

Necessary and Sufficient Conditions for Existence and Uniqueness of Recursive Utilities

JAROSLAV BOROVÍČKA and JOHN STACHURSKI*

ABSTRACT

We obtain exact necessary and sufficient conditions for existence and uniqueness of solutions of a class of homothetic recursive utility models postulated by Epstein and Zin. The conditions center on a single test value with a natural economic interpretation. The test sheds light on the relationship between valuation of cash flows, impatience, risk adjustment, and intertemporal substitution of consumption. We propose two methods to compute the test value when an analytical solution is not available. We further provide several applications.

RECURSIVE PREFERENCE MODELS SUCH AS those discussed in Koopmans (1960), Epstein and Zin (1989), and Weil (1990) play an important role in macroeconomic and financial modeling. For example, the long-run risk models analyzed in Bansal and Yaron (2004), Hansen, Heaton, and Li (2008), Bansal, Kiku, and Yaron (2012), and Schorfheide, Song, and Yaron (2018) employ such preferences in discrete-time infinite-horizon settings with a variety of consumption path specifications to help resolve long-standing empirical puzzles identified in the literature.

In recursive utility models, the lifetime value of a consumption stream from a given point in time is expressed as the solution to a nonlinear forward-looking equation. While this representation is convenient and intuitive, it can also be vacuous, in the sense that no finite solution to the forward-looking recursion exists. Moreover, even when a solution is found, this solution lacks predictive content unless some form of uniqueness can also be established. In general,

*Jaroslav Borovíčka is with New York University, Federal Reserve Bank of Minneapolis, and NBER. John Stachurski is with the Research School of Economics, Australian National University. The authors thank Anmol Bhandari, Tim Christensen, Ippei Fujiwara, Jinill Kim, Stefan Nagel, Daisuke Oyama, Guanlong Ren, and the anonymous referees for useful comments and suggestions. Special thanks are due to Mirosława Zima for her valuable input on local spectral radius conditions. Borovíčka gratefully acknowledges financial support from New York University and Federal Reserve Bank of Minneapolis. Stachurski gratefully acknowledges financial support from ARC grant FT160100423. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. We have read *The Journal of Finance's* disclosure policy and have no conflicts of interest to disclose.

Correspondence: Jaroslav Borovíčka, Department of Economics, New York University, 19 W 4th Street, 6th floor, New York, NY 10012; e-mail: jaroslav.borovicka@nyu.edu.

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identifying restrictions that imply both existence and uniqueness of a solution for an empirically relevant class of consumption streams is challenging.

The aim of this paper is to obtain existence and uniqueness results that are as tight as possible in a range of empirically plausible settings, while restricting attention to practical conditions that can be tested in applied work. To this end, we provide conditions for existence and uniqueness of solutions to the class of homothetic preferences studied in Epstein and Zin (1989). These conditions admit both stationary and nonstationary consumption paths. Moreover, they are necessary as well as sufficient, and hence as tight as possible in the setting we consider. In particular, if the conditions hold, then a unique, globally attracting solution exists, while if they do not, then no finite solution exists. Existence of a finite solution is equivalent to the existence of a finite wealth-consumption ratio, a central object of interest in asset pricing.

To describe the setting in more detail, let preferences be defined recursively by

$$V_t = \left[(1 - \beta)C_t^{1-1/\psi} + \beta \{ \mathcal{R}_t(V_{t+1}) \}^{1-1/\psi} \right]^{1/(1-1/\psi)}, \quad (1)$$

where $\{C_t\}$ is a consumption path, V_t is the utility value of the path extending from time t , and \mathcal{R}_t is the Kreps-Porteus certainty-equivalent operator

$$\mathcal{R}_t(V_{t+1}) = (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}. \quad (2)$$

The parameter $\beta \in (0, 1)$ is a time discount factor, while $\gamma \neq 1$ governs risk aversion and $\psi \neq 1$ is the elasticity of intertemporal substitution. We take the consumption stream as given and seek a solution for normalized utility V_t/C_t .

The first step in our approach is to associate with each consumption process the risk-adjusted long-run mean consumption growth rate

$$\mathcal{M}_C := \lim_{n \rightarrow \infty} \left[\mathcal{R} \left(\frac{C_n}{C_0} \right) \right]^{1/n}, \quad (3)$$

where \mathcal{R} is the unconditional version of the Kreps-Porteus certainty equivalent operator. Beginning with the case in which the state vector driving the conditional distribution of consumption growth takes values in a compact set—which is where the sharpest results obtain—we show that a unique solution exists if and only if $\Lambda < 1$, where

$$\Lambda := \beta \mathcal{M}_C^{1-1/\psi}. \quad (4)$$

Under the same compactness restriction, we also show that the condition $\Lambda < 1$ is both necessary and sufficient for global convergence of successive approximations associated with a natural fixed-point mapping. Indeed, our results establish that convergence of successive approximations itself implies that a unique solution exists, and that the limit produced through this process is equal to the solution. Furthermore, we prove that when the condition $\Lambda < 1$

fails, not only does existence and uniqueness of a solution fail, but existence itself fails more specifically.

The value Λ represents the risk-adjusted long-term consumption growth rate, modified by both the degree of impatience and the intertemporal substitutability of consumption. Thus, despite the fact that the preference recursion (1) intertwines the contributions of impatience, intratemporal risk aversion, and intertemporal elasticity of substitution to value, the condition $\Lambda < 1$ effectively separates these forces. Aspects of the consumption growth process, such as its persistence or higher moments of its innovations, matter only through the long-run distribution of consumption growth encoded in \mathcal{M}_C .

From an asset pricing perspective, the condition $\Lambda < 1$ imposes a bound on the average growth rate in the value of long-dated consumption strips as their maturity increases. In particular, in an IID growth setting, $\log \Lambda$ is exactly equal to this growth rate, which must be negative for a finite wealth-consumption ratio to exist. We provide additional discussion on the intuition behind condition (4) in Section II and the applications.

In addition to the preceding results, we use a local spectral theorem to show that

$$\mathcal{M}_C = r(K)^{1/(1-\gamma)}, \quad (5)$$

where $r(K)$ is the spectral radius of a valuation operator K determined by the primitives and clarified below. This result is useful on two levels. First, spectral radii and dominant eigenfunctions associated with valuation operators have been increasingly used to understand long-run risks and long-run values in macroeconomic and financial applications by inducing a decomposition of the stochastic discount factor (see, e.g., Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Qin and Linetsky (2017), or Christensen (2017)). The identification in (5) allows us to connect and draw insights from this literature. Second, computationally, when the state space for the state process is finite, the valuation operator K is just a matrix and the spectral radius is easy to compute, which allows us to compute the test statistic Λ via (5). When the state space is not finite, one can still implement this idea after discretization.

When the state space is high dimensional, accurate discretization is nontrivial and calculation of the spectral radius becomes computationally expensive. For these scenarios, we instead propose a Monte Carlo method to calculate the test value Λ that is based on the idea of simulating consumption paths from a given specification and calculating the risk-adjusted expectation on the right-hand side of (3) by averaging over these paths. This approach is straightforward to implement and relatively insensitive to the dimension of the state space. The routine is easy to parallelize by simulating independent consumption paths along multiple execution threads.

All of the theoretical results on existence, uniqueness, and convergence of successive approximations discussed above are stated in the context of a compact-valued state process, which drives the persistent component of consumption growth. In this setting, we apply a fixed-point theorem due to Du

(1990) that extends to abstract vector space the idea that an increasing concave function f from \mathbb{R}_+ to itself has at most one strictly positive fixed point—and at least one such fixed point if the slope conditions $f'(0) > 1$ and $f'(\infty) < 1$ are satisfied. In the case of the valuation problem considered in this paper, monotonicity and concavity are inherent in the preference specification, while the condition $\Lambda < 1$ is key to the slope conditions. The arguments required for the last step are nontrivial and compactness of the state space plays a significant role.

At the same time, we provide some guidance on the implications of the condition $\Lambda < 1$ when the state space is unbounded. In this setting, we show that under an auxiliary technical restriction related to compactness of the iterates of the operator K , the condition $\Lambda < 1$ is again necessary and sufficient for existence of a solution. As before, convergence of successive approximations to a given function implies that the limiting function is a solution, and this can occur only when $\Lambda < 1$. The identification in (5) continues to be valid when these conditions hold, and the Monte Carlo method described above can still be applied. The proofs use an approximation argument that bootstraps previously stated results from the compact case.

We provide several applications of the numerical and theoretical results in Section III. We start with the model of trend-stationary consumption from Alvarez and Jermann (2005), for which we can characterize our condition analytically. The result reveals that transitory uncertainty in the consumption process, such as stationary fluctuations around a deterministic time trend, is immaterial for the existence of the continuation value.

Next, we focus on the frequently used case in which the dynamics of conditional moments of consumption growth are encoded using a Markov chain. Specifically, we use the calibration from Johannes, Lochstoer, and Mou (2016) and consider two information structures, one in which the agent observes the realizations of the Markov chain and another in which the agent must learn about the underlying state from consumption growth realizations. In the former case, given the simple structure of the state space, evaluating the test condition involves computing the spectral radius in (5) as the largest eigenvalue of a small matrix. In the latter case, the state space is continuous and encodes subjective probabilities of unobserved states of the Markov chain, updated using Bayes's rule. Despite the fact that the state space and transition dynamics are now much more complex, our results reveal that the continuation value exists for exactly the same set of parameters as in the full information case. The underlying reason is that the transitory role of state uncertainty for the conditional distribution of future consumption growth has no effect on the value of \mathcal{M}_C in (3).

Our main quantitative application is the long-run risk model specified in Schorfheide, Song, and Yaron (2018). Existence of finite continuation values is a nontrivial issue in such models because (i) discounting is extremely small in order to amplify the importance of persistent risk components, pushing them toward the boundary between stability and instability, and (ii) the state dynamics are nonlinear and relatively high dimensional. Using numerical

methods, we show that the condition $\Lambda < 1$ is satisfied with arbitrarily small amounts of truncation and almost certainly satisfied in the original model without truncation. We provide several robustness checks that relate alternative approximation procedures and the impact of truncation. We also discuss computation of the wealth-consumption ratio over a range of parameterizations, which provides intuition on failure of the stability condition and nonexistence of solutions.

Finally, we show that our results can also be applied to production economies in which consumption is endogenously determined. In many applications, consumption is cointegrated with an exogenously specified process that drives uncertainty in the model. Since transitory fluctuations in the consumption process are irrelevant, the risk-adjusted long-run growth rate can be directly computed using the exogenous driving process, without knowledge of the specific details of the consumption process.

Turning to existing literature, Epstein and Zin (1989) and Marinacci and Montrucchio (2010) provide sufficient conditions for existence and uniqueness of recursive utilities. These conditions require a finite bound B_c on consumption growth C_{t+1}/C_t that holds asymptotically with probability 1. As a result, they cannot be applied to many recent specifications of the consumption processes, such as the long-run risk specification given in Section III.D, as consumption growth in those settings is unbounded above. Even if a finite bound is obtained by truncation of the shocks, we show that the resulting conditions are always stricter than those presented in this paper, and typically far too strict for realistic parameterizations. This is due to the fact that probability-one bounds restrict utility uniformly along every future consumption trajectory, while the results in this paper consider what happens on average across all paths. In other words, our results are sharper because recursive utility specifications, while nonlinear, are still defined using integration over future continuation values. Conversely, focusing only on the upper tail of the consumption growth process leads to excessively tight stability conditions.

Another condition for existence of recursive utilities can be found in Alvarez and Jermann (2005), who focus on the case in which consumption has a deterministic time trend. Our condition $\Lambda < 1$ is also weaker than their condition, as shown in Section IV. The intuition behind this is that Alvarez and Jermann (2005) use a fixed-point argument that requires contraction in one step. In contrast, the restriction $\Lambda < 1$ is an asymptotic condition that ignores short-run fluctuations in consumption.

Also related is Hansen and Scheinkman (2012), who study Epstein-Zin utility models with unbounded consumption growth. Their approach is to connect the solution to the Epstein-Zin utility recursion and the Perron-Frobenius eigenvalue problem associated with a linear operator, denoted in their paper by \mathbb{T} , which is proportional to the operator K discussed above. Consumption growth is a function of an unbounded exogenous state process. In this setting, they show that a solution exists when a joint restriction holds on the spectral radius of \mathbb{T} and the preference parameters, along with integrability conditions on the eigenfunctions of \mathbb{T} mentioned above. They also obtain a uniqueness result for some parameter values (although not the most empirically relevant ones).

For the case in which \mathbb{X} is compact, our approach has the following advantages. First, we obtain uniqueness of the solution for all parameterizations. Second, we show that our conditions are necessary as well as sufficient, both for existence and for uniqueness. Third, we obtain a globally convergent computational method and show that it converges if and only if a solution exists. Fourth, we provide multiple representations of the test value, strengthening the economic interpretation, as well as a method of computation that can be applied in high-dimensional settings. Fifth, we avoid the auxiliary conditions in Hansen and Scheinkman (2012) involving integrability restrictions on the eigenfunctions of the operator \mathbb{T} , which means that all of our conditions are straightforward to test in applied settings. For the case in which \mathbb{X} is not compact, our results also augment those of Hansen and Scheinkman (2012) by showing that the condition $\Lambda < 1$ is necessary as well as sufficient for existence of a solution.

In another related study, Guo and He (2018) consider an extension to the Epstein-Zin recursive utility model that includes utility measures for investment gains and losses. As part of that study, they obtain results for existence, uniqueness, and convergence of solutions to Epstein-Zin recursive utility models with consumption specifications analogous to those in Hansen and Scheinkman (2012), except that the state space is restricted to be finite. In comparison, we allow the state space to be countably or uncountably infinite and we establish not only sufficiency but also necessity.

Finally, the working paper of Pohl, Schmedders, and Wilms (2015) treats existence of Epstein-Zin recursive utilities in a long-run risk setting with stochastic volatility. They obtain a sufficient condition for existence for the case $\gamma > 1/\psi$ by comparing the recursive utility problem to a parallel problem that combines the same consumption process with constant relative risk aversion (CRRA) utility. In contrast, we cover all cases for the preference parameters and treat general consumption paths, rather than focus exclusively on long-run risk models. In addition, our use of monotone concave operator theory allows us to obtain results on uniqueness and convergence of successive approximations.

The rest of the paper is organized as follows. Section I studies the risk-adjusted long-run mean consumption growth rate in more detail. Section II presents our main results. Section III discusses applications. Section IV contrasts our results with alternative sufficient conditions in the previous literature. Section V treats the unbounded case. Section VI concludes. All proofs are deferred to the appendices.¹

I. Consumption Paths and Risk-Adjusted Growth

Before presenting our main results, we introduce our baseline model for consumption paths and address an important issue—since $\Lambda = \beta \mathcal{M}_C^{1-1/\psi}$, the practicality of our condition $\Lambda < 1$ depends on the ability with which one can

¹ Code that replicates our numerical results is provided as Supporting Information in the online version of the article on *The Journal of Finance* website.

accurately evaluate the risk-adjusted long-run mean consumption growth rate \mathcal{M}_C for each consumption path. For some specifications of the consumption path, an analytical expression for \mathcal{M}_C exists. For others, however, no such expression can be obtained. In the second case, we must turn to numerical methods. This section provides examples and an overview of techniques.

A. Consumption Paths

As in Hansen and Scheinkman (2012), we suppose that consumption growth has the generic specification

$$\ln(C_{t+1}/C_t) = \kappa(X_t, X_{t+1}, \varepsilon_{t+1}), \quad (6)$$

where κ is a continuous function, $\{X_t\}$ is an exogenous state process, and $\{\varepsilon_t\}$ is an IID innovation process supported on $\mathbb{Y} \subset \mathbb{R}^k$ and independent of $\{X_t\}$. The state process is assumed to be stationary and Markov, taking values in a subset \mathbb{X} of \mathbb{R}^n . The unconditional density of each X_t is denoted by π . The function $q(x, \cdot)$ represents the conditional density of X_{t+1} given $X_t = x$. All of our results include the case in which \mathbb{X} is finite and, in this case, the transition density $q(x, y)$ should be interpreted as a transition matrix. More generally, the term “density” should be understood as a synonym for “probability mass function.”

ASSUMPTION 1: *The function q is continuous and the ℓ -step transition density q^ℓ is everywhere positive at some $\ell \in \mathbb{N}$.*

Continuity can be ignored when \mathbb{X} is finite. Positivity of q^ℓ for some ℓ means that $\{X_t\}$ is both aperiodic and irreducible, guaranteeing uniqueness of the stationary distribution π and providing regularity for asymptotic values such as the risk-adjusted long-run mean consumption growth rate. Assumption 1 is either satisfied directly in our applications or can be validated after an arbitrarily small perturbation (as is the case for the learning application—see Section III.C).

For example, consider the consumption growth specification

$$\ln(C_{t+1}/C_t) = \mu_c + X_t + \sigma_c \varepsilon_{t+1}, \quad (7)$$

$$X_{t+1} = \rho X_t + \sigma \eta_{t+1}, \quad (8)$$

from Section I.A of Bansal and Yaron (2004). Here, $-1 < \rho < 1$ and $\{\varepsilon_t\}$ and $\{\eta_t\}$ are IID standard normal innovations.² With the state process set to $\{X_t\}$, we have $q(x, \cdot) = N(\rho x, \sigma^2)$ and Assumption 1 is satisfied with $\ell = 1$ whenever $\sigma > 0$.

For this model of consumption dynamics, we can obtain an analytical expression for \mathcal{M}_C , as defined in (3), even though the same is not true for the continuation value V_t . To see this, observe that $C_n/C_0 = \exp(n\mu_c + \sum_{t=1}^n H_t)$, where

² These calculations can be further extended to the case in which consumption growth is a component of a VAR, as in Hansen, Heaton, and Li (2008) or Bansal et al. (2014).

$H_t := X_t + \sigma_c \varepsilon_{t+1}$, so the risk-adjusted long-run mean consumption growth rate is

$$\mathcal{M}_C = \lim_{n \rightarrow \infty} \left\{ \mathcal{R} \exp \left(n\mu_c + \sum_{t=1}^n H_t \right) \right\}^{1/n}.$$

Here, \mathcal{R} is the unconditional Kreps-Porteus certainty-equivalent operator defined by $\mathcal{R}[Y] := \mathbb{E}[Y^{1-\gamma}]^{1/(1-\gamma)}$. Noting that $\{H_t\}$ is Gaussian and setting s_n equal to the variance of $\sum_{t=1}^n H_t$, we have

$$\mathcal{R} \exp \left(n\mu_c + \sum_{t=1}^n H_t \right) = \exp \left\{ n\mu_c + \frac{(1-\gamma)s_n^2}{2} \right\}. \quad (9)$$

Using the fact that $\sum_{t=1}^n H_t$ is the sum of the independent terms $\sum_{t=1}^n X_t$ and $\sigma_c \sum_{t=1}^n \varepsilon_{t+1}$, along with the AR(1) dynamics in (8), straightforward calculations lead to

$$s_n^2 = n\sigma_c + \frac{\sigma^2}{1-\rho^2} \left\{ n + 2(n-1)\frac{\rho}{1-\rho} - 2\rho^2 \frac{1-\rho^{n-1}}{(1-\rho)^2} \right\}.$$

Raising the right-hand side of (9) to the power of $1/n$ and taking the limit yields

$$\mathcal{M}_C = \exp \left\{ \mu_c + \frac{1}{2}(1-\gamma) \left(\sigma_c^2 + \frac{\sigma^2}{(1-\rho)^2} \right) \right\}. \quad (10)$$

While the unconditional variance of the one-period consumption growth rate is $\sigma_c + \sigma^2/(1-\rho^2)$, persistence in consumption growth implies that the long-run variance of the consumption growth rate is given by $\sigma_c + \sigma^2/(1-\rho)^2$. Typical calibrations of risk aversion set $\gamma > 1$ and higher persistence and volatility then contribute negatively to \mathcal{M}_C . For example, under the parameterization of the consumption process in table I of Bansal and Yaron (2004), the expression in (10) yields 1.00045 when $\gamma = 7.5$ and 0.99964 when $\gamma = 12.5$.³

B. The Finite-State Case

The finite-state setting gives us an alternative view on the risk-adjusted long-run mean consumption growth rate \mathcal{M}_C and an alternative way to compute it. In particular, if \mathbb{X} is finite with typical elements x and y , and if K is the matrix with $(x, y)^{\text{th}}$ element

$$K(x, y) := \int \exp[(1-\gamma)\kappa(x, y, \varepsilon)] \nu(d\varepsilon) q(x, y), \quad (11)$$

then (5) holds, that is,

$$\mathcal{M}_C = r(K)^{1/(1-\gamma)} \quad \text{when} \quad r(K) = \max_{\lambda \in E} |\lambda|. \quad (12)$$

³ The parameter values in question are $\mu_c = 0.0015$, $\rho = 0.979$, $\sigma = 0.00034$, and $\sigma_c = 0.0078$.

Here, E is the set of eigenvalues of K , so $r(K)$ is the spectral radius of the matrix K .

To gain some understanding as to why the alternative representation of \mathcal{M}_C in (12) is valid, note that for all x in \mathbb{X} and all n in \mathbb{N} , we have

$$K^n \mathbb{1}(x) = \mathbb{E}_x \exp \left\{ (1 - \gamma) \sum_{t=1}^n \kappa(X_{t-1}, X_t, \varepsilon_t) \right\}. \quad (13)$$

Here, K^n is the n^{th} power of K in (11) and $\mathbb{1}$ is a vector of ones. The expectation \mathbb{E}_x conditions on $X_0 = x$. That the identity in (13) is true at $n = 1$ follows immediately from the definition of K in (11), and the case of general n can be confirmed by induction. Let $\|\cdot\|$ be the L_1 vector norm defined by $\|h\| = \sum_{x \in \mathbb{X}} |h(x)|\pi(x)$. By (13) and the law of iterated expectations, we have

$$\|K^n \mathbb{1}\| = \sum_{x \in \mathbb{X}} K^n \mathbb{1}(x) \pi(x) = \mathbb{E} \exp \left\{ (1 - \gamma) \sum_{t=1}^n \kappa(X_{t-1}, X_t, \varepsilon_t) \right\}.$$

In other words,

$$\|K^n \mathbb{1}\| = \mathbb{E} \left[\left(\frac{C_n}{C_0} \right)^{1-\gamma} \right] = \left[\mathcal{R} \left(\frac{C_n}{C_0} \right) \right]^{1-\gamma}. \quad (14)$$

Gelfand's formula for the spectral radius tells us that $\|K^n\|^{1/n} \rightarrow r(K)$ as $n \rightarrow \infty$ whenever $\|\cdot\|$ is a matrix norm, and in the present context, this result can be modified to show that $\|K^n \mathbb{1}\|^{1/n} \rightarrow r(K)$ also holds.⁴ Connecting the last result with (14) gives

$$\mathcal{M}_C^{1-\gamma} = \lim_{n \rightarrow \infty} \left[\mathcal{R} \left(\frac{C_n}{C_0} \right) \right]^{\frac{1-\gamma}{n}} = \lim_{n \rightarrow \infty} \|K^n \mathbb{1}\|^{1/n} = r(K), \quad (15)$$

which justifies the claim in (12). In several applications below, we use this identity by calculating $r(K)$ numerically and then recovering \mathcal{M}_C via $\mathcal{M}_C = r(K)^{1/(1-\gamma)}$.

C. Numerical Methods

As discussed in the previous section, if the state space is finite, then we can use the identity $\mathcal{M}_C = r(K)^{1/(1-\gamma)}$ obtained in (15) to calculate \mathcal{M}_C , which allows us, in turn, to compute the stability exponent Λ . If, in contrast, \mathbb{X} is not finite, then one option is to discretize the model, leading to a finite state space and a finite set of transition probabilities $q(x, y)$, and then proceed as above. In Appendix A, we verify the accuracy of this methodology when the state space is relatively small.

⁴ This is a finite-dimensional version of the local spectral radius result discussed in the Introduction. A complete proof that extends to infinite-state settings is given in Theorem B1.

One potential issue with the discretization-based method is that the algorithm is computationally inefficient when the state space is large. Thus, we also propose a Monte Carlo method to calculate an approximation to \mathcal{M}_C in such settings. The first step is to replace the limit in the definition of \mathcal{M}_C by some finite but large n , which leads to

$$\mathcal{M}_C(n) := \left\{ \mathbb{E} \left(\frac{C_n}{C_0} \right)^{1-\gamma} \right\}^{\frac{1}{1-\gamma} \frac{1}{n}}. \quad (16)$$

Next, the expectation in (16) is replaced by a sample average over m independent consumption paths that is generated according to the specifications of the model in question. In particular, with $\{C_t^{(j)}\}$ as the j^{th} of the m consumption paths,⁵ we take the approximation

$$\mathcal{M}_C(m, n) := \left\{ \frac{1}{m} \sum_{j=1}^m \left(\frac{C_n^{(j)}}{C_0^{(j)}} \right)^{1-\gamma} \right\}^{\frac{1}{1-\gamma} \frac{1}{n}}. \quad (17)$$

With fixed n and $Y^{(j)} := (C_n^{(j)}/C_0^{(j)})^{1-\gamma}$, the strong law of large numbers yields $\frac{1}{m} \sum_{j=1}^m Y^{(j)} \rightarrow \mathbb{E} Y$ as $m \rightarrow \infty$ with probability 1. While this result is only asymptotic, in Appendix A, we show that the estimator $\mathcal{M}_C(m, n)$ is accurate even when m and n are moderate.

II. Existence and Uniqueness of Recursive Utilities

We now state our main theoretical results. In this section, we restrict attention to the case in which the state space is compact. This also covers the finite-state case—and hence any numerical representation of consumption dynamics. For the sake of brevity, most of our exposition focuses on the continuous-state case. Translations to the finite-state setting are straightforward.

We are interested in the existence, uniqueness, and computability of V_t/C_t in (1) and (2), but it turns out to be convenient to first solve for

$$G_t := \left(\frac{V_t}{C_t} \right)^{1-\gamma}. \quad (18)$$

Using (2), (6), and (18), we can rewrite the preference recursion (1) as

$$G_t = \left\{ 1 - \beta + \beta \left\{ \mathbb{E}_t G_{t+1} \exp \left[(1-\gamma) \kappa(X_t, X_{t+1}, \varepsilon_{t+1}) \right] \right\}^{1/\theta} \right\}^\theta, \quad (19)$$

where

$$\theta := \frac{1-\gamma}{1-1/\psi}.$$

⁵ Since \mathbb{E} in (16) is the unconditional expectation, we draw the initial state $X_0^{(j)}$ from its stationary distribution when computing $\{C_t^{(j)}\}$.

In terms of a stationary Markov solution $G_t = g(X_t)$, the restriction in (19) translates to

$$g(x) = \left\{ 1 - \beta + \beta \left\{ \int g(y) \int \exp[(1 - \gamma)\kappa(x, y, \varepsilon)] \nu(d\varepsilon) q(x, y) dy \right\}^{1/\theta} \right\}^\theta \quad (20)$$

for all $x \in \mathbb{X}$, where ν is the distribution of ε_{t+1} .⁶

We solve (20) for the unknown function g by converting it into a fixed-point problem. As a first step, we define Kg by

$$Kg(x) = \int g(y) \int \exp[(1 - \gamma)\kappa(x, y, \varepsilon)] \nu(d\varepsilon) q(x, y) dy.$$

The action of the linear operator K on g generalizes the idea of applying the matrix K in (11) to a column vector, as per our discussion of the finite-state case in Section I.B. Now let φ be the scalar function

$$\varphi(t) = \{1 - \beta + \beta t^{1/\theta}\}^\theta \quad (21)$$

on \mathbb{R}_+ .⁷ Next, define A as the operator mapping g into Ag , where

$$Ag(x) = \varphi(Kg(x)). \quad (22)$$

Expression (20) can then be written as $g(x) = Ag(x)$. Thus, fixed points of A coincide with solutions to the recursive utility problem.

ASSUMPTION 2: *The state space \mathbb{X} is compact.*

Assumption 2 includes the case in which \mathbb{X} is finite. It holds for any numerical application but is not always satisfied in the theoretical models we consider. A treatment of the unbounded case is provided in Section V.

Let \mathcal{C} be the set of positive functions g on \mathbb{X} such that $g(X_t)$ has finite first moment. With Λ as defined in (4), we can state our main findings as follows.

THEOREM 1 (Existence and Uniqueness): *If Assumptions 1 and 2 hold, then Λ is well defined and the following statements are equivalent:*

- (a) $\Lambda < 1$.
- (b) A has a fixed point in \mathcal{C} .
- (c) There exists a $g \in \mathcal{C}$ such that $\{A^n g\}_{n \geq 1}$ converges to an element of \mathcal{C} .⁸
- (d) A has a unique fixed point in \mathcal{C} .
- (e) A has a unique fixed point g^* in \mathcal{C} and $A^n g \rightarrow g^*$ as $n \rightarrow \infty$ for any g in \mathcal{C} .

⁶ These transformations use homotheticity of Epstein-Zin utility. In particular, existence and uniqueness of V_t/C_t is equivalent to existence and uniqueness of the wealth-consumption ratio, which is equal to $(1 - \beta)^{-1} G_t^{1/\theta}$ in this model.

⁷ We adopt the convention $\infty^\alpha = 0$ whenever $\alpha < 0$ so that, in particular, $\varphi(0) = 0$ when $\theta < 0$.

⁸ Here and below, convergence is in terms of absolute mean error (i.e., L_1 deviation). Thus, the statement that $g_n \rightarrow g$ in \mathcal{C} means that $\int |g_n(x) - g(x)| \pi(x) dx \rightarrow 0$ as $n \rightarrow \infty$.

From a practical perspective, the most useful result in Theorem 1 is that a unique solution exists precisely when $\Lambda < 1$. Another interesting result is the logical equivalence of (b) and (d), which tells us that at most one solution exists for every parameterization. A third is that, since (c) is equivalent to (e), convergence of successive approximations from any starting point in \mathcal{C} implies that a unique solution exists and this solution is equal to the limit of the successive approximations from every initial condition. Thus, if computing the solution to the model at a given set of parameters is the primary objective, then convergence of the iterative method itself justifies the claim that the limit is a solution and that no other solution exists in \mathcal{C} .

Condition (a) in Theorem 1, which translates to $\beta \mathcal{M}_C^{1-1/\psi} < 1$, separates the contributions of impatience, intratemporal risk adjustment, and intertemporal substitutability of consumption to the valuation of the consumption stream. More impatience (lower β) reduces Λ , promoting a finite valuation. Higher intratemporal risk adjustment γ decreases \mathcal{M}_C but its impact on Λ depends on the elasticity of intertemporal substitution. When preferences are elastic, with $\psi > 1$, the income effect of a higher value of future consumption arising from an increase in \mathcal{M}_C is stronger than the change in the marginal rate of substitution between current and future consumption. Since V_t/C_t is denominated in utils per unit of current consumption, it diverges to infinity as \mathcal{M}_C increases. The relative strength of the income and substitution effect switches when $\psi < 1$, and $\Lambda < 1$ is violated when \mathcal{M}_C is sufficiently small.

Note that since the test value Λ depends on consumption only through the risk-adjusted long-run mean consumption growth rate \mathcal{M}_C , transitory details of the consumption process are irrelevant for existence of the continuation value. This insight is used in the applications below, particularly when we compare models with and without learning (Sections III.B and III.C, respectively).

The condition $\Lambda < 1$ imposes a bound on the average growth rate in the value of long-dated consumption strips as their maturity increases. In an IID growth setting, $\log \Lambda$ is exactly equal to this (constant) growth rate, which must be negative for a finite wealth-consumption ratio, and hence the continuation value, to exist.⁹ In other words, the risk-adjusted growth rate of consumption \mathcal{M}_C has to be sufficiently low relative to the risk-free rate. In more general settings, Λ is tied to the principal eigenvalue of a valuation operator used in the pricing of long-dated consumption strips. See Borovička, Hansen, and Scheinkman (2016) and Qin and Linetsky (2017) for more details on these operators. We elaborate on the relationship between (non)existence of a solution and the behavior of the wealth-consumption ratio in the discussion of the quantitative model of Schorfheide, Song, and Yaron (2018) in Section III.E.

⁹ Section III in the Internet Appendix outlines these calculations. The Internet Appendix is available in the online version of the article on *The Journal of Finance* website.

The sufficiency component of Theorem 1 uses fixed-point theory for monotone operators, applied to the operator A and focusing on specific existence and uniqueness results originally due to Du (1990). In the Introduction above, we discuss these results briefly, with discussion limited to the monotone concave case. However, the results in Du (1990) can handle both monotone concave and monotone convex operators. The operator A always falls into one of these categories, being concave when $\theta < 0$ or $\theta \geq 1$ and convex otherwise, as follows from the convexity and concavity properties of the function φ defined in (21) combined with the linearity of K .

The proof also uses spectral results based on the Krein-Rutman theorem, which extends Perron-Frobenius theory to function spaces. In particular, both the sufficiency and the necessity arguments exploit the spectral radius identity (5) from the introduction, which relies on an extension of a local spectral radius result originally due to Vladislav Ya. Stetsenko. See Theorem B1 for the original result, Theorem B2 also for the extension, and Proposition C1 for the proof of (5).

III. Applications

In this section, we demonstrate how Theorem 1 can be used to obtain existence and uniqueness of solutions in applied settings. We begin with relatively simple applications and then progress to more realistic specifications of consumption streams.

A. Consumption with a Deterministic Time Trend

Consider a model in which consumption obeys the geometric trend specification

$$C_t = \tau^t X_t, \quad (23)$$

where τ is a positive scalar and $\{X_t\}$ is positive, compactly supported, and IID, as in, say, Alvarez and Jermann (2005). Then the risk-adjusted long-run mean consumption growth rate is

$$\mathcal{M}_C = \lim_{n \rightarrow \infty} \left\{ \mathcal{R} \left(\frac{\tau^n X_n}{X_0} \right) \right\}^{\frac{1}{n}} = \tau \lim_{n \rightarrow \infty} \left\{ \mathbb{E} \left(\frac{X_n}{X_0} \right)^{1-\gamma} \right\}^{1/(n(1-\gamma))} = \tau,$$

where the last equality follows from the fact that the expectation is constant (since the state process is IID). Appealing to Theorem 1, we find the exact necessary and sufficient condition for a unique solution to exist is

$$\Lambda = \beta \tau^{1-\frac{1}{\psi}} < 1. \quad (24)$$

Table I
Parameter Values for the Markov Switching Model

This table presents parameter values for the Markov switching model outlined in Section III.B. They originate from Johannes, Lochstoer, and Mou (2016).

Parameter	$\mu(1)$	$\mu(2)$	$\sigma(1)$	$\sigma(2)$	$q(1, 1)$	$q(2, 2)$	γ
Value	0.007	0.0013	0.0015	0.0063	0.93	0.83	10.0

Since the consumption process (23) is trend-stationary, its stochastic component is immaterial for the long-run distribution of consumption growth. This is why the risk-aversion parameter γ does not enter into the expression for Λ obtained in (24).¹⁰

B. Markov Switching Dynamics

To illustrate the finite-state case, consider the two-state Markov switching specification for consumption growth of Johannes, Lochstoer, and Mou (2016), with

$$\ln(C_{t+1}/C_t) = \mu(X_{t+1}) + \sigma(X_{t+1})\varepsilon_{t+1},$$

where $\mathbb{X} = \{1, 2\}$ and baseline parameter values are as in Table I. The process $\{\varepsilon_t\}$ is IID and standard normal. We set aside for now the learning component of Johannes, Lochstoer, and Mou (2016) and assume that X_t is fully observable. (The learning problem is treated in Section III.C.) The conditions of Assumption 1 are satisfied with $\ell = 1$.

Our first step is to compute \mathcal{M}_C for this model using the spectral radius method discussed in Section I.B. We take K from (11), which in the present setting reduces to the 2×2 matrix

$$K(i, j) = K_{ij} = \exp \left[(1 - \gamma)\mu(j) + \frac{1}{2}(1 - \gamma)^2\sigma(j)^2 \right] q(i, j). \tag{25}$$

After inserting the parameters from Table I, we calculate its eigenvalues, evaluate $r(K)$ as the maximal eigenvalue in modulus, and then apply $\mathcal{M}_C = r(K)^{1/(1-\gamma)}$ from (15). The result is $\mathcal{M}_C = 1.005$. Figure 1 shows how \mathcal{M}_C varies as γ and $\sigma(1)$ shift around their baseline values. Both decrease the

¹⁰ The absence of the risk-aversion parameter in Λ contrasts with an earlier condition provided by Alvarez and Jermann (2005) that does depend on γ . However, our condition $\Lambda < 1$ is both necessary and sufficient, from which we can conclude that the risk-aversion parameter is indeed irrelevant to existence and uniqueness of a solution. See Section IV.B for more discussion.

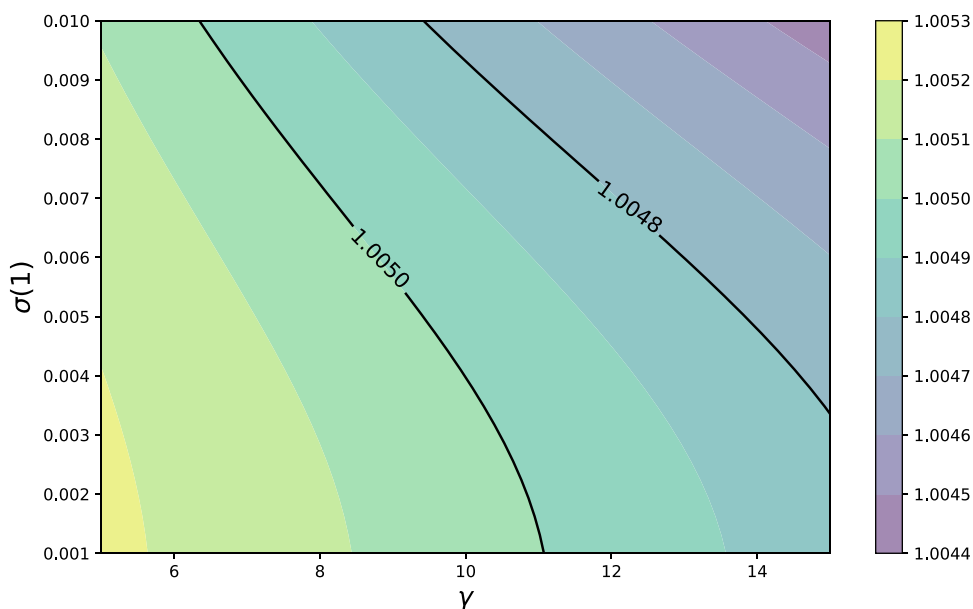


Figure 1. \mathcal{M}_C as a function of γ and $\sigma(1)$ in the Markov switching model. The remaining parameters are in Table I and originate from Johannes, Lochstoer, and Mou (2016). (Color figure can be viewed at wileyonlinelibrary.com)

risk-adjusted long-run mean consumption growth rate as they rise, with the negative effect of γ intensifying as volatility in the good state grows.

With \mathcal{M}_C in hand, we can calculate $\Lambda = \beta \mathcal{M}_C^{1-1/\psi}$. With preference parameters $\psi = 1.5$ and $\beta = 0.998$ from Johannes, Lochstoer, and Mou (2016), we obtain $\Lambda = 0.99567$. Hence, condition (a) in Theorem 1 is satisfied and statements (b) to (e) hold. In particular, a unique and globally attracting solution to the recursive utility problem exists. Conversely, if we (arbitrarily) pair the values $\psi = 1.97$ and $\beta = 0.999$ from Schorfheide, Song, and Yaron (2018) with this consumption process, then $\Lambda = 1.00147$ and, by part (b) of Theorem 1, no solution exists.

These results turn out to be significant even for the model that does include learning, as we show next.

C. Markov Switching with Learning

Consider again the Markov switching dynamics from the previous section but suppose as in Johannes, Lochstoer, and Mou (2016) that the investor does not observe the state process $\{X_t\}$. Instead, she forms a Bayesian posterior belief about X_t based on her time-0 prior and observations of consumption growth up to time t . The investor's filtering problem can be represented recursively, with $\bar{X}_t \in [0, 1]$ denoting the probability that $X_t = 1$ under the investor's information

set. This new state variable can be shown to follow the law of motion $\bar{X}_{t+1} = h(\bar{X}_t, Z_{t+1})$, where $Z_{t+1} = \ln(C_{t+1}/C_t)$ is observed consumption growth and

$$h(x, z) = \frac{[xq(1, 1) + (1 - x)q(2, 1)]\varphi_1(z)}{[xq(1, 1) + (1 - x)q(2, 1)](\varphi_1(z) - \varphi_2(z)) + \varphi_2(z)}. \quad (26)$$

Here, φ_j is the density of Z_{t+1} conditional on $X_t = j$. Details on the derivation of this standard filtering problem are provided in the Internet Appendix. The matrix K in (25), which served as the valuation operator in the Markov switching problem without learning, is replaced by the operator

$$\bar{K}g(x) = \int g(y) [\xi(1)y + \xi(2)(1 - y)] \bar{q}(x, y) dy, \quad (27)$$

where

$$\xi(i) = \exp\left((1 - \gamma)\mu(i) + \frac{1}{2}(1 - \gamma)^2\sigma^2(i)\right)$$

for $i = 1, 2$ and \bar{q} represents the transition density arising from the subjective belief of the Bayesian learner. In other words, $\bar{q}(x, \cdot)$ is the distribution of $h(x, Z_{t+1})$ when h is as defined in (26) and $Z_{t+1} = \ln(C_{t+1}/C_t)$.¹¹

We now evaluate the risk-adjusted mean consumption growth rate (3) under the model with learning, which we denote by $\bar{\mathcal{M}}_C$, to determine the test value Λ . Let $\bar{\mathbb{E}}_t$ be the expectations operator under the investor's time- t information set and \mathbb{E} be its unconditional version. Then,

$$\bar{\mathcal{M}}_C = \lim_{n \rightarrow \infty} \left\{ \bar{\mathbb{E}} \left[(C_n/C_0)^{(1-\gamma)} \right]^{\frac{1}{1-\gamma}} \right\}^{1/n}.$$

The law of iterated expectations implies that, for any $n \in \mathbb{N}$,

$$\min_{X_0} \mathbb{E}_0 \left[(C_n/C_0)^{(1-\gamma)} \right] \leq \bar{\mathbb{E}} \left[(C_n/C_0)^{(1-\gamma)} \right] \leq \max_{X_0} \mathbb{E}_0 \left[(C_n/C_0)^{(1-\gamma)} \right].$$

Raising this pair of inequalities to the power $1/(n(1 - \gamma))$ and taking the limit as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \left\{ \min_{X_0} \mathbb{E}_0 \left[(C_n/C_0)^{(1-\gamma)} \right]^{\frac{1}{1-\gamma}} \right\}^{1/n} \leq \bar{\mathcal{M}}_C \leq \left\{ \max_{X_0} \mathbb{E}_0 \left[(C_n/C_0)^{(1-\gamma)} \right]^{\frac{1}{1-\gamma}} \right\}^{1/n}.$$

However, the limits of the conditional expectations on the left- and right-hand sides of this relationship are independent of the initial state X_0 due to ergodicity of the two-state Markov chain. Consequently, $\bar{\mathcal{M}}_C = \mathcal{M}_C$, and the parameters for which unique continuation values exist coincide in the full- and partial-information models.

¹¹ While the normal densities $\varphi_j(z)$ specified in Johannes, Lochstoer, and Mou (2016) do not directly imply a transition density $\bar{q}(x, y)$ that satisfies Assumption 1, in the Internet Appendix, we construct an arbitrarily small perturbation of these densities such that Assumption 1 does hold.

Table II
Parameterization of the Consumption Process in the Long-Run Risk Model

This table presents parameter values for the model outlined in Section III.D. Parameter values are posterior median estimates from Schorfheide, Song, and Yaron (2018, table VII) estimation with consumption and financial markets data.

Parameter	μ_c	ρ	φ_z	$\bar{\sigma}$	φ_c	ρ_{h_z}	$\sigma_{h_z}^2$	ρ_{h_c}	$\sigma_{h_c}^2$
Value	0.0016	0.987	0.215	0.0035	1.0	0.992	0.0039	0.991	0.0096

This result holds despite the fact that the two models have fundamentally different state spaces and transition densities. The reason for the result is the transitory impact of state uncertainty on the conditional distribution of future consumption growth. Our test value Λ depends only on the long-run distribution of consumption growth under the investor's belief. Transitory deviations in the investor's belief about future consumption growth relative to the data-generating process, driven by learning about the unobserved state, are inconsequential in the long run and therefore irrelevant for existence of a finite continuation value, despite the fact that they affect the conditional expectations \mathbb{E}_t in each step of the continuation value recursion (20).

As a consequence, in this class of learning models, existence and uniqueness of the continuation value can be more easily evaluated by computing Λ in the full-information model as in Section III.B, which has a simpler state space and transition density.

D. Long-Run Risk

Next, consider the consumption specification adopted in Schorfheide, Song, and Yaron (2018), where

$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_{c,t} \eta_{c,t+1}, \quad (28)$$

$$z_{t+1} = \rho z_t + \sqrt{1 - \rho^2} \sigma_{z,t} \eta_{z,t+1}, \quad (29)$$

$$\sigma_{i,t} = \varphi_i \bar{\sigma} \exp(h_{i,t}) \quad \text{with} \quad h_{i,t+1} = \rho_{h_i} h_i + \sigma_{h_i} \eta_{h_i,t+1}, \quad i \in \{c, z\}. \quad (30)$$

The innovations $\{\eta_{i,t}\}$ and $\{\eta_{h_i,t}\}$ are IID and standard normal for $i \in \{c, z\}$. The state vector associated with these consumption dynamics can be represented as $X_t = (h_{c,t}, h_{z,t}, z_t)$. The consumption process parameters used by Schorfheide, Song, and Yaron (2018) are shown in Table II, while the preference parameters are $\gamma = 8.89$, $\beta = 0.999$, and $\psi = 1.97$.

All of the conditions of Theorem 1 hold apart from compactness of the state space, which fails because $X_t = (h_{c,t}, h_{z,t}, z_t)$ can take values in all of \mathbb{R}^3 . However, the fact that the correlation coefficients ρ , ρ_c , and ρ_z are less than one in absolute value means that the state space can be compactified by truncating the standard normal shocks η_z , η_{h_c} , and η_{h_z} . For now this is the path

Table III
Evaluation of Λ Using the Monte Carlo Method

This table presents descriptive statistics for the distribution of 1,000 draws of $\Lambda(m, n)$ when the number of consumption paths is fixed at $m = 5,000$ and the horizon n varies across rows. Parameter values are in Table II.

n	mean	SD	min	25%	50%	75%	max
500	0.999408	0.000111	0.998564	0.999385	0.999441	0.999473	0.999529
1,000	0.999384	0.000093	0.998551	0.999351	0.999409	0.999446	0.999517
1,500	0.999401	0.000061	0.999024	0.999386	0.999421	0.999441	0.999466

that we pursue. In this compactified setting, Theorem 1 applies and a unique and globally stable solution exists if and only if $\Lambda < 1$.

To evaluate Λ at a specific level of truncation, we begin with the Monte Carlo method introduced in Section I.C. This necessitates the use of a random number generator, so the compactification is implemented automatically by truncating the innovations in absolute value to the largest double-precision floating point number, without the need to impose further restrictions on the state space.¹² Table III reports summary statistics for 1,000 draws of $\Lambda(m, n)$ as we vary n , the time series length for consumption, across the rows, while m is held fixed at 5,000.¹³ In all cases, the mean estimate lies close to 0.9994 and the sample standard deviation is small. Other values of m and n produce similar numbers. Since $\Lambda(m, n)$ closely approximates Λ for moderate choices of m and n (see Appendix A, and Table AIII in particular), we can infer with a high degree of certainty that the valuation problem for this compactified model has a unique and globally stable solution.

While the values of $\Lambda(m, n)$ are all close to 1 at the given parameterization of Schorfheide, Song, and Yaron (2018), the outcome $\Lambda < 1$ survives reasonably large deviations in the parameters, which further supports the conclusion that the valuation problem is globally stable. Figure 2 illustrates this robustness by presenting the test value Λ at neighboring parameterizations. Values are computed by Monte Carlo, with $m = n = 1,000$. The neighboring parameterizations are obtained by varying two parameters while holding the others constant. Figure 2, Panel A, shows values obtained for Λ , represented by a contour plot, as we vary ψ and μ_c . Figure 2, Panel B, shows the same as we vary β and ψ . Parameterizations to the southwest of the 1.0 contour line are globally stable, while those to the northeast yield $\Lambda > 1$, and hence, by part (b) of Theorem 1, no solution exists. In both panels, the Schorfheide, Song, and Yaron (2018) parameterization falls well within the interior of the stable set.

¹² This evaluates to 2^{53} , which is $\approx 10^{16}$. A far smaller truncation point would be adequate but there are no obvious benefits to implementing such a modification, since any truncation of the innovations generates a compact state space, regardless of how large. Theorem 1 then applies. Further discussion of the effects of truncation is provided below.

¹³ See equations (17) and (A1) for the definition of $\Lambda(m, n)$. Consumption paths are generated according to (28) to (30), with independent normal variates supplied by a standard random number generator.

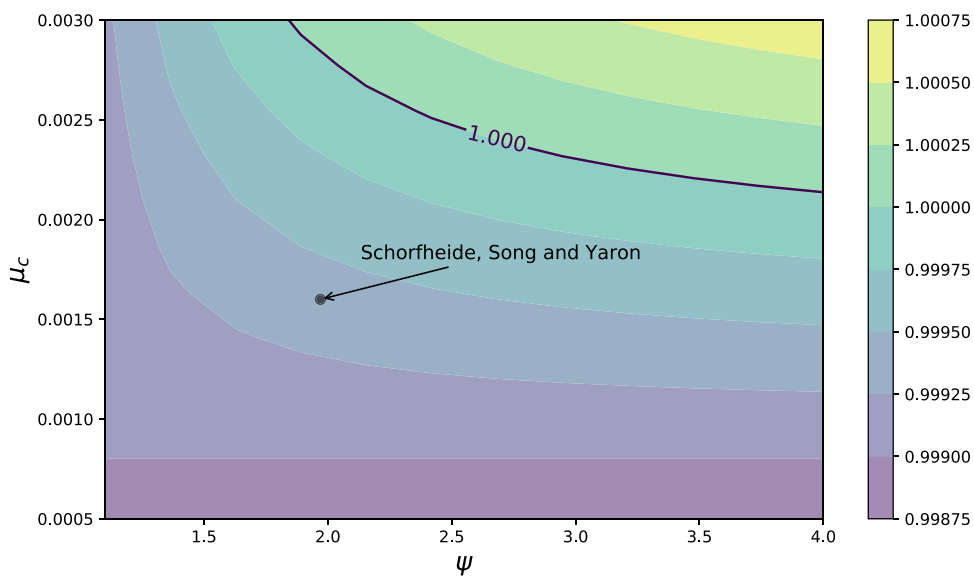
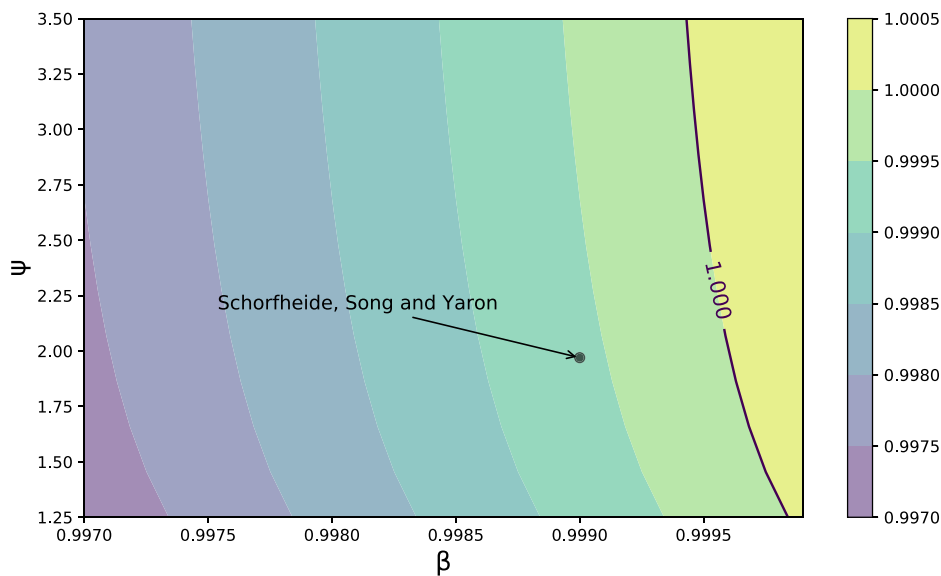
Panel A. Changes in test value, ψ vs μ_c Panel B. Changes in test value, β vs ψ

Figure 2. A contour map of Λ values in the long-run risk model. The figure displays the values of Λ near the parameterization from Schorfheide, Song, and Yaron (2018) as we vary the values of the preference parameters. The remaining parameters are fixed at their values from Table II. (Color figure can be viewed at wileyonlinelibrary.com)

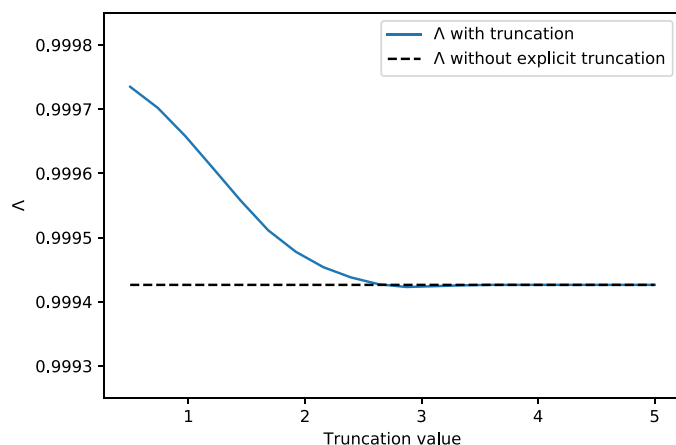


Figure 3. The impact of truncation on Λ in the long-run risk model. The truncation value is the number of standard deviations at which the normal shocks in (29) and (30) are truncated. (Color figure can be viewed at wileyonlinelibrary.com)

None of the results presented above speak directly to existence and uniqueness in the original theoretical model, where innovations are not truncated and the state space is unbounded. Two independent questions need to be considered here. First, what is the value of Λ at the Schorfheide, Song, and Yaron (2018) parameterization when shocks are not truncated? Second, what does the value of Λ imply for the unbounded state space, given that Theorem 1 is not applicable?

Neither question is fully answered here, although we can make reasonable conjectures. Regarding the first question, Figure 3 studies the impact of truncation of the innovations in the long-run risk model. The truncation value on the horizontal axis is the number of standard deviations at which the normal shocks in (29) and (30) are truncated. The values on the vertical axis are the corresponding values of $\Lambda(m, n)$, estimated using the Monte Carlo method with $n = m = 1,000$. The dashed line shows the value of $\Lambda(m, n)$ when no explicit truncation is imposed (although we are working with 64-bit floating point numbers, so truncation at around 10^{16} standard deviations is implicit). The first point made clear by the figure is that the impact of truncation becomes negligible once the truncation level for the standard normal innovations reaches three. This is consistent with our knowledge of the standard normal distribution (i.e., rapidly decreasing tails and 99.7% of probability mass within three standard deviations of the mean). The second point is that relaxing truncation shifts Λ down rather than up.¹⁴ Thus, there is strong evidence that $\Lambda < 1$ holds for the original unbounded model.

¹⁴ This is consistent with expectations because weakening truncation expands the tails of the shocks, increasing volatility. Given the relatively large value of γ adopted in Schorfheide, Song, and Yaron (2018), higher volatility tends to decrease the risk-adjusted mean consumption growth rate (see (10) for intuition) and hence Λ . We tested different combinations of m and n in these calculations but none alter our conclusions.

Turning to the second question—what does the condition $\Lambda < 1$ imply when the state space is not compact and Theorem 1 cannot be applied—some guidance in the unbounded setting is provided by Theorem 2 in Section V below, which shows that necessity and sufficiency of $\Lambda < 1$ for existence extends to the unbounded case. However, Theorem 2 relies on a side condition on K that is difficult to evaluate in a complex nonlinear model such as the specification of Schorfheide, Song, and Yaron (2018). At the same time, this condition is standard in the literature that uses eigendecompositions to analyze long-run valuations. Section V contains more details.

E. A Discretized Long-Run Risk Model

A common method for solving dynamic models with a continuous state space is discretization. We now apply this method to the long-run risk model to compute the equilibrium wealth-consumption ratio over a range of parameter values to shed light on the asset pricing consequences of moving in the parameter space toward the region in which a solution ceases to exist.¹⁵ Discretization is achieved by applying the Rouwenhorst method to the laws of motion for the state variables (equations (29) and (30)) in each of the three dimensions. The state space is then finite and the Markov chain for the state process aperiodic and irreducible, so Assumptions 1 and 2 hold. Hence, by Theorem 1, a unique and globally stable solution exists for the discretized model when $\Lambda < 1$. We found this to be true at the baseline parameterization for all levels of discretization we considered.¹⁶

The contour diagrams in Figure 2 suggest that if we shift the parameters of the model sufficiently far from the baseline setting of Schorfheide, Song, and Yaron (2018), we can find parameterizations for which $\Lambda \geq 1$. The same is true in the discretized setting. How should we interpret such outcomes? One way to understand failure of existence of a solution to the valuation problem in this model is to consider what happens to equilibrium quantities as we move close to the boundary between stability and instability. Consider, for example, Table IV, which shows how the mean wealth-consumption ratio varies with the parameters ψ and μ_c , while other parameters are held fixed at the default values in Table II. The equilibrium wealth-consumption ratio in this model is equal to $(1 - \beta)^{-1} g^*(X_t)^{1/\theta}$, where g^* is the fixed point of the operator A discussed

¹⁵ In practice, local methods such as log-linearization have, up to this point, been the standard approach for solving for endogenous objects in long-run risk models (see, e.g., Bansal and Yaron (2004)). These local approximations can be costly in terms of accuracy, as pointed out by Pohl, Schmedders, and Wilms (2018). The same reference contains an extensive discussion of computational methods.

¹⁶ To address the stochastic volatility component, we proceed as follows. First, we apply the Rouwenhorst method independently to $\{h_{c,t}\}$ and $\{h_{z,t}\}$, both of which are linear AR(1). Next, we translate the results into discretized dynamics for $\{\sigma_{c,t}\}$ and $\{\sigma_{z,t}\}$, with H and I possible values, respectively. Last, for each of the I possible values of σ_z , we again use the Rouwenhorst method to discretize $\{z_t\}$ across J possible states. With $H = I = J = 3$, we obtain $\Lambda = 0.99944$. For the finer discretizations, the value of Λ tends to decline, consistent with the intuition that we obtain from Figure 3.

Table IV
Mean of the Wealth-Consumption Ratio under the
Stationary Distribution

This table presents the mean of the wealth-consumption ratio under the stationary distribution of the state process in the long-run risk model, across different μ_c, ψ combinations. Remaining parameters are fixed at their values from Table II. Monthly consumption in the denominator. As $\psi \rightarrow 1$, the wealth-consumption ratio converges to the constant $(1 - \beta)^{-1} = 1,000$.

	$\psi = 1.1$	$\psi = 1.68$	$\psi = 2.26$	$\psi = 2.84$	$\psi = 3.42$	$\psi = 4.0$
$\mu_c = 0.0030$	1,290.3	46,604.4	NA	NA	NA	NA
$\mu_c = 0.0025$	1,219.3	4,610.7	4.6e+25	NA	NA	NA
$\mu_c = 0.0020$	1,155.7	2,423.3	4,986.7	12,840.7	3,596,674.7	1.7e+31
$\mu_c = 0.0015$	1,098.4	1,642.7	2,142.0	2,600.6	3,022.8	3,412.6
$\mu_c = 0.0010$	1,046.5	1,242.0	1,362.4	1,443.9	1,502.7	1,547.2
$\mu_c = 0.0005$	999.5	998.3	998.3	998.3	998.5	998.6

in Theorem 1 (see footnote 6). The function g^* is computed by iterating with A on the arbitrary initial choice $g \equiv 1$, a process for which Theorem 1 guarantees convergence whenever $\Lambda < 1$.¹⁷

Table IV covers some of the parameter values explored in Figure 2, Panel A, which shows that no solution exists when both μ_c and ψ are sufficiently large. For those pairs—or, more precisely, for any pair for which $\Lambda \geq 1$ —we print the string NA to indicate that no solution exists. For other pairs, we print the mean wealth-consumption ratio. Table IV shows that as the parameters approach the boundary between stability and instability, the mean wealth-consumption ratio explodes. This is the nature of nonexistence of solutions: parameters are such that wealth and forward-looking valuations are infinite.

F. Production Economies

So far, we have assumed that consumption follows an exogenously specified process, as given in (6). A large class of models with Epstein-Zin preferences studied in the macrofinance literature (see Tallarini (2000), Croce (2014), and Kaltenbrunner and Lochstoer (2010), among many others) involves endogenously determined consumption processes of the form

$$C_t = c(X_t)A_t, \\ \log A_{t+1} - \log A_t = \kappa_A(X_t, X_{t+1}, \varepsilon_{t+1}).$$

For example, suppose we are interested in a production economy where A_t is an exogenous technology process with stochastic growth and are solving for an endogenous function $c(X_t)$ that represents stationary deviations from the

¹⁷ The calculation was carried out for the discretized representation of the state process, which was, in turn, computed via multiple iterations of the Rouwenhorst technique, in the manner discussed above. In each case, we set $H = I = J = 3$, so the state space had 27 elements. Iteration continued until the maximal absolute deviation between successive iterates fell below 1e-6.

stochastic trend. Let the assumptions from Section I imposed on X_t and ε_t hold. Further assume that $c(x)$ is bounded above and away from zero, and define $\bar{\zeta} = \max_{x_0, x_n} (c(x_n)/c(x_0))^{1-\gamma}$ and $\underline{\zeta} = \min_{x_0, x_n} (c(x_n)/c(x_0))^{1-\gamma}$. In this case,

$$\underline{\zeta} E \left[\left(\frac{A_n}{A_0} \right)^{1-\gamma} \right] \leq E \left[\left(\frac{C_n}{C_0} \right)^{1-\gamma} \right] \leq \bar{\zeta} E \left[\left(\frac{A_n}{A_0} \right)^{1-\gamma} \right],$$

and thus, the risk-adjusted long-run mean consumption growth rate \mathcal{M}_C from (3) is given by

$$\mathcal{M}_C = \lim_{n \rightarrow \infty} \left[\mathcal{R} \left(\frac{A_n}{A_0} \right) \right]^{1/n}. \quad (31)$$

We can therefore infer the region of the parameter space for which the endogenously determined continuation value exists directly from the knowledge of the properties of the process A_t , without knowing the details of the function $c(x)$.¹⁸ This is another manifestation of the fact that our condition for existence and uniqueness depends only on long-run properties of the consumption process and transitory details of that process are irrelevant.

IV. Comparisons with Alternative Conditions

In the Introduction, we discuss a number of related tests for existence and uniqueness of recursive utilities. In this section, we provide a brief comparison of these alternative results with the necessary and sufficient condition $\Lambda < 1$.

A. Probability-One Bounds

Several conditions based on probability-one bounds have been proposed in the literature. A representative example is Theorem 3.1 of Epstein and Zin (1989), who show that a solution to the recursive utility problem exists whenever

$$\psi > 1 \text{ and } \beta B_c^{1-1/\psi} < 1, \quad (32)$$

where B_c is a probability-one upper bound on C_{t+1}/C_t . If such a B_c exists, then condition (32) is directly comparable with condition $\Lambda < 1$ because B_c always exceeds the risk-adjusted mean consumption growth rate. Indeed, $C_{t+1}/C_t \leq B_c$ for all t implies $C_n/C_0 \leq B_c^n$ for all n , and hence, by the definition of \mathcal{M}_C in (3),

$$\mathcal{M}_C \leq B_c. \quad (33)$$

¹⁸ Another sufficient condition for the result in (31) that is often satisfied in applications is an exponential decay rate in the correlation between $c(X_{t+n})^{1-\gamma}$ and $(A_{t+1}/A_t)^{1-\gamma}$ as $n \rightarrow \infty$. The argument can be further extended to cases in which the economy involves additional endogenous nonstationary state variables cointegrated with A_t , such as aggregate capital. In these cases, the transition law for X_t may not satisfy the regularity conditions imposed in Section I, but such economies can often be regularized using perturbation arguments similar to that described in Section IV of the Internet Appendix.

Thus, condition (32) implies $\Lambda < 1$. In other words, the condition $\Lambda < 1$ is weaker than the condition of Epstein and Zin (1989). Moreover, unless consumption growth is deterministic, the inequality in (33) is strict. This fact recalls the point made earlier that focusing only on the upper tail of the consumption growth distribution leads to overly pessimistic restrictions.

To illustrate, consider the case of the long-run risk model, where consumption growth is as specified in (28). Since consumption growth innovations are normally distributed and have no finite upper bound in this case, we have $B_c = +\infty$ and the Epstein-Zin stability coefficient $\beta B_c^{1-1/\psi}$ from condition (32) is also $+\infty$. In contrast, $\Lambda < 1$ at the baseline parameters, as discussed in Section III.D.

B. Contraction Arguments

Recall the specification for consumption used in Alvarez and Jermann (2005) and discussed in Section III.A, where $C_t = \tau^t X_t$ with $\tau > 0$ and $\{X_t\}$ is positive and IID. Let X_t have distribution F on $[a, b] \subset \mathbb{R}$ for some positive scalars $a < b$. When consumption obeys (23), Alvarez and Jermann (2005) show that a unique solution to the recursive utility problem exists whenever

$$\beta \tau^{1-\frac{1}{\psi}} \max_{a \leq x \leq b} \left\{ \int \left(\frac{y}{x} \right)^{1-\gamma} \pi(dy) \right\}^{\frac{1}{\theta}} < 1. \quad (34)$$

See Alvarez and Jermann (2005, Proposition 9 and Lemma A.1) for details.

By way of comparison, we know from Section III.A that a unique solution exists if and only if $\beta \tau^{1-\frac{1}{\psi}} < 1$. This condition is weaker than (34), since the additional maximized term in (34) always exceeds unity. The difference between the stability conditions arises because the condition from Alvarez and Jermann (2005) enforces contraction in one step. In contrast, the condition $\Lambda < 1$ is an asymptotic condition that ignores short-run fluctuations in consumption. This leads to a weaker condition because short-run fluctuations do not impinge on asymptotic outcomes.

V. Unbounded State Spaces

Lastly, we return to the setting of Theorem 1 but now drop the compactness assumption on the state space (i.e., Assumption 2) and replacing it with the following assumption.

ASSUMPTION 3: *The operator K is continuous and eventually weakly compact.*

An explanation of the terminology and discussion of sufficient conditions can be found in Appendix C.

THEOREM 2: *If Assumptions 1 and 3 both hold, then the following statements are equivalent:*

- (a) $\Lambda < 1$.
- (b) A has a fixed point in \mathcal{C} .
- (c) There exists a $g \in \mathcal{C}$ such that $\{A^n g\}_{n \geq 1}$ converges to an element of \mathcal{C} .

The proof of Theorem 2 relies heavily on a rather specific spectral continuity result for positive operators obtained by Schep (1980). This continuity is combined with an approximation argument that allows us to bootstrap some results from Theorem 1. Assumption 3 is used to show that the approximation step is valid.

Compared to Theorem 1, the calculation of the test value Λ in Theorem 2 is more problematic when no analytical solution exists, as even a flexible method such as Monte Carlo restricts the state space to a finite set of floating-point numbers. Moreover, the conclusions are weaker and the eventual compactness condition in Assumption 3 is nontrivial to test. We were unable to verify it for the noncompact models studied above. Nevertheless, such conditions are not uncommon in the literature (see, e.g., Assumption 2.1 in Christensen (2017)), suggesting that the core existence result extends to a significant class of unbounded theoretical models.

VI. Concluding Remarks

In this paper, we derive a simple condition that is both necessary and sufficient for existence and uniqueness of the continuation value and aggregate wealth-consumption ratio in recursive utility models with homothetic Epstein-Zin preferences. Despite the fact that the nonlinear preference recursion intertwines the role of risk aversion and intertemporal elasticity of substitution, our condition allows us to separate the two forces. What matters is the average long-run risk-adjusted consumption growth rate and its relationship with intertemporal substitutability of consumption and time discounting. In asset pricing terms, the condition imposes a bound on the long-horizon rate of decay of the value of consumption strips as their maturity increases.

We also provided a range of analytical examples, a globally stable method of computing solutions whenever they exist, and numerical methods that allow for efficient evaluation of our condition. Since transitory details of the consumption process do not affect the average long-run risk-adjusted consumption growth rate and hence are irrelevant for our stability condition, insights from this paper can also be applied in production economy settings without explicitly solving the model, as long as cointegration properties of consumption with an exogenous driver of uncertainty can be inferred directly.

While our sharpest results restrict the state space for the underlying Markov state to be compact, we also give extensions to unbounded state spaces and study sensitivity of numerical methods as we relax truncation of the state space. While our results for the unbounded setting are more limited than those for the

compact case, we conjecture that the combination of ever sharper numerical methods and the kinds of theoretical insights provided above forms the most promising road to further understanding stability and equilibrium properties of asset pricing models used in modern quantitative applications.

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Appendix A: Details of Numerical Methods

In this section, we provide more details on the numerical methods used to compute \mathcal{M}_C , and hence Λ , in the paper.

A.1 Discretization

If \mathbb{X} is infinite, then one option is to discretize to a finite state space combined with a finite set of transition probabilities $q(x, y)$, and then, use the finite-state methods from Section I.B. Here, we investigate the accuracy of this procedure. Our experiment is based on the Bansal-Yaron consumption growth dynamics in (7) and (8), where an analytical expression for \mathcal{M}_C is obtained in (10). We first discretize the Gaussian AR(1) state process (8) using Rouwenhorst’s method (Rouwenhorst (1995)). We then compute the matrix K in (11) corresponding to this discretized state process, calculate the spectral radius $r(K)$ using linear algebra routines, and compute the associated value for \mathcal{M}_C as $r(K)^{1/(1-\gamma)}$. Finally, we compare the result with the true value of \mathcal{M}_C obtained from the analytical expression (10).

Table AI provides results of such a comparison for a range of parameter values and levels of discretization. We vary the preference parameter γ across rows, as shown in the first column, while the remaining parameters are come from table I of Bansal and Yaron (2004), as in footnote 3. The second column shows the true value of \mathcal{M}_C for the nondiscretized model. The remaining columns show the value computed using the numerical procedure discussed in the previous paragraph at different levels of discretization. For example, $D = 5$

Table AI
True Value and Discrete Approximation of \mathcal{M}_C , Bansal and Yaron (2004) Model

This table presents a comparison of the true value of \mathcal{M}_C and its approximation computed by discretizing the model as the risk aversion γ varies across rows. The remaining parameters are from table I of Bansal and Yaron (2004).

	True value	$D = 5$	$D = 50$	$D = 100$	$D = 200$
$\gamma = 7.5$	1.0004504	1.0004998	1.0004549	1.0004527	1.0004516
$\gamma = 10.0$	1.0000466	1.0001658	1.0000584	1.0000525	1.0000496
$\gamma = 12.5$	0.9996430	0.9998662	0.9996673	0.9996552	0.9996491

Table AII
Evaluation of \mathcal{M}_C Using the Monte Carlo Method, Bansal and Yaron (2004) Model

This table presents the realizations of $\mathcal{M}_C(m, n)$ as the number of drawn consumption paths m and horizon n vary. Values shown are mean and standard deviation over 1,000 independent draws. The true value is $\mathcal{M}_C = 1.0004504$.

	$m = 1,000$	$m = 2,000$	$m = 3,000$	$m = 4,000$	$m = 5,000$
$n = 250$	1.0006940 (0.000076)	1.0006905 (0.000048)	1.0006940 (0.000032)	1.0006932 (0.000037)	1.0006934 (0.000029)
$n = 500$	1.0006208 (0.000084)	1.0006091 (0.000066)	1.0005813 (0.000068)	1.0005733 (0.000084)	1.0005775 (0.000062)
$n = 750$	1.0005979 (0.000112)	1.0005762 (0.000096)	1.0005764 (0.000076)	1.0005611 (0.000074)	1.0005523 (0.000092)

means that the AR(1) process (8) was discretized into a five-state Markov chain using the Rouwenhorst method. The results show that the discretization-based method is accurate for this model, even for relatively coarse approximations.

A.2 A Monte Carlo Method

Next, we consider the accuracy of the Monte Carlo method introduced in Section I.C. To evaluate it, we again use the lognormal consumption model from (7) and (8) and compare our approximations of \mathcal{M}_C with the true value obtained via the analytical expression given in (10).

Table AII illustrates our results. The consumption path parameters are again chosen to match Bansal and Yaron (2004), as in footnote 3. The parameter γ is set to 7.5, which matches the first row of Table AI. The true value of \mathcal{M}_C calculated from the analytical expression (10) is 1.0004504, as shown in the caption for the table. The interpretation of n and m in the table is consistent with the right-hand side of (17). For each n, m pair, we compute $\mathcal{M}_C(n, m)$ a total of 1,000 times using independent draws and then present the mean and standard deviation of the sample in the corresponding table cell. The Monte Carlo approximation is accurate up to three decimal places in all simulations. Standard deviations are small and decline with m .¹⁹

High accuracy in estimating \mathcal{M}_C translates into similarly high accuracy in estimating $\Lambda = \beta \mathcal{M}_C^{1-1/\psi}$ once we introduce the additional preference parameters β and ψ . Table AIII illustrates this point. Each (m, n) cell in the table gives the mean and standard deviation of 1,000 draws of

$$\Lambda(m, n) := \beta \mathcal{M}_C(m, n)^{1-1/\psi}. \quad (\text{A1})$$

¹⁹ One additional advantage of the Monte Carlo method centered on (17) is that simulation of the independent consumption processes can be parallelized. This leads to speed gains approaching two orders of magnitude in some of our implementations.

Table AIII
Evaluation of Λ Using the Monte Carlo Method, Bansal and Yaron
(2004) Model

This table presents the realizations of $\Lambda(m, n)$ as the number of drawn consumption paths m and horizon n vary. Values shown are mean and standard deviation over 1,000 independent draws. The true value is $\Lambda = 0.9981498$.

	$m = 1,000$	$m = 2,000$	$m = 3,000$	$m = 4,000$	$m = 5,000$
$n = 250$	0.9982308 (0.000025)	0.9982297 (0.000016)	0.9982308 (0.000011)	0.9982305 (0.000012)	0.9982306 (0.000010)
$n = 500$	0.9982065 (0.000028)	0.9982026 (0.000022)	0.9981934 (0.000022)	0.9981907 (0.000028)	0.9981921 (0.000021)
$n = 750$	0.9981989 (0.000037)	0.9981916 (0.000032)	0.9981917 (0.000025)	0.9981866 (0.000025)	0.9981837 (0.000031)

The draws of $\mathcal{M}_C(m, n)$ are computed as in Table AII, while the remaining parameters are set to $\beta = 0.998$ and $\psi = 1.5$, as in Bansal and Yaron (2004). The true value $\Lambda = 0.9981498$ from the caption of Table AIII is calculated as $\Lambda = \beta \mathcal{M}_C^{1-1/\psi}$, where \mathcal{M}_C is obtained from the analytical expression (10).

Appendix B: General Fixed-Point and Spectral Radius Results

In what follows, the state space \mathbb{X} is allowed to be any compact metric space. In particular, \mathbb{X} can be a compact subset of \mathbb{R}^n with the Euclidean distance or an arbitrary finite set endowed with the discrete metric $d(x, y) = \mathbb{1}\{x \neq y\}$. Note that in the latter case, κ and q are automatically continuous. The collection of Borel-measurable functions g from \mathbb{X} to \mathbb{R} such that $\|g\| := \int |g| d\pi < \infty$ is denoted by $L_1(\pi)$. Convergence is with respect to $\|\cdot\|$ unless otherwise stated. For $g, h \in L_1(\pi)$, the statement $g \leq h$ means that $g \leq h$ holds π -almost everywhere, while $g \ll h$ indicates that $g < h$ holds π -almost everywhere. The symbol \mathcal{C} denotes all $g \in L_1(\pi)$ such that $g \gg 0$.

Let M be a linear operator mapping $L_1(\pi)$ to itself. The *operator norm* and *spectral radius* of M are defined by $\|M\| := \sup\{\|Mg\| : g \in L_1(\pi), \|g\| \leq 1\}$, and $r(M) := \lim_{n \rightarrow \infty} \|M^n\|^{1/n}$, respectively. The operator M is called *positive* if $Mg \geq 0$ whenever $g \geq 0$. It is called *bounded* if $\|M\|$ is finite and *compact* if the image of the unit ball in $L_1(\pi)$ under M is precompact in the norm topology. A (possibly nonlinear) operator S mapping a convex subset E of $L_1(\pi)$ into itself is called *convex* on E if $S(\lambda f + (1 - \lambda)g) \leq \lambda S + (1 - \lambda)Sg$ for all $f, g \in E$ and all $\lambda \in [0, 1]$, and *concave* if the reverse inequality holds. It is called *isotone* if $f \leq g$ implies $Sf \leq Sg$.

The following is a local spectral radius result suitable for $L_1(\pi)$. The proof provided here is due to Mirosława Zima (private communication) and draws on earlier results from Zabreiko, Krasnosel'skii, and Stetsenko (1967).

THEOREM B1 (Zabreiko-Krasnosel'skii-Stetsenko-Zima): *Let h be an element of $L_1(\pi)$ and let M be a positive compact linear operator. If $h \gg 0$, then*

$$\lim_{n \rightarrow \infty} \|M^n h\|^{1/n} = r(M). \quad (\text{B1})$$

PROOF: Let h and M be as in the statement of the theorem. Recall that $r(h, M) = \limsup_{n \rightarrow \infty} \|M^n h\|^{1/n}$ is the local spectral radius of M at h . From the definition of $r(M)$, it suffices to show that $r(h, M) \geq r(M)$. To this end, let λ be a constant satisfying $\lambda > r(h, M)$ and let

$$h_\lambda := \sum_{n=0}^{\infty} \frac{M^n h}{\lambda^{n+1}}. \quad (\text{B2})$$

The point h_λ is a well-defined element of $L_1(\pi)$ by $\limsup_{n \rightarrow \infty} \|M^n h\|^{1/n} < \lambda$ and Cauchy's root test for convergence. It is also positive π -almost everywhere, since the sum in (B2) includes $h \gg 0$ and since M is a positive operator. Moreover, by standard Neumann series theory (e.g., (Krasnosel'skii et al., 1972, Theorem 5.1)), the point h_λ also has the representation $h_\lambda = (\lambda I - M)^{-1}h$, from which we obtain $\lambda h_\lambda - M h_\lambda = h$. Because $h \in \mathcal{C}$, this implies that $M h_\lambda \leq \lambda h_\lambda$. Applying this last inequality, compactness of M , quasi-interiority of h_λ , and Theorem 5.5(a) of Krasnosel'skii et al. (1972), we must have that $r(M) \leq \lambda$. Since this inequality was established for an arbitrary λ satisfying $\lambda > r(h, M)$, we conclude that $r(h, M) \geq r(M)$. Hence, $r(h, M) = r(M)$. Finally, since M is compact, Corollary 1 of Daneš (1987) implies that $r(h, M) = \lim_{n \rightarrow \infty} \|M^n h\|^{1/n}$, so (B1) holds. ■

The next result is an extension of Theorem B1, which weakens the compactness condition in that theorem while requiring additional positivity.

THEOREM B2: *Let h be an element of $L_1(\pi)$ and let M be a linear operator on $L_1(\pi)$. If M^i is compact for some $i \in \mathbb{N}$ and $Mf \gg 0$ whenever $f \in L_1(\pi)$ and $f \gg 0$, then*

$$\lim_{n \rightarrow \infty} \left\{ \int M^n h \, d\pi \right\}^{1/n} = r(M) \quad \text{for all } h \gg 0. \quad (\text{B3})$$

PROOF: Fix $h \in L_1(\pi)$ with $h \gg 0$ and choose $i \in \mathbb{N}$ such that M^i is a compact linear operator on $L_1(\pi)$. Fix $j \in \mathbb{N}$ with $0 \leq j \leq i - 1$. By our assumptions on M , we know that $M^j h \gg 0$, so Theorem B1 applied to M^i with initial condition $M^j h$ yields

$$\left\{ \int M^{in} M^j h \, d\pi \right\}^{1/n} = \left\{ \int M^{in+j} h \, d\pi \right\}^{1/n} \rightarrow r(M)^i \quad (n \rightarrow \infty).$$

But $r(M^i) = r(M)^i$, so

$$\left\{ \int M^{in+j} h \, d\pi \right\}^{1/(in)} \rightarrow r(M) \quad (n \rightarrow \infty).$$

It follows that

$$\left\{ \int M^{in+j} h \, d\pi \right\}^{1/(in+j)} \rightarrow r(M) \quad (n \rightarrow \infty).$$

As j is an arbitrary integer satisfying $0 \leq j \leq i-1$, we conclude that (B3) holds. ■

The next lemma is useful for detecting fixed points in $L_1(\pi)$.

LEMMA B1: *Let $\{g_n\}$ be a monotone increasing sequence in $L_1(\pi)$. If $\{g_n\}$ is bounded above by some h in $L_1(\pi)$, then there exists a g in $L_1(\pi)$ such that $g_n \rightarrow g$. Moreover, if $g_n = T^n g_0$ for some continuous operator T mapping a subset of $L_1(\pi)$ to itself, then g is a fixed point of T .*

PROOF: The first claim is a direct consequence of the monotone convergence theorem for integrals. Regarding the second, we have $g_n \rightarrow g$, and hence, by continuity, $Tg_n \rightarrow Tg$. But by the definition of the sequence $\{g_n\}$, we also have $Tg_n \rightarrow g$. Hence, $Tg = g$. ■

Appendix C: Remaining Proofs, Compact Case

As before, q is the transition density kernel for $\{X_t\}$. We define q^i recursively by $q^1 = q$ and $q^i(x, y) = \int q(x, z)q^{i-1}(z, y)dz$ for each $x, y \in \mathbb{X}$. By the conditions in Section I, we can take $\ell \in \mathbb{N}$ such that $q^\ell > 0$, and ℓ has this meaning throughout. Also, in what remains, we adopt the notation

$$k(x, y) = \int \exp[(1 - \gamma)\kappa(x, y, \varepsilon)]\nu(d\varepsilon)q(x, y), \quad (\text{C1})$$

so that the operator K satisfies $Kg(x) = \int k(x, y)g(y)dy$. Let k^i be defined on $\mathbb{X} \times \mathbb{X}$ by $k^1 = k$ and $k^i(x, y) = \int k(x, z)k^{i-1}(z, y)dz$. As is easily verified by induction, the i^{th} element k^i is the kernel of the i^{th} power of K . That is, for all $x \in \mathbb{X}$, $i \in \mathbb{N}$, and $g \in \mathcal{C}$,

$$K^i g(x) = \int k^i(x, y)g(y)dy.$$

Throughout this section, Assumptions 1 and 2 are taken to be true.

LEMMA C1: *The density π is the unique stationary density for q on \mathbb{X} . In addition, π is everywhere positive and continuous on \mathbb{X} .*

PROOF: Regarding uniqueness, the condition that $q^\ell > 0$ for some positive $\ell \in \mathbb{N}$ implies irreducibility of $\{X_t\}$, which, in turn, implies uniqueness of the stationary distribution. See, for example, Meyn and Tweedie (2009, Theorem 10.0.1).²⁰ Regarding positivity, suppose, to the contrary, that there

²⁰ To verify irreducibility, pick any A with $\int_A \pi(x)dx > 0$. Apparently, A has positive Lebesgue measure, so, with ℓ chosen such that q^ℓ is everywhere positive, we have $\int_A q^\ell(x, y)dy > 0$ for all $x \in \mathbb{X}$. Thus, A is visited with positive probability from any state in ℓ steps. In other words, π -irreducibility holds.

exists a y with $\pi(y) = 0$. Since π is a stationary density, this means that $\pi(y) = \int q^\ell(x, y)\pi(x)dx = 0$. But q^ℓ is everywhere positive and π is a density, and hence positive on a set of positive measure. Thus, the integral must be positive, a contradiction. Finally, take $y_n \rightarrow y$ in \mathbb{X} and observe that

$$\pi(y_n) = \int q(x, y_n)\pi(x)dx \rightarrow \int q(x, y)\pi(x)dx = \pi(y),$$

as $n \rightarrow \infty$. The convergence of the integrals is due to continuity of q and the dominated convergence theorem. The latter can be employed because q is bounded on $\mathbb{X} \times \mathbb{X}$ by compactness of \mathbb{X} and continuity of q . Hence, π is continuous on \mathbb{X} . ■

LEMMA C2: *Regarding the operator K , the following statements are true:*

- (a) K is a bounded linear operator on $L_1(\pi)$ that maps \mathcal{C} to itself.
- (b) Kg is continuous at each $g \in \mathcal{C}$.
- (c) $Kg \neq 0$ whenever $g \in \mathcal{C}$ and $g \neq 0$.
- (d) $Kg \gg 0$ whenever $g \in \mathcal{C}$ and $g \gg 0$.

PROOF: Regarding claim (a), k is continuous and hence bounded by some constant M on \mathbb{X} , while π is positive and continuous on a compact set, and hence bounded below by some positive constant δ . For arbitrary $f \in L_1(\pi)$, this yields:

$$|Kf(x)| = \left| \int k(x, y)f(y)dy \right| \leq M \int \frac{|f(y)|}{\pi(y)}\pi(y)dy \leq \frac{M}{\delta} \|f\|. \quad (\text{C2})$$

It follows directly that K is a bounded linear operator on $L_1(\pi)$.

Regarding continuity, fix $g \in \mathcal{C}$, $x \in \mathbb{X}$, and $x_n \rightarrow x$. Using the inequality from (C2), we have

$$k(x_n, y)g(y) \leq M \frac{g(y)}{\pi(y)}\pi(y) \leq \frac{M}{\delta} g(y)\pi(y).$$

Since $g \in L_1(\pi)$, the function on the right-hand side is integrable with respect to the Lebesgue measure, so we can apply the dominated convergence theorem to obtain

$$\lim_{n \rightarrow \infty} Kg(x_n) = \int \lim_{n \rightarrow \infty} k(x_n, y)g(y)dy = Kg(x).$$

In particular, Kg is continuous at any $x \in \mathbb{X}$.

Regarding claim (c), suppose that, to the contrary, that we have $Kg = 0$ for some nonzero $g \in \mathcal{C}$. Let $B = \{g > 0\}$. Since g is nonzero, we have $\pi(B) > 0$. Since $Kg = 0$, it must be the case that $\int_B k(x, y)dy = 0$ for any $x \in \mathbb{X}$. But then $\int_B q(x, y)dy = 0$ for any $x \in \mathbb{X}$. A simple induction argument shows that this extends to the n -step kernels, so that, in particular, $\int_B q^\ell(x, y)dy = 0$ for all $x \in \mathbb{X}$. The last equality contradicts $q^\ell > 0$, as guaranteed by Assumption 1.

Part (d) is immediate from $Kg(x) = \int g(y)k(x, y)dy$ and (C1). ■

The next lemma discusses irreducibility of K as a linear operator on $L_1(\pi)$. See page 262 of Meyer-Nieberg (2012) for the definition and further discussion.

LEMMA C3: *The operator K is irreducible and K^2 is compact.*

PROOF: Consider first irreducibility of K . By positivity of κ and the condition for irreducibility of kernel operators on page 262 of Meyer-Nieberg (2012), it suffices to show that, for any Borel set A of \mathbb{X} with $0 < \pi(A) < 1$, there exists a Borel set B of \mathbb{X} with

$$0 < \pi(B) < 1 \quad \text{and} \quad \int_B \int_A q(x, y) dx dy > 0. \quad (\text{C3})$$

To verify (C3), suppose to the contrary that we can choose $A \subset \mathbb{X}$ with $0 < \pi(A) < 1$ and

$$\int_{A^c} \int_A q(x, y) dx dy = \int_A \left[\int_{A^c} q(x, y) dy \right] dx = 0.$$

Then $\int_A \int_{A^c} q(x, y) dy \pi(x) dx = 0$. Hence, A is absorbing for q , in which case $\pi(A) = 1$ by π -irreducibility of q and Proposition 4.2.3 of Meyn and Tweedie (2009), a contradiction.

Regarding compactness, Theorem 9.9 of Schaefer (1974) implies that K^2 will be compact whenever K is weakly compact, which requires that the image of the unit ball B_1 in $L_1(\pi)$ under K is relatively compact in the weak topology. To prove this, it suffices to show that given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_A K|f| d\pi < \varepsilon$ whenever $f \in B_1$ and $\pi(A) < \delta$. This is true because k is continuous and hence bounded on \mathbb{X} , yielding

$$\int_A \int k(x, y) |f(y)| dy \pi(x) dx = \int \int_A k(x, y) \pi(x) dx |f(y)| dy \leq M \pi(A) \int |f(y)| dy$$

for some constant M . By continuity and positivity of π on \mathbb{X} , we have $\pi \geq a$ for some constant $a > 0$ and hence $\int |f(y)| dy \leq a \|f\| \leq a$. Thus, $\int_A K|f| d\pi < \varepsilon$ whenever $\pi(A) < \varepsilon/(aM)$. ■

PROPOSITION C1: *Λ is well defined and satisfies $\Lambda = \beta r(K)^{1/\theta}$.*

PROOF: Since K^i is compact for some i and maps positive functions into positive functions (see Lemmas C3 and C2), we can apply Theorem B2 to $\mathbb{1} \equiv 1$ to obtain $r(K) = \lim_{n \rightarrow \infty} \|K^n \mathbb{1}\|^{1/n}$. An inductive argument based on (6) shows that, for each n in \mathbb{N} , we have $K^n \mathbb{1}(x) = \mathbb{E}_x(C_n/C_0)^{1-\gamma}$. Hence, by the law of iterated expectations,

$$\begin{aligned} \|K^n \mathbb{1}\|^{1/n} &= \left\{ \mathbb{E} \left(\frac{C_n}{C_0} \right)^{1-\gamma} \right\}^{1/n} \\ &= \left\{ \mathcal{R} \left(\frac{C_n}{C_0} \right) \right\}^{(1-\gamma)/n}. \end{aligned}$$

Since $r(K) = \lim_{n \rightarrow \infty} \|K^n \mathbb{1}\|^{1/n}$, this yields

$$r(K) = \lim_{n \rightarrow \infty} \left\{ \mathcal{R} \left(\frac{C_n}{C_0} \right) \right\}^{(1-\gamma)/n} = \mathcal{M}_C^{1-\gamma}.$$

Because $\theta := (1 - \gamma)/(1 - 1/\psi)$, we now have

$$\beta r(K)^{1/\theta} = \beta \mathcal{M}_C^{1-1/\psi} = \Lambda.$$

■

THEOREM C1: *The spectral radius $r(K)$ of K is strictly positive. Moreover, there exists an eigenfunction e of K satisfying*

$$Ke = r(K)e \quad \text{and} \quad e \gg 0. \quad (\text{C4})$$

The function e is continuous everywhere on \mathbb{X} .

PROOF: Theorem 4.1.4 and Lemma 4.2.9 of Meyer-Nieberg (2012) together with the irreducibility and compactness properties of K obtained in Lemma C3 yield positivity of $r(K)$ and existence of the positive eigenfunction in (C4). Lemma C2 implies that e is continuous, since $e \in \mathcal{C}$ and $e = (Ke)/r(K)$. ■

In what follows, e in (C4) is called the Perron-Frobenius eigenfunction.

Since e is positive and continuous, the constants $\underline{e} := \min_{x \in \mathbb{X}} e(x)$ and $\bar{e} := \max_{x \in \mathbb{X}} e(x)$ are finite and strictly positive. These facts are now used to study A from (22).

LEMMA C4: *Let e be the Perron-Frobenius eigenfunction of K . If*

$$\lim_{t \downarrow 0} \frac{\varphi(t)}{t} r(K) > 1 \quad \text{and} \quad \lim_{t \uparrow \infty} \frac{\varphi(t)}{t} r(K) < 1, \quad (\text{C5})$$

then there exist positive constants $c_1 < c_2$ with the following properties:

- (a) *If $0 < c \leq c_1$ and $f = ce$, then there exists a $\delta_1 > 1$ such that $Af \geq \delta_1 f$.*
- (b) *If $c_2 \leq c < \infty$ and $f = ce$, then there exists a $\delta_2 < 1$ such that $Af \leq \delta_2 f$.*

PROOF: Let $\lambda := r(K)$ and let e be the Perron-Frobenius eigenfunction. Let \underline{e} and \bar{e} be the maximum and minimum of e on \mathbb{X} , as defined above. Regarding claim (a), observe that, in view of (C5), there exists a $\delta_1 > 1$ and an $\varepsilon > 0$ such that

$$\frac{\varphi(t)}{t} \lambda \geq \delta_1 \quad \text{whenever} \quad 0 < t < \varepsilon.$$

Choosing c_1 such that $0 < c_1 \lambda \bar{e} < \varepsilon$ and $c \leq c_1$, we have $c \lambda e(x) < \varepsilon$ for all $x \in \mathbb{X}$, and hence,

$$Ace(x) = \varphi(cKe(x)) = \varphi(c\lambda e(x)) = \frac{\varphi(c\lambda e(x))}{c\lambda e(x)} c\lambda e(x) \geq \delta_1 ce(x).$$

Turning to claim (b) and again using the hypotheses in (C5), we can choose a $\delta_2 < 1$ and finite constant M such that

$$\frac{\varphi(t)}{t} \lambda \leq \delta_2 \quad \text{whenever} \quad t > M.$$

Let c_2 be a constant strictly greater than $\max\{M/(\lambda \underline{e}), c_1\}$ and fix $c \geq c_2$. By the definition of \underline{e} , we have $c\lambda e(x) \geq c_2 \lambda \underline{e} > M$ for all $x \in \mathbb{X}$, so

$$Ace(x) = \varphi(c\lambda e(x)) = \frac{\varphi(c\lambda e(x))}{c\lambda e(x)} \lambda ce(x) \leq \delta_2 ce(x).$$

By construction, $0 < c_1 < c_2$, so all claims are now established. ■

LEMMA C5: *If the conditions in (C5) hold and A has a fixed point g^* in \mathcal{C} , then, given any $g \in \mathcal{C}$, there exist functions $f_1, f_2 \in \mathcal{C}$ such that*

$$f_1 \leq Ag, g^* \leq f_2, \quad Af_1 \geq f_1 + \varepsilon(f_2 - f_1), \quad \text{and} \quad Af_2 \leq f_2 - \varepsilon(f_2 - f_1).$$

PROOF: Fix $g \in \mathcal{C}$. Since Ag is continuous and \mathbb{X} is compact, Ag attains a finite maximum and strictly positive minimum on \mathbb{X} . The same is true of the fixed point $g^* = Ag^*$ and the Perron-Frobenius eigenfunction e . Hence, we can choose constants a_1 and a_2 such that $0 \ll a_1 e \leq g^*, Ag \leq a_2 e$. With a_1 chosen sufficiently small, Lemma C4 implies that $A(a_1 e) \geq \delta a_1 e$ for some $\delta > 1$. Setting $f_i := a_i e$, we then have $Af_1 \geq \delta a_1 e$. Since $\delta > 1$, we can write this as $Af_1 \geq a_1 e + \varepsilon(a_2 - a_1)e$ for some positive ε . In other words, $Af_1 \geq f_1 + \varepsilon(f_2 - f_1)$. The proof of the last inequality is similar. ■

THEOREM C2: *If $\Lambda < 1$, then A is globally stable on \mathcal{C} .*

PROOF: First, we show that if $\Lambda < 1$, then the conditions in (C5) hold. Throughout we use the fact that $\Lambda = \beta r(K)^{1/\theta}$, as shown in Proposition C1. To start, observe that

$$\frac{\varphi(t)}{t} = \left\{ \frac{1 - \beta}{t^{1/\theta}} + \beta \right\}^\theta. \quad (\text{C6})$$

If, on the one hand, $\theta < 0$, then $\Lambda < 1$ implies that $\beta^\theta r(K) > 1$ and, in addition, (C6) increases to β^θ as $t \rightarrow 0$. Thus, the first inequality in (C5) holds. The second inequality also holds because $\varphi(t)/t \rightarrow 0$ as $t \rightarrow \infty$. If, on the other hand, $\theta > 0$, then $\beta^\theta r(K) < 1$ and (C6) diverges to $+\infty$ as $t \rightarrow 0$, so the first inequality in (C5) holds. The second inequality also holds because $\varphi(t)/t \rightarrow \beta^\theta$ as $t \rightarrow \infty$.

To complete the proof of Theorem C2, note that φ is either concave or convex, depending on the value of θ . Suppose first that φ is concave, which implies that A is both isotone and concave on \mathcal{C} . Lemma C4 yields positive constants $c_1 < c_2$ such that $Ac_1 e \geq c_1 e$ and $Ac_2 e \leq c_2 e$. Theorem 2.1.2 of Zhang (2013), which is based on Du (1990), now implies that A has a fixed point $g^* \in L_1(\pi)$ satisfying $c_1 e \leq g^* \leq c_2 e$. Since $e \gg 0$ and $c_1 > 0$, we have $g^* \gg 0$.

Let g be a nonzero element of \mathcal{C} . Choose f_1, f_2 as in Lemma C5. Theorem 2.1.2 of Zhang (2013) now implies that every element of $[f_1, f_2]$ converges to g^* under

iteration of A . In particular, $A^n(Ag) \rightarrow g^*$ as $n \rightarrow \infty$. But then $A^n g \rightarrow g^*$ also holds. We conclude that A is globally asymptotically stable on \mathcal{C} .

The proof for the convex case is essentially identical. ■

The next result expands on a line of argument developed by Toda (2019), shifting up to infinite dimensions and allowing $\theta < 0$.

PROPOSITION C2: *If A has a nonzero fixed point in \mathcal{C} , then $\Lambda < 1$.*

PROOF: Let K^* be the adjoint operator associated with K . Since K is irreducible and K^2 is compact, we can employ the version of the Krein-Rutman's theorem presented in Lemma 4.2.11 of Meyer-Nieberg (2012). Combining this result with the Riesz representation theorem, there exists an $e^* \in L_\infty(\pi)$ such that

$$e^* \gg 0 \quad \text{and} \quad K^*e^* = r(K)e^*. \quad (\text{C7})$$

Let g be a nonzero fixed point of A in \mathcal{C} . In what follows, it is convenient to use the inner product notation $\langle f, h \rangle := \int fhd\pi$ for $f \in L_1(\pi)$ and $g \in L_\infty(\pi)$.

First, consider the case in which $\theta < 0$, so that by the definition of φ , we have $\varphi(t) \leq \beta^\theta t$, with strict inequality whenever $t > 0$. As a result, $g(x) = Ag(x) = \varphi(Kg(x)) \leq \beta^\theta Kg(x)$, with strict inequality whenever $Kg(x) > 0$. By part (c) of Lemma C2, we have $Kg \geq 0$ and $Kg \neq 0$. So, it must be the case that $g \leq \beta^\theta Kg$ and $g < \beta^\theta Kg$ on a set of positive π -measure. But then, taking e^* as in (C7), we have $\langle e^*, \beta^\theta Kg - g \rangle > 0$, or, equivalently, $\beta^\theta \langle e^*, Kg \rangle > \langle e^*, g \rangle$. Using the definition of the adjoint and (C7) gives $r(K)\langle e^*, g \rangle = \langle K^*e^*, g \rangle = \langle e^*, Kg \rangle$, so it must be the case that $\beta^\theta r(K)\langle e^*, g \rangle > \langle e^*, g \rangle$. Hence, $\Lambda = \beta^\theta r(K) > 1$. Because $\theta < 0$, this implies that $\Lambda = \beta r(K)^{1/\theta} < 1$.

Next, consider the case in which $\theta > 0$, so that $\varphi(t) > \beta^\theta t$ whenever $t > 0$. As a result, we have $g(x) = Ag(x) = \varphi(Kg(x)) \geq \beta^\theta Kg(x)$, with strict inequality whenever $Kg(x) > 0$. By part (c) of Lemma C2, we have $Kg \geq 0$ and $Kg \neq 0$. So, it must be the case that $g \geq \beta^\theta Kg$ and $g > \beta^\theta Kg$ on a set of positive π -measure. But then, taking e^* as in (C7), we have $\langle e^*, \beta^\theta Kg - g \rangle < 0$, or, equivalently, $\beta^\theta \langle e^*, Kg \rangle < \langle e^*, g \rangle$. As already shown, we have $r(K)\langle e^*, g \rangle = \langle e^*, Kg \rangle$, so it must be the case that $\beta^\theta r(K)\langle e^*, g \rangle < \langle e^*, g \rangle$. Hence, $\beta^\theta r(K) < 1$. Because $\theta > 0$, this implies that $\Lambda = \beta r(K)^{1/\theta} < 1$. ■

PROOF OF THEOREM 1: Clearly, (e) \Rightarrow (d), which, in turn, implies (c) since we can take g equal to the fixed point. In addition, (c) implies (b), since K is a bounded linear operator on $L_1(\pi)$ and φ is continuous on \mathbb{R}_+ , from which it follows that A is continuous on \mathcal{C} and hence any limit of a sequence of iterates $\{A^n g\}_{n \geq 1}$ of A is a fixed point of A . The implication (b) \Rightarrow (a) is due to Proposition C2. Finally, (a) \Rightarrow (e) by Theorem C2 and Proposition C1. ■

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix S1: Internet Appendix.
Replication code.