Solution 4

1. Consider a Markov Chain $\{X_n,\ n = 0, 1, 2, \ldots\}$ taking the values 1, 2 and 3 with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a). Calculate $\mathbf{P}^2$ and prove that all entries in $\mathbf{P}^n$ are positive for $n = 3, 4, \ldots$.

(b). Explain why the MC is (i) irreducible, (ii) aperiodic, (iii) positive recurrent.

(c). Suppose X_0 takes values 1, 2 and 3 with probabilities 1/2, 1/6 and 1/3 respectively. Prove that the probability distribution $\{P(X_n = j),\ j = 1, 2, 3\}$ is the same for all $n \geq 1$.

(d). *Write down* the limit of $\mathbf{P}^n$ as $n \to \infty$.

(e). What is the mean recurrence time of state 2?

(f). (**Adv. Mean hitting time**) What is the mean time that the MC first arrives at state 3 starting from the state 1?

(g). Suppose that, for the system modelled by the MC, its staying in states 1, 2 and 3 for one time unit incurs a cost of \$2, \$1 and \$5, respectively. What is the long-run average cost for the system per time unit?

(h). (**Adv. only**) Let $A = \{2, 3\}$. Prove that

$$P(X_{n+1} = 1 | X_n \in A) = 2/3 \quad \text{but that } P(X_{n+1} = 1 | X_n \in A, X_{n-1} \in A) = 1.$$

Note. This example highlights the need in the Markov property to have

$$P(X_{n+1} = j | X_n = i, X_{n-1} \in A_{n-1}, \ldots) = P(X_{n+1} = j | X_n = i)$$

rather than $P(X_{n+1} = j | X_n \in A, X_{n-1} \in A_{n-1}, \ldots) = P(X_{n+1} = j | X_n \in A)$.

Solution:

(a).

$$\mathbf{P^2} = \begin{pmatrix} 4/9 & 1/9 & 4/9 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

Multiplying this by $\mathbf{P}$ removes the 2 remaining 0's and further multiplication by P will not introduce any 0's.

(b). The elements in P^3 are all positive. So all states are communicate, and hence the MC is irreducible and positive recurrent. $p_{11}^{(n)} > 0$ as $p_{11} > 0$. This means that the period of the state 1 is 1, and hence all states are aperiodic.

(c). It only needs to check that $\pi = \{1/2, 1/6, 1/3\}$ is a stationary distribution.

(d). The limit of $\mathbf{P}^n$ is

$$\begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix},$$

as the MC $\{X_n, n \geq 0\}$ is irreducible, aperiodic and positive recurrent. (Recall Theorems 4.1 and 4.3)

(e). 6.

(f). (**Adv. Mean hitting time**) Using equation (9) in Lect 12, we need to find t_1 with $A = \{3\}$. Note that in our case, the equation (9) becomes: $t_3 = 0$,

$$\begin{aligned} t_1 &= 1 + t_1 p_{11} + t_2 p_{12} = 1 + t_1/3 + t_2/3 \\ t_2 &= 1 + t_1 p_{21} + t_2 p_{22} = 1 \end{aligned}$$

This yields the mean time that the MC first arrives at state 3 starting from the state 1 is $t_1 = 2$.

(g). Since $\pi = \{1/2, 1/6, 1/3\}$ is a stationary distribution, it follows that the long-run average cost

$$Ef(\pi) = \frac{1}{2} \times 2 + \frac{1}{6} \times 1 + \frac{1}{3} \times 5 = \frac{17}{6}.$$

(h). (**Adv.**) Let π as in (c). Then,

$$P(X_{n+1} = 1, X_n \in A) = p_{21}\pi_2 + p_{31}\pi_3 = 1/3$$

and $P(X_n \in A) = 1/6 + 1/3 = 1/2$ so $P(X_{n+1} = 1 \mid X_n \in A) = 2/3$. Similarly,

$$\begin{aligned} & P(X_{n+1} = 1, X_n \in A, X_{n-1} \in A) \\ = \; & P(X_{n+1} = 1, X_n \in A, X_{n-1} = 2) \\ = \; & P(X_{n+1} = 1, X_n = 3, X_{n-1} = 2) \\ = \; & P(X_{n+1} = 1 | X_n = 3, X_{n-1} = 2)\, P(X_n = 3 | X_{n-1} = 2)\, P(X_{n-1} = 2) \\ = \; & P(X_{n-1} = 2), \end{aligned}$$

and $P(X_n \in A, X_{n-1} \in A) = P(X_{n-1} = 2)$, So $P(X_{n+1} = 1 | X_n \in A, X_{n-1} \in A) = 1$.

2. In a Markov Chain with two states, 0 and 1, it is known that the mean recurrence time of state 0 is 3.

(a). Prove that the chain is irreducible and aperiodic.

(b). Prove that the chain is positive recurrent.

(c). Write down the stationary distribution.

(d). If it is also known that $p_{00}^{(2)} = 0.66$, find the transition matrix P.

Proof. (a). If state 0 were absorbing, then its mean recurrence time would be 1. If it were transient then its mean recurrence time would be ∞. So 0 and 1 communicate, i.e., it's irreducible. To be other than aperiodic would require $p_{01} = p_{10} = 1$ so the mean recurrence time of 0 is 2.

(b). Since there are only a finite number of states the MC must be positive recurrent.

(c) $\pi_0 = 1/3$ and $\pi_1 = 2/3$.

(d). Let $p_{00} = \alpha$ and $p_{10} = \beta$. Then, $\alpha^2 + (1-\alpha)\beta = 0.66$ (from $p_{00}^{(2)} = 0.66$) and $\alpha/3 + 2\beta/3 = 1/3$ (from the condition that the mean recurrence time of state 0 is 3) giving $\alpha = 0.8$ and $\beta = 0.1$.

3. (Adv only) As in Lectures, let $\{X_n, n \geq 0\}$ be a homogeneous MC and define

$$\begin{aligned}
p_{ij}^{(n)} &= P(X_n = j \mid X_0 = i), \quad p_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \\
f_{ij}^{(1)} &= P(X_1 = j \mid X_0 = i), \\
f_{ij}^{(n)} &= P(X_n = j, X_{n-1} \neq j, ..., X_1 \neq j \mid X_0 = i), \quad \text{for } n \geq 2.
\end{aligned}$$

The following identity is proved in Tutorial 3:

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}, \qquad \text{for all } n \geq 1.$$

Use this result to show that

(a). if $\sum_{k=1}^{\infty} p_{jj}^{(k)} < \infty$ for some $j \in S$, then $\sum_{k=1}^{\infty} p_{ij}^{(k)} < \infty$, for any $i \in S$, (Hence $p_{ij}^{(n)} \to 0$, as $n \to \infty$.)

(b). if C_0 is a closed class with finite states, then all states in C_0 are recurrent.

Proof. (a). Recall that $\sum_{k=1}^{\infty} f_{ij}^{(k)} \leq 1$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{ij}^{(n)} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_{ij}^{(k)} p_{jj}^{(n-k)} \\
&= \sum_{k=1}^{\infty} f_{ij}^{(k)} \sum_{n=0}^{\infty} p_{jj}^{(n)} < \infty.
\end{aligned}$$

(b). Recall that C_0 is a closed class, i.e.,

$$p_{jk}^{(n)} = 0, \quad \text{for } j \in C_0,\ k \notin C_0 \text{ and all } n \geq 1.$$

This implies that

$$\sum_{k \in C_0} p_{jk}^{(n)} = 1, \quad \text{for } j \in C_0 \text{ and all } n \geq 1.$$

Therefore, $\sum_{k \in C_0} \sum_{n=1}^{\infty} p_{jk}^{(n)} = \infty$, for $j \in C_0$. As C_0 is finite, there exists a $k \in C_0$ s.t.

$$\sum_{n=1}^{\infty} p_{jk}^{(n)} = \infty, \quad \text{for some } j \in C_0.$$

Now it must have $\sum_{n=1}^{\infty} p_{kk}^{(n)} = \infty$ (see part (a)). That is, k is recurrent and hence all states in C_0 are recurrent.