DP2 READING GROUP

Longye Tian

Sydney - October 2024

Available at https://github.com/longye-tian

ORDER CONVERGE

Definition

A sequence (v_n) in a Riesz space E is said to **order converge** to a point $v \in E$ if there exists $(d_n) \subset E$ such that

$$|v_n - v| \le d_n \ \forall n \in \mathbb{N}$$
 and $d_n \downarrow 0$

In this case, we write $v_n \stackrel{o}{\rightarrow} v$.

LEMMA 1.3.1ORDER CONVERGENCE GENERALIZE MONOTONE CONVERGENCE

Order limits are unique, in the sense that if $v_n \stackrel{o}{\to} v$ and $v_n \stackrel{o}{\to} u$ then u = v. Moreover,

- (i) if (v_n) is increasing, then $v_n \stackrel{o}{\rightarrow} v$ if and only if $v_n \uparrow v$
- (ii) if (v_n) is decreasing, then $v_n \stackrel{o}{\rightarrow} v$ if and only if $v_n \downarrow v$

Proof.

Let
$$|v_n - v| \le d_n \downarrow 0$$
, $|v_n - u| \le b_n \downarrow 0$. We have

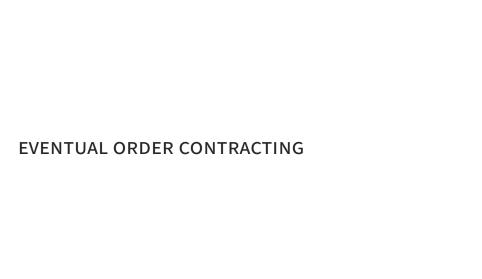
$$|v - u| \le |v - v_n + v_n - u| \le |v - v_n| + |v_n - u| \le d_n + b_n$$

By Exercsie 1.1.19, we have

$$|v-u| \le d_n + b_n \downarrow 0$$

Hence, we have v = u.





EVENTUAL ORDER CONTRACTING

Definition

Let *S* be a self-map on a subset *V* of a Riesz space *E*. We call *S* **eventual order contracting** on *V* if, there exists a order continuous linear operator $K : E \to E$ such that for any $v, w \in V$,

$$|Sv - Sw| \le K|v - w|$$
 and $K^n|v - w| \stackrel{o}{\to} 0$

DEFINITION

Definition

Let *S* be a eventual order contraction on *V* subset of a Riesz space *E*. We say *S* is **bounded by** *K* if $K : E \to E$ is a order continuous linear operator such that for any $v, w \in V$,

$$|Sv - Sw| \le K|v - w|$$
 and $K^n|v - w| \stackrel{o}{\to} 0$

LEMMA *

If S is a eventual order contraction on V a subset of Riesz space E bounded by K, then, S^m is also eventual order contraction on V and bounded by K^m .

Proof.

We have, for any $v, w \in V$,

$$|S^m v - S^m w| = |S(S^{m-1}v) - S(S^{m-1}w)| \le K|S^{m-1}v - S^{m-1}w|$$

Iterating this, we get

$$|S^m v - S^m w| \le K^m |v - w|$$
 and $(K^m)^n |v - w| \stackrel{o}{\rightarrow} 0$

Since composition of linear order continuous map is also linear order continuous, this shows the claim.

EXERCISE 1.1.29 REVISIT

Let *S* be an order preserving self-map on a subset *V* of a Riesz space *E*. Suppose there exists a order continuous linear operator $K: E \to E$ such that $|S v - S w| \le K|v - w|$ for all $v, w \in V$. Prove that *S* is order continuous on *V*.

Proof.

Fix $(v_n) \subset V$ with $v_n \uparrow v \in V$. We have

$$0 \le Sv - Sv_n \le K|v - v_n| \le Kd_n \downarrow K0 = 0$$

Hence $Sv - Sv_n \downarrow 0$. By Lemma 1.1.12, this gives $Sv_n \uparrow Sv$.

TARSKI-KANTOROVICH THEOREM REVISIT

Theorem

If S is an order continuous self-map on V and V is σ -chain complete, then S has a fixed point in V.

If V is a σ -chain complete subset of a Riesz space E and S is order preserving and eventually order contracting on V, then S has a unique fixed point $\bar{v} \in V$ and

$$S^n v \stackrel{o}{\to} \bar{v}$$
 for any $v \in E$.

Proof.

S is order continuous (by Exercise 1.1.29) on a σ -chain complete set V. TK implies S has a fixed point $\bar{v} \in V$. From proposition *, we have,

$$|S^n v - \overline{v}| = |S^n v - S^n \overline{v}| \le K^n |v - \overline{v}| \le d_n \downarrow 0$$

Uniqueness is from the uniqueness of the order limits.

EVENTUAL ORDER CONTRACTING ADP

Definition

Let E be a Riesz space and let (V, \mathbb{T}) be a ADP for some $V \subset E$. We call (V, \mathbb{T}) **eventually order contracting** if T_{σ} is eventually order contracting on V for all $\sigma \in \Sigma$.

THEOREM 1.2.14 REVISIT

Theorem

Let (V, \mathbb{T}) be regular, well-posed and order continuous. If V is σ -chain complete, then

- 1. the fundamental ADP optimality properties hold and
- 2. VFI, OPI and HPI all converge.

If V is a σ -chain complete subset of a Riesz space E and (V, \mathbb{T}) is regular and eventually order contracting, then

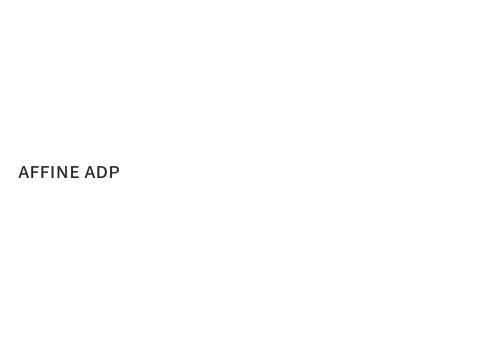
- 1. (V, \mathbb{T}) is well-posed,
- 2. the fundamental ADP optimality properties on page hold, and
- 3. VFI, OPI and HPI all converge.

Proof.

Well-posed from Theorem 1.3.2.

Order-continuity from Exercise 1.1.29.

Then, ADP optimality property and convergence from Theorem 1.2.14.



AFFINE ADP

Definition

Let (E, \mathbb{T}) be an ADP where E is an Archemidean Riesz space. We call (E, \mathbb{T}) affine if each $T_{\sigma} \in \mathbb{T}$ has the form

$$T_{\sigma} v = r_{\sigma} + K_{\sigma} v \qquad (v \in E)$$

for some $r_{\sigma} \in E$ and order continuous linear operator $K_{\sigma} : E \to E$.

Let S be a self-map on σ -Dedekind complete Archemidean Riesz space E. If

- 1. there exists a $h \in E$ and an order continuous linear operator K on E such that Sv = h + Kv for all $v \in E$, and
- 2. there is an $e \in E$ and $\rho \in [0,1)$ with $|h| \le e$ and $Ke \le \rho e$,

then S has a unique fixed point \bar{v} in the order interval

$$V := [-e/(1-\rho), e/(1-\rho)]$$
 and, moreover, $S^n v \stackrel{o}{\to} \bar{v}$ for any $v \in V$.

Proof.

- 1. Show S is eventual order contracting on V
- 2. Use Theorem 1.3.2



PROOF OF THEOREM 1.3.3

Proof.

First, we show that S is a self-map on V

$$|Sv| \le |h| + |Kv|$$
 $\le |h| + K|v|$ $\le e + K \frac{e}{1-\rho} \le e + \rho \frac{e}{1-\rho} = \frac{e}{1-\rho}$.

In Riesz space, we have for any $u, v \in E$ if $u \in E_+$, then $|v| \le u \Rightarrow v \in [-u, u]$.

This shows $Sv \in V$, hence S is a self-map

PROOF OF THEOREM 1.3.3 CONT.

Proof.

Fix $v, w \in V$, we have,

$$|Sv - Sw| = |h + Kv - h - Kw| = \underbrace{|Kv - Kw| = |K(v - w)|}_{linear} |\underbrace{\leq K|v - w|}_{positive}$$

Moreover,

$$|K^n|v - w| \le K^n \frac{2e}{1 - \rho} \le \underbrace{\rho^n \frac{2e}{1 - \rho} \downarrow 0}_{\rho \in [0, 1)}$$

Hence, *S* is eventually order contracting on *V*.

PROOF OF THEOREM 1.3.3 CONT.

Proof.

By Lemma 1.1.7, order interval V subset of σ -Dedekind complete space E is σ -chain complete.

Use Theorem 1.3.2, we get the result.

Let E be σ -Dedekind complete and let (E, \mathbb{T}) be an affine ADP with policy set Σ . If (E, \mathbb{T}) is regular and

$$\exists e \in E \text{ and } \rho \in [0,1) \text{ with } |r_{\sigma}| \leq e \text{ and } K_{\sigma} e \leq \rho e \text{ for all } \sigma \in \Sigma,$$

Let order interval $V := [-e/(1-\rho), e/(1-\rho)]$ then for the second ADP (V, \mathbb{T})

- 1. (V, \mathbb{T}) is well-posed,
- 2. the fundamental ADP optimality properties hold, and
- 3. VFI, OPI and HPI all converge.

PROOF OF THEOREM 1.3.5

Proof.

- *V* is σ -chain complete from Lemma 1.1.7.
- (E, \mathbb{T}) regular implies (V, \mathbb{T}) regular
- (V, \mathbb{T}) is eventual order contracting from Theorem 1.3.3.
- Use Theorem 1.3.4, we get all the results