Dynamic Programming

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Outline

Nonstationary policies

In this section, we will see that under some conditions, the lifetime value of any nonstationary policy will be weakly dominated by the lifetime value of a stationary policy. This ensures that we can focus on the stationary policies without loss of generality.

Comparison

Stationary policy

- Fixed a policy σ
- Lifetime value

$$v_\sigma = \lim_{j\to\infty} T_\sigma^j v$$

Nonstationary policy/Policy Plan

- a policy plan $\overline{\sigma} = (\sigma_t)_{t\geq 0} \in \bigtimes_{t\geq 0} \Sigma$
- Lifetime value of $v_{\overline{\sigma}}$

$$v_{\overline{\sigma}} = \lim_{j\to\infty} T_{\sigma_0} T_{\sigma_1} \cdots T_{\sigma_j} v$$

- Question, why not

$$v_{\overline{\sigma}} = \lim_{j\to\infty} T_{\sigma_j} \cdots T_{\sigma_1} T_{\sigma_0} v$$

Existence of Lifetime value of a policy plan

We want the limit to exist and, ideally, the limit is independent of v.

$$v_{\overline{\sigma}} = \lim_{j \to \infty} T_{\sigma_0} T_{\sigma_1} \cdots T_{\sigma_j} v$$

Set up

- $V = (V, \precsim)$ a partially ordered **space**
- $\mathbb{T} = \{T_\sigma : \sigma \in \Sigma\}$, family of order preserving self-map on V
- Metric d on V
    - d is complete (Every Cauchy sequence converges.)
    - $\exists \lambda \in (0, 1)$ such that

$$d(T_\sigma v, T_\sigma w) \leq \lambda d(v, w) \qquad \text{for all } v, w \in V, \sigma \in \Sigma$$

    - for all $v \in V$, we have

$$\sup_{\sigma \in \Sigma} d(v, T_\sigma v) < \infty$$

    - sup-nonexpansive, for any subsets (v_α) and (w_α) in V such that their supremum exists,

$$d\left(\bigvee_\alpha v_\alpha, \bigvee_\alpha w_\alpha\right) \leq \sup_\alpha d(v_\alpha, w_\alpha)$$

Lemma 1.4.1.(i)

Lemma 1

If the above conditions hold, then for each $v \in V$ and policy plan $\hat{\sigma} := (\sigma_t)_{t \geq 0}$, the limit

$$v_{\hat{\sigma}} = \lim_{n \to \infty} \bigtimes_{t=0}^{n} T_{\sigma_t} v$$

exists in V and is independent of v.

Proof for Existence.

To prove that (v_n) is Cauchy sequence, where $v_n := \bigtimes_{t=0}^{n} T_{\sigma_t} v$ □

Lemma 1.4.1.(i) Continue

Proof.

Fix $v \in V$, $\hat{\sigma} = (\sigma_t)_{t \geq 0}$, $\epsilon > 0$. Let $T_{m,n} := \bigtimes_{t=m}^{t=n} T_{\sigma_t}$, $v_n = T_{0,n} v$.
For $m \in \mathbb{N}$, we have

$$
\begin{aligned}
d(v_m, v_{m+1}) &= d\Big(T_0(T_{1,m} v), T_0(T_{1,m+1} v)\Big) \\
&\leq \lambda d\Big(T_1(T_{2,m} v), T_1(T_{2,m+1} v)\Big) && \text{(contraction)} \\
&\ \vdots \\
&\leq \lambda^{m+1} d(v, T_{m+1} v) \\
&\leq \lambda^{m+1} b_v && \text{(bounded)}
\end{aligned}
$$

□

Lemma 1.4.1.(i) Continue

Proof.

WLOG, let $m, n, j \in \mathbb{N}, n = m + j, j \geq 0$.

$$
\begin{aligned}
d(v_m, v_{m+j}) &\leq d(v_m, v_{m+1}) + d(v_{m+1}, v_{m+2}) + \cdots + d(v_{m+j-1}, v_{m+j}) && (\Delta \text{ inequality}) \\
&\leq \lambda^{m+1} b_v + \lambda^{m+2} b_v + \cdots + \lambda^{m+j} b_v && (\text{page } 7) \\
&\leq \lambda^{m+1} b_v (1 + \lambda + \cdots + \lambda^{m+j-1}) \\
&\leq \lambda^{m+1} b_v (1 + \lambda + \cdots + \lambda^{m+j-1} + \cdots) \\
&\leq \lambda^{m+1} b_v / (1 - \lambda) && (\text{geom sum})
\end{aligned}
$$

$\Longrightarrow$ Cauchy. By completeness, we get the limit exists. $\square$

Lemma 1.4.1.(i) Continue

The limit is independent of v.

Let $v, w \in V$. Then

$$\begin{aligned} d(v_n, w_n) &= d\Big(T_0(T_{1,n}v), T_0(T_{1,n}w)\Big) \\ &\leq \lambda d\Big(T_1(T_{2,n}v), T_1(T_{2,n}w)\Big) && \text{(contraction)} \\ &\vdots \\ &\leq \lambda^{n+1} d(v, w) \end{aligned}$$

So (v_n) and (w_n) have the same limit. $\square$

Lemma 1.4.1.(ii)

Lemma 2

If the conditions in page 5 holds, every $T_\sigma \in \mathbb{T}$ is continuous, globally stable on V, with unique fixed point v_σ satisfying

$$v_\sigma = \lim_{j \to \infty} T_\sigma^j v \qquad \textit{for all } v \in V$$

Proof.

From contraction and completeness. □

Lemma 1.4.1.(iii)

Lemma 3

If the conditions in page 5 holds, there exists a $v \in V$ *such that* $v := \bigvee_{\sigma \in \Sigma} T_\sigma v$

Proof.

If T is well-defined on V, then for $v, w \in V$, we have

$$\begin{aligned} d(Tv, Tw) &= d\left(\bigvee_{\sigma \in \Sigma} T_\sigma v, \bigvee_{\sigma \in \Sigma} T_\sigma w\right) \\ &\leq \sup_{\sigma \in \Sigma} d(T_\sigma v, T_\sigma w) && \text{(sup-nonexpansionary)} \\ &\leq \lambda d(v, w) && \text{(contraction)} \end{aligned}$$

Hence, T is a contraction, therefore, it has at least one fixed point in V. □

Question: Do we need $v \in V_G$? Is T continuous, globally stable with unique fixed point?

Review Theorem 1.3.3.

Theorem 4

Let V be a pospace, $(V, \mathbb{T})$ be regular, globally stable and T has a fixed point in V, then

- *the fundamental optimality properties hold*
- *VFI, HPI, OPI converge.*

Review of the Fundamental Optimality Properties

Let $(V, \mathbb{T})$ be regular and well-posed. We say **the fundamental optimality properties hold** for $(V, \mathbb{T})$ if

(B1) at least one optimal (stationary) policy exists

(B2) $v^* := \bigvee_\sigma v_\sigma$ is the unique solution to the Bellman equation

(B3) Bellman's principle of optimality holds (optimal policy is v^*-greedy)

A partial order $\precsim$ on topological space V is called **closed** if, given any two nets $(u_\alpha)_{\alpha\in\Lambda}$ and $(v_\alpha)_{\alpha\in\Lambda}$ contained in V,

$$u_\alpha \to u, v_\alpha \to v \qquad \text{and } u_\alpha \precsim v_\alpha \text{ for all } \alpha \in \Lambda \implies u \precsim v$$

A **partially ordered space**, is a Hausdorff topological space **endowed with a closed partial order**.

Proposition 1.4.2. Any policy plan is weakly dominated by a stationary policy

Proposition 1.4.2.

If $(V, \mathbb{T})$ is regular and conditions in page 5 holds, then

- the fundamental optimality properties hold
- Given any policy plan $\overline{\sigma}$, there exists a stationary policy plan σ such that $v_{\overline{\sigma}} \precsim v_\sigma$

$\square$

Proof.

Part One from Theorem 1.3.3. Part Two: Fix a policy plan $\overline{\sigma}$ and let σ be the optimal policy(B1). Then, for all $j \in \mathbb{N}$, we have

$$T_{\sigma_0} T_{\sigma_1} \cdots T_{\sigma_j} v_\sigma \to v_{\overline{\sigma}}, T^j v_\sigma \to v_\sigma, \qquad T_{\sigma_0} T_{\sigma_1} \cdots T_{\sigma_j} v_\sigma \precsim T^j v_\sigma = v_\sigma$$

The partial order is closed implies $v_{\overline{\sigma}} \precsim v_\sigma$. $\square$

Minimization Problem

For a given ADP $(V, \mathbb{T})$, a minimization problem can be converted to a maximization problem by reversing the partial order on V. Hence, we can focus on solving the maximization problem without loss of generality.

Definitions

Let $(V, \mathbb{T})$ be an ADP with policy set Σ. We define

- **Bellman min-operator** $T_\perp$ by

$$T_\perp v = \bigwedge_{\sigma \in \Sigma} T_\sigma v \qquad \text{whenever the infimum exists}$$

- $\sigma \in \Sigma$ is **v-min-greedy** if $T_\sigma v \precsim T_\tau v$ for all $\tau \in \Sigma$
- $(V, \mathbb{T})$ is **min-regular** if, for each $v \in V$, at least one v-min-greedy policy exists (V_G or V_G^{min})
- v satisfies the **Bellman min-equation** if $v = T_\perp v$

Definitions

Suppose $(V, \mathbb{T})$ is well-posed. We define

- **min-value function** by

$$v_\perp^* = \bigwedge_{\sigma \in \Sigma} v_\sigma \quad \text{whenever the infimum exists}$$

- $\sigma \in \Sigma$ is **min-optimal** for $(V, \mathbb{T})$ if $v_\sigma = v_\perp^*$
- $(V, \mathbb{T})$ obeys **Bellman's principle of min-optimality** if

$$\sigma \in \Sigma \text{ is min-optimal for } (V, \mathbb{T}) \iff \sigma \text{ is } v_\perp^*\text{-min-greedy}$$

Definitions

We say that the **fundamental min-optimality properties hold** if

(B1') at least one min-optimal policy exists

(B2') $v_\perp^*$ is the unique solution to the Bellman min-equation in V

(B3') Bellman's principle of min-optimality holds.

Definitions

When $(V, \mathbb{T})$ is min-regular, we define the **Howard policy min-operator** corresponding to $(V, \mathbb{T})$ via

$$H_\perp : V_G \to V_\Sigma, \quad H_\perp v = v_\sigma \quad \text{where } \sigma \text{ is } v\text{-min-greedy}$$

For each $m \in \mathbb{N}$, the **optimistic policy min-operator** via

$$W_\perp : V_G \to V, \quad W_\perp v = T_\sigma^m v \quad \text{where } \sigma \text{ is } v\text{-min-greedy}$$

Let V_D be all $v \in V$ with $T_\perp v \precsim v$. We say that

- **min-VFI converges** if $T_\perp^n \downarrow v_\perp^*$ for all $v \in V_D$
- **min-OPI converges** if $W_\perp^n v \downarrow v_\perp^*$ for all $v \in V_D$ and all $m \in \mathbb{N}$
- **min-HPI converges** if $H_\perp^n v \downarrow v_\perp^*$ for all $v \in V_D$.

Dual ADPs

How minimization problems can be converted to maximization problem in this abstract setting?

Order dual and Dual ADP

Definition 5

Given partially ordered set V, let $V^\partial = (V, \precsim^\partial)$ be the **order dual** (also called the **dual**), so that, for $u, v \in V$, we have

$$u \precsim^\partial v \iff v \precsim u$$

Definition 6

For ADP $(V, \mathbb{T})$, we call $(V, \mathbb{T})^\partial := (V^\partial, \mathbb{T})$ the **dual** of $(V, \mathbb{T})$. In other words, the dual ADP is created by replacing the poset V with its order dual V^∂.

Exercise 1.4.1

Show that $(V, \mathbb{T})^{\partial}$ is an ADP.

Proof.

We need to show that V^{∂} is a poset. And T_σ is order-preserving self-map on V^{∂} for any $\sigma \in \Sigma$. Let $u, v, w \in V$, we have

- (Reflexivity) $u \precsim u \implies u \precsim^{\partial} u$
- (Antisymmetry) $u \precsim^{\partial} v, v \precsim^{\partial} u \implies v \precsim u, u \precsim v \implies u = v$
- (Transitivity) $u \precsim^{\partial} v, v \precsim^{\partial} w \implies w \precsim v, v \precsim u \implies w \precsim u \implies u \precsim^{\partial} w$.

Hence V^{∂} is a poset. Let $u, v \in V$ and $u \precsim^{\partial} v$. We have $v \precsim u \implies T_\sigma v \precsim T_\sigma u \implies T_\sigma u \precsim^{\partial} T_\sigma v$ for any $\sigma \in \Sigma$. Hence, T_σ is order-preserving self map on V^{∂}. □

Notation for the dual ADP

For the dual ADP $(V, \mathbb{T})^\partial$,

- the Bellman max-operator will be denoted by T^∂
- the Bellman min-operator will be denoted by $T_\perp^\partial$
- the max-value function will be denoted by $(v^*)^\partial$

Self-Dual

Each ADP is a self-dual, i.e.,

$$((V, \mathbb{T})^{\partial})^{\partial} = (V, \mathbb{T})$$

This follows from the fact that all partially ordered sets are order self-dual.

Exercise 1.4.2 (i)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. Fix $v \in V$ and verify that:

$$\sigma \text{ is } v\text{-min-greedy for } (V, \mathbb{T}) \text{ if and only if } \sigma \text{ is } v\text{-max-greedy for } (V, \mathbb{T})^\partial$$

Proof ($\iff$)

$T_\sigma v \precsim T_\tau v$ for all $\tau \in \Sigma \iff T_\tau v \precsim^\partial T_\sigma v$ for all $\tau \in \Sigma$. $\square$

Exercise 1.4.2 (ii)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^{\partial}$. Fix $v \in V$ and verify that:

$$(V, \mathbb{T}) \text{ is min-regular if and only if } (V, \mathbb{T})^{\partial} \text{ is max-regular}$$

Proof.

By Exercise 1.4.2 (i) $\square$

Exercise 1.4.2.(iii)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. Fix $v \in V$ and verify that:

$$\text{If } T^\partial v \text{ exists then so does } T_\perp v \text{, and, moreover, } T_\perp v = T^\partial v$$

Proof.

By the definition of Bellman max-operator, we have $T_\sigma v \precsim^\partial T^\partial v$ for all $\sigma \in \Sigma$, i.e., $T^\partial v \precsim T_\sigma v$ for all $\sigma \in \Sigma$. Hence, $T_\perp v$ exists and equals to $T^\partial v$ by definition. □

Exercise 1.4.2.(iv)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. Fix $v \in V$ and verify that:

If $W^\partial v$ exists then so does $W_\perp v$, and, moreover, $W_\perp v = W^\partial v$

Proof.

By definition, we have $W^\partial v = T_\sigma v$ where σ is v-max-greedy for $(V, \mathbb{T})^\partial$. Hence by Exercise 1.4.2.(i), σ is v-min-greedy for $(V, \mathbb{T})$. Hence, we have $W^\partial v = W_\perp v$. □

Exercise 1.4.2.(v)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. Fix $v \in V$ and verify that:

If $H^\partial v$ exists then so does $H_\perp v$, and, moreover, $H_\perp v = H^\partial v$

Proof.

By definition, we have $H^\partial v = v_\sigma$ where σ is v-max-greedy for $(V, \mathbb{T})^\partial$. Hence by Exercise 1.4.2.(i), σ is v-min-greedy for $(V, \mathbb{T})$. Hence, we have $H^\partial v = H_\perp v$. $\square$

Exercise 1.4.2.(vi)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. Fix $v \in V$ and verify that: If the max-value function $(v^*)^\partial$ exists for $(V, \mathbb{T})^\partial$ then the min-value function $v^*_\perp$ exists for $(V, \mathbb{T})$, and, moreover,$v^*_\perp = (v^*)^\partial$.

Proof.

By definition,

$$(v^*)^\partial = \bigvee_{\sigma \in \Sigma}^{\partial} v_\sigma = \bigwedge_{\sigma \in \Sigma} v_\sigma = v^*_\perp$$

following Exercise A.1.15. □

Exercise 1.4.2.(vii)

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. Fix $v \in V$ and verify that:

$$\sigma \in \Sigma \text{ is min-optimal for } (V, \mathbb{T}) \text{ if and only if } \sigma \text{ is max-optimal for } (V, \mathbb{T})^\partial$$

Proof.

σ is min-optimal for $(V, \mathbb{T})$ if and only if $v_\sigma = v_\perp^* = (v^*)^\partial$, i.e., σ is max-optimal for $(V, \mathbb{T})^\partial$ from Exercise 1.4.2.(vi). $\square$

Optimality and Convergence

Theorem 7

Let $(V, \mathbb{T})$ be a well-posed ADP with dual $(V, \mathbb{T})^\partial$. The fundamental max-optimality properties hold for $(V, \mathbb{T})^\partial$ if and only if the fundamental min-optimality properties hold for $(V, \mathbb{T})$. Moreover,

(i) *max-VFI converges for $(V, \mathbb{T})^\partial$ if and only if min-VFI converges for $(V, \mathbb{T})$*

(ii) *max-OPI converges for $(V, \mathbb{T})^\partial$ if and only if min-OPI converges for $(V, \mathbb{T})$*

(iii) *max-HPI converges for $(V, \mathbb{T})^\partial$ if and only if min-HPI converges for $(V, \mathbb{T})$*