Solution 1

1. Let X be a r.v. such that $p_j = P(X=j) > 0, j = 0,1,2,...$ and $\sum_{j=0}^{\infty} p_j = 1$. Let $g(s)$ be the pgf of X, i.e., $g(s) = Es^X$.

(a). Show that $EX = g'(1)$ and $var(X) = g''(1) + g'(1) - [g'(1)]^2$.

(b). By applying (a), show that if $X \sim Poisson(\lambda)$, where $\lambda > 0$, then $EX = \lambda$ and $var(X) = \lambda$.

Solution: (a). Recall that $g^{(k)}(s) = \sum_{j=k}^{\infty} \frac{j!}{(j-k)!} s^{j-k} p_j$. Let $s = 1, k = 1, 2$. We obtain that

$$g'(1) = \sum_{j=1}^{\infty} j p_j = EX, \quad g''(1) = \sum_{j=1}^{\infty} j(j-1) p_j = EX(X-1).$$

Therefore, $EX = g'(1)$ and $var(X) = g''(1) + g'(1) - [g'(1)]^2$.

(b). Note that if $X \sim Poisson(\lambda)$, where $\lambda > 0$, then $g(s) = e^{\lambda(s-1)}$. We have

$$g'(1) = e^{\lambda(s-1)} \cdot \lambda\big|_{s=1} = \lambda, \quad g''(1) = e^{\lambda(s-1)} \cdot \lambda^2\big|_{s=1} = \lambda^2.$$

The results follows immediately.

2. The conditional expectation $E[X|Y] = \psi(Y)$, where $\psi(y) := E(X \mid Y = y)$, is a random variable (function of Y) and if $E|X| < \infty$, then $EX = E[E(X \mid Y)]$.

(a). Let Z and W be two discrete rvs, taking values $z = 0,1,2....$ and $w = 0,1,2,...$, respectively. Using the result given above, show that

$$\begin{aligned} P(Z = k) &= \sum_{j=0}^{\infty} P(Z = k \mid W = j) P(W = j), \\ EZ &= \sum_{j=0}^{\infty} E(Z \mid W = j) P(W = j). \end{aligned}$$

(b). Let Z and W be two independent continuous random variables with density $f(x)$ and $p(x)$, respectively. Using the result given above, calculate $P(Z \le W)$ and find the distribution of $Z + W$.

(c). Let (Z, W) be a continuous random vector with joint density $f(z, w)$. Calculate $P(Z \le W)$ and find the distribution of $Z + W$.

Solution:

(a). Let $X = I_{(Z=k)}$ and $Y = W$. Then $\psi(y) = E[X|Y = y] = P(Z = k \mid W = y)$. Hence

$$P(Z = k) = E\psi(W) = \sum_{j=0}^{\infty} \psi(j)\, P(W = j) = \sum_{j=0}^{\infty} P(Z = k \mid W = j)P(W = j).$$

(b). Let $X = I_{(Z\le W)}$ and $Y = W$. Then $\psi(y) = E[X|Y = y] = P(Z \le W \mid W = y) = P(Z \le y)$ from Lect 2. Hence

$$\begin{aligned} P(Z \le W) &= E\psi(W) = \int \psi(w)\, p(w)dw = \int P(Z \le w)p(w)dw \\ &= \int F(w)p(w)dw, \end{aligned}$$

where $F(w) = \int_{-\infty}^{w} f(x)dx$.

Note: In comparison with the direct calculation:

$$\begin{aligned} P(Z \le W) &= \int\int I_{(z\le w)} f(z)p(w)dzdw = \int\int_{-\infty}^{w} f(z)dzp(w)dz \\ &= \int F(w)p(w)dw. \end{aligned}$$

Similarly, we have

$$\begin{aligned} P(Z + W \le x) &= \int P(Z + W \le x \mid W = w)p(w)dw \\ &= \int P(Z + w \le x)p(w)dw = \int F(x - w)p(w)dw. \end{aligned}$$

(c)

$$\begin{aligned} P(Z \le W) &= \int\int I_{(z\le w)} f(z, w)dzdw = \int\int_{-\infty}^{w} f(z, w)dzdw, \\ P(Z + W \le x) &= \int\int I_{(z+w\le x)} f(z, w)dzdw = \int_{-\infty}^{\infty}\int_{-\infty}^{x-w} f(z, w)dzdw. \end{aligned}$$

3. (**Adv only**) Let $N := \{1, 2, \dots\}$ and let $\{X_k\}_{k\in N}$ be a sequence of iid random variables such that $\mathrm{Var}(X_1) > 0$, $E[|X_1|] < +\infty$ and $E[X_1] = 0$. Also, define $X_0 = X_{-1} = X_{-2} = 0$. Define a stochastic process $\{Z_k : k \in N\}$ in discrete time by

$$Z_k := X_k + X_{k-1} + X_{k-2} + X_{k-3}, \quad k \in N.$$

(a) Does the stochastic process $\{Z_k : k \in N\}$ have stationary increments?

(b) Does the stochastic process $\{Z_k : k \in N\}$ have independent increments?

(c) Compute the conditional expectation

$$E\left[Z_{k+1}|Z_k,\, Z_{k-1}, \dots, Z_1\right],$$

and express it first using $\{X_k\}_{k\in N}$ only, then next using Z_k if possible.

Solution: First, observe that the iid random variables $\{X_k\}_{k\in N}$ are not degenerate due to the assumption $\mathrm{Var}(X_1) > 0$. In particular, this assumption implies $P(X_0 = 0) < 1$.

(a) Observe that

$$\begin{aligned} Z_2 - Z_1 &= X_2, & (1) \\ Z_6 - Z_5 &= X_6 - X_2. & (2) \end{aligned}$$

Since $P(X_1 = 0) < 1$ and since two non-degenerate random variables X_2 and X_6 identically distributed, the random variable X_2 has a different distribution from the random variable $X_6 - X_2$. This is enough to claim that the stochastic process $\{Z_k : k \in N\}$ does not have stationary increments.

(b) The increments $Z_2 - Z_1$ and $Z_6 - Z_5$ have no overlap in time index, while as can be seen in (1) and (2), they have the non-degenerate random variable X_2 in common. Therefore, the increments are not independent. This is sufficient to conclude that the stochastic process $\{Z_k : k \in N\}$ does not have independent increments.

(c) Observe that $Z_1 = X_1$, $Z_2 = X_2 + X_1$, $Z_3 = X_3 + X_2 + X_1$ and $Z_4 = X_4 + X_3 + X_2 + X_1$. Therefore, the knowledge of Z_1, Z_2, Z_3 and Z_4 enables us to specify X_1, X_2, X_3 and X_4. By induction, we get

$$\begin{aligned} E\left[Z_{k+1} \middle| Z_k, Z_{k-1}, \ldots, Z_1\right] &= E\left[X_{k+1} \middle| Z_k, Z_{k-1}, \ldots, Z_1\right] + X_k + X_{k-1} + X_{k-2} \\ &= E\left[X_{k+1}\right] + X_k + X_{k-1} + X_{k-2} \\ &= X_k + X_{k-1} + X_{k-2} \\ &= Z_k - X_{k-3}. \end{aligned}$$