

# The wealth distribution in Bewley economies with capital income risk <sup>☆</sup>

Jess Benhabib <sup>a</sup>, Alberto Bisin <sup>a,b,\*</sup>, Shenghao Zhu <sup>c</sup>

<sup>a</sup> *New York University, United States*

<sup>b</sup> *NBER, United States*

<sup>c</sup> *National University of Singapore, Singapore*

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## Abstract

We study the wealth distribution in Bewley economies with idiosyncratic capital income risk. We show analytically that under rather general conditions on the stochastic structure of the economy, a unique ergodic distribution of wealth displays a fat tail.

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## 1. Introduction

*Bewley economies*, as e.g., in [Bewley \(1977, 1983\)](#) and [Aiyagari \(1994\)](#),<sup>1</sup> represent one of the fundamental workhorses of modern macroeconomics, its main tool when moving away from the

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<sup>\*</sup> Corresponding author.

E-mail address: [alberto.bisin@nyu.edu](mailto:alberto.bisin@nyu.edu) (A. Bisin).

<sup>1</sup> The *Bewley economy* terminology is rather generally adopted and has been introduced by [Ljungqvist and Sargent \(2004\)](#).

study of efficient economies with a representative agent to allow e.g., for incomplete markets.<sup>2</sup> In these economies each agent faces a stochastic process for labor earnings and solves an infinite horizon consumption-saving problem with incomplete markets. Typically, agents are restricted to save by investing in a risk-free bond and face a borrowing limit. The postulated process for labor earnings determines the dynamics of the equilibrium distributions for consumption, savings, and wealth.<sup>3</sup>

Models of Bewley economies have been successful in the study of several macroeconomic phenomena of interest. Calibrated versions of this class of models have been used to study welfare costs of inflation (Imrohoroglu, 1992), asset pricing (Mankiw, 1986; Huggett, 1993), unemployment benefits (Hansen and Imrohoroglu, 1992), fiscal policy (Aiyagari, 1995; Heathcote, 2005), and partial consumption insurance (Heathcote et al. 2008a, 2008b; Storesletten et al., 2001; Krueger and Perri, 2003).<sup>4</sup>

On the other hand, standard and plausible parametrizations of Bewley economies are hardly able to reproduce the observed distribution of wealth in many countries; see e.g., Aiyagari (1994) and Huggett (1993). More specifically, they cannot reproduce the high inequality and the fat right tail that empirical distributions of wealth tend to display.<sup>5</sup> This is because at high wealth levels, the incentives for precautionary savings taper off and the right tail of the wealth distribution remains thin; see Carroll (1997) and Quadrini (2000) for a discussion of these issues.<sup>6</sup>

In the present paper we analytically study the wealth distribution in the context of Bewley economies extended to allow for idiosyncratic capital income risk.<sup>7</sup> To this end we provide first an analysis of the standard *Income Fluctuation problem*, as e.g., in

<sup>2</sup> The assumption of complete markets is generally rejected in the data; see e.g., Attanasio and Davis (1996), Fisher and Johnson (2006) and Jappelli and Pistaferri (2006).

<sup>3</sup> More recent specifications of the model allow for aggregate risks and an equilibrium determination of labor earnings and interest rates; see Huggett (1993), Aiyagari (1994), Rios-Rull (1995), Krusell and Smith (1998, 2006); see also Ljungqvist and Sargent (2004), Ch. 17, for a review of results.

<sup>4</sup> See Heathcote et al. (2008b) for a recent survey of the quantitative implications of Bewley models.

<sup>5</sup> Large top wealth shares in the U.S. since the 60's are documented e.g., by Wolff (1987, 2004) and, more recently, by Kopczuk et al. (2014) using estate tax return data; Piketty and Zucman (2014) find large and increasing wealth-to-income ratios in the U.S. and Europe in 1970–2010 national balance sheets data. Fat tails for the distributions of wealth are also well documented, for example by Nirei and Souma (2004) for the U.S. and Japan from 1960 to 1999, by Clementi and Gallegati (2005) for Italy from 1977 to 2002, and by Dagsvik and Vatne (1999) for Norway in 1998. Restricting to the Forbes 400 richest U.S. individuals during 1988–2003, Klass et al. (2007) also find that the top end of the wealth distribution obeys a Pareto law.

<sup>6</sup> Stochastic labor earnings can in principle generate some skewness in the distribution of wealth, especially if the earnings process is itself skewed and persistent. Extensive evidence for the skewedness of the income distribution has been put forth in a series of papers by Emmanuel Saez and Thomas Piketty (some with co-authors), starting with Piketty and Saez (2003) on the U.S. We refer to Atkinson et al. (2011) for a survey and to the excellent website of the database they have collected (with Facundo Alvaredo), The World Top Incomes Database. However, most empirical studies of labor earnings find some form of stationarity of the earnings process; see Guvenen (2007) and e.g., the discussion of Primiceri and van Rens (2009) by Heathcote (2009). Persistent income shocks are often postulated to explain the cross-sectional distribution of consumption but seem hardly enough to produce fat tailed distributions of wealth; see e.g., Storesletten et al. (2004); see also Cagetti and De Nardi (2008) for a survey.

<sup>7</sup> Capital income risk has been introduced by Angeletos and Calvet (2005) and Angeletos (2007) and further studied by Panousi (2008) and by ourselves (Benhabib et al. 2011, 2013). Quadrini (1999, 2000) and Cagetti and De Nardi (2006) study entrepreneurial risk, one of the leading examples of capital income risk, explicitly. Jones and Kim (2014) study entrepreneurs in a growth context under risk introduced by creative destruction. Relatedly, Krusell and Smith (1998) introduce heterogeneous discount rates to numerically produce some skewness in the distribution of wealth. We refer to these papers and our previous papers, as well as to Benhabib and Bisin (2006) and Benhabib and Zhu (2008), for more general evidence on the macroeconomic relevance of capital income risk.

Chamberlain and Wilson (2000), extended to account for capital income risk.<sup>8</sup> As in Aiyagari (1994), the borrowing constraint together with stochastic incomes assures a lower bound to wealth acting as a reflecting barrier.<sup>9</sup> We analytically show that enough idiosyncratic capital income risk induces an ergodic stationary wealth distribution which is fat tailed, more precisely, a Pareto distribution in the right tail. Furthermore, we show that the consumption function under borrowing constraints is strictly concave at lower wealth levels, consistent with, e.g. Saez and Zucman (2014)'s evidence of substantial saving rate differentials across wealth levels. In this environment, therefore, the rich can get richer through savings, while the poor may not save enough to become rich. Such non-ergodicity however would imply no social mobility between rich and poor, which seems incompatible with observed levels of social mobility in income over time and across generations; see for example Chetty et al. (2014). In our analysis it is capital income risk that induces the necessary mobility across wealth levels to generate an ergodic stationary wealth distribution. This complements the results in our previous papers (Benhabib et al. 2011, 2013), which focus on overlapping generation economies. An alternative approach to generate fat tails without stochastic returns is to introduce a model with bequests, where the probability of death (and/or retirement) is independent of age. In these models, the stochastic component is not stochastic returns but the length of life. For models that embody such features, see Wold and Whittle (1957), Castaneda et al. (2003), and Benhabib and Bisin (2006). On the other hand, sidestepping the income fluctuation problem by assuming a constant savings rate, Nirei and Aoki (2015) shows that thick tails are a direct consequence of the linearity of the wealth equation.

The rest of the paper is organized as follows. We present the basic setup of our economy in Section 2. In Section 3 we obtain the characterization of the income fluctuation problem with idiosyncratic capital income risk. In Section 4 we show that the wealth accumulation process has a unique stationary distribution and the stationary distribution displays a fat right tail. In Section 5 we introduce a model of entrepreneurship which is embedded in our analysis of the wealth distribution induced by the income fluctuation problem.<sup>10</sup> In Section 6 we extend our analysis of Bewley economies to allow for a market for loans. In Section 7 we briefly conclude.

## 2. The economy

Consider an infinite horizon economy with a continuum of agents uniformly distributed with measure 1.<sup>11</sup> Let  $\{c_t\}_{t=0}^{\infty}$  denote an agent consumption process. Let  $\{y_t\}_{t=0}^{\infty}$  represent the agent's labor earnings process and  $\{R_{t+1}\}_{t=0}^{\infty}$  his/her idiosyncratic rate of return on wealth process, that is, capital income risk.

<sup>8</sup> The work by Levhari and Srinivasan (1969), Schechtman (1976), Schechtman and Escudero (1977), Chamberlain and Wilson (2000), Huggett (1993), Rabault (2002), Carroll and Kimball (2005) has been instrumental to provide several incremental pieces to our characterization of the solution of (various specifications of) the Income Fluctuation problem; see Ljungqvist and Sargent (2004), Ch. 16, as well as Rios-Rull (1995) and Krusell and Smith (2006), for a review of results regarding the standard Income Fluctuation problem.

<sup>9</sup> See also Achdou et al. (2015) and Gabaix et al. (2015) for a continuous time model with stochastic returns and borrowing constraints exploring, respectively, the interaction of aggregate shocks and inequality on the transition dynamics of the macroeconomy and the speed of convergence to the stationary wealth distribution.

<sup>10</sup> The NBER W.P. version of this paper, Benhabib et al. (2014), also contains some simulation results regarding the stationary wealth distribution and the social mobility of the wealth accumulation process.

<sup>11</sup> We avoid introducing notation to index agents in the paper.

The agent's budget constraint at time  $t$  is then

$$q_{t+1} = R_{t+1} (q_t + y_t - c_t),$$

where  $\{q_{t+1}\}_{t=0}^{\infty}$  is wealth before earnings. In the economy, each agent faces a no-borrowing constraint at each time  $t$ :

$$q_{t+1} \geq 0.$$

It is convenient however for our purposes to work with the process of wealth after earnings, that is  $a_t = q_t + y_t$ . In this case, the agent's budget constraint and his/her borrowing constraint take respectively the following form:

$$\begin{aligned} a_{t+1} &= R_{t+1}(a_t - c_t) + y_{t+1} \\ c_t &\leq a_t \end{aligned}$$

Each agent in the economy then solves the *Income Fluctuation (IF) problem* which is obtained under Constant Relative Risk Aversion (CRRA) preferences,

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma \geq 1,$$

constant discounting  $\beta < 1$ , and *capital income risk and earnings* processes,  $\{R_{t+1}\}_{t=0}^{\infty}$  and  $\{y_t\}_{t=0}^{\infty}$ :

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{\infty}, \{a_{t+1}\}_{t=0}^{\infty}} \quad & E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \quad (\text{IF}) \\ \text{s.t. } \quad & a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1} \\ & c_t \leq a_t \\ & a_0 \text{ given.} \end{aligned}$$

The following assumptions characterize formally the stochastic properties of the economic environment:

**Assumption 1.**  $R_t$  and  $y_t$  are stochastic processes, independent and identically distributed (i.i.d.) over time and across agents:  $y_t$  has probability density function  $f(y)$  on bounded support  $[\underline{y}, \bar{y}]$ , with  $\underline{y} > 0$  and  $R_t$  has probability density function  $g(R)$  with closed support  $[\underline{R}, \bar{R}]$ .<sup>12</sup>  $R_t$  and  $y_t$  are independent. Furthermore,  $y_t$  satisfies **i)**  $(\bar{y})^{-\gamma} < \beta E[R_t (y_t)^{-\gamma}]$ , while  $R_t$  satisfies: **ii)**  $\bar{R} > \underline{R} > 0$  and  $\bar{R}$  large enough, **iii)**  $\beta E R_t^{1-\gamma} < 1$ ; **iv)**  $(\beta E R_t^{1-\gamma})^{\frac{1}{\gamma}} E R_t < 1$ ; and **v)**  $\Pr(\beta R_t > 1) > 0$  and any finite moment of  $R_t$  exists.

<sup>12</sup> Note however we can allow the support of  $R$  to be the real numbers over the half-line,  $[\underline{R}, \infty)$ , which is closed in the real numbers. While  $\bar{R} = \infty$  is allowed for, a finite  $\bar{R}$ , as derived in the proof of Theorem 4, is sufficient for all our results. In the case  $R$  takes discrete values in state space  $\bar{R}$ , we also assume the elements of  $\bar{R}$  are not all integral multiples of each other; see Saporta (2005), Theorem 1. This non-arithmeticity assumption is immediately satisfied if the support of  $R$  contains an interval of real numbers; it assures that the discrete stochastic process for wealth results in a distribution with a continuous power tail without holes.

To induce a limit stationary distribution of wealth, these assumptions guarantee that the contractive and expansive components of the rate of return process  $\{R_t\}_{t=0}^{\infty}$  tend to balance and the earnings process  $\{y_t\}_{t=0}^{\infty}$  act as a reflecting barrier on wealth. The assumption that these processes are i.i.d. over time is restrictive as a positive correlation in earnings and returns would capture economic environments with limited social mobility (for example, environments in which returns economic opportunities are in part transmitted across generations); but it could possibly be relaxed.<sup>13</sup>

## 2.1. Outline

It is useful to briefly outline the role of our assumptions and our strategy to obtain the main results in the paper. [Assumptions 1.i\) and 1.ii\)](#) guarantee that an agent with zero wealth at some time  $t$  will not consume all his/her income at time  $t + 1$  for high enough realizations of earnings and rates of return; as a consequence, the lower bound of the wealth space is a reflecting barrier, i.e., the wealth accumulation process is not trapped in the lower part of the wealth space in which savings of the agent are zero (see [Proposition 6](#) in Section 4).

[Assumptions 1.iii\) and 1.iv\)](#) guarantee that the wealth accumulation process is stationary. In particular, [Assumption 1.iii\)](#) guarantees that the aggregate economy displays no unbounded growth in consumption and wealth.<sup>14</sup> [Assumption 1.iv\)](#) implies that

$$\beta E R_t < 1.$$

This is enough to guarantee that the economy contracts, giving rise to a stationary distribution of wealth. However, since we cannot obtain explicit solutions for consumption or savings policies, we have to explicitly show that under suitable assumptions there are no disjoint invariant sets or cyclic sets in wealth, so that agents do not get trapped in subsets of the support of the wealth distribution. In other words we have to show that the stochastic process for wealth is ergodic, and that a unique stationary distribution exists. We show this in [Theorem 3](#).

We then have to show that idiosyncratic capital income risk can give rise to a fat-tailed wealth distribution. Since in our economic environment policy functions are not linear and explicit solutions are not available even under CRRA preferences, we cannot use the results of [Kesten \(1973\)](#), for example as in [Benhabib et al. \(2011\)](#). We are nonetheless able to show that consumption and savings policies are asymptotically linear; a result which, under appropriate assumptions, in particular i.i.d. processes for  $R_t$  and  $y_t$ , allow us to apply [Mirek \(2011\)](#)'s generalization of [Kesten \(1973\)](#).<sup>15</sup> We do this in [Propositions 3, 4 and 5](#). The fat right tail of the stationary distribution of wealth, obtained in [Theorem 4](#), exploits crucially that  $\Pr(\beta R_t > 1) > 0$ , that is, [Assumption 1.v\)](#).

<sup>13</sup> See the next subsection for a detailed discussion of [Assumptions 1.i\)–1.v\)](#).

<sup>14</sup> We can allow for exogenous growth  $g > 1$  in earnings. To this end, we need to deflate the variables by the growth rate and let the borrowing constraint grow at growth rate. (In our context, since we allow for no borrowing, no modification of the constraint is needed. However, [Assumption 1.2.iii\)](#) would have to be modified so that  $\Pr(\frac{\beta R_t}{g} > 1) > 0$ .)

<sup>15</sup> We conjecture that the analysis could be extended to serially correlated earnings and returns processes along the lines of [Benhabib et al. \(2011\)](#), though this would require extending the main theorems of [Saporta \(2004, 2005\)](#) and [Roitershtein \(2007\)](#) to asymptotic Kesten processes. Furthermore our analysis can be generalized to the case in which returns follow an AR(1) process. In this case under some regularity conditions (the most important being that the additive term in the AR(1) has compact support and has a non-singular distribution); see [Collamore \(2009\)](#).

### 3. The income fluctuation problem with idiosyncratic capital income risk

In this section we show several technical results about the consumption function  $c(a)$  which solves the (IF) problem, as a build-up for its characterization of the wealth distribution in the next section. All proofs are in [Appendix A](#).

**Theorem 1.** *A consumption function  $c(a)$  which satisfies the constraints of the (IF) problem and furthermore satisfies*

i) *the Euler equation*

$$u'(c(a)) \geq \beta E R_{t+1} u'(c[R_{t+1}(a - c(a)) + y]) \text{ with equality if } c(a) < a, \quad (1)$$

and

ii) *the transversality condition*

$$\lim_{t \rightarrow \infty} E \beta^t u'(c_t) a_t = 0, \quad (2)$$

*represents a solution of the (IF) problem.*

By strict concavity of  $u(c)$ , there exists a unique  $c(a)$  which solves the (IF) problem.

The study of  $c(a)$  requires studying two auxiliary problems. The first is a version the (IF) problem in which the stochastic process for earnings  $\{y_t\}_0^\infty$  is turned off, that is,  $y_t = 0$ , for any  $t \geq 0$ . The second is a finite horizon version of the (IF) problem. In both cases we naturally maintain the relevant specification and assumptions imposed on our main (IF) problem.

#### 3.1. The (IF) problem with no earnings

The formal (IF) problem with no earnings is:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^\infty, \{a_{t+1}\}_{t=0}^\infty} E \sum_{t=0}^\infty \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} & \quad (\text{IF with no earnings}) \\ \text{s.t. } a_{t+1} = R_{t+1}(a_t - c_t) \\ c_t & \leq a_t \\ a_0 & \text{ given.} \end{aligned}$$

This problem can indeed be solved in closed form, following [Levhari and Srinivasan \(1969\)](#). Note that for this problem the borrowing constraint is never binding because Inada conditions are satisfied for CRRA utility.

**Proposition 1.** *The unique solution to the (IF with no earnings) problem is*

$$c^{no}(a) = \phi a, \text{ for } 0 < \phi = 1 - \left( \beta E (R_{t+1})^{1-\gamma} \right)^{\frac{1}{\gamma}} < 1. \quad (3)$$

### 3.2. The finite (IF) problem

For any  $\tau \in \mathbb{Z}$ ,  $T > 0$ , let the finite (IF) problem be:

$$\begin{aligned} & \max_{\{c_t\}_{t=\tau}^T, \{a_{t+1}\}_{t=\tau}^{T-1}} E \sum_{t=\tau}^T \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \\ & \text{s.t. } a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}, \quad \text{for } \tau \leq t \leq T-1 \\ & \quad c_t \leq a_t, \quad \text{for } \tau \leq t \leq T \\ & \quad a_\tau \text{ given.} \end{aligned} \tag{finite IF}$$

**Proposition 2.** *The unique solution to the (finite IF) problem is a consumption function  $c_{t,\tau}(a)$  which is continuous and increasing in  $a$ . Furthermore, let  $s_{t,\tau}(a)$  denote the induced savings function,*

$$s_{t,\tau}(a) = a - c_{t,\tau}(a).$$

*Then  $s_{t,\tau}(a)$  is also continuous and increasing in  $a$ .*

### 3.3. Characterization of $c(a)$

We can now derive a relation between  $c_{t,\tau}(a)$ ,  $c^{no}(a)$  and  $c(a)$ . The following Lemma is a straightforward extension of Proposition 2.3 and Proposition 2.4 in Rabault (2002).

**Lemma 1.**  *$\lim_{t,\tau \rightarrow -\infty} c_{t,\tau}(a)$  exists, it is continuous, and satisfies the Euler equation. Furthermore,*

$$\lim_{t,\tau \rightarrow -\infty} c_{t,\tau}(a) \geq c^{no}(a).$$

The main result of this section follows:

**Theorem 2.** *The unique solution to the (IF) problem is the consumption function  $c(a)$  which satisfies:*

$$c(a) = \lim_{t,\tau \rightarrow -\infty} c_{t,\tau}(a).$$

Let the induced savings function  $s(a)$  be

$$s(a) = a - c(a).$$

**Proposition 3.** *The consumption and savings functions  $c(a)$  and  $s(a)$  are Lipschitz continuous and increasing in  $a$ .*

Carroll and Kimball (2005) show that  $c_{t,\tau}(a)$  is concave.<sup>16</sup> But Lemma 2 guarantees that  $c(a) = \lim_{t,\tau \rightarrow -\infty} c_{t,\tau}(a)$  and thus  $c(a)$  is also a concave function of  $a$ .

<sup>16</sup> See also Carroll et al. (2014).

**Proposition 4.** *The consumption function  $c(a)$  is a concave function of  $a$ .*

The most important result of this section is that the optimal consumption function  $c(a)$ , in the limit for  $a \rightarrow \infty$ , is linear and has the same slope as the optimal consumption function of the income fluctuation problem with no earnings,  $\phi$ .

**Proposition 5.** *The consumption function  $c(a)$  satisfies  $\lim_{a \rightarrow \infty} \frac{c(a)}{a} = \phi$ .*

The proof, in [Appendix A](#), is non-trivial.

#### 4. The stationary distribution

In this section we study the distribution of wealth in the economy. The wealth accumulation equation of the (IF) problem is

$$a_{t+1} = R_{t+1}(a_t - c(a_t)) + y_{t+1}. \quad (4)$$

It is useful to compare it with the (IF with no earnings). Using [Lemma 1](#) we have:

$$\begin{aligned} a_{t+1} &= R_{t+1}(a_t - c(a_t)) + y_{t+1} \\ &\leq R_{t+1}(a_t - c^{no}(a_t)) + y_{t+1} \\ &= R_{t+1}(1 - \phi)a_t + y_{t+1}. \end{aligned}$$

Let

$$\mu = 1 - \phi = \left( \beta E R^{1-\gamma} \right)^{\frac{1}{\gamma}}.$$

Thus  $\mu < 1$  by [Assumption 1.iii](#). We have

$$a_{t+1} \leq \mu R_{t+1} a_t + y_{t+1}.$$

The main results in this section are the following two theorems.<sup>17</sup>

**Theorem 3.** *The process  $\{a_{t+1}\}_{t=0}^{\infty}$  is ergodic and hence there exists a unique stationary distribution for  $a_{t+1}$  which satisfies the stochastic wealth accumulation equation (4).*

The proof, in [Appendix A](#), requires two steps. First, we show that the wealth accumulation process  $\{a_{t+1}\}_{t=0}^{\infty}$  induced by equation (4) above is  $\varphi$ -irreducible, i.e., there exists a non-trivial measure  $\varphi$  on  $[y, \infty)$  such that if  $\varphi(A) > 0$ , the probability that the process enters the set  $A$  in finite time is strictly positive for any initial condition (see Chapter 4 of [Meyn and Tweedie, 2009](#)). Second, to show that there exists a unique stationary wealth distribution we exploit the results in [Meyn and Tweedie \(2009\)](#) and show that the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is ergodic.

The next proposition shows that the stationary wealth distribution of our model is critically different from that of [Aiyagari \(1994\)](#), in that it is unbounded.

**Proposition 6.** *The support of the unique stationary distribution for  $a_{t+1}$  is unbounded.*

<sup>17</sup> The result in [Theorem 3](#) can also be obtained as an application of Theorem 2 in [Kamihigashi and Stachurski \(2014\)](#) under slightly weaker assumptions. We thank a referee for pointing this out to us.



In the next theorem we show that the wealth accumulation process  $\{a_{t+1}\}_{t=0}^{\infty}$  has a fat tail.<sup>18</sup> More precisely,

**Definition 1.** A distribution  $X$  is said to have a right fat tail if there exists  $\alpha > 0$  such that

$$\liminf_{x \rightarrow +\infty} \frac{\Pr(X > x)}{x^{-\alpha}} \geq C,$$

where  $C$  is a positive constant.

We use the characterization of  $c(a)$  and  $s(a)$  in Section 3.3, and in particular the fact that  $\frac{s(a)}{a}$  is increasing in  $a$  and  $\frac{s(a)}{a}$  approaches  $\mu$  as  $a$  goes to infinity (see the discussion after the proof of Lemma 3 in Appendix A); this allows us to apply some results by Mirek (2011) regarding conditions for asymptotically Pareto stationary distributions for processes induced by non-linear stochastic difference equations.

**Theorem 4.** The unique stationary distribution for  $a_{t+1}$  which satisfies the stochastic wealth accumulation equation (4) has a fat tail.

**Proof.** We use a comparison method to show the result. Firstly, we construct an auxiliary process,  $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$ . Then we show that the tail of the stationary distribution for  $\tilde{a}_{t+1}$  is asymptotic to a Pareto law. Finally, we show that the stationary distribution for  $a_t$ , which satisfies the stochastic wealth accumulation equation (4) has a fat tail, through comparing processes  $\{a_{t+1}\}_{t=0}^{\infty}$  and  $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$ .

*Construction of  $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$ .* Since  $\frac{s(a)}{a}$  is increasing in  $a$  and  $\frac{s(a)}{a}$  approaches  $\mu$  as  $a$  goes to infinity (see the discussion after the proof of Lemma 3 in Appendix A), there exist an  $\epsilon > 0$  arbitrarily small such that we can pick a large  $a^\epsilon$  to satisfy

$$\mu - \frac{s(a^\epsilon)}{a^\epsilon} < \epsilon.$$

Let

$$\mu^\epsilon = \frac{s(a^\epsilon)}{a^\epsilon}.$$

Thus  $\mu - \epsilon < \mu^\epsilon \leq \mu$ .

<sup>18</sup> A simple definition of a power law, or fat tailed, distribution is as follows. Define a regularly varying function with index  $\alpha \in (0, \infty)$  as

$$\lim_{x \rightarrow \infty} \frac{L(tx)}{L(x)} = t^{-\alpha}, \forall t > 0$$

Then, a distribution with a differentiable cumulative distribution function (cdf)  $F(x)$  and counter-cdf  $1 - F(x)$  is defined as a power-law with tail index  $\alpha$  if  $1 - F(z)$  is regularly varying with index  $\alpha > 0$ . If  $\lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} = 1, \forall t > 0$ , this is a slowly-varying function. If  $\lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} = \infty, \forall t > 0$ , then the function is neither a slowly-varying function nor a power-law. For example, the counter-cdf of the Cauchy distribution is slowly varying (i.e.,  $\alpha = 0$  above), while for the lognormal and normal distributions, the limit is infinite. Intuitively,  $\alpha$  captures the number of moments:  $\alpha = 0$  means the Cauchy has no moments, while  $\alpha = \infty$  means the distribution has all the moments; see Soulier (2009).

Let

$$l(a) = \begin{cases} s(a), & a \leq a^\epsilon \\ \mu^\epsilon a, & a \geq a^\epsilon. \end{cases} \quad (5)$$

Note that  $l(a) \leq s(a)$  for  $\forall a \in [y, \infty)$ , since  $\frac{s(a)}{a}$  is increasing in  $a$ ; furthermore, the function  $l(a)$  in (5) is Lipschitz continuous, since  $s(a)$  is Lipschitz continuous.

Let  $\theta = (R, y)$  and

$$\psi_\theta(a) = Rl(a) + y. \quad (6)$$

The stochastic process  $\{\tilde{a}_{t+1}\}_{t=0}^\infty$  is induced by  $\tilde{a}_{t+1} = \psi_\theta(\tilde{a}_t)$ . Now we apply Theorem 1.8 of Mirek (2011) to show that  $\{\tilde{a}_{t+1}\}_{t=0}^\infty$  has a unique stationary distribution. From Proposition 6 we know that the support of the stationary distribution for  $a_{t+1}$  is unbounded. It is easy to see, from the construction of  $\psi_\theta(\cdot)$  and Assumptions 1.i) and 1.ii), that the support of the stationary distribution for  $\tilde{a}_{t+1}$  is also unbounded. Furthermore, Theorem 1.8 of Mirek (2011) implies that the tail of the stationary distribution for  $\tilde{a}_{t+1}$  is asymptotic to a Pareto law, i.e.

$$\lim_{a \rightarrow \infty} \frac{\Pr(\tilde{a}_\infty > a)}{a^{-\alpha}} = C,$$

where  $C$  is a positive constant.

In order to apply Theorem 1.8 of Mirek (2011), we need to verify Assumption 1.6 and Assumption 1.7 of Mirek (2011). Assumption 1.6 essentially guarantees that  $\psi_\theta(\cdot)$  is asymptotically linear. Assumption 1.7 instead is the standard assumption which induces fat tails in the stationary distribution of a Kesten (linear) process.

*Verification of Assumption 1.6 of Mirek (2011).* For every  $z > 0$ , let

$$\psi_{\theta,z}(a) = z\psi_\theta\left(\frac{1}{z}a\right).$$

$\psi_{\theta,z}$  are called dilatations of  $\psi_\theta$ . Let

$$\bar{\psi}_\theta(a) = \lim_{z \rightarrow 0} \psi_{\theta,z}(a).$$

By the definition of  $\psi_\theta(\cdot)$  we have

$$\bar{\psi}_\theta(a) = \lim_{z \rightarrow 0} \psi_{\theta,z}(a) = \lim_{z \rightarrow 0} \left[ z\psi_\theta\left(\frac{1}{z}a\right) \right] = \mu^\epsilon Ra, \quad \text{for } \forall a \in [y, \infty).$$

Let

$$M^\epsilon = \mu^\epsilon R.$$

Thus

$$\bar{\psi}_\theta(a) = M^\epsilon a.$$

Let

$$N_\theta = \Omega R + y$$

where

$$\Omega = \max_{a \in [y, a^\epsilon]} |s(a) - \mu^\epsilon a|.$$

It is easy to verify that

$$|\psi_\theta(a) - M^\epsilon a| < N_\theta, \quad \text{for } \forall a \in [\underline{y}, \infty),$$

and hence that Assumption 1.6 (Shape of the mappings) in Mirek (2011) is satisfied.

*Verification of Assumption 1.7 of Mirek (2011).* As for Assumption 1.7 in Mirek (2011), condition (H3) is satisfied since  $M^\epsilon = \mu^\epsilon R$ ,  $R_t$  is i.i.d. over time and the support of  $R_t$  is closed. The law of  $\log M^\epsilon$  is non-arithmetic by Assumption 1 (see footnote 12) so H(4) in Assumption 1.7 of Mirek is satisfied. Let  $h(d) = \log E(M^\epsilon)^d$ . By Assumption 1.iv) we have  $E(\mu R_t) < 1$ . Thus  $h(1) = \log E(M^\epsilon) \leq \log E(\mu R) < 0$ . We now show that Assumption 1.iv) and Assumption 1.v) imply that there exists  $\kappa > 1$  such that  $\mu^\kappa E(R_t)^\kappa > 1$ . By Jensen's inequality we have  $E(R_t)^{1-\gamma} \geq (E R_t)^{1-\gamma}$ . Also, Assumption 1.iv) implies that  $\beta E R_t < 1$ . Thus

$$\mu = \left( \beta E(R_t)^{1-\gamma} \right)^{\frac{1}{\gamma}} \geq \left[ \beta (E R_t)^{1-\gamma} \right]^{\frac{1}{\gamma}} \geq \left[ \beta \left( \frac{1}{\beta} \right)^{1-\gamma} \right]^{\frac{1}{\gamma}} = \beta.$$

Thus

$$E(\mu R_t)^\kappa \geq E(\beta R_t)^\kappa \geq \int_{\{\beta R_t > 1\}} (\beta R_t)^\kappa.$$

By Assumption 1.v),  $\Pr(\beta R_t > 1) > 0$ . Thus there exists  $\kappa > 1$  such that  $\mu^\kappa E(R_t)^\kappa > 1$ . We could pick  $\mu^\epsilon$  such that  $(\mu^\epsilon)^\kappa E(R_t)^\kappa > 1$ . Thus  $h(\kappa) = \log E(M^\epsilon)^\kappa > 0$ . By Assumption 1.v), any finite moment of  $R_t$  exists. Thus  $h(d)$  is a continuous function of  $d$ . Thus there exists  $\alpha > 1$  such that  $h(\alpha) = 0$ , i.e.  $E(M^\epsilon)^\alpha = 1$ . Also we know that  $h(d)$  is a convex function of  $d$ . Thus there is a unique  $\alpha > 0$ , such that  $E(M^\epsilon)^\alpha = 1$ .

Moreover,  $E[(M^\epsilon)^\alpha | \log M^\epsilon] < \infty$ , since  $M^\epsilon$  has a lower bound, and, by Assumption 1.v), any finite moment of  $R$  exists.

We also know that  $E(N_\theta)^\alpha < \infty$  since  $y$  has bounded support and, by Assumption 1.v), any finite moment of  $R$  exists.

Thus  $M^\epsilon$  and  $N$  satisfy Assumption 1.7 (Moments condition for the heavy tail) of Mirek (2011).

*The comparison method.* Applying Theorem 1.8 of Mirek (2011), we find that the stationary distribution of  $\{\tilde{a}_{t+1}\}_{t=0}^\infty$ ,  $\tilde{a}_\infty$ , has an asymptotic Pareto tail. Finally, we show that the stationary distribution of  $\{a_{t+1}\}_{t=0}^\infty$ ,  $a_\infty$ , has a fat tail.

Pick  $a_0 = \tilde{a}_0$ . The stochastic process  $\{a_{t+1}\}_{t=0}^\infty$  is induced by

$$a_{t+1} = R_{t+1}s(a_t) + y_{t+1}.$$

And the stochastic process  $\{\tilde{a}_{t+1}\}_{t=0}^\infty$  is induced by

$$\tilde{a}_{t+1} = R_{t+1}l(\tilde{a}_t) + y_{t+1}.$$

For a path of  $\{(R_{t+1}, y_{t+1})\}_{t=0}^\infty$ , we have  $a_t \geq \tilde{a}_t$ ,  $\forall t \geq 0$ . Thus for  $\forall a > \underline{y}$ , we have

$$\Pr(a_t > a) \geq \Pr(\tilde{a}_t > a), \quad \text{for } \forall t \geq 0.$$

This implies that

$$\Pr(a_\infty > a) \geq \Pr(\tilde{a}_\infty > a),$$

since stochastic processes  $\{a_{t+1}\}_{t=0}^{\infty}$  and  $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$  are ergodic. Thus

$$\lim_{a \rightarrow \infty} \inf \frac{\Pr(a_{\infty} > a)}{a^{-\alpha}} \geq \lim_{a \rightarrow \infty} \inf \frac{\Pr(\tilde{a}_{\infty} > a)}{a^{-\alpha}} = \lim_{a \rightarrow \infty} \frac{\Pr(\tilde{a}_{\infty} > a)}{a^{-\alpha}} = C. \quad \square$$

## 5. Investment risk and entrepreneurship

In this section we discuss how to embed the analysis of the distribution of wealth induced by the (IF) problem in an equilibrium economy of entrepreneurship, one of the leading examples of investment risk. Following Angeletos (2007) we assume that each agent acts as entrepreneur of his own individual firm. Each firm has a constant returns to scale neo-classical production function

$$F(k, n, A)$$

where  $k, n$  are, respectively, capital and labor, and  $A$  is an idiosyncratic productivity shock. Agents can only use their own savings as capital in their own firm. In each period  $t + 1$ , each agent observes his/her firm's productivity shock  $A_{t+1}$  and decides how much labor to hire in a competitive labor market,  $n_{t+1}$ . Therefore, each firm faces the same market wage rate  $w_{t+1}$ . The capital he/she invests is instead predetermined, but the agent can decide not to engage in production, in which case  $n_{t+1} = 0$  and the capital invested is carried over (with no return nor depreciation) to the next period. The firm's profits in period  $t + 1$  are denoted  $\pi_{t+1}$ :

$$\pi_{t+1} = \max \{F(k_{t+1}, n_{t+1}, A_{t+1}) - wn_{t+1} + (1 - \delta)k_{t+1}, k_{t+1}\}. \quad (7)$$

Letting each agent's earnings in period  $t + 1$  are denoted  $w_{t+1}e_{t+1}$ , where  $e_{t+1}$  is his/her idiosyncratic (exogenous) labor supply, we have

$$a_{t+1} = \pi_{t+1} + w_{t+1}e_{t+1}.$$

Furthermore,

$$k_{t+1} = a_t - c_t.$$

Given a sequence  $\{w_t\}_{t=0}^{\infty}$ , each agent solves the following modified (IF) problem:

$$\max_{\{c_t, n_t\}_{t=0}^{\infty}, \{k_{t+1}, a_{t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \quad (\text{IF with entrepreneurship})$$

s.t.  $a_{t+1} = \pi_{t+1} + w_{t+1}e_{t+1}$  where  $\pi_{t+1}$  is defined in (7)

$$k_{t+1} = a_t - c_t$$

$$c_t \leq a_t$$

$$k_0 \text{ given.}$$

A stationary equilibrium in our economy consists of a constant wage rate  $w$ , sequences  $\{c_t, n_t\}_{t=0}^{\infty}, \{k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$  which constitute a solution to the (IF with entrepreneurship) problem under  $w_t = w$  for any  $t \geq 0$ , and a distribution  $v(a_{t+1}; w)$ , such that the following conditions hold:

- (i) labour markets clear:  $En_t = Ee_t$ ;<sup>19</sup>
- (ii)  $v$  is a stationary distribution of  $a_{t+1}$ , given  $w$ .

We can now illustrate how such an equilibrium can be constructed, inducing a stationary distribution of wealth for a given wage  $w$ ,  $v(a_{t+1}; w)$ , with the same properties, notably the fat tail, as the one characterized in the previous section under appropriate assumptions for the stochastic processes  $\{A_{t+1}\}_{t=0}^{\infty}$  and  $\{e_t\}_{t=0}^{\infty}$ . The first order conditions of each agent firm's labor choice requires

$$\frac{\partial F}{\partial n}(k_{t+1}, n_{t+1}, A_{t+1}) = w_{t+1};$$

which, under constant returns to scale implies,

$$\frac{\partial F}{\partial n}\left(1, \frac{n_{t+1}}{k_{t+1}}, A_{t+1}\right) = w_{t+1}. \quad (8)$$

Equation (8) can be solved to give

$$\frac{n_{t+1}}{k_{t+1}} = g(w_{t+1}, A_{t+1}); \text{ or } n_{t+1} = g(w_{t+1}, A_{t+1})k_{t+1}.$$

The market clearing condition (i) is then satisfied by a constant wage rate  $w$  such that

$$En_{t+1} = E(g(w, A_{t+1}))Ek_{t+1},$$

as long as the process  $\{A_{t+1}\}_{t=0}^{\infty}$  is i.i.d. over time and in the cross-section and  $Ek_{t+1}$  is constant over time.

In the stationary equilibrium  $\frac{n_{t+1}}{k_{t+1}}$  is determined by  $A_{t+1}$  and  $w$ . From the constant returns to scale assumption, once again, we can write profits  $\pi_{t+1}$  as:

$$\pi_{t+1} = R_{t+1}k_{t+1}$$

where  $\{R_{t+1}\}_{t=0}^{\infty}$ , in the stationary equilibrium, is induced by the process  $\{A_{t+1}\}_{t=0}^{\infty}$  and  $w$  as follows:

$$R_{t+1} = \max \left\{ \frac{\partial F}{\partial k} \left( 1, \frac{n_{t+1}}{k_{t+1}}, A_{t+1} \right) + 1 - \delta, 1 \right\}.$$

Let  $y_{t+1} = we_{t+1}$ . Then the dynamic equation for wealth can be written as

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}.$$

We conclude that the solution to (IF with entrepreneurship) induces a stochastic process  $\{a_{t+1}\}_{t=0}^{\infty}$  which has the same properties as the one induced by the (IF) problem as long as i)  $Ek_{t+1}$  is constant and ii) the process  $\{R_{t+1}\}_{t=0}^{\infty}$  induced by  $\{A_{t+1}\}_{t=0}^{\infty}$  and the process  $\{y_t\}_{t=0}^{\infty}$  induced by  $\{e_t\}_{t=0}^{\infty}$  satisfy [Assumption 1](#).<sup>20</sup> In particular, in this case,  $\{a_{t+1}\}_{t=0}^{\infty}$  has a unique

<sup>19</sup> The usual abuse of the Law of Large Numbers guarantees that the market clearing condition as stated holds in the cross-section of agents.

<sup>20</sup> General conditions on  $\{A_{t+1}\}_{t=0}^{\infty}$  that induce a process  $\{R_{t+1}\}_{t=0}^{\infty}$  that satisfies [Assumption 1](#) are hard to characterize. Simulations might have to be used to have a better sense of the range of parameters which induces a stationary distribution of wealth with a fat right tail; see [Benhabib et al. \(2014\)](#).

stationary distribution. The stationary distribution of  $\{a_{t+1}\}_{t=0}^{\infty}$  induces in turn a stationary distribution of  $k_{t+1}$ . The aggregate capital  $Ek_{t+1}$  is the first moment of the stationary distribution of  $k_{t+1}$  and is therefore constant. As a consequence, the labor market indeed clear with a constant wage  $w$  as postulated. It is verified then that at a stationary general equilibrium, as long as ii) above is satisfied, the stochastic process  $\{a_{t+1}\}_{t=0}^{\infty}$  has the same properties as the one induced by the (IF) problem; it displays, in particular, a fat tail.

## 6. Market for loans

Our analysis of Bewley economies is constructed on the assumption that the agent's borrowing is restricted as in the (IF) problem. More specifically, the agent at  $t$  can only invest in a risky asset with idiosyncratic return  $R_{t+1}$  and no market for loans is active in the economy. In this section we show how to extend the analysis to relax this assumption.

Let  $b_{t+1}$  denote the agent's holdings of the riskless asset at time  $t + 1$ , while  $k_{t+1}$  denotes his/her risky asset holdings. Let then  $a_{t+1}$  denote the total wealth after earnings:  $a_{t+1} = R_f b_{t+1} + R_{t+1} k_{t+1} + y_{t+1}$ , where  $R_f$  is the rate of return of the riskless asset and  $R_{t+1}$  is the rate of return of the risky asset, as in Section 2.<sup>21</sup> We maintain Assumption 1 and we impose a negative borrowing limit on bond holdings:  $b_{t+1} \geq -L$ , where  $0 \leq L$ .<sup>22</sup>

The (IF) problem, after allowing for an active market for loans to complement the risky asset, takes the following form:

$$\begin{aligned} & E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} && \text{(IF with loan market)} \\ \text{s.t. } & b_{t+1} + k_{t+1} = a_t - c_t \\ & a_{t+1} = R_f b_{t+1} + R_{t+1} k_{t+1} + y_{t+1} \\ & k_{t+1} \geq 0 \\ & b_{t+1} \geq -L \\ & a_0 \text{ given.} \end{aligned}$$

We can now illustrate how the solution of the (IF with loan market) problem induces a stochastic process  $\{a_{t+1}\}_{t=0}^{\infty}$  which has the same properties as the one induced by the (IF) problem. Indeed, the key to this result is that, as  $a_t$  becomes large, the solution to the (IF with loan market) problem is characterized by asymptotically constant portfolio shares and as a consequence by an asymptotically linear consumption function, as in the (IF) problem (Proposition 5).<sup>23</sup>

More specifically, the policy functions of the (IF with loan market) problem can be written as

$$c_t = c(a_t), \quad k_{t+1} = k(a_t), \quad b_{t+1} = b(a_t).$$

<sup>21</sup> We assume  $R_f$  is constant and exogenous; though a constant  $R_f$  can be endogenously obtained at the stationary distribution by imposing market clearing in the market for loans.

<sup>22</sup> To guarantee that the constraints are binding and induce a reflecting barrier, it is enough for instance to assume that  $L < \frac{y}{R_f - 1}$ .

<sup>23</sup> Achdou et al. (2015) have an elegant analysis of a related problem in continuous time, using viscosity solutions. Their analysis is formulated in general equilibrium with an endogenously determined risk free rate clearing the market for loans.

Most importantly, they satisfy

$$\lim_{a \rightarrow \infty} \frac{k(a)}{a} = \omega(1 - \tilde{\phi}), \quad \lim_{a \rightarrow \infty} \frac{b(a)}{a} = (1 - \omega)(1 - \tilde{\phi})$$

and hence

$$\lim_{a \rightarrow \infty} \frac{c(a)}{a} = \tilde{\phi}$$

for some  $0 < \tilde{\phi}, \omega < 1$ .

In fact we can easily solve for  $\tilde{\phi}$  and  $\omega$ . For large  $a_t$ , the first order conditions for the problem are

$$c_t^{-\gamma} = \beta R_f E c_{t+1}^{-\gamma} \quad (9)$$

and

$$E c_{t+1}^{-\gamma} (R_f - R_{t+1}) = 0. \quad (10)$$

Equation (10) implies

$$E [R_f (1 - \omega) + R_{t+1} \omega]^{-\gamma} (R_f - R_{t+1}) = 0, \quad (11)$$

which determines  $\omega$ ; and in turn equation (9) implies

$$\left( \frac{c_t}{a_t} \right)^{-\gamma} = \beta R_f E \left( \frac{a_{t+1}}{a_t} \right)^{-\gamma} \left( \frac{c_{t+1}}{a_{t+1}} \right)^{-\gamma},$$

and thus

$$\tilde{\phi} = 1 - \left( \beta R_f E [R_f (1 - \omega) + R_{t+1} \omega]^{-\gamma} \right)^{\frac{1}{\gamma}}. \quad (12)$$

Note that the equation for  $\tilde{\phi}$  is analogous to equation (3) for  $\phi$ , the asymptotic slope of the consumption function in the (IF) problem we obtained in Proposition 5:

$$\phi = 1 - \left( \beta E (R_{t+1})^{1-\gamma} \right)^{\frac{1}{\gamma}},$$

once the rate of return on the risky asset  $R_{t+1}$  is substituted by the rate of return on the agent's portfolio,  $R_f (1 - \omega) + R_{t+1} \omega$ .

Assuming the upper bound on labor earnings,  $\bar{y}$ , is large enough, we obtain a reflecting barrier at the lower bound of the wealth accumulation process

$$a_{t+1} = R_f b_{t+1} + R_{t+1} k_{t+1} + y_{t+1},$$

as in the benchmark model with only the risky asset. It is straightforward now to proceed as in benchmark to construct a stationary wealth distributions with fat tails.<sup>24</sup>

<sup>24</sup> An interesting result in this context is that an increase in the volatility of the risky asset can cause wealth inequality to decrease because agents respond by holding a smaller share of the risky asset in their portfolio, and in effect the volatility of the overall portfolio declines; see Benhabib and Zhu (2008).

## 7. Conclusion

In this paper we construct an equilibrium model with idiosyncratic capital income risk in a Bewley economy and analytically demonstrate that the resulting wealth distribution has a fat right tail under well defined and natural conditions on the parameters and stochastic structure of the economy.

## Appendix A

**Proof of Theorem 2.** A feasible policy  $c(a)$  is said to overtake another feasible policy  $\hat{c}(a)$  if starting from the same initial wealth  $a_0$ , the policies  $c(a)$  and  $\hat{c}(a)$  yield stochastic consumption processes  $(c_t)$  and  $(\hat{c}_t)$  that satisfy

$$E \left[ \sum_{t=0}^T \beta^t (u(c_t) - u(\hat{c}_t)) \right] > 0 \quad \text{for all } T > \text{some } T_0.$$

Also, a feasible policy is said to be optimal if it overtakes all other feasible policies.

Proof: For an  $a_0$ , the stochastic consumption process  $(c_t)$  is induced by the policy  $c(a)$ . Let  $(\hat{c}_t)$  be an alternative stochastic consumption process, starting from the same initial wealth  $a_0$ . By the strict concavity of  $u(\cdot)$ , we have

$$E \left[ \sum_{t=0}^T \beta^t (u(c_t) - u(\hat{c}_t)) \right] \geq E \left[ \sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \right].$$

From the budget constraint we have

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}$$

and

$$\hat{a}_{t+1} = R_{t+1}(\hat{a}_t - \hat{c}_t) + y_{t+1}.$$

For a path of  $(R_t, y_t)$ , we have

$$\frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} = a_t - c_t - (\hat{a}_t - \hat{c}_t) \quad (13)$$

and

$$c_t - \hat{c}_t = a_t - \hat{a}_t - \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}}.$$

Therefore we have

$$\sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) = \sum_{t=0}^T \beta^t u'(c_t) \left( a_t - \hat{a}_t - \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} \right).$$

Using  $a_0 = \hat{a}_0$  and rearranging terms, we have



$$\begin{aligned} & \sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \\ &= - \sum_{t=0}^T \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} - \beta^T u'(c_T) \frac{a_{T+1} - \hat{a}_{T+1}}{R_{T+1}}. \end{aligned}$$

Using equation (13) we have

$$\begin{aligned} & \sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \\ &= - \sum_{t=0}^T \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} \\ & \quad - \beta^T u'(c_T) [a_T - c_T - (\hat{a}_T - \hat{c}_T)] \\ &\geq - \sum_{t=0}^T \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} - \beta^T u'(c_T) a_T. \end{aligned}$$

Thus we have

$$\begin{aligned} & E \left[ \sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \right] \\ &\geq -E \left( \sum_{t=0}^T \beta^t [u'(c_t) - \beta E R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} \right) - E \beta^T u'(c_T) a_T. \quad (14) \end{aligned}$$

By the Euler equation (1) we have  $u'(c_t) - \beta E R_{t+1} u'(c_{t+1}) \geq 0$ . If  $c_t < a_t$ , then  $u'(c_t) = \beta E R_{t+1} u'(c_{t+1})$ . If  $c_t = a_t$ , then  $a_t - c_t - (\hat{a}_t - \hat{c}_t) = -(\hat{a}_t - \hat{c}_t) \leq 0$ . Thus

$$-E \left( \sum_{t=0}^T \beta^t [u'(c_t) - \beta E R_{t+1} u'(c_{t+1})] \{a_t - c_t - (\hat{a}_t - \hat{c}_t)\} \right) \geq 0. \quad (15)$$

Combining equations (14) and (15) we have

$$E \left[ \sum_{t=0}^T \beta^t u'(c_t)(c_t - \hat{c}_t) \right] \geq -E \beta^T u'(c_T) a_T.$$

By the transversality condition (2) we know that for large  $T$ ,

$$E \left[ \sum_{t=0}^T \beta^t (u(c_t) - u(\hat{c}_t)) \right] \geq 0. \quad \square$$

**Proof of Proposition 3.** The Euler equation of this problem is

$$c_t^{-\gamma} = \beta E R_{t+1} c_{t+1}^{-\gamma}. \quad (16)$$

Guess  $c_t = \phi a_t$ . From the Euler equation (16) we have

$$\phi = 1 - \left( \beta E R^{1-\gamma} \right)^{\frac{1}{\gamma}},$$

which is  $> 0$  by Assumption 1.iii).

It is easy to verify the transversality condition,

$$\lim_{t \rightarrow \infty} E \left( \beta^t c_t^{-\gamma} a_t \right) = 0. \quad \square$$

In the finite (IF) problem, let  $V_t(a)$  be the optimal value function of an agent who has wealth  $a$  in period  $t$ . Thus we have

$$V_t(a) = \max_{c \leq a} \{u(c) + \beta E V_{t+1}(R(a - c) + y)\} \quad \text{for } t \geq \tau$$

and

$$V_T(a) = \max_{c \leq a} u(c).$$

We have then the Euler equation for this problem, for  $t > 1$ :

$$u'(c_t(a)) \geq \beta E [R u'(c_{t+1}(R(a - c_t(a)) + y))] \text{ with equality if } c_t(a) < a.$$

**Proof of Proposition 2.** Continuity is a consequence of the Theorem of the Maximum and mathematical induction. The proof that  $c_{t,\tau}(a)$  and  $s_{t,\tau}(a)$  are increasing can be easily adapted from the proof of Theorem 1.5 of Schechtman (1976); it makes use of the fact that  $c_{t,\tau}(a) > 0$ , a consequence of Inada conditions which hold for CRRA utility functions.  $\square$

**Proof of Theorem 2.** By Lemma 1 we know that  $c(a)$  satisfies the Euler equation. Now we verify that  $c(a)$  satisfies the transversality condition (2).

By Lemma 1 and Theorem 2 we have

$$c_t \geq \phi a_t.$$

Note that  $a_t \geq y$  for  $t \geq 1$ . We have

$$u'(c_t) a_t \leq \phi^{-\gamma} \left( \underline{y} \right)^{1-\gamma} \quad \text{for } t \geq 1.$$

Thus

$$\lim_{t \rightarrow \infty} E \beta^t u'(c_t) a_t = 0. \quad \square$$

**Proof of Proposition 3.** By Lemma 1,  $c(a)$  is continuous. Thus  $s(a)$  is continuous since  $s(a) = a - c(a)$ .

Also, by Theorem 2,  $\lim_{t,\tau \rightarrow -\infty} s_{t,\tau}(a) = s(a)$ , since  $\lim_{t,\tau \rightarrow -\infty} c_{t,\tau}(a) = c(a)$ ,  $s_{t,\tau}(a) = a - c_{t,\tau}(a)$ , and  $s(a) = a - c(a)$ . The conclusion that  $c(a)$  and  $s(a)$  are increasing in  $a$  follows from Proposition 2.

For  $\tilde{a}, \hat{a} > 0$ , without loss of generality, we assume that  $\tilde{a} < \hat{a}$ . We have  $c(\tilde{a}) \leq c(\hat{a})$  and  $s(\tilde{a}) \leq s(\hat{a})$ . Also  $c(\tilde{a}) + s(\tilde{a}) = \tilde{a}$  and  $c(\hat{a}) + s(\hat{a}) = \hat{a}$ . Thus

$$c(\hat{a}) - c(\tilde{a}) + s(\hat{a}) - s(\tilde{a}) = \hat{a} - \tilde{a}.$$

Thus we have

$$0 \leq c(\hat{a}) - c(\tilde{a}) \leq \hat{a} - \tilde{a}$$

and

$$0 \leq s(\hat{a}) - s(\tilde{a}) \leq \hat{a} - \tilde{a}.$$

Thus

$$|c(\hat{a}) - c(\tilde{a})| \leq |\hat{a} - \tilde{a}|$$

and

$$|s(\hat{a}) - s(\tilde{a})| \leq |\hat{a} - \tilde{a}|.$$

Therefore,  $c(a)$  and  $s(a)$  are Lipschitz continuous.  $\square$

**Proof of Proposition 5.** The proof involves several steps, producing a characterization of  $\frac{c(a)}{a}$ .

**Lemma 2.**  $\exists \zeta > \underline{y}$ , such that  $s(a) = 0$ ,  $\forall a \in (0, \zeta]$ .

**Proof.** Suppose that  $s(a) > 0$  for  $a > \underline{y}$ . Pick  $a_0 > \underline{y}$ . For any finite  $t \geq 0$ , we have  $a_t > \underline{y}$  and  $u'(c_t) = \beta E R_{t+1} u'(c_{t+1})$ . Thus

$$u'(c_0) = \beta^t E R_1 R_2 \cdots R_{t-1} R_t u'(c_t). \quad (17)$$

By Lemma 1 and Theorem 2 we have

$$c_t \geq \phi a_t > \phi \underline{y}.$$

Thus equation (17) implies that

$$u'(c_0) \leq \left(\phi \underline{y}\right)^{-\gamma} (\beta E R)^t. \quad (18)$$

Thus the right hand side of equation (18) approaches 0 as  $t$  goes to infinity. A contradiction. Thus  $s(\zeta) = 0$  for some  $\zeta > \underline{y}$ . By the monotonicity of  $s(a)$ , we know that  $s(a) = 0$ ,  $\forall a \in (0, \zeta]$ .  $\square$

We can now show the following:

**Lemma 3.**  $\frac{c(a)}{a}$  is decreasing in  $a$ .

**Proof.** By Lemma 2 we know that  $c(\underline{y}) = \underline{y}$ . For  $\forall a > \underline{y}$ ,  $\frac{c(a)}{a} \leq 1 = \frac{c(\underline{y})}{\underline{y}}$ . Note that  $-c(a)$  is a convex function of  $a$ , since  $c(a)$  is a concave function of  $a$ . For  $\hat{a} > \tilde{a} > \underline{y}$ , we have<sup>25</sup>

$$\frac{c(\hat{a}) - c(\underline{y})}{\hat{a} - \underline{y}} \leq \frac{c(\tilde{a}) - c(\underline{y})}{\tilde{a} - \underline{y}}.$$

This implies that

<sup>25</sup> See Lemma 16 on p. 113 of Royden (1988).

$$c(\hat{a})\tilde{a} \leq c(\tilde{a})\hat{a} - [\hat{a} - \tilde{a} - (c(\hat{a}) - c(\tilde{a}))]\underline{y}. \quad (19)$$

From the Proof of [Proposition 3](#) we know that

$$c(\hat{a}) - c(\tilde{a}) \leq \hat{a} - \tilde{a}. \quad (20)$$

Combining inequalities (19) and (20) we have

$$c(\hat{a})\tilde{a} \leq c(\tilde{a})\hat{a},$$

i.e.

$$\frac{c(\hat{a})}{\hat{a}} \leq \frac{c(\tilde{a})}{\tilde{a}}. \quad \square$$

By [Lemma 1](#), [Theorem 2](#) and [Proposition 1](#) we know that  $\frac{c(a)}{a} \geq \phi$ . Thus we have

$$\lim_{a \rightarrow \infty} \frac{c(a)}{a} \text{ exists.}$$

Let

$$\lambda = \lim_{a \rightarrow \infty} \frac{c(a)}{a}. \quad (21)$$

Note that  $\lambda \leq 1$  since  $c(a) \leq a$ . This furthermore implies that  $\frac{s(a)}{a}$  is increasing and converges to a limit as  $a$  goes to infinity.

The Euler equation of this problem is

$$c_t^{-\gamma} \geq \beta E R_{t+1} c_{t+1}^{-\gamma} \text{ with equality if } c_t < a_t. \quad (22)$$

**Lemma 4.**  $\lambda \in [\phi, 1)$ .

**Proof.** Suppose that  $\lambda = 1$ . Thus

$$\lim_{a_t \rightarrow \infty} \inf \frac{c(a_t)}{a_t} = \lim_{a_t \rightarrow \infty} \frac{c(a_t)}{a_t} = 1.$$

From the Euler equation (22) we have

$$c_t^{-\gamma} \geq \beta E R_{t+1} c_{t+1}^{-\gamma} \geq \beta E R_{t+1} a_{t+1}^{-\gamma}$$

since  $c_{t+1} \leq a_{t+1}$  and  $\gamma \geq 1$ .

Thus

$$\left( \frac{c(a_t)}{a_t} \right)^{-\gamma} \geq \beta E R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma}.$$

By Fatou's lemma we have

$$\begin{aligned} \lim_{a_t \rightarrow \infty} \inf E R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \\ \geq E \lim_{a_t \rightarrow \infty} \inf \left[ R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \right]. \end{aligned}$$

Thus

$$\begin{aligned}
 1 &= \lim_{a_t \rightarrow \infty} \left( \frac{c(a_t)}{a_t} \right)^{-\gamma} \\
 &\geq \beta \lim_{a_t \rightarrow \infty} E R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \\
 &= \beta \lim_{a_t \rightarrow \infty} \inf E R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \\
 &\geq \beta E \lim_{a_t \rightarrow \infty} \inf \left[ R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \right] \\
 &= \beta E \lim_{a_t \rightarrow \infty} \left[ R_{t+1} \left( R_{t+1} \left( 1 - \frac{c(a_t)}{a_t} \right) + \frac{y_{t+1}}{a_t} \right)^{-\gamma} \right] \\
 &= \infty.
 \end{aligned}$$

A contradiction.  $\square$

From [Lemma 4](#) we know that  $c_t < a_t$  when  $a_t$  is large enough. Thus the equality of the Euler equation holds

$$c_t^{-\gamma} = \beta E R_{t+1} c_{t+1}^{-\gamma}.$$

Thus

$$\left( \frac{c_t}{a_t} \right)^{-\gamma} = \beta E R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma}. \quad (23)$$

Taking limits on both sides of equation (23) we have

$$\lim_{a_t \rightarrow \infty} \left( \frac{c_t}{a_t} \right)^{-\gamma} = \beta \lim_{a_t \rightarrow \infty} E R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma}.$$

Thus

$$\lambda^{-\gamma} = \beta \lim_{a_t \rightarrow \infty} E R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma}. \quad (24)$$

We turn to the computation of  $\lim_{a_t \rightarrow \infty} E R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma}$ .

In order to compute  $\lim_{a_t \rightarrow \infty} E R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma}$ , we first show a lemma.

**Lemma 5.** For  $\forall H > 0$ ,  $\exists J > 0$ , such that  $a_{t+1} > H$  for  $a_t > J$ . Here  $J$  does not depend on realizations of  $R_{t+1}$  and  $y_{t+1}$ .

**Proof.** Note that

$$\frac{a_{t+1}}{a_t} = \frac{R_{t+1}(a_t - c_t) + y_{t+1}}{a_t} \geq R_{t+1} \left( 1 - \frac{c_t}{a_t} \right).$$

From equation (21) we know that for some  $\varepsilon > 0$ ,  $\exists J_1 > 0$ , such that

$$\frac{c_t}{a_t} < \lambda + \varepsilon$$

for  $a_t > J_1$ . Thus

$$\frac{a_{t+1}}{a_t} \geq R_{t+1} \left( 1 - \frac{c_t}{a_t} \right) \geq R_{t+1}(1 - \lambda - \varepsilon). \quad (25)$$

And

$$\frac{a_{t+1}}{a_t} \geq R_{t+1}(1 - \lambda - \varepsilon) \geq \underline{R}(1 - \lambda - \varepsilon).$$

We pick  $J > J_1$  such that  $\underline{R}(1 - \lambda - \varepsilon) \geq \frac{H}{J}$ . Thus for  $a_t > J$ , we have

$$\frac{a_{t+1}}{a_t} \geq \frac{H}{J}.$$

This implies that

$$a_{t+1} \geq \frac{H}{J} a_t > H. \quad \square$$

From equation (21) we know that for some  $\eta > 0$ ,  $\exists H > 0$ , such that

$$\frac{c_{t+1}}{a_{t+1}} > \lambda - \eta \quad (26)$$

for  $a_{t+1} > H$ .

From Lemma 5 and equations (25) and (26) we have

$$R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma} = R_{t+1} \left( \frac{c_{t+1}}{a_{t+1}} \frac{a_{t+1}}{a_t} \right)^{-\gamma} \leq (\lambda - \eta)^{-\gamma} (1 - \lambda - \varepsilon)^{-\gamma} R_{t+1}^{1-\gamma}$$

for  $a_t > J$ . And

$$(\lambda - \eta)^{-\gamma} (1 - \lambda - \varepsilon)^{-\gamma} E R_{t+1}^{1-\gamma} < \infty$$

since  $\gamma \geq 1$ . Thus when  $a_t$  is large enough,  $(\lambda - \eta)^{-\gamma} (1 - \lambda - \varepsilon)^{-\gamma} R_{t+1}^{1-\gamma}$  is a dominant function of  $R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma}$ .

Note that

$$\lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_{t+1}} = \lim_{a_t \rightarrow \infty} \frac{c(a_{t+1})}{a_{t+1}} = \lambda \quad a.s.$$

by Lemma 5 and equation (21). And

$$\lim_{a_t \rightarrow \infty} \frac{a_{t+1}}{a_t} = \lim_{a_t \rightarrow \infty} \left( \frac{R_{t+1}(a_t - c_t) + y_{t+1}}{a_t} \right) = R_{t+1}(1 - \lambda) \quad a.s.$$

since  $y_{t+1} \in [\underline{y}, \bar{y}]$ . Thus

$$\lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_t} = \lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_{t+1}} \frac{a_{t+1}}{a_t} = \lambda(1 - \lambda) R_{t+1} \quad a.s.$$

Thus by the Dominated Convergence Theorem, we have

$$\lim_{a_t \rightarrow \infty} E R_{t+1} \left( \frac{c_{t+1}}{a_t} \right)^{-\gamma} = E R_{t+1} \left( \lim_{a_t \rightarrow \infty} \frac{c_{t+1}}{a_t} \right)^{-\gamma} = \lambda^{-\gamma} (1 - \lambda)^{-\gamma} E R_{t+1}^{1-\gamma}. \quad (27)$$

Combining equations (24) and (27) we have

$$\lambda^{-\gamma} = \beta \lambda^{-\gamma} (1 - \lambda)^{-\gamma} E R_{t+1}^{1-\gamma}. \quad (28)$$

By Lemma 4 we know that  $\lambda \geq \phi > 0$ . Thus we find  $\lambda$  from equation (28)

$$\lambda = 1 - \left( \beta E R^{1-\gamma} \right)^{\frac{1}{\gamma}}.$$

Thus  $\lambda = \phi$ .  $\square$

**Proof of Theorem 3.** The proof requires two steps.

**Lemma 6.** *The wealth accumulation process  $\{a_{t+1}\}_{t=0}^{\infty}$  is  $\psi$ -irreducible.*

**Proof.** First we show that the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is  $\varphi$ -irreducible. We construct a measure  $\varphi$  on  $[\underline{y}, \infty)$  such that

$$\varphi(A) = \int_A f(y) dy,$$

where  $f(y)$  is the density of labor earnings  $y_t$ . Note that the borrowing constraint binds in finite time with a positive probability for  $\forall a_0 \in [\underline{y}, \infty)$ . Suppose not. For any finite  $t \geq 0$ , we have  $a_t > \underline{y}$  and  $u'(c_t) = \beta E R_{t+1} u'(c_{t+1})$ . Following the same procedure as in the proof of Lemma 2, we obtain a contradiction. If the borrowing constraint binds at period  $t$ , then  $a_{t+1} = y_{t+1}$ . Thus any set  $A$  such that  $\int_A f(y) dy > 0$  can be reached in finite time with a positive probability. The process  $\{a_{t+1}\}_{t=0}^{\infty}$  is  $\varphi$ -irreducible.

By Proposition 4.2.2 in Meyn and Tweedie (2009), there exists a probability measure  $\psi$  on  $[\underline{y}, \infty)$  such that the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is  $\psi$ -irreducible, since it is  $\varphi$ -irreducible.  $\square$

To show that there exists a unique stationary wealth distribution, we have to show that the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is ergodic. Actually, we can show that it is geometrically ergodic.

**Lemma 7.** *The process  $\{a_{t+1}\}_{t=0}^{\infty}$  is geometrically ergodic.*

**Proof.** To show that the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is geometrically ergodic, we use part (iii) of Theorem 15.0.1 of Meyn and Tweedie (2009). We need to verify that

- a the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is  $\psi$ -irreducible;
- b the process  $\{a_{t+1}\}_{t=0}^{\infty}$  is aperiodic;<sup>26</sup> and

<sup>26</sup> For the definition of aperiodic, see p. 114 of Meyn and Tweedie (2009).

c there exists a petite set  $C$ ,<sup>27</sup> constants  $b < \infty$ ,  $\rho > 0$  and a function  $V \geq 1$  finite at some point in  $[\underline{y}, \infty)$  satisfying

$$E_t V(a_{t+1}) - V(a_t) \leq -\rho V(a_t) + b I_C(a_t), \quad \forall a_t \in [\underline{y}, \infty).$$

By Lemma 6, the process  $\{a_{t+1}\}_{t=0}^\infty$  is  $\psi$ -irreducible.

For a  $\varphi$ -irreducible Markov process, when there exists a  $v_1$ -small set  $A$  with  $v_1(A) > 0$ ,<sup>28</sup> then the stochastic process is called strongly aperiodic; see Meyn and Tweedie (2009, p. 114). We construct a measure  $v_1$  on  $[\underline{y}, \infty)$  such that

$$v_1(A) = \int_A f(y) dy.$$

By Lemma 2, we know that  $s(a) = 0$ ,  $\forall a \in [\underline{y}, \zeta]$ . Thus  $[\underline{y}, \zeta]$  is  $v_1$ -small and  $v_1([\underline{y}, \zeta]) = \int_{\underline{y}}^{\zeta} f(y) dy > 0$ . The process  $\{a_{t+1}\}_{t=0}^\infty$  is strongly aperiodic.

We now show that an interval  $[\underline{y}, B]$  is a petite set for  $\forall B > \underline{y}$ . To show this, we first show that  $\mathbb{R}s(a) + \underline{y} < a$  for  $a \in (\underline{y}, \infty)$ . For  $s(a) = 0$ , this is obviously true. For  $s(a) > 0$ , suppose that  $\mathbb{R}s(a) + \underline{y} \geq a$ , we have

$$u'(c(a)) = \beta E R_t u'(c(R_t s(a) + y)) \leq \beta E R_t u'(c(a)).$$

We obtain a contradiction since Assumption 1.iv) implies that  $\beta E R_t < 1$ . Also by Lemma 2, there exists an interval  $[\underline{y}, \zeta]$ , such that  $s(a) = 0$ ,  $\forall a \in [\underline{y}, \zeta]$ . For an interval  $[\underline{y}, B]$ ,  $\forall a_0 \in [\underline{y}, B]$ , there exists a common  $t$  such that the borrowing constraint binds at period  $t$  with a positive probability. Then for any set  $A \subset [\underline{y}, \bar{y}]$ ,  $\Pr(a_{t+1} \in A | s(a_t) = 0) = \int_A f(y) dy$ . Note that a  $t$ -step probability transition kernel is the probability transition kernel of a specific sampled chain. Thus we construct a measure  $v_a$  on  $[\underline{y}, \infty)$  such that  $v_a$  has a positive measure on  $[\underline{y}, \bar{y}]$  and  $v_a((\bar{y}, \infty)) = 0$ . The  $t$ -step probability transition kernel of a process starting from  $\forall a_0 \in [\underline{y}, B]$  is greater than the measure  $v_a$ . An interval  $[\underline{y}, B]$  is a petite set for  $\forall B > \underline{y}$ .

We pick a function  $V(a) = a + 1$ ,  $\forall a \in [\underline{y}, \infty)$ . Thus  $V(a) > 1$  for  $a \in [\underline{y}, \infty)$ . Pick  $0 < q < 1 - \mu E R_{t+1}$ . Let  $\rho = 1 - \mu E R_{t+1} - q > 0$  and  $b = 1 - \mu E R_{t+1} + E y_{t+1}$ . Pick  $B > \underline{y}$ , such that  $B + 1 \geq \frac{b}{q}$ . Let  $C = [\underline{y}, B]$ . Thus  $C$  is a petite set. Therefore, for  $\forall a_t \in [\underline{y}, \infty)$ , we have

$$\begin{aligned} E_t V(a_{t+1}) - V(a_t) &= E(a_{t+1}) - a_t \\ &\leq -(1 - \mu E R_{t+1}) V(a_t) + 1 - \mu E R_{t+1} + E y_{t+1} \\ &\leq -\rho V(a_t) + b I_C(a_t) \end{aligned}$$

where  $I_C(\cdot)$  is the indicator function of the set  $C$ .

By Theorem 15.0.1 of Meyn and Tweedie (2009) the process  $\{a_{t+1}\}_{t=0}^\infty$  is geometrically ergodic.  $\square$

This concludes the proof of Theorem 3.  $\square$

**Proof of Proposition 6.** From the proof of Lemma 6 we know that the borrowing constraint binds in finite time with a positive probability for  $\forall a_0 \in [\underline{y}, \infty)$ . If the borrowing constraint

<sup>27</sup> For the definition of petite sets, see p. 117 of Meyn and Tweedie (2009).

<sup>28</sup> For the definition of small sets, see p. 102 of Meyn and Tweedie (2009).



binds at period  $t$ , then  $a_{t+1} = y_{t+1}$ . By [Assumption 1.ii](#)),  $\bar{R}$  is large enough. Thus to show that the support of the stationary distribution is unbounded, it is sufficient to show that  $s(\bar{y}) > 0$ . Suppose that  $s(\bar{y}) = 0$ . Then  $s(a) = 0$  for  $a \in [y, \bar{y}]$ . Thus by the Euler equation we have

$$(\bar{y})^{-\gamma} \geq \beta E [R_t (y_t)^{-\gamma}].$$

This is impossible under [Assumption 1.i](#)). Thus  $s(\bar{y}) > 0$  and the support of the stationary distribution is unbounded.  $\square$

## References

- Achdou, Y., Lasry, J-M, Lions, P-L., B. Moll, 2015. Heterogeneous agent models in continuous time. Mimeo, Princeton University. Available at: <http://www.princeton.edu/~moll/HACT.pdf>.
- Aiyagari, S.R., 1994. Uninsured idiosyncratic risk and aggregate saving. *Q. J. Econ.* 109 (3), 659–684.
- Aiyagari, S.R., 1995. Optimal capital income taxation with incomplete markets and borrowing constraints. *J. Polit. Econ.* 103 (6), 1158–1175.
- Angeletos, G., 2007. Uninsured idiosyncratic investment risk and aggregate saving. *Rev. Econ. Dyn.* 10, 1–30.
- Angeletos, G., Calvet, L.E., 2005. Incomplete-market dynamics in a neoclassical production economy. *J. Math. Econ.* 41, 407–438.
- Atkinson, T., Saez, E., Piketty, T., 2011. Top incomes in the long run of history. *J. Econ. Lit.* 49 (1), 3–71.
- Attanasio, O.P., Davis, S.J., 1996. Relative wage movements and the distribution of consumption. *J. Polit. Econ.* 104 (6), 1227–1262.
- Benhabib, J., Bisin, A., 2006. The distribution of wealth and redistributive policies. Mimeo, New York University.
- Benhabib, J., Zhu, S., 2008. Age, luck and inheritance. NBER W.P. 14128.
- Benhabib, J., Bisin, A., Zhu, S., 2011. The distribution of wealth and fiscal policy in economies with finitely lived agents. *Econometrica* 79 (1), 122–157.
- Benhabib, J., Bisin, A., Zhu, S., 2013. The distribution of wealth in the Blanchard–Yaari model. *Macroecon. Dyn.* <http://dx.doi.org/10.1017/S1365100514000066>. Forthcoming, special issue on Complexity in Economic Systems.
- Benhabib, J., Bisin, A., Zhu, S., 2014. The wealth distribution in Bewley models with investment risk. NBER W.P. 20157.
- Bewley, T., 1977. The permanent income hypothesis: a theoretical formulation. *J. Econ. Theory* 16 (2), 252–292.
- Bewley, T., 1983. A difficulty with the optimum quantity of money. *Econometrica* 51 (5), 1485–1504.
- Cagetti, M., De Nardi, M., 2006. Entrepreneurship, frictions, and wealth. *J. Polit. Econ.* 114 (5), 835–870.
- Cagetti, M., De Nardi, M., 2008. Wealth inequality: data and models. *Macroecon. Dyn.* 12 (52), 285–313.
- Carroll, C.D., 1997. Buffer-stock saving and the life cycle/permanent income hypothesis. *Q. J. Econ.* 112, 1–56.
- Carroll, C.D., Kimball, M.S., 2005. Liquidity constraints and precautionary saving. Mimeo, Johns Hopkins University.
- Carroll, C.D., Slacalek, J., Tokunaka, K., 2014. The distribution of wealth and the marginal propensity to consume. Mimeo, Johns Hopkins University.
- Castaneda, A., Diaz-Gimenez, J., Rios-Rull, J.V., 2003. Accounting for the US earnings and wealth inequality. *J. Polit. Econ.* 111 (4), 818–857.
- Chamberlain, G., Wilson, C.A., 2000. Optimal intertemporal consumption under uncertainty. *Rev. Econ. Dyn.* 3 (3), 365–395.
- Chetty, R., Hendren, N., Kline, P., Saez, E., 2014. Where is the land of opportunity? The geography of intergenerational mobility in the United States. Mimeo, University of California, Berkeley. Available at: [http://obs.rc.fas.harvard.edu/chetty/mobility\\_geo.pdf](http://obs.rc.fas.harvard.edu/chetty/mobility_geo.pdf).
- Clementi, F., Gallegati, M., 2005. Power law tails in the Italian personal income distribution. *Physica A* 350, 427–438.
- Collamore, J.F., 2009. Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.* 19 (4), 1404–1458.
- Dagsvik, J.K., Vatne, B.H., 1999. Is the distribution of income compatible with a stable distribution? Discussion Paper 246. Research Department, Statistics Norway.
- Fisher, J., Johnson, D., 2006. Consumption mobility in the United States: Evidence from two panel data sets. *B.E. J. Econ. Anal. Policy* 6 (1), 16.
- Gabaix, X., Lasry, M., Lions, P-L., Moll, B., 2015. The dynamics of inequality. Mimeo, NYU.
- Guvenen, F., 2007. Learning your earning: are labor income shocks really very persistent? *Am. Econ. Rev.* 97 (3), 687–712.
- Hansen, G.D., Imrohoroglu, A., 1992. The role of unemployment insurance in an economy with liquidity constraints and moral hazard. *J. Polit. Econ.* 100 (1), 118–142.

- Heathcote, J., 2005. Fiscal policy with heterogeneous agents and incomplete markets. *Rev. Econ. Stud.* 72 (1), 161–188.
- Heathcote, J., 2009. Discussion of “Heterogeneous life-cycle profiles, income risk and consumption inequality” by Giorgio Primiceri and Thijs van Rens. *J. Monet. Econ.* 56, 40–42.
- Heathcote, J., Storesletten, K., Violante, G.L., 2008a. Insurance and opportunities: a welfare analysis of labor market risk. *J. Monet. Econ.* 55 (3), 501–525.
- Heathcote, J., Storesletten, K., Violante, G.L., 2008b. The macroeconomic implications of rising wage inequality in the United States. Mimeo, New York University.
- Huggatt, M., 1993. The risk-free rate in heterogeneous-agent incomplete-insurance economies. *J. Econ. Dyn. Control* 175 (6), 953–969.
- Imrohoroglu, A., 1992. The welfare cost of inflation under imperfect insurance. *J. Econ. Dyn. Control* 16 (1), 79–91.
- Jappelli, T., Pistaferri, L., 2006. Intertemporal choice and consumption mobility. *J. Eur. Econ. Assoc.* 4 (1), 75–115.
- Jones, C.I., Kim, J., 2014. A Schumpeterian model of top income inequality. Mimeo, Stanford University. Available at: <http://web.stanford.edu/~chadj/inequality.pdf>.
- Kamihigashi, T., Stachurski, J., 2014. Stochastic stability in monotone economies. *Theor. Econ.* 9 (2), 383–407.
- Kesten, H., 1973. Random difference equations and renewal theory for products of random matrices. *Acta Math.* 131, 207–248.
- Klass, O.S., Biham, O., Levy, M., Malcai, O., Solomon, S., 2007. The Forbes 400, the Pareto power-law and efficient markets. *Eur. Phys. J. B* 55, 143–147.
- Kopczuk, W., Saez, E., Top, 2014. Wealth shares in the United States, 1916–2000: Evidence from estate tax returns. *Natl. Tax J.* 57 (2), 445–487. Long NBER Working Paper No. 10399, March 2004.
- Krueger, D., Perri, F., 2003. On the welfare consequences of the increase in inequality in the United States. In: Gertler, M., Rogoff, K. (Eds.), *NBER Macroeconomics Annual 2003*. MIT Press, Cambridge, MA, pp. 83–121.
- Krusell, P., Smith, A.A., 1998. Income and wealth heterogeneity in the macroeconomy. *J. Polit. Econ.* 106 (5), 867–896.
- Krusell, P., Smith, A.A., 2006. Quantitative macroeconomic models with heterogeneous agents. In: Blundell, R., Newey, W., Persson, T. (Eds.), *Advances in Economics and Econometrics: Theory and Applications*. Cambridge University Press, Cambridge.
- Levhari, D., Srinivasan, T.N., 1969. Optimal savings under uncertainty. *Rev. Econ. Stud.* 36, 153–163.
- Ljungqvist, L., Sargent, T.J., 2004. *Recursive Macroeconomic Theory*. MIT Press, Cambridge, MA.
- Mankiw, N.G., 1986. The equity premium and the concentration of aggregate shocks. *J. Financ. Econ.* 17 (1), 211–219.
- Meyn, S.P., Tweedie, R.L., 2009. *Markov Chains and Stochastic Stability*. Cambridge University Press.
- Mirek, M., 2011. Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps. *Probab. Theory Relat. Fields* 151, 705–734.
- Nirei, M., Aoki, S., 2015. Pareto distribution of income in neoclassical growth models. Mimeo, Hitotsubashi University. Available at: [http://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=2436858](http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2436858).
- Nirei, M., Souma, W., 2004. Two factor model of income distribution dynamics. Mimeo, Utah State University.
- Panousi, V., 2008. Capital taxation with entrepreneurial risk. Mimeo, MIT.
- Piketty, T., Saez, E., 2003. Income inequality in the United States, 1913–1998. *Q. J. Econ.* 118, 1–39. <http://dx.doi.org/10.1162/00335530360535135>.
- Piketty, T., Zucman, G., 2014. Capital is back: wealth–income ratios in rich countries, 1700–2010. *Q. J. Econ.* 129, 1155–1210.
- Primiceri, G., van Rens, T., 2009. Heterogeneous life-cycle profiles, income risk and consumption inequality. *J. Monet. Econ.* 56, 20–39.
- Quadrini, V., 1999. The importance of entrepreneurship for wealth concentration and mobility. *Rev. Income Wealth* 45, 1–19. <http://dx.doi.org/10.1111/j.1475-4991.1999.tb00309.x>.
- Quadrini, V., 2000. Entrepreneurship, savings and social mobility. *Rev. Econ. Dyn.* 3, 1–40. <http://dx.doi.org/10.1006/redo.1999.0077>.
- Rabault, G., 2002. When do borrowing constraints bind? Some new results on the income fluctuation problem. *J. Econ. Dyn. Control* 26, 217–245.
- Rios-Rull, J.V., 1995. Models with heterogeneous agents. In: Cooley, T.F. (Ed.), *Frontiers of Business Cycle Research*. Princeton University Press, Princeton, NJ.
- Roitershtein, A., 2007. One-dimensional linear recursions with Markov-dependent coefficients. *Ann. Appl. Probab.* 17, 572–608. <http://dx.doi.org/10.1214/1050516060000000844>. mr=2308336.
- Royden, H.L., 1988. *Real Analysis*, third ed. Prentice-Hall, Inc., Upper Saddle River, New Jersey.
- Saez, E., Zucman, G., 2014. The distribution of US wealth, capital income and returns since 1913. University of California, Berkeley. Unpublished slides available at: <http://gabriel-zucman.eu/files/SaezZucman2014Slides.pdf>.
- Saporta, B., 2004. Etude de la Solution Stationnaire de l'Equation  $Y_n + 1 = a_n Y_n + b_n$ , a Coefficients Aleatoires. Thesis. Université de Rennes I. Available at <http://tel.archivesouvertes.fr/docs/00/04/74/12/PDF/tel-00007666.pdf>.

- Saporta, B., 2005. Tail of the stationary solution of the stochastic equation  $Y_{n+1} = a_n Y_n + \gamma_n$  with Markovian coefficients. *Stoch. Process. Appl.* 115 (12), 1954–1978.
- Schechtman, J., 1976. An income fluctuation problem. *J. Econ. Theory* 12 (2), 218–241.
- Schechtman, J., Escudero, V.L.S., 1977. Some results on an income fluctuation problem. *J. Econ. Theory* 16 (2), 151–166.
- Soulier, P., 2009. Some applications of regular variation in probability and statistics. *Escuela Venezolana de Matemáticas*. <http://evm.ivic.gob.ve/LibroSoulier.pdf>.
- Storesletten, K., Telmer, C.L., Yaron, A., 2001. The welfare cost of business cycles revisited: finite lives and cyclical variation in idiosyncratic risk. *Eur. Econ. Rev.* 45 (7), 1311–1339.
- Storesletten, K., Telmer, C.L., Yaron, A., 2004. Consumption and risk sharing over the life cycle. *J. Monet. Econ.* 51 (3), 609–633.
- Wold, H.O.A., Whittle, P., 1957. A model explaining the Pareto distribution of wealth. *Econometrica* 25, 591–595.
- Wolff, E., 1987. Estimates of household wealth inequality in the U.S., 1962–1983. *Rev. Income Wealth* 33, 231–256.
- Wolff, E., 2004. Changes in household wealth in the 1980s and 1990s in the U.S. Mimeo, New York University.