Week 8

Poisson processes

The Poisson process is one of the most widely-used counting processes. It is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate, but completely at random (without a certain structure). Typical example includes: the timings of earthquakes; the number of car accidents at a site or in an area; the requests for individual documents on a web server; the location of users in a wireless network, etc.

22 Definition of Poisson process

A random process $\{N_t, t \geq 0\}$ with state space $S = \{0, 1, 2, ...\}$ is said to be a Poisson process with rate (or intensity) $\lambda > 0$ if

(i). $\{N_t, t \geq 0\}$ is a process with $N_0 = 0$ and independent increments, i.e., for all $n \geq 1$ and time points $0 \leq t_0 < t_1 < ... < t_n$,

$$N_{t_0},\ N_{t_1} - N_{t_0},\ ...,\ N_{t_n} - N_{t_{n-1}},$$

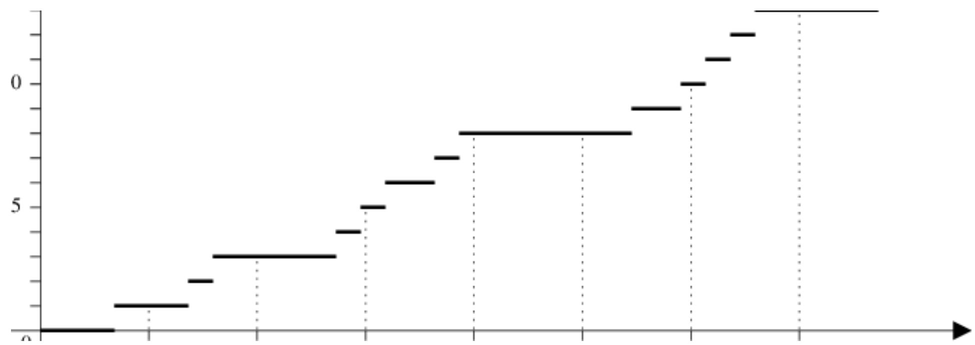
are independent r.vs;

(ii). for all $s \geq 0$ and $t > 0$,

$$P(N_{t+s} - N_s = j) = \frac{(\lambda t)^j e^{-\lambda t}}{j!}, \quad j = 0, 1, 2, ...$$

Note: The Poisson process has stationary increments, due to the fact that the distribution of the number of arrivals in any interval depends only on the length of the interval.

The sample pass of a Poisson process



Example 8.1. The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity $\lambda = 10$ customers per hour.

- Find the probability that there are 2 customers between 10:00 and 10:20.
- Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:20 and 11.

Solution: Let N_t be a Poisson process with $\lambda = 10$ (per hour).

- If N is the number of arrivals in that interval, then $N = N_{t+1/3} - N_t \sim Poisson(10/3)$. The required probab is

$$P(N = 2) = \frac{e^{-\frac{10}{3}} \left(\frac{10}{3}\right)^2}{2!} \approx 0.2$$

- Here, we have two non-overlapping intervals I_1 =(10:00 a.m., 10:20 a.m.] and I_2 = (10:20 a.m., 11 a.m.]. The required probab is

$$\begin{aligned} p &= P\Big(3 \text{ arrivals in } I_1 \quad \text{and} \quad 7 \text{ arrivals in } I_2\Big) \\ &= P(N_{t+1/3} - N_t = 3, N_{t+1} - N_{t+1/3} = 7) \\ &= P(N_{t+1/3} - N_t = 3)\, P(N_{t+1} - N_{t+1/3} = 7). \end{aligned}$$

The lengths of the intervals are $\tau_1 = 1/3$ and $\tau_2 = 2/3$ respectively. Thus,

$$p = \frac{e^{-\frac{10}{3}} \left(\frac{10}{3}\right)^3}{3!} \cdot \frac{e^{-\frac{20}{3}} \left(\frac{20}{3}\right)^7}{7!} \approx 0.0325$$

Poisson process is generally used to describe the rare events.

Let N_t be a Poisson process with rate λ. Consider a very short interval of length δ. Then, the number of arrivals in this interval has the same distribution as N_δ. In particular, we can write

$$\begin{aligned} P(N_\delta = 0) &= e^{-\lambda\delta} \\ &= 1 - \lambda\delta + \frac{\lambda^2}{2}\delta^2 - \cdots \text{ (Taylor Series).} \end{aligned}$$

As a consequence, as $\delta \to 0$, we have

$$P(N_\delta = 0) = 1 - \lambda\delta + o(\delta)$$

Here $o(\delta)$ shows a function that is negligible compared to δ, as $\delta \to 0$.

Now, let us look at the probability of having one arrival in an interval of length δ.

$$\begin{aligned} P(N_\delta = 1) &= e^{-\lambda\delta}\lambda\delta \\ &= \lambda\delta\left(1 - \lambda\delta + \frac{\lambda^2}{2}\delta^2 - \cdots\right) \text{ (Taylor Series)} \\ &= \lambda\delta + \left(-\lambda^2\delta^2 + \frac{\lambda^3}{2}\delta^3 \cdots\right) \\ &= \lambda\delta + o(\delta). \end{aligned}$$

We conclude that

$$P(N_\delta = 1) = \lambda\delta + o(\delta).$$

Similarly, we can show that

$$P(N_\delta \geq 2) = o(\delta).$$

Due to above equations, we may provide another way to define a Poisson process.

The Second Definition of the Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N_t, t \in [0, \infty)\}$ is called a **Poisson process** with **rate** λ if all the following conditions hold:

- $N_0 = 0$;
- N_t has independent and stationary increments;
- we have

$$\begin{aligned} &P(N_\delta = 0) = 1 - \lambda\delta + o(\delta), \\ &P(N_\delta = 1) = \lambda\delta + o(\delta), \\ &P(N_\delta \geq 2) = o(\delta). \end{aligned}$$

Note: due to the independent and stationary increments, we have

$$P(N_\delta = k) \ = \ P(N_{t+\delta} = k + i \mid N_t = i), \quad \text{for any } t \geq 0, i \geq 0.$$

Draft proof: it suffices to show that

$$P(N_t = j) = \frac{(\lambda t)^j e^{-\lambda t}}{j!}, \quad j = 0, 1, 2, ...$$

In fact, by defining $g_k(t) = P(N_t = k)$, we have:

1. for $k = 0$, $g_0(t+\delta) = g_0(t)[1 - \lambda\delta + o(\delta)]$, indicating that $g_0(t) = e^{-\lambda t}$;
2. for $k \geq 1$,

   $$g_k(t+\delta) = g_k(t)(1 - \lambda\delta) + g_{k-1}(t)\lambda\delta + o(\delta).$$

   i.e.,

   $$g_k'(t) = -\lambda g_k(t) + \lambda g_{k-1}(t),$$

   which is equivalent to

   $$\frac{d}{dt}\left[e^{\lambda t} g_k(t)\right] = \lambda e^{\lambda t} g_{k-1}(t);$$

   indicating $g_k(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, k \geq 1$.

Markov property of Poisson process:

Let $\{N_t : t \geq 0\}$ be a Poisson process with rate λ. Let $0 \leq t_1 < t_2 < ... < t_m < s$ and let $k_1 \leq k_2 \leq ... \leq k_m \leq j$. Then,

$$P(N_s = j | N_{t_m} = k_m, ..., N_{t_1} = k_1) = P(N_s = j | N_{t_m} = k_m)$$

Proof. Note that, for any event A depending only on $N_{t_{m-1}}, ...$, we have

$$\begin{aligned} P(N_s = j | N_{t_m} = k_m, A) &= \frac{P(N_s - N_{t_m} = j - k_m, N_{t_m} = k_m, A)}{P(N_{t_m} = k_m, A)} \\ &= P(N_s - N_{t_m} = j - k_m). \end{aligned}$$

Conditional distribution of Poisson process

Let $\{N_t : t \geq 0\}$ be a Poisson process with rate λ. Let $s, t \geq 0$. Then,

$$P(N_t = k \mid N_{s+t} = n) = \binom{n}{k} \big(\frac{t}{t+s}\big)^k \big(\frac{s}{t+s}\big)^{n-k}, \quad k = 0, 1, ..., n$$

Proof.

$$\begin{aligned} P(N_t = k \mid N_{s+t} = n) &= \frac{P(N_{s+t} - N_t = n - k, N_t = k)}{P(N_{s+t} = n)} \\ &= \frac{P(N_{s+t} - N_t = n - k)\, P(N_t = k)}{P(N_{s+t} = n)} \\ &= \frac{n!}{k!(n-k)!} \big(\frac{t}{t+s}\big)^k \big(\frac{s}{t+s}\big)^{n-k}. \end{aligned}$$

Ex8.2. Customers arrive at a bank according with a Poisson process with a rate 20 customers per hour. Suppose that two customer arrived during the first hour. What is the probability that at least one arrived during the first 20 minutes?

Solution: Let $\{N_t : t \geq 0\}$ be a Poisson process with rate $\lambda = 20$. We need to find:

$$P(N_{1/3} \geq 1 \mid N_1 = 2)$$

23 Interarrival and arrival (waiting) times

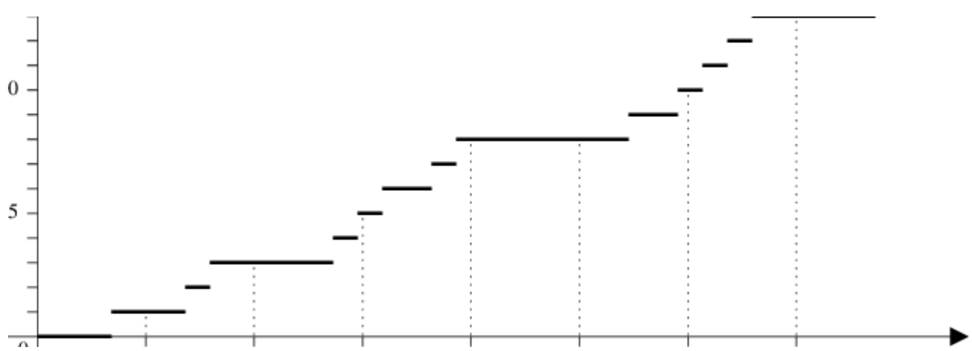
Let N_t be a Poisson process with rate λ.

Write $T_0 = 0$,

$$T_n = \inf\{s > 0 : N_s = n\}, \quad n = 1, 2, ...,$$

and $E_n = T_n - T_{n-1}$, $n = 1, 2, ...$ (hence $T_n = \sum_{j=1}^{n} E_j$). Then,

- T_n is called **arrival** (or **waiting**) time of the n-th jump in the process $\{N_t, t \geq 0\}$;
- $E_n, n \geq 1$, are called **interarrival times**, representing the time intervals between two successive occurrences (inter-occurrence time).



We are interested in the distributions of E_k and $T_k, k \geq 1$.

First note that $E_1 > t$ iff no arrival in$(0, t]$, i.e.,

$$\begin{aligned}P(E_1 > t) &= P\big(\text{no arrival in } (0, t]\big) \\ &= P(N_t = 0) = e^{-\lambda t}, \quad t > 0\end{aligned}$$

We conclude that $E_1 \sim Exp(\lambda)$.

We next show that E_1 and E_2 are independent with $E_2, E_1 \sim Exp(\lambda)$. In fact, for any $s, t \geq 0$, we have

$$\begin{aligned}P(E_1 > s, E_2 > t) &= P(\text{no arrival in } (E_1, E_1 + t] \cap \{E_1 > s\}) \\ &= \int_s^\infty P(\text{no arrival in } (u, u + t])\lambda e^{-\lambda u} du \\ &= \int_s^\infty P(N_{t+u} - N_u = 0)\lambda e^{-\lambda u} du \\ &= P(N_t = 0)e^{-\lambda s} = e^{-\lambda t} e^{-\lambda s},\end{aligned}$$

indicating $E_2 \sim Exp(\lambda)$ and E_2 is independent of E_1.

In general, by induction, we may prove $E_j, j \geq 1$, are mutually independent with $E_j \sim Exp(\lambda), j \geq 1$.

Example 8.2. Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.

(a) What is the expected time until the tenth immigrant arrives?

(b) What is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two days?

Solution: Let N_t be a Poisson process with rate $\lambda = 1$ (per day). Write $T_0 = 0$,

$$T_n = \inf\{s > 0 : N_s = n\}, \quad n = 1, 2, ...,$$

and $E_n = T_n - T_{n-1}$, $n = 1, 2, ...$ We have that $E_n \sim Exp(1)$ are iid random variables.

(a) $E[T_{10}] = \sum_{j=1}^{10} E(E_j) = 10/\lambda = 10$ (days).

(b) $P(E_{11} > 2) = e^{-2\lambda} = e^{-2} = 0.133$.

Example 8.3. Let N_t be a Poisson process with intensity $\lambda = 2$, and let E_1, E_2, $\cdots$ be the corresponding interarrival times.

- Find the probability that the first arrival occurs after $t = 0.5$, i.e., $P(E_1 > 0.5)$.
- Given that we have had no arrivals before $t = 1$, find the probability of $E_1 > 3$.
- Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.
- I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find ET and $\mathrm{Var}(T)$.
- I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. Find the conditional expectation and the conditional variance of T given that I am informed that the last arrival occurred at time $t = 9$.

Solution.

- Since $E_1 \sim Exp(2)$, we can write

$$P(E_1 > 0.5) = e^{-(2\times 0.5)} \approx 0.37.$$

- We can write

$$\begin{aligned} P(E_1 > 3|E_1 > 1) &= P(E_1 > 2) \text{ (memoryless property)} \\ &= e^{-2\times 2}. \end{aligned}$$

- The time between the third and the fourth arrival is $E_4 \sim Exp(2)$. Thus, the desired conditional probability is equal to

$$\begin{aligned} P(E_4 > 2|E_1 + E_2 + E_3 = 2) &= P(E_4 > 2) \text{ (independence of the } E_i\text{'s)} \\ &= e^{-2\times 2} \\ &\approx 0.0183 \end{aligned}$$

- When I start watching the process at time $t = 10$, I will see a Poisson process. Thus, the time of the first arrival from $t = 10$ is $Exp(2)$. In other words, we can write

$$T = 10 + X,$$

where $X \sim Exp(2)$. Thus,

$$\begin{aligned} ET &= 10 + EX \\ &= 10 + \frac{1}{2} = \frac{21}{2}, \end{aligned}$$

$$\begin{aligned} \text{Var}(T) &= \text{Var}(X) \\ &= \frac{1}{4}. \end{aligned}$$

- Arrivals before $t = 10$ are independent of arrivals after $t = 10$. Thus, knowing that the last arrival occurred at time $t = 9$ does not impact the distribution of the first arrival after $t = 10$. Thus, if A is the event that the last arrival occurred at $t = 9$, we can write

$$\begin{aligned} E[T|A] &= E[T] \\ &= \frac{21}{2}, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(T|A) &= \mathrm{Var}(T) \\ &= \frac{1}{4}. \end{aligned}$$

We next consider the distribution for arrival times $T_n, n \geq 1$. Note that $P(T_n > t) = P(N_t < n)$. We have

$$P(T_n \leq t) = P(N_t \geq n) = \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

This yields that T_n has a density:

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad \text{for } t > 0,$$

i.e., $T_n \sim Gamma(n, \lambda)$.

Recall that $T_n = E_1 + E_2 + \cdots + E_n$, where the E_i's are independent $Exp(\lambda)$ random variables. We have proved:

$$T_n = E_1 + E_2 + \cdots + E_n \sim Gamma(n, \lambda).$$

Since

$$E[E_1] = \frac{1}{\lambda}, \quad \text{Var}(E_1) = \frac{1}{\lambda^2}$$

we conclude that

$$E[T_n] = nE(E_1) = \frac{n}{\lambda}, \quad \text{Var}(T_n) = n\text{Var}(E_1) = \frac{n}{\lambda^2}.$$

Note that the arrival times $T_n, n \geq 1$, are not independent. In particular, we must have $T_1 \leq T_2 \leq T_3 \leq \cdots$.

24 Conditional distribution of arrival times

Example 8.4. A cable TV company collects \$1/unit time for each subscriber. Subscribers sign up in accordance with a Poisson process with rate λ. What is the expected total revenue received in $(0, t]$.

Solution. Let N_t denote the total number of customers that sign up in $(0, t]$, which is a Poisson process with rate λ. Let T_i denote the arrival time of the ith customer. The revenue generated by this customer in $(0, t]$ is $t - T_i$. Hence the expected total revenue received in $(0, t]$ is

$$
\begin{aligned}
\text{Total} &= E\big(\sum_{i=1}^{N_t}(t - T_i)\big) \\
&= \sum_{k=1}^{\infty} E\Big[(\sum_{i=1}^{k}(t - T_i) \mid N_t = k\Big] P(N_t = k).
\end{aligned}
$$

to be continued ...

In this example, we have to calculate $E(\sum_{i=1}^{k} T_i \mid N_t = k)$, i.e.,

given that k events of a Poisson process have taken place by time t, we are asked to determine the distribution of the arrival times $(T_1, \cdots, T_k)$.

Th8.1. Let $\{N_t, t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Then, given $N_t = k$, $(T_1, T_2, ..., T_k)$ has the same distribution as $(U_{(1)}, U_{(2)}, ..., U_{(k)})$, where $U_{(1)} \leq U_{(2)} \leq ... \leq U_{(k)}$ are the order statistics of iid r.vs $U_1, U_2, ..., U_k$ with uniform distribution over $[0, t]$.

Proof. only for $k = 1$.

$$\begin{aligned}
P(T_1 < s \mid N_t = 1) &= \frac{P(T_1 < s, N_t = 1)}{P(N_t = 1)} \\
&= \frac{P\big(1 \text{ event in } (0, s], \text{no event in } (s, t]\big)}{P(N_t = 1)} \\
&= \frac{P\big(N_s = 1, N_t - N_s = 0\big)}{P(N_t = 1)} \\
&= \frac{\lambda s e^{-\lambda s}\, e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = s/t, \quad 0 \leq s \leq t,
\end{aligned}$$

i.e., given $N_t = 1$, T_1 has the same distribution as $U_1 \sim U(0, t)$.

Note: In general situation, using the similar idea, we may prove that, given $N_t = k$, $(T_1, T_2, ..., T_k)$ has the joint density:

$$f(t_1, ..., t_k) = \frac{k!}{t^k}, \quad 0 < t_1 < t_2 < ... < t_k \leq t,$$

which is the same as $(U_{(1)}, U_{(2)}, ..., U_{(k)})$.

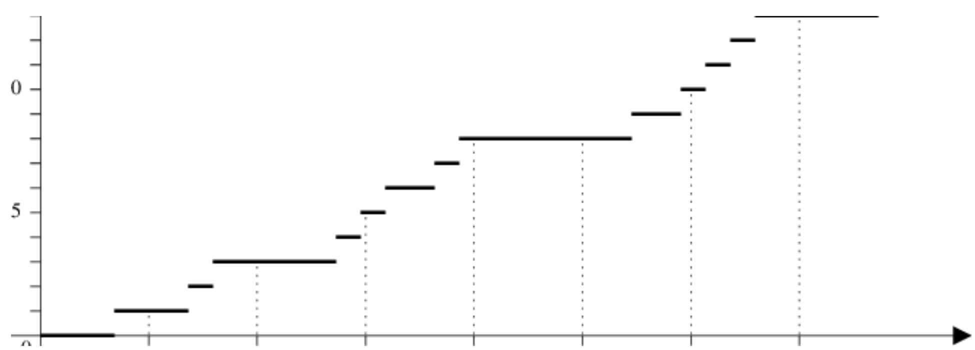
Example 8.4 (continue): by using Theorem 8.1, we have

$$\begin{aligned}
\text{Total} &= E\big(\sum_{i=1}^{N_t}(t-T_i)\big) \\
&= \sum_{k=1}^{\infty} E\Big[(\sum_{i=1}^{k}(t-T_i) \mid N_t = k\Big] P(N_t = k) \\
&= \sum_{k=1}^{\infty} P(N_t = k)\left(tk - E\sum_{i=1}^{k} U_{(i)}\right) \\
&= \frac{t}{2}\sum_{k=1}^{\infty} k\, P(N_t = k) \, = \frac{t}{2} E N_t = \frac{1}{2}\lambda t^2.
\end{aligned}$$

Let N_t be a Poisson process with rate λ.

In summary, we have that

- arrival time $T_n = \inf\{t > 0 : N_t = n\}$ can be decomposed as a partial sum of iid interarrival times, i.e., $T_n = \sum_{k=1}^{n} E_k$, where $E_k \sim Exp(\lambda)$ are iid random variables.



In reverse, if $T_n = \sum_{k=1}^{n} \xi_k$, where ξ_k are iid positive random variables with $0 < E\xi_1 < \infty$, we may define a process N_t:

$$N_0 = 0, \quad N_t = \max\{n \geq 1 : T_n \leq t\}, \quad t > 0,$$

This process is called a **Renewal process**, which is a generalization of Poisson process.

In particular, if $\xi_k \sim Exp(\lambda)$, then N_t is a Poisson process with rate λ.

Elementary renewal theorem

Let $m(t) = EN_t$, which is called *renewal function*. The following result is called the elementary renewal theorem:

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{E\xi_1}.$$

Proof is omitted. It is also true to say that $\lim N_t/t = \frac{1}{E\xi_1}$, i.e., there is a strong law for renewal process. It would be interesting to **note that**:

$$m(t) = EN_t = \sum_{n=1}^{\infty} P(N_t \geq n) = \sum_{n=1}^{\infty} F_n(t),$$

where $F_n(t) = P(T_n \leq t)$.