Week 6

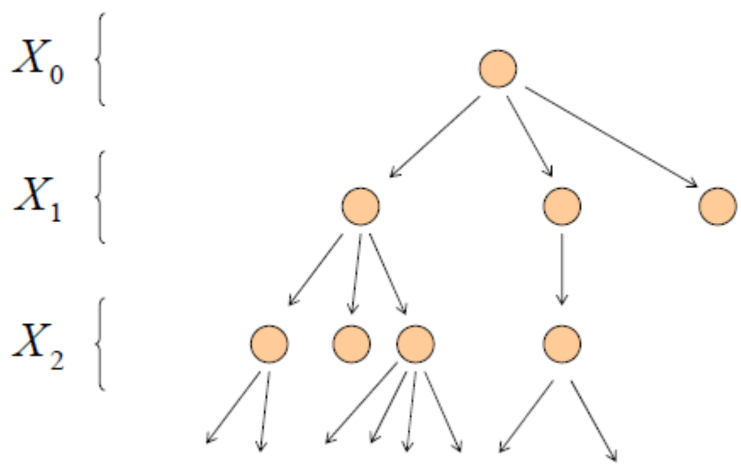
Branching processes: extinction probability, expectation and variance.

16 Branching process: extinction probability

Let ξ_{jk}, $k \geq 0, j \geq 1$, are iid random variables with distribution:

$$P(\xi_{jk} = i) = f_i, \quad i = 0, 1, 2..., \quad \sum_{i=0}^{\infty} f_i = 1, \quad 0 < f_0 < 1,$$

Let $X_0 = 1$, $X_1 = \xi_{10}$ and $X_{n+1} = \sum_{j=1}^{X_n} \xi_{jn} I_{(X_n \geq 1)}$.



This lecture is interested in

- the probability that the process eventually dies out or the extinction probability:

$$
\begin{aligned}
P(\text{extinction}) &= P(X_n = 0,\ \text{for some } n \geq 1) \\
&= P\Big(\cup_{n=1}^{\infty} \{X_n = 0\}\Big) = \lim_{n\to\infty} P(X_n = 0).
\end{aligned}
$$

Let $\mu = EX_1 = E\xi_{10} = \sum_{k=1}^{\infty} kf_k$ be the expectation of the offspring distribution and $q = \lim_{n\to\infty} P(X_n = 0)$. For the extinction probability, we have following fundamental result:

Th6.1. (1). If $\mu \le 1$, then $q = 1$. (2). If $\mu > 1$, then $q < 1$ and q is the unique non-negative solution to the equation:

$$F(s) = s, \quad 0 < s < 1, \tag{1}$$

where $F(s) = Es^{X_1} = \sum_{k=0}^{\infty} s^k f_k$, which is the common pgf of ξ_{jk}.

Note. The equation (1) is usually called the **extinction equation** and $F(x)$ is the generating function of the offspring distribution.

We have $\mu = F'(1)$. It follows from the Th6.1 that

- If $\mu \le 1$, then the probability that the population will terminate before finite generations tends to one;
- If $\mu > 1$, the limit of the probability that the population terminates finitely after many generations is less than one. In this case, the quantity $1 - q$ is the chance of infinitely prolonged population. In practice, it may happen that $q_n := P(X_n = 0)$ converge to q fairly quickly. This may be interpreted that the population which is going to become extinct is likely to do so after only a few generations, and hence we may interpret q as the chance of rapid extinction.

Example 6.3. Consider a BP, where the offspring has a Poisson distribution with mean λ, i.e.,

$$P(\xi_{10} = k) \ = \ e^{-\lambda}\lambda^k/k!, \quad k = 0, 1, 2, ...$$

We have $\mu = EX_1 = E\xi_{10} = \lambda$ and $F(s) = Es^{X_1} = e^{\lambda(s-1)}$.

Example 6.4. Consider a BP where the offspring has the dist. function:

$$P(\xi_{10} = k) \;=\; \alpha\beta^k, \quad k = 0, 1, 2, ..., \quad \alpha + \beta = 1,\; 0 < \alpha < 1.$$

We have $\mu = EX_1 = E\xi_{10} = \beta/\alpha$ and $F(s) = Es^{X_1} = \alpha/(1 - \beta s)$.

Proof of Th6.1. Let $F_n(s) = Es^{X_n}$. It follows from Wald's equation that

$$F_{n+1}(s) = F_n(F(s)) = ... = F(F_n(s)), \quad n \geq 1.$$

Step 1. q is a solution of $F(s) = s$.

Let $q_n = P(X_n = 0)$. Since $0 \leq q_n \leq 1$ and $q_n \uparrow$ as $n \to \infty$, there exists a $0 \leq q \leq 1$ such that $q_n \to q$. Now, by noting $q_n = P(X_n = 0) = F_n(0)$, we have

$$q_{n+1} = F(q_n),$$

indicating $F(q) = q$ as $F(s)$ is continuous.

Step 2. q is the smallest solution of $F(s) = s$. In fact, if there exists $0 \leq q_0 \leq 1$ such that $F(q_0) = q_0$, then

$$\begin{aligned}
q_1 &= F_1(0) = F(0) \leq F(q_0) = q_0 \\
q_2 &= F_2(0) = F(F_1(0)) = F(q_1) \leq F(q_0) = q_0 \\
&.... \\
q_n &= F_n(0) \leq ... \leq q_0,
\end{aligned}$$

i.e., $q \leq q_0$.

Step 3. Note that $F(s)$ is a convex function. We must have at most two points in common for $y = F(s)$ and $y = s$ and one of the two points is $s = 1$. For another one, if $\mu \leq 1$, i.e, $\mu = F'(1) \leq 1$, we must have $s = 1$, i.e., $q = 1$. If $\mu > 1$, we must have another q so that $q = F(q)$.

17 Branching process: Expectation and other properties

Let ξ_{jk}, $k \geq 0, j \geq 1$, are iid random variables with distribution:

$$P(\xi_{jk} = i) = f_i, \quad i = 0, 1, 2..., \quad \sum_{i=0}^{\infty} f_i = 1, \quad 0 < f_0 < 1,$$

Let $X_0 = 1$, $X_1 = \xi_{10}$ and $X_{n+1} = \sum_{j=1}^{X_n} \xi_{jn} I_{(X_n \geq 1)}$.

Th6.2. (1). If $EX_1 = \mu$ and $var(X_1) = \sigma^2$, then

$$\begin{aligned} EX_n &= \mu^n \\ var(X_n) &= \begin{cases} \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}, & \text{if } \mu \neq 1, \\ n\sigma^2, & \text{if } \mu = 1. \end{cases} \end{aligned}$$

Proof. For the mean, it follows from the Wald's equation. The proof of $vax(X_n)$ is omitted.

We next consider a branching process with $X_0 = i$, i.e., the BP starts with i individuals in the zero generation.

Let ξ_{jk}, $k \geq 0, j \geq 1$, are iid random variables with distribution:

$$P(\xi_{jk} = i) = f_i, \quad i = 0, 1, 2..., \quad \sum_{i=0}^{\infty} f_i = 1, \quad 0 < f_0 < 1,$$

Let $X_0 = i \geq 1$ and $X_{n+1} = \sum_{j=1}^{X_n} \xi_{jn} I_{(X_n \geq 1)}, n \geq 0$.

We have the following results.

Th6.3. Suppose that $X_0 = i$. Then

$$P(\text{extinction} \mid X_0 = i) \;=\; \lim_{n\to\infty} P(X_n = 0 \mid X_0 = i) = q^i,$$

where $q = P(\text{extinction} \mid X_0 = 1)$. We further have

$$\begin{aligned} E(X_n \mid X_0 = i) &= iE(X_n \mid X_0 = 1) = i\,\mu^n \\ var(X_n \mid X_0 = i) &= i\,var(X_n \mid X_0 = 1). \end{aligned}$$

Note: $E(X_{n+r} \mid X_n = k) = E(X_r \mid X_0 = k)$ by the Markov property.

Proof. Let A_j denote the event that the j th individual in X_0 eventually dies out. It follows that A_j are independent $P(A_j) = q$ and

$$\begin{aligned} P(\text{extinction} \mid X_0 = i) &= P(A_1 \cdot A_2 \cdots A_i) \\ &= \prod_{j=1}^{n} P(A_j) = q^i. \end{aligned}$$

Let ξ_{jk}, $k \geq 0, j \geq 1$, are iid random variables with distribution:

$$P(\xi_{jk} = i) = f_i, \quad i = 0, 1, 2..., \quad \sum_{i=0}^{\infty} f_i = 1, \quad 0 < f_0 < 1,$$

Consider $X_{n+1} = \sum_{j=1}^{X_n} \xi_{jn} I_{(X_n \geq 1)}, n \geq 0$, where X_0 is a r.v. with distribution:

$$P(X_0 = i) = a_i, \quad i = 0, 1, 2, ..$$

where $\sum_{i=0}^{\infty} a_i = 1$.

Th6.4. We have

$$P(\text{extinction}) \;=\; \lim_{n \to \infty} P(X_n = 0) = \sum_{i=0}^{\infty} a_i \, q^i,$$

where $q = P(\text{extinction} \mid X_0 = 1)$. We further have

$$E(X_n) \;=\; \mu^n \, EX_0 \,.$$

Proof.

$$P(\text{ext.}) \;=\; \sum_{i=0}^{\infty} P(\text{ext.} \mid X_0 = i) \, P(X_0 = i) = \sum_{i=0}^{\infty} a_i \, q^i,$$

and

$$\begin{aligned} EX_n &= \sum_{i=0}^{\infty} E(X_n \mid X_0 = i) \, P(X_0 = i) \\ &= \sum_{i=0}^{\infty} a_i \, i\mu^n = \mu^n EX_0. \end{aligned}$$

Th6.5. For $r = 1, 2, ...,$ we have

$$E(X_{n+r} \mid X_n = k) = \mu^r \, k, \quad \text{where } \mu = EX_1.$$

(Hint: By induction to r and note that $\{X_n, n \geq 0\}$ is a MC)

Proof is given in Adv tutorial.

Example 6.5. The spread of coronavirus in population forms a BP with the offspring distribution $\xi_{10} \sim Poisson(\lambda)$ and it is known that the coronavirus will produce a new generation infected people each 4 days. Suppose that the initial number of infected persons in some country obeys $Poission(1)$ and, 8 weeks later, this country has over 10000 infected persons. Find the mean of the offspring distribution.

Solution: The BP process under investigation is that $X_{n+1} = \sum_{k=1}^{X_n} \xi_{kn}$ and $X_0 \sim Poi(1)$, where $\xi_{kn} \sim Poi(\lambda)$ and are iid random variables.

In 8 weeks, the coronaviras produces $2 \times 7 = 14$ generations infected people. It follows from $EX_n = EX_0\, \mu^n$, where $\mu = EX_1 = E\xi_{10} = \lambda$, that

$$10000 = 1 \times \lambda^{14}, \quad i.e., \quad \lambda = (10000)^{1/14} \sim 1.8$$