Solution 5

1. Two players A and B match pennies. A starts with three cents and B starts with four. The game ends whenever one of the players has all seven pennies. Suppose A and B have the same chance to win in each game. What is the prob. that A wins? What is the average duration of the game?

Solution. It is easy by using Thms 5.4 and 5.5. 3/7; 12.

2. An ant is assumed to move along a straight line. Suppose, at each step, the ant moves one unit length left or right with probability q or p respectively. Suppose also that the ant will be eaten by a snake if it arrives in the location "0", and the ant will go back if it arrives in the location with "N". (This is an example of random walk with one barrier and one reflecting barrier).

(a). Describe the behavior of the ant by a MC, find the transition matrix and classify the states (recurrent, aperiodic, etc).

(b). Find the probability that the ant is eaten by a snake when it starts to move in the location "3".

(c). (**3921/4021**) Find the average steps that the ant is eaten by a snake when it starts to move in the location "3" if $p = 1/2$.

Solution. (a) Let $X_0 = m$ and

$$X_k = \begin{cases} 1, & \text{the ant moves right in the } k\text{-th step,} \\ -1, & \text{the ant moves left in the } k\text{-th step.} \end{cases}$$

Then, $S_n = \sum_{k=0}^{n} X_k$ denotes the location of the ant after the n-step movements starting from location m. It is clear that $\{S_n, n \geq 0\}$, with restrictive conditions that

$$\begin{aligned} P(S_{n+1} = 0 \mid S_n = 0) &= 1, \\ P(S_{N+1} = N - 1 \mid S_N = N) &= 1, \end{aligned}$$

models the behaviour of the ant. As a MC, $\{S_n, n \geq 0\}$ has transition matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & \cdots & \cdots & \cdots & & & & \\ & \cdots & \cdots & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Clearly, $\{0\}$ is absorbing, all other states are communicate, transient and with a period 2.

To establish (b) and (c), let $\tau = \min\{n \geq 1 : S_n = 0\}$. We want to find y_3, where $y_k = P(S_\tau = 0 \mid X_0 = k)$ (the probab. that the ant is eaten by a snake from location 3) and m_k, where $m_k = E(\tau \mid X_0 = k)$ (the average steps that the ant is eaten by a snake from location 3).

(b). As in Lect 12, y_k satisfies that $y_0 = 1$,

$$\begin{aligned}
y_0 &= q + p\, y_1, \\
y_1 &= q + p\, y_2, \\
y_k &= q\, y_{k-1} + p\, y_{k+1}, \quad 2 \leq k \leq N-1, \\
y_N &= y_{N-1}.
\end{aligned}$$

This yields that $y_k = 1, k = 1, 2, ..., N$.

(c). As in Lect 12, m_k satisfies that

$$\begin{aligned}
m_1 &= 1 + p\, m_2, \\
m_k &= 1 + q\, m_{k-1} + p\, m_{k+1}, \quad 2 \leq k \leq N-1, \\
m_N &= 1 + m_{N-1}.
\end{aligned}$$

When $p = q = 1/2$, by letting $y_k = m_k - m_{k-1}$ ($m_0 = 0$), we have for $k = 1, 2, ..., N-1$

$$y_k = 2 + y_{k+1} = ... = 2(N-k) + y_N = 2(N-k) + 1.$$

This gives that

$$m_k = \sum_{j=1}^{k} y_j = (2N+1)k - 2\sum_{j=1}^{k} j = k(2N-k).$$

What is the result for $p \neq q$? Similarly as above, we have

$$\begin{aligned}
y_k &= 1/q + \theta y_{k+1} = ... \\
&= \frac{1}{q}(1 + \theta + ... + \theta^{N-k-1}) + \theta^{N-k} y_N = \frac{1 - 2p\theta^{N-k}}{q-p}. \quad (\theta = p/q)
\end{aligned}$$

This gives that

$$m_k = \sum_{j=1}^{k} y_k = \frac{1}{q-p}\Big[k - 2p\theta^{N-k}(1-\theta^k)/(1-\theta)\Big].$$

3. A machine produces two items per day. The probability that an item is non-defective is p (all the items are independent), and defective items are thrown away instantly. The demand is one item per day, and any demand that can not be satisfied by the end of the day is lost, while any extra item is stored. Let X_n be the number of items in storage just before the beginning of the $(n+1)$-th day (This is an example of random walk with one barrier).

(a). Explain why $\{X_n, n \geq 0\}$ is a RW. As a MC, find the transition matrix and classify the states.

(b). (**3921/4021**) Let $p < 1/2$. Show that $\pi = (\pi_j, j \geq 0)$, where

$$\pi_j = (1-r)r^j, \quad r = p^2/(1-p)^2, \quad j = 0, 1, 2, ...$$

is a stationary distribution of the MC. Suppose it costs \$$c$ to store an item for one night and \$$d$ for every demand that cannot be fulfilled. Compute the long run cost rate of the production facility when it is stable.

Solution. (a). $\{X_n, n \geq 0\}$ is a RW as we may write $X_0 = 0, X_n = \sum_{j=1}^{n} Y_j$ with restrictive conditions that $P(X_{n+1} = 1 \mid X_n = 0) = p^2$ and $P(X_{n+1} = 0 \mid X_n = 0) = 1 - p^2$, where

$$Y_j = \begin{cases} 1, & \text{Two items are non-defective in the } j\text{-th day,} \\ 0, & \text{An item is non-defective in the } j\text{-th day,} \\ -1, & \text{Two items are defective in the } j\text{-th day.} \end{cases}$$

As a MC, $S = \{0, 1, 2, ...\}$ and transition matrix is

$$P = \begin{pmatrix} 1-p^2 & p^2 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ (1-p)^2 & 2p(1-p) & p^2 & 0 & 0 & \cdots & \cdots & \\ 0 & (1-p)^2 & 2p(1-p) & p^2 & 0 & \cdots & & \\ & \cdots & \cdots & \cdots & & & & \\ & \cdots & \cdots & \cdots & & & & \end{pmatrix}$$

as for $k \geq 1$,

$$\begin{aligned} P(X_{n+1} = k+1 \mid X_n = k) &= p^2, \\ P(X_{n+1} = k-1 \mid X_n = k) &= (1-p)^2, \\ P(X_{n+1} = k \mid X_n = k) &= 2p(1-p). \end{aligned}$$

(b). For $p < 1/2$, $r < 1$ and it is to check that $\pi = \pi P$. Total cost on day n:

$$d\, I_{X_n=0}\, I_{\text{both are defective}} + c\, X_{n+1},$$

so the expected value (in the long run) is

$$
\begin{aligned}
& d\,(1-p)^2\,P(X_n = 0) + dEX_{n+1} \\
= \;& d(1-p)^2\pi_0 + c\sum_{j=0}^{\infty} j\pi_j \\
= \;& (1-r)[d(1-p)^2 + c\sum_{j=1}^{\infty} jr^j \\
= \;& (1-r)[d(1-p)^2 + cr/(1-r)^2] \\
= \;& d(1-2p) + cp^2/(1-2p),
\end{aligned}
$$

where $\pi = \{\pi_j, j = 0, 1, 2...\}$ is the stationary distribution and $r = p^2/(1-p)^2$.