Solution 12

1. A supermarket has three girls ringing up sales at the counts. Service time for each customer is exponential with mean 5 min. and people arrive in a Poisson process at the rate of 30/per hour.

   (a) Under steady-state conditions, what is the probability that all the girls will be busy?

   (b) What is the expected number of people waiting to be serviced?

   (c) What is the expected length of time customers have to wait for service?

   (d) If a customer has to wait, what is the expected length of his waiting time?

   Solution: This is a M/M/3 queue with arrival rate $\lambda = 30/60 = 1/2\text{min}$ and service rate $\mu = 1/5\text{min}$. Hence $\rho = \lambda/\mu = 5/2$ and $\rho_1 = 5/6$.

   (a) According to lecture, the probability that all the girls will be busy is $P(L \geq 3) = \pi_3(1-\rho/3)^{-1} = 125/178$, since

   $$\pi_0^{-1} = 1 + \rho + \rho^2/2 + \frac{\rho^3}{6(1-\rho/3)} = 89/4, \quad \pi_3 = \rho^3\pi_0/6 = 125/1068.$$

   (b) $E\max\{L-3, 0\} = \pi_3(\rho/3)(1-\rho/3)^{-2} = 625/178$.

   (c) $EW_1 = [\pi_3/(3\mu)](1-\rho/3)^{-2} = 1250/178 \approx 7$ min

   (d) This is a conditional expectation of W_1, given that a customer has to wait, i.e., we have to calculate $E(W_1 \mid L \geq 3)$. In fact, by noting $W_1 > 0$ iff $L \geq 3$,

   $$E(W_1 \mid L \geq 3) = \frac{E(W_1 I_{L\geq 3})}{P(L \geq 3)} = \frac{EW_1}{P(L \geq 3)} = 10 \ \text{ min.}$$

2. Consider an infinite capacity queueing system with Poisson arrivals with parameter λ and 3 servers, service times being independent exponential variables with mean $1/\mu$. Use results from lectures to show that if the queue is in stationary mode, find the expression of the probability all servers are idle.

   Solution: From lectures, $\pi_1 = \rho\pi_0$, $\pi_2 = \frac{\rho^2}{2}\pi_0$ and $\pi_n = \frac{9(\rho/3)^n}{2}\pi_0$ for $n \geq 3$. Since $\sum \pi_n = 1$,

   $$\pi_0 = \frac{1}{1 + \rho + \rho^2/2 + \frac{9(\rho/3)^3}{2}(1 + \rho/3 + (\rho/3)^2 + \cdots)} = \frac{6-2\rho}{6+4\rho+\rho^2}.$$

3. (**3921/4021**) Customers arrive at a bank according to the Poisson process with the rate λ. The service times are iid exponential random variables with the rate μ. The banks follows this policy: when there are fewer than four customers in the bank, only one teller is active; for four to nine customers, there are two tellers; and beyond nine customers there are three tellers. Model the number of customers in the bank as a birth and death process. When is this system stable? Assuming stability, compute the steady state distribution.

   Solution: As in Lecture 35, let L_t denote the number of customers (either waiting or being served) in the bank.

   - $\{L_t, t \geq 0\}$ is a birth and death chain with birth rate $\lambda_i = \lambda > 0, i \geq 0$, and death rate $\mu_0 = 0$ and

     $$\mu_i = \begin{cases} \mu, & \text{if } i = 1, 2, 3 \\ 2\mu, & \text{if } i = 4, ..., 9 \\ 3\mu, & \text{if } i > 9 \end{cases}$$

     i.e.,

     $$\begin{aligned} & P(A_{t+h} = k \mid A_t = i) \\ = & \begin{cases} 1 - (\lambda + \mu_i)\, h + o(h) & \text{for } k = i, \\ \lambda\, h + o(h) & \text{for } k = i + 1, \\ \mu_i h + o(h) & \text{for } k = i - 1, \\ o(h) & \text{for } |k - i| \geq 2\ . \end{cases} \end{aligned}$$

The Q-matrix is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \ldots & & & \\ \mu_1 & -(\lambda+\mu_1) & \lambda & 0 & 0 & \ldots & & & \\ 0 & \mu_1 & -(\lambda+\mu_1) & \lambda & 0 & 0 & \ldots & & \\ 0 & 0 & \mu_1 & -(\lambda+\mu_1) & \lambda & 0 & 0 & \ldots & \\ 0 & 0 & 0 & \mu_2 & -(\lambda+\mu_2) & \lambda & 0 & \ldots & \\ 0 & 0 & 0 & 0 & \mu_2 & -(\lambda+\mu_2) & \lambda & 0 & \ldots \\ . & . & \ldots & \ldots & & & & & \\ . & . & . & \ldots. & & & & & \end{pmatrix}.$$

Using $\pi Q = 0$, i.e.,

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 \\ (\lambda+\mu)\pi_j &= \lambda\pi_{j-1} + \mu\,\pi_{j+1}, \quad 1 \le j \le 2 \\ (\lambda+\mu)\pi_3 &= \lambda\pi_2 + 2\mu\,\pi_4, \\ (\lambda+2\mu)\pi_j &= \lambda\pi_{j-1} + 2\mu\,\pi_{j+1}, \quad 4 \le j \le 8 \\ (\lambda+2\mu)\pi_9 &= \lambda\pi_8 + 3\mu\,\pi_{10}, \\ (\lambda+3\mu)\pi_j &= \lambda\pi_{j-1} + 3\,\mu\,\pi_{j+1}, \quad j > 9, \end{aligned}$$

indicating, for $\rho = \lambda/\mu$,

$$\pi_k = \begin{cases} \rho\pi_{k-1}, & \text{if } 1 \le k \le 3, \\ \rho\pi_{k-1}/2, & \text{if } 4 \le k \le 9, \\ \rho\pi_{k-1}/3, & \text{if } k \ge 10 \end{cases}$$

Since $\sum_{j\ge 0}\pi_j < \infty$ iff $\sum_{j\ge 10}\pi_j = c\sum_{j\ge 10}(\rho/3)^j < \infty$, the system is stable iff $\rho < 3$. Steady-state distribution: $\pi_j = K_j\pi_0, j \ge 1$, where,

$$K_k = \begin{cases} \rho^k, & \text{if } k = 1,2,3 \\ 2^{-(k-3)}\rho^k, & \text{if } k = 4,\ldots,9, \\ 2^{-6}\rho^9(\rho/3)^{k-9}, & \text{if } k > 9 \end{cases}$$

$$\pi_0^{-1} = \sum_{j\ge 0} K_j = \frac{1-\rho^4}{1-\rho} + \frac{\rho^4(1-(\rho/2)^6)}{2-\rho} + \frac{\rho^{10}}{2^6(3-\rho)}.$$