Solution 9

1. For a nonhomogenous Poisson process the intensity function is given by

$$\lambda(t) = \begin{cases} 10, & \text{if } t \in (0, 1/2];\ (1, 3/2]; ... \\ 2, & \text{if } t \in (1/2, 1];\ (3/2, 2]; ... \end{cases}$$

- How many occurrences are expected in the time period $(0, 1]$? During $(0, 3/2]$?
- Let T_n be the time of the nth occurrence. Given $T_{10} = 0.45$, calculate the probability that $T_{11} > 0.75$.

Solution:

- How many occurrences are expected in the time period $(0, 1]$? During $(1, 3/2]$?
  Let $m(u) = \int_0^u \lambda(t)dt$. The expected number in the time period $(0, 1]$ is

  $$m(1) = \int_0^{1/2} 10dt + \int_{1/2}^{1} 2dt = 6.$$

  The expected number in the time period $(1, 3/2]$ is

  $$m(3/2) - m(1) = \int_1^{3/2} 10dt = 5.$$

- Let T_n be the time of the nth occurrence. Given $T_{10} = 0.45$, calculate the probability that $T_{11} > 0.75$.
  Note that, given $T_{10} = 0.45$, $T_{11} > 0.75$ iff there are no jumps during $(0.45, 0.75]$. Hence the required probability is

  $$\begin{aligned} p &= P(N_{0.75} - N_{0.45} = 0) \\ &= e^{-\int_{0.45}^{0.75} \lambda(t)dt} = e^{-1}. \end{aligned}$$

2. The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of $\lambda = 50$ envelopes per week. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

$$P(X = 1) = 1/5, P(X = 2) = 1/4, P(X = 3) = 2/5, P(X = 4) = 0.15.$$

Using the normal approximation, calculate the 90th percentile of the number of claims received in 13 weeks.

Solution. Let $Z_t = \sum_{j=1}^{N_t} X_j$, where N_t is a Poisson process with rate $\lambda = 50$ per week, and $X, X_1, X_2, ...$ are iid rvs. We need to find the q so that

$$0.9 = P(Z_{13} \leq q).$$

Note that $EZ_{13} = 50 \times 13 \times EX = 1625$ ($EX = 2.5$) and $var(Z_{13}) = 50 \times 13 \times EX^2 = 4680$ ($EX^2 = 7.2$). Hence the q satisfies that

$$\begin{aligned} 0.9 &= P(Z_{13} \le q) = P\Big((Z_{13} - EZ_{13})/\sqrt{varZ_{13}} \le (q - 1625)/\sqrt{4680}\Big) \\ &\sim P\Big(N(0,1) \le (q - 1625)/\sqrt{4680}\Big) = P(N(0,1) \le 1.28) \end{aligned}$$

So, $q \sim 1712.5$.

3. Consider an elevator that starts in the basement and travels upward. Let N_i denote the number of people that get in the elevator at floor i. Assume those N_i's are independent and that N_i is Poisson with mean λ_i. Each person entering at floor i will, independent of everything else, get off at j with probability p_{ij}, satisfying $\sum_{j>i} p_{ij} = 1$. Let O_j denote the number of people getting off the elevator at floor j.

(a) What is the distribution of O_j?

(b) What is the joint distribution of O_j and O_k if $j \neq k$?

Solution: Let $N_{i,j}$ denote the number of people that get on at floor i and off at floor j. Clearly, $N_i = N_{i,i+1} + N_{i,i+2} + \cdots$. Since each person gets off independent of everything else, the splitting of a Poisson process (at unit time) ensures that these N_{ij}'s are independent Poisson, respectively, with mean $\lambda_i p_{ij}$.

(a) Notice that $O_j = N_{1,j} + N_{2,j} + \cdots + N_{j-1,j}$, which is a sum of independent Poisson random variables, since N_i's are mutually independent. Hence, the sum is again Poisson with mean $\sum_i \lambda_i p_{ij}$. (The index of the summation can be left unspecified here, since $p_{ij} = 0$ if it is impossible to get in at i and off at j.)

(b) Comparing $O_j = N_{1,j} + N_{2,j} + \cdots + N_{j-1,j}$ and $O_k = N_{1,k} + N_{2,k} + \cdots + N_{k-1,k}$, all the elements are independent, so O_j and O_k are independent as well.

4. (**Adv**) Consider a Poisson process $\{X_t, t \ge 0\}$ with rate λ. We consider the case in which an arrival occurring at time s is a type I arrival with prob. $p(s)$ and a type II arrival with prob. $1 - p(s)$. Let N_t denote the number of type I arrivals until time t. Show that, for each $t \ge 0$,

$$N_t \sim Poisson(\lambda\, p\, t), \qquad \text{where } p = \tfrac{1}{t}\int_0^t p(s)ds.$$

Solution: We verify the definition.

- clearly $N_0 = 0$ and $\{N_t, t > 0\}$ has indep. increments since $\{X_t, t \ge 0\}$ has;

- for any $t > 0$,

$$\begin{aligned}
& P(N_{t+h} = i+1 \mid N_t = i) = P(N_{t+h} - N_t = 1) \quad \text{(indep. increments)} \\
= \; & \sum_{k=1}^{\infty} P(N_{t+h} - N_t = 1 \mid X_{t+h} - X_t = k) P(X_{t+h} - X_t = k) \\
= \; & P(N_{t+h} - N_t = 1 \mid X_{t+h} - X_t = 1) P(X_{t+h} - X_t = 1) + o(h) \\
= \; & \lambda h \, e^{-\lambda h} \, p(t) + o(h) \\
= \; & \lambda h \, p(t) + o(h);
\end{aligned}$$

- for any $t > 0$,

$$\begin{aligned}
& P(N_{t+h} \geq i+2 \mid N_t = i) = P(N_{t+h} - N_t \geq 2) \quad \text{(indep. increments)} \\
= \; & \sum_{k=2}^{\infty} P(N_{t+h} - N_t \geq 2 \mid X_{t+h} - X_t = k) P(X_{t+h} - X_t = k) \\
\leq \; & \sum_{k=2}^{\infty} P(X_{t+h} - X_t = k) \\
= \; & P(X_{t+h} - X_t \geq 2) = o(h).
\end{aligned}$$

These facts imply that $\{N_t, t \geq 0\}$ is a nonhomogeneous Poisson process with rate $\lambda(t) = \lambda p(t), t > 0$.