Solution 7

A. Let $\{X_n,\ n = 0, 1, 2, \ldots\}$ be a branching process with offspring distribution $f_j = 1/4,\ j = 0, 1, 2, 3$.

1. Write down an expression for the offspring pgf, $F(s)$.
2. Prove that the offspring mean $\mu = 1.5$ and find the offspring variance σ^2 (either by differentiating $F(s)$, or from first principles).
3. *Write down* the mean and variance of X_n with $X_0 = 1$.
4. Calculate the probability of the event $\{X_1 > 0, X_2 = 0\}$.
5. What is the value of the extinction probability, q, if $X_0 = 1$?
6. What is the value of the extinction probability if $X_0 = 2$?
7. What is the *exact* value of the extinction probability if X_0 has the distribution $P(X_0 = 0) = 1/4$, $P(X_0 = 1) = 1/2$, $P(X_0 = 2) = 1/4$?

Solution:

1. $F(s) = (1 + s + s^2 + s^3)/4$.
2. $\mu = (1 + 2 + 3)/4 = 1.5$. $E(X^2) = (1 + 4 + 9)/4 = 3.5$ so $\sigma^2 = var(X) = 5/4$.
3. $E(X_n) = \mu^n$, and since $\mu \neq 1$, $var(X_n) = \frac{\sigma^2\mu^{n-1}(\mu^n-1)}{\mu-1} = \frac{5}{2}\left(\frac{3}{2}\right)^{n-1}\left(\left(\frac{3}{2}\right)^n - 1\right)$.
4. (Assume $X_0 = 1$)
$$\begin{aligned} P(X_1 > 0, X_2 = 0) &= P(X_1 = 1, X_2 = 0) + P(X_1 = 2, X_2 = 0) + P(X_1 = 3, X_2 = 0) \\ &= \frac{1}{4}\left(\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3\right). \end{aligned}$$
5. $F(s) = s$ gives $s^3 + s^2 - 3s + 1 = 0$ and since $s = 1$ is always a root we get $s^2 + 2s - 1 = 0$ with roots $-1 \pm \frac{\sqrt{8}}{2}$. So the extinction probability is $q = \sqrt{2} - 1 = 0.4142$ to 4 d.p.
6. Now extinction involves the extinction of two independent branching processes each with a single ancestor so extinction probability is $q^2 = 3 - 2\sqrt{2} = 0.1716$.
7. $\frac{1}{4} + \frac{1}{2}q + \frac{1}{4}q^2 = \frac{1}{2}$.

B. Let $\{X_n, n \geq 0\}$ be a branching process and assume $X_0 = 1$.

1. Suppose the offspring ξ_{10} has the pgf $F(s) = Es^{\xi_{10}} = \frac{1}{2}(1+s)$. Find the offspring distribution, the extinction probability and the average population size at the n-th generation.

2. (**Adv**) Show that, for $r = 1, 2, \ldots,$

$$E(X_{n+r} \mid X_n = k) = \mu^r\, k, \quad \text{where } \mu = EX_1. \tag{1}$$

(Hint: By induction to r and note that $\{X_n, n \geq 0\}$ is a MC)

Solution 1. (a) $f_0 = P(\xi_{10} = 0) = F(0) = 1/2$, $f_1 = P(\xi_{10} = 1) = F'(0) = 1/2$, and $f_k = 0, k \geq 2$.

(b) $\mu = \sum_{k=0}^{\infty} k f_k = 1 \times \frac{1}{2} = 1/2$. Since $1/2 < 1$, the distinction prob $q \equiv 1$.

(c) $EX_n = \mu^n = (1/2)^n$.

2. **Proof.** Step 1. Prove that the result holds true for $r = 1$ and all $n \geq 1$ and all k. In fact,

$$\begin{aligned} E(X_{n+1} \mid X_n = k) &= E\Big[\sum_{j=1}^{X_n} \xi_{jn} I_{(X_n \geq 1)} \mid X_n = k\Big] \\ &= E\Big(\sum_{j=1}^{k} \xi_{jn}\Big) = k E\xi_{10} = k\,\mu. \end{aligned}$$

Step 2. Suppose that (1) holds true for $r = m$ and all $n \geq 1$ and all k. We prove (1) still holds true for $r = m+1$ and all $n \geq 1$ and all k. Indeed,

$$\begin{aligned} E(X_{n+m+1} \mid X_n = k) &= \sum_{i=0}^{\infty} i\, P(X_{n+m+1} = i, X_n = k)/P(X_n = k) \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} i\, P(X_{n+m+1} = i, X_n = k, X_{n+1} = l)/P(X_n = k) \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} i\, P(X_{n+m+1} = i \mid X_n = k, X_{n+1} = l)\, P(X_{n+1} = l \mid X_n = k) \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} i\, P(X_{n+m+1} = i \mid X_{n+1} = l)\, P(X_{n+1} = l \mid X_n = k) \\ &= \sum_{l=0}^{\infty} E(X_{n+m+1} \mid X_{n+1} = l)\, P(X_{n+1} = l \mid X_n = k) \\ &= \sum_{l=0}^{\infty} \mu^m\, l\, P(X_{n+1} = l \mid X_n = k) \qquad \text{(by induction)} \\ &= \mu^m\, E(X_{n+1} \mid X_n = k) = \mu^{m+1}\, k. \end{aligned}$$

This implies the result holds true for $r = m+1$.