Solution 3

1. Let $\{X_n, n \geq 0\}$ be a MC on $S = \{0, 1, 2, 3, 4, 5\}$ with transition matrix:

$$\begin{pmatrix} 0.1 & 0.9 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 \\ 0.1 & 0 & 0.2 & 0.3 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

(a). Verify that all states are aperiodic.

(b). Draw the transition diagrams and decompose the state S into the recurrent classes and transient states.

(c). Identify the ergodic states.

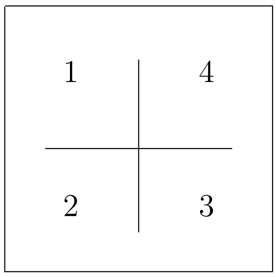
Solution.

(a). Obviously as for all state i and all $n \geq 1$, $p_{ii}^{(n)} \geq p_{ii} \cdot p_{ii} \cdots p_{ii} > 0$.

(b). $C_1 = \{0, 1\}$ and $C_2 = \{4, 5\}$ are closed classes. Hence C_1 and C_2 are positive recurrent classes. $\{2, 3\}$ is a communicate class, but not closed. So, $S = T \cup C_1 \cup C_2$ where $T = \{2, 3\}$ consists of the transient states.

(c). $\{0, 1, 4, 5\}$ are ergodic states.

2. A rat is put into the following maze:



At each stage, it moves from its present compartment to any adjacent compartment with probability $1/k$ if there are k accessible adjacent compartments (suppose that the rat can access 1 and 3 from 2, and 2 and 4 from 3, but cannot access 4 directly from 1). Successive moves have the Markov property. At time n the rat is in compartment number X_n.

(a). Write down all the elements of the transition matrix P and of P^2 and P^3.

(b). Find $p_{11}^{(k)}, k = 1, 2, 3, 4$ and $f_{ii}^{(2)}, i = 1, 2$.

(c). Classify the states and give the period.

(d). Explain briefly why $\{X_0, X_2, X_4, \ldots\}$ is also a Markov chain with transition matrix P^2 and with two recurrent classes.

Solution.

(a).

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0 & 3/4 & 0 & 1/4 \\ 3/8 & 0 & 5/8 & 0 \\ 0 & 5/8 & 0 & 3/8 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}$$

(b). $p_{11}^{(1)} = 0$, $p_{11}^{(2)} = 1/2$, $p_{11}^{(3)} = 0$, $p_{11}^{(4)} = 3/8$,

$$\begin{aligned} f_{11}^{(2)} &= P(X_2 = 1, X_1 \neq 1 \mid X_0 = 1) = P(X_2 = 1, X_1 = 2 \mid X_0 = 1) = 1/2, \\ f_{22}^{(2)} &= 1/2 + (1/2)^2 = 3/4. \end{aligned}$$

(c). Single class, period 2, recurrent;

(d). Markov with transition matrix P^2 since

$$\begin{aligned} & P(X_{2(k+1)} = j \mid X_{2k} = i, X_{2(k-1)} = i_{k-1}, ..., X_0 = i_0) \\ = \; & P(X_{2(k+1)} = j \mid X_{2k} = i) = p_{ij}^{(2)}. \end{aligned}$$

$\{1, 3\}$ and $\{2, 4\}$ both are closed classes. Both are recurrent.

3. (Adv only) As in Lectures, let $\{X_n, n \geq 0\}$ be a homogeneous MC and define

$$\begin{aligned} p_{ij}^{(n)} &= P(X_n = j \mid X_0 = i), \quad p_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \\ f_{ij}^{(1)} &= P(X_1 = j \mid X_0 = i), \\ f_{ij}^{(n)} &= P(X_n = j, X_{n-1} \neq j, ..., X_1 \neq j \mid X_0 = i), \quad \text{for } n \geq 2. \end{aligned}$$

Show that the identity:

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \qquad \text{for all } n \geq 1.$$

Proof. (Main idea) Let

$$C_1 = \{X_1 = j\}, \quad C_k = \{X_1 \neq j, ..., X_{k-1} \neq j, X_k = j, \}, \quad k = 2, ..., n$$

Then, $\cup_{k=1}^n C_k \supset \{X_n = j\}$. Hence,

$$\begin{aligned} p_{ij}^{(n)} &= P(X_n = j \mid X_0 = i) \\ &= \sum_{k=1}^{n} P(X_m \neq j, \text{all } m < k, X_k = j, X_n = j \mid X_0 = i). \end{aligned}$$

The result follows by the MC property and a simple calculation (finish it by yourself!!).