Solution 11

1. Arrivals at a public telephone booth form a Poisson process with a rate of 12 per hour. The duration of a phone call made from the booth is an exponential RV with average 2 minutes.

   (a) Explain why the phone calls in the tooth is a $M/M/1$ queueing system and find the traffic intensity.

   (b) Find the probability that an arrival will find the phone occupied.

   (c) Find the average length of the queue (including the person speaking) when it forms in the lung run.

   (d) It is the policy of the telephone company to install additional booth if the customers wait on the average at least 3 min for the phone. By how much must the flow of arrivals increase in order to justify the second booth?

Solution:

(a) $\lambda = 12/$ per hour=1/5 per min and $\mu = 1/2$ per min. So $\rho = 2/5$.

(b) $1 - \pi_0 = \rho = 0.4$

(c) In the lung run,

$$\begin{aligned} E(L \mid L > 0) &= \sum_{k=1}^{\infty} kP(L = k \mid L > 0) \\ &= \sum_{k=1}^{\infty} kP(L = k)/P(L > 0) = EL/P(L > 0) \\ &= 1/(1 - \rho) = 5/3. \end{aligned}$$

(d) We have the average waiting time:

$$EW = \rho/[\mu(1 - \rho)] = \lambda/[\mu(\mu - \lambda)].$$

Given $\mu = 0.5$. the new booths will be installed if

$$EW = \lambda/[0.5(0.5 - \lambda)] \geq 3.$$

Solving this equation, we find that $\lambda \geq 0.3$/min. With $\lambda = 0.3$, the average number of arrivals per hour is 18. Therefore, for an additional booth to be justified the arrivals rate should increase by 6 per hour.

2. Assume that the arrival of cars at a service station with one service-man is a Poisson process with rate λ (cars/per hour) and the service times are exponential with mean $1/\mu$ (hours/per car). Also assume that the cost incurred by the service station due to waiting cars (including being served) is $\$C_1$ per car per hour and the extra operating and service costs are $\$\mu C_2$ per hour when the service rate is μ. Determine the service rate μ that results in the least expected cost in the lung run.

   Solution: As "arrival-service" of the cars is a M/M/1 system with the traffic intensity $\rho = \lambda/\mu$, the average number of cars waiting (including being served) at the service station is $EL = \rho/(1-\rho) = \lambda/(\mu - \lambda)$. Hence, the total cost per hour to the station can be given as

$$C(\mu) = \lambda C_1/(\mu - \lambda) + \mu C_2.$$

   Minimizing $C(\mu)$ through standard techniques, we get

$$\mu = \lambda + \sqrt{\frac{\lambda C_1}{C_2}} \quad \text{or} \quad \lambda - \sqrt{\frac{\lambda C_1}{C_2}}.$$

   Since for stability we need $\lambda < \mu$, we get the optimal value

$$\mu^* = \lambda + \sqrt{\frac{\lambda C_1}{C_2}}.$$

3. (**3921/4021**) In a single-server queuing system with random arrivals and exponential service times with parameter β, customers only arrive in pairs, the probability of two arrivals in the interval $(t, t+h)$ being $\alpha h + o(h)$. Show that the stationary distribution of this system satisfies

$$\begin{aligned} \alpha\pi_0 &= \beta\pi_1 \\ (\alpha+\beta)\pi_1 &= \beta\pi_2 \\ (\alpha+\beta)\pi_n &= \alpha\pi_{n-2} + \beta\pi_{n+1}, \quad n \geq 2. \end{aligned}$$

   By multiplying both sides by s^n and summing over n, show that the pgf $\Pi(s) = \sum_{n=0}^{\infty} \pi_n s^n$ satisfies the equation

$$\Pi(s) = \frac{2 - 2\gamma}{2 - \gamma s - \gamma s^2},$$

   where $\gamma = 2\alpha/\beta$. Hence or otherwise find the mean of the distribution and the variance.

   Solution: Let X_t be the number of customers at time t. It is P+D P with transition probability:

For $i = 0$,

$$\begin{aligned} p_{ij}(h) &= 1 - \alpha h + o(h), \quad j = i, \\ &= \alpha h + o(h), \quad j = i + 2, \\ &= o(h), \quad \text{otherwise} \end{aligned}$$

For $i \geq 1$,

$$\begin{aligned} p_{ij}(h) &= \beta h + o(h), \quad j = i - 1, \\ &= \alpha h + o(h), \quad j = i + 2, \\ &= 1 - (\alpha + \beta)h + o(h), \quad j = i, \\ &= o(h), \quad \text{otherwise} \end{aligned}$$

Hence the stationary distribution satisfies ($\pi\, Q = 0$):

$$\begin{aligned} \alpha\pi_0 &= \beta\pi_1 \\ (\alpha+\beta)\pi_1 &= \beta\pi_2 \\ (\alpha+\beta)\pi_n &= \alpha\pi_{n-2} + \beta\pi_{n+1}, \quad n \geq 2. \end{aligned}$$

Multiplying the first line by s^0, the second line by s^1 and so on, and adding, gives

$$\alpha\Pi(s) + \beta(\Pi(s) - \pi_0) = \beta\left(\frac{\Pi(s) - \pi_0}{s}\right) + \alpha s^2 \Pi(s)$$

so that

$$\begin{aligned} \Pi(s) &= \frac{\pi_0\beta(1 - 1/s)}{\alpha + \beta - \frac{\beta}{s} - \alpha s^2} \\ &= \frac{\pi_0\beta}{\beta - \alpha s - \alpha s^2}. \end{aligned}$$

π_0 is chosen to make $\sum \pi_n = 1$ giving $\pi_0 = 1 - 2\alpha/\beta$.
Differentiating $\Pi(s)$ and putting $s = 1$ shows the mean to be $\mu = \frac{3\gamma}{2(1-\gamma)}$.
Differentiating twice and putting $s = 1$ gives

$$\sigma^2 = \Pi''(1) + \Pi'(1) - \Pi'(1)^2 = \frac{\gamma(10-\gamma)}{4(1-\gamma)^2}.$$