Solution 10

1. Potential customers arrive at a single-server station in accordance with a Poisson process with rate λ. However, if the arrival finds n customers already in the station, he will enter the queue with probability α_n. Assuming an exponential service rate μ, set the number of customers in the queue at time t as a birth and death process, find its generator Q and the transition matrix P of its embedded chain.

Solution: Let X_t be the number of customers in the queue at time t. We need to consider the probability: for small h,

$$P(X_{t+h} = j | X_t = i), \quad i, j = 0, 1, 2, ...$$

Recall that the arrival during $[t, t+h)$ is Poisson with rate λh, whereas a customer enters the queue only with probability α_n when there are already n customers ahead of him/her. Hence, when there are i customers in the system (i.e., $X_t = i$), the next actual entrance can be thought of as the first arrival of Poisson λh with thinning probability α_i, or equivalently the first arrival of Poisson $N \sim Poi(\lambda \alpha_i h)$. So, for any $i \geq 0$

$$P(X_{t+h} = i+1 | X_t = i) = P(N = 1) = \lambda \alpha_i h + o(h).$$

Similarly, for any $i \geq 0$ and $k \geq 2$,

$$P(X_{t+h} = i+k | X_t = i) \leq P(N \geq 2) \leq \sum_{i=2}^{\infty} P(N = i) = o(h).$$

Since it is a single-server station with exponential service rate μ, it is strightforward (from lectures) that

$$\begin{aligned} P(X_{t+h} = i-1 | X_t = i) &= \mu h + o(h), \quad i \geq 1, \\ P(X_{t+h} = i-k | X_t = i) &= o(h), \quad i \geq 2, k \geq 2. \end{aligned}$$

Furthermore, we have (why?)

$$\begin{aligned} P(X_{t+h} = 0 | X_t = 0) &= 1 - \lambda \alpha_0 h + o(h); \\ P(X_{t+h} = i | X_t = i) &= 1 - (\lambda \alpha_i + \mu) h + o(h), \quad i \geq 1. \end{aligned}$$

Combining all these facts, it is easy to see that $\{X_t\}_{t \geq 0}$ is a B&D process with the birth rates $\mu_i = \lambda \alpha_i, i = 0, 1, 2, ..$ and death rates $v_0 = 0$ and $v_i = \mu, i \geq 1$.

For this process, its generator is

$$Q = \begin{pmatrix} -\lambda\alpha_0 & \lambda\alpha_0 & 0 & 0 & ... \\ \mu & -(\lambda\alpha_1+\mu) & \lambda\alpha_1 & 0 & ... \\ 0 & \mu & -(\lambda\alpha_2+\mu) & \lambda\alpha_2 & ... \\ . & . & . & & \\ . & . & . & . & ... \end{pmatrix}.$$

and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & ... \\ \mu/(\mu+\lambda\alpha_1) & 0 & \lambda\alpha_1/(\mu+\lambda\alpha_1) & 0 & ... \\ 0 & \mu/(\lambda\alpha_2+\mu) & 0 & \lambda\alpha_2/(\lambda\alpha_2+\mu) & ... \\ . & . & . & & \\ . & . & . & . & ... \\ . & . & . & . & ... \end{pmatrix}.$$

2. Consider a telephone switchboard with N lines. Assume that there will be exactly one incoming phone call within the time interval $(t, t+h]$ with probability $\lambda h + o(h)$ and $\lambda > 0$. The probability is the same for all $t = 0$. The probability that two or more calls will come within the same interval is $o(h)$. No phone call will come with probability $1 - \lambda h + o(h)$. All the incoming calls are independent of each other. If all the N lines are engaged, then the incoming phone call is lost. The length of one phone call is exponentially distributed with the expected value $1/\mu, \mu > 0$

(a) Model the number of engaged lines at time t by a homogeneous continuous time Markov chain.

(b) Find the generator Q and the transition matrix P of the embedded chain.

Solution: Let X_t be the the number of engaged lines at time t. $X = \{X_t\}_{t\geq 0}$ is a continuous time process with the state space $S = \{0, 1, ..., N\}$. We need to consider the probability: for small h,

$$P(X_{t+h} = j|X_t = i), \quad i, j = 0, 1, 2, ..., N.$$

It is strightforward to see that

$$\begin{aligned} P(X_{t+h} = i+1|X_t = i) &= \lambda h + o(h), \quad i = 0, 1, ..., N-1; \\ P(X_{t+h} = i-1|X_t = i) &= i\mu\, h + o(h), \quad i = 1, 2, ..., N; \end{aligned}$$

(there are i calls at time t *and* exact one of them ends at time $t+h$)

Furthermore, we have $P(X_{t+h} = j|X_t = i) = o(h)$ if $|j - i| \geq 2$ (why?) and hence

$$\begin{aligned} P(X_{t+h} = 0|X_t = 0) &= 1 - \lambda h + o(h); \\ P(X_{t+h} = i|X_t = i) &= 1 - (\lambda + i\mu)h + o(h), \quad i = 1, ..., N-1; \\ P(X_{t+h} = N|X_t = N) &= 1 - N\mu h + o(h); \end{aligned}$$

As a consequence, its generator is

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & ... & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & ... & 0 & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & ... & 0 & 0 & 0 \\ . & . & . & & & & & \\ 0 & 0 & 0 & 0 & ... & (N-1)\mu & -(\lambda+(N-1)\mu) & \lambda \\ 0 & 0 & 0 & 0 & ... & 0 & N\mu & -N\mu \end{pmatrix}.$$

and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & ... & 0 & 0 & 0 \\ 0 & 2\mu/(\lambda+2\mu) & 0 & \lambda/(\lambda+2\mu) & ... & 0 & 0 & 0 \\ . & . & . & & & & & \\ 0 & 0 & 0 & 0 & ... & (N-1)\mu/(\lambda+(N-1)\mu) & 0 & \lambda/(\lambda+(N-1)\mu) \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 0 \end{pmatrix}.$$