi(a,z)$ that solves (14).*

Lemma A.3. *T maps $\mathcal{P}$ into itself.*

Lemma A.4. *T is order preserving on $\mathcal{P}$ when $\mathcal{P}$ is endowed with the usual pointwise order.*

Lemma A.5. *The set $\mathcal{P}$ endowed with the metric $d(p,q) = \|p-q\|$ is a complete metric space, and T is a contraction on $(\mathcal{P}, d)$ of modulus βR.*

Proof of Lemma A.2. Fix $p \in \mathcal{P}$ and $(a,z) \in \mathbb{S}$. Let $h_p(r) =: h_p(r; a, z)$ be the function on $[\varphi(a,z), \infty)$ defined by

$$h_p(r) := \max\left\{\beta R \int p[Ra + y(z) - (u')^{-1}(r), \hat{z}]\Pi(z, d\hat{z}),\ \varphi(a,z)\right\}. \tag{15}$$

The function h_p is decreasing, and sends $[\varphi(a,z), \infty)$ into itself. Of these claims, the only nonobvious one is that $h_p(r)$ is always finite. It suffices to check that $h_p(r)$ is finite at $r = \varphi(a,z)$, which in turn reduces to the claim that $\int p(0,\hat{z})\Pi(z,\hat{z}) < \infty$. To see this, recall that $\|p-\varphi\|$ is finite, and hence there exists a $K < \infty$ with $p \leq \varphi + K$. This leads to the bound

$$h_p(r) \leq \int \varphi(0,\hat{z})\Pi(z,\hat{z}) + \varphi(0,z) + K = \int u'[y(\hat{z})]\Pi(z,\hat{z}) + u'[y(z)] + K. \tag{16}$$

Finiteness of the right-hand side is guaranteed by Assumptions 2.3 and 2.5.

In addition, a simple application of the dominated convergence theorem shows that h_p is continuous. Clearly $h_p(\varphi(a,z)) \geq \varphi(a,z)$, and $h_p(r)$ converges to a finite constant as $r \to \infty$. Existence of a fixed point in $[\varphi(a,z), \infty)$ now follows from the intermediate value theorem. Since h_p is decreasing, the fixed point is unique. □

Proof of Lemma A.3. Fix $p \in \mathcal{P}$. Let $h_p(r; a, z)$ be as in (15). To see that Tp is continuous, recall that, by the proof of Lemma A.2, $r \mapsto h_p(r; a, z)$ takes values in a closed interval $I(a,z) := [\varphi(a,z), \Phi(a,z)]$, where $\Phi(a,z)$ is the right-hand side of (16). Since the correspondence $(a,z) \mapsto I(a,z)$ is nonempty, compact-valued and continuous, and since

$$\text{gr}\, I \ni (a,z,r) \mapsto h_p(r; a, z) \in I(a,z)$$

is continuous, the fixed point of $r \mapsto h_p(r; a, z)$ is also continuous in (a,z).[8] In other words, Tp is continuous on $\mathbb{S}$.

To show that Tp is decreasing on $\mathbb{S}$, let (a_1, z_1) and (a_2, z_2) be points in $\mathbb{S}$ with $(a_1, z_1) \leq (a_2, z_2)$. Let $h_i(r) := h_p(r; a_i, z_i)$ for $i = 1, 2$. Let r_i be the fixed point of h_i. Given their definition, to show that $r_2 \leq r_1$, it suffices to show that $h_2 \leq h_1$ pointwise. To see the latter, pick any r and observe that

$$\begin{aligned} h_2(r) &= \max\left\{\beta R \int p[Ra_2 + y(z_2) - (u')^{-1}(r), \hat{z}]\Pi(z_2, d\hat{z}),\ \varphi(a_2, z_2)\right\} \\ &\leq \max\left\{\beta R \int p[Ra_1 + y(z_1) - (u')^{-1}(r), \hat{z}]\Pi(z_2, d\hat{z}),\ \varphi(a_2, z_2)\right\}. \end{aligned}$$

Since p is decreasing and Π is an increasing kernel, it follows that $\int p(x,\hat{z})\Pi(z_2, d\hat{z}) \leq \int p(x,\hat{z})\Pi(z_1, d\hat{z})$ for all $x \in \mathbb{R}_+$, and therefore

$$h_2(r) \leq \max\left\{\beta R \int p[Ra_1 + y(z_1) - (u')^{-1}(r), \hat{z}]\Pi(z_1, d\hat{z}), \varphi(a_1, (z_1))\right\} = h_1(r).$$

In conclusion, we have $h_2 \leq h_1$, and hence $r_2 \leq r_1$. That is, $Tp(a_2, z_2) \leq Tp(a_1, z_1)$, and Tp is decreasing as claimed.

To complete our proof of the claim that $Tp \in \mathcal{P}$ whenever $p \in \mathcal{P}$, it remains to show that $p \in \mathcal{P}$ implies $\|Tp - \varphi\| < \infty$. To see this, pick any $(a,z) \in \mathbb{S}$. Since $Tp \geq \varphi$ (Lemma A.2),

$$\begin{aligned} |Tp(a,z) - \varphi(a,z)| &= Tp(a,z) - \varphi(a,z) \\ &\leq \beta R \int p[Ra + y(z) - (u')^{-1}(p(a,z)), \hat{z}]\Pi(z, d\hat{z}). \\ &\leq \int p(0,\hat{z})\Pi(z, d\hat{z}). \end{aligned}$$

[8] See, for example, Theorem B.1.4 in Stachurski (2009).

Since $p \in \mathcal{P}$ we have $\|p-\varphi\| < \infty$, and hence there is a finite K with

$$|Tp(a,z)-\varphi(a,z)| \le \int \varphi(0,\hat{z})\Pi(z,d\hat{z})+K = \int u'[y(\hat{z})]\Pi(z,d\hat{z})+K.$$

The right-hand side is bounded by Assumption 2.5. Hence $\|Tp-\varphi\| < \infty$. □

Proof of Lemma A.4. Pick any $p_1,p_2 \in \mathcal{P}$ with $p_1 \le p_2$. We claim that $Tp_1 \le Tp_2$. To see this, fix any $(a,z) \in \mathbb{S}$. Let $h_i = h_{p_i}$, so that, in particular,

$$h_i(r) := \max\left\{\beta R \int p_i[Ra+y(z)-(u')^{-1}(r),\hat{z}]\Pi(z,d\hat{z}), \varphi(a,z)\right\}$$

for $i=1,2$. By definition, $Tp_i(a,z)$ is the fixed point of h_i, and $h_1 \le h_2$ clearly holds. Hence $Tp_1(a,z) \le Tp_2(a,z)$, and, more generally, $Tp_1 \le Tp_2$ on $\mathbb{S}$. □

Proof of Lemma A.5. Regarding completeness of $(\mathcal{P},d)$, let $\mathcal{P}_0$ be all $p:\mathbb{S}\to\mathbb{R}$ such that $d(p,\varphi)=\|p-\varphi\|<\infty$, and let $b\mathbb{S}$ be all bounded functions from $\mathbb{S}$ to $\mathbb{R}$. Let τ be an operator on $\mathcal{P}_0$ defined by $\tau(p)=p-\varphi$. Evidently $\tau(p)$ is bounded. In fact it is straightforward to show that τ is a bijection from $\mathcal{P}_0$ to $b\mathbb{S}$. Moreover, for any $p,q \in \mathcal{P}_0$ we have

$$d(p,q) = \|p-q\| = \|p-\varphi-(q-\varphi)\| = \|\tau(p)-\tau(q)\|.$$

Hence τ is an isometric isomorphism between $(\mathcal{P}_0,d)$ and $(b\mathbb{S},d)$. Since the latter is complete, so is the former. Finally, $\mathcal{P}$ is a closed subset of $(\mathcal{P}_0,d)$, and hence also complete. □

Contractivity of T will be shown using Blackwell's condition. To apply Blackwell's condition, we need to show that if $p \in \mathcal{P}$ and $\lambda \in \mathbb{R}_+$, then $p+\lambda\mathbb{1}_{\mathbb{S}} \in \mathcal{P}$, and

$$T(p+\lambda\mathbb{1}_{\mathbb{S}}) \le Tp+\beta R\lambda\mathbb{1}_{\mathbb{S}} \tag{17}$$

holds pointwise on $\mathbb{S}$. Fix $p \in \mathcal{P}$ and $\lambda \in \mathbb{R}_+$. That $p+\lambda\mathbb{1}_{\mathbb{S}} \in \mathcal{P}$ is immediate from the definition of $\mathcal{P}$. Regarding (17), let p and λ be as above, and let $q:=p+\lambda\mathbb{1}_{\mathbb{S}}$. Pick any $(a,z) \in \mathbb{S}$. Let r_p stand for $Tp(a,z)$ and let r_q stand for $Tq(a,z)$. Using the fact that $r_p \le r_q$ (since $p \le q$ and T is monotone), we have

$$\begin{aligned}
r_q &= \max\left\{\beta R \int q(Ra+y(z)-(u')^{-1}(r_q),\hat{z})\Pi(z,d\hat{z}), \varphi(a,z)\right\}\\
&\le \max\left\{\beta R \int q(Ra+y(z)-(u')^{-1}(r_p),\hat{z})\Pi(z,d\hat{z}), \varphi(a,z)\right\}\\
&= \max\left\{\beta R \int p(Ra+y(z)-(u')^{-1}(r_p),\hat{z})\Pi(z,d\hat{z})+\beta R\lambda, \varphi(a,z)\right\}\\
&= \max\left\{\beta R \int p(Ra+y(z)-(u')^{-1}(r_p),\hat{z})\Pi(z,d\hat{z}), \varphi(a,z)\right\}+\beta R\lambda\\
&= r_p+\beta R\lambda.
\end{aligned}$$

Since (a,z) was arbitrary we have shown that (17) holds. Given that T is order preserving on $\mathcal{P}$, (see Lemma A.4), we conclude that T is a contraction of modulus βR on the space $(\mathcal{P},\|\cdot\|)$.[9] □

A.3. Proofs from Section 4: Proposition 4.1

We now turn towards the proof of Proposition 4.1. To begin, let U be the mapping from $\mathcal{C}$ to $\mathcal{P}$ defined by $Uc=u'\circ c$.

Lemma A.6. *U is a bijection from $\mathcal{C}$ to $\mathcal{P}$.*

Proof. Fix $c \in \mathcal{C}$. Our first claim is that $Uc \in \mathcal{P}$. Since u' is strictly decreasing, c is increasing and both are continuous functions, Uc is continuous and decreasing. Furthermore, $c \in \mathcal{C}$ implies $0 < c(a,z) \le Ra+y(z)$, and hence $u'(Ra+y(z)) \le u'(c(a,z)) < \infty$. In other words, $Uc \in \mathcal{P}$. That U is one-to-one follows immediately from the strict monotonicity of u'. To show that U is onto, take any $p \in \mathcal{P}$ and let $c:=(u')^{-1}p$. The function c is in $\mathcal{C}$, because $u'(Ra+y(z)) \le p(z,a) < \infty$, and hence $0 < c(a,z) \le Ra+y(z)$. Moreover, $Uc = u'\circ(u')^{-1}\circ p = p$. Hence U is onto as claimed. □

Lemma A.7. *The operators T and K are topologically conjugate, in the sense that $U\circ K = T\circ U$ on $\mathcal{C}$.*

Proof. For $p \in \mathcal{P}$, $(a,z) \in \mathbb{S}$ and $r \ge \varphi(a,z)$, define

$$F(r,p,a,z) := r-\max\left\{\beta R \int p(Ra+y(z)-(u')^{-1}(r),\hat{z})\Pi(z,d\hat{z}(,\varphi(a,z)\right\}.$$

Observe that

$$F(r,p,a,z)=0 \quad \Leftrightarrow \quad r=Tp(a,z), \tag{18}$$

[9] See, for example, Stachurski (2009, Theorem 6.3.8).

and, moreover,

$$F(u'(t), u'\circ c, a, z) = 0 \quad \Leftrightarrow \quad t = Kc(a, z). \tag{19}$$

To show that $UK = TU$ it suffices to show that $K = U^{-1}TU$ on $\mathcal{C}$. To this end, fix $c \in \mathcal{C}$ and $(a, z) \in \mathbb{S}$. Let

$$t := (U^{-1}TUc)(a, z) = (u')^{-1}(T(u'\circ c))(a, z).$$

We claim that, $t = Kc(a, z)$. To see this, observe that by the definition of t we have $u'(t) = T(u'\circ c)(a, z)$. By (18), this is equivalent to $F(u'(t), u'\circ c, a, z) = 0$. In view of (19), this is also equivalent to $t = Kc(a, z)$, as was to be shown. Since $(a, z) \in \mathbb{S}$ was arbitrary, we have established that $Kc = U^{-1}TUc$ on $\mathbb{S}$. Since $c \in \mathcal{C}$ was arbitrary, we have $K = U^{-1}TU$ on $\mathcal{C}$. This is equivalent to the statement in the lemma. □

Proof of Proposition 4.1. Beginning with the claim that $(\mathcal{C}, \rho)$ is a complete metric space, recall from Lemma A.6 that U is a bijection from $\mathcal{C}$ to $\mathcal{P}$. Since we can write the metric ρ as $\rho(c_1, c_2) = \| Uc_1 - Uc_2 \| = d(Uc_1, Uc_2)$, the map U is also an isometry from $(\mathcal{C}, \rho)$ to $(\mathcal{P}, d)$. Hence $(\mathcal{C}, \rho)$ is isometrically isomorphic to $(\mathcal{P}, d)$. As the latter is complete (recall Lemma A.5), so is the former.

To see that K is a well-defined mapping from $\mathcal{C}$ to itself, observe from Lemma A.7 that $K = U^{-1}TU$. Since U is a bijection from $\mathcal{C}$ to $\mathcal{P}$ and T is a well-defined map from $\mathcal{P}$ to itself, it follows immediately that K is a well-defined mapping from $\mathcal{C}$ to itself.

To see that K is a ρ-contraction of modulus βR, observe that, using Lemmas A.5 and A.7, we have

$$\rho(Kc_1, Kc_2) = \| UKc_1 - UKc_2 \| = \| TUc_1 - TUc_2 \| \le \beta R \| Uc_1 - Uc_2 \| = \beta R\, \rho(c_1, c_2).$$

Thus, K is a ρ-contraction of modulus βR as claimed. □

A.4. *Proofs from Section 4: Theorems 4.1 and 4.2*

Proof of Theorem 4.1. That the pathwise Euler equation holds is trivial: Fix $(a_0, z_0) \in \mathbb{S}$ and let $\{c_t\}$ be the path generated by c^*. Since c^* satisfies (5) for any $(a, z) \in \mathbb{S}$, it satisfies (5) at (a_t, z_t) in particular. Using the definition $c_t = c^*(a_t, z_t)$ yields (3).

Regarding the transversality condition (4), let $\{c_t\}$ again be the path generated by c^*, and let $\{a_t\}$ be the corresponding asset path. Since $c^* \in \mathcal{C}$, there exists a finite constant N such that $u'\circ c^* \le \varphi + N$, and hence

$$\mathbb{E} u'(c_t) a_{t+1} = \mathbb{E} u'(c^*(a_t, z_t)) a_{t+1} \le \mathbb{E} u'(Ra_t + y(z_t)) a_{t+1} + N \mathbb{E} a_{t+1}.$$

Since $a_{t+1} \le w_t := Ra_t + y(z_t)$, the right-hand side of this expression is dominated by $\mathbb{E} u'(w_t) w_t + N \mathbb{E} w_t$. Letting $J \in (0, \infty)$ Decomposing the first expectation in this expression over $\{w_t < J\}$ and $\{w_t \ge J\}$, we obtain

$$\mathbb{E} u'(c_t) a_{t+1} \le J \mathbb{E} u'(w_t) + u'(J) \mathbb{E} w_t + N \mathbb{E} w_t.$$

From (10) we have $w_t \le R^{t+1} a_0 + R^t \sum_{j=0}^t R^{-j} y_j$, and hence

$$\mathbb{E} w_t \le R^{t+1} a_0 + R^t M(z_0). = R^t [Ra_0 + M(z_0)].$$

Moreover, by Assumption 2.5, there is a constant N such that $\mathbb{E}_{t-1} u'(y(z_t)) \le N$, and hence, by iterated expectations

$$\mathbb{E} u'(w_t) \le \mathbb{E} u'(y(z_t)) = \mathbb{E}\mathbb{E}_{t-1} u'(y(z_t)) \le N.$$

Combining these last three bounds, we get

$$\beta^t \mathbb{E} u'(c_t) a_{t+1} \le \beta^t JN + (\beta R)^t (u'(J) + N)[Ra_0 + M(z_0)].$$

Since $\beta < 1$ and $\beta R < 1$, this term converges to zero. In other words, the transversality condition holds. □

Proof of Theorem 4.2. Let F be a bounded subset of $\mathbb{S}$, and let $c \in \mathcal{C}$. From the definitions of r_1 and r_2 above Theorem 4.2, it is immediate that

$$r_1 \le u'\circ K^n c(a, z) \le r_2 \quad \text{and} \quad r_1 \le u'\circ c^*(a, z) \le r_2, \quad \forall (a, z) \in F, \tag{20}$$

As in the statement of the theorem, let m denote the inverse of the function u'. By the inverse function theorem and the fact that u'' exists, is continuous and strictly negative, the derivative m' exists on $(0, \infty)$ and is given by $m'(x) = 1/u''(m(x))$. Since $r_1 > 0$ and $r_2 < \infty$ and m' is continuous and strictly negative on $(0, \infty)$, it follows that the constant

$$L(R, y, u, F) := \max_{r_1 \le r \le r_2} |m'(r)| = \max_{r_1 \le r \le r_2} \frac{-1}{u''(m(r))}$$

exists and is finite. If we now fix $(a, z) \in F$ and apply the bound (20), we obtain

$$\begin{aligned} |K^n c(a, z) - c^*(a, z)| &= |m \circ u' \circ K^n c(a, z) - m \circ u' \circ c^*(a, z)| \\ &\le \max_{r_1 \le r \le r_2} |m'(r)| |u' \circ K^n c(a, z) - u' \circ c^*(a, z)| \\ &\le L(R, y, u, F) \rho(K^n c, c^*). \end{aligned}$$

Now observe that, by the triangle inequality,

$$\rho(K^n c, c^*) \le \rho(K^n c, KK^n c) + \rho(KK^n c, Kc^*) \le \beta R \rho(K^{n-1} c, K^n c) + \beta R \rho(K^n c, c^*).$$

$$\therefore \quad \rho\left(K^n c, c^*\right) \le \frac{\beta R}{1 - \beta R} \rho\left(K^{n-1} c, K^n c\right).$$

Combining the last two estimates gives the claim in the theorem. □

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