Solution 2

1. Consider the following three matrices:

$$(i) \quad \begin{pmatrix} \alpha & 1-\alpha & 0 \\ 0 & \beta & 1-\beta \\ 1 & 0 & 0 \end{pmatrix} \qquad (ii) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha & 0 & \beta & 0 & 0 \\ 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (iii) \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & \beta & 0 & 0 \\ 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Assume $0 < \alpha < 1$ and $0 < \beta < 1$. Label the states 1–3 or 1–5.

(a). Specify the values of α and β which make these matrices the transition matrices of Markov Chains.

(b). For all chains, draw the transition diagrams, classify the states and specify all communicating classes.

(c). For chain (iii), calculate $P(X_3 = 4 \mid X_2 = 3)$ and $P(X_5 = 3 \mid X_1 = 1, X_3 = 3)$.

(d). For chain (iii), let $\pi_0(i) = P(X_0 = i)$, $i = 1, \ldots, 5$ and $\pi^{(0)} = (\pi_0(1), ..., \pi_0(5))$ with

$$\pi^{(0)} = \frac{1}{2 - 4\alpha + 4\alpha^2} \left(\alpha^3, \alpha^2, \alpha(1-\alpha), (1-\alpha)^2, (1-\alpha)^3\right).$$

Show that $P(X_n = i) = \pi_0(i)$, $i = 1, \ldots, 5$ for $n = 1, 2, \ldots$.
(Hint: only need to show that $\pi^{(0)}P = \pi^{(0)}$, where P is the transition matrix in (iii))

Solution:

(a). Any $0 < \alpha < 1$ and $0 < \beta < 1$ for (i). Any $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta = 1$ for (ii) and (iii).

(b). (i), Single class (irreducible); (ii). $\{1\}$ and $\{5\}$ are absorbings, $\{2, 3, 4\}$ is a class, not closed; (iii). Single class (irreducible), period 2.

(c). For chain (iii), $P(X_3 = 4 \mid X_2 = 3) = \beta$ and

$$\begin{aligned} P(X_5 = 3 \mid X_1 = 1, X_3 = 3) &= P(X_5 = 3 \mid X_3 = 3) = p_{33}^{(2)} \\ &= p_{32}^{(1)} p_{23}^{(1)} + p_{34}^{(1)} p_{43}^{(1)} \\ &= 2\alpha\beta. \end{aligned}$$

(d). Let $\pi^{(n)} = \big(P(X_n = 1), ..., P(X_n = 5)\big)$. It follows from Lect 5 that $\pi^{(n)} = \pi^{(0)}P^n$, and so, if $\pi^{(0)}P = \pi^{(0)}$, then

$$\pi^{(n)} = \pi^{(0)}P^n = \pi^{(0)}P^{n-1} = ... = \pi^{(0)}P = \pi^{(0)}.$$

The result $\pi^{(0)}P = \pi^{(0)}$ follows from a simple matrix multiplication.

2. Suppose the chances of rain tomorrow depends on the previous weather conditions (“fine” or ”rain”) only through whether or not it rains today (and only on past weather conditions). Suppose that, given “fine” today, it is “fine” tomorrow with probability 0.7, and given ”rain” today, it is fine tomorrow with probability 0.4.

(a). Describe the weather conditions (“fine” or ”rain”) by a MC.

(b). Find the probability that it will rain *4 days in a row* from today, given that it is raining today.

(c). If we start keeping record of weather on a day when the probability of rain is 0.4, what is the probability that it will rain *4 days in a row* after we start keeping records?

(d) What is the probability of raining *3 days later* given that it is raining today?

Solution. (a) Denote “fine” by state 0 and “rain” by state 1. A MC $\{X_n\}_{n\geq 0}$ with state space $S = \{0, 1\}$ and the transition matrix:

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix},$$

describes the weather conditions.

(b) We need to find the probability:

$$P(X_1 = 1, X_2 = 1, X_3 = 1 \mid X_0 = 1) = p_{11}p_{11}p_{11} = (0.6)^3 = 0.216.$$

(c) In this case, the initial dist of X_0 is $P(X_0 = 1) = 0.4$ and $P(X_0 = 0) = 0.6$. We need to find

$$\begin{aligned} & P(X_0 = 1, X_1 = 1, X_2 = 1, X_3 = 1) \\ = \; & P(X_0 = 1)\, P(X_1 = 1, X_2 = 1, X_3 = 1 \mid X_0 = 1) = 0.4\, 0.6^3 = 0.0864. \end{aligned}$$

(d) We need to find $P(X_3 = 1 \mid X_0 = 1)$. By the Chapman-Kolmogorov equation, we need to compute

$$P^{(3)} = \left(P^{(1)}\right)^3 = \left[\begin{array}{cc} 0.7 & 0.3 \\ 0.4 & 0.6 \end{array}\right]^3 = \left[\begin{array}{cc} 0.583 & 0.417 \\ 0.556 & 0.444 \end{array}\right].$$

Here, the $(2, 2)$-entry, 0.444, represents the probability of raining 3 days later given that it is raining today.

3. (**Adv**) Let $\{X_n, n \geq 0\}$ is a Markov Chain with state space S. Show that for all $A \subset S$, $i, j \in S$,

$$\begin{aligned} P(X_2 = j \mid X_0 \in A, X_1 = i) &= P(X_2 = j \mid X_1 = i), \\ P(X_{k+1} = j \mid X_0 \in A, X_1 = i) &= P(X_{k+1} = j \mid X_1 = i), \quad \text{for any } k \geq 1. \end{aligned}$$

Proof. In fact, we have

$$
\begin{aligned}
P(X_2 = j \mid X_0 \in A, X_1 = i) &= \frac{P(X_2 = j, X_0 \in A, X_1 = i)}{P(X_0 \in A, X_1 = i)} \\
&= \frac{\sum_{k \in A} P(X_2 = j, X_0 = k, X_1 = i)}{P(X_0 \in A, X_1 = i)} \\
&= \frac{\sum_{k \in A} P(X_2 = j \mid X_0 = k, X_1 = i)\, P(X_0 = k, X_1 = i)}{P(X_0 \in A, X_1 = i)} \\
&= \frac{P(X_2 = j \mid X_1 = i) \sum_{k \in A} P(X_0 = k, X_1 = i)}{P(X_0 \in A, X_1 = i)} \\
&= P(X_2 = j \mid X_1 = i).
\end{aligned}
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