Dynamic Programming

Thomas J. Sargent and John Stachurski

Longye Tian
longye.tian@anu.edu.au

Australian National University
School of Economics

March 7th, 2025

Outline

Order Isomorphism

Order Isomorphism

Definition 1

A surjective map F from poset $(V, \precsim)$ to poset $(\hat{V}, \leq)$ is called an

- **Order isomorphism** if $v \precsim w \iff Fv \leq Fw$
- **Order anti-isomorphism** if $v \precsim w \iff Fw \leq Fv$

Comment: F under this definition is bijective.

Exercise 3.1.1.

Given $h \in \mathbb{R}^X$, let $Fh = \exp(\theta h)$. Show that F is an order isomorphism from $\mathbb{R}^X$ to $(0, \infty)^X$ whenever $\theta > 0$.

Proof.

Fix $\theta > 0$. We know that $\exp(\theta x)$ is a bijective function. Let $h_1, h_2 \in \mathbb{R}^X$ such that $h_1 \leq h_2$. This implies

$$\theta h_1 \leq \theta h_2$$

As $\exp(\cdot)$ is order preserving, hence, we have

$$Fh_1 = \exp(\theta h_1) \leq \exp(\theta h_2) = Fh_2$$

Let $k_1, k_2 \in (0, \infty)^X$, $k_1 \leq k_2$ by surjectivity, we have

$$k_1 = F(q_1) = \exp(\theta q_1), k_2 = F(q_2) = \exp(\theta q_2), q_1, q_2 \in \mathbb{R}^X$$

$\square$

Exercise 3.1.1. Continue

Proof.

We have

$$q_1 = \frac{\ln k_1}{\theta} \leq \frac{\ln k_2}{\theta} = q_2$$

as ln is order preserving. Therefore, by definition, F is an order isomorphism from $\mathbb{R}^X$ to $(0, \infty)^X$ $\square$

Exercise 3.1.2.

Let $V = M^X$ and $\hat{V} = \hat{M}^X$, $M, \hat{M} \subset \mathbb{R}$. Let φ be a map from M onto $\hat{M}$ and let $Fv = \varphi \circ v$. Prove if φ is an order isomorphism from M to $\hat{M}$, then F is an order isomorphism from V to $\hat{V}$.

Proof.

φ is order isomorphism then φ is bijective, order preserving with order preserving inverse. Hence apply this $\dim X$ times, we get F is bijective, order preserving with order preserving inverse. $\square$

Exercise 3.1.3

Let $V, \hat{V}$ be posets. Show that every order isomorphism F is a bijection. Show that every order anti-isomorphism is also a bijection.

Proof.

Let $v_1, v_2 \in \hat{V}$ such that $v_1 = v_2$. By surjectivity, we have

$$v_1 = F(w_1), \quad v_2 = F(w_2), \quad w_1, w_2 \in V$$

Hence, we have

$$F(w_1) \leq F(w_2) \implies w_1 \precsim w_2$$

$$F(w_2) \leq F(w_1) \implies w_2 \precsim w_1$$

Hence,$w_1 = w_2$. This proves that F is injective. $\square$

Exercise 3.1.4.

Let F be a bijection from $(V, \precsim)$ to $(\hat{V}, \leq)$. Show that

1. F is an order isomorphism if and only if F and F^{-1} are order preserving
2. F is an order anti-isomorphism if and only if F and F^{-1} are order reversing.

Proof.

Skip $\square$

Lemma 3.1.1.

Let F be an order isomorphism from $(V, \precsim)$ to $(\hat{V}, \leq)$. If the supremum of $\{v_\alpha\}_{\alpha\in\Lambda} \subset V$ exists in V, then

$$\bigvee_\alpha Fv_\alpha \text{ exists in } \hat{V} \text{ and } \bigvee_\alpha Fv_\alpha = F\bigvee_\alpha v_\alpha$$

Proof.

Let $v := \bigvee_\alpha v_\alpha \in V$. Let $\hat{w}$ be any upper bound of $\{Fv_\alpha\}$, i.e., $Fv_\alpha \leq \hat{w}$ for all $\alpha \in \Lambda$. By surjectivity, we let $\hat{w} = F(w)$, and by order isomorphism, we have

$$v_\alpha \precsim w \quad \text{for all } \alpha \in \Lambda$$

Hence, w is an upper bound of $\{v_\alpha\}$, this implies $v \precsim w$. Hence,

$$F(v) \leq F(w) = \hat{w}$$

This implies $F(v) = F\bigvee_\alpha v_\alpha$ is the least upper bound of $\{Fv_\alpha\}$. □

Exercise 3.1.6

Let $V, \hat{V}$ be posets and let (v_n) be a sequence in V. And let F be a map from V to $\hat{V}$. Prove the following

1. If F is an order isomorphism, then $v_n \uparrow v$ if and only if $Fv_n \uparrow Fv$ in $\hat{V}$
2. If F is an order anti-isomorphism, then $v_n \uparrow v$ if and only if $Fv_n \downarrow Fv$ in $\hat{V}$.

Proof.

$v_n \uparrow v \implies v_1 \leq v_2 \leq \cdots \leq v$ and $v = \bigvee_n v_n$.
Hence, by order isomorphism, we have

$$Fv_1 \leq Fv_2 \leq \cdots \leq Fv$$

and $Fv = \bigvee_n Fv_n$ Moreover, F is order isomorphism implies F^{-1} is order isomorphism.
Hence, the other direction follows. □

Exercise 3.1.7

Prove

- If $V, \hat{V}$ are order isomorphic, then V is totally ordered if and only if $\hat{V}$ is totally ordered
- F is an order anti-isomorphism from V to $\hat{V}$ if and only if F is an order isomorphism from V to its dual $\hat{V}^\partial$

Proof.

Skip. $\square$

Conjugate Dynamics

- We start with the definition of conjugacy between dynamical systems ((V, T_σ) is a dynamical system with state space V and evolution T_σ).
- Then, we go to the most basic structure, V as a poset, and upgrade conjugacy to order conjugacy.
- This prepares for the later upgrade from dynamical system to ADP

Definitions

Definition 2

We call a **discrete time dynamical system** is a pair (V, S), where V is any set, and S is a self-map on V.

Definition 3

Two dynamical systems (V, S) and $(\hat{V}, \hat{S})$ are said to be **conjugate under** F if

$$F \text{ is a bijection from } V \text{ into } \hat{V} \text{ and } F \circ S = \hat{S} \circ F \text{ on V}$$

or we can write it as

$$S = F^{-1} \circ \hat{S} \circ F$$

Proposition 3.1.2.

If (V, S) and $(\hat{V}, \hat{S})$ are conjugate, then

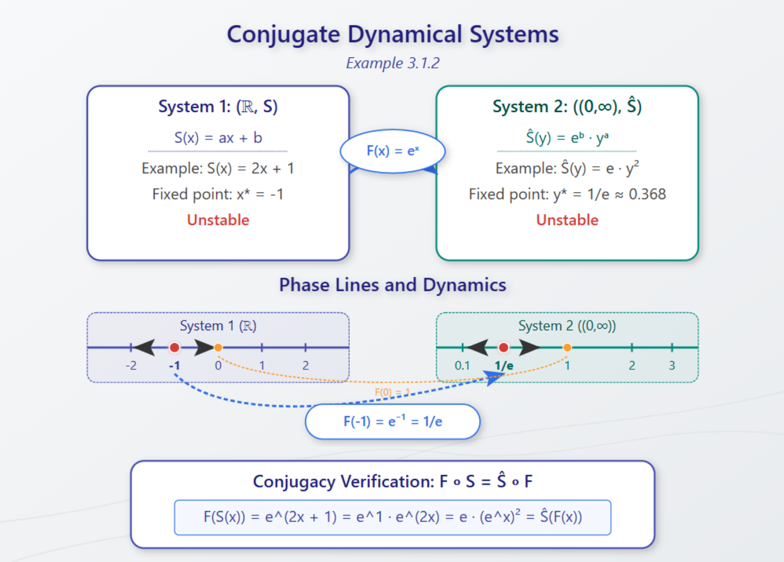
1. $S^n = F^{-1}\hat{S}^n F$ for all $n \in \mathbb{N}$
2. v is a fixed point of S if and only if Fv is a fixed point of $\hat{S}$
3. $\hat{v}$ is fixed point of $\hat{S}$ if and only if $F^{-1}\hat{v}$ is a fixed point of S
4. v is the unique fixed point of S in V if and only if Fv is the unique fixed point of $\hat{S}$ in $\hat{V}$.

Proof.

Let v be the unique fixed point of S, i.e., $Sv = v$. Hence,

$$F(v) = F(Sv) = \hat{S}(Fv)$$

Hence, $F(v)$ is a fixed point of $\hat{S}$. Let $\hat{w} = F(w)$ be a fixed point $\hat{S}$, then by part 2, w is the fixed point of S. Hence, $w = v$, and this implies $F(w) = F(v)$. □


Conjugate Dynamical Systems
Example 3.1.2
System 1: (ℝ, S)
S(x) = ax + b
Example: S(x) = 2x + 1
Fixed point: x* = -1
Unstable
F(x) = eˣ
System 2: ((0,∞), Ŝ)
Ŝ(y) = eᵇ · yᵃ
Example: Ŝ(y) = e · y²
Fixed point: y* = 1/e ≈ 0.368
Unstable
Phase Lines and Dynamics
System 1 (ℝ)
-2
-1
0
1
2
System 2 ((0,∞))
0.1
1/e
1
2
3
F(0) = 1
F(-1) = e⁻¹ = 1/e
Conjugacy Verification: F ∘ S = Ŝ ∘ F
F(S(x)) = e^(2x + 1) = e^1 · e^(2x) = e · (e^x)² = Ŝ(F(x))

Order conjugacy

Definition 4

Consider two dynamical systems (V, S) and $(\hat{V}, \hat{S})$, where $V, \hat{V}$ are posets. We call these systems **order conjugate under** F if they are conjugate under F, and, F is an order isomorphism.

Exercise 3.1.9.

Prove that order conjugacy is an equivalence relation on the set of dynamical systems over partially ordered set.

Proof.

We denote $(V, S) \sim (\hat{V}, \hat{S})$ is they are order conjugate. We need to show this relation is reflexive, symmetric and transitive.

- (Reflexivity) Let $F = Id$ which is a bijection. We have

  $$F \circ S = S = S \circ F$$

  Moreover, we have $F = F^{-1}$ is order preserving. Hence,

  $$(V, S) \sim (V, S)$$

□

Exercise 3.1.9 Continue

Proof.

- (Symmetry) Let $(V, S) \sim (\hat{V}, \hat{S})$ under F.
  - F is bijection implies F^{-1} is bijection
  - $F \circ S = \hat{S} \circ F \implies F^{-1} \circ \hat{S} = S \circ F^{-1}$

  Hence, (V, S) and $(\hat{V}, \hat{S})$ are conjugate under F^{-1}.
  - F is order preserving with order preserving inverse implies F^{-1} is order preserving with order preserving inverse

  Hence, F^{-1} is order isomorphism. Hence $(\hat{V}, \hat{S}) \sim (V, S)$

$\square$

Exercise 3.1.9 Continue

Proof.

- (Transitive) Let $(V_1, S_1) \sim (V_2, S_2)$ under F and $(V_2, S_2) \sim (V_3, S_3)$ under G.
  - F, G are bijective implies $H := (G \circ F)$ is bijective
  - $F \circ S_1 = S_2 \circ F, G \circ S_2 = S_3 \circ G \implies (G \circ F) \circ S_1 = G \circ S_2 \circ F = S_3 \circ (G \circ F)$

  Hence, (V_1, S_1) and (V_3, S_3) are conjugate under H.
  - F, G are order preserving with order perserving inverses
  - $G \circ F$ are order preserving with order preserving inverses

  Hence, $(V_1, S_1) \sim (V_3, S_3)$ under $(G \circ F)$.

$\square$

Lemma 3.1.3.

If (V, S) and $(\hat{V}, \hat{S})$ are order conjugate under F, then S is order stable on V if and only if $\hat{S}$ is order stable on $\hat{V}$.

Proof.

$(\implies)$ Suppose S is order stable on V. This implies

(S1) S has a unique fixed point $v^* \in V$

(S2) $v \in V, v \precsim v^* \implies v \precsim Sv$ and $v \in V, v^* \precsim v \implies Sv \precsim v$

$(S1)$ implies $\hat{S}$ has a unique fixed point $\hat{v}^* := F(v) \in \hat{V}$ by Proposition 3.1.2. Moreover we have

- For $\hat{v} := F(v) \in \hat{V}, \hat{v} \leq \hat{v}^* \underbrace{\implies}_{o.i.} v \precsim v^* \underbrace{\implies}_{S2} v \precsim Sv \underbrace{\implies}_{o.i.} Fv \leq \underbrace{FSv = \hat{S}\hat{v}}_{conjugate}$

$\square$

Isomorphic ADP

In this section, we will see how one ADP can be transformed into another ADP (or ADPs are equivalent up to a transformation).
Such transformation will result

- Simpler form of Bellman equation
- Tailored to solve for some problems (Exponential BE)
- etc.

Definition 5

Let $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ be ADPs with policy sets $\mathbb{T} := \{T_\sigma : \sigma \in \Sigma\}$ and $\hat{\mathbb{T}} := \{\hat{T}_\sigma : \sigma \in \Sigma\}$. We call these ADPs **isomorphic** under F if

1. F is an order isomorphism from V to $\hat{V}$
2. these two ADPs have the same policy set Σ
3. (V, T_σ) and $(V, \hat{T}_\sigma)$ are order conjugate under F for all $\sigma \in \Sigma$.

Example 3.1.3. Fei et al. (2021) Exponential Bellman Equation

Exponential risk-sensitive Q-factor Bellman equation (ADP: $((0, \infty)^G, \mathbb{M})$)

$$M_\sigma h = \exp(\theta r + \beta \ln P_\sigma h), \quad P_\sigma h(x, a) := \sum_{x'} h(x', \sigma(x')) P(x, a, x')$$

Risk-sensitive Q-factor policy operator (ADP: $(\mathbb{R}^G, \mathbb{T})$)

$$T_\sigma f = r + \frac{\beta}{\theta} \ln \left[P_\sigma \exp(\theta f) \right], \quad P_\sigma \exp(\theta f)(x, a) := \sum_{x'} \exp(\theta f(x', \sigma(x')) P(x, a, x')$$

Example 3.1.3. Continue

Let $\theta > 0$, and

$$Fh = \exp(\theta h)$$

is an order isomorphism from $\mathbb{R}^G$ to $(0, \infty)^G$.
For conjugacy, we have

$$\begin{aligned}
(F \circ T_\sigma)(h) &= \exp\left(\theta(r + \frac{\beta}{\theta}) \ln\left[P_\sigma \exp(\theta h)\right]\right) \\
&= \exp\left(\theta r + \beta \ln P_\sigma(Fh)\right) \\
&= (M_\sigma \circ F)(h)
\end{aligned}$$

Hence, $((0, \infty)^G, \mathbb{M})$ and $(\mathbb{R}^G, \mathbb{T})$ are isomorphic

Example 3.1.4. RDP

Let (Γ, V, B) and $(\Gamma, \hat{V}, \hat{B})$ be two RDPs with identical state sapce X, action space A, and feasible correspondence Γ. Let $V = M^X, \hat{V} = \hat{M}^X$, where $M, \hat{M} \subset \mathbb{R}$. If there exists an order isomorphism φ from M to $\hat{M}$ such that

$$B(x, a, v) = \varphi^{-1}[\hat{B}(x, a, \varphi \circ v)] \quad \text{for all } v \in V \text{ and } (x, a) \in G$$

then $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ are isomorphic. From exercise 3.1.2, F is an order isomorphism from V to $\hat{V}$, and

$$T_\sigma = F^{-1} \circ \hat{T}_\sigma \circ F$$

Lemma 3.1.4.

Lemma 6

Isomorphism between ADPs is an equivalence relation on the set of ADPs.

Proof.

Let $\mathbb{A}$ be the set of ADPs. We denote $(V_1, \mathbb{T}_1) \sim (V_2, \mathbb{T}_2)$ if there are isomorphic. We need to prove that $\sim$ is reflexive, symmetric and transitive.

- (Reflexivity) Let $(V, \mathbb{T}) \in \mathbb{A}$, as the ADP has the same policy set as itself and by Exercise 3.1.9, we get reflexivity.
- (Symmetry) Let $(V_1, \mathbb{T}_1) \sim (V_2, \mathbb{T}_2)$, then they have the same policy set. We use Exercise 3.1.9 get symmetry
- (Transitivity) Let $(V_1, \mathbb{T}_1) \sim (V_2, \mathbb{T}_2)$ and $(V_2, \mathbb{T}_2) \sim (V_3, \mathbb{T}_3)$, hence these three ADPs have the same policy set. We use Exercise 3.1.9. get transitivity.

$\square$

Isomorphisms and Optimality

Notations

We take

- $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ be two ADPs
- $\mathbb{T} := \{T_\sigma : \sigma \in \Sigma\}$; $\hat{\mathbb{T}} := \{\hat{T}_\sigma : \sigma \in \Sigma\}$
- $v_\sigma(resp. \hat{v}_\sigma)$ be the unique fixed point of $T_\sigma(resp. \hat{T}_\sigma)$
- $T(resp. \hat{T})$ be the Bellman operator of $(V, \mathbb{T})$ (resp. $(\hat{V}, \hat{\mathbb{T}})$)
- v^* (resp. $\hat{v}^*$) be the value function of $(V, \mathbb{T})$ (resp. $(\hat{V}, \hat{\mathbb{T}})$)

Theorem 3.1.5.

Theorem 7

If $(V, \mathbb{T})$ *and* $(\hat{V}, \hat{\mathbb{T}})$ *are isomorphic under* F*, then*

1. σ *is* v*-greedy for* $(V, \mathbb{T}) \iff \sigma$ *is* Fv*-greedy for* $(\hat{V}, \hat{\mathbb{T}})$
2. σ *is optimal for* $(V, \mathbb{T}) \iff \sigma$ *is optimal for* $(\hat{V}, \hat{\mathbb{T}})$
3. *Regularity, well-posedness, and order stability is preserved under isomorphism.*