DP2 READING GROUP

Longye Tian

September 2024

Available at `https://github.com/longye-tian`

Definition

Let V be a poset.

A sequence $(v_n)_{n\geq 1}$ is called **increasing** if $v_n \precsim v_{n+1}$ for all $n \in \mathbb{N}$.

We write

$$v_n \uparrow v = \bigvee_n v_n$$

A sequence $(v_n)_{n\geq 1}$ is called **decreasing** if $v_{n+1} \precsim v_n$ for all $n \in \mathbb{N}$.

We write

$$v_n \downarrow v = \bigwedge_n v_n$$

EXERCISE 1.1.11

Let $(u_n), (v_n)$ be sequence in V. Prove

1) $v_n \uparrow v, v_n \precsim u_n \precsim v \implies \bigvee_n u_n = v$

Proof.

Define (w_n) be a sequence in V such that $w_n = v$ for all $n \in \mathbb{N}$.

Then the claim follows from Exercise 1.1.6. □

2) $v_n \downarrow v, v \precsim u_n \precsim v_n \implies \bigwedge_n u_n = v$

CLOSED UNDER POINTWISE SUPREMA

Definition

We say that V is **closed under pointwise suprema** if, for every increasing sequence $(v_n) \subset V$ that is bounded above, the pointwise supremum

$$s(x) = \sup_n v_n(x)$$

is an element of V.

Lemma

Let $V \subset \mathbb{R}^X$ be closed under pointwise suprema, let (v_n) be a sequence in V and let $v \in V$. Then,

$$v_n(x) \uparrow v(x) \text{ in } \mathbb{R} \text{ for all } x \in X \iff v_n \uparrow v$$

Proof.

$(\implies)$

$v_n(x) \uparrow v(x) \implies v_n(x) \le v_{n+1}(x),\ \forall x \in X \implies v_n \le v_{n+1}$

By Exercise 1.1.7., we know $\bigvee_n v_n = v$.

$(\impliedby)$

$v_n \uparrow v \implies v_n \le v_{n+1} \implies v_n(x) \le v_{n+1}(x)\ \forall x \in X$

Let s be the pointwise supremum, by closed under pointwise suprema, $s \in V$.

We have $v_n \le s \le v \implies s = v$ $\square$

MAPPING OVER POSETS - ORDER PRESERVING MAPS

Definition

A self-map S on poset $V = (V, \precsim)$ is called **order preserving** on V if

$$v, w \in V, v \precsim w \implies Sv \precsim Sw$$

EXERCISE 1.1.12

Let $(V, \precsim)$ be a poset and let $\mathcal{S}$ be the set of all order preserving self-map on V.Let $\precsim$ be the pointwise order on $\mathcal{S}$, i.e., $Sv \precsim Tv \implies S \precsim T$

Prove

1) $S \in \mathcal{S} \implies S^k \in \mathcal{S}\, \forall k \in \mathbb{N}$

Proof.

Let $v, w \in V, v \precsim w$.

$S \in \mathcal{S} \implies Sv \precsim Sw \implies S(Sv) \precsim S(Sw) \implies S^2 v \precsim S^2 w$. Then by induction. $\square$

EXERCISE 1.1.12 CONTINUE

2) $S, T \in \mathcal{S}, S \lesssim T \implies S^k \lesssim T^k$ for all $k \in \mathbb{N}$

Proof.

From 1), $S^k, T^k \in \mathcal{S}\ \forall k \in \mathbb{N}$.

We have,

$$\underbrace{S \lesssim T \implies Sv \lesssim Tv}_{\text{definition}} \implies \underbrace{S(Sv) \lesssim S(Tv)}_{S \in \mathcal{S}} \underbrace{\lesssim T(Tv)}_{S \lesssim T} \implies S^2 \lesssim T^2$$

Then by induction. $\square$

ORDER STABILITY

Definition

Let V be a poset and S be a self-map on V and has a unique fixed point in $\bar{v} \in V$. We call S

- **upward stable** on V if $v \in V, v \precsim Sv \implies v \precsim \bar{v}$
- **strongly upward stable** on V if $v \in V, v \precsim Sv \implies S^n v \uparrow \bar{v}$
- **downward stable** on V if $v \in V, Sv \precsim v \implies \bar{v} \precsim v$
- **strongly upward stable** on V if $v \in V, Sv \precsim v \implies S^n v \downarrow \bar{v}$
- **order stable** on V if S is both upward and downward stable.
- **strongly order stable** on V if S is both strongly upward and strongly downward stable.

EXERCISE 1.1.13

Consider the self-map on $\mathbb{R}^k$ defined by $Sv = r + Av$ for some $r \in \mathbb{R}^k$ and A is a bounded positive linear operator with $\rho(A) < 1$. Prove that S is strongly order stable on $\mathbb{R}^k$

Proof.

First, we find the unique fixed point of S, $v = Sv = r + Av \implies \underbrace{v = (I - A)^{-1} r}_{NSL}$

Second, we show S is strongly upward stable. Let $w \in \mathbb{R}^k, w \leq Sw$

$$w \leq Sw = r + Aw \underbrace{\implies Aw \leq Ar + A^2 W}_{A \text{ is positive}} \implies Sw = r + Aw \leq r + Ar + A^2 W = S^2 w$$

We have $S^n w \leq S^{n+1} w$, and we have $S^n w \to v$ hence $S^n w \uparrow v$. $\square$

ORDERED VECTOR SPACE

Definition

Let E be a vector space and let $\leq$ be a partial order on E. We call $(E, \leq)$ an **ordered vector space** if the order is preserved under addition and nonnegative scalar multiplication that is if

1. $u \leq v, \alpha \in \mathbb{R}$ with $\alpha \geq 0 \implies \alpha u \leq \alpha v$
2. $u \leq v \implies u + b \leq v + b$ for any $b \in E$

POSITIVE CONE

Definition

The **positive cone** of E, typically denoted as E_+ is all $v \in E$ with $v \geq 0$.

EXERCISE 1.1.16

Exercise

Let S be the vector space of all $n \times n$ matrices (with addition and scalar multiplication defined in the obvious way) and let N be the negative semidefinite matrices in S. As in §2.1.4.3, we impose the Loewner partial order, writing $A \geq B$ when $A - B \in N$. Show that $(S, \geq)$ is an ordered vector space.

Proof.

To show that $(S, \geq)$ is an ordered vector space, we need to prove that the Loewner partial order preserves addition and nonnegative scalar multiplication.

(i) For $\alpha \geq 0$ and $A \geq B$: $\alpha(A - B) \in N$ (since N is closed under nonnegative scalar multiplication) Thus, $\alpha A - \alpha B \in N$, which means $\alpha A \geq \alpha B$

(ii) For $A \geq B$ and $C \in S$: $(A + C) - (B + C) = A - B \in N$ Thus, $A + C \geq B + C$

Therefore, $(S, \geq)$ is an ordered vector space. □

EXERCISE 1.1.17

Exercise

Let X be any nonempty set and let $\mathbb{R}^X$ be the vector space of real-valued functions on X. Let $\leq$ be the pointwise partial order. Show that $(\mathbb{R}^X, \leq)$ is an ordered vector space.

Proof.

To show that $(\mathbb{R}^X, \leq)$ is an ordered vector space:

(i) For $\alpha \geq 0$ and $f \leq g$: $(\alpha f)(x) = \alpha(f(x)) \leq \alpha(g(x)) = (\alpha g)(x)$ for all $x \in X$

(ii) For $f \leq g$ and $h \in \mathbb{R}^X$: $(f + h)(x) = f(x) + h(x) \leq g(x) + h(x) = (g + h)(x)$ for all $x \in X$

Therefore, $(\mathbb{R}^X, \leq)$ is an ordered vector space. □

EXERCISE 1.1.18

Exercise

Let $(E, \leq)$ be an ordered vector space and fix $u, v, w \in E$. Prove that

(i) $u \geq 0$ *and* $v \geq 0$ *implies* $u + v \geq 0$,

(ii) $u \geq v$ *implies* $-v \geq -u$,

(iii) $(u \vee v) + w = (u + w) \vee (v + w)$, *and*

(iv) $\alpha(u \vee v) = (\alpha u) \vee (\alpha v)$ *whenever* $\alpha \geq 0$.

Proof.

(i) $u \geq 0$ and $v \geq 0$ implies $u + v \geq 0$: By the properties of ordered vector spaces, $u \geq 0$ implies $u + v \geq 0 + v = v \geq 0$ (ii) $u \geq v$ implies $-v \geq -u$: $u \geq v \implies u - u \geq v - u \implies 0 \geq v - u \implies 0 - v \geq v - u - v \implies -v \geq -u$ $\square$

EXERCISE 1.1.18 CONTINUE

Proof.

(iii) $(u \vee v) + w = (u + w) \vee (v + w)$: Let $z = (u \vee v) + w$. Then $z \geq u + w$ and $z \geq v + w$. Also, for any $y \geq u + w$ and $y \geq v + w$, we have $y - w \geq u \vee v$, so $y \geq z$. Thus, $z = (u + w) \vee (v + w)$

(iv) $\alpha(u \vee v) = (\alpha u) \vee (\alpha v)$ for $\alpha \geq 0$:

First, we show $\alpha(u \vee v) \geq (\alpha u) \vee (\alpha v)$ for $\alpha \geq 0$.

We have, $u \vee v \geq u, u \vee v \geq v \implies \alpha(u \vee v) \geq \alpha u, \alpha(u \vee v) \geq \alpha v \implies \alpha(u \vee v) \geq (\alpha u) \vee (\alpha v)$.

Second, we show $\alpha(u \vee v) \leq (\alpha u) \vee (\alpha v)$ for $\alpha \geq 0$:

We have

$(\alpha u) \vee (\alpha v) \geq \alpha u; (\alpha u) \vee (\alpha v) \geq \alpha v \implies \frac{1}{\alpha}[(\alpha u) \vee (\alpha v)] \geq u, \frac{1}{\alpha}[(\alpha u) \vee (\alpha v)] \geq v \implies \frac{1}{\alpha}[(\alpha u) \vee (\alpha v)] \geq u \vee v \implies (\alpha u) \vee (\alpha v) \geq \alpha(u \vee v)$ □

EXERCISE 1.1.19

Exercise

Prove that

(i) *if* $u_n \uparrow 0$ *and* $v_n \uparrow 0$, *then* $u_n + v_n \uparrow 0$

Proof.

$u_n \leq u_{n+1}, v_n \leq v_{n+1} \implies u_n + v_n \leq u_{n+1} + v_n \leq u_{n+1} + v_{n+1} \implies (u_n + v_n)$ *is increasing.*

Then, we show $\bigvee_n (u_n + v_n) \leq 0$*:*

$u_n \leq 0 \implies u_n + v_n \leq v_n \implies \bigvee_n (u_n + v_n) \leq \bigvee_n v_n = 0$

Now, we show that 0 *is the least upper bound of* $(u_n + v_n)$. *Let* w *be an upper bound of* $u_n + v_n$, *i.e.,* $u_n + v_n \leq w \, \forall n \implies (\bigvee_n w_n) + v_n \leq w \, \forall n \implies \bigvee_n u_n + \bigvee_n v_n \leq w \implies 0 \leq w.$ ☐

Exercise

(ii) *if $u_n \uparrow u$ and $b \in E$, then $u_n + b \uparrow u + b$.*

Proof.

$u_n \uparrow u \implies u_n \leq u_{n+1} \implies u_n + b \leq u_{n+1} + b \implies (u_n + b)$ *is increasing*

Now we prove $u + b$ is the least upper bound of $(u_n + b)$:

Let w be any upper bound of $(u_n + b)$, then we have

$u_n + b \leq w \; \forall n \implies (\bigvee_n u_n) + b \leq w \implies u + b \leq w$ □

LEMMA 1.1.12

Let $(u_n), (v_n)$ be sequences in the ordered vector space E and let α, β be nonnegative constants. Then

1. $u_n \uparrow u, v_n \uparrow v \implies \alpha u_n + \beta v_n \uparrow \alpha u + \beta v$
2. $u_n \uparrow u \implies -u_n \downarrow -u$

Proof.

From Exercise 1.1.19 $\square$

PARTIAL ORDER INDUCED BY CONE

Definition

Let E be any vector space. A nonempty subset C is called a **cone** if

1. C is convex
2. $x \in C, -x \in C \implies x = 0$
3. $x \in C, \alpha \geq 0 \implies \alpha x \in C$

A partial order is introduced into a vector space E by first choosing a pointed convex cone C on E and stating that

$$u \leq v \iff v - u \in C$$

EXERCISE 1.1.20

Exercise

With $\leq$ defined as above, show that $(E, \leq)$ is an ordered vector space and that C is the positive cone of $(E, \leq)$.

Proof.

Part 1. Let $b \in E$ and $\alpha \in \mathbb{R}_+$.

1. $u \leq v \implies v - u \in C \implies (v+b)-(u-b) = v-u \in C \implies u+b \leq v+b$
2. $u \leq v \implies v - u \in C \implies \alpha(v-u) \in C \implies \alpha v - \alpha u \in C \implies \alpha u \leq \alpha v$

Part 2: WTS $v \in C \implies v \geq 0$:

$v \in C \implies v - 0 \in C \implies 0 \leq v$ $\square$

CLOSED PARTIAL ORDER

Definition

A partial order $\precsim$ on topological space V is called **closed** if, given any two nets $(u_\alpha)_{\alpha\in\Lambda}$ and $(v_\alpha)_{\alpha\in\Lambda}$ contained in V,

$$u_\alpha \to u, v_\alpha \to v, u_\alpha \precsim v_\alpha \ \forall \alpha \in \Lambda \implies u \precsim v$$

EXERCISE 1.1.21

Exercise

Continuing the previous exercise, show that if E is a normed linear space and C is closed in E, then $\leq$ is a closed partial order (see page 31).

Proof.

Let $(u_n), (v_n)$ be two sequences in E such that $u_n \to u, v_n \to v$ and $u_n \leq v_n$ for all n.

Then, we have $v_n - u_n \in C = E_+ \ \forall n$, we want to show $v - u \in E_+$.

Since C is closed,

$\lim_{n\to\infty}(v_n - u_n) = \lim_{n\to\infty} v_n - \lim_{n\to\infty} u_n = v - u \in C$. □

EXERCISE 1.1.22

Exercise

Show conversely that if $(E, \leq)$ is an ordered vector space, then the positive cone in E is a (pointed convex) cone.

Proof.

We first show that E_+ is convex. Let $\lambda \in [0,1]$, $u, v \in E_+$, we have

$u, v \in E_+ \implies u, v \geq 0 \implies \lambda u, (1-\lambda)v \geq 0 \implies \lambda u + (1-\lambda)v \geq 0$

Secondly, we show $x \in E_+, -x \in E_+ \implies x = 0$:

$x \in E_+ \implies x \geq 0, -x \in E_+ \implies x \leq 0 \implies x = 0$

Last, we show that $x \in E_+, \alpha \in \mathbb{R}_+ \implies \alpha x \in E_+$.

$x \in E_+ \implies x \geq 0 \implies \alpha x \geq 0 \implies \alpha x \in E_+$ □

POSITIVE OPERATOR

Definition

A linear operator T mapping ordered vector space E to itself is called **positive** if T is invariant on the positive cone. That is, if

$u \in E, u \geq 0 \implies Tu \geq 0$

EXERCISE 1.1.23

Exercise

Prove that a linear operator mapping E to itself is positive if and only if it is order preserving.

Proof.

Let $u, v \in E$ and $u \geq v$

$(\implies)$

$$u \geq v \implies u - v \geq 0 \implies T(u-v) \geq 0 \underbrace{\implies Tu - Tv \geq 0}_{\text{linear}} \implies Tu \geq Tv.$$

$(\impliedby)$

$$u \geq v \implies Tu \geq Tv \implies Tu - Tv \geq 0 \implies T(u-v) \geq 0$$ $\square$

EXERCISE 1.1.24

Exercise

Show that, for a linear operator $A : E \to E$, TFAE

(i) *A is order continuous on E.*

(ii) *$Av_n \downarrow Av$ whenever $(v_n) \subset E$ and $v_n \downarrow v \in E$.*

(iii) *$Av_n \downarrow 0$ whenever $(v_n) \subset E$ and $v_n \downarrow 0$.*

Proof.

$(i) \Rightarrow (ii)$ $v_n \downarrow v \in E \Rightarrow -v_n \uparrow -v \in E$ by [L1.1.12]. Then,

$-Av_n = A(-v_n) \uparrow A(-v) = -Av \implies Av_n \downarrow Av$

$(ii) \Rightarrow (iii)$ since $A0 = 0$, we get (iii).

$(iii) \Rightarrow (i)$ $v_n \downarrow 0 \implies -v_n + v \uparrow v$. Let $u_n = -v_n + v$. Then

$Av_n \downarrow 0 \implies -Av_n + Av \uparrow Av \implies Au_n \uparrow Av$ $\square$

ORDER CONTINUITY

Definition

We call a map S from poset V to poset W **order continuous** on V if

$$v_n \uparrow v \implies Sv_n \uparrow Sv$$

In other words, if $(v_n) \subset V$ with $v_n \uparrow v \in V$, then $Sv = \bigvee_n Sv_n \in W$

Definition

A self-map S on a convex subset C of ordered vector space $E := (E, \leq)$ is **convex** on C if

$$S(\lambda v + (1-\lambda)v') \leq \lambda S v + (1-\lambda) S v' \text{ whenever } v \leq v' \in C \text{ and } 0 \leq \lambda \leq 1$$

The self-map S is called **concave** on C if

$$\lambda S v + (1-\lambda) S v' \leq S(\lambda v + (1-\lambda)v') \text{ whenever } v \leq v' \in C \text{ and } 0 \leq \lambda \leq 1$$