Week 5

This week discusses random walks and branching processes, which can be considered as special examples of general MC.

Let $X_k, k \geq 1$ be i.i.d r.vs only taking integer values. Let $X_0 = 0$ or some i, $S_n = \sum_{k=0}^{n} X_k$. The process $\{S_n, n \geq 1\}$ is called a **Random walk** (RW).

Random walk is a discrete homogenous MC with state space $S \subseteq \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$. Indeed, for any A depending only on $S_{n-1}, S_{n-2}, ..., S_1$ and any $n \geq 1$, we have

$$\begin{aligned} P(S_{n+1} = k \mid S_n = j, A) &= \frac{P(S_{n+1} = k, S_n = j, A)}{P(S_n = j, A)} \\ &= \frac{P(X_{n+1} = k - j, S_n = j, A)}{P(S_n = j, A)} \\ &= P(X_{n+1=k-j}) = P(X_1 = k - j), \end{aligned}$$

provided without further restriction is imposed.

A random walk is said to be **unrestricted** if $S = \mathbb{Z}$.

In the case that S is a proper subset of $\mathbb{Z}$, the endpoints of S are called **barriers**. An **absorbing barrier** is an endpoint that the MC never leaves once it reaches the endpoint.

Example 5.1. (Simple random walk) Let $X_k, k \geq 1$ be i.i.d r.vs with

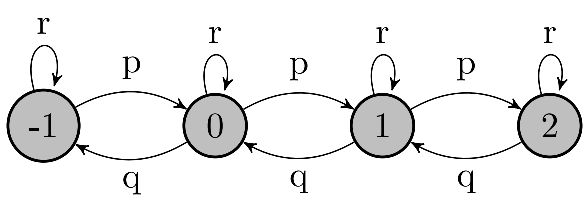
$$P(X_1 = 1) = p, \quad P(X_1 = 0) = r, \quad P(X_1 = -1) = q,$$

where $p + r + q = 1$. Let $X_0 = 0$ and $S_n = \sum_{k=0}^{n} X_k$.

$\{S_n, n \geq 1\}$ is a simple random walk with $S = \mathbb{Z}$ (unrestricted). $\{S_n, n \geq 1\}$ is a MC with one-step transition probability:

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ r & \text{if } j = i \\ q & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

The state transition diagram:



We say a simple RW **symmetric** if $r = 0$ and $p = 1/2$.

Example 5.2. (Simple random walk with barriers) Let $X_k, k \geq 1$ be i.i.d r.vs with

$$P(X_1 = 1) = p, \qquad P(X_1 = -1) = 1 - p,$$

where $0 < p < 1$.

Let $X_0 = m > 0$ and $S_n = \sum_{k=0}^{n} X_k$.

Also let $S_{n+1} = 0$ if $S_n = 0$, and $S_{n+1} = k$ if $S_n = k$ where $k > m$.

$\{S_n, n \geq 1\}$ is a simple Random walk with $S = \{0, 1, 2, ..., k\}$. The absorbing barriers are $\{0, k\}$.

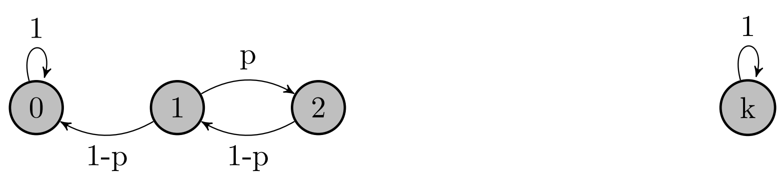
The one-step transition probability:

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \text{ and } 0 < i < k \\ 1-p & \text{if } j = i-1 \text{ and } 0 < i < k \\ 1 & \text{if } i = j = 0 \text{ or } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

The transition matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ . \\ . \\ k-1 \\ k \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & ... & k-1 & k \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 1-p & 0 & p & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & 1-p & 0 & p \\ 0 & 0 & ... & 0 & 1 \end{pmatrix} \end{array}$$

The state transition diagram:



Note: **Ex 5.2** is known from a Gambler's ruin problem. Consider a gambler who has a initial capital m and play a game that wins \$1 or lose \$1 with probabilities p and q, respectively. The game ends whenever the gambler reaches \$ k or goes broke. In the **Ex 5.2**, S_n stand for the cumulative fortune in the n-step play.

13 Unrestricted random walk

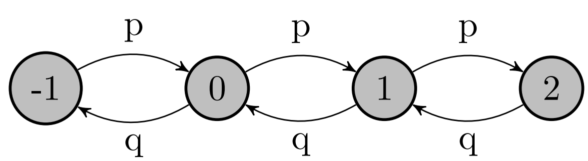
This lecture assumes that $\{S_n, n \geq 1\}$ is a RW defined as in **Ex5.1** with $r = 0$, $X_0 = 0$ and $q = 1 - p$, i.e., $S_n = \sum_{k=0}^{n} X_k$, where $X_k, k \geq 1$ are i.i.d r.vs with

$$P(X_1 = 1) = p, \quad P(X_1 = -1) = q = 1 - p.$$

As a MC, $\{S_n, n \geq 1\}$ has transition probability: for all $i, j = 0, \pm 1, \pm 2, ...,$

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

and the state transition diagram:



and it is irreducible.

Th5.1 Simple RW with $r = 0$ is recurrent (transient) if and only if $p = 1/2$ ($p \neq 1/2$).

Proof. It suffices to note that the RW is recurrent if and only if $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$, where

$$\begin{aligned} p_{00}^{(n)} &= P(S_{n+1} = 0 \mid S_1 = 0) = P(S_n = 0) \\ &= \begin{cases} \binom{2k}{k} p^k (1-p)^k & \text{if } n = 2k \\ 0 & \text{if } n = 2k-1. \end{cases} \end{aligned} \tag{1}$$

Stirling formula:

$$m! \;\sim\; \sqrt{2\pi}\, m^{m+1/2} e^{-m}, \quad \text{as } m \to \infty.$$

We further have that if $p = 1/2$, then

- all states are null recurrent; the chain has a period 2;

  Proof. In fact, if $p = q$, then $p_{00}^{(n)} \to 0$ as $n \to \infty$, as

$$p_{00}^{(2k)} \sim (4pq)^k/\sqrt{\pi k}.$$

- the RW returns to state 0 (original starting state) infinitely often with probability one: i.e.,

$$\begin{aligned} f_{00} &= P(S_n = 0 \text{ for some } n \geq 2 \mid S_1 = 0) \\ &= P(\cup_{n=2}^{\infty}\{S_n = 0\} \mid S_1 = 0) = 1. \end{aligned}$$

  See Lect 8 for more details.

- there is no stationary distribution. See Lect 11.

If $p \neq 1/2$, it may be proved that

$$\begin{aligned} & P(\cup_{n=k}^{\infty}\{S_n = 0\} \mid S_1 = 0) \\ \leq & \sum_{n=k}^{\infty} P(S_n = 0 \mid S_1 = 0) \to 0, \end{aligned}$$

as $k \to \infty$, i.e., in the long run, the RW do not return to starting point with probability one.

First passage time (extended knowledge for advanced students)

Define the first passage time from the origin 0 to point $k \geq 0$ as

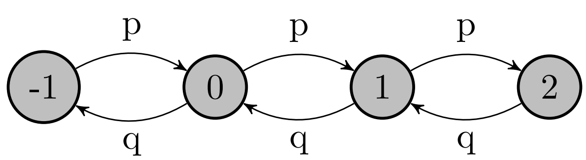
$$T_k^0 = \inf\{n \geq 1 : S_n = k\}.$$

If $k = 0$, T_0^0 is also called the first time of return to 0.

We are interested in the probability in relation to the first passage time and the average first passage time. Note that for $k \geq 1$,

$$T_k^0 = T_1^0 + T_2^1 + \cdots + T_k^{k-1},$$

where T_j^i is the first passage time from state i to state j, and these first passage times are iid r.vs.



We only need to find the distributions of T_0^0 and T_1^0, ET_0^0 and ET_1^0.

Recall: For a MC $\{X_n\}_{n\geq 0}$, we define

$$\begin{aligned} p_{ii}^{(n)} &= P(X_n = i \mid X_0 = i), \\ f_{ij}^{(1)} &= P(X_1 = j \mid X_0 = i), \\ f_{ij}^{(n)} &= P(X_n = j, X_{n-1} \neq j, ..., X_1 \neq j \mid X_0 = i),\ n \geq 2, \\ f_{ij} &= \sum_{n=1}^{\infty} f_{ij}^{(n)} = P(X_n = j \text{ for some } n \geq 1 \mid X_0 = i). \end{aligned}$$

Define

$$\begin{aligned} f_{ij}(s) &= \sum_{n=1}^{\infty} f_{ij}^{(n)} s^n, \\ p_{ij}(s) &= \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n, \end{aligned}$$

where $p_{ij}^{(0)} = 1$ if $i = j$ and $p_{ij}^{(0)} = 0$ if $i \neq j$. Using the identity equation:

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \qquad \text{for all } n \geq 1,$$

we may prove

$$p_{ii}(s) = 1 + f_{ii}(s) p_{ii}(s), \quad \text{for } 0 < s < 1.$$

Note: we may prove $p_{ij}(s) = f_{ij}(s) p_{jj}(s)$ for $i \neq j$.

Th5.2. Let T_0^0 be the time of the first return to 0. Then,

(1). The pgf of T_0^0 is

$$Es^{T_0^0} = 1 - (1 - 4pqs^2)^{1/2}.$$

Proof. Note that

$$P(T_0^0 = n) \;=\; P(S_n = 0, S_{n-1} \neq 0, ..., S_1 \neq 0 \mid S_0 = 0) = f_{00}^{(n)}.$$

It follows that

$$\begin{aligned} Es^{T_0^0} &= \sum_{n=1}^{\infty} f_{00}^{(n)} s^n \qquad\qquad = f_{00}(s) \\ &= 1 - 1/p_{00}(s) = 1 - (1 - 4pqs^2)^{1/2}, \end{aligned}$$

since

$$\begin{aligned} p_{00}(s) &= \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n \\ &= 1 + \sum_{k=1}^{\infty} p_{ij}^{(2k)} s^{2k} \\ &= 1 + \sum_{k=1}^{\infty} \binom{2k}{k} (pqs^2)^k = (1 - 4pqs^2)^{-1/2}. \end{aligned}$$

(2). $P(T_0^0 < \infty) = f_{00}(1) = 1 - \sqrt{1 - 4pq}$.

Consequently, the probability that the RW never returns to 0 is $(1 - 4pq)^{1/2}$.

When $p = q = 1/2$, the RW will return to 0 with probability one, but $ET_0^0 = \infty$.

(3). The distribution of T_0^0 is given by: for any $n \geq 1$, $P(T_0^0 = 2n-1) = 0$ and

$$P(T_0^0 = 2n) = \begin{cases} 2pq & \text{if } n = 1 \\ \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-3}{2} \frac{(4pq)^n}{2\,n!} & \text{if } n \geq 2. \end{cases}$$

Proof. Recall **Th1.2.** (Lect 1): Let X be a r.v. only takes integer values. Then the dist of X is uniformly determined by its pgf $g(s) = Es^X, 0 \leq s \leq 1$. In fact, we have

$$P(X = k) = g^{(k)}(0)/k!, \quad k = 0, 1, 2, ...$$

The details are omitted.

Th5.3. Let T_1^0 be the first passage time from 0 to 1. Then,

(1). The pgf of T_1^0 is

$$Es^{T_1^0} = [1 - (1 - 4pqs^2)^{1/2}]/(2qs).$$

Proof. Omitted.

(2). $P(T_1^0 < \infty) = 1$ for $p \geq 1/2$, and $P(T_1^0 < \infty) = p/q$ for $p < 1/2$.

Proof. Recall $q = 1 - p$.

$$\begin{aligned} P(T_1^0 < \infty) &= f_{01}(1) = [1 - (1 - 4pq)^{1/2}]/(2q) \\ &= \left(1 - \left[(2p-1)^2\right]^{1/2}\right)/(2q) \\ &= 1, \quad \text{if } p \geq 1/2 \\ &= p/q, \quad \text{if } p < 1/2. \end{aligned}$$

(3). The distribution of T_1^0 is given by: for any $n \geq 1$, $P(T_1^0 = 2n) = 0$ and

$$P(T_1^0 = 2n-1) = \frac{1}{2n-1}\binom{2n-1}{n}p^n(1-p)^{n-1} = \frac{1}{2n-1}P(S_{2n-1} = 1).$$

(4). $ET_1^0 = \infty$ for $p \le 1/2$, and $ET_1^0 = 1/(p-q)$ for $p > 1/2$.

Proof. Note that $S_{T_1^0} = 1$. The result can be proved by using the Wald's equation.

> **Wald's Identity:** Let $X_k, k \ge 1$ be iid r.vs with $E|X_1| < \infty$. Let N be a **stopping time** defined by: $N \ge 0$ is an integer value r.v. such that $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, ...$ Let $S_N = \sum_{k=1}^N I_{(N \ge 1)}$. Then we have
>
> $$E(N) < \infty \Rightarrow ES_N = EN\, EX_1.$$

Indeed, by noting $(T_1^0 = n)$ depending only on $X_n, X_{n-1}, ...$, $EX_1 = p - q$ and $ES_{T_1^0} = 1$,

- if $p > q$, i.e., $p > 1/2$, then $ET_1^0 = 1/(p-q)$;
- when $p \le q$, $EX_1 \le 0$, we must have $ET_1^0 = \infty$, since
  if $ET_1^0 < \infty$, then $1 = ET_1^0(p-q) \le 0$, which is a contradiction.

14 Simple random walk with barriers

In this section we consider the RW defined as in **Ex5.2**, i.e.,

Ex5.2. (Simple random walk with barriers) Let $X_k, k \geq 1$ be i.i.d r.vs with

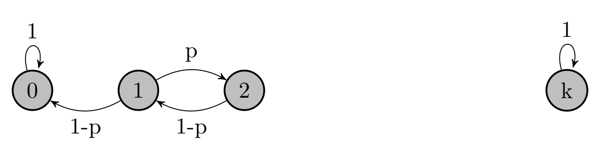
$$P(X_1 = 1) = p, \qquad P(X_1 = -1) = 1 - p,$$

where $0 < p < 1$. Let $X_0 = m > 0$ and $S_n = \sum_{k=0}^{n} X_k$. Also let $S_{n+1} = 0$ if $S_n = 0$, and $S_{n+1} = k$ if $S_n = k$ where $k > m$. $\{S_n, n \geq 1\}$ is a simple Random walk with $S = \{0, 1, 2, ..., k\}$. The absorbing barriers are $\{0, k\}$. The one-step transition probability:

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1 \text{ and } 0 < i < k \\ 1 - p & \text{if } j = i - 1 \text{ and } 0 < i < k \\ 1 & \text{if } i = j = 0 \text{ or } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

(Recall that a gambler has an initial capital m and play a game that win \$1 or lose \$1 with probabilities p and q, respectively. The game ends whenever the gambler reaches \$ k or goes broke. S_n stands for the cumulative fortune after n-steps of play.)

Clearly, as a MC, 0 and k are absorbing states and $T = \{1, 2, ..., k - 1\}$ is a set of transients.



Define *the time of absorption* or *the time of ruin* N from the original capital $X_0 = m$:

$$N = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = k\},$$

(Recall the game will stop when the MC reaches 0 or k).

Th5.4. (Probability of a Gambler's ruin) If $p = q$, we have

$$\begin{aligned} P(S_N = 0) &= P(\text{the gambler is ruined}) = (k-m)/k; \\ P(S_N = k) &= P(\text{the gambler reaches } k\$) = m/k. \end{aligned}$$

If $p \neq q$, we have

$$\begin{aligned} P(S_N = 0) &= 1 - (1-\theta^m)/(1-\theta^k); \\ P(S_N = k) &= (1-\theta^m)/(1-\theta^k), \quad \text{where } \theta = q/p. \end{aligned}$$

This thm is a generalization of Example 4.5.

Proof. Let

$$a_i = P(\cup_{n=1}^{\infty}(S_n = 0) \mid S_1 = i), \quad i = 0, 1, ..., k.$$

It follows from (6) in Week 4 notes (Lect 12) that $a_0 = 1, a_k = 0$ and for $1 \leq i \leq k-1$

$$\begin{aligned} a_1 &= q a_0 + p a_2, \\ a_i &= q a_{i-1} + p a_{i+1}, \quad 2 \leq i \leq k-2, \\ a_{k-1} &= q a_{k-2} + p a_k. \end{aligned}$$

Hence

$$p(a_i - a_{i+1}) = q(a_{i-1} - a_i),$$

i.e., with $\theta = q/p$,

$$a_i - a_{i+1} = \theta\,(a_{i-1} - a_i) = ... = \theta^i (a_0 - a_1), \quad i = 1, 2, ..., k-1,$$

indicating that

$$\begin{aligned} a_0 - a_j &= \sum_{i=0}^{j-1} (a_i - a_{i+1}) \\ &= (a_0 - a_1)(1 + \theta + ... + \theta^{j-1}), \quad j = 1, 2, ..., k, \end{aligned}$$

Taking $j = k$, we obtain

$$1 = (1 - a_1)(1 + \theta + ... + \theta^{k-1}).$$

It follows that

$$\begin{aligned} a_j &= 1 - \frac{1 + \theta + ... + \theta^{j-1}}{1 + \theta + ... + \theta^{k-1}} \\ &= \begin{cases} 1 - j/k & \text{if } p = q \\ 1 - \frac{1-\theta^j}{1-\theta^k} & \text{if } p \neq q \end{cases} \end{aligned}$$

Theorem follows by noting $a_m = P(S_N = 0)$ and

$$P(S_N = k) = 1 - P(S_N = 0).$$

Th5.5. (The expected duration of the game)

$$
\begin{aligned}
EN &= m(k-m), \qquad \text{if } p = q; \\
EN &= \frac{1}{q-p}\big[m - k(1-\theta^m)/(1-\theta^k)\big], \quad \text{where } \theta = q/p.
\end{aligned}
$$

Proof. Let $\tau = \min\{n \geq 1 : S_n = 0 \text{ or } S_n = k\}$ and

$$
m_i = E(\tau \mid S_1 = i), \ i = 1, 2, ..., k-1.
$$

It follows from (8) in Lect 12 that (define $m_0 = m_k = 0$)

$$
\begin{aligned}
m_1 &= 1 + qm_0 + pm_2 \\
m_i &= 1 + qm_{i-1} + pm_{i+1}, \quad 1 \leq i \leq k-2, \\
m_{k-1} &= 1 + qm_{k-2} + pm_k
\end{aligned}
$$

Solving these equations, if $p = q$, we get

$$
m_i = i(k-i), \quad i = 1, ..., k;
$$

if $p \neq q$, then

$$
m_i = \frac{k}{p-q}\frac{1-\theta^i}{1-\theta^k} - \frac{i}{p-q}, \quad i = 1, 2, ..., k.
$$

Theorem follows by noting $EN = E(\tau \mid S_1 = m)$.

In the gambling scheme, the case $k = \infty$ corresponds to a game against an infinitely rich adversary. The random walk describing this situation has only 0 as an absorbing barrier.

Th5.6. Letting $k \to \infty$ in Th5.4 and Th5.5, we obtain that

$$\begin{aligned} P(S_N = 0) &= \begin{cases} 1 & \text{when } p \le 1/2 \\ (q/p)^m & \text{when } p > 1/2. \end{cases} \\ EN &= \begin{cases} \infty & \text{when } p \ge 1/2 \\ m/(q-p) & \text{when } p < 1/2. \end{cases} \end{aligned}$$

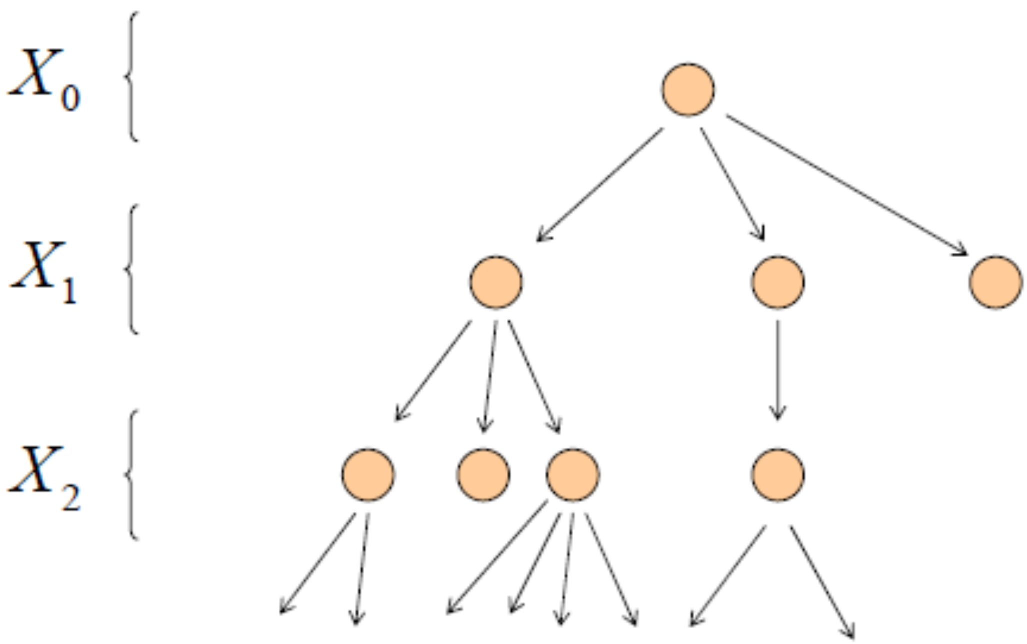
15 Branching process

Consider a population consisting of individuals that are able to produce offspring of the same kind.

Suppose that each individual will, by the end of its lifetime, have produced i new offspring with probability $f_i, i \geq 0$, (assume $0 < f_0 < 1$) independently of the number produced by any other individual.

Let X_0 be the number of individual initially present (zeroth generation). $X_n, n \geq 1$, be the size of the n-th generation.

$\{X_n, n \geq 0\}$ is called a **Branching process** (BP).



We next assume $X_0 = 1$. Let ξ_{jk} denote the offspring of the j-th number in the k-th generation.

According to the assumption above, it is readily seen that

- ξ_{jk}, $k \geq 0, j \geq 1$, are iid random variables with distribution:

$$P(\xi_{jk} = i) = f_i, \quad i = 0, 1, 2..., \quad \sum_{i=0}^{\infty} f_i = 1, \quad 0 < f_0 < 1,$$

- $X_1 = \xi_{10}$, $X_{n+1} = \sum_{j=1}^{X_n} \xi_{jn} I_{(X_n \geq 1)}$, $n \geq 1$ and X_n is independent of ξ_{jn} for $n \geq 1$.

Note that X_n is a random sum as considered in Lecture 3:

Random sum: Let $N, Z_k, k \geq 1$, be iid r.vs taking non-negative integer values $0, 1, 2, ...$ N is independent of $Z_k, k \geq 1$. Let $T_0 = 0$ and $T_n = Z_1 + Z_2 + ... + Z_n$. **A random sum** is defined as $T_N = \sum_{j=1}^{N} Z_j I_{N\geq 1}$.

For the random sum, we have the following results:

- $ET_N = EN\,\mu$, where $\mu = EZ_1$,
- If Z_i has pgf $A(s)$ and N has pgf $B(s)$, then T_N has pgf $B(A(s))$.

Proof. We have

$$\begin{aligned} Es^{T_N} &= E\big[E\big(e^{T_N} \mid N\big)\big] = \sum_{k=0}^{\infty} Es^{T_k} P(N = k) \\ &= \sum_{k=0}^{\infty} A^k(s) P(N = k) \\ &= E\big[A(s)\big]^N = B[A(s)], \quad 0 \leq s \leq 1. \end{aligned}$$

Application. Let $N \sim Poisson(\lambda)$ and $P(Z_1 = 1) = 1 - P(Z_1 = 0) = p$. Then $T_N \sim Poisson(\lambda\, p)$.

Proof. Recall that $N \sim Poisson(\lambda)$ iff $B(s) = Es^N = e^{\lambda(s-1)}$.

Note that $A(s) = Es^{Z_1} = q + ps$, where $q = 1 - p$. It follows that

$$Es^{T_N} = B[A(s)] = e^{\lambda(q+ps-1)} = e^{\lambda p(s-1)},$$

i.e., $T_N \sim Poisson(\lambda\, p)$.

We also have the following result:

- $\{X_n, n \geq 0\}$ is a MC with state space $S \subset (0, 1, 2, ...)$ and

$$p_{ij} := P(X_{n+1} = j \mid X_n = i) = P\left(\xi_{1n} + \xi_{2n} + ... + \xi_{in} = j\right).$$

- $\{0\}$ is absorbing and $S - \{0\}$ is a transient class: in fact,

$$\begin{aligned} f_{kk} &= P\big(\cup_{j=n+1}^{\infty} \{X_j = k\} \mid X_n = k\big) \\ &\leq P(X_{n+1} \neq 0 \mid X_n = k) \\ &= 1 - P(X_{n+1} = 0 \mid X_n = k) = 1 - f_0^k < 1. \end{aligned}$$

- We may prove

$$P(X_n \to 0) + P(X_n \to \infty) = 1.$$