Solution 8

1. (i) Let X and Y be two independent r.v. such that X has an exponential distribution with mean λ_1 and Y has an exponential distribution with mean λ_2. Then,

$$P(X < Y) = \frac{1/\lambda_1}{1/\lambda_1 + 1/\lambda_2}.$$

(ii) Let $X_1, ..., X_n$ be independent r.v.s such that X_i has an exponential distribution with mean λ_i. Then,

$$P(X_1 = \min_{1 \le k \le n}\{X_k\}) = \frac{1/\lambda_1}{\sum_{k=1}^n 1/\lambda_k}.$$

(iii) You have a stereo system consisting of two main parts, a radio and a speaker. Suppose the lifetime of the radio is exponential with mean 1000 hours and the lifetime of the speaker is exponential with mean 500 hours. Also, their lifetimes are independent. What is the probability that the failure of the system (when it occurs) will be caused by the radio failing?

Solution: (i) It follows that

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_s^\infty \frac{1}{\lambda_1}\frac{1}{\lambda_2} e^{-s/\lambda_1} e^{-t/\lambda_2} ds dt \\ &= \int_0^\infty \frac{1}{\lambda_1} e^{-s/\lambda_1} e^{-s/\lambda_2} ds = \frac{1/\lambda_1}{1/\lambda_1 + 1/\lambda_2} \end{aligned}$$

(ii) Note that $Y := \min_{2 \le k \le n}\{X_k\} \sim Exp(\sum_{k=2}^n 1/\lambda_k)$ and X_1 is independent of Y. It follows that

$$P(X_1 = \min_{1 \le k \le n}\{X_k\}) = P(X_1 < \min_{2 \le k \le n}\{X_k\}) = \frac{1/\lambda_1}{\sum_{k=1}^n 1/\lambda_k}$$

(iii) Let X and Y denote the lifetimes of radio and speaker respectively. It follows from (i) that

$$p = P(X < Y) = \frac{1/1000}{1/1000 + 1/500} = \frac{1}{3}$$

2. Let $\{E_k\}_{k=1,\cdots,n}$ be a sequence of independent exponential random variables with respective rates λ_k.

(a) Compute $E[(E_1 - c)_+]$ for $c > 0$, where $X_+ = \max\{X, 0\}$.

(b) Compute $E[\min(E_1, c)]$ for $c > 0$.

(c) Derive the distribution of the random variable $\min(E_1, \cdots, E_n)$.

(d) Derive the distribution of the random variable $\max(E_1, \cdots, E_n)$.

Solution:

- (a) It holds that for $c > 0$,

$$\begin{aligned} E\left[(E_1 - c)_+\right] &= E \max\{E_1 - c, 0\} \\ &= \int_{x \geq c} (x - c)\lambda_1 e^{-\lambda_1 x} dx = \frac{1}{\lambda_1} e^{-\lambda_1 c}. \end{aligned}$$

  or

$$\begin{aligned} E\left[(E_1 - c)_+\right] &= E\left[(E_1 - c)_+ \middle| E_1 > c\right] P(E_1 > c) + E\left[(E_1 - c)_+ \middle| E_1 \leq c\right] P(E_1 \leq c) \\ &= E\left[(E_1 - c)_+ \middle| E_1 > c\right] P(E_1 > c) = E[E_1] P(E_1 > c) = \frac{1}{\lambda_1} e^{-\lambda_1 c}, \end{aligned}$$

  where the third equality holds true by the memoryless property of the exponential distribution.

- (b) Along with the identity $\min(x, c) = x - (x - c)_+$, we further obtain $E[\min(E_1, c)] = E[E_1] - E[(E_1 - c)_+] = 1/\lambda_1 - e^{-\lambda_1 c}/\lambda_1$.

- (c) It holds that for $x > 0$,

$$\begin{aligned} P\left(\min(E_1, \cdots, E_n) > x\right) &= P\left(\cap_{k=1}^n \{E_k > x\}\right) \\ &= \prod_{k=1}^n P(E_k > x) = \prod_{k=1}^n e^{-\lambda_k x} = \exp\left[-x \sum_{k=1}^n \lambda_k\right], \end{aligned}$$

  which indicates that the random variable $\min(E_1, \ldots, E_n)$ has the exponential distribution with rate $\sum_{k=1}^n \lambda_k$.

- (d) It holds that for $x > 0$,

$$P\left(\max(E_1, \cdots, E_n) \leq x\right) = P\left(\cap_{k=1}^n \{E_k \leq x\}\right) = \prod_{k=1}^n P(E_k \leq x) = \prod_{k=1}^n \left(1 - e^{-\lambda_k x}\right).$$

  Unlike in (c), there exists no particular name for this distribution, whereas this equality characterizes the distribution of the random variable $\max(E_1, \cdots, E_n)$ uniquely anyway.

3. The number of claims received each day by a claims center has a Poisson distribution. On Mondays, the center expects to receive 2 claims but on other days of the week, the claims center expects to receive 1 claim per day. The numbers of claims received on separate days are mutually independent of one another. Find the probability that the claim center receives at least 2 claims in a week (Monday to Friday).

Solution: Let X_j be the number of claims received in j-th day in a week. It follows that $X_1 \sim Poi(2)$ and $X_j \sim Poi(1), j = 2, ..., 5$ and X_j are independent. Hence

$$Y := \sum_{j=1}^5 X_j \sim Poi(6).$$

The required probability is

$$p = P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1) = 1 - e^{-6} - 6e^{-6} = 1 - 7e^{-6}.$$

4. (**Adv**) Let X be a memoryless random variable, i.e., for any $s, t \geq 0$,

$$P(X \geq s + t) = P(X \geq s)P(X \geq t).$$

Show that $X \sim Exp(\lambda)$ for some $\lambda > 0$.

Proof. Let $g(t) = P(X \geq t)$. We have

$$g(s + t) = g(s)g(t), \quad s, t \geq 0.$$

It follows that

$$g(1) = g(1/n + ... + 1/n) = g^n(1/n) \quad \text{or} \quad g(1/n) = g^{1/n}g(1).$$

Hence $g(m/n) = g^{m/n}(1)$, implying that $g(x) = g^x(1)$ since g is right continuous. This yields that $g(x) = e^{-\lambda x}$ with $\lambda = -log g(1)$.