DP2 READING GROUP

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ORDER CONVERGE

Definition

A sequence (v_n) in a Riesz space E is said to **order converge** to a point $v \in E$ if there exists $(d_n) \subset E$ such that

$$|v_n - v| \leq d_n \ \forall n \in \mathbb{N} \quad \text{and } d_n \downarrow 0$$

In this case, we write $v_n \xrightarrow{o} v$.

LEMMA 1.3.1ORDER CONVERGENCE GENERALIZE MONOTONE CONVERGENCE

Order limits are unique, in the sense that if $v_n \overset{o}{\to} v$ and $v_n \overset{o}{\to} u$ then $u = v$. Moreover,

(i) if (v_n) is increasing, then $v_n \overset{o}{\to} v$ if and only if $v_n \uparrow v$

(ii) if (v_n) is decreasing, then $v_n \overset{o}{\to} v$ if and only if $v_n \downarrow v$

Proof.

Let $|v_n - v| \leq d_n \downarrow 0$, $|v_n - u| \leq b_n \downarrow 0$. We have

$|v - u| \leq |v - v_n + v_n - u| \leq |v - v_n| + |v_n - u| \leq d_n + b_n$

By Exercsie 1.1.19, we have

$|v - u| \leq d_n + b_n \downarrow 0$

Hence, we have $v = u$. □

EVENTUAL ORDER CONTRACTING

EVENTUAL ORDER CONTRACTING

Definition

Let S be a self-map on a subset V of a Riesz space E. We call S **eventual order contracting** on V if, there exists a order continuous linear operator $K : E \to E$ such that for any $v, w \in V$,

$$|Sv - Sw| \leq K|v - w| \quad \text{and } K^n|v - w| \xrightarrow{o} 0$$

DEFINITION *

Definition

Let S be a eventual order contraction on V subset of a Riesz space E. We say S is **bounded by** K if $K : E \to E$ is a order continuous linear operator such that for any $v, w \in V$,

$$|Sv - Sw| \leq K|v - w| \quad \text{and } K^n|v - w| \xrightarrow{o} 0$$

LEMMA *

If S is a eventual order contraction on V a subset of Riesz space E bounded by K, then, S^m is also eventual order contraction on V and bounded by K^m.

Proof.
We have, for any $v, w \in V$,

$$|S^m v - S^m w| = |S(S^{m-1}v) - S(S^{m-1}w)| \leq K|S^{m-1}v - S^{m-1}w|$$

Iterating this, we get

$$|S^m v - S^m w| \leq K^m|v - w| \text{ and } (K^m)^n|v - w| \overset{o}{\to} 0$$

Since composition of linear order continuous map is also linear order continuous, this shows the claim. □

EXERCISE 1.1.29 REVISIT

Let S be an order preserving self-map on a subset V of a Riesz space E. Suppose there exists a order continuous linear operator $K\colon E \to E$ such that $|Sv - Sw| \leq K|v - w|$ for all $v, w \in V$. Prove that S is order continuous on V.

Proof.

Fix $(v_n) \subset V$ with $v_n \uparrow v \in V$.We have

$0 \leq Sv - Sv_n \leq K|v - v_n| \leq Kd_n \downarrow K0 = 0$

Hence $Sv - Sv_n \downarrow 0$. By Lemma 1.1.12 , this gives $Sv_n \uparrow Sv$. $\square$

TARSKI-KANTOROVICH THEOREM REVISIT

Theorem

If S is an order continuous self-map on V and V is σ-chain complete, then S has a fixed point in V.

THEOREM 1.3.2

If V is a σ-chain complete subset of a Riesz space E and S is order preserving and eventually order contracting on V, then S has a unique fixed point $\bar{v} \in V$ and

$$S^n v \xrightarrow{o} \bar{v} \quad \text{for any } v \in E.$$

Proof.

S is order continuous (by Exercise 1.1.29) on a σ-chain complete set V. TK implies S has a fixed point $\bar{v} \in V$. From proposition *, we have,

$$|S^n v - \bar{v}| = |S^n v - S^n \bar{v}| \leq K^n |v - \bar{v}| \leq d_n \downarrow 0$$

Uniqueness is from the uniqueness of the order limits. □

EVENTUAL ORDER CONTRACTING ADP

Definition

Let E be a Riesz space and let $(V, \mathbb{T})$ be a ADP for some $V \subset E$. We call $(V, \mathbb{T})$ **eventually order contracting** if T_σ is eventually order contracting on V for all $\sigma \in \Sigma$.

THEOREM 1.2.14 REVISIT

Theorem

Let $(V, \mathbb{T})$ be regular, well-posed and order continuous. If V is σ-chain complete, then

1. *the fundamental ADP optimality properties hold and*
2. *VFI, OPI and HPI all converge.*

THEOREM 1.3.4

If V is a σ-chain complete subset of a Riesz space E and $(V, \mathbb{T})$ is regular and eventually order contracting, then

1. $(V, \mathbb{T})$ is well-posed,
2. the fundamental ADP optimality properties on page hold, and
3. VFI, OPI and HPI all converge.

Proof.

Well-posed from Theorem 1.3.2.

Order-continuity from Exercise 1.1.29.

Then, ADP optimality property and convergence from Theorem 1.2.14.

$\square$

AFFINE ADP

AFFINE ADP

Definition

Let $(E, \mathbb{T})$ be an ADP where E is an Archemidean Riesz space. We call $(E, \mathbb{T})$ **affine** if each $T_\sigma \in \mathbb{T}$ has the form

$$T_\sigma v = r_\sigma + K_\sigma v \qquad (v \in E)$$

for some $r_\sigma \in E$ and order continuous linear operator $K_\sigma \colon E \to E$.

THEOREM 1.3.3

Let S be a self-map on σ-Dedekind complete Archemidean Riesz space E. If

1. there exists a $h \in E$ and an order continuous linear operator K on E such that $Sv = h + Kv$ for all $v \in E$, and
2. there is an $e \in E$ and $\rho \in [0, 1)$ with $|h| \leq e$ and $Ke \leq \rho e$,

then S has a unique fixed point $\bar{v}$ in the order interval $V := [-e/(1-\rho), e/(1-\rho)]$ and, moreover, $S^n v \xrightarrow{o} \bar{v} \quad$ for any $v \in V$.

Proof.

1. Show S is eventual order contracting on V
2. Use Theorem 1.3.2

□

Proof.

First, we show that S is a self-map on V

$$|Sv| \underbrace{\leq |h| + |Kv|}_{\Delta ineq} \quad \underbrace{\leq |h| + K|v|}_{o.c \Rightarrow o.p \Rightarrow positive} \quad \leq e + K\frac{e}{1-\rho} \leq e + \rho\frac{e}{1-\rho} = \frac{e}{1-\rho}.$$

In Riesz space, we have for any $u, v \in E$ if $u \in E_+$, then $|v| \leq u \Rightarrow v \in [-u, u]$.

This shows $Sv \in V$, hence S is a self-map □

Proof.

Fix $v, w \in V$, we have,

$$|Sv - Sw| = |h + Kv - h - Kw| = \underbrace{|Kv - Kw| = |K(v - w)|}_{\textit{linear}} \underbrace{\leq K|v - w|}_{\textit{positive}}$$

Moreover,

$$K^n|v - w| \leq K^n \frac{2e}{1-\rho} \leq \underbrace{\rho^n \frac{2e}{1-\rho} \downarrow 0}_{\rho \in [0,1)}$$

Hence, S is eventually order contracting on V. □

Proof.

By Lemma 1.1.7, order interval V subset of σ-Dedekind complete space E is σ-chain complete.

Use Theorem 1.3.2, we get the result. $\square$

THEOREM 1.3.5

Let E be σ-Dedekind complete and let $(E, \mathbb{T})$ be an affine ADP with policy set Σ. If $(E, \mathbb{T})$ is regular and

$$\exists\, e \in E \text{ and } \rho \in [0,1) \text{ with } |r_\sigma| \leq e \text{ and } K_\sigma e \leq \rho e \text{ for all } \sigma \in \Sigma,$$

Let order interval $V := [-e/(1-\rho), e/(1-\rho)]$ then for the second ADP $(V, \mathbb{T})$

1. $(V, \mathbb{T})$ is well-posed,
2. the fundamental ADP optimality properties hold, and
3. VFI, OPI and HPI all converge.

PROOF OF THEOREM 1.3.5

Proof.

- V is σ-chain complete from Lemma 1.1.7.
- $(E, \mathbb{T})$ regular implies $(V, \mathbb{T})$ regular
- $(V, \mathbb{T})$ is eventual order contracting from Theorem 1.3.3.
- Use Theorem 1.3.4, we get all the results

$\square$