Week 7

Quiz, exponential and Poisson distribution

19 Quiz

20 Exponential distribution

A random variable X is called **exponential** with parameter (or rate) $\lambda > 0$), often written as $X \sim Exp(\lambda)$, if it has probability density function:

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

For $X \sim Exp(\lambda)$, we have

$$P(X > x) = e^{-\lambda x}, \quad E[X] = 1/\lambda, \quad Var(X) = 1/\lambda^2,$$

and the mgf of X is $Ee^{tX} = \lambda/(\lambda - t)$, which is well defined when $t < \lambda$.

Such a random variable is memoryless:

$$P(X > s + t | X > t) = P(X > s), \qquad \text{for all } s, t \geq 0,$$

or

$$P(X > s + t) = P(X > s)\, P(X > t), \qquad \text{for all } s, t \geq 0.$$

We further have the following result: the strong memoryless property.

Th7.1. Suppose that $X \sim Exp(\lambda)$ and $Y \geq 0$ is a random variable independent of X. Then, for all $s \geq 0$, we have

$$P[X > s + Y | X > Y] = P[X > s]. \tag{1}$$

Note: If $X \sim Exp(\lambda)$, $Z, Y \geq 0$ and X, Y, Z are mutually independent, then

$$P[X > Z + Y | X > Y] = P[X > Z]. \tag{2}$$

In particular, $P(X > Z + t \mid X > t) = P(X > Z)$ for any $t \geq 0$.

It follows from (2) that

$$P[X > Z + Y] = P[X > Z]\, P(X > Y).$$

Example 7.1 Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes.

(a) What is the probability that a customer will spend more than 15 minutes in the bank?

(b) What is the probability that a customer will spend more than 15 minutes in the bank given that he is still in the bank after 10 minutes?

Example 7.2 A post office has 2 clerks. The customer service time of each clerk is $Exp(\lambda)$. Neither clerk is busy. One customer arrives at a random time and, while that customer is still being served, another customer arrives at a random time and begins service with the other clerk. What is the chance that the first customer finishes first?

The answer is 1/2 by the (strong) memoryless property and symmetry. Note that we assume the arrival times and service times are mutually independent.

Proof. Let X_1, X_2 be the arrival times and S_1 and S_2 the service times. We have

$$\begin{aligned} & P[X_1 + S_1 > X_2 + S_2 | X_1 < X_2 < X_1 + S_1] \\ = \; & P[S_1 > S_2 + (X_2 - X_1) | 0 < X_2 - X_1 < S_1] \\ = \; & P[S_1 > S_2] = 1/2. \end{aligned}$$

So, the expected probability is

$$\begin{aligned} & P[X_1 + S_1 \leq X_2 + S_2 | X_1 < X_2 < X_1 + S_1] \\ = \; & 1 - P[X_1 + S_1 > X_2 + S_2 | X_1 < X_2 < X_1 + S_1] = 1/2. \end{aligned}$$

Example 7.3. A post office has 2 clerks. The customer service time of each clerk is $Exp(\lambda)$. You enter and are first in line, with both clerks already serving customers. What is the chance that both customers currently being served will be finished before you are?

Answer: 1/2 by the strong memoryless property.

Proof. Let S_1 and S_2 be the service times of both previous customers and your service time be X. S_1, S_2 and X are mutually independent with $Exp(\lambda)$. We want to show that

$$I := P[X + \min\{S_1, S_2\} > max\{S_1, S_2\}] = 1/2. \tag{3}$$

In fact,

$$\begin{aligned} & P[max\{S_1, S_2\} > X + \min\{S_1, S_2\}] \\ = \ & P(S_2 > S_1 + X, S_2 > S_1) + P(S_1 > S_2 + X, S_1 > S_2) \\ = \ & P(S_2 > X)P(S_2 > S_1) + P(S_1 > X)P(S_1 > S_2) = 1/2, \end{aligned}$$

indicating (3), where we have used (2) in page 103.

Let $X_i \sim Exp(\lambda_i), 1 \leq i \leq n$, be independent random variables. We have

$$P\big(\min\{X_1, ..., X_n\} > x\big) = \exp(-\sum_{k=1}^{n} \lambda_k), \quad x > 0,$$

i.e., $\min\{X_1, ..., X_n\} \sim Exp(\sum_{k=1}^{n} \lambda_k)$.

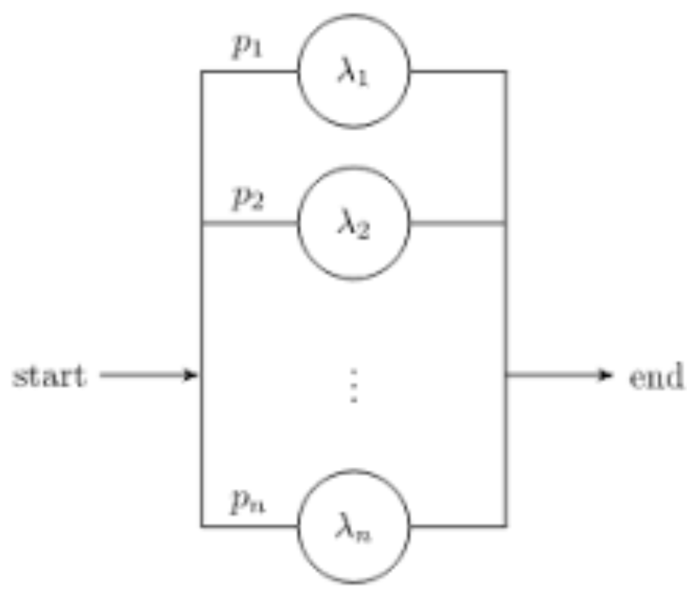
This property is key to superposition of Poisson processes.

Let $X_1, X_2, ..., X_n$ be independent exponential rv with respective rate $\lambda_1, \lambda_2, ..., \lambda_n$, where $\lambda_i \neq \lambda_j$ when $i \neq j$. Let N be a random variable independent of $X_1, ..., X_n$ such that

$$\sum_{j=1}^{n} p_j = 1, \quad \text{where } p_j = P(N = j).$$

X_N is called a **hyperexponential** random variable,$\sum_{j=1}^{n} X_j$ is called **hypoexponential** and $Y = \sum_{j=1}^{N} X_j$ is a **Coxian** random variable.

Hyperexponential, hypoexponential and Coxian rvs are usually used in querying (network) system.



The hyperexponential random variable X_N has the density function:

$$f(t) \;=\; \sum_{j=1}^{n} p_j \lambda_j e^{-\lambda_j t}, \quad t \geq 0.$$

The Coxian random variable has the density:

$$f_Y(t) \;=\; \sum_{j=1}^{n} p_j f_{\sum_{k=1}^{j} X_k}(t), \quad t \geq 0,$$

where $f_{\sum_{k=1}^{j} X_k}(t)$ is the density of $X_1 + ... + X_n$.

21 Poisson distribution

A discrete random variable X is said to be a **Poisson** random variable with parameter μ, shown as $X \sim Poisson(\mu)$, if

$$P(X = k) = \frac{e^{-\mu}\mu^k}{k!}, \quad k = 0, 1, 2, ...$$

Here are some useful facts that we have seen before:

- If $X \sim Poisson(\mu)$, then $EX = \mu$, and $\text{Var}(X) = \mu$.
- If $X_i \sim Poisson(\mu_i)$, for $i = 1, 2, \cdots, n$, and the X_i's are independent, then

$$X_1 + X_2 + \cdots + X_n \sim Poisson(\mu_1 + \mu_2 + \cdots + \mu_n).$$

- If $N \sim Poisson(\mu)$ and, conditional on $N = n$, $X \sim Binomial\big(n, p\big)$, i.e.,

$$P(X = k \mid N = n) \;\; = \;\; \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad 0 \le k \le n,$$

then $X \sim Poisson(\mu p)$.

The Poisson distribution can be viewed as the limit of binomial distribution.

Th7.2. Let $Y_n \sim Binomial\big(n, p\big)$. Let $\mu > 0$ be a fixed real number, and $\lim_{n\to\infty} np = \mu$. Then, for any $k = 0, 1, 2, ...$, we have

$$\lim_{n\to\infty} P(Y_n = k) = \frac{e^{-\mu}\mu^k}{k!}.$$